# A geometric approach to calculating the limit set of eigenvalues for banded Toeplitz matrices

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**Abstract**

This thesis is about the limiting eigenvalue distribution of $n \times n$ Toeplitz matrices as $n \to \infty$. The two classical questions we want to answer are: what is the limit set of the eigenvalues, and what is the limiting distribution of the eigenvalues. Our main result is a new approach to calculate the limit set $\Lambda(b)$ for a Laurent polynomial $b$, i.e. for banded Toeplitz matrices. The approach is geometrical and based on the formula $\Lambda(b) = \cap_{\rho \in (0,\infty)} \operatorname{sp} T(b_\rho)$. We show that the full intersection can be approximated by the intersection for a finite number of $\rho$'s, and that $\operatorname{sp} T(b_\rho)$ can be well approximated by a polygon. This results in an algorithm whose output we show converge to $\Lambda(b)$ in the Hausdorff metric. We implement the algorithm in python and test it. It performs on par to and better in some cases than existing algorithms. We argue, but do not prove, that the average time complexity of the algorithm is $O(n^2 + mn)$, where $n$ is the number of $\rho$'s and $m$ is the number of vertices for the polygons approximating $\operatorname{sp} T(b_\rho)$. Further, we present the theory for Toeplitz matrices for symbols in the Wiener algebra, with an emphasis on the limiting eigenvalue distribution. In particular, we derive the limiting measure and limit set for hermitian Toeplitz matrices and banded Toeplitz matrices.

# Preface

This is my Master's thesis for the program engineering mathematics that I have written during the fall of 2022. First and foremost I would like to thank my supervisors, Jacob Stordal Christiansen and Mikael Persson Sundqvist. They have given me great support, and have always been excited and had time to discuss the current problem.

I would also like to thank my friends and family for their encouragement and support during my time at LTH.

# Contents

# Glossary

| **Term** | **Definition** |
|---|---|
| $A$ | The Lebesgue measure on $\mathbb{C}$, when nothing else stated. |
| $b$ | Typically a Laurent polynomial. |
| Banded Toeplitz matrix | A Toeplitz matrix whose symbol is a Laurent polynomial. |
| $b_\rho$ | A Laurent polynomial defined by $b_\rho(t) = b(\rho t)$. |
| $\mathcal{B}(X)$ | The set of all bounded linear operators on the Banach space $X$. |
| $\operatorname{conv} A$ | The convex hull of a set $A$, that typically is a subset of $\mathbb{C}$. |
| $\Delta$ | The distributional Laplacian. |
| $d_H$ | The Hausdorff metric. |
| $\lvert E\rvert$ | The Lebesgue measure of $E$, assuming $E$ is Borel and $E \subset \mathbb{R}$. |
| $H(a)$ | The infinite Hankel matrix generated by the symbol $a$. |
| $\operatorname{Ind} A$ | The Fredholm index of an Fredholm operator $A$. |
| $\Lambda(b)$ | The limit set of eigenvalues for a banded Toeplitz matrix. |
| Laurent polynomial | $\sum_{n=-r}^{s} a_n x^n$ i.e. a regular polynomial but negative powers can occur. |
| $l^p$ | The Banach space of all sequences $(x_n)_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} \lvert x_n\rvert^p < \infty$. |
| $Q(z, \lambda)$ | Polynomial defined by a Laurent polynomial: $Q(z, \lambda) = z^r(b(z) - \lambda)$. |
| $\operatorname{sp} A$ | The spectrum of $A$. |
| $\operatorname{sp}_{\text{ess}} A$ | The essential spectrum of $A$. |
| $\operatorname{sp} T(b_\rho^D)$ | A polygon approximation of $\operatorname{sp} T(b_\rho)$. |
| $\mathbb{T}$ | The unit circle. |
| $T(a)$ | The infinite Toeplitz matrix generated by the symbol $a$. |
| $T_n(a)$ | The finite $n \times n$ Toeplitz matrix generated by the symbol $a$. |
| $W$ | The Wiener algebra, see section 2.1. |
| $(X)_\epsilon$ | The $\epsilon$-fattening of $X$: $(X)_\epsilon = \cup_{x \in X} \{z \in \mathbb{C} : \lvert z - x\rvert \leq \epsilon\}$. |

# Chapter 1

# Introduction

A finite Toeplitz matrix is an $n \times n$ matrix on the form $(a_{j-k})_{j,k=0}^{n-1}$, i.e.

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \dots & a_{-n+2} \\ a_2 & a_1 & a_0 & \dots & \dots \\ \dots & \dots & \dots & \dots & a_{-1} \\ a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \end{pmatrix}. \tag{1.1}$$

The matrix is completely determined by the complex doubly infinite sequence $(a_j)_{j=-\infty}^{\infty}$ that we in this paper assume to be absolutely summable. Let

$$a(t) := \sum_{j=-\infty}^{\infty} a_j t^j$$

be the function defined on the unit circle that has $(a_j)_{j=-\infty}^{\infty}$ as its Fourier coefficients. We will also study the infinite Toeplitz matrix $T(a)$, which is defined as $(a_{j-k})_{j,k=0}^{\infty}$. The function $a$ is called the *symbol* for the matrix $T(a)$. We denote the matrix in (1.1) by $T_n(a)$.

Toeplitz matrices arise in several different applications, such as stochastic processes and time series analysis, signal processing, numerical methods for differential and integral equations, image processing, and quantum mechanics [6].

A crucial part of understanding linear operators is to understand the spectrum. In the case $T_n(a)$, we wish to understand how the eigenvalues are distributed. For $n$ of order up to 100 the spectrum of $T_n(a)$ can be calculated numerically using standard methods, but in some applications such as statistical physics, $n$ can be of order $10^8 - 10^{12}$ [6], which is way beyond what numerical methods can handle. Hence, theory is required to know how the eigenvalues of $T_n(a)$ are distributed. There are two main questions that we want to answer in order to understand the spectrum of $T_n(a)$, how the eigenvalues are distributed on average, and where they accumulate. We proceed by specifying further what this means.

Denote by $\lambda_j^{(n)},\ j = 1, 2, \dots, n$ the eigenvalues of $T_n(a)$. Define the measures

$$\mu_n(E) := \frac{\#\{j : \lambda_j^{(n)} \in E\}}{n}.$$

They describe what fraction of the eigenvalues of $T_n(a)$ are located in $E$. The first main question we want to answer is if there exists a limiting measure that $\mu_n$ converge to as $n \to \infty$, and if it exists, we want to compute the limiting measure. For the second main question, define the strong and weak limit set

$$\Lambda_s = \{\lambda \in \mathbb{C} : \lambda_{j_n}^{(n)} \to \lambda \text{ for some sequence } j_n \to \infty\},$$
$$\Lambda_w = \{\lambda \in \mathbb{C} : \lambda_{j_k}^{(n_k)} \to \lambda \text{ for some sequences } j_k, n_k \to \infty\}.$$

The second question is: how are these limit sets structured?

The general answer to the two main questions remains to be discovered, but a lot of research has been done, and in some cases a lot is known. The case when $a$ is real-valued, which corresponds to $T_n(a)$ being hermitian is well understood, the theory for it is presented in Section 2.7.

The case of non-hermitian matrices is more tricky, but for the case of banded Toeplitz matrices, i.e. when only a finite number of Fourier coefficients of $a$ are non-zero, the two questions can be answered. The pioneering paper for the eigenvalue distribution of banded Toeplitz matrices is due to Schmidt and Spitzer [18]. It was published in 1960, and they showed that $\Lambda_s$ and $\Lambda_w$ are equal and consists of a finite union of analytic arcs, equal to

$$\bigcap_{\rho \in (0,\infty)} \operatorname{sp} T(a_\rho), \tag{1.2}$$

where $a_\rho(t) := a(\rho t)$, with $a$ being extended to $\mathbb{C} \setminus \{0\}$ in the natural way. Hirschman improved on their result in 1967 and showed that there exists a limiting measure with support on the limit set [15]. Widom later simplified their proofs in [23] and [24]. We present the theory for banded matrices in Section 2.8.

In Chapter 3 we present a novel approach to calculating the limit set in the banded case. Our approach is geometric and based on (1.2). First, we show that only a countable number of $\rho$'s in (1.2) actually are needed, and that the full intersection can be well approximated by a finite number of $\rho$'s. Theory presented in Chapter 2 provides a geometric description of $\operatorname{sp} T(a_\rho)$, so our approach consists of approximating $\operatorname{sp} T(a_\rho)$ with polygons which we intersect for a finite number of $\rho$'s. Further, we show that polygon approximation is justified, and that our method converges to the limit set in the Hausdorff metric. Also, we have implemented the algorithm in python, which outputs a polygon which approximates the limit set. Several examples of limit set calculations are provided.

The aim of this thesis is twofold. The first part is to present the core parts of the existing theory for asymptotic eigenvalue behavior for Toeplitz matrices and to make the proofs easier to read than those in the monographic literature. We briefly summarize what is consider to be the “core parts”: What do we know about general Toeplitz matrices, how does the spectrum look for infinite matrices, what can we say in general about the limiting eigenvalue distribution, and how does the eigenvalues of the finite matrices relate to the spectrum of the infinite matrix? What is known in the special cases of Hermitian and banded Toeplitz matrices is also considered a core part. The second part of the over all aim is to make an attempt at contributing with new knowledge about the limiting eigenvalue distribution. The second part of the aim turned into the major result of this thesis, which is the geometrically based algorithm for calculating the limit set of eigenvalues for banded Toeplitz matrices.

The workflow consisted of first reading up on and understanding the theory, and writing it down. After that, practical examples of how $\operatorname{sp} T(a_\rho)$ for a banded Toeplitz matrix behaves when $\rho$ varies was considered. This turned into the idea of calculating the limit set geometrically with intersections. Further, this investigation is motivated by the recent article [7], where it is stated that a good algorithm for calculating the limit set for banded matrices is missing.

The outline of this thesis is as follows: In Chapter 2, theory about Toeplitz matrices with an emphasis on spectral behavior is presented. The material in Chapter 2 is condensed from [5] and [8], but more elaborate explanations have been added. Chapter 3 is about computing the limit set for banded Toeplitz matrices. An existing algorithm from literature is presented, and then our geometric approach is introduced. The chapter ends with a discussion on how to practically implement the algorithm along with some illustrating examples.

# Chapter 2

# Theory

## 2.1 Introduction

A general infinite *Toeplitz* matrix has the form

$$(a_{j-k})_{j,k=0}^{\infty} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{2.1}$$

The defining property of Toeplitz matrices is that they are constant along the diagonals parallel to the main diagonal. One can see that the matrix in (2.1) is completely determined by the doubly infinite sequence

$$(a_k)_{k=-\infty}^{\infty} = (\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots). \tag{2.2}$$

All the $a_k$s are assumed to be complex numbers. We will regard (2.1) as an operator on $l^p$ for $1 \leq p \leq \infty$ defined in the natural way. The matrix in (2.1) is called banded if and only if (2.2) contains a finite number of non-zero elements. Finite Toeplitz matrices of the form (1.1) can be viewed as truncations of (2.1).

Another important type of matrices are *Hankel* matrices that have the form

$$(a_{j+k+1})_{j,k=0}^{\infty} = \begin{pmatrix} a_1 & a_2 & a_3 & \dots \\ a_2 & a_3 & \dots & \dots \\ a_3 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \tag{2.3}$$

We note that similarly to (2.1) it is determined completely by the sequence in (2.2).

We will work a lot with a class of functions called the *Wiener algebra* $W$ that consists of all the functions $a : \mathbb{T} \to \mathbb{C}$ with absolutely convergent Fourier series, i.e. all functions that can we written as

$$a(t) = \sum_{n=-\infty}^{\infty} a_n t^n \quad \text{where} \quad ||a||_W := \sum_{-\infty}^{\infty} |a_n| < \infty. \tag{2.4}$$

By $\mathbb{T}$ we mean the complex unit circle. The Fourier coefficients $a_n$ of $a$ can be calculated using the formula

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-in\theta} \, d\theta. \tag{2.5}$$

There is a close connection between a function $a$ on the form given in (2.4) and the Toeplitz matrix and Hankel matrix given by (2.1) and (2.3) respectively. If we let $a$ have the Fourier coefficients in (2.2), we will get a very rich connection between the properties of the matrix and the function. This is one of the main points that make Toeplitz matrices so interesting to study. We say that $a$ is the *symbol* for (2.1) if (2.2) are the Fourier coefficients for $a$. We will henceforth use the notation $T(a)$ for the infinite Toeplitz matrix having $a$ as symbol and analogously $H(a)$ for the infinite Hankel matrix having $a$ as symbol.

**Example 2.1.1.** Define for all $n \in \mathbb{Z}$, $\chi_n(t) = t^n$, $t \in \mathbb{T}$. We see that $T(\chi_n)$ corresponds to the one-shift operator on $l^p$.

$$T(\chi_1) = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

so

$$T(\chi_1)(x_0, x_1, x_2, \dots) = (0, x_0, x_1, \dots).$$

We also see that all $H(\chi_n)$ does is set all the $x_i, i \geq n$ to zero, and reverse the first $n$ numbers.

## 2.2 Toeplitz matrices as operators

An important question for linear operators is of course if they are bounded. There is an easy exact classification of when (2.1) is bounded on $l^2$, namely if and only if there is a function $a \in L^\infty$ that have (2.2) as Fourier coefficients. This result was presented in 1911 by Otto Toeplitz in [20], and this is the reason for matrices of type (2.1) being called Toeplitz matrices.

In this thesis however, we will only work with $a \in W$ so we can prove boundedness quite easily.

**Proposition 2.2.1.** *Assume $a \in W$, then $T(a)$ and $H(a)$ induce bounded operators on $l^p$, $1 \leq p \leq \infty$. Additionally we have the bounds*

$$||T(a)||_p \leq ||a||_W, \quad ||H(a)||_p \leq ||a||_W.$$

*Proof.* We will use the representations

$$T(a) = \sum_{n=-\infty}^{\infty} a_n T(\chi_n), \quad H(a) = \sum_{n=1}^{\infty} a_n H(\chi_n).$$

Since $T(\chi_n)$ is nothing more than the shift operator and $H(\chi_n)$ essentially is only a projection onto $\mathbb{C}^n$ we see that $||T(\chi_n)||_p = 1$ for $n \in \mathbb{Z}$ and $||H(\chi_n)||_p = 1$ for $n \geq 1$. Hence, the triangle inequality gives us

$$||T(a)||_p = \left\| \sum_{n=-\infty}^{\infty} a_n T(\chi_n) \right\|_p \leq \sum_{n=-\infty}^{\infty} |a_n| ||T(\chi_n)||_p = \sum_{n=-\infty}^{\infty} |a_n| = ||a||_W.$$

And in the same way we get $\|H(a)\|_p \leq \|a\|_W$. □

There are more useful bounds for the norm of a Toeplitz matrix. The next one we will prove stems from the fact that taking $T(a)x$ almost is the same as multiplying the function that has $x$ as its Fourier coefficients with $a$. Consider $L^2(\mathbb{T})$, with the usual $L^2$ norm. In $L^2$, every function $f$ can be represented by a Fourier series $f(t) = \sum_{n=-\infty}^{\infty} f_n t^n$. There is a subspace $H^2(\mathbb{T})$ of $L^2(\mathbb{T})$ called the *Hardy space* of $L^2$, defined as $\{f \in L^2 : f_n = 0, n < 0\}$. Let $P$ be the orthogonal projection $L^2 \to H^2$. Because of Parseval's formula, we can represent functions in $H^2$ as the vector in $l^2$ having the Fourier coefficients of $H^2$ as elements, in fact the operator $\Phi : H^2 \to l^2,\ f \mapsto (f_n)_{n=0}^{\infty}$ is unitary, since it preserves the scalar products. Now it is not difficult to see that if $a \in W$, then

$$\Phi^{-1} T(a) \Phi : H^2 \to H^2, \text{ operates by } f \mapsto P(af). \tag{2.6}$$

Using (2.6) we can prove the following Lemma.

**Lemma 2.2.2.** *Let $a \in W$. Then $\|T(a)\|_2 \leq \|a\|_\infty$.*

*Proof.* The proof is quite straight forward. We can write $\left\|\Phi^{-1} T(a) \Phi f\right\|_2 = \|P(af)\|_2 \leq \|af\|_2 \leq \|a\|_\infty \|f\|_2$. So $\|T(a)\|_2 = \left\|\Phi^{-1} T(a) \Phi\right\|_2 \leq \|a\|_\infty$ □

We will also show a useful property about Hankel matrices that we will use later.

**Proposition 2.2.3.** *Assume $a \in W$. Then $H(a)$ induces a compact operator on $l^p$, $1 \leq p \leq \infty$.*

*Proof.* We will use a standard technique for proving compactness, namely to show that a sequence of finite rank operators converge to $H(a)$ in operator norm. Assume that $a$ is given on the form (2.4). Let $S_N a = \sum_{n=-N}^{N} a_n t^n$. We see that $H(S_N a)$ of course is of finite rank since it only has $N$ non-zero columns. Also, we see that

$$\|H(a) - H(S_N a)\|_p = \left\| \sum_{n=N+1}^{\infty} a_n H(\chi_n) \right\|_p \leq \sum_{n=N+1}^{\infty} |a_n| \to 0,\ N \to \infty,$$

since $a \in W$. □

We will now introduce some new notation and insights. $W$ is called the Wiener algebra because it is an algebra under pointwise algebraic operations. One can also easily see that $(W, ||\cdot||_W)$ is a Banach space, and that for $a, b \in W$ we have $\|ab\|_W \leq \|a\|_W \|b\|_W$. Hence, $W$ is a Banach algebra. For $a \in W$ we let $\tilde{a} = a(1/t),\ t \in \mathbb{T}$. We directly see that if $a$ is in $W$, so is also $\tilde{a}$, since

$$\tilde{a} = \sum_{n=-\infty}^{\infty} a_{-n} t^n \Rightarrow \|\tilde{a}\|_W = \|a\|_W .$$

We also see that $T(\tilde{a})$ is the transpose of $T(a)$, whilst $H(\tilde{a})$ is completely unrelated to $H(a)$.

One can quite easily prove

$$a, b \in W \ \Rightarrow\ T(ab) = T(a)T(b) + H(a)H(\tilde{b}) \tag{2.7}$$

by comparing the left and right hand side element wise.

There are two subalgebras of $W$, called $W_+$ and $W_-$. $W_+$ consists of all the $a \in W$ that only have non-zero Fourier coefficients for non-negative indices. $W_-$ is defined analogously, but for non-positive indices instead. We also see that for $a$ in $W_+$, $T(a)$ becomes lower-triangular and $H(\tilde{a}) = 0$, but if $a \in W_-$, we instead get $T(a)$ as upper-triangular and $H(a) = 0$. Using these observations along with (2.7) we can derive the following important factorization.

**Proposition 2.2.4.** *Let $a_- \in W_-, b \in W, a_+ \in W_+$. Then*

$$T(a_- b a_+) = T(a_-)T(b)T(a_+).$$

*Proof.* We use that $H(a_-) = H(\tilde{a}_+) = 0$ and (2.7).

$$\begin{aligned} T(a_- b a_+) &= T(a_-)T(ba_+) + H(a_-)H(\tilde{b}\tilde{a}_+) \\ &= T(a_-)T(ba_+) \\ &= T(a_-)(T(b)T(a_+) + H(b)H(\tilde{a}_+)) \\ &= T(a_-)T(b)T(a_+). \end{aligned}$$

□

## 2.3 Wiener–Hopf factorization

Next we will introduce more subsets of $W$ and classify them in theorems 2.3.1–2.3.3. To fully prove them would require results from the theory on commutative Banach algebras, which we will not provide in full detail here. A good reference for commutative Banach algebras and the Wiener algebra can be found in chapter 2 of [12]. After the results we need have been presented we will sketch the main ideas.

Let $GW$ consists of all the invertible elements of $W$ (with respect to multiplication), this is exactly all $a \in W$ such that there is a $b \in W$, fulfilling

$a(t)b(t) = 1$ for all $t \in \mathbb{T}$. One can directly note that functions in $GW$ can't have any zeros. But it is not immediately obvious that all functions in $W$ that are non-zero on the unit circle are in $GW$. But the following theorem originally due to Wiener is in fact true.

**Theorem 2.3.1.** *$GW = \{a \in W : a(t) \neq 0 \; \forall t \in \mathbb{T}\}$.*

The next subset we introduce is $\exp W$ which consists of all the $a \in W$ that have a logarithm, i.e. every $a \in W$ such that there is a $b \in W$ with $a = e^b$. There is a very convenient geometric classification of $\exp W$, but for it we need the concept of *winding number*. Since all $a \in W$ are continuous, $a(t),\ t \in \mathbb{T}$ will trace out a closed curve when $t$ moves in positive direction around the unit circle, further assume that $a : \mathbb{T} \to \mathbb{C} \setminus \{0\}$. The number of times this curve revolts around the origin is the definition of the winding number. A less geometric and equivalent definition is as follows: Every $a \in W$ such that $a : \mathbb{T} \to \mathbb{C} \setminus \{0\}$ can be written on the form $a(e^{i\theta}) = |a(e^{i\theta})|e^{ic(\theta)}$, for a continuous $c : [0, 2\pi) \to \mathbb{R}$. The winding number now becomes

$$\frac{1}{2\pi}(c(2\pi - 0) - c(0 + 0)). \tag{2.8}$$

Note that (2.8) is independent from what $c$ is chosen, also the winding number is of course always an integer, and we denote it $\operatorname{wind} a$. The next theorem is the awaited classification.

**Theorem 2.3.2.** $\exp W = \{a \in GW : \operatorname{wind} a = 0\}$

We earlier introduced the subalgebras $W_+$ and $W_-$. A natural extension is to look at the invertible elements in them as well as the elements with logarithms. Therefore, let $GW_\pm$ consist of all the functions $a_\pm \in W_\pm$ such that there exists $b_\pm \in W_\pm$ for which $a_\pm(t)b_\pm(t) = 1$ for all $t \in \mathbb{T}$. Similarly let $\exp W_\pm$ be the functions $a_\pm$ in $GW_\pm$ that can be represented $a_\pm = e^{b_\pm}$ for $b_\pm \in GW_\pm$.

All the subalgebras just defined also have convenient classifications, for which we will look at the analytic continuation. Let $\mathbb{D}$ be the open unit disk, i.e. the set $\{z \in \mathbb{C} : |z| < 1\}$. We can extend every $a_+ \in W_+$ analytically to $\mathbb{D}$ by $a_+(z) = \sum_{n=0}^{\infty} a_n z^n,\ z \in \mathbb{D}$ since $a_+(z)$ must have convergence radius at least 1, since it converges on the unit circle. In the same manner we see that we can extend every $a_- \in W_-$ analytically to $\{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$ by $a_-(z) = \sum_{n=-\infty}^{0} a_n z^n,\ 1 < z \leq \infty$. Now the full classification comes.

**Theorem 2.3.3.** *It holds that*

$$\begin{aligned}
GW_+ &= \{a \in W : a(z) \neq 0 \; \forall \; |z| \leq 1\}, \\
GW_- &= \{a \in W : a(z) \neq 0 \; \forall \; |z| \geq 1 \textit{ and } z = \infty\}, \\
\exp W_+ &= GW_+, \\
\exp W_- &= GW_-.
\end{aligned}$$

*Sktech of proof for 2.3.1–2.3.3.* To find the invertible elements of $W$, $W_-$ and $W_+$ we use a result from [12] which states that for each non-invertible element $a$ in a commutative Banach algebra there is a multiplicative linear functional $\varphi$ such that $\varphi(a) = 0$. To use this result we need to find all the multiplicative linear functionals on $W$. So assume that $\varphi$ is a multiplicative linear functional. Another result from [12] says that $\|\varphi\| = 1$, so we have

$$1 = \|t\|_W \geq |\varphi(t)| = \frac{1}{|\varphi(t^{-1})|} \geq \frac{1}{\|t^{-1}\|_W} = 1.$$

Hence $|\varphi(t)| = 1$. Let $t^* := \varphi(t) \in \mathbb{T}$. We now see that for any $W \ni a = \sum_{n=-\infty}^{\infty} a_n t^n$ we have

$$\varphi\left(\sum_{n=-\infty}^{\infty} a_n t^n\right) = \sum_{n=-\infty}^{\infty} a_n \varphi(t)^n = a(t^*).$$

Hence all multiplicative linear functionals on $W$ have the form $\varphi(a) = a(t^*)$ for some $t^*$. Also note that $a \mapsto a(t^*)$ indeed is a multiplicative linear functional, and that they are different for different $t^*$, this is easily checked.

We wanted to prove that $a \in W$ not having any zeros implies that $a \in GW$. So assume that $a \notin GW$, then there is a multiplicative linear functional $\varphi$ with $\varphi(a) = 0$, but then $\varphi(a) = a(t^*) = 0$, which is a contradiction.

A similar approach as for $W$ and $GW$ can be used to handle $W_-$ and $W_+$. Note that we only need to do the argument for one of them, e.g. $W_+$, the result for $W_-$ follows by the change of variables $z^* = 1/z$. So, we need to find the multiplicative linear functionals on $W_+$. We see that $|\varphi(t)| \leq 1$, so all multiplicative linear functionals are of the form $\varphi(a_+) = a(z_+^*)$ for $\left|z_+^*\right| \leq 1$. Hence, we have the classification of $GW, GW_-$ and $GW_+$.

To find the exponential sets we utilize another result from [12] saying that $\exp \mathfrak{B}$ for a commutative Banach algebra $\mathfrak{B}$ is the connected component of the invertible elements that contains the identity. So we want to show that $\{a \in GW : \operatorname{wind} a = 0\}$ is the connected component of $GW$ that contains 1. Of course the winding number of the identity function is 0. We can visualize how we find a connected component of $GW$ by deforming the curve $a(\mathbb{T})$ continuously. $1(\mathbb{T})$ is just a point. But we can stretch it out to any curve as long as the curve does not go through the origin, since we then would leave $GW$. So, we can deform all curves that have zero winding number into each other without the curve going though the origin, but to get a non-zero winding number the curve would have to pass through the origin, which would mean leaving $GW$.

We can find $\exp W_+$ and $\exp W_-$ in a similar manner. Again, we only need to show it for $\exp W_+$, the result for $\exp W_-$ follows by symmetry. Take $a \in GW_+$ and create a family of functions in $GW_+$ by $a_\lambda(z) = a(\lambda z)$. We see that $a_1 = a$, and $a_0$ is a constant function, which clearly is in the same connected component of $GW_+$ as the identity. Also, note that $a_\lambda(z) \neq 0$ for $|z| \leq 1$ for all $\lambda \in [0, 1]$, so $a_\lambda$ never leaves $GW_+$ as we vary $\lambda$ from 1 to 0. Hence, all of $GW_+$ is connected, and the result follows. $\square$

We are now ready to prove a very important Theorem that gives a great factorization of $a \in W$, namely the *Wiener–Hopf factorization.*

**Theorem 2.3.4.** *Let $a \in GW$, and suppose that* $\operatorname{wind} a = m$. *Then we can factorize $a$ as*

$$a(t) = a_-(t)t^m a_+(t) \;\; \textit{where} \;\; t \in \mathbb{T}, \;\; \textit{and} \;\; a_\pm \in GW_\pm.$$

*Proof.* We first note that because of the winding number definition given in (2.8) we have the property $\operatorname{wind} ab = \operatorname{wind} a + \operatorname{wind} b$ for $a, b \in W$. Also note that $\operatorname{wind} t^m = m$. This gives us $\operatorname{wind}(at^{-m}) = m - m = 0$. Therefore, by Theorems 2.3.1 and 2.3.2 we get that $at^{-m} = e^b$ for some $b = \sum_{-\infty}^{\infty} b_n t^n \in W$. Now define $b_- = \sum_{n=-\infty}^{-1} b_n t^n$ and $b_+ = \sum_{n=0}^{\infty} b_n t^n$. Because of Theorem 2.3.3 we see that $e^{b_\pm} \in GW_\pm$, and we now see that $at^{-m} = e^b = e^{b_-}e^{b_+} \Leftrightarrow a = e^{b_-}t^m e^{b_+}$. □

## 2.4 The spectrum of $T(a)$

We will momentarily turn to a more general setting than before. Let $X$ be a Banach space, we will denote by $\mathcal{B}(X)$ the set of all bounded linear operators $A : X \to X$. A generalization of spectra of matrices to spectra of bounded linear operators on Banach spaces can be made using the following definition:

$$\operatorname{sp} A = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}.$$

Here $\operatorname{sp} A$ denotes the *spectrum* of the bounded linear operator $A$. It can be proved that $\operatorname{sp} A$ is always a compact set. An operator $A \in \mathcal{B}(X)$ is called a Fredholm operator if its kernel and cokernel ($:= X/\mathrm{Im}\ A$) both have finite dimension. It can be shown that a bounded operator $A$ is Fredholm if and only if there is a $B \in \mathcal{B}(X)$ such that $AB - I$ and $BA - I$ are compact operators. One could loosely say that a Fredholm operator is "almost" invertible. This leads to the definition of the *essential spectrum*:

$$\operatorname{sp}_{\mathrm{ess}} A = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}.$$

Of course $\operatorname{sp}_{\mathrm{ess}} A \subset \operatorname{sp} A$, since all invertible operators are Fredholm (invertible operators have both kernel and cokernel of dimension 0). An important concept connected to Fredholm operators is the index, it is defined through

$$\mathrm{Ind}\ A := \dim \mathrm{Ker}\ A - \dim \mathrm{Coker}\ A.$$

A central property of Fredholm operators is that if $A \in \mathcal{B}(X)$ is Fredholm then all $B \in \mathcal{B}(X) : \|A - B\| < \epsilon$ for a sufficiently small $\epsilon$ are also Fredholm and $\mathrm{Ind}\ A = \mathrm{Ind}\ B$. Proofs as well as a thorough treatise of all the facts presented about Fredholm operators and the spectrum can be found in [10].

We will now begin to develop the theory that will give us a clear geometric picture of the spectrum of $T(a)$. We will begin by classifying when $T(a)$ is Fredholm.

**Theorem 2.4.1.** *Assume $a \in W$. $T(a)$ as an operator on $l^p$ $(1 \leq p \leq \infty)$ is Fredholm if and only if $a \in GW$. If $T(a)$ is Fredholm, we have* $\operatorname{Ind} T(a) = -\operatorname{wind} a$.

*Proof.* We begin with the if part, so assume that $a \in GW$ and that $\operatorname{wind} a = m$. We can now use the Wiener-Hopf factorization to write $a = a_- t^m a_+$, where $a_\pm \in GW_\pm$. Using Proposition 2.2.4 we now see that

$$T(a) = T(a_-)T(t^m)T(a_+).$$

We will now use the fact that $T(a_\pm)$ are invertible with inverses $T(a_\pm^{-1})$. This follows from Proposition 2.2.4. We know that $a_-$ is invertible, and $a_-^{-1} \in W_- \subset W$ so from 2.2.4 we get $I = T(a_-^{-1}a_-) = T(a_-^{-1})T(a_-) = T(a_-a_-^{-1}) = T(a_-)T(a_-^{-1})$. An analogous argument works for $a_+$. Since $T(a_\pm)$ are invertible, $T(a)$ has the same dimension for its kernel and cokernel as $T(t^m)$. So all that remains to prove for the if part is that $T(t^m)$ is Fredholm and that it has index $-m$. But this is quite easily seen. If we keep in mind that $T(t^m)$ just is the shift operator, we see that

$$\dim \operatorname{Ker} T(t^m) = \begin{cases} 0 & \text{if } m \geq 0, \\ -m & \text{if } m < 0, \end{cases} \quad \dim \operatorname{Coker} T(t^m) = \begin{cases} m & \text{if } m \geq 0, \\ 0 & \text{if } m < 0. \end{cases}$$

And this implies exactly what we want, that $T(t^m)$ is Fredholm with index $-m$, and hence that $T(a)$ also is Fredholm with index $-m$.

Now to the only if part. Assume that $T(a)$ is Fredholm and assume to the contrary that there is a $t_0$ such that $a(t_0) = 0$. We will now construct a contradiction that builds upon the fact that Fredholmness and index is constant under small perturbations. We can construct $b, c \in GW$ by adding a constant to $a$, we can view this as translating the graphs slightly. We can construct $b$ and $c$ in such a way that the origin no longer intersects the graphs, and the origin lies on different sides of the graphs. This implies that $|\operatorname{wind} b - \operatorname{wind} c| \geq 1$, but we can make the translation arbitrarily small, which means that $\|a - b\|_W$ and $\|a - c\|_W$ can be made small, this in turn implies by Proposition 2.2.1 that $\|T(a) - T(b)\|$ and $\|T(a) - T(c)\|$ can be made arbitrarily small. But now we have a contradiction, since we proved in the if part that $T(b)$ and $T(c)$ are Fredholm and because of $|\operatorname{wind} b - \operatorname{wind} c| \geq 1$ their index must differ by one. But the index is constant for sufficiently small perturbations, which implies that $\operatorname{Ind}\, T(b) = \operatorname{Ind}\, T(c)$. Therefore the assumption that $a$ has a zero on the unit circle is false. $\square$

Theorem 2.4.1 now directly gives us the essential spectrum. If we apply it to $a - \lambda$ instead of $a$ we immediately get the following corollary.

**Corollary 2.4.2.** *Assume $a \in W$, then* $\operatorname{sp_{ess}} T(a) = a(\mathbb{T})$.

We proceed to classify the invertibility in terms of the winding number.

**Theorem 2.4.3.** *Let $a \in W$. $T(a)$ as an operator on $l^p$ $(1 \leq p \leq \infty)$ is invertible if and only if $a \in \exp W$.*

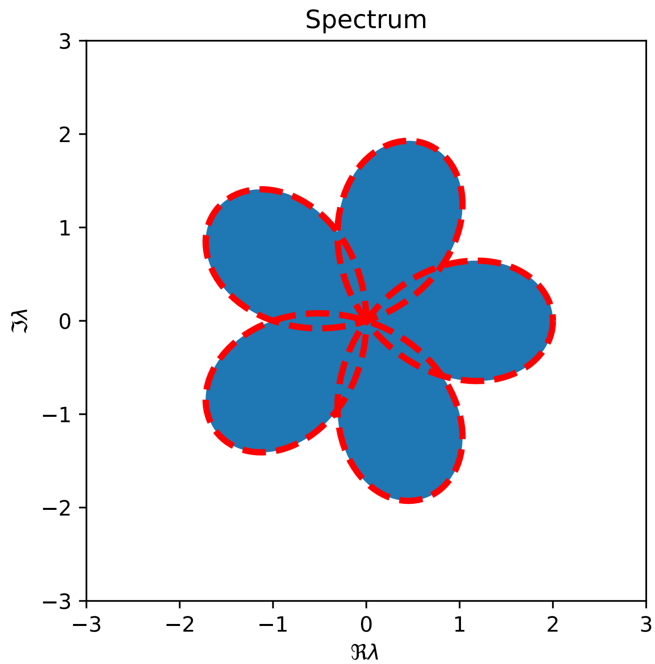



Figure 2.1: The spectrum and essential spectrum of $T(a)$ for $a(t) = t^{-4} + t$. The essential spectrum is the dashed red line, and the blue area is the spectrum.

*Proof.* If $T(a)$ is invertible, it is Fredholm with index 0. Theorem 2.4.1 then gives that $a$ has no zeros on $\mathbb{T}$ and that $\operatorname{wind} a = 0$. The other direction follows easily from the Wiener–Hopf factorization. If $a$ has no zeros on $\mathbb{T}$ and $\operatorname{wind} a = 0$, we know that we can write $a = a_- a_+$ for $a_\pm \in GW_\pm$. As we noted in the proof of 2.4.1, the inverses of $T(a_\pm)$ are $T(a_\pm^{-1})$. Hence, the inverse of $T(a) = T(a_- a_+) = T(a_-)T(a_+)$ is $T(a_+^{-1})T(a_-^{-1})$. □

We are now ready to present an amazing geometric description of a Toeplitz operator.

**Corollary 2.4.4.** *Let $a \in W$, then*

$$\operatorname{sp} T(a) = a(\mathbb{T}) \cup \{\lambda \in \mathbb{C} \setminus a(\mathbb{T}) : \operatorname{wind}(a - \lambda) \neq 0\}$$

*Proof.* Apply Theorem 2.4.3 to $a - \lambda$. □

**Example 2.4.5.** Let $a(t) = t^{-4} + t$. We plot the spectrum of $T(a)$ in Figure 2.1 by first plotting the essential spectrum and then filling in the area with non-zero winding number.

## 2.5 Stability of truncations of $T(a)$

The next topic we will discuss is stability, and how we approximately can solve $T(a)x = y$ for $x, y \in l^2$. We will use the results to get information about how the eigenvalues of the partial matrices converge. Throughout this subsection we will assume that the underlying Banach space is $l^2$, and we will only use the $l^2$ norm.

We momentarily take a step back to more generality and consider

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{pmatrix}. \tag{2.9}$$

A naive and natural approach to solving (2.9) is to truncate the system and instead solve

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1^{(n)} \\ \vdots \\ x_n^{(n)} \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}. \tag{2.10}$$

Intuitively, if they exists, the solutions to (2.10) should approach the solution of (2.9), but this needs to be formalized. We begin by some notation. Let $P_n$ be the projection $l^2 \to l^2$ which acts by

$$P_n : (x_1, x_2, x_3, \dots) \mapsto (x_1, \dots, x_n, 0, 0, \dots).$$

We can write (2.10) on the form $P_n A x^{(n)} = P_n y$, where $x^{(n)} \in \operatorname{Im} P_n$. We can naturally identify $\operatorname{Im} P_n$ with $\mathbb{C}^n$, which we will do freely throughout this section. We also identify the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

with the restriction $P_n A P_n | \operatorname{Im} P_n$. If $A = T(a), a \in W$ we denote the truncation $P_n T(a) P_n | \operatorname{Im} P_n$ by $T_n(a)$. We will now study how, if, and when the solutions to (2.10) (if they exist) approach the solution to (2.9).

Before getting into more details we need to define different modes of convergence. Let $(A_n)_{n=1}^{\infty}$ be a sequence of bounded operators on $X$. We say that it *converges strongly* to $A \in \mathcal{B}(X)$ if $\|A_n x - Ax\| \to 0$ as $n \to \infty$ for all $x \in X$. Further, we say that the sequence $A_n$ *converge uniformly* or *in norm* to $A \in \mathcal{B}(X)$ if $\|A_n - A\| \to 0$ as $n \to \infty$.

We call a sequence of matrices $(A_n)_{n=1}^{\infty}$ an *approximating sequence* for an operator $A \in \mathcal{B}(l^2)$ if $A_n P_n$ converges strongly to $A$, where $A_n$ is a $n \times n$ matrix. A typical example of an approximating sequence for $A$ is $P_n A P_n | \operatorname{Im} P_n$. But just that we have an approximating sequence does not mean we can apply it to solve (2.9). We therefore write $A \in \Pi(A_n)$ and say that the *approximation method* $(A_n)$ is *applicable* to $A$ if there exists an $n_0 \in \mathbb{N}$ such that

1. the matrices $A_n$ are invertible for all $n \geq n_0$;
2. for all $y \in l^2$, the solutions $x^{(n)} \in \operatorname{Im} P_n$ to the systems $A_n x^{(n)} = P_n y$, where $n \geq n_0$ converge in $l^2$ to a solution $x \in l^2$ to $Ax = y$.

The other concept we need, and that is very common in numerical analysis, is that of stability. We say that a sequence of matrices $(A_n)_{n=1}^{\infty}$ is *stable* if there is a $n_0$ such that for all $n \geq n_0$, $A_n$ is invertible and $\sup_{n \geq n_0} \left\|A_n^{-1}\right\| < \infty$. We will show that an approximation method being applicable implies stability, and stability, together with invertibility of $A$ implies $A \in \Pi(A_n)$.

Next we formulate a version of the uniform boundedness principle, a well known theorem of functional analysis that we need to prove the result just mentioned.

**Theorem 2.5.1.** *Assume that $(A_n)_{n=1}^{\infty}$ is a sequence of bounded operators on $l^2$ such that $A_n x$ is a convergent sequence in $l^2$ for all $x \in l^2$. Then $\sup_{n \geq 1} ||A_n|| < \infty$, and the operator $A$ which is defined by $Ax := \lim_{n\to\infty} A_n x$ is bounded on $l^2$, and the operator norm of $A$ satisfies $\|A\| \leq \liminf_{n\to\infty} \|A_n\|$.*

A proof can be found in [10]. We use the result to prove the announced equivalence.

**Proposition 2.5.2.** *Let $A \in \mathcal{B}(l^2)$ and let $(A_n)_{n=1}^{\infty}$ be an approximating sequence. Then $A \in \Pi(A_n)$ if and only if $A$ is invertible and $(A_n)_{n=1}^{\infty}$ is stable.*

*Proof.* We start by proving the only if part. First note that $A_n^{-1} P_n$ is a sequence of bounded operators on $l^2$, and $A_n^{-1} P_n y$ converges for all $y \in l^2$. Uniform boundedness now tells us that $\sup_{n \geq n_0} ||A_n^{-1} P_n|| < M$ for some constant $M$, and since $\left\|A_n^{-1} P_n\right\| = \left\|A_n^{-1}\right\|$, $(A_n)_{n=1}^{\infty}$ is stable. Next, we prove that $A$ is invertible. The second requirement for $A \in \Pi(A_n)$ implies that $A$ is surjective, so we only need to prove that $A$ in injective, which is equivalent to the kernel of $A$ only consisting of 0. Using the bound $||A_n^{-1} P_n|| < M$ we can for an arbitrary $x \in l^2$ write

$$\|P_n x\| = \left\|A_n^{-1} A_n P_n x\right\| = \left\|A_n^{-1} P_n A_n P_n x\right\| \leq \left\|A_n^{-1} P_n\right\| \|A_n P_n x\|.$$

If we let $n \to \infty$ we get $\|x\| \leq M \|Ax\|$ since $P_n x$ converge to $x$ and $A_n P_n x$ converge to $Ax$ since $A_n P_n$ converges strongly to $A$. But now we see that 0 must be the only element in the kernel of $A$, so the only if part is done.

Now to the if part. The fact that $(A_n)$ is stable implies the first requirement of $A \in \Pi(A_n)$, so we only need to prove that $A_n^{-1} P_n y \to A^{-1} y$ as $n \to \infty$, since this would imply second requirement. We can use the bound we get from stability and write

$$\left\|A_n^{-1} P_n y - A^{-1} y\right\| \leq \left\|A_n^{-1} P_n y - P_n A^{-1} y\right\| + \left\|P_n A^{-1} y - A^{-1} y\right\|. \qquad (2.11)$$

We see that the second term on the right in (2.11) goes to zero, since $P_n \to I$ strongly. Also, we can rewrite the first term on the right to

$$\left\|A_n^{-1}P_n y - P_n A^{-1}y\right\| = \left\|A_n^{-1}(P_n y - A_n P_n A^{-1}y)\right\| \leq M \left\|P_n y - A_n P_n A^{-1}y\right\|.$$

Since $P_n y \to y$, and $A_n P_n$ converges strongly to $A$ we see that $A_n P_n(A^{-1}y) \to AA^{-1}y = y$. Hence $\left\|A_n^{-1}P_n y - P_n A^{-1}y\right\| \to 0$, which through (2.11) implies $\left\|A_n^{-1}P_n y - A^{-1}y\right\| \to 0$, which is what we wanted to prove. □

Proposition 2.5.2 gives us equivalence between two important concepts but we still have no tool that allows to show that a sequence is stable. Our next step is to prove that $T_n(a)$ is stable if $T(a)$ is invertible, and $a \in W$. In order to do this we need some lemmas.

**Lemma 2.5.3.** *Let $(B_n)_{n=1}^{\infty}$ be a sequence of bounded operators on $l^2$ that converge strongly to a bounded operator $B \in \mathcal{B}(l^2)$. Further, let $K$ be a compact operator on $l^2$. Then $B_n K$ converges to $BK$ uniformly, i.e. $\|B_n K - BK\| \to 0$.*

*Proof.* We want to show that $\sup_{\|x\|=1} \|B_n Kx - BKx\| \to 0$ as $n \to \infty$. Since $K$ is compact, it sends the unit sphere of $l^2$ into a relatively compact set. Therefore, we can for each $\epsilon > 0$ find $x_1, x_2, \dots x_N$ in the unit sphere of $l^2$ such that for each $x$ with $\|x\| \leq 1$ there is a $j \in \{1, 2, \dots N\}$ fulfilling $\|Kx_j - Kx\| < \epsilon$. Using this insight, we can bound $\|B_n Kx - BKx\|$. Take an arbitrary $\epsilon > 0$ and $x$ with $\|x\| = 1$. Choose a $x_j$ such that $\|Kx_j - Kx\| < \epsilon$. Then

$$\begin{aligned}\|B_n Kx - BKx\| &\leq \|B_n Kx - B_n Kx_j\| + \|B_n Kx_j - BKx_j\| + \|BKx_j - BKx\| \\ &\leq \|B_n\| \|Kx - Kx_j\| + \|B_n Kx_j - BKx_j\| + \|B\| \|Kx_j - Kx\|.\end{aligned} \tag{2.12}$$

Uniform boundedness gives ut that $\|B_n\| \leq M$ for some constant $M$, and by assumption, $B_n \to B$ strongly, so there is a $n_0$ for which $\|B_n Kx_j - BKx_j\| < \epsilon$ for all $n \geq n_0$ and all $j = 1, \dots, N$. All in all, we just showed that

$$\|B_n Kx - BKx\| \leq (M + 1 + \|B\|)\epsilon$$

for all $n \geq n_0$. Since $x$ was arbitrary, we have shown that $\|B_n K - BK\| \to 0$. □

The next lemma is quite general and very useful when dealing with truncated matrices.

**Lemma 2.5.4.** *Assume that $X$ is a linear space, and that $P$ and $Q$ are projections ($P^2 = P$ and $Q^2 = Q$) such that $P + Q = I$. Let $A$ be an invertible operator on $X$. Then $A$ "truncated" to $\operatorname{Im} P$, $PAP|\operatorname{Im} P$ is invertible on $\operatorname{Im} P$ if and only if $QA^{-1}Q|\operatorname{Im} Q$ is invertible on $\operatorname{Im} Q$. Further the inverse is given by*

$$(PAP|\operatorname{Im} P)^{-1} = (PA^{-1}P - PA^{-1}Q(QA^{-1}Q)^{-1}QA^{-1}P|\operatorname{Im} P).$$

*Proof.* The proof is just direct calculation. Assume that $QA^{-1}Q|\operatorname{Im} Q$ is invertible (we will simply write the inverse as $(QA^{-1}Q)^{-1}$). We have

$$\begin{aligned}
&PAP(PA^{-1}P - PA^{-1}Q(QA^{-1}Q)^{-1}QA^{-1}P)\\
&\quad= PA(I-Q)A^{-1}P - PA(I-Q)A^{-1}Q(QA^{-1}Q)^{-1}QA^{-1}P\\
&\quad= P - PAQA^{-1}P - PQ(QA^{-1}Q)^{-1}QA^{-1}P + PAQA^{-1}P\\
&\quad= P - PAQA^{-1}P - 0 + PAQA^{-1}P\\
&\quad= P.
\end{aligned}$$

The equality $PQ(QA^{-1}Q)^{-1}QA^{-1}P = 0$ is due to the fact that $PQ = 0$ for projections with the property $P + Q = I$. Similar calculations give that $(PA^{-1}P - PA^{-1}Q(QA^{-1}Q)^{-1}QA^{-1}P)PAP = P$. Note that we have proved the if part, and due to symmetry, the only if part follows, so we are done. □

Now we are ready to prove the promised result, which was first presented by Gohlberg and Feldman in [14].

**Theorem 2.5.5.** *Let $a \in W$, and assume that $T(a)$ is invertible. Then $(T_n(a))_{n=1}^{\infty}$ is stable.*

*Proof.* We begin by noting that $T(a)$ being invertible implies that $a$ is invertible. We can therefore use (2.7) and write $I = T(aa^{-1}) = T(a)T(a^{-1}) + H(a)H(\tilde{a})$, from which we obtain

$$T(a)^{-1} = T(a^{-1}) + T(a)^{-1}H(a)H(\tilde{a}) =: T(a^{-1}) + K, \tag{2.13}$$

where $K$ is a compact operator, this follows from the fact that $H(\tilde{a})$ is compact. We will now aim to use Lemma 2.5.4. We want information about the invertibility of $P_nT(a)P_n|\operatorname{Im} P_n$, and we know that it is invertible precisely when $Q_nT(a)^{-1}Q_n|\operatorname{Im} Q_n$ is invertible, where $Q_n$ is a projection that fulfills $P_n + Q_n = I$. From (2.13) we get that

$$Q_nT(a)^{-1}Q_n|\operatorname{Im} Q_n = Q_nT(a^{-1})Q_n|\operatorname{Im} Q_n + Q_nKQ_n|\operatorname{Im} Q_n. \tag{2.14}$$

We now use Lemma 2.5.3 to see that $\|Q_nKQ_n|\operatorname{Im} Q_n\| \to 0$. This follows from the fact that $\|Q_nKQ_n|\operatorname{Im} Q_n\| = \|Q_nKQ_n\|$ and that $Q_n \to 0$ strongly, so $Q_nK \to 0$ uniformly. Due to this we see that

$$\left\|Q_nT(a)^{-1}Q_n|\operatorname{Im} Q_n - Q_nT(a^{-1})Q_n|\operatorname{Im} Q_n\right\| \to 0$$

uniformly. Also, we note that $Q_nT(a^{-1})Q_n|\operatorname{Im} Q_n$ has the same matrix as $T(a^{-1})$, due to the Toeplitz structure. So $Q_nT(a^{-1})Q_n|\operatorname{Im} Q_n$ is always invertible, and its norm is constant. One can prove using series expansion of the inverse that if $A$ is invertible and $\|A - B\| < 1/\|A\|$, then $B$ is also invertible. Therefore we get that $Q_nT(a)^{-1}Q_n|\operatorname{Im} Q_n$ is invertible for all sufficiently large $n$. Additionally, using the series expansion of the inverse we can for each $\epsilon > 0$ find an $n_0$ such that

$$\left\|(Q_nT(a)^{-1}Q_n|\operatorname{Im} Q_n)^{-1}\right\| < (1+\epsilon)\left\|Q_nT(a^{-1})Q_n|\operatorname{Im} Q_n\right\| = (1+\epsilon)\left\|T(a^{-1})\right\|$$

for all $n > n_0$. We are now ready to apply Lemma 2.5.4. Since $Q_n T(a)^{-1} Q_n | \operatorname{Im} Q_n$ is invertible for sufficiently large $n$, so is $P_n T(a) P_n | \operatorname{Im} P_n$, and the inverse is $P_n T(a)^{-1} P_n - P_n T(a)^{-1} Q_n (Q_n T(a)^{-1} Q_n)^{-1} Q_n T(a)^{-1} P_n | \operatorname{Im} P_n$, whose operator norm does not exceed

$$\left\|T(a)^{-1}\right\| + (1+\epsilon)\left\|T(a)^{-1}\right\|\left\|T(a^{-1})\right\|\left\|T(a)^{-1}\right\|$$

for all sufficiently large $n$, which shows that $(P_n T(a) P_n | \operatorname{Im} P_n) = (T_n(a))$ is stable. $\square$

Theorem 2.5.5 gives us the first link between $T(a)$ and $T_n(a)$ that we need for investigating the eigenvalues of $T_n(a)$. We utilize this link in the next section.

## 2.6 The Szegő limit theorem

In this subsection we will show a version of the first Szegő limit theorem, which gives an estimate of how the determinants of $T_n(a)$ grows. This will in turn give us information about how the eigenvalues of $T_n(a)$ are distributed. The main result in this subsection will provide an asymptotic estimate for

$$\frac{1}{n}\sum_{i=1}^{n} f(\lambda_i^{(n)})$$

where $\lambda_i^{(n)}$ are the eigenvalues, counted with multiplicity of $T_n(a)$ and $f$ is a function that is analytic in an open neighborhood of the spectrum of $T(a)$.

We begin by proving an intuitively reasonable estimate of where the eigenvalues of $T_n(a)$ are located.

**Lemma 2.6.1.** *Let $a \in W$ and let $\Omega$ be an open set containing $\operatorname{sp} T(a)$. Then there is a $n_0 = n_0(a, \Omega)$ such that $\operatorname{sp} T_n(a) \subset \Omega$ for all $n \geq n_0$.*

*Proof.* We remind ourselves that the spectrum of $T_n(a)$ are all the $\lambda$ such that $T_n(a-\lambda)$ is not invertible. Gershgorin's theorem gives that there is an $R > 0$ such that $T_n(a-\lambda)$ is invertible for all $n$ and for all $|\lambda| > R$ since the row sums of non-diagonal entries in $T_n(a-\lambda)$ is bounded by $\|a\|_W$. Assume W.L.O.G. that $\{z \in \mathbb{C} : |z| \leq R\} =: B(0,R)$ contains $\Omega$. Now we just need to show that there is an $n_0$ such that $T_n(a-\lambda)$, $n \geq n_0$ is invertible for all $\lambda \in (\mathbb{C} \setminus \Omega) \cap B(0,R)$. Take an arbitrary $\lambda \in (\mathbb{C} \setminus \Omega) \cap B(0,R)$. Theorem 2.5.5 gives that $T_n(a-\lambda)$ is invertible for all $n \geq m_0 = m_0(a,\lambda)$, and that

$$\left\|T_n^{-1}(a-\lambda)\right\| \leq M(\lambda) < \infty,\ n \geq m_0.$$

Using the same invertibility argument as in the proof of Theorem 2.5.5 we see that for all $\mu$ with $|\mu-\lambda| < 1/M(\lambda)$, $T_n(a-\mu)$ is invertible for all $n \geq m_0$, since $\|T(a-\lambda) - T(a-\mu)\| = \|T(\mu-\lambda)\| = |\mu-\lambda|$. Now we use the compactness of $(\mathbb{C} \setminus \Omega) \cap B(0,R)$ to finish the proof. For all $\lambda \in (\mathbb{C} \setminus \Omega) \cap B(0,R)$, construct an open ball with radius $1/M(\lambda)$, this forms an open covering, so we can pick a finite number of the balls and pick the greatest $m_0$ from them, and use that value as our $n_0$. $\square$

Lemma 2.6.1 is quite a substantial step in our understanding of the eigenvalues of $T_n(a)$, since we have a very good understanding of how the spectrum of $T(a)$ looks. But the lemma does not give any information of where in the spectrum of $T(a)$ the eigenvalues of $T_n(a)$ are located. We will, however, develop the theory further to partially answer that question.

We will now present a version of the first Szegő limit theorem.

**Theorem 2.6.2.** *Let $a \in W$ and let $D_n(a) := \det T_n(a)$. Assume that $T(a)$ is invertible. Then*

$$\lim_{n\to\infty} \frac{D_n(a)}{D_{n-1}(a)} = G(a), \tag{2.15}$$

*where $G(a)$ is defined by $\exp \frac{1}{2\pi}\int_0^{2\pi} \log(a(e^{i\theta}))\,d\theta$.*

*Proof.* We will first show that $\lim_{n\to\infty} \frac{D_{n-1}(a)}{D_n(a)} = P_1T^{-1}(a)P_1$ where $P_1T^{-1}(a)P_1$ means the $(1,1)$ entry of $T^{-1}(a)$. We first show that $\frac{D_{n-1}(a)}{D_n(a)} = P_1T_n^{-1}(a)P_1$. This follows from solving the system $T_n(a)x = E_1$ where $E_1$ is a vector in $\mathbb{C}^n$ or $l^2$ with a 1 in its first entry and with all other entries 0. We see that $P_1T_n^{-1}(a)P_1$ is exactly the first entry of the solution $T_n^{-1}(a)E_1$. Solving the system using Cramer's rule we also see that the first entry of the solution is $\frac{D_{n-1}(a)}{D_n(a)}$, hence we know that $\frac{D_{n-1}(a)}{D_n(a)} = P_1T_n^{-1}(a)P_1$. Now we apply Theorem 2.5.5 to get that $(T_n(a))$ is stable, which by Proposition 2.5.2 implies that $T_n^{-1}(a)E_1 \to T^{-1}(a)E_1$, which implies that $P_1T_n^{-1}(a)P_1 \to P_1T^{-1}(a)P_1$. So we now have that $\lim_{n\to\infty} \frac{D_{n-1}(a)}{D_n(a)} = P_1T^{-1}(a)P_1$, and the only step remaining is to show that $G(a) = 1/P_1T^{-1}(a)P_1$.

Note that due to Theorem 2.4.3, $T(a)$ being invertible implies $a \in \exp W$. Let $b \in W$ be such that $e^b = a$. Using the Wiener-Hopf factorization we can write

$$T^{-1}(a) = (T(a_-)T(a_+))^{-1} = T(a_+^{-1})T(a_-^{-1}).$$

Note that $T(a_+^{-1})$ is lower triangular and $T(a_-^{-1})$ is upper triangular, so the $(1,1)$ entry of $T^{-1}(a)$ is just the product of the $(1,1)$ entries of $T(a_-^{-1})$ and $T(a_+^{-1})$, which equals $(a_+^{-1})_0(a_-^{-1})_0$ where $(a)_0$ means the 0th Fourier coefficient. Note that $a_\pm = e^{b_\pm}$, we can now write

$$\begin{aligned}
(a_+^{-1})_0(a_-^{-1})_0 &= (e^{-b_+})_0(e^{-b_-})_0 \\
&= e^{(-b_+)_0+(-b_-)_0} \\
&= e^{(-b)_0} = e^{-(b)_0} \\
&= \frac{1}{e^{(b)_0}} \\
&= \frac{1}{G(a)}.
\end{aligned}$$

The second equality can be justified by doing a series expansion of $e^x$. Now we know that $P_1T^{-1}(a)P_1 = \frac{1}{G(a)}$ so $P_1T^{-1}(a)P_1$ is nonzero, which means we get

$$\lim_{n\to\infty} \frac{D_n(a)}{D_{n-1}(a)} = \frac{1}{P_1T^{-1}(a)P_1} = G(a). \quad \square$$

Next, we will utilize Theorem 2.6.2 to get information about the eigenvalues of $T_n(a)$. The intuition behind this is that the determinants is the product of the eigenvalues, so we should be able to get information about how they change and are distributed as $n$ grows.

**Theorem 2.6.3.** *Let $a \in W$ and let $\Omega \supset \operatorname{conv}\operatorname{sp} T(a)$ be an open subset of $\mathbb{C}$. Let $f$ be a function that is holomorphic on $\Omega$. Then*

$$\frac{1}{n}\sum_{j=1}^{n} f(\lambda_j^{(n)}) \to G_f(a) := \frac{1}{2\pi}\int_0^{2\pi} f(a(e^{i\theta}))d\theta$$

*as $n \to \infty$.*

*Proof.* The idea behind the proof is to take the logarithm of both sides in (2.15), show that the limit still holds, use the "trick" that if $D_n/D_{n-1} \to g$ then $\sqrt[n]{D_n} \to g$, then take the derivative, multiply with $f(\lambda)$ and integrate to use Cauchy's integral formula.

Let $\Omega_1$ be an open subset of $\Omega$ such that $\operatorname{conv}\operatorname{sp} T(a) \subset \Omega_1 \subset \Omega$ and $\Omega \setminus \overline{\Omega}_1$ contains a smooth curve surrounding $\Omega_1$. This is possible since $\operatorname{conv}\operatorname{sp} T(a)$ is closed, while $\Omega$ is open. Let $D$ be an open disk in $(\mathbb{C} \setminus \Omega_1) \cap \Omega$ such that $\overline{D} \cap \overline{\Omega}_1 = \emptyset$, and let $\lambda \in D$. We will now look at

$$\frac{D_{n-1}(a-\lambda)}{D_n(a-\lambda)}.$$

Since we are only interested in what happens at the limit as $n \to \infty$ and Lemma 2.6.1 gives us that $\operatorname{sp} T_n(a) \subset \Omega_1$ for sufficiently large $n$, we can W.L.O.G. assume that the eigenvalues of $T_n(a)$ are inside $\Omega_1$. We need this for showing that

$$\frac{D_{n-1}(a-\lambda)}{D_n(a-\lambda)} \to \frac{1}{G(a-\lambda)} \tag{2.16}$$

uniformly on compact subsets of $D$. Since $D$ does not touch $\operatorname{sp} T(a)$, we know that $T(a-\lambda)$ is invertible for all $\lambda \in D$, so (2.16) holds pointwise. As before, denote the eigenvalues of $T_n(a)$ by $\lambda_j^{(n)}$. Since $\frac{D_{n-1}(a-\lambda)}{D_n(a-\lambda)} = \frac{\Pi_{j=1}^{n-1}(\lambda_j^{(n-1)}-\lambda)}{\Pi_{j=1}^{n}(\lambda_j^{(n)}-\lambda)}$ is a rational function it is holomorphic, which means that we only need to prove that the sequence $\phi_n(\lambda) := \frac{D_{n-1}(a-\lambda)}{D_n(a-\lambda)}$ is locally bounded, i.e. for each $\lambda \in D$, there is a neighborhood $N$ of $\lambda$ such that $\sup_{n\in\mathbb{N},\lambda\in N} |\phi_n(\lambda)| < \infty$. This follows from Vitali's theorem, which is presented in e.g. [17, p. 150]. So we need to prove that $(\phi_n)$ is locally bounded. To do this we utilize the fact that $\phi_n(\lambda) = P_1T_n^{-1}(a-\lambda)P_1$. As in the proof of Theorem 2.6.2, let $E_1$ be a vector in $\mathbb{C}^n$ with first entry 1 and the rest being 0. $P_1T_n^{-1}(a-\lambda)P_1$

is the first entry of $T_n^{-1}(a-\lambda)E_1$, which is the solution $x_\lambda$ to the equation $T_n(a-\lambda)x_\lambda = E_1$. So it suffices (and is stronger) to find a neighborhood where $x_\lambda$ is bounded. Therefore, fix $\lambda_0 \in D$. Consider $x_\lambda$ close to $x_{\lambda_0}$, i.e. solutions to $T_n(a-\lambda)x_\lambda = E_1 \Leftrightarrow T_n(a-\lambda_0+(\lambda_0-\lambda))x_\lambda = E_1$, which we rewrite as

$$T_n(a-\lambda_0)x_\lambda+(\lambda_0-\lambda)x_\lambda = E_1 \Leftrightarrow x_\lambda = (\lambda-\lambda_0)T_n^{-1}(a-\lambda_0)x_\lambda+T_n^{-1}(a-\lambda_0)E_1.$$

Theorem 2.5.5 gives us that for all $n \geq n_0(\lambda_0)$, $T_n(a-\lambda_0)$ is invertible and $\left\|T_n^{-1}(a-\lambda_0)\right\| < M(\lambda_0)$. Let $|\lambda-\lambda_0| < 1/2M(\lambda_0)$. Taking the norm of $x_\lambda$ now yields

$$\|x_\lambda\| \leq \|x_\lambda\|/2 + M(\lambda_0) \Rightarrow \|x_\lambda\| \leq 2M(\lambda_0).$$

This holds for all $\lambda$ with $|\lambda-\lambda_0| < 1/2M(\lambda_0)$ and all $n \geq n_0$. So we have shown that $\phi_n(\lambda)$ is locally bounded, since $|\phi_n(\lambda)| \leq \|x_\lambda\|$. Therefore (2.16) holds uniformly for compact subsets of $D$. This implies that the limit function $\frac{1}{G(a-\lambda)}$ is holomorphic on $D$, and even on a neighborhood of $\overline{D}$, since we could have extended $D$ slightly, since $\overline{D}\cap\overline{\Omega}_1 = \emptyset$.

The next step in our proof is to show that

$$\log\frac{D_{n-1}(a-\lambda)}{D_n(a-\lambda)} \to \log\frac{1}{G(a-\lambda)} \tag{2.17}$$

uniformly on compact subsets of $D$, with a suitably chosen branch. Uniformly continuous functions preserve uniform convergence, so if we can find a branch defined on a compact subset which contains the ranges of $\phi_n(\lambda)$ and $\frac{1}{G(a-\lambda)} =: \phi(\lambda)$ we are done. To define a continuous branch we need the domain to be compact and not enclose 0. Since $D$ is bounded, $\overline{D}$ is compact. As $1/G(a-\lambda)$ is holomorphic on a neighborhood of $\overline{D}$, we can look at the image $1/G(a-\overline{D}) =: K$, which will be a compact set not containing 0. Let $d_K$ be the distance between $K$ and 0. We see that for sufficiently large $n$, $\phi_n(\lambda)$ is contained in the compact set $\{z\in\mathbb{C} : \inf_{k\in K}|z-k| < d_k/2\} =: K_e$, since $\phi_n$ converges uniformly to $\phi$ on $\overline{D}$. Notice that

$$G(a-\lambda) = \exp\frac{1}{2\pi}\int_0^{2\pi}\log\left(a(e^{i\theta})-\lambda\right)d\theta.$$

Because of convexity there is a line in $\mathbb{C}$ separating $\operatorname{conv}\operatorname{sp}T(a)\supset a(\mathbb{T})$ and $D$. Hence, the imaginary part of $\log\left(a(e^{i\theta})-\lambda\right)$ (which is a continuous function) will describe angles in some half plane. The intermediate value theorem then tells us that $G(a-\lambda)$ must be in the same half plane for all $\lambda\in D$, and the same must hold for $\frac{1}{G(a-\lambda)}$, which in turn implies that $K_e$ does not surround 0. So we can in fact construct a branch such that (2.17) holds uniformly on compact subsets of $D$.

We now have that

$$\sum_{j=1}^{n-1}\log\left(\lambda_j^{(n-1)}-\lambda\right)-\sum_{j=1}^{n}\log\left(\lambda_j^{(n)}-\lambda\right)+\log G(a-\lambda)-2\pi i m(n,\lambda) \to 0 \tag{2.18}$$

uniformly on compact subsets, where $m(n, \lambda)$ is an integer. Next, we want to switch the branches such that all terms on the form $\log(\lambda_j^{(n)} - \lambda)$ are holomorphic in $D$, which we can do by compensating for the phase, so (2.18) still holds, but with a modified $m(n, \lambda)$. We now see that $m(n, \lambda)$ only depends on $n$ for sufficiently large $n$, since all the other terms in (2.18) are continuous in $\lambda$ a jump in $m(n, \lambda)$ would violate the uniform convergence on compact subsets.

Introduce $s_n$ such that $s_n - s_{n-1} = m(n)$ and define

$$H_n(\lambda) = \sum_{j=1}^{n} \log(\lambda_j^{(n)} - \lambda) + 2\pi i s_n.$$

Let $K$ be any compact subset of $D$. From (2.18) we now have for any $\epsilon > 0$ an $n_0$ such that

$$|H_n(\lambda) - H_{n-1}(\lambda) - \log G(a - \lambda)| < \epsilon \ \ \forall \lambda \in K, \ n \geq n_0.$$

To prove the theorem we are interested in how $H_n/n$ behaves. For $n \geq n_0$ we can write a telescopic sum to get the estimate

$$\begin{aligned}\left|\frac{H_n}{n} - \log G(a - \lambda)\right| \leq \sum_{j=1}^{n_0-1} \left|\frac{H_j - H_{j-1} - \log G(a - \lambda)}{n}\right| \\ + \sum_{j=n_0}^{n} \left|\frac{H_j - H_{j-1} - \log G(a - \lambda)}{n}\right|\end{aligned}$$

for all $\lambda \in K$, where we let $H_0 := 0$. The second term is bounded by $\epsilon$ and the first can be made arbitrarily small by increasing $n$. So we see that $\frac{H_n}{n} \to \log G(a - \lambda)$ uniformly on $K$. Which means that

$$\frac{\sum_{j=1}^{n} \log(\lambda_j^{(n)} - \lambda)}{n} - \log G(a - \lambda) + \frac{2\pi i s_n}{n} \to 0$$

uniformly on compact subsets of $D$. Differentiating, we get that

$$\frac{1}{n} \sum_{j=1}^{n} \frac{1}{\lambda - \lambda_j^{(n)}} - \frac{d}{d\lambda} \frac{1}{2\pi} \int_0^{2\pi} \log(a(e^{i\theta}) - \lambda) d\theta \to 0$$

uniformly on compact subsets of $D$. We interchange differentiation and integration in the second term, and multiply the expression with $f(\lambda)/2\pi i$, and retain uniform convergence on compact subsets, since $f$ is bounded on compact sets. Furthermore, let $\gamma$ be a closed curve in $\Omega \setminus \overline{\Omega_1}$ that surrounds $\Omega_1$ once. Cover $\gamma$ with a finite number of open disks $D_k$ ($\gamma$ is compact). Collecting all facts we can now integrate around $\gamma$ and get

$$\frac{1}{2\pi i} \frac{1}{n} \sum_{j=1}^{n} \int_\gamma \frac{f(\lambda)}{\lambda - \lambda_j^{(n)}} d\lambda - \frac{1}{2\pi i} \int_\gamma \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\lambda)}{\lambda - a(e^{i\theta})} d\theta d\lambda \to 0$$

as $n \to \infty$. We swap the order of integration in the second term and use Cauchy's integral formula to rewrite it as

$$\frac{1}{n}\sum_{j=1}^{n} f(\lambda_j^{(n)}) \to \frac{1}{2\pi}\int_0^{2\pi} f(a(e^{i\theta}))d\theta,\ n \to \infty,$$

which is what we wanted to prove. □

Theorem 2.6.3 looks very strong, like it would give is a lot of information about the eigenvalue distribution, all we have to do is select $f$ as some suitable test function. But this does not work. Since harmonic functions do not have local minima or maxima we are very restricted when choosing $f$, so Theorem 2.6.3 is actually not as strong as one could wish. But there is a special case where Theorem 2.6.3 indeed gives us a lot of information, which we will explore in the next section.

## 2.7 Eigenvalue distribution for Hermitian Toeplitz matrices

In the case of Hermitian Toeplitz matrices, we can in fact say a lot more about the eigenvalues. $T(a)$ is Hermitian if and only if $a$ is real-valued, so $\operatorname{conv}\operatorname{sp} T(a)$ is just a line segment. To begin our treatment we will need an auxiliary result to Lemma 2.6.1.

**Lemma 2.7.1.** *Let $a \in W$, then $\operatorname{sp} T_n(a) \subset \operatorname{conv}\operatorname{sp} T(a)$ for all $n \geq 1$.*

*Proof.* Take a $\lambda \in \mathbb{C}\backslash\operatorname{conv}\operatorname{sp} T(a)$. We want to show that $T_n(a-\lambda)$ is invertible. We will do this by transforming $T_n(a-\lambda)$ linearly to something that we can show is invertible. Let $b = a - \lambda$ and study $\operatorname{conv}\operatorname{sp} T(b)$. This set does not contain 0 because of how $\lambda$ is chosen. Therefore, there is a minimum distance $d > 0$ between $\operatorname{conv}\operatorname{sp} T(b)$ and 0. Because of convexity, there is a $\gamma \in \mathbb{T}$ such that $\gamma \operatorname{conv}\operatorname{sp} T(b)$ ends up in the cut off disk $\{z \in \mathbb{C} : \Re z \geq d, |z| \leq \|b\|_\infty\}$, now we can scale $\gamma \operatorname{conv}\operatorname{sp} T(b)$ with $d/\left\|b\right\|_\infty^2$, which moves the bounds to

$$\left\{z \in \mathbb{C} : \Re z \geq \frac{d^2}{\|b\|_\infty^2}, |z| \leq \frac{d}{\|b\|_\infty}\right\} \subset \left\{z \in \mathbb{C} : |z-1| \leq \sqrt{1 - \frac{d^2}{\|b\|_\infty^2}}\right\},$$

which can be verified by a simple geometric argument. Now we see that the spectrum of $\frac{\gamma d}{\|b\|_\infty^2}T(b)$ is close to 1, so $\frac{\gamma d}{\|b\|_\infty^2}T(b)$ should almost be the identity matrix, and hence invertible. In fact we have

$$\left\|\frac{\gamma d}{\|b\|_\infty^2}T_n(b) - I\right\|_2 = \left\|T_n(\frac{\gamma d}{\|b\|_\infty^2}b - 1)\right\|_2 \leq \left\|\frac{\gamma d}{\|b\|_\infty^2}b - 1\right\|_\infty \leq \sqrt{1 - \frac{d^2}{\|b\|_\infty^2}} < 1$$

which implies that $\frac{\gamma d}{\|b\|_\infty^2}T_n(b)$ and $T_n(b)$ are invertible. In the second inequality we used Lemma 2.2.2 and the fact that $\|T_n(b)\| = \|P_nT(b)P_n\| \leq \|T(b)\|$. □

We are now ready to prove a version of Theorem 2.6.3 that is applicable in the Hermitian case.

**Theorem 2.7.2.** *Let* $a \in W$ *and assume that* $\operatorname{conv}\operatorname{sp} T(a)$ *is a real line segment. Let* $f$ *be a function that is continuous on* $\operatorname{conv}\operatorname{sp} T(a)$. *Then*

$$\frac{1}{n}\sum_{j=1}^{n} f(\lambda_j^{(n)}) \to G_f(a) := \frac{1}{2\pi}\int_0^{2\pi} f(a(e^{i\theta}))d\theta, \quad n \to \infty.$$

*Proof.* We can prove such a strong result thanks to holomorphic functions being dense in the continuous functions on line segments. Thanks to Weierstraß' approximation Theorem we have for each $\epsilon > 0$ a polynomial $p$ defined such that $|f(z) - p(z)| < \epsilon$ for all $z \in \operatorname{conv}\operatorname{sp} T(a)$. We now see the bounds

$$\left|\frac{1}{n}\sum_{j=1}^{n} f(\lambda_j^{(n)}) - p(\lambda_j^{(n)})\right| < \epsilon$$

$$\left|\frac{1}{2\pi}\int_0^{2\pi} f(a(e^{i\theta})) - p(a(e^{i\theta}))d\theta\right| < \epsilon$$

where the first is justified by Lemma 2.7.1. Now we can use Theorem 2.6.3 with our polynomial $p$ as the holomorphic function and get

$$\left|\frac{1}{n}\sum_{j=1}^{n} p(\lambda_i^{(n)}) - \frac{1}{2\pi}\int_0^{2\pi} p(a(e^{i\theta}))d\theta\right| < \epsilon$$

for sufficiently large $n$. Combining all the bounds, we see that

$$\left|\frac{1}{n}\sum_{j=1}^{n} f(\lambda_i^{(n)}) - \frac{1}{2\pi}\int_0^{2\pi} f(a(e^{i\theta}))d\theta\right| < 3\epsilon$$

for sufficiently large $n$, so we are done. ◻

There are multiple useful corollaries to Theorem 2.7.2 that give us more of a hands on understanding of the distribution of eigenvalues for $T_n(a)$. We will present two different but similar ways to view Theorem 2.7.2. We begin by viewing two collections $((\lambda_j^{(n)})_{j=1}^n)_{n=1}^\infty$ and $((\mu_j^{(n)})_{j=1}^n)_{n=1}^\infty$ of real numbers. We say that the collections are *equally distributed in the sense of Weyl* if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \Big(f(\lambda_j^{(n)}) - f(\mu_j^{(n)})\Big) = 0 \tag{2.19}$$

for all $f \in C_0(\mathbb{R})$, i.e. all continuous functions with compact support. Note that (2.19) is weaker than

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} f(\lambda_j^{(n)}) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} f(\mu_j^{(n)}), \quad f \in C_0(\mathbb{R}). \tag{2.20}$$

But (2.20) implies (2.19). We are now ready to formulate our first corollary.

**Corollary 2.7.3.** *Let $a \in W$ be a real-valued symbol. Then the collection of eigenvalues of $T_n(a)$, $((\lambda_j^{(n)})_{j=1}^n)_{n=1}^\infty$ are equally distributed in the sense of Weyl to $((a(e^{\frac{2\pi i j}{n}}))_{j=1}^n)_{n=1}^\infty$.*

*Proof.* We need to prove something with the sums $\frac{1}{n}\sum_{j=1}^n f(a(e^{\frac{2\pi i j}{n}}))$, which are Riemann sums. Since $a \in W$, it is continuous and therefore Riemann integrable. Because of this we see that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n f(a(e^{\frac{2\pi i j}{n}})) = \frac{1}{2\pi}\int_0^{2\pi} f(a(e^{i\theta}))d\theta$$

for all $f \in C_0(\mathbb{R})$. Now Theorem 2.7.2 gives us (2.20), which in turn gives us (2.19). $\square$

Now we have a better idea of the consequences of Theorem 2.7.2. And there is yet another interesting viewpoint. Consider a sequence of measures $(\mu_n)_{n=1}^\infty$ and a measure $\mu$. We say that $\mu_n$ converges weakly to $\mu$ if

$$\int_{\mathbb{R}} f d\mu_n \to \int_{\mathbb{R}} f d\mu \tag{2.21}$$

for all $f \in C_0(\mathbb{R})$. As usual, let $a \in W$, let $(\lambda_j^{(n)})_{j=1}^n$ be the eigenvalues of $T_n(a)$, and denote for Borel sets $E$ the usual Lebesgue measure as $|E|$. We define Borel measures $\mu_n$ and $\mu$ on $\mathbb{R}$ as

$$\mu_n(E) = \frac{1}{n}\sum_{j=1}^n \chi_E(\lambda_j^{(n)}), \tag{2.22}$$

$$\mu(E) = \frac{1}{2\pi}|\phi^{-1}(E)|, \tag{2.23}$$

where $\phi$ is the function $\phi : [0, 2\pi) \to \mathbb{R}$ that takes $\theta \mapsto a(e^{i\theta})$. $\mu_n(E)$ is the fraction of eigenvalues for $T_n(a)$ in $E$, while $\mu(E)$ on the other hand can be interpreted as the probability that a uniformly random point on the unit circle is mapped to $E$ by $a$. We are now ready for our next corollary.

**Corollary 2.7.4.** *Let $a \in W$ be a real-valued symbol. Then the sequence of measures defined by* (2.22) *converges weakly to the measure defined by* (2.23)*. Furthermore, for Borel sets $E \subset \mathbb{R}$ with $|\phi^{-1}(\partial E)| = 0$ we have $\mu_n(E) \to \mu(E)$.*

*Proof.* Let $f$ be an arbitrary function in $C_0(\mathbb{R})$. We see that

$$\int_{\mathbb{R}} f d\mu_n = \frac{1}{n}\sum_{j=1}^n f(\lambda_j^{(n)})$$

directly from the definition of $\mu_n$. We also see that

$$\int_{\mathbb{R}} f d\mu = \frac{1}{2\pi}\int_0^{2\pi} f(a(e^{i\theta}))d\theta$$

by a standard identity from integration theory. Now Theorem 2.7.2 implies the first part of the corollary. For the second statement, we will show that

$$\liminf_{n\to\infty} \mu_n(E) = \limsup_{n\to\infty} \mu_n(E) = \mu(E).$$

The second statement is equivalent to (2.21) being true for $f = \chi_E$, so the idea is to use the first part of the Corollary and approximate $\chi_E$ with functions in $C_0(\mathbb{R})$. However, for approximations to have compact support we need $E$ to be bounded, and because of Lemma 2.7.1 we can assume W.L.O.G. that $E \subset \operatorname{Im} a$ since $\mu_n$ and $\mu$ have support in $\operatorname{Im} a$. Introduce $f_m, g_m : \mathbb{R} \to \mathbb{R}$ by $f_m(x) = 1 - \min\{1, md(x, E)\}$ and $g_m(x) = \min\{1, md(x, E^c)\}$ for all $m \in \mathbb{Z}^+$, where $d$ is the usual distance function between a point and a set. We have $g_m(x) \leq \chi_E(x) \leq f_m(x)$ for all $x \in a(\mathbb{T})$. Further, $g_m$ converges pointwise to $\chi_{\text{Int } E}$, and that $f_m$ converges pointwise to $\chi_{\overline{E}}$. Equipped with these facts we can write

$$\limsup_{n\to\infty} \mu_n(E) = \limsup_{n\to\infty} \int_{a(\mathbb{T})} \chi_E \, d\mu_n \leq \limsup_{n\to\infty} \int_{a(\mathbb{T})} f_m \, d\mu_n = \int_{a(\mathbb{T})} f_m \, d\mu.$$

In the last equality we use the fact that (2.21) holds for $C_0(\mathbb{R})$-functions. Taking the limit as $m$ goes to $\infty$, and using the dominated converge theorem we get that

$$\limsup_{n\to\infty} \mu_n(E) \leq \int_{a(\mathbb{T})} \chi_{\overline{E}} \, d\mu = \mu(\overline{E}).$$

Doing analogous calculations with $\liminf_{n\to\infty} \mu_n(E)$ and $g_m$ yields $\mu(\text{Int } E) \leq \liminf_{n\to\infty} \mu_n(E)$. In total we have

$$\mu(\text{Int } E) \leq \liminf_{n\to\infty} \mu_n(E) \leq \limsup_{n\to\infty} \mu_n(E) \leq \mu(\overline{E}).$$

But because of the assumption $|\phi^{-1}(\partial E)| = 0 \Leftrightarrow \mu(\partial E) = 0$ we have that $\mu(\text{Int } E) = \mu(\overline{E}) = \mu(E)$, so we are done. $\square$

Now we have an even clearer picture of how the eigenvalues of $T_n(a)$ behave as $n$ grows, especially the property $\mu_n(E) \to \mu(E)$ is quite easy to visualize and gives us a lot of insight. Further, this property makes it easy for us to find the limit sets $\Lambda_w$, $\Lambda_s$ defined in the introduction.

**Corollary 2.7.5.** *Let $a \in W$ be a real-valued symbol, and let $\Lambda = \operatorname{conv}\operatorname{sp} T(a)$. Then we have*

$$\Lambda_s = \Lambda_w = \Lambda.$$

*Proof.* Because of the definition, we always have $\Lambda_s \subset \Lambda_w$, and thanks to Lemma 2.7.1 we have $\Lambda_w \subset \Lambda$. So we only need to show $\Lambda \subset \Lambda_s$. Since $a \in W$, $\phi$, defined by $\theta \mapsto a(e^{i\theta})$ is continuous. Since $a$ is real valued, we know that $a(\mathbb{T}) = \operatorname{conv} \operatorname{sp} T(a)$. So pick $\lambda \in \Lambda$. Then there is an open interval around $\lambda$, $E$ of width $\epsilon > 0$. Because of continuity $\phi^{-1}(E)$ is open, and it cannot be empty, since $a(\mathbb{T}) \ni \lambda \in E$. This implies that $\mu(E) > 0$, and since $\mu_n(E) \to \mu(E)$, $\mu_n(E) > 0$ for sufficiently large $n$. Hence, there is an eigenvalue of $T_n(a)$ in $E$ for all sufficiently large $n$. $\square$

We proceed with an example where we can analytically calculate the limiting measure.

**Example 2.7.6.** Let $a(t) = (t^{-1} + t)/2$. This corresponds to $a(e^{i\theta}) = \cos\theta$. We derive an analytic expression for $\mu$ by a change of variables.

$$
\begin{aligned}
\mu(E) &= \frac{1}{2\pi} \int_0^{2\pi} \chi_E(\cos\theta)\, d\theta \\
&= \frac{1}{\pi} \int_0^{\pi} \chi_E(\cos\theta)\, d\theta \\
&= \begin{bmatrix} \cos\theta = \lambda \\ \theta = \arccos(\lambda) \\ d\theta = -\frac{1}{\sqrt{1-\lambda^2}}\, d\lambda \\ \lambda : 1 \to -1 \end{bmatrix} \\
&= \frac{1}{\pi} \int_{-1}^{1} \chi_E(\lambda) \frac{1}{\sqrt{1-\lambda^2}}\, d\lambda \\
&= \frac{1}{\pi} \int_{[-1,1]\cap E} \frac{1}{\sqrt{1-\lambda^2}}\, d\lambda.
\end{aligned}
$$

So $d\mu = 1/\pi(1-\lambda^2)^{-1/2} d\lambda$ with support on $[-1, 1]$. We numerically calculate the eigenvalues of $T_{500}(a)$ and plot the histogram compared to the density $1/\pi(1 - \lambda^2)^{-1/2}$. The comparison can be seen in Figure 2.2.

## 2.8 Banded Toeplitz matrices

We saw that there is an extensive theory in the case of Hermitian Toeplitz matrices, and the limiting measure could be derived. Next we investigate the case of banded Toeplitz matrices, i.e. how the eigenvalues of $T_n(b)$ behave when $b$ is a Laurent polynomial,

$$
b(t) = \sum_{n=-r}^{s} b_n t^n. \tag{2.24}
$$

The cases where $r$ or $s$ is $\leq 0$ are trivial, since we in those cases get triangular matrices. So we henceforth assume $r, s \geq 1$ and that $b_{-r}$ and $b_s$ are non-zero.

As mentioned in the introduction, one goal in understanding the eigenvalue distribution is finding the strong and weak limit set, $\Lambda_s$ and $\Lambda_w$. From the

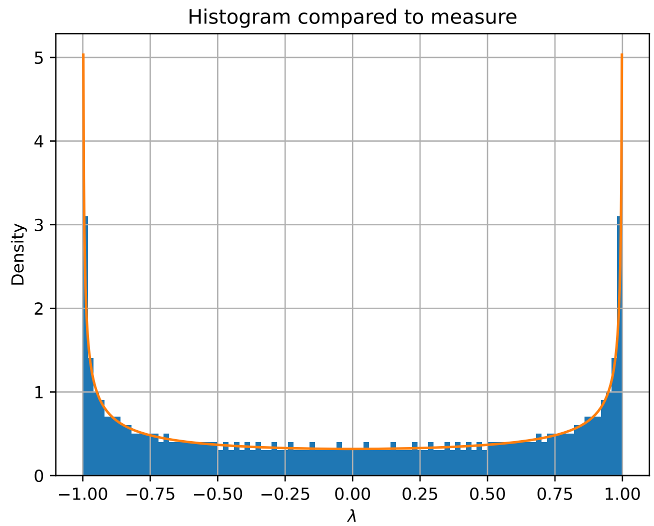



Figure 2.2: A histogram of the eigenvalues for $T_{500}(a)$ with $a = (t^{-1} + t)/2$ along with the theoretical density of the eigenvalues $1/\pi(1 - \lambda^2)^{-1/2}$.

definition we have that $\Lambda_s(b) \subset \Lambda_w(b)$. Also, thanks to Lemma 2.6.1 we see that if we take $\lambda_0 \notin \operatorname{sp} T(b)$, there is a set $\Omega \supset \operatorname{sp} T(b)$ such that $\inf_{z\in\Omega} |z - \lambda_0| > 0$ and $\operatorname{sp} T_n(b) \subset \Omega$ for sufficiently large $n$. So we have actually shown that

$$\Lambda_s(b) \subset \Lambda_w(b) \subset \operatorname{sp} T(b). \tag{2.25}$$

The article that begun the investigation of banded Toeplitz matrices is written by Schmidt and Spitzer, [18]. We will present their results and further theory in this section. Schmidt and Spitzer had the insight that if $\Lambda_s(b)$ and $\Lambda_w(b)$ are to "mimic" the behavior of $\operatorname{sp} T(b)$, then they are forced to also mimic the behavior or $\operatorname{sp} T(b_\rho)$ where $b_\rho(t) := b(\rho t)$, since $T_n(b_\rho)$ and $T(b)$ only differ by a similarity transform,

$$T_n(b_\rho) = \operatorname{diag}(\rho, \rho^2, \cdots, \rho^n)\ T_n(b)\ \operatorname{diag}(\rho^{-1}, \rho^{-2}, \cdots, \rho^{-n}). \tag{2.26}$$

So $\Lambda_s(b_\rho) = \Lambda_s(b)$ and $\Lambda_w(b_\rho) = \Lambda_w(b)$. Using these observations we can strengthen (2.25) into

**Lemma 2.8.1.** *Let $b$ be a Laurent polynomial, then*

$$\Lambda_s(b) \subset \Lambda_w(b) \subset \bigcap_{\rho\in(0,\infty)} \operatorname{sp} T(b_\rho).$$

The big result of Schmidt and Spitzer is that all the sets in Lemma 2.8.1 are equal, and they are a finite union of analytic arcs. To prove these claims we need to investigate how $\operatorname{sp} T(b_\rho)$ behaves when $\rho$ varies, specifically we want to find the intersection. We will begin by calculating the winding number for a Laurent polynomial $b$. Clearly we can view $b$ as a rational function on $\mathbb{C}$ that has one pole of order $r$ in 0 and $r+s$ zeros in $\mathbb{C} \setminus \{0\}$. Assume that $0 \notin b(\mathbb{T})$ and that $b$ has exactly $p$ zeros in the open unit disk $\mathbb{D}$. Then we can factor $b$ as

$$b(t) = b_s t^{-r} \prod_{j=1}^{p} (t - z_j) \prod_{j=p+1}^{r+s} (t - z_j),$$

with $|z_j| < 1$ for $1 \leq j \leq p$ and $|z_j| > 1$ for $p+1 \leq j \leq r+s$. $t^{-r}$ has winding number $-r$ and a factor $(t-z)$ has winding number 1 if $|z| < 1$ and 0 if $|z| > 1$. So the winding number for a Laurent polynomial $b$ is the number of zeros minus the number of poles in $\mathbb{D}$, with our notation $p - r$. We can now use this fact to deduce that $T(b)$ is invertible if and only if $0 \notin b(\mathbb{T})$ and $b$ has $r$ zeros in $\mathbb{D}$. If we instead look at the invertibility of $T(b_\rho)$ we see that is it invertible when $b$ has no zeros on $\rho\mathbb{T}$ and exactly $r$ zeros inside $\rho\mathbb{D}$.

When looking at the spectrum of $T(b_\rho)$ we are interested in when $b_\rho(t) - \lambda$ is invertible, which makes us interested in where the zeros of $b(z) - \lambda$ are located. We therefore introduce

$$Q(\lambda, z) := z^r (b(z) - \lambda) = b_{-r} + b_{-r+1} z + \cdots (b_0 - \lambda) z^r + \cdots b_{s-1} z^{r+s-1} + b_s z^{r+s}.$$

It is clear that $Q(\lambda, z)$ has the same zeros as $b(z) - \lambda$. We are interested when exactly $r$ of the zeros are inside $\rho\mathbb{D}$, so it makes sense to order the zeros. Let $z_j(\lambda), j = 1, 2, \cdots, r+s$ be the zeros of $Q(\lambda, z)$ for a fixed $\lambda$, also order them so that $|z_j(\lambda)| \leq |z_{j+1}(\lambda)|, j = 1, 2, \cdots r+s-1$. Now we can classify $\cap_{\rho \in (0,\infty)} \operatorname{sp} T(b_\rho)$ in terms of the zeros of $Q(\lambda, z)$ by defining the set

$$\Lambda(b) = \{\lambda \in \mathbb{C} : |z_r(\lambda)| = |z_{r+1}(\lambda)|\}.$$

In fact, we can show the following Lemma

**Lemma 2.8.2.** *Let $b$ be a Laurent polynomial, then*

$$\bigcap_{\rho \in (0,\infty)} \operatorname{sp} T(b_\rho) = \Lambda(b).$$

*Proof.* Assume first that $\lambda \notin \cap_{\rho \in (0,\infty)} \operatorname{sp} T(b_\rho)$. Then these is a $\rho$ such that $T(b_\rho - \lambda)$ is invertible. This means that $b_\rho(z) - \lambda$ has no zeros on $\rho\mathbb{T}$ and exactly $r$ zeros inside $\rho\mathbb{D}$. This implies that $|z_r(\lambda)| < \rho < |z_{r+1}(\lambda)|$ and hence that $\lambda \notin \Lambda(b)$.

The other direction can be proved in a similar manner. Assume $\lambda \notin \Lambda(b)$. Then there is a $\rho$ such that $|z_r(\lambda)| < \rho < |z_{r+1}(\lambda)|$ but this implies that $T(b_\rho - \lambda)$ is invertible and hence $\lambda \notin \cap_{\rho \in (0,\infty)} \operatorname{sp} T(b_\rho)$. □

**Example 2.8.3.** Let $b(t) = t^{-1} + 2t$. To calculate $\Lambda(b)$ algebraically we solve $z^{-1} + 2z = \lambda$, which gives $z^2 - \lambda/2z + 1/2 = 0$. If we call the roots $z_1$ and

$z_2$, we see from Vieta's formulas $z_1 z_2 = 1/2$, which implies $z_1 = 1/\sqrt{2}e^{i\theta}$ and $z_2 = 1/\sqrt{2}e^{-i\theta}$, as well as $z_1 + z_2 = \lambda/2$, which implies that $2\sqrt{2}\cos\theta = \lambda$. So it is exactly for $\lambda \in [-2\sqrt{2}, 2\sqrt{2}]$ that the roots could have equal absolute value. Additionally, if $\lambda$ is real and $|\lambda| \leq 2\sqrt{2}$ the roots must be complex conjugated and therefore have equal moduli. Hence $\Lambda(b) = [-2\sqrt{2}, 2\sqrt{2}]$.

We can also look at

$$b(\rho e^{i\theta}) = 2\rho e^{i\theta} + \rho^{-1}e^{-i\theta} = (2\rho + \rho^{-1})\cos\theta + (2\rho - \rho^{-1})i\sin\theta.$$

So $b(\rho\mathbb{T})$ is an ellipse with half axes $2\rho + \rho^{-1}$ and $\left|2\rho - \rho^{-1}\right|$. So the spectrum of $T(b_\rho)$ is exactly the filled in ellipses. We see that all ellipses contain the strip $[-2\sqrt{2}, 2\sqrt{2}]$, since $2\rho + \rho^{-1} \geq 2\sqrt{2\rho\rho^{-1}} = 2\sqrt{2}$. Also, for $\rho = 1/\sqrt{2}$ the ellipse is exactly the strip $[-2\sqrt{2}, 2\sqrt{2}]$, so $\cap_{\rho\in(0,\infty)} \operatorname{sp} T(b_\rho) = [-2\sqrt{2}, 2\sqrt{2}]$.

We will now investigate the structure of $\Lambda(b)$. To do this we will consider how the zeros $z_j(\lambda)$ of $b(z) - \lambda$ (and $Q(\lambda, z)$) vary with $\lambda$. If $b - \lambda$ only has simple zeros, then $b - \lambda$ is locally invertible with analytic inverse around each $z_j(\lambda)$, which implies that $z_j(\lambda)$ depends analytically on $\lambda$. In the case of multiple zeros the dependence of the roots on $\lambda$ is a bit more intricate. But we can introduce a uniformization parameter $t$ such that $\lambda = \lambda_0 + t^m$, for some positive integer $m$ so that the zeros of $Q(\lambda, z)$ depend analytically on $t$. The uniformization parameter comes from the fact the equation $f(z) = 0$ for an arbitrary holomorphic $f$ around a critical point $z_0$ of order $m$ can be written as $f(z) = g(z)^m + f(z_0)$ where $g$ is holomorphic with a simple zero at $z_0$ and therefore is locally invertible. One can then introduce $t := g(z)$ and see that the zeros close to $z_0$ of $b - \lambda$ depend analytically on $t$. If there are different multiple zeros we can choose $m$ as the least common multiple and the statement still holds. An exposition of the local behavior of holomorphic functions can be found in chapter VIII of [13]. All in all we have that for all $\lambda_0 \in \mathbb{C}$ there is a parameter $t$ and integer $m \geq 1$ so that $\lambda = \lambda_0 + t^m$ and the zeros of $Q(\lambda, z)$, $z_j(t)$ depend analytically on $t$, for $t$ in some neighborhood of 0.

A fact we will use later is that $Q(\lambda, z)$ only have multiple zeros for a finite number of $\lambda$'s. This is due to the fact that $Q(\lambda, z)$ has a double zero in $z_0$ exactly when $Q(\lambda, z_0) = 0$ and $\frac{\partial}{\partial z}Q(\lambda, z_0) = 0$, i.e. when the determinant of the resultant of $Q$ and $\frac{\partial}{\partial z}Q$ is zero. This determinant is itself a polynomial in $\lambda$, which of course has a finite number of zeros.

We are currently investigating the structure of $\Lambda(b)$. For a coming argument we will need the fact that the gcd of $\{n : b_n \neq 0\}$ is 1. However, that is not true for all Laurent polynomials $b$ but we can construct a new Laurent polynomial $b^\#$ which fulfills this criterion, and such that $\Lambda(b) = \Lambda(b^\#)$. If the gcd is not 1, it is equal to $d > 1$, this means that $b$ can be written $b(z) = \sum_{j=-r/d}^{s/d} b_{dj} z^{dj}$, we now introduce $b^\#(z) = \sum_{j=-r/d}^{s/d} b_{dj} z^j$ and see that $\Lambda(b) = \Lambda(b^\#)$ since the zeros of $b - \lambda$ are exactly the $d$th roots of the zeros of $b^\# - \lambda$, since $b^\#(z^d) = b(z)$. So we will henceforth without loss of generality assume that the gcd of $\{n : b_n \neq 0\}$ is 1.

It is of interest to consider the ratios $z_l(t)/z_k(t)$ of the zeros of $Q(\lambda, z)$, since $\Lambda(b)$ is given precisely by $|z_r(t)/z_{r+1}(t)| = 1$. We will therefore need the following lemma.

**Lemma 2.8.4.** *The ratios of two different zeros of $Q(\lambda, z)$ cannot be constant, i.e. there is no neighborhood around any $\lambda_0$ so that $z_k(t)/z_l(t) = \gamma$, where $l \neq k$.*

*Proof.* Assume the opposite. Then we have $z_k(t) = \gamma z_l(t)$ with $\gamma \neq 0$ since $Q(\lambda, 0) \neq 0$, which implies

$$0 = \gamma^{-r} Q(\lambda, \gamma z_l(t)) - Q(\lambda, z_l(t)) = \sum_{j=0}^{r+s} b_{j-r}(\gamma^{j-r} - 1) z_l(t)^j.$$

So if we show that $z_l(t)$ cannot be constant, we have that $\gamma^{j-r} = 1$ for all $b_{j-r} \neq 0$. Which with the gcd-assumption implies $\gamma = 1$. So assume that $z_l(t)$ is constant, then we have

$$\begin{aligned} 0 &= Q(\lambda_0 + t^m, z_l(t)) \\ &= b_{-r} + b_{-r+1} z_l(t) + \cdots + (b_0 - (\lambda_0 + t^m)) z_l(t)^r + \cdots b_s z_l(t)^{r+s} \end{aligned}$$

It follows that

$$(\lambda_0 + t^m) = z_l(t)^{-r} Q(0, z_l(t))$$

But this gives us a contradiction since $(\lambda_0 + t^m)$ cannot be constant. By the earlier remarks we now have $\gamma = 1$, but this means that $Q(\lambda, z)$ has multiple zeros for infinitely many $\lambda$, which is a contradiction. $\square$

We will next show the Lemma leading up to the main result on the structure of $\Lambda(b)$.

**Lemma 2.8.5.** *For each $\lambda_0 \in \Lambda(b)$ there is an open neighborhood $U$ of $\lambda_0$ such that $\Lambda(b) \cap U$ is a union of a finite number of analytic arcs. Also, $\Lambda(b)$ contains no isolated points.*

*Proof.* As before we use the notation $\lambda = \lambda_0 + t^m$ with the zeros depending analytically on $t$. Also let $U$ be an open neighborhood around $\lambda$ and let $V$ be the corresponding open neighborhood of 0 for $t$. Since $\lambda_0 \in \Lambda(b)$ there are $p, q \geq 1$ such that

$$\begin{gathered} |z_1(0)| \leq \cdots \leq |z_{r-p}(0)| < \\ |z_{r-p+1}(0)| = \cdots = |z_{r+q}(0)| < \\ |z_{r+q+1}(0)| \leq \cdots \leq |z_{r+s}(0)| . \end{gathered} \tag{2.27}$$

We choose $V$ small enough so that the strict inequalities in (2.27) hold for all $t \in V$. Now consider $\varphi_{jk}(t) := \frac{z_j(t)}{z_k(t)}$ for $j \neq k$ and $r - p + 1 \leq j, k \leq r + q$. Thanks to Lemma 2.8.4 we have that $\varphi_{jk}$ is not constant, so if we make $V$ small enough, the solutions to $|\varphi_{jk}(t)| = 1$ is given by a finite number of analytic arcs originating in $t = 0$ and ending on the boundary of $V$. Now consider the set

$\Gamma = \{t \in V : |\varphi_{jk}(t)| = 1, r-p+1 \leq j,k \leq r+q, j \neq k\}$, i.e. all the arcs for all different $j,k$. We can choose $V$ such that $\Gamma$ consists of a finite number of analytic arcs originating in 0 and ending on the boundary of $V$ without intersecting. This is true since if two arcs are different, their series expansions must differ in some term, we can therefore for each pair of arcs make $V$ small enough so that the differing term makes the arcs distinct throughout $V$ (except for $t = 0$ of course).

We want to know when the $r$th and $(r+1)$th smallest zeros (ordered by absolute value) have the same absolute value. The interesting zeros must have indices in the range $[r-p+1, r+q]$ because of (2.27), and since the arcs in $\Gamma$ only intersect in $t = 0$ the absolute value ordering of the zeros $z_{r-p+1}(0), \cdots, z_{r+q}(0)$ cannot change in $\Gamma \setminus \{0\}$. So the set where the $r$th and $(r+1)$th smallest zeros (ordered by absolute value) have the same absolute value are exactly some or none of the arcs in $\Gamma$. We will now show that it is not none of the arcs, i.e. that $\lambda_0$ is not an isolated point of $\Lambda(b)$. If $\lambda_0$ is isolated we have for some ordering of the zeros

$$|z_j(t)| < |z_k(t)|\,, \quad j = 1, 2, \cdots, r \quad k = r+1, r+2, \cdots r+s$$

for all $t \in V \setminus \{0\}$. But we also have $|z_r(0)| = |z_{r+1}(0)|$. So the function $\varphi(t) := \frac{z_r(t)}{z_{r+1}(t)}$ is analytic on $V$ with $|\varphi(0)| = 1$ and $|\varphi(t)| < 1$ on the rest of $V$, but this violates the maximum modulus principle, so $\lambda_0$ is not isolated in $\Lambda(b)$. $\square$

We are almost ready to formulate our final theorem describing the structure of $\Lambda(b)$. We first introduce the notion of *end point*, which are all the points $\lambda_0 \in \Lambda(b)$ that either are such that $Q(\lambda_0, z)$ have multiple zeros or such that there is no neighborhood $U$ of $\lambda_0$ such that $\Lambda(b) \cap U$ consists of one analytic arc beginning and ending on $\partial U$.

**Theorem 2.8.6.** *The set $\Lambda(b)$ consists of a finite number of pairwise disjoint analytical arcs without endpoints as well as a finite number of end points. Also, $\Lambda(b)$ has no isolated points.*

*Proof.* Thanks to Lemma 2.8.2 we have that $\Lambda(b)$ is compact, since it is the intersection of closed bounded sets. So cover each point in $\Lambda(b)$ with the neighborhoods given by Lemma 2.8.5. Now we have that $\Lambda(b)$ is covered by a finite number of open sets each containing a finite number of analytic arcs without end points and a finite number of end points. Also, Lemma 2.8.5 says that $\Lambda(b)$ does not have any isolated points. $\square$

We conclude that $\Lambda(b)$ and hence $\Lambda_w(b)$ and $\Lambda_s(b)$, must be quite small, they all have measure 0 for the usual Lebesgue measure on $\mathbb{C}$. A natural question that Theorem 2.8.6 raises is whether $\Lambda(b)$ is connected. It turns out that it is, but we will need more tools before proving it.

Out next endeavor is to construct a measure on $\Lambda(b)$ that describes the limiting distribution of eigenvalues. The result we are about to present is due to Hirschman, [15]. But the proofs we present are based on refinements of the original proof due to Widom's work in [23] and [24]. An outline is as follows: We use the Szegő limit theorem to obtain

$$\log |D_n(b_\rho - \lambda)|^{1/n} \to \log |G(b_\rho - \lambda)| \,. \tag{2.28}$$

Then we use potential theory, which says that the generalized Laplacian of $\log|f|$ is a sum of $2\pi$-point measures located at the zeros of $f$, so if we take the Laplacian of (2.28) we obtain that the limiting measure of the eigenvalues is $\frac{1}{2\pi}\Delta_\lambda \log|G(b_\rho - \lambda)|$, which we can simplify using Green's identities. But this is only an outline and now a rigorous proof is presented.

The first result we need is that $\log|D_n(b_\rho - \lambda)|^{1/n}$ converges.

**Lemma 2.8.7.** *There exists a continuous function*

$$g : \mathbb{C} \setminus \Lambda(b) \to \mathbb{R}^+$$

*such that*

$$|D_n(b - \lambda)|^{1/n} \to g(\lambda)$$

*locally uniformly. In fact, $g$ is given by*

$$g(\lambda) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|b_\rho(e^{i\theta}) - \lambda\right| d\theta\right), \tag{2.29}$$

*where $\rho$ is chosen so that*

$$|z_r(\lambda)| < \rho < |z_{r+1}(\lambda)| \tag{2.30}$$

*locally.*

*Proof.* Take $\lambda \in \mathbb{C} \setminus \Lambda(b)$. Then we can choose a $\rho$ that satisfies (2.30) locally. Equation (2.26) implies that $D_n(b_\rho - \lambda) = D_n(b - \lambda)$, so we can shift our focus to $D_n(b_\rho - \lambda)$. Clearly, $T(b_\rho - \lambda)$ is invertible, since the winding number of $b_\rho - \lambda$ is 0. We can therefore use Theorem 2.6.2, and the same reasoning as in the proof of Theorem 2.6.3 to get that

$$\frac{D_{n-1}(b_\rho - \lambda)}{D_n(b_\rho - \lambda)} \to \frac{1}{G(b_\rho - \lambda)}$$

locally uniformly. We earlier remarked that both sides stay away from zero, so

$$\log|D_{n-1}(b_\rho - \lambda)| - \log|D_n(b_\rho - \lambda)| \to -\log|G(b_\rho - \lambda)|$$

locally uniformly. Using the same method with the telescopic sum as in the proof of Theorem 2.6.3 we get that

$$\frac{1}{n}\log|D_n(b_\rho - \lambda)| \to \log|G(b_\rho - \lambda)|$$

locally uniformly. Moving in the $\frac{1}{n}$, and taking the exponent, and noting that both sides are locally bounded we get that

$$|D_n(b - \lambda)|^{1/n} \to |G(b_\rho - \lambda)| =: g(\lambda)$$

locally uniformly. Now it is not difficult to see that some manipulations involving $\left|e^f\right| = e^{\Re f}$ and $\Re \log f = \log|f|$ gives the explicit formula for $g(\lambda)$. □

We shall next derive an alternate expression for $g(\lambda)$ involving the roots $z_j(\lambda)$ of $Q(\lambda, z)$. Essentially, we will rewrite $\exp(\frac{1}{2\pi}\int_0^{2\pi}\log(b_\rho(e^{i\theta})-\lambda)d\theta)$ with the help of logarithmic identities and some algebraic manipulation. However, first we need to discuss our new setting and make some definitions. Take a $\lambda_0 \in \Lambda(b)$ that is not an end point. Then there is an open neighborhood of $\lambda_0$, $U$ such that $\Lambda(b) \cap U$ consists of one analytic arc. Furthermore, $\Lambda(b)$ divides $U$ into two disconnected components that we call $D_1$ and $D_2$. Also, we can choose $U$ small enough so that there exists $p, q \geq 1$, such that for $\lambda \in \Lambda(b) \cap U$

$$\begin{aligned} &|z_1(\lambda)| \leq \cdots \leq |z_{r-p}(\lambda)| < \\ &|z_{r-p+1}(\lambda)| = \cdots = |z_{r+q}(\lambda)| < \\ &|z_{r+q+1}(\lambda)| \leq \cdots \leq |z_{r+s}(\lambda)|. \end{aligned}$$

After reordering we can assume that for $\lambda \in D_1$ we have

$$\max_{1\leq j\leq r} |z_j(\lambda)| < \min_{r+1\leq j\leq r+s} |z_j(\lambda)|.$$

Let $N_1 = \{r+1, r+2, \cdots, r+s\}$. Now we will define $N_2$ similarly for $D_2$ by taking the $s$ zeros with largest modulus. Since $D_2 \cap \Lambda(b) = \emptyset$, the largest $s$ zeros are strictly larger than the smallest $r$ zeros. Hence we have for $\lambda \in D_2$

$$\max_{j\notin N_2} |z_j(\lambda)| < \min_{j\in N_2} |z_j(\lambda)|.$$

Since the zeros depend analytically on $\lambda$ (we are not at an end point) we see that $N_2$ is a union of $\{r+q+1, \cdots, r+s\}$ and $q$ of the values $\{r-p+1, \cdots, r+q\}$. Note that this specifically implies that for $\lambda \in \Lambda(b) \cap U$ we have

$$\prod_{j\in N_1} |z_j(\lambda)| = \prod_{j\in N_2} |z_j(\lambda)|. \tag{2.31}$$

Now we are ready to formulate out next Lemma.

**Lemma 2.8.8.** *For $\lambda \in D_k$ we have*

$$g(\lambda) = |b_s| \prod_{j\in N_k} |z_j(\lambda)|,$$

*for $k = 1, 2$.*

*Proof.* Fix $\lambda \in D_1$ and choose a $\rho$ that satisfies (2.30). We can now factor $b_\rho(e^{i\theta}) - \lambda$ as

$$\begin{aligned} b_\rho(e^{i\theta}) - \lambda &= b_s\rho^{-r}e^{-ri\theta} \prod_{j\notin N_1} (\rho e^{i\theta} - z_j(\lambda)) \prod_{j\in N_1} (\rho e^{i\theta} - z_j(\lambda)) \\ &= b_s \prod_{j\notin N_1} (1 - \rho^{-1}e^{-i\theta} z_j(\lambda)) \prod_{j\in N_1} (\rho e^{i\theta} - z_j(\lambda)). \end{aligned}$$

Now we use the complex logarithm identity to obtain

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \log(b(\rho e^{i\theta}) - \lambda) d\theta \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \log b_s d\theta + \sum_{j \notin N_1} \frac{1}{2\pi}\int_0^{2\pi} \log(1 - \rho^{-1} e^{-i\theta} z_j(\lambda)) d\theta \\
&\quad + \sum_{j \in N_1} \frac{1}{2\pi}\int_0^{2\pi} \log(-z_j(\lambda)) + \log(1 - \rho e^{i\theta}/z_j(\lambda)) d\theta + 2\pi i m,
\end{aligned}
\tag{2.32}
$$

for some integer $m$. Now we use the analytic series expansion $\log(1 - z) = \sum_{j=1}^{\infty} \frac{z^j}{j}$, and the fact that $\int_0^{2\pi} e^{j\theta i} d\theta = 0$ for integers $j \neq 0$. We see that most of (2.32) cancels out. In fact the only terms remaining are

$$
\frac{1}{2\pi}\int_0^{2\pi} \log(b(\rho e^{i\theta}) - \lambda) d\theta = \log b_s + \sum_{j \in N_1} \log(-z_j(\lambda)) + 2\pi i m.
$$

Taking the exponent, and then absolute value of both sides, and taking the formula (2.29) of $g(\lambda)$ into account we obtain our desired result. The argument is the same for $D_2$ and $N_2$. □

We introduce the functions

$$
G_k(\lambda) = \prod_{j \in N_k} (-z_j(\lambda))
$$

for $k = 1, 2$. We note that $G_k$ is analytic in $U$ and $g(\lambda) = |b_s|\,|G_k(\lambda)|$ for $\lambda \in D_k$, and see that we can extend $g$ continuously to all of $U$ since $\overline{D_1} \cup \overline{D_2} \supset U$ and (2.31) holds for $\lambda \in \Lambda(b) \cap U$.

The next lemma gives important information about the size of $g$ near $\Lambda(b)$.

**Lemma 2.8.9.** *For some neighborhood $U$ for a non-endpoint $\lambda \in \Lambda(b)$ we have*

$$
\begin{aligned}
|G_1(\lambda)| &= |G_2(\lambda)| \;\; \text{for } \lambda \in \Lambda(b) \cap U, \\
|G_1(\lambda)| &> |G_2(\lambda)| \;\; \text{for } \lambda \in D_1, \\
|G_1(\lambda)| &< |G_2(\lambda)| \;\; \text{for } \lambda \in D_2.
\end{aligned}
$$

*$D_1$ and $D_2$ is defined as above.*

*Proof.* Equality of the moduli in $\Lambda(b)$ has already been proven, so let us prove that $|G_1(\lambda)| > |G_2(\lambda)|$ for $\lambda \in D_1$. If we use the same ordering as before we see that $N_1 = \{r+1, \cdots, r+s\}$, while $N_2$ is a union of $\{r+q+1, \cdots, r+s\}$ and $q$ of the values $\{r-p+1, \cdots, r+q\}$. So it suffices to show that $N_2$ contains at least one of the indices $\{r-p+1, \cdots, r\}$. If we assume the contrary, this would mean that it is exactly the same roots that have the largest moduli in $D_1$ and $D_2$, i.e. that $N_1 = N_2$. But now we can look at the analytic function $\varphi(\lambda) := \frac{z_r(\lambda)}{z_{r+1}(\lambda)}$. We have $|\varphi(\lambda)| = 1$ on $\Lambda(b) \cap U$ and $|\varphi(\lambda)| < 1$ on $D_1$ and $D_2$.

But this means that $|\varphi(\lambda)| \leq 1$ on $U$ which by the maximum modulus principle implies that $\varphi(\lambda)$ is constant in $U$, which is a contradiction. The corresponding statement for $D_2$ follows by symmetry. $\square$

Let us summarize and add some properties of $g$ that we need to construct the limiting measure on $\Lambda(b)$.

**Lemma 2.8.10.** *The function $g$ can be extended to a function that is positive and continuous on $\mathbb{C}$ except for the end points of $\Lambda(b)$. For non-end points $\lambda \in \Lambda(b)$, let $U$ be a neighborhood give by Lemma 2.8.9 and $n_1 = n_1(\lambda)$ be the outer normal vector of $D_1$ and $n_2 = n_2(\lambda)$ the outer normal vector for $D_2$. Then both $\frac{\partial g}{\partial n_1}$ and $\frac{\partial g}{\partial n_2}$ exist and they fulfill*

$$\frac{\partial g}{\partial n_1}(\lambda) + \frac{\partial g}{\partial n_2}(\lambda) < 0.$$

*Proof.* We already know that $g$ is positive on $\mathbb{C} \setminus \Lambda(b)$ and because all zeros of $Q(\lambda, z)$ are nonzero in the whole of $U$, Lemma 2.8.8 gives the desired statement.

For the statement regarding the normal derivatives, take $\lambda_0 \in \Lambda(b)$ and write $G_1 = u + iv$, we then have $g(\lambda) = \sqrt{u^2 + v^2}$ for $\lambda \in D_1$, and since $G_1(\lambda) \neq 0$ for $\lambda \in U$ we get that $\frac{\partial g}{\partial n_1}(\lambda_0)$ exists and if we write $n_1 = n_{1x} + i n_{1y}$ we get that

$$\begin{aligned}
\frac{\partial g}{\partial n_1}(\lambda) &= (n_{1x}, n_{1y}) \cdot \left( \frac{u_x u + v_x v}{\sqrt{u^2 + v^2}}, \frac{u_y u + v_y v}{\sqrt{u^2 + v^2}} \right) \\
&= \frac{1}{|G_1(\lambda)|} (n_{1x}(u_x u + v_x v) + n_{1y}(v_y v + u_y u)) \\
&= \frac{\Re(\overline{G_1} G_1' n_1)}{|G_1(\lambda)|},
\end{aligned}$$

where the last step is just algebra and an application of the Cauchy–Riemann equations. This implies that we can write

$$\frac{\partial g}{\partial n_1}(\lambda_0) + \frac{\partial g}{\partial n_2}(\lambda_0) = \frac{\Re(\overline{G_1(\lambda_0)} G_1'(\lambda_0) n_1 + \overline{G_2(\lambda_0)} G_2'(\lambda_0) n_2)}{|G_1(\lambda_0)|}. \tag{2.33}$$

It turns out that the right hand term in (2.33) is similar to $(G_1/G_2)'(\lambda_0)$. Observe how $G_1/G_2$ transforms $U$, it sends $D_1$ outside the unit circle, $\Lambda(b)$ to the unit circle and $D_2$ inside the unit circle. This implies that $(G_1/G_2)'(\lambda_0) \neq 0$ since elsewise, the solutions to $|G_1/G_2(\lambda)| = 1$ near $\lambda_0$ would form multiple arcs. This means that the tangent to $\Lambda(b)$ at $\lambda_0$ is transformed to a tangent to the unit circle, i.e. $(G_1/G_2)'(\lambda_0) n_1 i$ is tangent to the unit circle, which means that $(G_1/G_2)'(\lambda_0) n_1$ is orthogonal to the tangent to the unit circle, i.e. $(G_1/G_2)'(\lambda_0) n_1 = -k G_1/G_2(\lambda_0)$ for $k \in \mathbb{R}^+$, since the absolute value of $G_1/G_2(\lambda)$ decreases along $n_1$. Now, noting that $n_2 = -n_1$ we get

$$\begin{aligned}
\left(\frac{G_1}{G_2}\right)'(\lambda_0) n_1 = \frac{G_1' G_2 n_1 + G_1 G_2' n_2}{G_2^2} &= -k \frac{G_1}{G_2} \\
\Leftrightarrow \overline{G_1} G_1' n_1 + \overline{G_2} G_2' n_2 &= -k \left| G_2 \right|^2.
\end{aligned}$$

Now we get that the nominator on the right in (2.33) is negative, which proves exactly what we want. □

It turns out that the final measure actually will involve $\frac{\partial g}{\partial n_1}(\lambda) + \frac{\partial g}{\partial n_2}(\lambda)$, which is why we have payed so much attention to it.

We have now done all the preliminary work, and can take the Laplacian of $\log |D_n(b-\lambda)|^{1/n}$, which was our original idea. Introduce Borel measures $\mu_n$ in a similar way as before, i.e. like (2.22), but now our measure space is $(\mathbb{C}, B(\mathbb{C}))$. We want to take the Laplacian of $\log g$, since it is the limit of $\log |D_n(b-\lambda)|^{1/n} = \frac{1}{n} \log |D_n(b) - \lambda|$. To justify this, we need to show that $\log g$ is subharmonic in $\mathbb{C}$. We haven't yet considered the end points of $\Lambda(b)$. Define $g(\lambda)$ to be 0 here. In the proof of Theorem 2.6.2 we showed that for each $\lambda_0 \in \mathbb{C} \setminus \Lambda(b)$, and $\rho$ satisfying (2.30) there is an open neighborhood around $\lambda_0$ where $G(b_\rho - \lambda)$ is holomorphic, hence $\log g(\lambda) = \log |G(b_\rho - \lambda)|$ is harmonic in a neighborhood of $\lambda_0$ and consequently, in the whole of $\mathbb{C} \setminus \Lambda(b)$. For a neighborhood $U$ of $\lambda_0 \in \Lambda(b)$, not being an end point we showed that $\log g(\lambda) = \max(\log |G_1(\lambda)|, \log |G_2(\lambda)|)$, so $\log g$ is subharmonic for all of $\mathbb{C}$, since $\log g(\lambda) = -\infty$ at the end points. Let $A$ denote the Lebesgue measure on $\mathbb{C}$. Since $\log g \neq -\infty$ for all of $\mathbb{C}$ except the end points we get that the Radon measure $\Delta \log g$ exists and it is the unique Radon measure that fulfills

$$\int_{\mathbb{C}} \varphi \Delta \log g = \int_{\mathbb{C}} \log g \Delta \varphi dA$$

for all $\varphi \in C_0^\infty(\mathbb{C})$, where $C_0^\infty(\mathbb{C})$ is the set of real valued infinitely differentiable functions on $\mathbb{C}$ with compact support. A good exposition about subharmonicity and the Laplacian can be found in [16].

**Lemma 2.8.11.** *The measures $\mu_n$ converge in distributional sense to the Radon measure $\frac{1}{2\pi}\Delta \log g$, i.e.*

$$\int_{\mathbb{C}} \varphi d\mu_n \to \int_{\mathbb{C}} \frac{1}{2\pi} \log g \Delta \varphi dA \tag{2.34}$$

*for all $\varphi \in C_0^\infty(\mathbb{C})$.*

*Proof.* As remarked earlier, potential theory tells us that $\Delta \frac{1}{n} \log |D_n(b-\lambda)| = 2\pi\mu_n$, i.e.

$$\int_{\mathbb{C}} 2\pi \varphi d\mu_n = \int_{\mathbb{C}} \frac{1}{n} \log |D_n(b-\lambda)| \Delta \varphi dA$$

for all $\varphi \in C_0^\infty(\mathbb{C})$. So in order to prove (2.34) we need to show that

$$\int_{\mathbb{C}} \frac{1}{n} \log |D_n(b-\lambda)| \Delta \varphi dA \to \int_{\mathbb{C}} \log g \Delta \varphi dA \tag{2.35}$$

for all $\varphi \in C_0^\infty(\mathbb{C})$. Let $K \supset \operatorname{supp} \varphi$ be a compact set, and let $A_K$ denote the restriction of $A$ to $K$, i.e. $A_K(E) = A(K \cap E)$. Now $A_K$ is a finite Borel measure on $\mathbb{C}$, and since $\Delta\varphi$ has support inside $K$, it is enough to prove (2.35) with $A$ replaced by $A_K$. Furthermore, $\frac{1}{n}\log|D_n(b-\lambda)| \to \log g(\lambda)$, $A$-a.e. since $\Lambda(b)$ has Lebesgue measure 0, and hence also $A_K$-a.e. Because $A_K$ is a finite measure we now get that $\frac{1}{n}\log|D_n(b-\lambda)| \to \log g(\lambda)$ in measure, i.e.

$$A_K\left(\left\{\lambda \in K : \left|\frac{1}{n}\log|D_n(b-\lambda)| - \log g(\lambda)\right| > \epsilon\right\}\right) \to 0$$

for all $\epsilon > 0$. To ease notation let

$$B_\epsilon := \left\{\lambda \in K : \left|\frac{1}{n}\log|D_n(b-\lambda)| - \log g(\lambda)\right| > \epsilon\right\}.$$

Further, let $f_n(\lambda) := \frac{1}{n}\log|D_n(b-\lambda)|\,\Delta\varphi(\lambda)$ and $f(\lambda) = \log g(\lambda)\Delta\varphi(\lambda)$. We show (2.35) by writing

$$\left|\int_{\mathbb{C}} f_n - f dA_K\right| = \left|\int_{B_\epsilon} f_n - f dA_K + \int_{K\setminus B_\epsilon} f_n - f dA_K\right| \leq$$
$$\int_{B_\epsilon} |f_n|\, dA_K + \int_{B_\epsilon} |f|\, dA_K + \epsilon A(K).$$

Now it is sufficient to prove that $\int_{B_\epsilon}|f_n|\,dA_K$ and $\int_{B_\epsilon}|f|\,dA_K$ get small as $A_K(B_\epsilon)$ gets small. It is known from measure theory that if $h$ is an $L_1$ function, then there for all $\epsilon > 0$ exists $\delta > 0$ such that $\int_B |h|\,dA_K < \epsilon$ for all measurable sets $B \subset K$ with $A_K(B) < \delta$, see e.g. chapter 4.2 in [9]. Now, the only thing that could cause trouble is $\int_{B_\epsilon}|f_n|\,dA_K$ not being uniformly small, or, more correctly, $\{f_n\}$ not being uniformly integrable. But $f_n = \Delta\varphi\frac{1}{n}\sum_{j=1}^n \log|\lambda_j^{(n)} - \lambda|$, is the mean of translated logarithms times $\Delta\varphi$, and since $\log|\lambda|$ is locally integrable, i.e. integrable on all compact sets and $\Delta\varphi$ is bounded we get that that $\{f_n\}$ is uniformly integrable, and hence for all $\epsilon > 0$ there is a $\delta > 0$ such that $\int_B |f_n|\,dA_K < \epsilon$ for all $n \in \mathbb{N}$ and $B$ with $A_K(B) < \delta$. Now we can conclude our proof, for each $\epsilon > 0$, pick $N$ large enough so that $A_K(B_\epsilon)$ is small enough to make $\int_{B_\epsilon}|f_n|\,dA_K$ and $\int_{B_\epsilon}|f|\,dA_K < \epsilon$ for all $n > N$. We then have

$$\left|\int_{\mathbb{C}} f_n - f dA_K\right| < (2 + A(K))\epsilon.$$

□

The next step is to simplify $\Delta\log g$ and show that it only has support on $\Lambda(b)$. It makes intuitive sense that it should only have support on $\Lambda(b)$ since as we remarked earlier, $\log g$ is harmonic on $\mathbb{C}\setminus\Lambda(b)$.

**Lemma 2.8.12.** *We have for all $\varphi \in C_0^\infty(\mathbb{C})$*

$$\int_{\mathbb{C}} \varphi\Delta\log g = \int_{\Lambda(b)} \varphi\frac{1}{g}\left|\frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2}\right| ds, \tag{2.36}$$

*where $s$ is the Lebesgue length measure.*

*Proof.* Note that we proved in Lemma 2.8.10 that $\frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2}$ exists and is negative for the non-end points of $\Lambda(b)$, and that $g > 0$ for non-end points on $\Lambda(b)$. Our method is to show (2.36) for a small neighborhood $U$ of some point $\lambda$ in the support of $\varphi$ and then write $\varphi$ as a partition of smooth functions with support on these sets. The most interesting case is to look at a neighborhood $U$ of $\lambda \in \Lambda(b)$ such that $\Lambda(b) \cap U$ just consists of an analytic arc without end points, use the same notation as before with $D_1$ and $D_2$ as the two components of $U \setminus \Lambda(b)$ and $n_1$ and $n_2$ as the outer normal vectors to $D_1$ and $D_2$ at $\Lambda(b)$. Now, Greens formula says

$$\int_\Omega (u\Delta v - v\Delta u)\,dA = \int_{\partial\Omega} \left( u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) ds.$$

Setting $\Omega = D_1$, $u = \log g$ and $v = \varphi$ we get

$$\int_{D_1} (\log g \Delta\varphi - \varphi\Delta \log g)\,dA = \int_{\partial D_1} \left( \log g\frac{\partial \varphi}{\partial n} - \varphi\frac{\partial \log g}{\partial n} \right) ds.$$

Since $\log g$ is harmonic in $D_1$, we get

$$\int_{D_1} \log g \Delta\varphi\,dA = \int_{\partial D_1} \log g\frac{\partial \varphi}{\partial n}\,ds - \int_{\partial D_1} \varphi\frac{\partial \log g}{\partial n}\,ds. \tag{2.37}$$

Similarly for $D_2$ we get

$$\int_{D_2} \log g \Delta\varphi\,dA = \int_{\partial D_2} \log g\frac{\partial \varphi}{\partial n}\,ds - \int_{\partial D_2} \varphi\frac{\partial \log g}{\partial n}\,ds.$$

Adding the two together we get

$$\begin{aligned}\int_{U} \log g \Delta\varphi\,dA &= \int_{\partial U} \log g\frac{\partial \varphi}{\partial n}\,ds - \int_{\partial U} \varphi\frac{\partial \log g}{\partial n}\,ds \\ &+ \int_{U\cap\Lambda(b)} \log g\frac{\partial \varphi}{\partial n_1}\,ds + \int_{U\cap\Lambda(b)} \log g\frac{\partial \varphi}{\partial n_2}\,ds \\ &- \int_{U\cap\Lambda(b)} \varphi\frac{\partial \log g}{\partial n_1}\,ds - \int_{U\cap\Lambda(b)} \varphi\frac{\partial \log g}{\partial n_2}\,ds.\end{aligned}$$

Noting that $\frac{\partial \log g}{\partial n} = \frac{1}{g}\frac{\partial g}{\partial n}$ and $\frac{\partial \varphi}{\partial n_1} + \frac{\partial \varphi}{\partial n_2} = 0$ we can simplify to

$$\begin{aligned}\int_{U} \log g \Delta\varphi\,dA = \int_{\partial U} \log g\frac{\partial \varphi}{\partial n}\,ds - \int_{\partial U} \varphi\frac{1}{g}\frac{\partial g}{\partial n}\,ds \\ - \int_{U\cap\Lambda(b)} \varphi\frac{1}{g}\left(\frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2}\right) ds.\end{aligned} \tag{2.38}$$

From Lemma 2.8.10 we know that $\frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2} < 0$ so

$$-\int_{U\cap\Lambda(b)} \varphi\frac{1}{g}\left(\frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2}\right) ds = \int_{U\cap\Lambda(b)} \varphi\frac{1}{g}\left|\frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2}\right| ds.$$

If $\varphi$ has compact support in $U$, we see that two of the terms in (2.38) vanish. Hence we are left with

$$\int_U \log g \Delta\varphi \, dA = \int_{U\cap\Lambda(b)} \varphi \frac{1}{g} \left| \frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2} \right| ds. \tag{2.39}$$

So (2.39) holds if $\varphi$ has compact support in $U$, where $U$ is a neighborhood of some $\lambda \in \Lambda(b)$ such that $U \cap \Lambda(b)$ just consists of an analytic arc without endpoints. If instead $U$ was just an open subset of $\mathbb{C} \setminus \Lambda(b)$ and $\varphi$ had compact support in $U$, we see from (2.37) that $\int_U \log g \Delta\varphi \, dA = 0$.

Now, we need to argue that (2.39) implies (2.36). Let $K$ be a compact set such that $\operatorname{supp} \varphi \subset K$, assume further that $K \supset \Lambda(b)$, and construct an open covering for $K$ in the following way: For each $\epsilon > 0$ construct open balls with radius $\epsilon$ around all the end points, for the other $\lambda \in K$, if $\lambda \notin \Lambda(b)$, take an open ball around $\lambda$ that is a subset of $\mathbb{C} \setminus \Lambda(b)$, and for $\lambda \in \Lambda(b)$ choose as usual an open neighborhood $U$ so that $U \cap \Lambda(b)$ just consists of an analytic arc. Since $K$ is compact we get that there are finite index sets $I_\epsilon$ and $J_\epsilon$ such that $K \subset (\cup_{\alpha\in I_\epsilon} U_\alpha) \cup (\cup_{\beta\in J_\epsilon} B_\beta)$, where $J_\epsilon$ consists of all discs around the end points and $I_\epsilon$ the rest of the covering sets. Now choose a partition of unity $1 = \sum_{\alpha\in I_\epsilon} \psi_\alpha + \sum_{\beta\in J_\epsilon} \psi_\beta$, where all $\psi_\alpha$ and $\psi_\beta$ are in $C_0^\infty(\mathbb{C})$ with compact support in $U_\alpha$ and $B_\beta$ respectively, and fulfill $0 \le \psi_\alpha, \psi_\beta \le 1$. We first want to show that

$$\int_{\Lambda(b)} \frac{1}{g} \left| \frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2} \right| ds < \infty.$$

Define $\psi_n = \sum_{\alpha\in I_{\frac{1}{n}}} \psi_\alpha$, $n \in \mathbb{Z}^+$. We can choose the $\psi_\alpha$'s such that $\psi_1 \le \psi_2 \le \ldots$, and write

$$\begin{aligned} \int_{\mathbb{C}} \psi_n \Delta \log g &= \sum_{\alpha\in I_{\frac{1}{n}}} \int_{\mathbb{C}} \psi_\alpha \Delta \log g \\ &= \sum_{\alpha\in I_{\frac{1}{n}}} \int_{\Lambda(b)} \psi_\alpha \frac{1}{g} \left| \frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2} \right| ds \\ &= \int_{\Lambda(b)} \psi_n \frac{1}{g} \left| \frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2} \right| ds. \end{aligned} \tag{2.40}$$

In the second equality we used (2.39) since $\psi_\alpha$ have compact support on well behaving sets. Note that $\psi_n$ converges a.e. to 1 on $K$, and that $\frac{1}{g} \left| \frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2} \right| > 0$, and that $\int_{\mathbb{C}} \psi_n \Delta \log g$ can be bounded by $\Delta \log g(K_e)$ where $K_e$ is some compact set with $K_e \supset K$ such that $\psi_\alpha$ all have support inside $K_e$. Hence we can use the monotone convergence theorem on (2.40) and see that

$$\lim_{n\to\infty} \int_{\Lambda(b)} \psi_n \frac{1}{g} \left| \frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2} \right| ds = \int_{\Lambda(b)} \frac{1}{g} \left| \frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2} \right| ds \le \Delta \log g(K_e) < \infty.$$

Now we can use a similar argument with $\varphi_n := \varphi\psi_n$, but with the dominated convergence theorem instead. Since $|\varphi_n| \leq |\varphi|$ and $\varphi$ is bounded we get that

$$\begin{aligned}\int_{\mathbb{C}} \varphi \Delta \log g &= \lim_{n\to\infty} \int_{\mathbb{C}} \varphi_n \Delta \log g \\ &= \lim_{n\to\infty} \int_{\Lambda(b)} \varphi_n \frac{1}{g} \left| \frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2} \right| ds \\ &= \int_{\Lambda(b)} \varphi \frac{1}{g} \left| \frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2} \right| ds.\end{aligned}$$

Which concludes our proof. For a reference on how to construct partitions of unity, see Theorem 1.8.4 in [19].

□

Combining Lemmas 2.8.11 and 2.8.12 we get our main theorem about the distribution of the eigenvalues on $\Lambda(b)$. The theorem was first presented by Hirschman in [15].

**Theorem 2.8.13.** *The measures $\mu_n$ converge in the weak sense to the measure*

$$\frac{1}{2\pi} \frac{1}{g} \left| \frac{\partial g}{\partial n_1} + \frac{\partial g}{\partial n_2} \right| ds.$$

*supported on $\Lambda(b)$, i.e.*

$$\frac{1}{n} \sum_{j=1}^{n} \varphi(\lambda_j^{(n)}) \to \frac{1}{2\pi} \int_{\Lambda(b)} \varphi(\lambda) \frac{1}{g(\lambda)} \left| \frac{\partial g}{\partial n_1}(\lambda) + \frac{\partial g}{\partial n_2}(\lambda) \right| ds(\lambda) \tag{2.41}$$

*for all continuous $\varphi : \mathbb{C} \to \mathbb{R}$ with compact support.*

*Proof.* We have $\int_{\mathbb{C}} \phi \, d\mu_n = \frac{1}{n} \sum_{j=1}^{n} \varphi(\lambda_j^{(n)})$, so from Lemmas 2.8.11 and 2.8.12 we get that (2.41) holds for all $\phi \in C_0^\infty(\mathbb{C})$, but since $C_0^\infty(\mathbb{C})$ is dense in $C_0(\mathbb{C})$, it follows that (2.41) holds for all continuous $\varphi$ with compact support. □

We next give a result that was first proven by Schmidt and Spitzer in [18]. Schmidt and Spitzer proved their result in 1960 before Hirschman proved his result in 1967. The proof we give here builds upon the result of Hirschman, so it differs from the techniques Schmidt and Spitzer used.

**Theorem 2.8.14.** *Using the previous notation with limiting sets of eigenvalues we have*

$$\Lambda_s(b) = \Lambda_w(b) = \Lambda(b)$$

*Proof.* In Lemma 2.8.1 we showed that $\Lambda_s(b) \subset \Lambda_w(b) \subset \Lambda(b)$, so to prove equality we only need to prove $\Lambda(b) \subset \Lambda_s(b)$. We do this by for each $\lambda_0 \in \Lambda(b)$ showing that for each neighborhood $U$ of $\lambda_0$ there is a $N$ such that for all $n \geq N$

there is an eigenvalue of $T_n(b)$ in $U$. Essentially, we want to put in $\chi_U$ as $\varphi$ in Theorem 2.8.13, but we can't do it directly, since $\chi_U$ is not continuous. Instead, choose a continuous $\varphi \leq 1$ with support in $U$ with $\varphi(\lambda_0) = 1$. Then we have

$$\frac{\#\{j : \lambda_j^{(n)} \in U\}}{n} = \frac{1}{n}\sum_{j=1}^{n} \chi_U(\lambda_j^{(n)}) \geq \frac{1}{n}\sum_{j=1}^{n} \varphi(\lambda_j^{(n)}) \to$$
$$\frac{1}{2\pi}\int_{\Lambda(b)} \varphi(\lambda)\frac{1}{g(\lambda)}\left|\frac{\partial g}{\partial n_1}(\lambda) + \frac{\partial g}{\partial n_2}(\lambda)\right| ds(\lambda) > 0.$$

This shows that $\#\{j : \lambda_j^{(n)} \in U\} > 0$ for all $n$ sufficiently large, so we are done. $\square$

We have before promised a result on the connectedness of $\Lambda(b)$, which now follows. The theorem was first proved by Ullman in [21].

**Theorem 2.8.15.** *The limiting set* $\Lambda(b)$ *is connected.*

*Proof.* Assume that $\Lambda(b)$ is not connected. Then there is a $K \subset \Lambda(b)$ such that $K$ and $\Lambda(b) \setminus K$ form two disconnected components. We now construct open sets $\Omega_1$ and $\Omega_2$ such that $K \subset \Omega_1$ and $\Lambda(b) \setminus K \subset \Omega_2$, where $\Omega_1$ and $\Omega_2$ are simply connected and have smooth boundaries. We can construct $\Omega_1$ and $\Omega_2$ sufficiently far away from each other (since $K$ in closed) such that there exists a $\varphi \in C_0^\infty(\mathbb{C})$ such that $\varphi$ is 1 on $\Omega_1$ and 0 on $\Omega_2$. Denote the limiting measure from Theorem 2.8.13 by $\mu$. Since no points in $\Lambda(b)$ are isolated, we must have $0 < \mu(K) < 1$. Our goal is to calculate $\mu(K)$ in a different way and get that it must be an integer, and hence get a contradiction. From Lemma 2.8.12 we get that

$$\mu(K) = \int_K d\mu = \int_{\Lambda(b)} \varphi\, d\mu = \frac{1}{2\pi}\int_{\mathbb{C}} \log g \Delta\varphi\, dA. \tag{2.42}$$

We now wish to use Green's formula and notice that $\Delta\varphi = 0$ for $\Omega_1$ and $\Omega_2$, which gives us useful boundary conditions. Hence we create $\widetilde{\Omega}_1 \supset K$ and $\widetilde{\Omega}_2 \supset \Lambda(b) \setminus K$, both open simply connected sets with smooth boundaries whose closures are subsets of $\Omega_1$ and $\Omega_2$ respectively. Let $\Omega$ be a bounded subset of $\mathbb{C} \setminus (\widetilde{\Omega}_1 \cup \widetilde{\Omega}_2)$ such that $\Delta\varphi$ has support in $\Omega$. As we have noted earlier, $\log g$ is harmonic in $\mathbb{C} \setminus \Lambda(b) \supset \Omega$, so Green's formula gives us

$$\int_\Omega \log g \Delta\varphi\, dA = \int_{\partial\Omega} \log g \frac{\partial\varphi}{\partial n} - \varphi\frac{\partial \log g}{\partial n}\, ds. \tag{2.43}$$

But $\varphi$ is constant on $\Omega_1$ and $\Omega_2$, and 0 on $\Omega_2$. So (2.42) and the right hand side in (2.43) gives

$$\mu(K) = \frac{1}{2\pi}\int_{\partial\widetilde{\Omega}_1} \frac{\partial \log g}{\partial n}\, ds.$$

Recall from the proof of Lemma 2.8.7 that $g(\lambda) = |G(b_\rho - \lambda)|$ and that

$$G(b_\rho - \lambda) = \lim_{n\to\infty} \frac{D_n(b_\rho - \lambda)}{D_{n-1}(b_\rho - \lambda)} = \lim_{n\to\infty} \frac{D_n(b - \lambda)}{D_{n-1}(b - \lambda)} =: G(b - \lambda),$$

so $g(\lambda) = |G(b-\lambda)|$, where $G(b-\lambda)$ is holomorphic in $\mathbb{C} \setminus \Lambda(b)$. We choose a branch of $\log G(b-\lambda)$ that is analytic in a neighborhood of $\partial\widetilde{\Omega}_1 \setminus \{\lambda_0\}$, where $\lambda_0$ an arbitrary element of $\partial\widetilde{\Omega}_1$, this is possible, because if zero is inside $\widetilde{\Omega}_1$ we draw the branch through $\lambda_0$. This gives

$$\log G(b-\lambda) = \log g + i \arg G(b-\lambda) =: u + iv.$$

Note that we can use the Cauchy–Riemann equations in the neighborhood of $\partial\widetilde{\Omega}_1 \setminus \{\lambda_0\}$, so here $u_x = v_y$ and $u_y = -v_x$. Let $(x(t), y(t))$, $t \in (0,1)$ be a parametrization of $\partial\widetilde{\Omega}_1 \setminus \{\lambda_0\}$ in counter-clockwise direction. With this we can write

$$\begin{aligned}
\mu(K) &= \frac{1}{2\pi} \int_{\partial\widetilde{\Omega}_1} \frac{\partial \log g}{\partial n}\, ds \\
&= \frac{1}{2\pi} \int_0^1 (u_x, u_y) \cdot \frac{(\dot{y}, -\dot{x})}{|(\dot{x}, \dot{y})|}\, |(\dot{x}, \dot{y})|\, dt \\
&= \frac{1}{2\pi} \int_0^1 u_x \dot{y} - u_y \dot{x}\, dt \\
&= \frac{1}{2\pi} \int_0^1 v_x \dot{x} + v_y \dot{y}\, dt \\
&= \frac{1}{2\pi} (v(x(1), y(1)) - v(x(0), y(0))) \in \mathbb{Z}.
\end{aligned}$$

The last step is due to the fact that $v$ is the imaginary part of a branch of $\log G(b-\lambda)$. We got our contradiction, so we are done. $\square$

It is due time for an example of $\Lambda(b)$ that is not a subset of $\mathbb{R}$, which demonstrates the results of Theorems 2.8.13 – 2.8.15. It also presents one of the few types of limit sets that are possible to compute analytically.

**Example 2.8.16.** Let $b(t) = t^{-4} + t^1$. Then

$$\Lambda(b) = \{\lambda \in \mathbb{C} : \lambda = re^{\frac{2\pi i}{5}}, 0 \le r \le 5 \cdot 4^{-4/5}\}.$$

In [18], $b$ on the form $t^{-k} + t^h$, $k, h \ge 1$ were stated as one of the only examples where it is relatively easy to compute $\Lambda(b)$ analytically. However, the proof relies on "careful analysis of trinomials on the form $1 + az^k + z^{k+h}$", and the calculations are not presented in [18], instead they reference [4]. In [18], it also stated that in fact $\operatorname{sp} T_n(b) \subset \Lambda(b)$ for all $n$, so one could say that our specific choice of $b$ is quite well behaving. We plot $\Lambda(b)$ along with $T_{100}(b)$ in Figure 2.3.

## 2.9 Beyond banded matrices

The problem of finding the limit sets $\Lambda_s$ and $\Lambda_w$, and a limiting measure for the eigenvalues of $T_n(a)$ for an arbitrary symbol $a$ is an open question. In

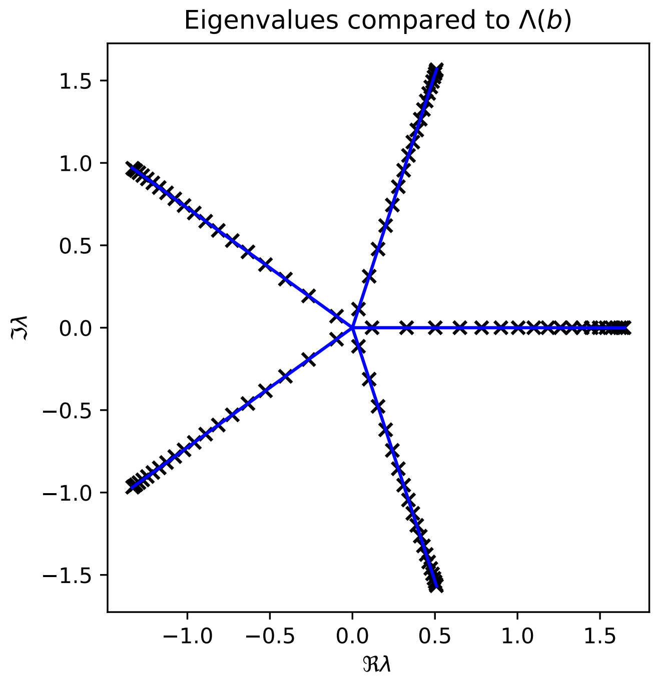



Figure 2.3: $\Lambda(b)$ in blue lines compared to $T_{100}(b)$ for $b(t) = t^{-4} + t$.

1994 Widom summarized the current knowledge on the topic [24]. As we have done in this thesis, Widom begins by describing real symbols and then Laurent polynomials. It should be mentioned that in the case of real symbols, one can generalize the results we have presented to any real symbol in $L^\infty$ [5]. The results for Laurent polynomials can be generalized to rational symbols without poles on the unit circle. This was done by Day in 1975 [11].

Using the same notation as in (2.22) and (2.23) for $\mu_n$ and $\mu$, but defining them on $\mathbb{C}$ instead of $\mathbb{R}$, we say that the eigenvalues of $T_n(a)$ are *canonically distributed* if $\mu_n \to \mu$ weakly. Theorem 2.6.3 indicates that canonically distributed eigenvalues should be the usual case. Of course, as we have seen is the case for Laurent polynomials there are very basic counter examples to this. One of the things we used when dealing with the Laurent polynomials was (2.26) which "forces" $\operatorname{sp} T_n(a)$ to behave similarly to not only $\operatorname{sp} T(a)$, but $\operatorname{sp} T(a_\rho)$ for all $\rho \in (0, \infty)$. A similar argument can be done for any symbol that can be analytically extended into an annulus $r < |z| < 1$ or $1 < |z| < r$, so we should not expect canonical distribution from those symbols [23]. Therefore, Widom claims in [24] that there is "some merit" to the assertion that for a "normal" symbol $a$ that can not be analytically continued to $r < |z| < 1$ nor $1 < |z| < r$, the eigenvalues of $T_n(a)$ are canonically distributed. In [23], Widom proved that for a symbol $a$ that is continuous and piecewise $C^\infty$ but not $C^\infty$ with exactly one singularity point (points where $a$ is not $C^\infty$ in any neighborhood), the eigenvalues of $T_n(a)$ are canonically distributed, and $\Lambda_s = \lambda_w = a(\mathbb{T})$. The proof involves the following result, from [23].

**Theorem 2.9.1.** *Suppose that $a \in L^\infty$ and*

$$|D_n(a-\lambda)|^{1/n} \to G(|a-\lambda|) \tag{2.44}$$

*in measure. Then the eigenvalues of $T_n(a)$ are canonically distributed.*

The proof of 2.9.1, is almost identical to the proof of Lemma 2.8.11, but uses the identity

$$\frac{1}{2\pi}\Delta\frac{1}{2\pi}\int \log\left|a(e^{i\theta})-\lambda\right| d\theta = \mu.$$

If $T(a-\lambda)$ is invertible, then we know that (2.44) holds, thanks to Theorem 2.6.2. But this is only true when $\operatorname{wind}(a-\lambda) = 0$, which is not always the case. So to handle $\operatorname{wind}(a-\lambda) = m$, Widom defines $a_0(t) := t^{-m}(a(t)-\lambda)$. Now, $a_0$ has winding number 0, so the asymptotic formula for $D_n(a_0)$ is known, and using techniques related to what we did with Cramer's rule in the proof of Theorem 2.6.2, it is possible to express $D_n(a)$ in $D_n(a_0)$ and the elements of $T_n^{-1}(a_0)$. To do this, a good understanding of the elements of the inverse matrix is needed, and that is why it is difficult to generalize the result to discontinuous functions [24].

# Chapter 3

# Finding $\Lambda(b)$

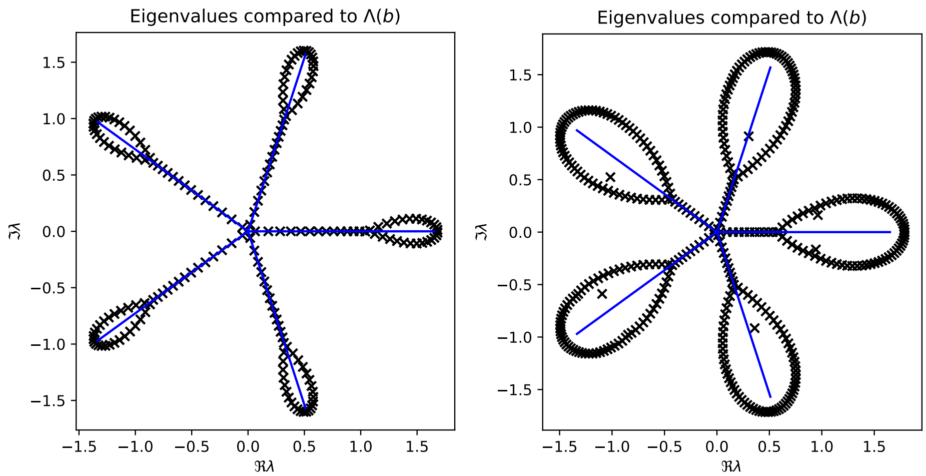
As demonstrated in Section 2.8, there is for banded Toeplitz matrices an extensive theory that describes the structure of the limit set $\Lambda(b)$ and how dense the eigenvalues cluster on $\Lambda(b)$. But we have as of yet not presented a practical way to calculate it. A natural approach to find $\Lambda(b)$ is to calculate $\operatorname{sp} T_n(b)$ for $n$ of increasing size and simply see where the eigenvalues seem to cluster. However, this approach does not work. If we study the $b$ from Example 2.8.16 we know $\Lambda(b)$ exactly, but if we try computing $\operatorname{sp} T_{200}(b)$ and $\operatorname{sp} T_{400}(b)$ using numpy, see Figure 3.1, the eigenvalues do not cluster around $\Lambda(b)$. Also, as we noted in Example 2.8.16, all the eigenvalues of $T_n(b)$ belong to $\Lambda(b)$, so what we see in Figure 3.1 is purely a result of numerical errors. A better method to calculate $\Lambda(b)$ is definitely needed.

In the recent article [7], it is stated that "Finding this limiting set nevertheless remains a challenge". Then the authors propose an algorithm that is based on the article [3], which essentially looks at the roots of $Q(z, \lambda)$. This algorithm gives some points contained in the limit set, and the more sample points used, the better the representation will be. However, this algorithm requires finding zeros of polynomials, which is a hard problem if the degree of the polynomial is big, and for some cases a large number of sampling points are needed. Instead of searching for $\Lambda(b)$ algebraically, we present a novel approach to find it geometrically.

## 3.1 Previous work

A naive approach to calculating $\Lambda(b)$ is to sample a grid in $\mathbb{C}$ and check what points almost are in the set, by solving $b(z) = \lambda$ and looking at the root sizes. However, this approach is time consuming and it is not obvious how a point almost being in $\Lambda(b)$ should be interpreted. In [7] a better algorithm for calculating $\Lambda(b)$ is presented. The idea is: instead of sampling $\lambda$, sample $\varphi \in (0, 2\pi)$ and solve

$$b(z) - b(ze^{i\varphi}) = 0. \tag{3.1}$$




(a) $\Lambda(b)$ and the eigenvalues of $T_{200}(b)$ for $b(t) = t^{-4} + t$.

(b) $\Lambda(b)$ and the eigenvalues of $T_{400}(b)$ for $b(t) = t^{-4} + t$.

Figure 3.1: Examples of bad numerical behavior for computed eigenvalues.

This approach works since if $\lambda \in \Lambda(b)$ then two of the roots of $b(z)-\lambda = 0$, $z_r(\lambda)$ and $z_{r+1}(\lambda)$ have the same modulus, so $z_{r+1}(\lambda)/z_r(\lambda) = e^{i\varphi}$, which implies that $z_r(\lambda)$ is a root of (3.1). Note that in the case $z_r(\lambda) = z_{r+1}(\lambda)$, (3.1) is the equation $0 = 0$. But for double roots, the derivative also has a zero, so to find the cases where $z_r(\lambda) = z_{r+1}(\lambda)$, we solve

$$b'(z) = 0. \tag{3.2}$$

From solving (3.2), and (3.1) for the sampled $\varphi$ we get some candidates $z_k$. For each of these candidates we can calculate the corresponding $\lambda_k := b(z_k)$ and check each of these $\lambda_k$ for membership in $\Lambda(b)$ by solving the equation

$$b(z) = \lambda_k$$

and checking if the $r$'th and $(r+1)$'th smallest root ordered by modulus have the same absolute value. The output from this algorithm is a set of points belonging to $\Lambda(b)$, and it involves numerically finding the roots of multiple polynomials that have degree $r + s$. For fixed $r$ and $s$, the computational time is at most $O(N)$, where $N$ is the amount of sampled $\varphi$.

## 3.2 A geometric approach

One way to look at the problem of finding $\Lambda(b)$ is Lemma 2.8.2, which says

$$\bigcap_{\rho\in(0,\infty)} \operatorname{sp} T(b_\rho) = \Lambda(b).$$

We can either work with the left hand side and try to compute it geometrically, or look at the right hand side and compute it algebraically. In this subsection we will investigate how to compute it geometrically.

The first observation that speaks in favor of $\cap_{\rho \in (0,\infty)} \operatorname{sp} T(b_\rho)$ even being possible to compute numerically is that not all $\rho$'s are needed.

**Theorem 3.2.1.** *Given a Laurent polynomial $b$, there exists $\rho_l, \rho_h$ with $0 < \rho_l < \rho_h < \infty$ such that*

$$\Lambda(b) = \bigcap_{\rho \in (0,\infty)} \operatorname{sp} T(b_\rho) = \bigcap_{\rho \in [\rho_l, \rho_h]} \operatorname{sp} T(b_\rho). \tag{3.3}$$

*Furthermore, $\rho_l$ and $\rho_h$ can be found by solving a polynomial equation with real coefficients.*

*Proof.* Intuitively, (3.3) is clear. We know that $\Lambda(b)$ is bounded, i.e. $|\lambda| \leq K \; \forall \lambda \in \Lambda(b)$ for some constant $K$. As $\rho$ tends to zero, the lowest order term in $b$ will dominate and $b_\rho(\mathbb{T})$ will approximately trace out a circle of radius $b_{-r}\rho^{-r}$ revolving $r$ times in negative direction. Similarly, as $\rho$ tends to $\infty$, $b_\rho(\mathbb{T})$ will approximately trace out a circle of radius $b_s\rho^s$ revolving $s$ times in positive direction. So intersecting with the spectrum for small or large enough $\rho$ will not change the result.

Let $\rho_l$ be the smallest real positive solution to $|b_{-r}|\,\rho^{-r} - \sum_{n=-r+1}^{s} |b_n|\,\rho^n = K$. Hence $|b_{-r}|\,\rho^{-r} - \sum_{n=-r+1}^{s} |b_n|\,\rho^n \geq K$ for all $\rho \leq \rho_l$. We now estimate the size of $b_\rho(t)$. For $\rho$ in $0 < \rho \leq \rho_l$:

$$\begin{aligned} |b_\rho(t)| = \left| \sum_{n=-r}^{s} b_n \rho^n t^n \right| &\geq \left| |b_{-r}|\,\rho^{-r} - \left| \sum_{n=-r+1}^{s} b_n \rho^n t^n \right| \right| \\ &\geq \left| |b_{-r}|\,\rho^{-r} - \sum_{n=-r+1}^{s} |b_n|\,\rho^n \right| \\ &\geq K. \end{aligned}$$

Hence $\Lambda(b) \subset \operatorname{sp} T(b_\rho)$ for $\rho$ in $0 < \rho \leq \rho_l$. Analogously we define $\rho_h$ as the largest positive root to $|b_s|\,\rho^s - \sum_{n=-r}^{s-1} |b_n|\,\rho^n = K$ and do similar calculations to get that $\Lambda(b) \subset \operatorname{sp} T(b_\rho)$ for $\rho \geq \rho_h$, which concludes our proof. $\square$

So, only a compact set of $\rho$'s are needed to obtain $\Lambda(b)$. A natural approach to calculate $\Lambda(b)$ is to select a *partition* $\rho_l = \rho_0 < \rho_1 < \ldots < \rho_{n-1} < \rho_n = \rho_h$ and compute

$$\bigcap_{j=0}^{n} \operatorname{sp} T(b_{\rho_j}). \tag{3.4}$$

The *granularity* of the partition $(\rho_j)_{j=0}^n$ is defined as $\Delta := \max_{0 \leq j \leq n-1}(\rho_{j+1} - \rho_j)$. A natural question now is if (3.4) will approach $\Lambda(b)$, and in what way? We will show that (3.4) approach $\Lambda(b)$ in the Hausdorff metric as the granularity of $(\rho_j)_{j=0}^n$ tends to 0. By the *Hausdorff metric* we mean the metric $d_H$ operating on the set $\mathcal{C}$ consisting of all the bounded non-empty closed subsets of $\mathbb{C}$ defined by

$$d_H(A,B) := \inf\{\epsilon : A \subset (B)_\epsilon, B \subset (A)_\epsilon\},$$

where $(X)_\epsilon$ denotes the $\epsilon$–*fattening* of $X$, defined by

$$(X)_\epsilon := \bigcup_{x\in X} \{z \in \mathbb{C} : |z-x| \leq \epsilon\}.$$

One can show that $(\mathcal{C}, d_H)$ forms a complete metric space. To show that (3.4) approach $\Lambda(b)$ we need the following Theorem.

**Theorem 3.2.2.** *For a Laurent polynomial $b$, the map*

$$\rho \mapsto \operatorname{sp} T(b_\rho)$$

*from $(\mathbb{R}^+, |\cdot|) \to (\mathcal{C}, d_H)$ is continuous.*

*Proof.* Throughout the proof we will view $b$ as a function from $\mathbb{C}\setminus\{0\} \to \mathbb{C}$, but when we write $b_\rho$ we view it as a function from $\mathbb{T} \to \mathbb{C}$. Given $\epsilon > 0$ and $\rho^* > 0$ we want to find $\delta > 0$ such that $d_H(\operatorname{sp} T(b_{\rho^*}), \operatorname{sp} T(b_\rho)) < \epsilon$ for all $\rho$ with $|\rho - \rho^*| < \delta$. Fix an $\epsilon > 0$ and a $\rho^* > 0$ that we will work with throughout the proof. We will work with $\operatorname{wind}(b_\rho - \lambda)$ and use the formula

$$\operatorname{wind}(b_\rho - \lambda) = \frac{1}{2\pi i}\int_{b_\rho(\mathbb{T})} \frac{1}{z-\lambda}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\rho i e^{iv} b'(\rho e^{iv})}{b(\rho e^{iv}) - \lambda}\,dv. \tag{3.5}$$

Hence, it is of interest how the function $zb'(z)/(b(z)-\lambda)$ behaves. We will frequently work with annuli, so we introduce the notation $A(\rho,\delta) = \{z \in \mathbb{C} : ||z| - \rho| \leq \delta\}$, where $\rho$ and $\delta$ are real positive numbers. Since $b$ is continuous on $\mathbb{C}\setminus\{0\}$, there is due to uniform continuity a $\delta_0 > 0$ such that for $z_1, z_2 \in A(\rho^*, \delta_0)$ with $|z_1 - z_2| \leq 2\delta_0$ we have $|b(z_1) - b(z_2)| < \epsilon$. Now, we want to study the poles of $zb'(z)/(b(z)-\lambda)$. Therefore we introduce the set

$$R_\epsilon := \bigcup_{\rho\in[\rho^*-\delta_0,\rho^*+\delta_0]} \left( \bigcup_{\lambda\notin(\operatorname{sp} T(b_\rho))_\epsilon} \{r \in \mathbb{C} : b(r) = \lambda\} \right).$$

In words, $R_\epsilon$ consists of all the roots of $b(r) = \lambda$ for some $\lambda \notin (\operatorname{sp} T(b_\rho))_\epsilon$ for some $\rho \in [\rho^* - \delta_0, \rho^* + \delta_0]$. Note that $R_\epsilon \cap A(\rho^*, \delta_0) = \emptyset$ since if $r \in R_\epsilon$ and $r \in A(\rho^*, \delta_0)$ then there is a $\rho \in [\rho^* - \delta_0, \rho^* + \delta_0]$ such that $\left|b(r) - b(\rho e^{it})\right| > \epsilon$ for all $t \in [0, 2\pi)$, but this implies $\left|r - \rho e^{it}\right| > 2\delta_0$ for all $t \in [0, 2\pi)$, which in turn implies $r \notin A(\rho^*, \delta_0)$. So $R_\epsilon \cap A(\rho^*, \delta_0) = \emptyset$.

For ease of notation, let

$$f(z,\lambda) := zb'(z)/(b(z)-\lambda)$$
$$\Lambda_0 := \bigcup_{\rho\in[\rho^*-\delta_0/2,\rho^*+\delta_0/2]} ((\operatorname{sp} T(b_\rho))_\epsilon)^C$$

To estimate the rightmost integral in (3.5) we roughly want $f$ to be uniformly continuous on $A(\rho^*, \delta_0/2) \times \Lambda_0$. Note that $R_\epsilon \cap A(\rho^*, \delta_0) = \emptyset$ implies that $f$ is continuous on $A(\rho^*, \delta_0/2) \times \Lambda_0$. Formally we want to show that there exists $\delta_r > 0$ such that $z_1, z_2 \in A(\rho^*, \delta_0/2)$ and $\lambda \in \Lambda_0$ with $|z_1 - z_2| < \delta_r$ implies $|f(z_1, \lambda) - f(z_2, \lambda)| < 1/2$. To prove this, assume the opposite, for all $\delta_r > 0$, i.e. there exists $\lambda_0 \in \Lambda_0$ and $z_1, z_2 \in A(\rho^*, \delta_0/2)$ with $|z_1 - z_2| < \delta_r$ but with $|f(z_1, \lambda_0) - f(z_2, \lambda_0)| \geq 1/2$. We now proceed with a standard "passing to a subsequence" argument. Let $\delta_n = 1/n$, and construct $z_{1n}, z_{2n} \in A(\rho^*, \delta_0/2)$ with $|z_{1n} - z_{2n}| < 1/n$, such that both $(z_{1n})$ and $(z_{2n})$ are convergent sequences. Additionally there are $\lambda_n$ such that $|f(z_{1n}, \lambda_n) - f(z_{2n}, \lambda_n)| \geq 1/2$. If $(\lambda_n)$ is unbounded, we get a contradiction since $\sup_{z \in A(\rho^*, \delta_0/2)} |f(z, \lambda)| \to 0$ as $\lambda \to \infty$. So $(\lambda_n)$ must be bounded. Hence we can choose $(\lambda_n)$ to be a convergent sequence. We have $z_{1n} \to z_1$, $z_{2n} \to z_2$ and $\lambda_n \to \lambda$. But $|z_{1n} - z_{2n}| < 1/n$ so $z_1 = z_2$. Because of continuity, $f(z_{1n}, \lambda_n) \to f(z_1, \lambda)$ and $f(z_{2n}, \lambda_n) \to f(z_2, \lambda)$, but this is a contradiction, and so a $\delta_r$ with the specified requirements must exist.

Choose $\delta := \min(\delta_r, \delta_0/2)$. Next we show that $|\rho - \rho^*| < \delta$ implies that $d_H(\operatorname{sp} T(b_{\rho^*}), \operatorname{sp} T(b_\rho)) \leq \epsilon$, i.e. we want to show $(\operatorname{sp} T(b_\rho))_\epsilon \supset \operatorname{sp} T(b_\rho^*)$ and $(\operatorname{sp} T(b_\rho^*))_\epsilon \supset \operatorname{sp} T(b_\rho)$. To do this, take an arbitrary $\lambda_0 \notin (\operatorname{sp} T(b_\rho))_\epsilon$. Hence, $\lambda_0 \in \Lambda_0$. Now we have

$$\begin{aligned} &|\operatorname{wind}(b_\rho - \lambda_0) - \operatorname{wind}(b_{\rho^*} - \lambda_0)| \\ &\qquad = \left| \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{it}, \lambda_0) - f(\rho^* e^{it}, \lambda_0)\, dt \right| \leq \frac{1}{2}, \end{aligned}$$

since $|\rho - \rho^*| < \delta_r$. Because $\operatorname{wind}(b_\rho - \lambda_0)$ always is integer valued, $\operatorname{wind}(b_\rho - \lambda_0) = \operatorname{wind}(b_{\rho^*} - \lambda_0) = 0$ since $\lambda_0 \notin \operatorname{sp} T(b_\rho)$ , and this implies $\lambda_0 \notin \operatorname{sp} T(b_{\rho^*})$. So $(\operatorname{sp} T(b_\rho))_\epsilon \supset \operatorname{sp} T(b_{\rho^*})$ holds. A similar argument can be made in the opposite direction since the integral bound holds for all $\lambda \in \Lambda_0$, which concludes our proof. □

Theorem 3.2.2 is quite intuitively reasonable, we expect $b_\rho(\mathbb{T})$ to vary continuously with $\rho$, and so also the spectrum. We can use Theorem 3.2.2 to show the promised result, that (3.4) approaches $\Lambda(b)$ in the limit.

**Theorem 3.2.3.** *Let $\rho_l$ and $\rho_h$ be the bounds given by Theorem 3.2.1 and let $(\rho_j)_{j=0}^n$ be a partition of $[\rho_l, \rho_h]$, and let $\Delta$ be the granularity of the partition. Then we have*

$$\lim_{\Delta \to 0} \bigcap_{j=0}^{n} \operatorname{sp} T(b_{\rho_j}) = \Lambda(b)$$

*in the Hausdorff metric, i.e. for each $\epsilon > 0$ there exists a $\delta > 0$ such that for all partitions with $\Delta < \delta$ it holds that $d_H(\cap_{j=0}^n \operatorname{sp} T(b_{\rho_j}), \Lambda(b)) < \epsilon$.*

*Proof.* From Theorem 3.2.2 we have that the map $\rho \mapsto \operatorname{sp} T(b_\rho)$ from $[\rho_l, \rho_h] \to \mathcal{C}$ is uniformly continuous. Given an $\epsilon > 0$ there is a $\delta_s > 0$ such that for all

$\rho^* \in [\rho_l, \rho_h]$; $|\rho - \rho^*| < \delta_s$ implies $d_H(\operatorname{sp} T(b_{\rho^*}), \operatorname{sp} T(b_\rho)) < \epsilon$. We now show that for all partitions $(\rho_j)_{j=0}^n$ with granularity $\Delta < 2\delta_s$ we have

$$d_H\left(\bigcap_{j=0}^{n} \operatorname{sp} T(b_{\rho_j}), \bigcap_{\rho\in[\rho_l,\rho_h]} \operatorname{sp} T(b_\rho)\right) \leq \epsilon.$$

Naturally we always have $\bigcap_{\rho\in[\rho_l,\rho_h]} \operatorname{sp} T(b_\rho) \subset \left(\bigcap_{j=0}^{n} \operatorname{sp} T(b_{\rho_j})\right)_\epsilon$. So we need to prove the opposite inclusion. To do this, take an arbitrary

$$\lambda \notin \left(\bigcap_{\rho\in[\rho_l,\rho_h]} \operatorname{sp} T(b_\rho)\right)_\epsilon = \bigcap_{\rho\in[\rho_l,\rho_h]} (\operatorname{sp} T(b_\rho))_\epsilon .$$

So there is a $\rho^* \in [\rho_l, \rho_h]$ such that $\lambda \notin (\operatorname{sp} T(b_{\rho^*}))_\epsilon$. Because $\Delta < 2\delta_s$ there is a $\rho_k$ in the partition with $|\rho^* - \rho_k| < \delta_s$. This implies $d_H(\operatorname{sp} T(b_{\rho^*}), \operatorname{sp} T(b_\rho)) < \epsilon$, which means $(\operatorname{sp} T(b_{\rho^*}))_\epsilon \supset \operatorname{sp} T(b_{\rho_k})$ so $\lambda \notin \operatorname{sp} T(b_{\rho_k})$, and therefore not in $\cap_{j=0}^n \operatorname{sp} T(b_{\rho_j})$ either. Hence our proof is finished. □

Theorem 3.2.3 hints at great possibilities for computing $\Lambda(b)$ numerically. But it is difficult to represent $\operatorname{sp} T(b_\rho)$ numerically. A possible solution is to approximate $\operatorname{sp} T(b_\rho)$ as a polygon with vertices sampled along $b_\rho(\mathbb{T})$. The next theorem will show that polygon approximation indeed gives the desired result. We will partition $[0, 2\pi]$ with $0 = v_0 < c_1 < \ldots < v_{n-1} < v_n = 2\pi$. The granularity for $(v_j)_{j=0}^n$ is defined as before. A partition of $[0, 2\pi]$ defines a polygon discretization $b_\rho^D$ approximating $b_\rho$, on the form $b_\rho^D : [0, 2\pi) \to \mathbb{C}$,

$$b_\rho^D(v) = \frac{v - v_j}{v_{j+1} - v_j}\left(b(\rho e^{iv_{j+1}}) - b(\rho e^{iv_j})\right) + b(\rho e^{iv_j}),\ v \in [v_j, v_{j+1}).$$

We define $\operatorname{sp} T(b_\rho^D)$ for a polygon as

$$b_\rho^D([0, 2\pi)) \cup \left\{\lambda \in \mathbb{C} : \operatorname{wind}(b_\rho^D - \lambda) \neq 0\right\}.$$

Now we are ready to formulate our next Theorem.

**Theorem 3.2.4.** *Let $\rho_l$ and $\rho_h$ be the bounds given by Theorem 3.2.1. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that for all partitions $(v_j)_{j=0}^n$ of $[0, 2\pi]$ with granularity $\Delta < \delta$ we have*

$$d_H\left(\operatorname{sp} T(b_\rho), \operatorname{sp} T(b_\rho^D)\right) < \epsilon$$

*for all $\rho \in [\rho_l, \rho_h]$.*

*Proof.* This proof is quite similar to that of Theorem 3.2.2, we will bound the difference winding number integral

$$\begin{aligned}&\operatorname{wind}(b_\rho - \lambda) - \operatorname{wind}(b_\rho^D - \lambda)\\ &\quad= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\frac{d}{dv} b(\rho e^{iv})}{b(\rho e^{iv}) - \lambda} - \frac{\frac{d}{dv} b_\rho^D(v)}{b_\rho^D(v) - \lambda}\, dv.\end{aligned} \tag{3.6}$$

To do that we need some bounds.

Comparing $b_\rho^D$ with $b_\rho$ it is convenient to write

$$b(\rho e^{iv}) = x_\rho(v) + iy_\rho(v).$$

Further, we see

$$\begin{aligned} x_\rho(v) &= \Re b(\rho e^{iv}) = \frac{b(\rho e^{iv}) + \overline{b}(\rho e^{-it})}{2}, \\ y_\rho(v) &= \Im b(\rho e^{iv}) = \frac{b(\rho e^{iv}) - \overline{b}(\rho e^{-iv})}{2i}, \end{aligned}$$

with $\overline{b}(z) = \sum_{n=-r}^{s} \overline{b_n} z^n$. By differentiating we get

$$\begin{aligned} \dot{x}_\rho(v) &= \frac{i\rho e^{iv} b'(\rho e^{iv}) - i\rho e^{iv} \overline{b}'(\rho e^{-iv})}{2}, \\ \dot{y}_\rho(v) &= \frac{i\rho e^{iv} b'(\rho e^{iv}) + i\rho e^{iv} \overline{b}'(\rho e^{-iv})}{2i}. \end{aligned}$$

Specifically we see that $\dot{x}_\rho(v)$ and $\dot{y}_\rho(v)$ are uniformly continuous on $[\rho_l, \rho_h] \times [0, 2\pi]$, and that $\left|\frac{d}{dv} b(\rho e^{iv})\right| < M < \infty$ on $[\rho_l, \rho_h] \times [0, 2\pi]$.

Next we wish to compare $\frac{d}{dv} b_\rho^D(v)$ and $\frac{d}{dv} b(\rho e^{iv}) = \dot{x}_\rho(v) + i\dot{y}_\rho(v)$. For $v \in (v_j, v_{j+1})$ we have

$$\frac{d}{dv} b_\rho^D(v) = \frac{b(\rho e^{iv_{j+1}}) - b(\rho e^{iv_j})}{v_{j+1} - v_j} = \dot{x}_\rho(v_{jx}) + i\dot{y}_\rho(v_{jy}),$$

for some $v_{jx}, v_{jy} \in (v_j, v_{j+1})$ by the mean value theorem. Hence

$$\left|\frac{d}{dv} b(\rho e^{iv}) - \frac{d}{dv} b_\rho^D(v)\right| \le |\dot{x}_\rho(v) - \dot{x}_\rho(v_{jx})| + |\dot{y}_\rho(v) - \dot{y}_\rho(v_{jy})|, \tag{3.7}$$

for $v \in [v_j, v_{j+1})$. A similar argument can be made for $b(\rho e^{iv}) - \lambda$ and $b_\rho^D(v) - \lambda$ with $v \in (v_j, v_{j+1})$:

$$\begin{aligned} \left|b_\rho^D(v) - b(\rho e^{iv})\right| &= \left|(v - v_j)(\dot{x}_\rho(v_{jx}) + \dot{y}_\rho(v_{jy})) + b(\rho e^{iv_j}) - b(\rho e^{iv})\right| \\ &= \left|(v - v_j)(\dot{x}_\rho(v_{jx}) + \dot{y}_\rho(v_{jy})) - (v - v_j)(\dot{x}_\rho(v_{jx}^*) + \dot{y}_\rho(v_{jy}^*))\right| \\ &\le |v - v_j| \left(\left|(\dot{x}_\rho(v_{jx}) - (\dot{x}_\rho(v_{jx}^*)\right| + \left|(\dot{y}_\rho(v_{jy}) - (\dot{y}_\rho(v_{jy}^*)\right|\right), \end{aligned} \tag{3.8}$$

for some $v_{jx}^*, v_{jy}^* \in (v_j, v_{j+1})$.

Now we can begin assembling the different estimates. To simplify reasoning, we will bound

$$\frac{\frac{d}{dv}b(\rho e^{iv})}{b(\rho e^{iv})-\lambda}-\frac{\frac{d}{dv}b_\rho^D(v)}{b_\rho^D(v)-\lambda}=\frac{\frac{d}{dv}b(\rho e^{iv})}{b(\rho e^{iv})-\lambda}-\frac{\frac{d}{dv}b(\rho e^{iv})}{b_\rho^D(v)-\lambda}+\frac{\frac{d}{dv}b(\rho e^{iv})}{b_\rho^D(v)-\lambda}-\frac{\frac{d}{dv}b_\rho^D(v)}{b_\rho^D(v)-\lambda}$$
$$=\frac{d}{dv}b(\rho e^{iv})\left(\frac{1}{b(\rho e^{iv})-\lambda}-\frac{1}{b_\rho^D(v)-\lambda}\right)$$
$$+\frac{1}{b_\rho^D(v)-\lambda}\left(\frac{d}{dv}b(\rho e^{iv})-\frac{d}{dv}b_\rho^D(v)\right). \tag{3.9}$$

Now, let $\epsilon > 0$ be as given in the statement of the theorem. We want to show that $(\operatorname{sp} T(b_\rho))_\epsilon \supset \operatorname{sp} T(b_\rho^D)$ and $(\operatorname{sp} T(b_\rho^D))_\epsilon \supset \operatorname{sp} T(b_\rho)$, so we will assume $\lambda \notin (\operatorname{sp} T(b_\rho^D))_\epsilon$ or $\notin (\operatorname{sp} T(b_\rho))_\epsilon$. Let $\Delta_1 > 0$ be given by the uniform continuity of $\dot{x}_\rho(v)$ and $\dot{y}_\rho(v)$, such that $|v_1 - v_2| < \Delta_1$ implies $|\dot{x}_\rho(v_1) - \dot{x}_\rho(v_2)| < \epsilon/4$ and $|\dot{y}_\rho(v_1) - \dot{y}_\rho(v_2)| < \epsilon/4$. Further, let $\Delta_1 < 1$. From (3.8) we see that $\left|b_\rho^D(v) - b(\rho e^{iv})\right| \leq \epsilon/2$ for partitions with granularity $< \Delta_1$. This means that since either $\lambda \notin (\operatorname{sp} T(b_\rho^D))_\epsilon$ or $\notin (\operatorname{sp} T(b_\rho))_\epsilon$, we have $\left|b(\rho e^{iv}) - \lambda\right| \geq \epsilon/2$ and $\left|b_\rho^D(v) - \lambda\right| \geq \epsilon/2$. The map $z \mapsto 1/z$ is uniformly continuous on $\{z \in \mathbb{C} : |z| \geq \epsilon/2\}$. Let $\delta_2$ be such that $z_1, z_2 \in \{z \in \mathbb{C} : |z| \geq \epsilon/2\}$ with $|z_1 - z_2| < \delta_2$ implies $|1/z_1 - 1/z_2| < 1/4M$. Now, let $\Delta_2$ be less than $\Delta_1$ and such that $\left|b_\rho^D(v) - b(\rho e^{iv})\right| < \delta_2$ for partitions with granularity $< \Delta_2$. We have now managed to get the bound

$$\left|\frac{d}{dv}b(\rho e^{iv})\left(\frac{1}{b(\rho e^{iv})-\lambda}-\frac{1}{b_\rho^D(v)-\lambda}\right)\right| < \frac{1}{4}, \tag{3.10}$$

for partitions with granularity $< \Delta_2$.

Next, let $\Delta_3$ be $\leq \Delta_2$ and such that $\left|\frac{d}{dv}b_\rho^D(v) - \frac{d}{dv}b(\rho e^{iv})\right| < \epsilon/8$ for partitions of granularity $< \Delta_3$, this is possible thanks to the uniform continuity of $\dot{x}_\rho(v)$ and $\dot{y}_\rho(v)$, and (3.7). From this we get the bound

$$\left|\frac{1}{b_\rho^D(v)-\lambda}\left(\frac{d}{dv}b(\rho e^{iv})-\frac{d}{dv}b_\rho^D(v)\right)\right| < \frac{2}{\epsilon}\frac{\epsilon}{8} = \frac{1}{4}. \tag{3.11}$$

Now let $\delta = \Delta_3$. Combining (3.10) and (3.11) with (3.9) we see from (3.6) that $\left|\operatorname{wind}(b_\rho - \lambda) - \operatorname{wind}(b_\rho^D - \lambda)\right| < 1/2$ for any $\lambda \notin (\operatorname{sp} T(b_\rho^D))_\epsilon$ or $\notin (\operatorname{sp} T(b_\rho))_\epsilon$ for all partitions with granularity less than $\delta$. Hence, using the same reasoning as in the end of the proof for Theorem 3.2.3 we get that $d_H\left(\operatorname{sp} T(b_\rho), \operatorname{sp} T(b_\rho^D)\right) < \epsilon$ which concludes our proof. $\square$

With this result we are ready to prove a stronger version of Theorem 3.2.3.

**Theorem 3.2.5.** *Let $\rho_l$ and $\rho_h$ be the bounds given by Theorem 3.2.1, and let $(\rho_j)_{j=0}^n$ denote partitions of $[\rho_l, \rho_h]$ with granularity $\Delta_\rho$, and let $(v_j)_{j=0}^m$ denote partitions of $[0, 2\pi]$ with granularity $\Delta_v$. Then*

$$\lim_{\Delta_\rho, \Delta_v \to 0} \bigcap_{j=0}^{n} \operatorname{sp} T(b_{\rho_j}^D) = \Lambda(b)$$

*in the Hausdorff metric, i.e. for each $\epsilon > 0$ there exists $\delta_\rho > 0$ and $\delta_v > 0$ such that $\Delta_\rho < \delta_\rho$ and $\Delta_v < \delta_v$ implies $d_H\left(\cap_{j=0}^n \operatorname{sp} T(b_\rho^D), \Lambda(b)\right) < \epsilon$.*

*Proof.* We can use the triangle inequality for $d_H$ to estimate

$$d_H\left(\bigcap_{j=0}^n \operatorname{sp} T(b_{\rho_j}^D), \Lambda(b)\right) \leq d_H\left(\bigcap_{j=0}^n \operatorname{sp} T(b_{\rho_j}^D), \bigcap_{j=0}^n \operatorname{sp} T(b_{\rho_j})\right) + d_H\left(\bigcap_{j=0}^n \operatorname{sp} T(b_{\rho_j}), \Lambda(b)\right).$$

Let $\delta_\rho$ be such that $d_H\left(\cap_{j=0}^n \operatorname{sp} T(b_{\rho_j}), \Lambda(b)\right) < \epsilon/2$ for all partitions of $[\rho_l, \rho_h]$ with granularity less than $\delta_\rho$, which exists by Theorem 3.2.3. Now, choose $\delta_v$ such that $d_H\left(\operatorname{sp} T(b_\rho), \operatorname{sp} T(b_\rho^D)\right) < \epsilon/2$ for all partitions of $[0, 2\pi]$ with granularity less than $\delta_v$. Using this we can prove that $d_H\left(\cap_{j=0}^n \operatorname{sp} T(b_{\rho_j}^D), \cap_{j=0}^n \operatorname{sp} T(b_{\rho_j})\right) < \epsilon/2$, by noting that

$$\begin{aligned}
\left(\bigcap_{j=0}^n \operatorname{sp} T(b_{\rho_j}^D)\right)_{\epsilon/2} &= \bigcap_{j=0}^n \left(\operatorname{sp} T(b_{\rho_j}^D)\right)_{\epsilon/2} \supset \bigcap_{j=0}^n \operatorname{sp} T(b_{\rho_j}), \\
\left(\bigcap_{j=0}^n \operatorname{sp} T(b_{\rho_j})\right)_{\epsilon/2} &= \bigcap_{j=0}^n \left(\operatorname{sp} T(b_{\rho_j})\right)_{\epsilon/2} \supset \bigcap_{j=0}^n \operatorname{sp} T(b_{\rho_j}^D),
\end{aligned}$$

since $(\operatorname{sp} T(b_{\rho_j}))_{\epsilon/2} \supset \operatorname{sp} T(b_{\rho_j}^D)$ and $(\operatorname{sp} T(b_{\rho_j}^D))_{\epsilon/2} \supset \operatorname{sp} T(b_{\rho_j})$ for all $j$ since $d_H\left(\operatorname{sp} T(b_\rho), \operatorname{sp} T(b_\rho^D)\right) < \epsilon/2$ for all $\rho \in [\rho_l, \rho_h]$. This concludes our proof since we now have $d_H\left(\bigcap_{j=0}^n \operatorname{sp} T(b_{\rho_j}^D), \Lambda(b)\right) \leq \epsilon/2 + \epsilon/2 = \epsilon$. □

Thanks to Theorem 3.2.5 we have the outlines of an algorithm that we know will converge to $\Lambda(b)$. Given $b$ we sample $\rho$'s in $[\rho_l, \rho_h]$, and for each sampled $\rho$ construct an estimating polygon. Then, taking the intersection with respect to non-zero winding number of all the sampled polygons yields an estimate of $\Lambda(b)$. By Theorem 3.2.5 we know that the intersection of the polygons will approach $\Lambda(b)$ as we decrease the granularities. Pseudocode for the algorithm can be seen in Algorithm 1.

## 3.3 Implementation details

The main operation we need in Algorithm 1 is polygon intersection, which is a common problem in the field computer graphics. Hence, it has been thoroughly researched, and there exists efficient implementations. For this thesis, we use the python library pyclipper [2], which is a python wrapper library for the C++ library Clipper2 [1]. The library is based on Vatti's clipping algorithm, which is

**Algorithm 1** Basic approach to calculating $\Lambda(b)$ geometrically

**procedure** CALCLIMITSET$(b, n, m)$
$b$: the symbol,
$n$: the number of sampled $\rho$'s,
$m$: number of sampled $v$'s.
  $\rho_l, \rho_h \leftarrow$ bounds from theorem 3.2.1
  rhos $\leftarrow$ sample $n+1$ $\rho$'s in $[\rho_l, \rho_h]$
  vs $\leftarrow$ sample $m+1$ $v$'s in $[0, 2\pi]$
  $\Lambda \leftarrow b^D_{rhos[1]}(vs)$
  **for** $i \leftarrow 2$ to $n+1$ **do**
    $\Lambda \leftarrow \Lambda \cap b^D_{rhos[i]}(vs)$
  **end for**
  **return** $\Lambda$
**end procedure**

described in his paper [22]. The processing time for intersecting two polygons using Vatti's algorithm scales linearly in the number of vertices of the polygons being intersected.

Besides finding a good polygon intersection algorithm, the most important details about the implementation are how to choose $(v_j)_{j=0}^m$ and $(\rho_j)_{j=0}^n$, i.e. where to choose the vertices of $b^D_{\rho_j}$ on $b_{\rho_j}(\mathbb{T})$ and what $\rho$'s to choose for the intersections. We know from Theorem 3.2.5 that if we just choose the granularities finer and finer, the intersection will approach $\Lambda(b)$, but we should be able to get a more computationally efficient algorithm if the partitions are chosen in a good way.

For simplicity, the only investigated partition for $v$ is the uniform one, i.e. $v_j = 2\pi j/m$, it is quite difficult to imagine some strategy working much better. Choosing $\rho$'s is more interesting. A natural approach is to first calculate $[\rho_l, \rho_h]$ using the estimate from Theorem 3.2.1. Then, a naive solution is to sample $\rho$'s uniformly in this interval. There are two major points of critique for this method of sampling $\rho$'s. First, the bounds $[\rho_l, \rho_h]$ are not strict. For the $\rho$ close to the end points, $\operatorname{sp} T(b_\rho)$ probably contains the entirety of the currently intersected polygon. Secondly, from the proof of Theorem 3.2.3 we see that ideally, we want to choose more $\rho$'s where "the Hausdorff metric of $\operatorname{sp} T(b_\rho)$ varies rapidly with $\rho$".

To get an intuition of good heuristics for choosing stricter bounds for $\rho$ it is helpful to plot $b_\rho(\mathbb{T})$ and vary $\rho$. An insight from experimenting is that it seems like $\Lambda(b)$ is formed by the points where $b_\rho(\mathbb{T})$ intersects itself, and when $\rho$ is varied, this intersection moves along $\Lambda(b)$ in a continuous manner. This would suggest that we only want to find the intervals for which $b_\rho(\mathbb{T})$ intersects itself, and where these intersections are a part of $\Lambda(b)$. In essence, there should be a finite number of interesting intervals for $\rho$. Actually, one can see from (3.1) that all $\lambda \in \Lambda(b)$ except for a finite number must arise from an intersection of $b_\rho(\mathbb{T})$ with itself, since (3.1) exactly describes this type of intersection.

To find the interesting intervals of $\rho$'s is not obvious analytically. But we can use the fact that when we intersect our partial polygon with $\operatorname{sp} T(b_\rho)$ for a good $\rho$, the area of the partial polygon decreases. So we could sweep over $[\rho_l, \rho_h]$, and calculate the decrease in area when intersecting with different $\operatorname{sp} T(b_\rho)$. Also, note that area of a polygon can be calculated in linear time with respect to the number of vertices. Pseudocode calculating $\Lambda(b)$ with the sweep improvement can be seen in Algorithm 2. How to choose the intervals of $\rho$'s where the area is reduced the most deserves some attention. If some $\rho$ is in both rhos and sweeprhos in Algorithm 2, the area reduced will be 0. To avoid missing out on good $\rho$ because of this a moving average filter is used for areareduce, and then all the good intervals are selected as the ones where areareduce$[\rho]$ $\geq (\max_\rho \ \text{areareduce}[\rho])/10^6$.

The problem of sampling the same $\rho$ also applies to when we sample rhos for the next intersection-run. Typically, we will find the exact same interval of good $\rho$ that reduces area the most in each iteration. Therefore a randomly chosen $\epsilon$ with $0 \leq \epsilon \leq \text{sweeprhos}[i+1] - \text{sweeprhos}[i]$ is added to the left boundaries of the good intervals, and subtracted from the right boundaries.

A significant optimization can be done when we calculate areareduce. The important observation is that areareduce$[\rho]$ is non-increasing. When updating areareduce we begin by calculating $\Lambda_s$ for the intervals of $\rho$ that we just intersected with, then if areareduce for the $\rho$ that did not reduce area the most in the previous iteration still are smaller then the threshold, we don't need to update those values. Practically, this drastically reduces the number of intersections done in total. Almost all polygon intersections used for calculating $\Lambda_s$ are done in the first iteration, when we have no prior values in areareduce.

The second points of improvement for Algorithm 1 is that we want to sample $\rho$ denser where $\operatorname{sp} T(b_\rho)$ varies rapidly. Ideally, we would want to differentiate $\operatorname{sp} T(b_\rho)$ with respect to the Hausdorff metric, but there is to the knowledge of the author not any way of doing this. However, one could make the observation that $b$ can be seen as a sum of two polynomials, one in $z$ and one in $1/z$. The standard approach if nothing better is known is to sample uniformly, but because of $1/z$, it would make sense to sample $\rho_j$ uniformly for $\rho \geq 1$ and sampling $\rho < 1$ in such a way that $1/\rho_j$ becomes uniformly distributed.

### 3.3.1 Testing suggested improvements

To investigate the two suggested improvements, i.e. using Algorithm 2 instead of 1 and not sampling $\rho$ uniformly we study an example. We compute $\Lambda(b)$ for

$$b(t) = -2t^{-1} + 4(1-i) + 7it - 3(1+i)t^2 + t^3. \tag{3.12}$$

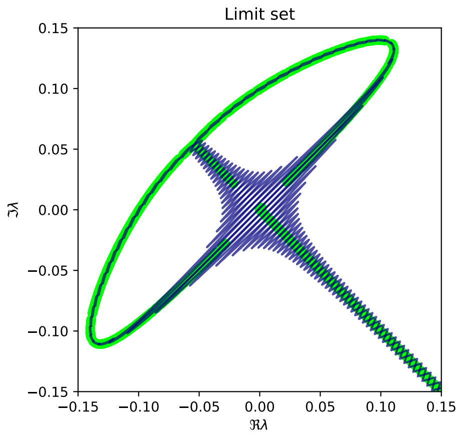
The symbol is presented in [7] as an example of where the numerically computed eigenvalues for $T_n(b_\rho)$ do not converge to the true $\Lambda(b)$, and the computed eigenvalues vary with $\rho$, which they theoretically do not.

To test Algorithms 2 and 1, we run Algorithm 1 and 2 with $b$ as in (3.12), and $n = 2500$, $m = 500$, for Algorithm 1 and $n = 2000$, $m = 500$, $l = 500$,

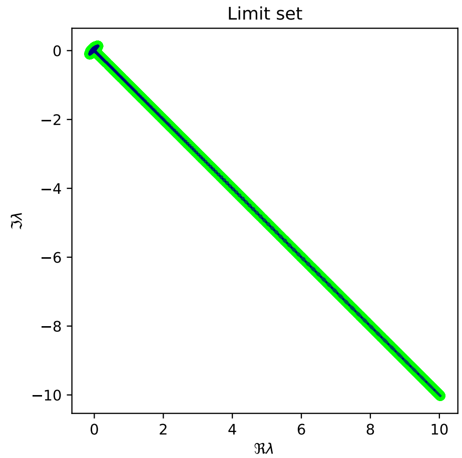
**Algorithm 2** Improved $\Lambda(b)$–calculator using area sweeps

**procedure** CALCLIMITSET($b, n, m, l, \#sweeps$)
$b$: the symbol,
$n$: the number of sampled $\rho$,
$m$: number of sampled $v$,
$l$: the number of sample $\rho$ for area sweeping,
$\#sweeps$: how many times new $\rho$ should be generated.
    $\rho_l, \rho_h \leftarrow$ bounds from theorem 3.2.1
    sweeprhos $\leftarrow$ sample $l$ points for area sweeping in $[\rho_l, \rho_h]$.
    rhos $\leftarrow$ sample $n/\#sweeps$ points in $[\rho_l, \rho_h]$
    vs $\leftarrow$ sample $m+1$ points in $[0, 2\pi]$
    $\Lambda \leftarrow b^D_{rhos[1]}(vs)$
    **for** $i \leftarrow 1$ to $\#sweeps$ **do**
        **for** each $\rho$ in rhos **do**
            $\Lambda \leftarrow \Lambda \cap b^D_\rho(vs)$
        **end for**
        **for** each $\rho$ in sweeprhos **do**
            $\Lambda_s \leftarrow \Lambda \cap b^D_\rho(vs)$
            areareduce[$\rho$] $\leftarrow$ Area($\Lambda$) - Area($\Lambda_s$)
        **end for**
        rhos $\leftarrow$ sample $n/s$ points in the intervals that reduces area the most
    **end for**
    **return** $\Lambda$
**end procedure**

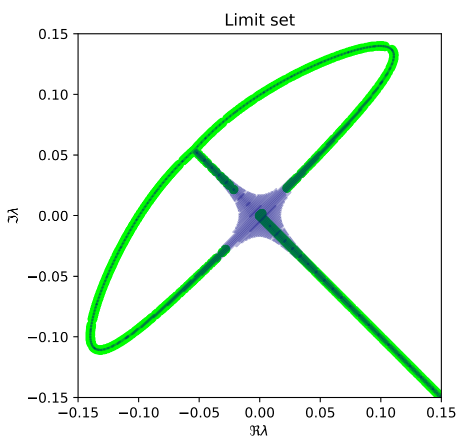
$\#sweeps = 2$ for Algorithm 2. The parameters are chosen so that both algorithms use 2500 polygon intersections. Also, we calculate $\Lambda(b)$ using the algebraic approach from [7], to see that the result is close to the limit set. The results can be seen in Figure 3.2. If one looks closely in Figures 3.2b and 3.2d, one sees that the line segment part is much more narrow in 3.2d than 3.2b. This is because we have not wasted polygon intersections where they made no difference. An even bigger difference can be seen if one zooms in at the top of $\Lambda(b)$, which is done in 3.2a and 3.2c. So for this example our suggested improvement really does help in computing $\Lambda(b)$. And it should generalize to different examples, since it is never good to "waste" polygon intersections.



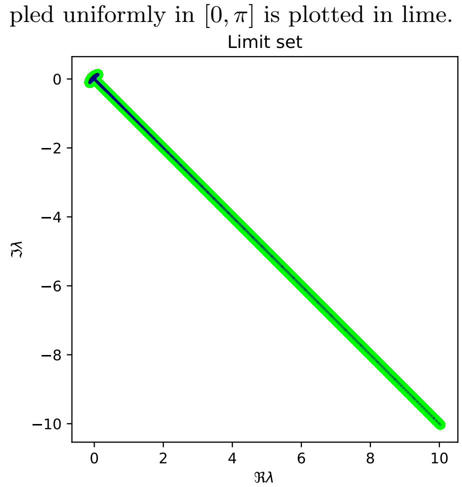

(a) A zoomed in version of 3.2b.




(b) $\Lambda(b)$ calculated from Algorithm 1 using $n = 2500$, $m = 500$, and $b$ from (3.12), in blue. Also the result from the algorithm described in section 3.1 with $10^5$ $\varphi$ sampled uniformly in $[0, \pi]$ is plotted in lime.



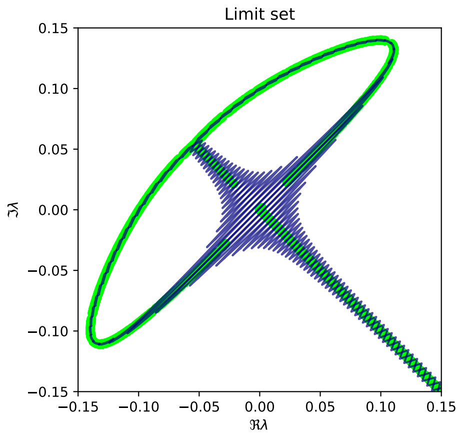

(c) A zoomed in version of 3.2d.



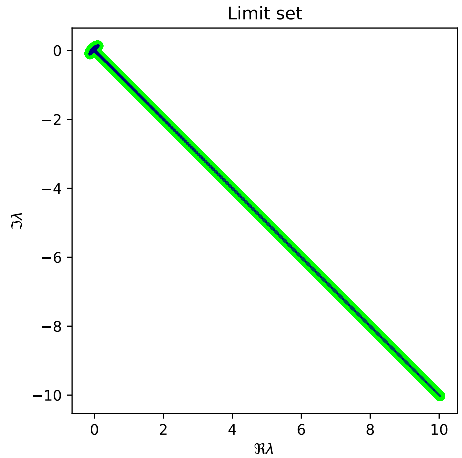

(d) $\Lambda(b)$ calculated from Algorithm 2 using $n = 2000$, $m = 500$, $l = 500$, $\#sweeps = 2$ and $b$ from (3.12), in blue. Also the result from the algorithm described in section 3.1 with $10^5$ $\varphi$ sampled uniformly in $[0, \pi]$ is plotted in lime.

Figure 3.2: A comparison of Algorithms 1 and 2. Both algorithms have calculated $\Lambda(b)$ for $b$ as in (3.12) and have used 2500 polygon intersections.

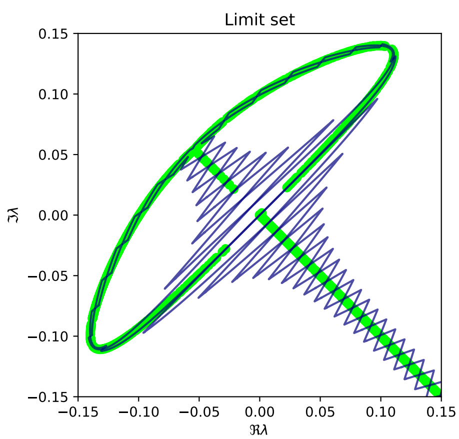
To compare the different $\rho$ sampling strategies we proceed by comparing the results of Algorithm 1 with $b$ as given by (3.12), with $n = 2500$, $m = 500$, and vary how we sample $\rho$. The first sampling strategy is uniform sampling in $[\rho_l, \rho_h]$. The second is sampling uniformly in $[1, \infty) \cap [\rho_l, \rho_h]$, and sampling $\rho_j$ so that $1/\rho_j$ are distributed uniformly for $\rho_j$ in $(0, 1] \cap [\rho_l, \rho_h]$. A comparison can be seen in Figure 3.3. If one looks closely it can be seen from Figures 3.3b and 3.3d that the first sampling strategy is worse for the line segment part of $\Lambda(b)$, but if you look at 3.3a and 3.3c, the second strategy far under performs on the zoomed in area. Overall, for this $\Lambda(b)$, it looks like the first strategy performed better, even though we had an argument for the second being better. So the optimal $\rho$ sampling strategy probably depends a lot on the specific symbol.



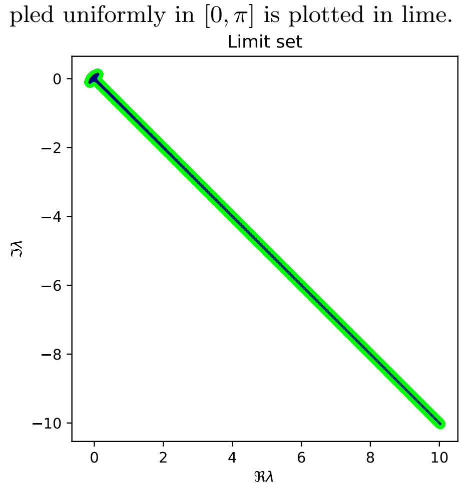

(a) A zoomed in version of 3.3b.




(b) $\Lambda(b)$ calculated using Algorithm 1 with $n = 2500$, $m = 500$, and $b$ from (3.12), in blue. The first sampling strategy is used. Also the result from the algorithm described in section 3.1 with $10^5$ $\varphi$ sampled uniformly in $[0, \pi]$ is plotted in lime.




(c) A zoomed in version of 3.3d.

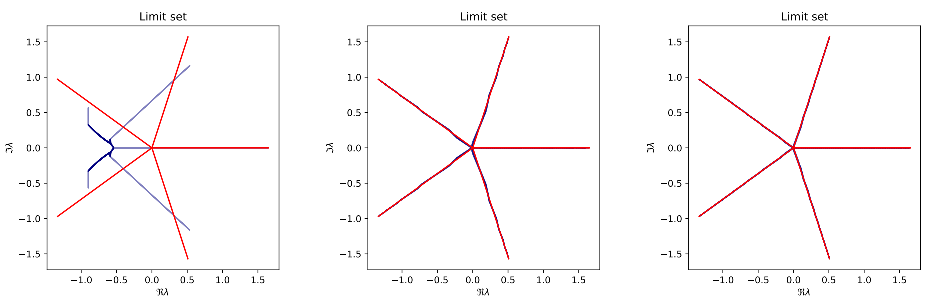



(d) $\Lambda(b)$ calculated using Algorithm 1 with $n = 2500$, $m = 500$, and $b$ from (3.12), in blue. The second sampling strategy is used. Also the result from the algorithm described in section 3.1 with $10^5$ $\varphi$ sampled uniformly in $[0, \pi]$ is plotted in lime.

Figure 3.3: A comparison of sampling strategies for $\rho$. Both runs have calculated $\Lambda(b)$ for $b$ as in (3.12) and have used 2500 polygon intersections.

### 3.3.2 Convergence for known examples

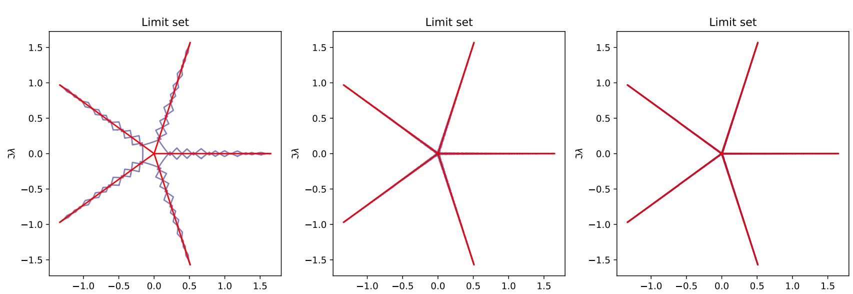
To investigate convergence when increasing $n$ and $m$ for Algorithm 2 we study the example $b(t) = t^{-4} + t$. This is the same symbol as in example 2.8.16, so we know $\Lambda(b)$. We first run Algorithm 2 with $n = 500$, $\#sweeps = 2$, $l = \lfloor n/4 \rfloor$ and $m = [10, 50, 100]$. The results can be seen in Figure 3.4. For $m = 10$ the results are quite rubbish, but already for $m = 50$ one could hardly see any difference, and for $m = 100$ one cannot see any difference.




(a) $m = 10$. (b) $m = 50$ (c) $m = 100$

Figure 3.4: In blue, the result of Algorithm 2 for inputs $b(t) = t^{-4} + t$, $n = 500$, $\#sweeps = 2$, $l = \lfloor n/4 \rfloor$, and $m$ being varied. The true $\Lambda(b)$ is plotted in red.

We conduct a similar experiment, but we vary $n$ instead. For $b(t) = t^{-4} + t$, $n = [50, 250, 500]$, $\#sweeps = 2$, $l = \lfloor n/4 \rfloor$ and $m = 500$ we run Algorithm 2. The results can be seen in Figure 3.5. As expected, as $n$ grows the polygon becomes finer and finer. For $n = 50$ the polygon structure is clearly visible, for $n = 250$ it is barely visible, and for $n = 500$ it is difficult to see a difference between the polygon and $\Lambda(b)$.




(a) $n = 50$. (b) $n = 250$ (c) $n = 500$

Figure 3.5: In blue, the result of Algorithm 2 for inputs $b(t) = t^{-4} + t$, $m = 500$, $\#sweeps = 2$, $l = \lfloor n/4 \rfloor$, and $n$ being varied. The true $\Lambda(b)$ is plotted in red.

## 3.4 Reasoning about time complexity

To analyze the time complexity of Algorithm 2 rigorously is difficult, but we will provide an argument and an example to back up that the running time on average probably is $O(nm + n^2)$.

The most computationally complex parts of Algorithm 2 is the polygon intersections which we know are linear in the the number of vertices of the incoming polygons. So what we really want to analyze is how the number of vertices grows in $\Lambda$. Let $w(i)$ be the number of vertices of $\Lambda$ after $i$ intersections. We will now argue that $w(i)$ should increase at most linearly. To see this, view Figure 3.6. When we intersect $\Lambda$ with $b_\rho^D(vs)$, the typical situation is described in Figure 3.6, specifically Figure 3.6 is zoomed in on one of the spots where $b_\rho^D(vs)$ intersects itself, and the intersection is part of $\Lambda(b)$, which happens a bounded number of times. In Figure 3.6 we see that 3 vertices are added to $\Lambda$ for the specific intersection of $b_\rho^D(vs)$ with itself. Since the number of self intersections of $b_\rho^D(vs)$ is limited, $w(i)$ should increase at most linearly.

To test our hypothesis about $w(i)$ growing linearly we generate a random Laurent polynomial with $r = s = 5$. We get the polynomial

$$\begin{aligned} b(t) = (9.563 - 6.844i)t^5 + (-2.246 + -4.182j)t^4 + (0.8478 + 0.4667j)t^3 \\ + (-4.125 + 1.539j)t^2 + (9.114 + 2.07j)t + (6.679 + 7.744j) \\ + (5.21 - 3.819j)t^{-1} + (-4.62 - 3.149j)t^{-2} + (-6.911 - 1.115j)t^{-3} \\ + (4.486 + 4.44j)t^{-4} + (4.279 + 0.9657j)t^{-5}. \end{aligned} \tag{3.13}$$

We then run Algorithm 2 with $m = 500$, $n = 3000$, $l = 500$, $\#sweeps = 6$, $\rho$ sampled uniformly, and keep track of $w(i)$, and plot it in Figure 3.7. The resulting limit set can be seen in Figure 3.8. Figure 3.7 looks like it should according to our argument. It seems like $w(i)$ grows piecewise linearly, with the slope changing. The changes corresponds to the alterations of the amount of intersections with itself. The algorithm finds the same interval of good $\rho$'s after each intersection-run, which explains the repetition of the pattern. Also note that the repetitive pattern we see is exactly what should happen if our reasoning in Figure 3.6 is correct, since we intersect over the same interval of $\rho$'s each time.

Assuming $w(i) = O(i + m)$, the total complexity is

$$\begin{aligned} &\sum_{i=1}^{n} O(w(i) + m) + \sum_{s=1}^{\#sweeps} \sum_{i=1}^{l} O(w(s \cdot n/\#sweeps) + m) \\ &\quad = O\left(\frac{n(n + m + 2m)}{2}\right) + O\left(\frac{l\#sweeps(n/\#sweeps + 2m + n + 2m)}{2}\right) \\ &\quad = O(n^2 + nm + nl\#sweeps + ml\#sweeps). \end{aligned}$$

However, as we remarked earlier, we can practically optimize away the *#sweeps*-factor if it is of moderate size, and in regular use cases it should never be set

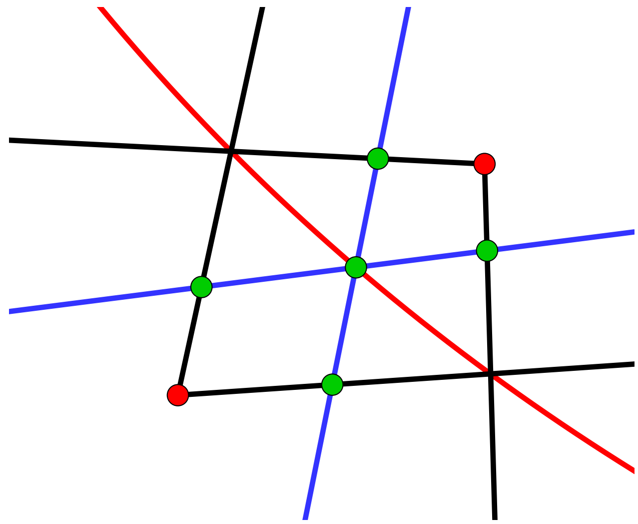


Figure 3.6: A probable illustration of how it looks when $\Lambda$ is intersected with $b_\rho^D(vs)$ in Algorithm 2. The red line is the true $\Lambda(b)$, the black lines are $\Lambda$ prior to being intersected, and the blue lines are $b_\rho^D(vs)$. The figure is zoomed in on one of the spots where $b_\rho^D(vs)$ intersects itself. The green points mark the new vertices in $\Lambda$, and the red ones are the vertices being removed from $\Lambda$.

higher than about 10. Also, typically $l$ is chosen as smaller than $n$, else we would "waste" too many intersections on area sweeping. So with these reasonable restrictions on $\#sweeps$ and $l$, the time complexity of Algorithm 2 should on average be $O(n^2 + nm)$.

## 3.5 Conclusion

A new approach to calculate $\Lambda(b)$ geometrically has been presented. Pseudocode for the approach can be found in Algorithm 2. Compared to the previous approaches that are algebraically based, it has the advantage of not requiring root finding of arbitrary complex polynomials. Furthermore, one could argue that the output from the geometric algorithm is more natural than the output for the algebraic one, since you get a connected subset of the complex plane, instead of points sampled on $\Lambda(b)$. In some case the geometric algorithm yields better information on $\Lambda(b)$. An example of this can be seen in Figure 3.2c. It is not clear from the lime dots whether $\Lambda(b)$ separates the plane or not, but from the blue polygon, we see that this is very likely the case.

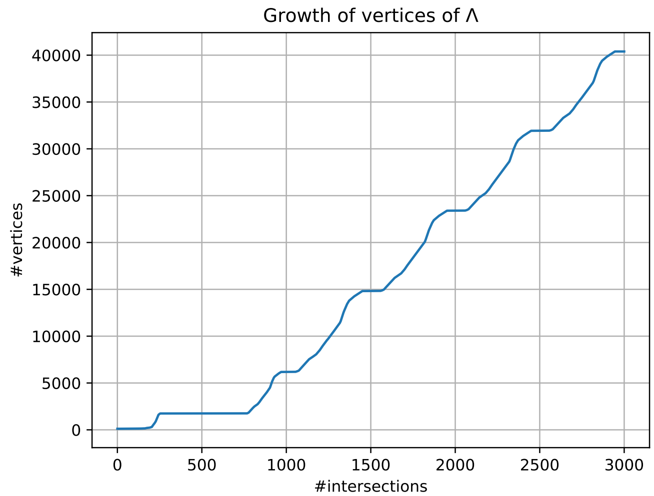



Figure 3.7: The number of vertices of $\Lambda$ in Algorithm 2 as a function of the number of intersections.

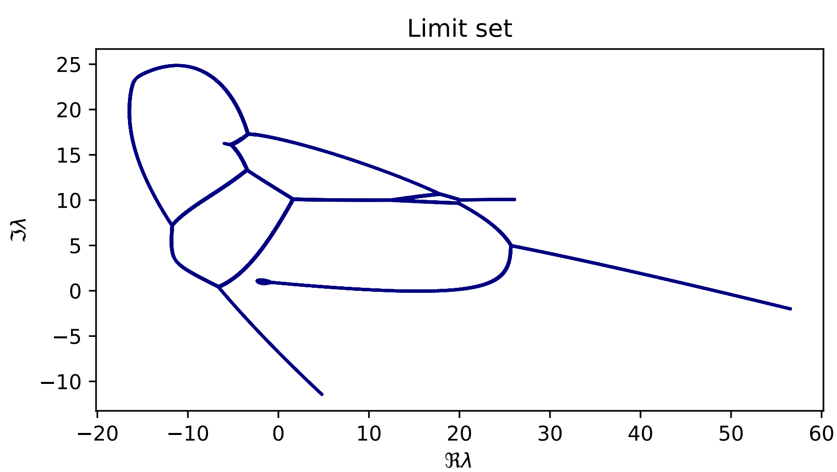



Figure 3.8: The limit set given by Algorithm 2 $b$ as in (3.13), $m = 500$, $n = 3000$, $l = 500$, $\#sweeps = 6$.

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