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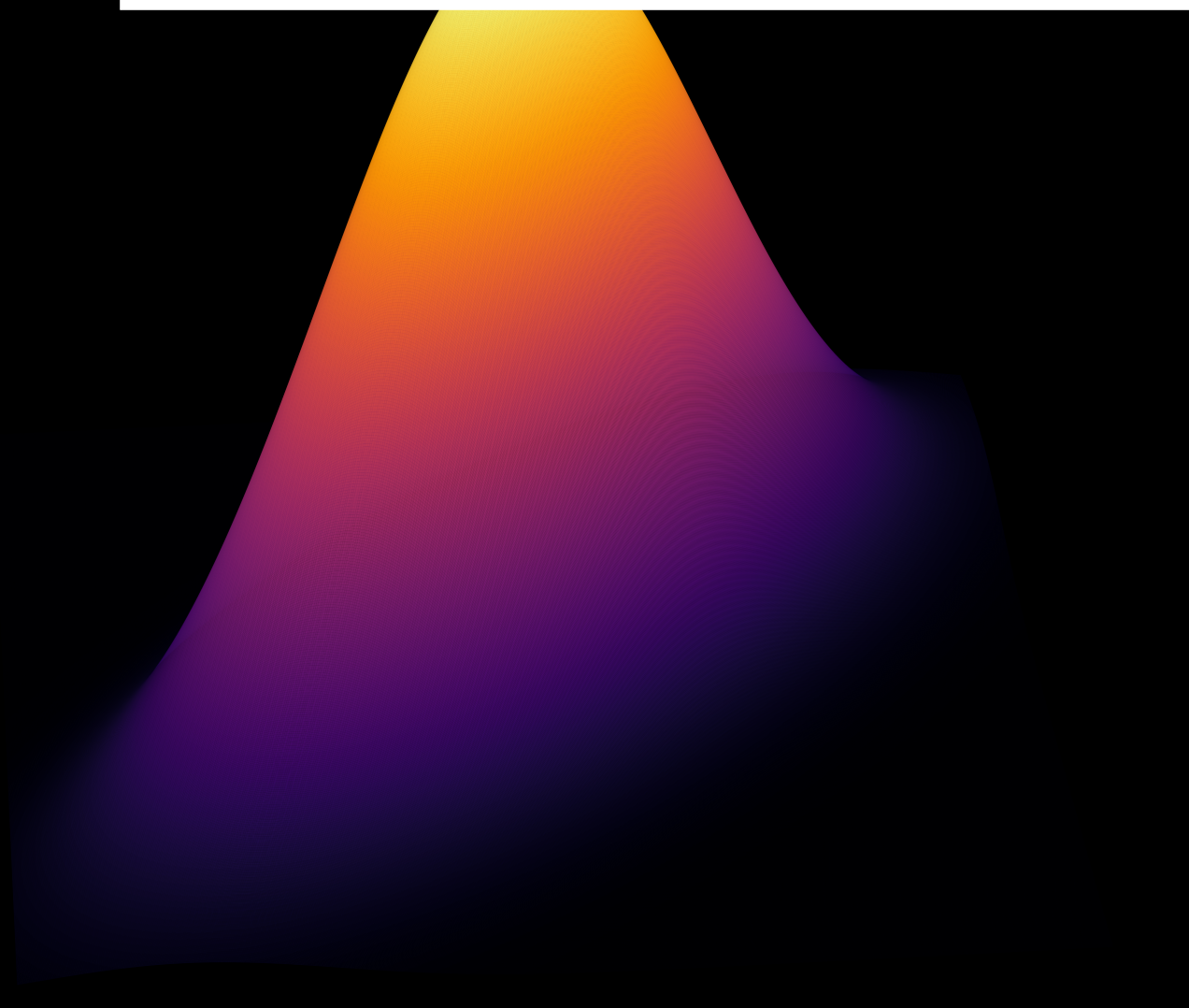
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Hybrid Microwave Resonator- Nanoscale Conductor Systems

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by Daniel Larsson Persson



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List of Abbreviations

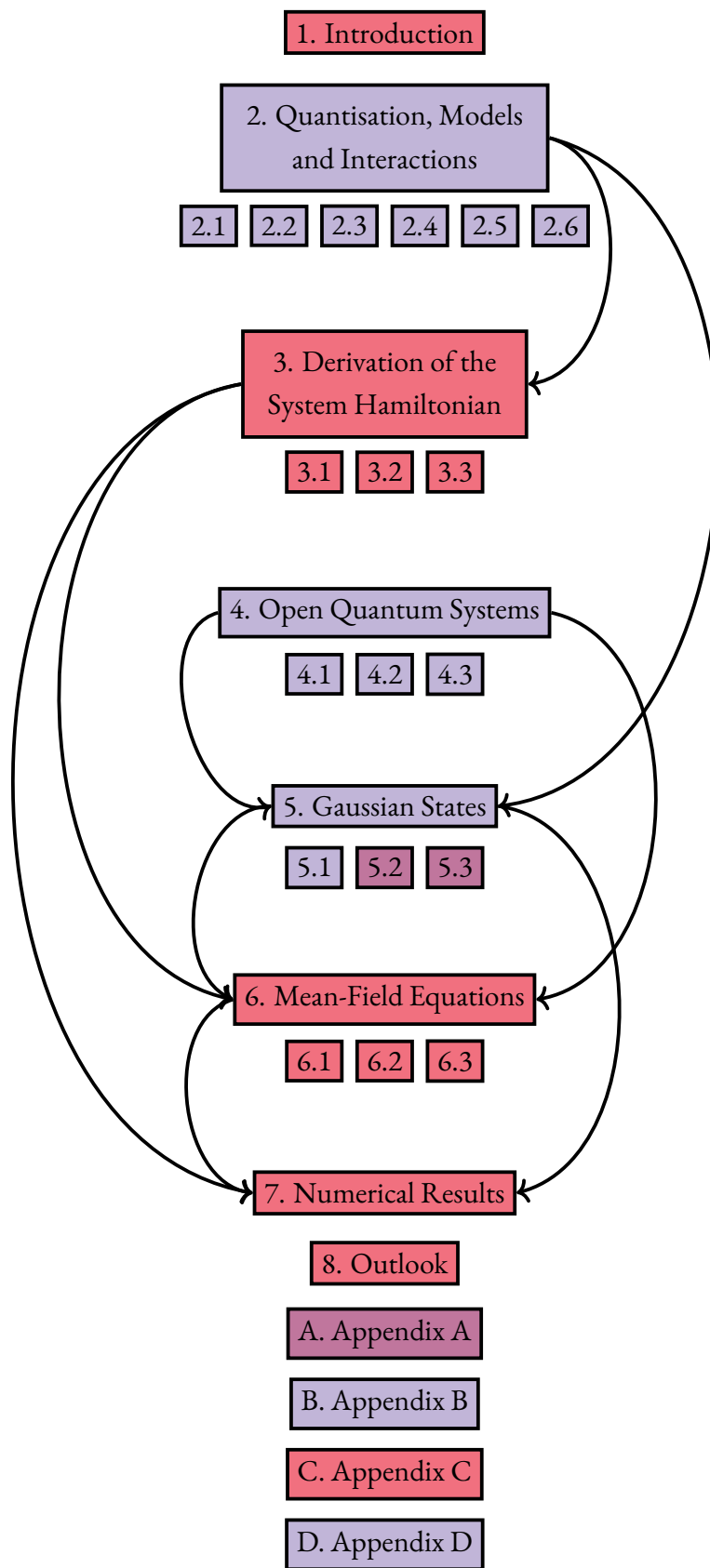
▶ CALE	–	Continuous Algebraic Lyapunov Equation
▶ CDLE	–	Continuous Differential Lyapunov Equation
▶ CM	–	Covariance Matrix
▶ CPTP	–	Completely Positive and Trace-Preserving
▶ cQED	–	Circuit Quantum Electrodynamics
▶ DQD	–	Double Quantum Dot
▶ EPR	–	Einstein–Podolski-Rosen
▶ GKSL	–	Gorini–Kossakowski–Sudarshan–Lindblad
▶ HO	–	Harmonic Oscillator
▶ MF	–	Mean Field
▶ RWA	–	Rotating Wave Approximation
▶ STS	–	Squeezed Thermal State
▶ SW	–	Schrieffer-Wolff

Abstract

Entanglement is a fundamental aspect of quantum computation and quantum information processes. In this work, we analyse the entanglement of two microwave resonators when connected to a semiconductor double quantum dot. The two-mode squeezing operation is a commonly used tool in entanglement creation and, based on this, we utilise the related two-mode two-photon Jaynes-Cummings model in the theoretical description. We derive the Hamiltonian starting from circuit quantum electrodynamics, and consider it as an open quantum system described by the Gorini-Kossakowski-Sudarshan-Lindblad equation. Applying a mean-field approximation in the model, the Hamiltonian becomes quadratic, enabling the use of symplectic and phase space methods. In this framework, we derive the continuous differential Lyapunov equation as the equation of motion for the microwave resonators. The mean-field equations are not solvable analytically in the general case, and instead numerical methods are employed. Using the Duan criterion, we can calculate how the entanglement of the two resonators depends on the system. It is found, among other things, that the driving of the resonators generates entanglement within the mean-field system. We also find a good analytical approximation for the populations of the resonators.

Overview of Contents

● Literature Review ● Original Content ● Mixed



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1. Introduction

The field of hybrid light-matter quantum networks has seen significant growth in interest and importance over the last few decades with the advent of quantum computing and quantum information technologies [1–8]. This includes areas such as single-photon detection and photonic control of electrons, including state initiation.

The aim of the thesis is to study phenomena arising from resonant coupling between a semiconductor double quantum dot (DQD) and two microwave resonators, mainly entanglement between the single modes of each resonator. This is motivated by the recent experimental work by W. Khan et al. [9], as well as related theoretical work by D. Zenelaj et al. [10] concerning a semiconductor DQD, resonantly coupled to a single microwave resonator, which has been shown to be able to act as a continuous photodetector with high efficiency [9]. The inclusion of two modes allows for the utilisation of two-photon interactions which can be controlled in an effective way. In particular, it is possible to tune the electronic energy levels in the DQD to match the sum of photon energies, one from each of the microwave resonators. If the resonators are also made to have distinguishably different energies, making them non-degenerate, then we expect resonant two-photon exchange with a single resonator to vanish in favour of correlated exchange of photon pairs, instead formed by one photon from each resonator. Such a setup would also allow for greater control of the DQD system via the two resonators, while also introducing effects stemming from interactions between the resonators.

The DQD-resonator system can then be connected via the DQD to nanoscale lead electrodes, and by extension to other identical DQD-resonator systems, forming a larger network, where the DQDs act as controllable qubits, and the resonators as controls. The resonators are controlled through a laser being directed at them, called a drive, forming a coherent source of photons which are itinerant on the resonators, modulating the photon population, and by extension influencing the DQD-resonator interaction. As an alternative, the DQD could act as control if it is driven by a current.

The effective Hamiltonian we would like to use for the DQD coupled to two resonators is that of the two-mode two-photon Jaynes-Cummings model (based on the interaction term involving two-photon processes for two modes)

$$\hat{H} = \hbar\omega_1\hat{a}_1^\dagger\hat{a}_1 + \hbar\omega_2\hat{a}_2^\dagger\hat{a}_2 + \frac{1}{2}\hbar\Omega\hat{\sigma}_z + \hbar\Lambda\left(\hat{\sigma}_+\hat{a}_1\hat{a}_2 + \hat{\sigma}_-\hat{a}_1^\dagger\hat{a}_2^\dagger\right), \quad (1.1)$$

where $\omega_{1(2)}$ is the characteristic frequency of the first or second resonator, Ω is the energy level splitting between the states $|e\rangle$ and $|g\rangle$ in Fig. 1.1, Λ is the coupling strength between the resonators and the DQD, $\hat{a}_{1(2)}^\dagger$ and $\hat{a}_{1(2)}$ are bosonic creation and annihilation operators respectively, and the $\hat{\sigma}$ are Pauli operators. The system described above, including the relevant quantities appearing in the Hamiltonian in Eq. (1.1) is illustrated in Fig. 1.1.

The two-mode two-photon Jaynes-Cummings model was first investigated by S.-C. Gou in 1989 [11] for two-mode uncorrelated coherent states and two-mode squeezed vacuum states. This model was then subsequently investigated with correlated pair coherent states [12], correlated $SU(1, 1)$ coher-

1. Introduction

ent states [13], and two-mode uncorrelated coherent states with atomic motion [14]. We also consider it as an open quantum system, modelling it at the level of the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) equation with local dissipators as the environment.

Correlated photon exchange between the DQD and the resonators also introduce the possibility of entanglement between the modes. It is of interest to investigate to what extent this entanglement can be used for the purposes of quantum information and computation. Thus, the starting point is to investigate a simpler system consisting of two coupled resonators. The DQD is then added with the assumption that correlations between the states of the DQD and the resonator are sufficiently small as to be neglected, constituting a mean-field (MF) approximation, where the DQD and resonators respectively, only influence each other through the expectation values of their operators. This approximation makes the states of the resonators Gaussian and enables the use of a set of phase space tools, including the Lyapunov equation.

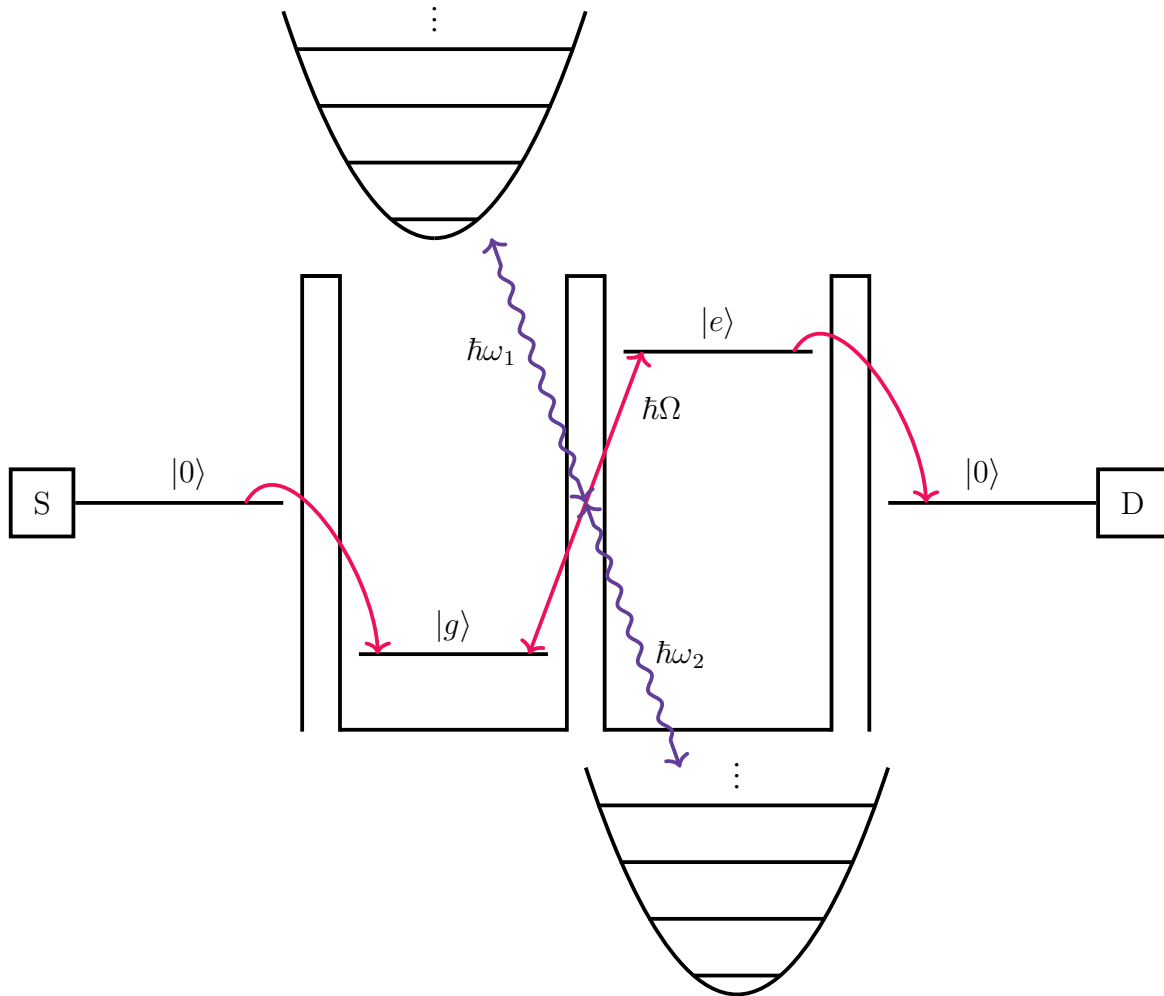


Figure 1.1: Illustration of the DQD system, with levels $|e\rangle$ and $|g\rangle$, resonantly coupled to two microwave cavities according to Eq. (1.1). The DQD is also connected to leads, where the leads represent the empty state, $|0\rangle$, of the DQD in the Coulomb blockade regime. S is the source port, and D the drain port.

The thesis is organised as follows: Sec. 2 covers much of the theoretical basis for the analytical results presented in Sec. 3, including circuit quantum electrodynamics and transformations. The third

section contains the derivation of the effective Hamiltonian in Eq. 1.1. Section 4 introduces open quantum systems and gives a microscopic derivation of the GKSL equation. The next section, Sec. 5, presents the theoretical framework used for treating Gaussian states. Sections 6 and 7, finalises the setup used for calculations and presents the obtained numerical results, respectively. Section 8 then discusses the outlook of hybrid networks. Finally, some additional details of the derivations and results are presented in Appendices A through D.

2. Quantisation, Models and Interactions

2.1 Canonical Quantisation

The early developments of quantum physics were heavily focused on quantising the motions of particles at the atomic scale, where classical mechanics seemed to no longer be valid [15]. This necessitated a new theory of mechanics to be developed, which W. Heisenberg and M. Born called “quantum mechanics” [15], first used in the literature by M. Born in 1924 [16]. In these early stages, however, the electromagnetic field was still treated as classical [15]. Not long after, P.A.M. Dirac introduced what he called the “method of classical analogy” in his 1925 PhD thesis [17]. This method is used for quantising a classical theory, specifically by supplanting Poisson brackets in classical mechanics by commutators according to [18]

$$\{A, B\} \mapsto \frac{1}{i\hbar} [\hat{A}, \hat{B}], \quad (2.1)$$

which is often called the Dirac rule.

2.1.1 First Quantisation

The Dirac rule was first used in relation to position and momentum, with Poisson bracket $\{x, p\} = 1$ [18], for which it gives [19]

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad (2.2)$$

generalised to several components with the Kronecker delta, δ_{ij} . This later became known as first quantisation, as only the motions of the particles were quantised.

2.1.2 Second Quantisation

Not long after the quantisation of position and momentum, Dirac quantised the electromagnetic field [20]. His work on the quantisation of fields was later built upon by the likes of P. Jordan [21] and V. Fock [22]. Quantisation of a field involves moving to the formalism of occupation number representation, more commonly called second quantisation, in which the occupation of single-particle states are considered, collectively forming a Fock state. All Fock states combined form a complete basis for the Fock space $\mathcal{F}(\mathcal{H})$, as well as the many-body Hilbert space $\mathcal{H}(N)$. The operators acting on the Fock space are the creation and annihilation operators, also called ladder operators, defined in the case of bosons as [19]

$$\hat{a}_i^\dagger |\dots, n_i, \dots\rangle \equiv \sqrt{n_i + 1} |\dots, n_i + 1, \dots\rangle, \quad (2.3)$$

and

$$\hat{a}_i |\dots, n_i, \dots\rangle \equiv \sqrt{n_i} |\dots, n_i - 1, \dots\rangle, \quad (2.4)$$

respectively, obeying $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$. Here, n_i is the occupation of the single-particle state $\phi_i \in \mathcal{H}(1)$. These operators constitute mappings between many-body Hilbert spaces $\mathcal{H}(N)$, each with a different total particle number N [23]. Bosonic wavefunctions are defined in symmetric Fock space, $\mathcal{F}_+(\mathcal{H})$,

while fermionic wavefunctions are defined in antisymmetric Fock space, $\mathcal{F}_-(\mathcal{H})$. A field can then be expanded in terms of these operators to create the field operators [24]

$$\hat{\Psi}(\vec{r}) = \sum_i \psi_i(\vec{r}) \hat{a}_i, \quad \text{and} \quad \hat{\Psi}^\dagger(\vec{r}) = \sum_i \psi_i^*(\vec{r}) \hat{a}_i^\dagger, \quad (2.5)$$

where the $\psi_i(\vec{r})$ are wave functions of a single-particle state and the sum runs over the complete set of single-particle quantum numbers. This is a crude introduction to quantum field theory, and the interested reader is referred to e.g. Refs. [24, 25]. For quantisation of the electromagnetic field, see e.g. Ref. [26]. In representing the electric and magnetic field operators, the operators

$$\hat{x}_i = \frac{1}{\sqrt{2}}(\hat{a}_i^\dagger + \hat{a}_i), \quad \text{and} \quad \hat{p}_i = \frac{i}{\sqrt{2}}(\hat{a}_i^\dagger - \hat{a}_i), \quad (2.6)$$

are often introduced, which are still subject to Eq. (2.2) when $\hbar = 1$. These are commonly called quadrature operators in quantum optics [26], and throughout this work, we will use this name to refer more generally to position and momentum operators.

2.2 Quantum Network Theory and Circuit Quantum Electrodynamics

The Lagrangian description of circuit theory has been well known for at least close to one hundred years [27], but saw significant developments by L.O. Chua in the 1970's [28]. An extension of circuit theory to quantum mechanics through canonical quantisation was done by B. Yurke and J.S. Denker in their seminal paper from 1984 [29]. A decade later, M.H. Devoret built upon their work to generalise it to Hamiltonian mechanics [30], creating the field of circuit quantum electrodynamics (cQED), a review of which can be found in Ref. [1]. Here, we employ cQED to find the quantum Hamiltonian for the system described in Sec. 1. The classical Lagrangian and Hamiltonian can be written as [31]

$$\mathcal{L} = T - U, \quad \text{and} \quad \mathcal{H} = T + U, \quad (2.7)$$

respectively, where T is the kinetic and U the potential energy.

2.2.1 Nodal Analysis of Circuits

Nodal analysis is one of two standard methods of circuit analysis, the other being loop analysis [32, 33]. Nodal analysis is done using Kirchhoff's current law, and loop analysis is done through Kirchhoff's voltage law¹. In the method of nodal analysis employed here, the node flux, ϕ , being the time integral of the voltage V across a capacitor, takes the role of the position coordinate. The conjugate momentum is the charge q , resulting from the electrochemical potential difference present between the two capacitor plates [32, 33]. Using this convention for position and momentum, canonical quantisation through the Dirac rule gives the commutation relation [32, 33]

$$\left[\hat{\Phi}_\mu, \hat{q}_\nu \right] = i\hbar \delta_{\mu\nu}. \quad (2.8)$$

¹Both methods of analysis can be derived from graph theory.

2. Quantisation, Models and Transformations

The classical Lagrangian can be found using matrix notation through [32, 33]

$$\mathcal{L} = \frac{1}{2} \vec{\Phi}^\top \mathbf{C} \vec{\Phi} - \frac{1}{2} \vec{\Phi}^\top \mathbf{L}^{-1} \vec{\Phi}, \quad (2.9)$$

where \mathbf{C} is the capacitance matrix and \mathbf{L}^{-1} is the inverse² inductance matrix. From the Lagrangian, it is possible to define the momenta conjugate to the node fluxes, from [32, 33]

$$q_v = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_v} = \sum_{\mu} C_{v\mu} \dot{\Phi}_\mu, \quad \text{or} \quad \vec{q} = \mathbf{C} \vec{\dot{\Phi}}, \quad (2.10)$$

in explicit vector form [33]. The connection between the classical Lagrangian and the classical Hamiltonian is the Legendre transformation which in this case reads as [33]

$$\mathcal{H} = \sum_v \dot{\Phi}_v q_v - \mathcal{L}. \quad (2.11)$$

Using the relation in Eq. (2.10) the matrix form of the classical Hamiltonian becomes [32, 33]

$$\mathcal{H} = \frac{1}{2} \vec{q}^\top \mathbf{C}^{-1} \vec{q} + \frac{1}{2} \vec{\Phi}^\top \mathbf{L}^{-1} \vec{\Phi}. \quad (2.12)$$

In arriving at this expression, we have ignored any static flux created by the inductors. In case there are voltage offsets connected to nodes, the offsets do not need to be counted as nodes in themselves. They instead give terms in the Lagrangian of the form

$$\mathcal{L}_{v;n} = C_{v;n} (\dot{\Phi}_v - V_{v;n})^2, \quad (2.13)$$

where $V_{v;n}$ is the n :th voltage offset to $\dot{\Phi}_v$, and $C_{v;n}$ its connecting capacitance. The conjugate variable, q_v , will then be modified according to

$$\begin{aligned} q_v &= \frac{\partial}{\partial \dot{\Phi}_v} \left(\mathcal{L} + \sum_n \mathcal{L}_{v;n} \right) \\ \implies q_v + \sum_n C_{v;n} V_{v;n} &= \sum_{\mu} C_{v\mu} \dot{\Phi}_\mu + \sum_n C_{v;n} \dot{\Phi}_v, \end{aligned} \quad (2.14)$$

offsetting the charge at the node. Thus, appropriate modification to the charge vector \vec{q} needs to be made when offset voltages are used. It also means that the contribution of the offset voltage to the capacitance matrix is simply along the diagonal at the location of the node it is connected to. Explicitly, for inversion of the matrix, the modifications that need to be made are

$$q_v \longrightarrow q_v + \sum_n C_{v;n} V_{v;n}, \quad \text{and} \quad C_{vv} \longrightarrow C_{vv} + \sum_n C_{v;n} \dot{\Phi}_v, \quad (2.15)$$

for the elements of the charge vector and capacitance matrix respectively. We note however, that a full treatment of the voltage offsets as nodes will give the same Hamiltonian as above when the charge is substituted by a voltage at the end and V^2 terms set to zero.

²It is only a true inverse matrix in some special cases. In general, it is merely a matrix of reciprocal inductances.

The Euler-Lagrange equation is [31, 32]

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_\nu} - \frac{\partial \mathcal{L}}{\partial \Phi_\nu} = 0, \quad (2.16)$$

which gives the equation of motion for the circuit [31–33]. The equation can also be used as a verification for the Lagrangian being correct. When the classical Hamiltonian has been found, the quantum Hamiltonian is found through canonical quantisation and the Dirac rule from Eq. (2.8), giving the quadrature operators [33]

$$\hat{q}_\nu = i q_j^{\text{ZPF}} (\hat{a}_j^\dagger - \hat{a}_j) \quad \text{and} \quad \hat{\Phi}_\nu = \Phi_j^{\text{ZPF}} (\hat{a}_j^\dagger + \hat{a}_j), \quad (2.17)$$

where the pre-factors are

$$q_j^{\text{ZPF}} = \sqrt{\frac{\hbar}{2Z_j}} \quad \text{and} \quad \Phi_j^{\text{ZPF}} = \sqrt{\frac{\hbar Z_j}{2}}. \quad (2.18)$$

Here, Z_j is the impedance and ω_j the resonance frequency of the circuit. The index j refers to a given LC -oscillator in the circuit with an adjacent node ν . That is, we have the correspondence $j \longleftrightarrow \nu$. The definitions of impedance and frequency can be generalised to [33]

$$\omega \equiv \frac{1}{\sqrt{LC}} \longrightarrow \omega_j^2 \equiv \frac{1}{L_j} (C^{-1})_{\nu\nu}, \quad \text{and} \quad Z \equiv \sqrt{\frac{L}{C}} \longrightarrow Z_j \equiv \sqrt{L_j (C^{-1})_{\nu\nu}}, \quad (2.19)$$

respectively.

2.2.2 Quantisation of the LC -Oscillator

In the picture where the node flux Φ is the position coordinate, as shown in Fig. 2.1, it can be defined as [32, 33]

$$\Phi(t) = \int_{t_0}^t V(\tau) d\tau, \quad (2.20)$$

which makes $V(t) = \dot{\Phi}$. The conjugate momentum is then taken to be the charge q . The LC -oscillator is analogous to a mechanical mass-and-spring oscillator, and can also be shown to correspond to a resonant cavity. The latter correspondence is described in Sec. D.1 of Appendix D. The role of the mass in the analogy with the mass-and-spring oscillator in the nodal picture is played by the capacitance C , and the spring constant becomes $\frac{1}{L}$. This analogy suggests that the energy of the inductor is $\frac{(\Phi - \Phi_{\text{ext.}})^2}{2L}$, where $\Phi_{\text{ext.}}$ represents any external flux. The classical Lagrangian of an LC -oscillator can be found through identifying the potential energy stored on the capacitor as [33]

$$U = \frac{1}{2} C \dot{\Phi}^2, \quad (2.21)$$

which looks like the kinetic energy when Φ is taken as the position coordinate. Similarly, we

2. Quantisation, Models and Transformations

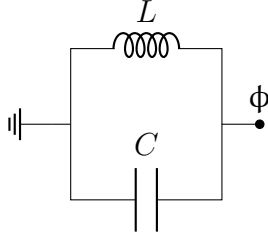


Figure 2.1: The circuit of a single LC -oscillator in the nodal picture, where ϕ is taken as the position coordinate. A single LC -oscillator corresponds to a single HO. The intuition for this correspondence is explained in Sec. D.1.

identify the kinetic energy which is stored in the inductor as

$$T = \frac{1}{2L}\dot{\phi}^2. \quad (2.22)$$

This in turn looks like the potential energy. With potential and kinetic energy interchanged compared to Eq. (2.7), the classical Lagrangian for the LC -oscillator becomes

$$\mathcal{L}_{LC} = \frac{1}{2}C\dot{\phi}^2 - \frac{1}{2L}\phi^2. \quad (2.23)$$

From Eq. (2.11), the classical Hamiltonian becomes

$$\mathcal{H}_{LC} = q\dot{\phi} - \mathcal{L}_{LC} = \frac{q^2}{2C} + \frac{\phi^2}{2L}, \quad (2.24)$$

and subsequent canonical quantisation according to Eq. (2.17) gives the quantum Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \quad (2.25)$$

which is the Hamiltonian for the quantum harmonic oscillator (HO). Using the above tools, one can quantise more intricate circuits. Quantisation of two capacitively coupled LC -oscillators with the same tools as above is done in Appendix A.

2.3 Two-Level Atoms and Double Quantum Dots

The simplest non-trivial quantum system is comprised of two levels, $|e\rangle$ and $|g\rangle$, the excited and ground state respectively. These span a Hilbert space equivalent to a spin- $\frac{1}{2}$ system [34]. The Pauli operators can then be written as [34]

$$\hat{\sigma}_x = |e\rangle\langle g| + |g\rangle\langle e|, \quad \hat{\sigma}_y = -i|e\rangle\langle g| + i|g\rangle\langle e|, \quad \text{and} \quad \hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|, \quad (2.26)$$

which satisfy the commutation relations of the $\mathfrak{su}(2)$ algebra [35]

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i \sum_k \varepsilon_{ijk} \hat{\sigma}_k, \quad \text{and} \quad \{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij}, \quad (2.27)$$

where ε_{ijk} is the Levi-Civita symbol. The corresponding raising and lowering operators are then

$$\hat{\sigma}_+ = \hat{\sigma}_-^\dagger = |e\rangle\langle g| = \frac{1}{2}(\hat{\sigma}_x + i\hat{\sigma}_y), \quad \text{and} \quad \hat{\sigma}_- = \hat{\sigma}_+^\dagger = |g\rangle\langle e| = \frac{1}{2}(\hat{\sigma}_x - i\hat{\sigma}_y). \quad (2.28)$$

In the basis $(|e\rangle, |g\rangle)$, the matrix representations of the above Pauli operators are simply the Pauli matrices [34]. The Hamiltonian of a two-level system which is diagonal in the basis $(|e\rangle, |g\rangle)$ can then be written as [34]

$$\hat{H} = \frac{1}{2}\hbar\omega\hat{\sigma}_z \quad (2.29)$$

where ω is the transition frequency between the two levels. In the 2×2 case, the Pauli- z matrix can also be rewritten as

$$\frac{1}{2}\hat{\sigma}_z = \hat{\sigma}_+\hat{\sigma}_- - \frac{1}{2}\mathbb{1}_2, \quad (2.30)$$

which includes an energy shift of the Hamiltonian, but is a form which resembles the number operator for fermions or bosons, namely $\hat{a}^\dagger\hat{a}$.

A DQD is then described by just two levels according to Fig. 1.1. We also wish to include electronic leads which are modelled by the empty state of the DQD, $|0\rangle$, connected to source and drain, represented by S and D, respectively, in Fig. 1.1. This modifies the Pauli operators of the two-level system to include a third level, now in the basis $(|e\rangle, |g\rangle, |0\rangle)$. For all of the above operators, this simply amounts to adding a row and column of zeros to their matrix representations. We also introduce the operators

$$\hat{s}_g^\dagger = |g\rangle\langle 0|, \quad \text{and} \quad \hat{s}_e = |0\rangle\langle e|, \quad (2.31)$$

for tunnelling into and out of the DQD, respectively. Their matrix representations in the $(|e\rangle, |g\rangle, |0\rangle)$ basis are

$$\hat{s}_g^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \hat{s}_e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad (2.32)$$

These operators will be used in Sec. 6 when setting up the equations for the system discussed in Sec. 1.

2.4 The Baker-Campbell-Hausdorff Theorem

The Baker-Campbell-Hausdorff (BCH) theorem describes the form of a solution for Z to the equation [36–39]

$$e^Z = e^X e^Y, \quad (2.33)$$

where X and Y are some non-commuting operators as part of a Lie algebra. The pioneering work which lead to the theorem and its solution was due to³ H.F. Baker, J.E. Campbell, F. Hausdorff and E.B. Dynkin, the latter often being left out, despite important contributions [36]. The theorem and solution can be stated in different ways (including an integral solution given in Ref. [37]). Here we are interested in its most common form, which is the series solution in terms of Lie polynomials, known as the BCH formula⁴.

2.4.1 The Baker-Campbell-Hausdorff Formula and the Hadamard Lemma

Of interest in most applications to quantum mechanics is the special case where X and Y are Hilbert-Schmidt operators, which are denoted here by a hat (\hat{X}, \hat{Y}). The BCH formula to the first few orders is then given by [36, 37, 39]

$$\hat{Z} = \ln\left(e^{\hat{X}}e^{\hat{Y}}\right) = \hat{X} + \hat{Y} + \frac{1}{2}[\hat{X}, \hat{Y}] + \frac{1}{12}[\hat{X}, [\hat{X}, \hat{Y}]] - \frac{1}{12}[\hat{Y}, [\hat{X}, \hat{Y}]] + \dots \quad (2.34)$$

³Related contributions were also made by F. Schur, H. Poincaré and E. Pascal [36].

⁴For details, the reader is referred to e.g. Refs. [36, 37], and the references therein.

2. Quantisation, Models and Transformations

The most important consequence of the BCH theorem to quantum mechanics is the Hadamard lemma (or sometimes just the BCH lemma) because of its use in evaluating transformations as discussed in Sec. 2.5. The Hadamard Lemma gives the series solution of $e^{\hat{X}}\hat{Y}e^{-\hat{X}}$ to be [19, 37, 40]

$$e^{\hat{X}}\hat{Y}e^{-\hat{X}} = \sum_{n=0}^{\infty} \frac{[(\hat{X})^n, \hat{Y}]}{n!} = \hat{Y} + [\hat{X}, \hat{Y}] + \frac{1}{2!} [\hat{X}, [\hat{X}, \hat{Y}]] + \dots, \quad (2.35)$$

since $[(\hat{X})^0, \hat{Y}] \equiv \hat{Y}$. For clarification, $(\hat{X})^n$ refers to n nested commutators with \hat{X} .

2.4.2 The Zassenhaus Formula and the Kermack-McCrae Identities

The Zassenhaus formula can be described as a dual of the BCH formula where instead of giving a solution to Eq.(2.33), it gives the solution to $e^{\hat{X}+\hat{Y}}$. The formula was first published in a paper by W. Magnus in 1954 [41] in which he cites unpublished work of H. Zassenhaus. The formula is the following unique decomposition [39, 41]

$$e^{\hat{X}+\hat{Y}} = e^{\hat{X}}e^{\hat{Y}} \prod_{n=2}^{\infty} e^{C_n(\hat{X}, \hat{Y})}, \quad (2.36)$$

where C_n are Lie polynomials. For example, the first two Lie polynomials are given by

$$C_2 = -\frac{1}{2}[\hat{X}, \hat{Y}], \quad C_3 = \frac{1}{6} \left(2[\hat{Y}, [\hat{X}, \hat{Y}]] + [\hat{X}, [\hat{X}, \hat{Y}]] \right). \quad (2.37)$$

An earlier discovery in 1931 by W.O. Kermack and W.H. McCrae [42], is a special case of the Zassenhaus formula when $[\hat{X}, \hat{Y}] = 1$. We can generalise this result slightly by reformulating it as requiring that

$$[\hat{Y}, [\hat{X}, \hat{Y}]] = [\hat{X}, [\hat{X}, \hat{Y}]] = 0, \quad (2.38)$$

meaning that $C_n = 0$ for $n > 2$. This reduces Eq. (2.36) to

$$e^{\hat{X}+\hat{Y}} = e^{\hat{X}}e^{\hat{Y}}e^{-\frac{1}{2}[\hat{X}, \hat{Y}]}, \quad (2.39)$$

for normal ordering of the operators, or

$$e^{\hat{X}+\hat{Y}} = e^{\hat{Y}}e^{\hat{X}}e^{\frac{1}{2}[\hat{X}, \hat{Y}]}, \quad (2.40)$$

for anti-normal ordering⁵. Equations (2.39) and (2.40) are referred to as the Kermack-McCrae identities or as the disentangling theorem⁶ [26].

⁵Proofs or derivations of the normal-ordered Kermack-McCrae identity can be found in Refs. [33, 35, 37].

⁶Sometimes it is also referred to as Feynman's disentangling theorem [43], based on a 1951 paper by R.P. Feynman [44].

2.5 Transformations and Symplectic Form

2.5.1 The Schrieffer-Wolff Transformation

The Schrieffer-Wolff (SW) transformation is a unitary transformation, used to diagonalise a Hamiltonian perturbatively to first order in the interacting part of the Hamiltonian, often to remove higher excitations and thus obtain an effective Hamiltonian for the lower energy states [45]. In Sec. 3 it will enable us to derive the two-mode two-photon Jaynes-Cummings model. Though the SW transformation was first employed by J.M. Luttinger and W. Kohn in 1955 [46], it is commonly attributed to J.R. Schrieffer and P.A. Wolff based on its use in a paper of theirs from 1966 [47]. With a Hamiltonian composed of a non-interacting and interacting part \hat{H}_0 and \hat{V} respectively, written as

$$\hat{H} = \hat{H}_0 + \hat{V}, \quad (2.41)$$

the Hamiltonian is then transformed using the BCH formula in Eq. (2.35) to be [48]

$$e^{\hat{S}} \hat{H} e^{-\hat{S}} = \hat{H} + [\hat{S}, \hat{H}] + \frac{1}{2!} [\hat{S}, [\hat{S}, \hat{H}]] + \dots, \quad (2.42)$$

where \hat{S} is the generator of the transformation and is assumed to be of the same energy scale as \hat{V} . Because of the unitarity of $e^{\hat{S}}$, \hat{S} is anti-hermitian, $\hat{S}^\dagger = -\hat{S}$ [48]. If \hat{S} is chosen in such a way that its commutator with \hat{H}_0 cancels the interaction \hat{V} , that is

$$\hat{V} + [\hat{S}, \hat{H}_0] = 0, \quad (2.43)$$

then the transformed Hamiltonian becomes [48]

$$\hat{H}' = \hat{H}_0 + \frac{1}{2} [\hat{S}, \hat{V}] + \mathcal{O}(\hat{V}^3), \quad (2.44)$$

to lowest order in the interaction.

2.5.2 The Bogoliubov-Valatin Transformation

The Bogoliubov-Valatin transformation was developed independently by N.N. Bogoliubov and J.G. Valatin [49, 50] in 1958, and is a canonical transformation of the canonical commutation (bosonic) or anticommutation (fermionic) relations. Here we will consider the case of bosonic operators. The commutation relation for a single field mode is

$$[\hat{a}, \hat{a}^\dagger] = 1. \quad (2.45)$$

Introducing new operators as superpositions of the above operators, we have⁷ [24, 35, 52, 53]

$$\hat{b} = \mu \hat{a} + \nu \hat{a}^\dagger \quad \text{and} \quad \hat{b}^\dagger = \mu^* \hat{a}^\dagger + \nu^* \hat{a}. \quad (2.46)$$

⁷This step precedes the developments of N.N. Bogoliubov and J.G. Valatin in 1958 and was used by T. Holstein and H. Primakoff in a paper from 1940 [51].

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The canonical transformation which maps the operators of the first mode $(\hat{a}, \hat{a}^\dagger)$ to the operators $(\hat{b}, \hat{b}^\dagger)$ is called the Bogoliubov-Valatin transformation. The transformation is canonical if the canonical commutation relation is isomorphic, which provides conditions on μ and ν through

$$\left[\hat{b}, \hat{b}^\dagger \right] = \left[\mu \hat{a} + \nu \hat{a}^\dagger, \mu \hat{a}^\dagger + \nu^* \hat{a} \right] = (|\mu|^2 - |\nu|^2) \left[\hat{a}, \hat{a}^\dagger \right]. \quad (2.47)$$

The condition is thus $|\mu|^2 - |\nu|^2 = 1$, which can be identified as the hyperbolic identity up to a phase factor

$$\cosh(x)^2 - \sinh(x)^2 = 1. \quad (2.48)$$

Choosing to put a phase on ν , the parametrisation of the constants becomes [35, 52, 53]

$$\mu = \cosh(r), \quad \text{and} \quad \nu = e^{i\psi} \sinh(r), \quad (2.49)$$

where $r \in \mathbb{R}$ and $\phi \in \mathbb{R}$. We can also write the transformation in matrix form as

$$\begin{bmatrix} \hat{b} \\ \hat{b}^\dagger \end{bmatrix} = \mathbf{S}_1 \begin{bmatrix} \hat{a} \\ \hat{a}^\dagger \end{bmatrix}, \quad \mathbf{S}_1 = \begin{bmatrix} \mu & \nu \\ \nu^* & \mu^* \end{bmatrix}. \quad (2.50)$$

Now, we can identify \mathbf{S}_1 along with the condition as the group element of the $SU(1, 1)$ group, the definition of which is [35]

$$SU(1, 1) = \left\{ \mathbf{S}_1 \in \mathbb{C}^{2 \times 2} : \det(\mathbf{S}_1) = |\mu|^2 - |\nu|^2 = 1 \right\}. \quad (2.51)$$

This condition can also be written as

$$\mathbf{S}_1^\dagger \hat{\sigma}_z \mathbf{S}_1 = \hat{\sigma}_z, \quad (2.52)$$

where $\hat{\sigma}_z$ is the Pauli- z spin matrix. The generators of the $SU(1, 1)$ group, that form the $\mathfrak{su}(1, 1)$ Lie algebra, are the operators $\hat{K}_i, i \in \{0, 1, 2\}$, satisfying the commutation relations [35, 52]

$$\left[\hat{K}_0, \hat{K}_1 \right] = i\hat{K}_2, \quad \left[\hat{K}_1, \hat{K}_2 \right] = -i\hat{K}_0 \quad \text{and} \quad \left[\hat{K}_2, \hat{K}_0 \right] = i\hat{K}_1. \quad (2.53)$$

One can also introduce corresponding raising and lowering operators

$$\hat{K}_\pm = \hat{K}_1 \pm \hat{K}_2, \quad \text{with} \quad \hat{K}_+ = \hat{K}_-^\dagger, \quad (2.54)$$

giving the new commutation relations

$$\left[\hat{K}_0, \hat{K}_\pm \right] = \pm i\hat{K}_\pm \quad \text{and} \quad \left[\hat{K}_+, \hat{K}_- \right] = -2\hat{K}_0. \quad (2.55)$$

One set of operators satisfying the commutation relations in Eq. (2.53) is [35, 52]

$$\hat{K}_0 = \frac{1}{4}(2\hat{a}^\dagger \hat{a} + 1), \quad \hat{K}_+ = \frac{1}{2}\hat{a}^\dagger \hat{a}^\dagger \quad \text{and} \quad \hat{K}_- = \frac{1}{2}\hat{a} \hat{a}. \quad (2.56)$$

The type of canonical transformations generated by \hat{K}_0 and \hat{K}_\pm are [35]

$$\hat{S}_K^\dagger \begin{bmatrix} \hat{a} \\ \hat{a}^\dagger \end{bmatrix} \hat{S}_K = \begin{bmatrix} \hat{S}_K^\dagger \hat{a} \hat{S}_K \\ \hat{S}_K^\dagger \hat{a}^\dagger \hat{S}_K \end{bmatrix} = \begin{bmatrix} \hat{b} \\ \hat{b}^\dagger \end{bmatrix}, \quad (2.57)$$

with the evolution operator \hat{S}_K being

$$\hat{S}_K = \exp\left(-i\psi_0 \hat{K}_0 + \beta \hat{K}_+ - \beta^* \hat{K}_-\right), \quad (2.58)$$

where we introduced the polar coordinate $\beta = r e^{i\psi}$, giving $r = |\beta|$ and $\psi = \arg(\beta)$. The transformation generated by \hat{K}_0 amounts to simple multiplication of the operators with a phase, while the transformation generated by \hat{K}_\pm with evolution operator

$$\hat{S}_1(\beta) = \exp\left(\beta \hat{K}_+ - \beta^* \hat{K}_-\right), \quad (2.59)$$

is the one that mixes the operators. The explicit transformations for one mode are [52]

$$\hat{S}_1^\dagger(\beta) \hat{a} \hat{S}_1(\beta) = \mu \hat{a} + \nu \hat{a}^\dagger, \quad \text{and} \quad \hat{S}_1^\dagger(\beta) \hat{a}^\dagger \hat{S}_1(\beta) = \mu \hat{a}^\dagger + \nu^* \hat{a}. \quad (2.60)$$

It is possible to disentangle the evolution operator, \hat{S}_1 , using Gauss decomposition for the $SU(1, 1)$ group, giving [35]

$$\hat{S}_1(\beta) = \exp\left(\beta_+ \hat{K}_+\right) \exp\left(\beta_0 \hat{K}_0\right) \exp\left(-\beta_- \hat{K}_-\right), \quad (2.61)$$

where

$$\beta_\pm = e^{\pm i\psi} \tanh(r), \quad \text{and} \quad \beta_0 = -2 \ln(\cosh(r)). \quad (2.62)$$

This disentangled form can then be written as [52, 53]

$$\hat{S}_1(\beta) = \exp\left(\frac{\nu}{2\mu} \hat{a}^\dagger \hat{a}^\dagger\right) \mu^{-\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)} \exp\left(-\frac{\nu^*}{2\mu} \hat{a} \hat{a}\right). \quad (2.63)$$

Transformation of Two Modes

In the case of two field modes, the operators

$$\hat{K}_0^{(2)} = 2(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1), \quad \hat{K}_+^{(2)} = \hat{a}_1^\dagger \hat{a}_2^\dagger, \quad \text{and} \quad \hat{K}_-^{(2)} = \hat{a}_1 \hat{a}_2, \quad (2.64)$$

satisfy the commutation relations of the $\mathfrak{su}(1, 1)$ algebra in Eq. (2.53) [52], and give a two-mode evolution operator

$$\hat{S}_2(\beta) = \exp\left(\beta \hat{K}_+^{(2)} - \beta^* \hat{K}_-^{(2)}\right), \quad (2.65)$$

which can be disentangled in the same way as before, yielding [52, 53]

$$\hat{S}_2(\beta) = \exp\left(\frac{\nu}{\mu} \hat{a}_1^\dagger \hat{a}_2^\dagger\right) \mu^{-\left(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2}\right)} \exp\left(-\frac{\nu^*}{\mu} \hat{a}_1 \hat{a}_2\right). \quad (2.66)$$

Once again, we can utilise a matrix description, giving the transformation as being [52]

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$$\hat{S}_2^\dagger(\beta) \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{bmatrix} \hat{S}_2(\beta) = \mathbf{S}_1 \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{bmatrix}, \quad (2.67)$$

with the same matrix as before. For several modes, we can also represent the transformations with the complex form of the symplectic group $\text{Sp}(2N, \mathbb{R})$. The matrix giving Bogoliubov-Valatin transformations for two modes in the basis of mode operators introduced later in Eq. (2.77) is

$$\mathbf{S}_2 = \begin{bmatrix} \mu & 0 & 0 & \nu \\ 0 & \mu^* & \nu^* & 0 \\ 0 & \nu & \mu & 0 \\ \nu^* & 0 & 0 & \mu^* \end{bmatrix}, \quad (2.68)$$

still subject to the condition $|\mu|^2 - |\nu|^2 = 1$, though now through $\det(\mathbf{S}_2) = (|\mu|^2 - |\nu|^2)^2 = 1$, which can also be written as [40]

$$\mathbf{S}_2^\dagger \mathbf{K}_\zeta \mathbf{S}_2 = \mathbf{K}_\zeta, \quad \mathbf{K}_\zeta = \bigoplus_{k=1}^2 \hat{\sigma}_z, \quad (2.69)$$

along with the condition

$$\boldsymbol{\mu} \boldsymbol{\nu}^\top = (\boldsymbol{\mu} \boldsymbol{\nu}^\top)^\top, \quad (2.70)$$

where

$$\boldsymbol{\mu} = \begin{bmatrix} \mu & 0 \\ 0 & \mu^* \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\nu} = \begin{bmatrix} 0 & \nu \\ \nu^* & 0 \end{bmatrix}, \quad (2.71)$$

giving the block form of \mathbf{S}_2 as being

$$\mathbf{S}_2 = \begin{bmatrix} \boldsymbol{\mu} & \boldsymbol{\nu} \\ \boldsymbol{\nu} & \boldsymbol{\mu} \end{bmatrix}. \quad (2.72)$$

We also note that the groups $\text{Sl}(2, \mathbb{R})$, $\text{Sp}(2, \mathbb{R})$ and $\text{SU}(1, 1)$ are isomorphic [54].

2.5.3 Symplectic Form

The real symplectic group is defined as [40, 55]

$$\text{Sp}(2N, \mathbb{R}) = \left\{ \mathbf{S}_R \in \mathbb{R}^{2N \times 2N} : (\mathbf{S}_R)^\top \boldsymbol{\Omega} \mathbf{S}_R = \boldsymbol{\Omega} \right\}, \quad (2.73)$$

where the matrix $\boldsymbol{\Omega}$ has the form [40, 56–58]

$$\boldsymbol{\Omega} = \bigoplus_{k=1}^N i \hat{\sigma}_y, \quad \text{or conventionally} \quad \boldsymbol{\Omega} = \bigoplus_{k=1}^N \boldsymbol{\omega}, \quad \text{where} \quad \boldsymbol{\omega} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad (2.74)$$

and N is the number of field modes. Writing the quadrature operators in vector form as

$$\hat{\mathbf{R}} = \left[\hat{x}_1, \hat{p}_1, \dots, \hat{x}_N, \hat{p}_N \right]^\top, \quad (2.75)$$

the elements of Ω are defined through the commutation relations of the elements of $\hat{\mathbf{R}}$ as

$$\left[\hat{\mathbf{R}}_i, \hat{\mathbf{R}}_j \right] = i\Omega_{ij}. \quad (2.76)$$

From the quadrature operators, we can move to the basis of the mode operators, which is the basis known as the complex form of $\text{Sp}(2N, \mathbb{R})$ ⁸, and is formed by the mode operators [40, 55]

$$\hat{\zeta} = \left[\hat{a}_1, \hat{a}_1^\dagger, \dots, \hat{a}_N, \hat{a}_N^\dagger \right]^\top, \quad (2.77)$$

which obey the commutation relation

$$\left[\hat{\zeta}_i, \hat{\zeta}_j^\dagger \right] = K_{ij}. \quad (2.78)$$

From this, we can construct the corresponding matrix \mathbf{K}_ζ as

$$\mathbf{K}_\zeta = \bigoplus_{k=1}^N \hat{\sigma}_z, \quad \text{or} \quad \mathbf{K}_\zeta = \bigoplus_{k=0}^{2N-1} (-1)^k \mathbb{1}_1, \quad (2.79)$$

where $\hat{\sigma}_z$ is the third Pauli spin matrix, and $\mathbb{1}_1$ is the 1×1 identity matrix. This complex form alters the matrices entering into the definition of $\text{Sp}(2N, \mathbb{R})$ to being

$$\text{Sp}(2N, \mathbb{R}) = \left\{ \mathbf{S} \in \mathbb{C}^{2N \times 2N} : \mathbf{S}^\dagger \mathbf{K}_\zeta \mathbf{S} = \mathbf{K}_\zeta \right\}. \quad (2.80)$$

The matrix \mathbf{S} has the form [40]

$$\mathbf{S} = \begin{bmatrix} \mathbf{s}_{11} & \mathbf{s}_{12} & \cdots & \mathbf{s}_{1N} \\ \mathbf{s}_{21} & \mathbf{s}_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{s}_{N1} & \cdots & \cdots & \mathbf{s}_{NN} \end{bmatrix}, \quad (2.81)$$

where the elements \mathbf{s}_{ij} are 2×2 sub-blocks containing the transformation between modes i and j . The mode operators can also be ordered as [40]

$$\hat{\xi} = \left[\hat{a}_1, \dots, \hat{a}_N, \hat{a}_1^\dagger, \dots, \hat{a}_N^\dagger \right]^\top, \quad (2.82)$$

⁸This should not be seen as moving to the group $\text{Sp}(2N, \mathbb{C})$, but simply as a change of basis.

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giving the corresponding matrix \mathbf{K}_ξ as [40]

$$\mathbf{K}_\xi = \begin{bmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{bmatrix}. \quad (2.83)$$

The reordering of the basis is achieved through [40]

$$\hat{\xi} = \Delta \hat{\zeta}, \quad \text{with} \quad \Delta_{ij} = \delta_{j,2i-1} + \delta_{j+2N,2i}, \quad (2.84)$$

for $i, j \in \{1, \dots, 2N\}$. The same reordering of the quadrature operators will make the symplectic matrix Ω have the block form [40, 55]

$$\Omega_\Delta = \begin{bmatrix} \mathbf{0} & \mathbb{1}_N \\ -\mathbb{1}_N & \mathbf{0} \end{bmatrix}. \quad (2.85)$$

A symplectic matrix subject to the condition in Eq. (2.73) with Ω_Δ or Eq. (2.80) with \mathbf{K}_ξ , can be written in block form as

$$\mathbf{S}_{(R)} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \quad (2.86)$$

and has the following conditions for the block matrices [40, 55]

$$\mathbf{A}^{\dagger(\top)} \mathbf{C} = (\mathbf{A}^{\dagger(\top)} \mathbf{C})^{\dagger(\top)}, \quad \mathbf{B}^{\dagger(\top)} \mathbf{D} = (\mathbf{B}^{\dagger(\top)} \mathbf{D})^{\dagger(\top)},$$

and $\mathbf{A}^{\dagger(\top)} \mathbf{D} - \mathbf{C}^{\dagger(\top)} \mathbf{B} = \mathbb{1}_{2N}$. (2.87)

Moving between the quadrature basis, and the mode operator basis can be done through [56]

$$\begin{cases} \hat{\mathbf{R}} = \mathbf{T} \hat{\zeta} \\ \hat{\zeta} = \mathbf{T}^\dagger \hat{\mathbf{R}} \end{cases}, \quad \text{where} \quad \mathbf{T} = \frac{1}{\sqrt{2}} \bigoplus_{k=1}^N \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, \quad (2.88)$$

and for a symplectic matrix through

$$\mathbf{S}_R = \mathbf{T} \mathbf{S} \mathbf{T}^\dagger, \quad (2.89)$$

or easily the other way since \mathbf{T} is unitary, $\mathbf{T}^\dagger \mathbf{T} = 1$. If we introduce the matrix $\mathbf{W} = -i \mathbf{K}_\zeta \mathbf{H}$, where \mathbf{H} is the Hermitian Hamiltonian matrix corresponding to a quadratic Hamiltonian

$$\hat{H} = \hat{\zeta}^\dagger \mathbf{H} \hat{\zeta}, \quad (2.90)$$

it can be shown that the commutation relation of the Hamiltonian with the mode operators is [40]

$$-i [\hat{H}, \hat{\zeta}] = \mathbf{W} \hat{\zeta}. \quad (2.91)$$

Using the Hadamard lemma in Eq. (2.35) to transform the mode operators with the Hamiltonian gives

$$\exp(-i\hat{H})\hat{\zeta}\exp(i\hat{H}) = e^{\mathbf{W}}\hat{\zeta}. \quad (2.92)$$

From this, it is possible to identify that a quadratic unitary evolution operator and its corresponding symplectic matrix are related through [59]

$$\hat{S}_{\zeta} = \exp(-i\hat{\zeta}^{\dagger}\mathbf{H}\hat{\zeta}) \longrightarrow \mathbf{S} = e^{\mathbf{W}}. \quad (2.93)$$

The matrix \mathbf{W} will thus have the same 2×2 sub-block structure as \mathbf{S} given in Eq. (2.81). This sub-block structure stems from the quadratic exponent of the unitary evolution operator. A table of the correspondences between the Hilbert space and phase space description is given in Appendix D.

2.6 Displacement, Squeezing and Beam-Splitting

2.6.1 Displacement

The displacement operator, sometimes also called the Weyl or Glauber operator, is a shift operator similar to the translation operator and the time-evolution operator. For one bosonic field mode, it is given by [26, 53, 60, 61]

$$\hat{D}(\alpha) = \exp(\alpha\hat{a}^{\dagger} - \alpha^*\hat{a}), \quad (2.94)$$

where $\alpha \in \mathbb{C}$. It is a shift operator because transforming the ladder operators results in a displacement in phase space of the creation operator by α^* , and of the annihilation operator by α as

$$\hat{D}^{\dagger}(\alpha)\hat{a}^{\dagger}\hat{D}(\alpha) = \hat{a}^{\dagger} + \alpha^*, \quad \text{and} \quad \hat{D}^{\dagger}(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha. \quad (2.95)$$

Its action on the vacuum state creates what is called a coherent state, first introduced by J.R. Klauder (see e.g. Ref. [62]), and later applied to the quantum theory of light by R.J. Glauber [63] and E.C.G. Sudarshan [64]. Explicitly,

$$\hat{D}(\alpha)|0\rangle = |\alpha\rangle \quad (2.96)$$

where $|\alpha\rangle$ is the formed coherent state.

The coherent state is also an eigenstate of the annihilation operator $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. The creation operator has no eigenket, and similarly the annihilation operator has no eigenbra. The coherent state-ket is thus not an eigenstate of the creation operator and the coherent state-bra is not an eigenstate of the annihilation operator. Their actions are instead given by⁹ [53]

$$\hat{a}^{\dagger}|\alpha\rangle = \left(\frac{\partial}{\partial\alpha} + \frac{\alpha^*}{2}\right)|\alpha\rangle, \quad \text{and} \quad \langle\alpha|\hat{a} = \langle\alpha|\left(\frac{\partial}{\partial\alpha^*} + \frac{\alpha}{2}\right), \quad (2.97)$$

where the derivative in the second equation acts to the left on the bra.

⁹The action of the creation operator can also generate an Agarwal-Tara state (or simply Agarwal state), also called a photon-added coherent state through $\hat{a}^{\dagger}|\alpha\rangle = \sqrt{1+|\alpha|^2}|\alpha, 1\rangle$, which can also be generalised to higher orders [65, 66].

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The reason for $|\alpha\rangle$ being called a coherent state is that its coordinate representation is the minimum-uncertainty Gaussian wave packet, which oscillates back and forth in a HO potential [61]. During this process, it does not change its shape and can be said to cohere.

Using the Kermack-McCrae identity on the displacement operator gives its normal ordered

$$\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger) \exp(-\alpha^* \hat{a}) \exp\left(-\frac{|\alpha|^2}{2}\right), \quad (2.98)$$

and anti-normal ordered

$$\hat{D}(\alpha) = \exp(-\alpha^* \hat{a}) \exp(\alpha \hat{a}^\dagger) \exp\left(\frac{|\alpha|^2}{2}\right), \quad (2.99)$$

form. For N field modes, we can introduce a vector of displacement coefficients as

$$\vec{\lambda} = [\alpha_1, \alpha_1^*, \dots, \alpha_N, \alpha_N^*]^\top, \quad (2.100)$$

which we use to generalise the displacement operator. The generalised operator simply becomes the products of displacements of different field modes and can be written in symplectic form as

$$\hat{D}(\vec{\lambda}) = \exp(-\hat{\zeta}^\dagger \mathbf{K}_\zeta \vec{\lambda}). \quad (2.101)$$

We also note that the displacement operator is part of the group element of the Heisenberg–Weyl Group $H(1)$, since \hat{a}^\dagger and \hat{a} are two of the generators of $H(1)$ which form the $\mathfrak{h}(1)$ algebra [35].

2.6.2 Single-Mode Squeezing

The operator in Eq. (2.61) of Sec. 2.5.2 is called the single-mode squeezing operator. If we look at the symplectic representation of this operator in the basis of the quadrature operators, \hat{x} and \hat{p} , we have [40]

$$\mathbf{S}_1^R(r, \varphi) = \begin{bmatrix} \cosh(r) + \cos(\varphi) \sinh(r) & \sin(\varphi) \sinh(r) \\ \sinh(\varphi) \sinh(r) & \cosh(r) - \cos(\varphi) \sinh(r) \end{bmatrix}, \quad (2.102)$$

which reduces to $\text{diag}(e^r, e^{-r})$ for $\varphi = 0$ [40, 58]. This is where the term squeezing stems from, since when $r > 0$, this squeezes the momentum, \hat{p} , reducing its variance proportional to the factor e^{-r} , while also increasing the variance of the position by e^r . For the angle $\varphi = \frac{\pi}{2}$, the opposite is true. For a review of squeezed states of light, see Ref. [67]. A derivation of the single-mode squeezing interaction from cQED can be found in Ref. [68].

2.6.3 Two-Mode Squeezing

As for single-mode squeezing, we introduced the two-mode squeezing operator in Sec. 2.5.2, specifically Eq. (2.65). This has a similar interpretation as for a single mode, where it squeezed the variance

of an operator exponentially. For two modes however, the operators are now superposition operators of the two modes of position and momentum [26]. Two-mode squeezing is also commonly used to create continuous-variable entanglement [40].

2.6.4 Beam Splitting

The beam splitter represents the simplest case of a two-mode interaction, which is linear mixing of the modes according to [40, 52]

$$\hat{B}^\dagger(\kappa)\hat{a}_1\hat{B}(\kappa) = \cos(\phi)\hat{a}_1 + \exp(i\theta)\sin(\phi)\hat{a}_2, \quad (2.103)$$

and

$$\hat{B}^\dagger(\kappa)\hat{a}_2\hat{B}(\kappa) = \cos(\phi)\hat{a}_2 - \exp(-i\theta)\sin(\phi)\hat{a}_1, \quad (2.104)$$

where [52]

$$\hat{B}(\kappa) = \exp\left(\kappa\hat{a}_1^\dagger\hat{a}_2 - \kappa^*\hat{a}_1\hat{a}_2^\dagger\right), \quad (2.105)$$

with $\kappa = \phi \exp(i\theta)$. Often the quantity $\tau = \cos^2(\phi)$ is introduced, called the transmissivity, or alternatively as $\phi = \arccos(\sqrt{\tau})$, which represents a rotation in phase space. $\phi = \frac{\pi}{4}$, or $\tau = \frac{1}{2}$ gives symmetric beam splitting. With $\theta = 0$, the symplectic transformation in the basis of Eq. (2.77) is [40, 69]

$$\mathbf{B} = \begin{bmatrix} \sqrt{\tau} & 0 & \sqrt{1-\tau} & 0 \\ 0 & \sqrt{\tau} & 0 & \sqrt{1-\tau} \\ \sqrt{1-\tau} & 0 & -\sqrt{\tau} & 0 \\ 0 & \sqrt{1-\tau} & 0 & -\sqrt{\tau} \end{bmatrix}. \quad (2.106)$$

One can also perform a similar decomposition as before using the Schwinger two-mode boson representation of the $\mathfrak{su}(2)$ algebra [52].

2.6.5 Phase Shift

Rotation or phase shift of a single mode in phase space by an angle $\frac{\theta}{2}$ is performed through the unitary operator [40, 70]

$$\hat{U}_k = \exp\left(-i\theta\hat{a}_k^\dagger\hat{a}_k\right), \quad (2.107)$$

for a single mode k . Transforming the ladder operators with this gives [70]

$$\hat{U}_k^\dagger(\theta)\hat{a}_k\hat{U}_k(\theta) = \exp(-i\theta)\hat{a}_k, \quad \text{and} \quad \hat{U}_k^\dagger(\theta)\hat{a}_k^\dagger\hat{U}_k(\theta) = \exp(i\theta)\hat{a}_k^\dagger. \quad (2.108)$$

For the basis $\hat{\mathbf{R}}$, the symplectic matrix for a rotation is [40]

$$\hat{\mathcal{R}}(\theta) = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}. \quad (2.109)$$

3. Derivation of the System Hamiltonian

The first step in deriving our desired effective Hamiltonian in Eq. (1.1), is to derive a first quantum Hamiltonian from a circuit equivalent of the schematic in Fig. 1.1 using cQED. Subsequently, we apply approximations and unitary transformations to get the desired form, which is the two-mode two-photon Jaynes-Cummings model described in Sec. 1, and discussed in relation to other quantum models in Appendix B. Important to note is that *quantum tunnelling cannot be obtained from a classical model*, and has to be added into the model from outside the framework of cQED.

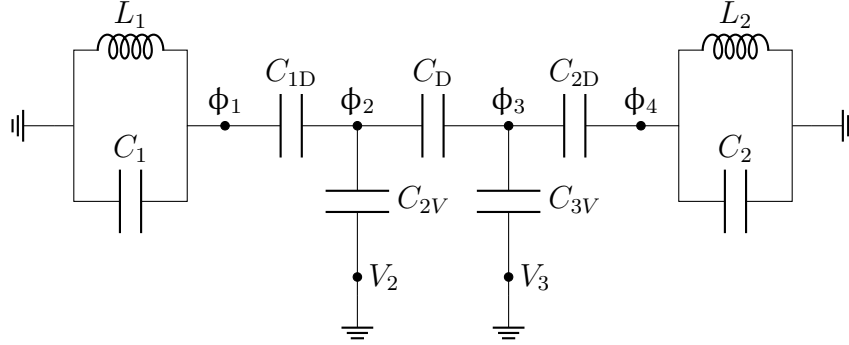


Figure 3.1: Circuit equivalent of a two-level DQD coupled to two cavities. The two LC -oscillators correspond to two HO's, and the nodes connected to the capacitor C_D , to a DQD. C_{1D} and C_{2D} are the capacitive couplings between the oscillators and the DQD. The ϕ are the node fluxes between the elements, taking the role of position coordinates.

3.1 Circuit Quantum Electrodynamics

Our system, described in Sec. 1 and illustrated in Fig. 3.1, has the classical Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{sys.}} = & \frac{1}{2}C_1\dot{\phi}_1^2 - \frac{1}{2L_1}\phi_1^2 + \frac{1}{2}C_{1D}(\dot{\phi}_1 - \dot{\phi}_2)^2 + \frac{1}{2}C_{2V}(\dot{\phi}_2 - V_2)^2 + \frac{1}{2}C_D(\dot{\phi}_2 - \dot{\phi}_3)^2 \\ & + \frac{1}{2}C_{3V}(\dot{\phi}_3 - V_3)^2 + \frac{1}{2}C_{2D}(\dot{\phi}_3 - \dot{\phi}_4)^2 + \frac{1}{2}C_2\dot{\phi}_4^2 - \frac{1}{2L_2}\phi_4^2. \end{aligned} \quad (3.1)$$

Using the matrix form in Eq. (2.9), the capacitance matrix can be written as

$$\mathbf{C} = \begin{bmatrix} C_1 + C_{1D} & -C_{1D} & 0 & 0 \\ -C_{1D} & C_{1D} + C_D + C_{2V} & -C_D & 0 \\ 0 & -C_D & C_D + C_{2D} + C_{3V} & -C_{2D} \\ 0 & 0 & -C_{2D} & C_{2D} + C_2 \end{bmatrix}, \quad (3.2)$$

and the inverse inductance matrix as $\mathbf{L}^{-1} = \text{diag}(\frac{1}{L_1}, 0, 0, \frac{1}{L_2})$. To find the classical Hamiltonian, we make use of Eq. (2.12), which requires finding the inverse of the capacitance matrix. Importantly, when finding the conjugate variables q_v , from $\dot{\phi}_v$, the voltage offsets will now modify the charge according to Eq. (2.15). We expand the inverse capacitance matrix to first order in the coupling strengths

C_{1D} , C_D and C_{2D} , by introducing a small shared parameter η , thus neglecting second order processes in η . This gives the inverse capacitance matrix

$$\mathbf{C}^{-1} \approx \begin{bmatrix} \frac{C_1 - \eta C_{1D}}{C_1^2} & \frac{\eta C_{1D}}{C_1 C_{2V}} & 0 & 0 \\ \frac{\eta C_{1D}}{C_1 C_{2V}} & \frac{C_{2V} - \eta(C_{1D} + C_D)}{C_{2V}^2} & \frac{\eta C_D}{C_{2V} C_{3V}} & 0 \\ 0 & \frac{\eta C_D}{C_{2V} C_{3V}} & \frac{C_{3V} - \eta(C_{2D} + C_D)}{C_{3V}^2} & \frac{\eta C_{2D}}{C_2 C_{3V}} \\ 0 & 0 & \frac{\eta C_{2D}}{C_2 C_{3V}} & \frac{C_2 - \eta C_{2D}}{C_2^2} \end{bmatrix} \quad (3.3)$$

From Eq. (2.12), keeping only leading order terms in η , the Hamiltonian is then

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V}, \quad (3.4)$$

with

$$\mathcal{H}_0 = \frac{q_1^2}{2C_1} + \frac{\Phi_1^2}{2L_1} + \frac{(q_2 - \tilde{q}_{2V})^2}{2C_{2V}} + \frac{(q_3 - \tilde{q}_{3V})^2}{2C_{3V}} + \frac{q_4^2}{2C_2} + \frac{\Phi_4^2}{2L_2}, \quad (3.5)$$

and

$$\mathcal{V} = \frac{C_{1D} q_1 (q_2 - \tilde{q}_{2V})}{C_1 C_{2V}} + \frac{C_D (q_2 - \tilde{q}_{2V}) (q_3 - \tilde{q}_{3V})}{C_{2V} C_{3V}} + \frac{C_{2D} (q_3 - \tilde{q}_{3V}) q_4}{C_2 C_{3V}}. \quad (3.6)$$

Here, we also introduced the two rewritings of the offset charges,

$$\tilde{q}_{2V} \equiv -C_{2V} V_2, \quad \text{and} \quad \tilde{q}_{3V} \equiv -C_{3V} V_3, \quad (3.7)$$

for simplification.

For the charges of the DQD, we introduce dimensionless offset charges for the left and right dot along with corresponding integer-valued number operators

$$n_L \equiv \frac{\tilde{q}_{2V}}{2e}, \quad n_R \equiv \frac{\tilde{q}_{3V}}{2e}, \quad \hat{n}_L \equiv \frac{\hat{q}_2}{2e}, \quad \text{and} \quad \hat{n}_R \equiv \frac{\hat{q}_3}{2e}. \quad (3.8)$$

We now perform canonical quantisation on the flux and charge from Eq. (2.17), giving that the quantum Hamiltonian, $\hat{H} = \hat{H}_0 + \hat{V}$, has the contributions

$$\hat{H}_0 = \hbar\omega_1 \left(\hat{a}_1^\dagger \hat{a}_1 + \frac{1}{2} \right) + \hbar\omega_2 \left(\hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2} \right) + 4E_L (\hat{n}_L - n_L)^2 + 4E_R (\hat{n}_R - n_R)^2, \quad (3.9)$$

and

$$\begin{aligned} \hat{V} = E_L C_L i (\hat{a}_1^\dagger - \hat{a}_1) (\hat{n}_L - n_L) + E_L E_R \frac{16C_D}{e^2} (\hat{n}_L - n_L) (\hat{n}_R - n_R) \\ + E_R C_R i (\hat{a}_2^\dagger - \hat{a}_2) (\hat{n}_R - n_R), \end{aligned} \quad (3.10)$$

where we used the generalised definitions in Eq. (2.19), making the oscillator frequencies and impedances

$$\omega_1 = \frac{1}{\sqrt{L_1 C_1}}, \quad \omega_2 = \frac{1}{\sqrt{L_2 C_2}}, \quad Z_1 = \sqrt{\frac{L_1}{C_1}}, \quad \text{and} \quad Z_2 = \sqrt{\frac{L_2}{C_2}}, \quad (3.11)$$

3. Derivation of the System Hamiltonian

respectively. We also defined the charging energies for the left and right dot as

$$E_L \equiv \frac{e^2}{2C_{2V}}, \quad \text{and} \quad E_R \equiv \frac{e^2}{2C_{3V}}, \quad (3.12)$$

and further introduced

$$C_L = \frac{C_{1D}}{e} \sqrt{\frac{2\hbar\omega_1}{C_1}}, \quad \text{and} \quad C_R = \frac{C_{2D}}{e} \sqrt{\frac{2\hbar\omega_2}{C_2}}. \quad (3.13)$$

We restrict the DQD to be in the Coulomb blockade regime. This means that we allow only one excess electron to reside on the DQD. If we then insert complete sets of eigenstates for the two dots, $(|0\rangle_{L(R)}, |1_1\rangle_{L(R)}, |1_2\rangle_{L(R)}, \dots)$, into our non-interacting Hamiltonian using fermionic occupation number representation, while at first neglecting the empty state, we have for the first DQD part,

$$\begin{aligned} \hat{H}_0^{\text{DQD}} = & \left(\sum_{k=0}^{\infty} |1_k\rangle_L |0\rangle_{RL} \langle 1_k|_R \langle 0| + |0\rangle_L |1_k\rangle_{RL} \langle 0|_R \langle 1_k| \right) \left(4E_L (\hat{n}_L - n_L)^2 \right. \\ & \left. + 4E_R (\hat{n}_R - n_R)^2 \right) \left(\sum_{k'=0}^{\infty} |1_{k'}\rangle_L |0\rangle_{RL} \langle 1_{k'}|_R \langle 0| + |0\rangle_L |1_{k'}\rangle_{RL} \langle 0|_R \langle 1_{k'}| \right), \quad (3.14) \end{aligned}$$

which reduces to

$$\begin{aligned} \hat{H}_0^{\text{DQD}} = & 4 \left(\sum_{k=0}^{\infty} \left(E_L (1 - 2n_L + n_L^2) + E_R n_R^2 \right) |1_k\rangle_L |0\rangle_{RL} \langle 1_k|_R \langle 0| \right. \\ & \left. + \left(E_L n_L^2 + E_R (1 - 2n_R + n_R^2) \right) |0\rangle_L |1_k\rangle_{RL} \langle 0|_R \langle 1_k| \right), \quad (3.15) \end{aligned}$$

where we used orthogonality, ${}_i \langle 1_k | 1_{k'} \rangle_i = \delta_{kk'}$, and $\hat{n}_i |1_k\rangle_i = |1_k\rangle_i$, $i \in \{L, R\}$. For clarification, the notation $|1_k\rangle_L$ in this case means that the k :th energy level in the left dot is occupied. Restricting the sum to run over only the ground state, and simplifying the notation through

$$|1_0\rangle_L |0\rangle_{RL} \langle 1_0|_R \langle 0| \longrightarrow |L\rangle \langle L|, \quad \text{and} \quad |0\rangle_L |1_0\rangle_{RL} \langle 0|_R \langle 1_0| \longrightarrow |R\rangle \langle R|, \quad (3.16)$$

we get

$$\begin{aligned} \hat{H}_0^{\text{DQD}} = & 4(E_L(1 - n_L)^2 + E_R n_R^2) |L\rangle \langle L| + 4(E_R(1 - n_R)^2 + E_L n_L^2) |R\rangle \langle R| \\ & + 4(E_L n_L^2 + E_R n_R^2) |0\rangle \langle 0|, \quad (3.17) \end{aligned}$$

where we now also included the empty state. Using the fact that

$$\mathbf{1}_3 = |0\rangle \langle 0| + |L\rangle \langle L| + |R\rangle \langle R|, \quad (3.18)$$

we can rewrite this as

$$\hat{H}_0^{\text{DQD}} = 4E_L(1 - 2n_L) |L\rangle \langle L| + 4E_R(1 - 2n_R) |R\rangle \langle R| + 4(E_L n_L^2 + E_R n_R^2) \mathbf{1}_3. \quad (3.19)$$

Doing the same thing for \hat{V} gives

$$\begin{aligned}\hat{V} &= E_L C_L i(\hat{a}_1^\dagger - \hat{a}_1)(|L\rangle\langle L| - n_L \mathbb{1}_3) + E_R C_R i(\hat{a}_2^\dagger - \hat{a}_2)(|R\rangle\langle R| - n_R \mathbb{1}_3) \\ &\quad + E_L E_R \frac{16C_D}{e^2}(n_L n_R \mathbb{1}_3 - n_R |L\rangle\langle L| - n_L |R\rangle\langle R|).\end{aligned}\quad (3.20)$$

Here, we get terms proportional to $(\hat{a}_i^\dagger - \hat{a}_i)\mathbb{1}_3$, which can be viewed as displacements of the oscillators in phase space. To get rid of these, we perform a unitary transformation

$$\hat{H}' = \hat{D}^\dagger(\alpha_1, \alpha_2)\hat{H}\hat{D}(\alpha_1, \alpha_2),\quad (3.21)$$

using the displacement operator for two modes from Eq. (2.101). Neglecting the above constants in energy, we get with $\alpha_i = -\alpha_i^* = iE_L C_L n_L / (\hbar\omega_i)$ that the terms $(\hat{a}_i^\dagger - \hat{a}_i)\mathbb{1}_3$ cancel, giving

$$\begin{aligned}\hat{H}' &= \hbar\omega_1 \hat{a}_1^\dagger \hat{a}_1 + \hbar\omega_2 \hat{a}_2^\dagger \hat{a}_2 + 4E_L(1 - 2n_L)|L\rangle\langle L| + 4E_R(1 - 2n_R)|R\rangle\langle R| \\ &\quad + E_L C_L i(\hat{a}_1^\dagger - \hat{a}_1 - 2i \operatorname{Im}(\alpha_1))|L\rangle\langle L| + E_R C_R i(\hat{a}_2^\dagger - \hat{a}_2 - 2i \operatorname{Im}(\alpha_2))|R\rangle\langle R| \\ &\quad - E_L E_R \frac{16C_D}{e^2}(n_R |L\rangle\langle L| + n_L |R\rangle\langle R|),\end{aligned}\quad (3.22)$$

where we again neglected the constant terms resulting from the displacement. Using $(|R\rangle, |L\rangle, |0\rangle)$ as our local basis, we can introduce the Pauli operators

$$\hat{\tau}_z = |R\rangle\langle R| - |L\rangle\langle L|, \quad \text{and} \quad \hat{\tau}_x = |L\rangle\langle R| + |R\rangle\langle L|.\quad (3.23)$$

From Eq. (3.22) with the above Pauli operators, we get the non-interacting Hamiltonian

$$\hat{H}_0 = \hbar\omega_1 \hat{a}_1^\dagger \hat{a}_1 + \hbar\omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{\hbar\varepsilon_z}{2}\tau_z,\quad (3.24)$$

with $\varepsilon_z = \varepsilon_R - \varepsilon_L$, where we introduced the parametrisation

$$\varepsilon_L = -8E_L \left(1 - 2n_L + \frac{1}{2}C_L \operatorname{Im}(\alpha_1) - \frac{4E_R C_D n_R}{e^2}\right) \hbar^{-1},\quad (3.25)$$

and

$$\varepsilon_R = 8E_R \left(1 - 2n_R + \frac{1}{2}C_R \operatorname{Im}(\alpha_2) - \frac{4E_L C_D n_L}{e^2}\right) \hbar^{-1}.\quad (3.26)$$

The Pauli operators in the above local basis can then also be used for the following rewritings of $|L\rangle\langle L|$ and $|R\rangle\langle R|$, where we for convenience set $\mathbb{1}_2 \equiv \operatorname{diag}(1, 1, 0)$

$$|L\rangle\langle L| = -\frac{1}{2}\tau_z + \frac{1}{2}\mathbb{1}_2, \quad \text{and} \quad |R\rangle\langle R| = \frac{1}{2}\tau_z + \frac{1}{2}\mathbb{1}_2.\quad (3.27)$$

This we then insert into Eq. (3.22), giving

$$\hat{V} = \hbar g_1 i(\hat{a}_1^\dagger - \hat{a}_1)(\tau_z + \mathbb{1}_2) + \hbar g_2 i(\hat{a}_2^\dagger - \hat{a}_2)(\tau_z + \mathbb{1}_2),\quad (3.28)$$

3. Derivation of the System Hamiltonian

where $g_1 = E_L C_L$, and $g_2 = E_R C_R$. We then wish to change the phase of the creation and annihilation operators to change the coupling to be with position. A transformation with $\hat{U}_{1(2)}\left(\frac{3\pi}{2}\right)$ from Eq. (2.107), according to Eq. (2.108), leaves Eq. (3.24) unchanged since the operator commutes, and Eq. (3.28) becomes

$$\hat{V} = \hbar g_1 (\hat{a}_1^\dagger + \hat{a}_1) (\tau_z + \mathbb{1}_2) + \hbar g_2 (\hat{a}_2^\dagger + \hat{a}_2) (\tau_z + \mathbb{1}_2). \quad (3.29)$$

Now, we would like to get rid of the terms proportional to $(\hat{a}_i^\dagger + \hat{a}_i) \mathbb{1}_2$, though this time for $\mathbb{1}_2$. We thus modify the displacement operator to be

$$\hat{D}(\Xi_1, \Xi_2) = \exp\left(\mathbb{1}_2(\Xi_1 \hat{a}_1^\dagger - \Xi_1^* \hat{a}_1)\right) \exp\left(\mathbb{1}_2(\Xi_2 \hat{a}_2^\dagger - \Xi_2^* \hat{a}_2)\right). \quad (3.30)$$

Carrying out the transformation with $\Xi_i = \Xi_i^* = -g_i/\omega_i$, we eliminate the drive to obtain

$$\begin{aligned} \hat{H}' = \hbar\omega_1 \hat{a}_1^\dagger \hat{a}_1 + \hbar\omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2} \hbar \varepsilon_z \tau_z + 2\hbar(g_1 \Xi_1 + g_2 \Xi_2) \tau_z \\ + \hbar g_1 (\hat{a}_1^\dagger + \hat{a}_1) \tau_z + \hbar g_2 (\hat{a}_2^\dagger + \hat{a}_2) \tau_z, \end{aligned} \quad (3.31)$$

where we again neglected the constants. Tunnelling cannot be obtained from a classical circuit model, and as such, needs to be added after the fact. We thus add a term $t\tau_x$ to the Hamiltonian. With these considerations, taking $\varepsilon = \varepsilon_z + 2(g_1 \Xi_1 + g_2 \Xi_2)$, we have arrived at our quantum Hamiltonian in the $(|R\rangle, |L\rangle)$ basis

$$\hat{H} = \hbar\omega_1 \hat{a}_1^\dagger \hat{a}_1 + \hbar\omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{\hbar\varepsilon}{2} \tau_z + \hbar t \tau_x + \hbar \left[g_1 (\hat{a}_1^\dagger + \hat{a}_1) + g_2 (\hat{a}_2^\dagger + \hat{a}_2) \right] \tau_z. \quad (3.32)$$

The coupling τ_z can be seen as an electric dipole, since there can be only a single excess electron, either on $|L\rangle$ or $|R\rangle$.

3.2 Rotation and Schrieffer-Wolff Transformation

To proceed in finding the type of two-photon interactions we are after, we want to rotate the basis $(|R\rangle, |L\rangle)$ into the eigenstate basis of the DQD, spanned by the non-interacting ground state $|g\rangle$ and the excited state $|e\rangle$. This rotation is obtained from the matrix in Eq. (2.109) as

$$\begin{bmatrix} |g\rangle \\ |e\rangle \end{bmatrix} = \hat{\mathcal{R}}(\theta) \begin{bmatrix} |L\rangle \\ |R\rangle \end{bmatrix}, \quad (3.33)$$

where θ is the mixing angle, given by $\cos(\theta)/2 = \arctan(2t/(\varepsilon + \Omega))$, and $\Omega = \sqrt{\varepsilon^2 + 4t^2}$ is the energy splitting between the eigenstates of the DQD after rotation. With the rotation, the Hamiltonian in the new basis is

$$\begin{aligned} \hat{H} = \hbar\omega_1 \hat{a}_1^\dagger \hat{a}_1 + \hbar\omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2} \hbar \Omega \hat{\sigma}_z \\ + \hbar (g_1^x \hat{\sigma}_x + g_1^z \hat{\sigma}_z) (\hat{a}_1^\dagger + \hat{a}_1) + \hbar (g_2^x \hat{\sigma}_x + g_2^z \hat{\sigma}_z) (\hat{a}_2^\dagger + \hat{a}_2), \end{aligned} \quad (3.34)$$

with new coupling strengths $g_i^x = -g_i \sin(\theta)$, and $g_i^z = g_i \cos(\theta)$. $\hat{\sigma}_z$ and $\hat{\sigma}_x$ are now Pauli operators in the $(|e\rangle, |g\rangle)$ basis. Next, we wish to transform the obtained Hamiltonian into an effective Hamiltonian, which is the two-mode two-photon Jaynes-Cummings Hamiltonian in Eq. (B.3), and take inspiration from Ref. [71]. In order to do this, we utilise the SW transformation in Sec. 2.5.1. To perform the transformation, we introduce the new operators

$$\hat{X}_{\pm}^i = \hat{a}_i^\dagger \hat{\sigma}_- \pm \hat{\sigma}_+ \hat{a}_i, \quad \hat{Y}_{\pm}^i = \hat{a}_i^\dagger \hat{\sigma}_+ \pm \hat{\sigma}_- \hat{a}_i, \quad \text{and} \quad \hat{Z}_{\pm}^i = \hat{\sigma}_z (\hat{a}_i^\dagger \pm \hat{a}_i), \quad (3.35)$$

where $i \in \{1, 2\}$. The interaction Hamiltonian, \hat{V} , in terms of these operators, is

$$\hat{V} = \sum_{i=1,2} g_i^x (\hat{X}_+^i + \hat{Y}_+^i) + g_i^z \hat{Z}_+^i. \quad (3.36)$$

We make the ansatz that the generator of the transformation, \hat{S} , is

$$\hat{S} = \sum_{i=1,2} (x_i \hat{X}_-^i + y_i \hat{Y}_-^i + z_i \hat{Z}_-^i). \quad (3.37)$$

Now, we can calculate the commutator, $[\hat{S}, \hat{H}_0]$, in the hopes that this will become $-\hat{V}$ with the correct choice of the parameters x_i, y_i and z_i

$$[\hat{S}, \hat{H}_0] = \sum_{i=1,2} [x_i \hat{X}_-^i + y_i \hat{Y}_-^i + z_i \hat{Z}_-^i, \hat{H}_0] \quad (3.38)$$

$$= \sum_{i=1,2} x_i (-\omega_i + \Omega) \hat{X}_+^i + y_i (-\omega_i - \Omega) \hat{Y}_+^i + z_i \omega_i \hat{Z}_+^i. \quad (3.39)$$

The appropriate choice of parameters is thus

$$x_i = \frac{g_i^x}{\omega_i - \Omega}, \quad y_i = \frac{g_i^x}{\omega_i + \Omega}, \quad \text{and} \quad z_i = \frac{g_i^z}{\omega_i}. \quad (3.40)$$

In requiring that \hat{S} is small, an extended requirement is thus for x_i, y_i , and z_i to be $\ll 1$. This also means that the difference $\omega_i - \Omega$ is large, and thus that the detuning is large between the DQD and the oscillators. Having found \hat{S} , the Hamiltonian from the SW transformation is

$$\begin{aligned} \hat{H}' = \hat{H}_0 + \frac{1}{2} [\hat{S}, \hat{V}] &= \hbar \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \hbar \omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2} \hbar \Omega \hat{\sigma}_z \\ &+ \frac{1}{2} \hbar \sum_{\substack{i=1,2 \\ j=1,2}} [x_i \hat{X}_-^i + y_i \hat{Y}_-^i + z_i \hat{Z}_-^i, g_j^x (\hat{X}_+^j + \hat{Y}_+^j) + g_j^z \hat{Z}_+^j]. \end{aligned} \quad (3.41)$$

Through evaluation of the commutators arising from the above Hamiltonian, and application of the rotating wave approximation (RWA), when $\Omega \approx \omega_1 + \omega_2$, neglecting counter-rotating terms, we

3. Derivation of the System Hamiltonian

have

$$\begin{aligned}
\hat{H}' = & \hbar\omega_1\hat{a}_1^\dagger\hat{a}_1 + \hbar\omega_2\hat{a}_2^\dagger\hat{a}_2 + \hbar\frac{\Omega}{2}\hat{\sigma}_z - \frac{\hbar}{2}\hat{\sigma}_z \sum_{i=1,2} g_i^x (x_i - y_i) \left(2\hat{a}_i^\dagger\hat{a}_i + 1 \right) \\
& + \hbar(x_1g_2^z + x_2g_1^z - z_1g_2^x - z_2g_1^x)\hat{\sigma}_+\hat{a}_1\hat{a}_2 + \text{h.c.} \\
& + \hbar(x_1g_1^z - z_1g_1^x)\hat{\sigma}_+\hat{a}_1\hat{a}_1 + \hbar(x_1g_1^z - z_1g_1^x)\hat{\sigma}_+\hat{a}_2\hat{a}_2 + \text{h.c.}
\end{aligned} \tag{3.42}$$

3.3 Frequency Tuning

The last row in Eq. (3.42) is negligible under the assumption that the detuning between the oscillators, $\omega_1 - \omega_2$, is sufficiently large such that $|\Omega - 2\omega_{1(2)}| \gg |\Omega - \omega_1 - \omega_2|$, as well as $|\Omega - 2\omega_{1(2)}| \gg \Lambda$, where Λ is the effective coupling constant for the two-mode two-photon interaction

$$\Lambda = x_1g_2^z + x_2g_1^z - z_1g_2^x - z_2g_1^x. \tag{3.43}$$

With the large detuning between the cavities, we have that $\omega_1 \gg \omega_2$. This implies that

$$\Omega \gg \Omega - \omega_2 \approx \omega_1 > \frac{\Omega}{2} > \Omega - \omega_1 \approx \omega_2 \gg \Omega - \omega_1 - \omega_2. \tag{3.44}$$

As a consequence of this, the effective coupling constant, Λ , becomes $\Lambda \approx x_1g_2^z - z_2g_1^x$. We may then set the relation between the frequencies to be $\Omega = \omega_1 + \omega_2$, giving

$$\Lambda \approx -\frac{2g_1^xg_2^z}{\omega_2} = -\frac{g_1g_2 \sin(2\theta)}{\omega_2}, \tag{3.45}$$

where θ is the mixing angle defined in the rotation. Thus, we can see that the coupling depends on the splitting of the levels in the DQD. The effective Hamiltonian in this special case is ($x_{1(2)} \gg y_{1(2)}$)

$$\begin{aligned}
\hat{H}' \approx & \hbar\omega_1\hat{a}_1^\dagger\hat{a}_1 + \hbar\omega_2\hat{a}_2^\dagger\hat{a}_2 + \hbar \left[\frac{\tilde{\Omega}}{2} + \frac{(g_1^x)^2}{\omega_2}\hat{a}_1^\dagger\hat{a}_1 + \frac{(g_2^x)^2}{\omega_1}\hat{a}_2^\dagger\hat{a}_2 \right] \hat{\sigma}_z \\
& + \hbar\Lambda \left(\hat{\sigma}_+\hat{a}_1\hat{a}_2 + \hat{\sigma}_-\hat{a}_1^\dagger\hat{a}_2^\dagger \right), \tag{3.46}
\end{aligned}$$

where

$$\tilde{\Omega} = \Omega + \frac{(g_1^x)^2}{\omega_2} + \frac{(g_2^x)^2}{\omega_1}, \tag{3.47}$$

becomes the slightly renormalised eigenfrequency of the DQD. In some cases, the effect of the terms involving $\frac{(g_1^x)^2}{\omega_2}$ and $\frac{(g_2^x)^2}{\omega_1}$ are small enough to be neglected, giving the desired effective Hamiltonian for the two-mode two-photon Jaynes-Cummings model in Eq. (1.1)

$$\hat{H} = \hbar\omega_1\hat{a}_1^\dagger\hat{a}_1 + \hbar\omega_2\hat{a}_2^\dagger\hat{a}_2 + \frac{1}{2}\hbar\Omega\hat{\sigma}_z + \hbar\Lambda \left(\hat{\sigma}_+\hat{a}_1\hat{a}_2 + \hat{\sigma}_-\hat{a}_1^\dagger\hat{a}_2^\dagger \right). \tag{3.48}$$

We give justification for why these dynamical renormalisations can be neglected in our case in Sec. 7.

4. Open Quantum Systems

The formalism of open quantum systems is useful in providing a simplified model of the environment, which in general is very difficult to include in full. Thus, we seek to utilise this formalism in constructing a simplified model of our system in Sec. 1.

4.1 Quantum Master Equations

In the theory of open quantum systems, quantum master equations are used to describe reduced system dynamics, where the environment is traced out. An exact quantum master equation is the Nakajima–Zwanzig equation [34, 72–74]. The first approximation to this is the Redfield equation [75], obtained after making a Born-Markov approximation [72]. Subsequently applying the RWA and a second Markov approximation gives the GKSL equation [72]. The majority of the theory concerning the GKSL equation is however independent of the higher level descriptions.

4.1.1 Closed Quantum Systems

Closed quantum systems are systems which are isolated from their environment. For these systems, the equation of motion for pure states, forming the basis of quantum mechanics, is the Schrödinger equation. The extension of this equation to mixed states can be obtained from considering the time-evolution of the density operator $\hat{\rho}$ [34]

$$\hat{\rho} = \hat{U}^\dagger(t, t_0)\hat{\rho}(t_0)\hat{U}(t, t_0), \quad (4.1)$$

from time t_0 to t , where \hat{U} is the time-evolution operator. Differentiating this gives the quantum mechanical analogue of the Liouville equation, the Liouville-von Neumann equation

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}, \hat{\rho}] = \hat{\mathcal{L}}(\hat{\rho}), \quad (4.2)$$

where $\hat{\mathcal{L}}$ is the Liouvillian. The Schrödinger and Liouville-von Neumann equations describe the full system dynamics and are exact in doing so. Solving the Nakajima-Zwanzig equation for the reduced system dynamics is in general as difficult as solving the Liouville-von Neumann equation for the full system dynamics, where the environment is taken as part of the closed system, and subsequently tracing over the environment [34, 72].

4.1.2 The GKSL Equation

The GKSL equation (also commonly referred to simply as the Lindblad equation) was independently discovered by G. Lindblad [76] and V. Gorini et al. [77] and is given by [34, 72]

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}, \hat{\rho}] + \sum_k \gamma_k \mathcal{D}[\hat{L}_k](\hat{\rho}) = \hat{\mathcal{L}}_{\text{GKSL}}(\hat{\rho}), \quad (4.3)$$

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where $\hat{\mathcal{L}}_{\text{GKSL}}$ is the Lindbladian (or quantum Liouvillian), $\gamma_k \geq 0$, and

$$\mathcal{D}[\hat{L}_k](\hat{\rho}) = \hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2} \left\{ \hat{L}_k^\dagger \hat{L}_k, \hat{\rho} \right\}, \quad (4.4)$$

is called the dissipator, with $\hat{L}_k^{(\dagger)}$ being arbitrary operators which are specific to the chosen environment coupled to the system. This equation is significant because a generator of the same form as $\hat{\mathcal{L}}_{\text{GKSL}}$ guarantees that the map it generates is completely positive and trace-preserving (CPTP) [72].

4.2 Time-Evolution of the Expectation Value of Operators

The expectation value of an arbitrary time-independent Hilbert-Schmidt operator, $\hat{\mathcal{O}}$, in a quantum system with a given density matrix $\hat{\rho}$ is

$$\langle \hat{\mathcal{O}} \rangle_{\hat{\rho}} = \text{tr} \left\{ \hat{\mathcal{O}} \hat{\rho} \right\}. \quad (4.5)$$

We are interested in the time-evolution of this operator, and thus differentiate the above expression with respect to time, which gives

$$\frac{d}{dt} \langle \hat{\mathcal{O}} \rangle_{\hat{\rho}} = \text{tr} \left\{ \hat{\mathcal{O}} \frac{d}{dt} \hat{\rho} \right\}. \quad (4.6)$$

Specifically when the GKSL equation given in Eq (4.3) is used to replace $\frac{d}{dt} \hat{\rho}$, Eq. (4.6) becomes

$$\frac{d}{dt} \langle \hat{\mathcal{O}} \rangle_{\hat{\rho}} = -i \text{tr} \left\{ \hat{\mathcal{O}} [\hat{H}, \hat{\rho}] \right\} + \sum_k \gamma_k \text{tr} \left\{ \hat{\mathcal{O}} \mathcal{D}[\hat{L}_k](\hat{\rho}) \right\}, \quad (4.7)$$

where we used the linearity of the trace

$$\text{tr}\{\mathbf{X} + \mathbf{Y}\} = \text{tr}\{\mathbf{X}\} + \text{tr}\{\mathbf{Y}\}, \quad (4.8)$$

with \mathbf{X} and \mathbf{Y} being square matrices. Expanding the commutator and using the cyclic property for square matrices \mathbf{X} , \mathbf{Y} and \mathbf{Z} of the same size,

$$\text{tr}\{\mathbf{X}\mathbf{Y}\mathbf{Z}\} = \text{tr}\{\mathbf{Z}\mathbf{X}\mathbf{Y}\} = \text{tr}\{\mathbf{Y}\mathbf{Z}\mathbf{X}\}, \quad (4.9)$$

we get that the first term is

$$-i \text{tr} \left\{ \hat{\mathcal{O}} \hat{H} \hat{\rho} \right\} - i \text{tr} \left\{ -\hat{\mathcal{O}} \hat{\rho} \hat{H} \right\} = i \text{tr} \left\{ [\hat{H}, \hat{\mathcal{O}}] \hat{\rho} \right\} = i \left\langle [\hat{H}, \hat{\mathcal{O}}] \right\rangle_{\hat{\rho}}. \quad (4.10)$$

Thus, the first term is analogous to the Heisenberg equation of motion

$$\frac{d}{dt} \langle \hat{\mathcal{O}} \rangle_{\hat{\rho}} = i \left\langle [\hat{H}, \hat{\mathcal{O}}] \right\rangle_{\hat{\rho}} + \sum_k \gamma_k \text{tr} \left\{ \hat{\mathcal{O}} \mathcal{D}[\hat{L}_k](\hat{\rho}) \right\}. \quad (4.11)$$

Using the dissipator form in Eq. (4.4), and performing similar steps as in Eq. (4.10), the expression in the sum of the second term becomes

$$\text{tr} \left\{ \hat{\mathcal{O}} \left(\hat{L}_k \hat{\varrho} \hat{L}_k^\dagger - \frac{1}{2} \{ \hat{L}_k^\dagger \hat{L}_k, \hat{\varrho} \} \right) \right\} = \left\langle \hat{L}_k^\dagger \hat{\mathcal{O}} \hat{L}_k - \frac{1}{2} \{ \hat{L}_k^\dagger \hat{L}_k, \hat{\mathcal{O}} \} \right\rangle_{\hat{\varrho}}. \quad (4.12)$$

We now define the expression in the expectation value as the adjoint dissipator, a superoperator acting on the operator instead of on the density matrix. Explicitly, this is

$$\mathcal{D}_{\text{ad}}[\hat{L}_k](\hat{\mathcal{O}}) \equiv \hat{L}_k^\dagger \hat{\mathcal{O}} \hat{L}_k - \frac{1}{2} \{ \hat{L}_k^\dagger \hat{L}_k, \hat{\mathcal{O}} \}, \quad (4.13)$$

or in the alternate form

$$\mathcal{D}_{\text{ad}}[\hat{L}_k](\hat{\mathcal{O}}) = \frac{1}{2} \hat{L}_k^\dagger [\hat{\mathcal{O}}, \hat{L}_k] + \frac{1}{2} [\hat{L}_k^\dagger, \hat{\mathcal{O}}] \hat{L}_k. \quad (4.14)$$

The final expression for the time-evolution is then

$$\frac{d}{dt} \langle \hat{\mathcal{O}} \rangle_{\hat{\varrho}} = i \langle [\hat{H}, \hat{\mathcal{O}}] \rangle_{\hat{\varrho}} + \sum_k \gamma_k \langle \mathcal{D}_{\text{ad}}[\hat{L}_k](\hat{\mathcal{O}}) \rangle. \quad (4.15)$$

Analogous to this is the adjoint quantum master equation for a Heisenberg operator $\hat{\mathcal{O}}_{\text{H}}$, which in the case of the Lindblad generator being time-independent, is [34]

$$\frac{d}{dt} \hat{\mathcal{O}}_{\text{H}} = i [\hat{H}, \hat{\mathcal{O}}_{\text{H}}] + \sum_k \gamma_k \mathcal{D}[\hat{L}_k](\hat{\mathcal{O}}_{\text{H}}) = \mathcal{L}_{\text{GKSL}}^\dagger(\hat{\mathcal{O}}_{\text{H}}), \quad (4.16)$$

where \mathcal{L}^\dagger is the adjoint Lindblad generator.

4.3 Microscopic Derivations of the GKSL Equation

There are a multitude of ways to derive the GKSL equation. For many of the possible microscopic derivations of the GKSL equation, see Refs. [34, 72]. Here, we derive the GKSL equation from the basic collision model.

4.3.1 The Basic Collision Model

Defining the Basic Collision Model

The derivation mostly follows that given in Ref. [78], and to a lesser extent that given in Ref. [79]. We consider an open quantum system, S , with a Hilbert-space of arbitrary dimension. The system is coupled to a quantum bath, B , constituted of a number of smaller ancillae, which are all identical and prepared in the same initial state $\hat{\eta}$. The system is prepared in the arbitrary state, $\hat{\varrho}_0$, and together, the initial joint state is the product state of the system with the ancillae through

$$\hat{\sigma}_0 = \hat{\varrho}_0 \bigotimes_{i=1}^N \hat{\eta}. \quad (4.17)$$

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Since $\hat{\sigma}_0$ is a product state, there are no correlations between S and B , or between any of the ancillae. The single-body Hamiltonian of the system and any given ancilla, which we label n , an integer, is

$$\hat{H}_0 = \hat{H}_S + \hat{H}_n, \quad (4.18)$$

and the operator for the S - n interaction is denoted \hat{V}_n . The interaction, combined with Eq. (4.18), gives the unitary time-evolution operator for a single interaction

$$\hat{U}_n(\tau) = \exp\left(-i\left[\hat{H}_0 + \hat{V}_n\right]\tau\right), \quad (4.19)$$

where τ defines the duration of the collision, and is independent of the interaction. For illustration, we consider the system to be placed in a gas of moving particles, where each particle corresponds to an ancilla. This illustration is shown in Fig. 4.1. The particles are then made to collide with the system one by one. Next, we impose restrictions on the interactions.

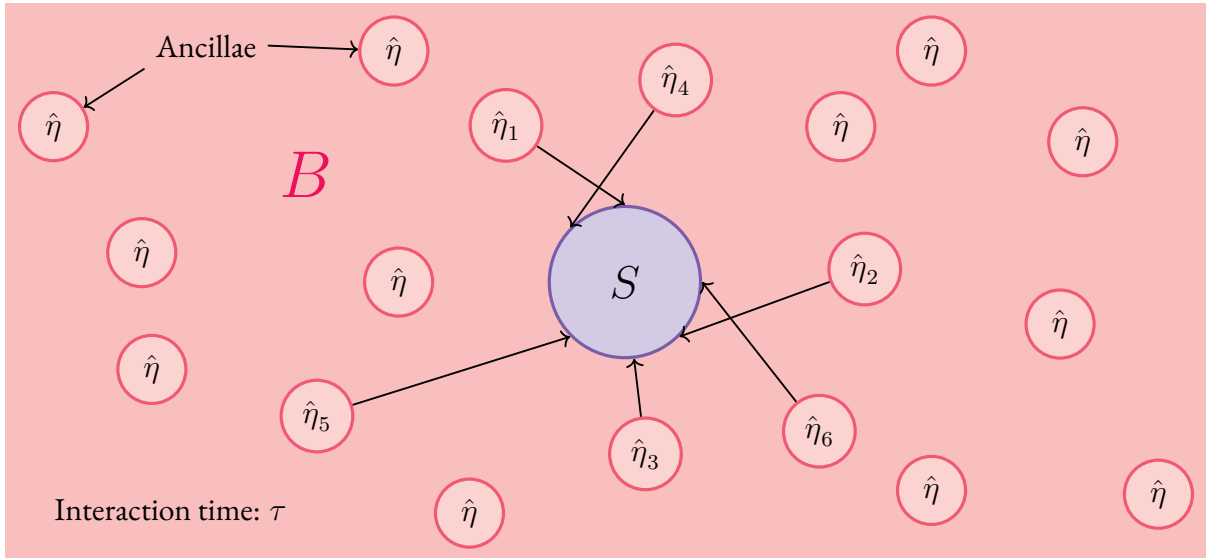


Figure 4.1: Example illustration of the basic collision model. The system, S , is surrounded by a bath, B , made up of ancillae, $\hat{\eta}$, here shown as a gas of particles. The ancillae interact sequentially, and only once with the system. All collisions have the same interaction/collision duration τ . The ancillae are not initially correlated and cannot collide with each other.

Markovian Conditions

- (i) There are no ancilla-ancilla interactions;
- (ii) There are no initial correlations between ancillae;
- (iii) Each ancilla is allowed to collide only once with the system.

These conditions, (i)-(iii), form the basis for the Markovian behaviour of the properties of collision models. A non-interrupted sequence of collisions with the same ancilla can be seen as not violating condition (iii). An example of this is the sequence S -1, S -1, S -2, S -2, \dots , if the collisions in this case are redefined to be double collisions. Condition (iii) can be understood by considering, that

after each collision, the ancilla goes back to interacting with the environment and effectively loses the history of the interaction. Thus, it can be seen as if the ancilla cannot collide with the system again.

Dynamics of the Open Quantum System.

After a single interaction of the initial system with an ancilla, the joint state is

$$\hat{\sigma}_1 = \hat{U}_1(\tau)\hat{\sigma}_0\hat{U}_1^\dagger(\tau). \quad (4.20)$$

Neglecting to write out the time-dependence in the evolution operator explicitly, we get after n successive interactions between the system and the ancillae, that the joint state is

$$\hat{\sigma}_n = \left(\prod_{i=1}^n \hat{U}_{n-i+1} \right) \hat{\sigma}_0 \left(\prod_{i=1}^n \hat{U}_i^\dagger \right). \quad (4.21)$$

Inserting the initial state $\hat{\sigma}_0$ in Eq. (4.17) into Eq (4.21), we have

$$\hat{\sigma}_n = \left(\prod_{i=1}^n \hat{U}_{n-i+1} \right) \left(\hat{\rho}_0 \otimes_{i=1}^n \hat{\eta} \right) \left(\prod_{i=1}^n \hat{U}_i^\dagger \right) \left(\otimes_{i=n+1}^N \hat{\eta} \right), \quad (4.22)$$

or with most of the compact notation removed as

$$\hat{\sigma}_n = \left(\hat{U}_n \cdots \left(\hat{U}_2 \left(\hat{U}_1 \hat{\rho}_0 \hat{\eta}_1 \hat{U}_1^\dagger \right) \hat{\eta}_2 \hat{U}_2^\dagger \right) \cdots \hat{\eta}_n \hat{U}_n^\dagger \right) \otimes_{i=n+1}^N \hat{\eta}. \quad (4.23)$$

By tracing over the bath as

$$\hat{\rho}_n = \text{tr}_B \{ \hat{\sigma}_n \} = \text{tr}_N \{ \cdots \text{tr}_n \{ \cdots \text{tr}_1 \{ \hat{\sigma}_n \} \cdots \} \cdots \}, \quad (4.24)$$

we see that the ancillae with label m such that $N \geq m > n$ simply vanish and can thus be ignored henceforth. Inserting $\hat{\sigma}_n$ then gives

$$\hat{\rho}_n = \text{tr}_n \left\{ \hat{U}_n \cdots \text{tr}_2 \left\{ \hat{U}_2 \text{tr}_1 \left\{ \hat{U}_1 \hat{\rho}_0 \hat{\eta}_1 \hat{U}_1^\dagger \right\} \hat{\eta}_2 \hat{U}_2^\dagger \right\} \cdots \hat{\eta}_n \hat{U}_n^\dagger \right\}. \quad (4.25)$$

This can alternatively be expressed in the compact form

$$\hat{\rho}_n = \mathcal{C}[\cdots [\mathcal{C}[\hat{\rho}_0]]] = \mathcal{C}^n[\hat{\rho}_0], \quad (4.26)$$

meaning that the evolution of the system in the environment follows the quantum collision map \mathcal{C} on S

$$\hat{\rho}_n = \mathcal{C}[\hat{\rho}_{n-1}] = \text{tr}_n \left\{ \hat{U}_n (\hat{\rho}_{n-1} \otimes \hat{\eta}_n) \hat{U}_n^\dagger \right\}. \quad (4.27)$$

This form of the map also ensures that \mathcal{C} is CPTP.

Equation of Motion for the System State

We are interested in the case when τ is small enough that we can get an continuous equation of mo-

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tion. For small τ we can thus make the approximation

$$\hat{U}_n \simeq \mathbb{1} - i\tau(\hat{H}_0 + \hat{V}_n) - \frac{1}{2}\tau^2\hat{V}_n^2. \quad (4.28)$$

Using the Hadamard lemma in Eq. (2.35) on the joint state with this approximation, it evolves as

$$\hat{\sigma}_n = \hat{U}\hat{\sigma}_{n-1}\hat{U}^\dagger = \hat{\sigma}_{n-1} - i\tau[\hat{H}_0 + \hat{V}_n, \hat{\sigma}_n] + \tau^2\left(\hat{V}_n\hat{\sigma}_n\hat{V}_n - \frac{1}{2}\{\hat{V}_n^2, \hat{\sigma}_n\}\right). \quad (4.29)$$

If we now subtract $\hat{\sigma}_{n-1}$ on both sides, take the partial trace over all ancillae, and divide by τ , we get

$$\begin{aligned} \frac{\hat{\varrho}_n - \hat{\varrho}_{n-1}}{\tau} = & -i\left[\hat{H}_S + \text{tr}_n\left\{\hat{V}_n\hat{\eta}_n\right\}, \hat{\varrho}_{n-1}\right] \\ & + \tau\text{tr}_n\left\{\left(\hat{V}_n\hat{\varrho}_{n-1}\hat{\eta}_n\hat{V}_n - \frac{1}{2}\{\hat{V}_n^2, \hat{\varrho}_{n-1}\hat{\eta}_n\}\right)\right\}. \end{aligned} \quad (4.30)$$

One can then spectrally decompose the initial density operator state for the ancillae as

$$\hat{\eta}_n = \sum_k p_k |k\rangle_n \langle k|, \quad \text{where} \quad \sum_k p_k = 1. \quad (4.31)$$

Inserting this decomposition into Eq. (4.30) gives

$$\begin{aligned} \frac{\hat{\varrho}_n - \hat{\varrho}_{n-1}}{\tau} = & -i\left[\hat{H}_S + \text{tr}_n\left\{\hat{V}_n\hat{\eta}_n\right\}, \hat{\varrho}_{n-1}\right] \\ & + \sum_{k,k'} \left(\hat{L}_{k,k'}\hat{\varrho}_{n-1}\hat{L}_{k,k'}^\dagger - \frac{1}{2}\{\hat{L}_{k,k'}^\dagger\hat{L}_{k,k'}, \hat{\varrho}_{n-1}\}\right), \end{aligned} \quad (4.32)$$

where the raising and lowering operators are given by

$$\hat{L}_{k,k'} = \sqrt{p_k\tau} {}_n\langle k'|\hat{V}_n|k\rangle_n, \quad (4.33)$$

with $|k\rangle$ and $|k'\rangle$ being eigenstates of the ancillae $\hat{\eta}$. Finally approximating this with the backward difference

$$\frac{d}{dt}\hat{\varrho} \simeq \frac{\hat{\varrho}_n - \hat{\varrho}_{n-1}}{\tau}, \quad (4.34)$$

we get

$$\frac{d}{dt}\hat{\varrho} \simeq -i\left[\hat{H}_S + \text{tr}_n\left\{\hat{V}_n\hat{\eta}_n\right\}, \hat{\varrho}_{n-1}\right] + \sum_{k,k'} \left(\hat{L}_{k,k'}\hat{\varrho}_{n-1}\hat{L}_{k,k'}^\dagger - \frac{1}{2}\{\hat{L}_{k,k'}^\dagger\hat{L}_{k,k'}, \hat{\varrho}_{n-1}\}\right), \quad (4.35)$$

which has the form of Eq. (4.3). This expression now depends on the operator $\text{tr}_n\{\hat{V}_n\hat{\eta}_n\}$, which is Hermitian. By choosing an appropriate form of this operator, one can derive the correct dissipator for a specific systems. Two forms of dissipators are used in Sec. 6.

5. Gaussian States

5.1 Phase Space Description of Quantum Mechanics

The phase space description represents one of a number of formulations of quantum mechanics (see e.g. Ref. [80] for a review of some of these). The development of the theory was finalised by H.J. Groenewold in 1946 [81] and later independently by J.E. Moyal in 1949 [82], both basing their work on previous developments by E.P. Wigner [83] and H. Weyl [84]. A quantum state in this formalism is represented by a quasiprobability distribution. We utilise this formalism specifically for the resonators, which simplifies the description considerably.

5.1.1 Quasiprobability Distributions

One quasiprobability distribution, usually referred to simply as a function, stems from the Wigner-Ville W [83, 85] representation of the phase space description. The Wigner-Ville function for N modes is [40]

$$W_N(\hat{\rho}, \vec{z}) = \frac{1}{\pi^{2N}} \int_{\mathbb{R}^{2N}} \text{tr} \left\{ \hat{\rho} \hat{D}(\vec{\lambda}) \right\} \exp\left(-(\vec{z})^\dagger \mathbf{K}_\zeta \vec{\lambda}\right) d^{2N} \vec{\lambda}. \quad (5.1)$$

where $\vec{\lambda}$ and $\hat{D}(\vec{\lambda})$ are given in Eqs. (2.100) and (2.101) respectively, and $\vec{z} \in \mathbb{C}^{2N}$. This function is part of an infinite family of distribution functions [86]. There are three prominent representations used, with their corresponding distribution functions. Besides the Wigner-Ville function, these are the Glauber-Sudarshan P [63, 64], and Husimi-Kano Q [87, 88] functions. A given function is related to the other two through convolution with a Gaussian function, called a Weierstrass transform. These three transforms for one field mode are [89]

$$W_1(z, z^*) = \frac{2}{\pi} \int \exp\left(-2|z - \alpha|^2\right) P_1(\alpha, \alpha^*) d^2 \alpha, \quad (5.2a)$$

$$Q_1(z, z^*) = \frac{2}{\pi} \int \exp\left(-2|z - \alpha|^2\right) W_1(\alpha, \alpha^*) d^2 \alpha, \quad (5.2b)$$

$$Q_1(z, z^*) = \frac{1}{\pi} \int \exp\left(-|z - \alpha|^2\right) P_1(\alpha, \alpha^*) d^2 \alpha, \quad (5.2c)$$

where Eq. (5.2a) relates W_1 to P_1 , Eq. (5.2b) relates Q_1 to W_1 , and finally Eq. (5.2c) relates Q_1 to P_1 . These functions relate to a density operator $\hat{\rho}$, but can be generalised to any system operator \hat{O} and a corresponding function $F_{\hat{O}}(\alpha, \alpha^*)$ which is anti-normal, symmetric or normal ordered [89].

5.1.2 Characteristic Functions

The family of s -ordered characteristic functions is given by [40, 53]

$$\chi_s(\hat{\rho}, \vec{\lambda}) = \text{tr} \left\{ \hat{\rho} \hat{D}(\vec{\lambda}) \right\} \exp\left(\frac{1}{2} s \|\vec{\lambda}\|^2\right), \quad (5.3)$$

where $\|\cdot\|$ denotes the extension of the Euclidean norm to \mathbb{C}^{2N}

$$\|\vec{\lambda}\| = \sqrt{|\alpha_1|^2 + \cdots + |\alpha_N|^2}. \quad (5.4)$$

These were introduced by K.E. Cahill and R.J. Glauber in 1969 [90], and are related to the family of quasi-probability distribution through the complex Fourier transform. We see that the symmetric case of the characteristic function, $s = 0$, appears in Eq. (5.1), which is thus the complex Fourier transform in the case of the Wigner function. The normally ordered case, $s = 1$, then corresponds to the P -function, and finally the antinormally ordered case, $s = -1$, to the Q -function [26]. The characteristic function in the Wigner-Ville representation can then be retrieved by taking the complex inverse Fourier transform of the Wigner-Ville function. These relations hold for all the distribution functions and their corresponding characteristic function. The characteristic function for a Gaussian state can be written as [40, 56, 57]

$$\chi_s = \exp\left(-\frac{1}{2}\vec{\lambda}^\dagger \mathbf{K}_\zeta \Theta \mathbf{K}_\zeta \vec{\lambda} + \langle \hat{\zeta} \rangle^\dagger \mathbf{K}_\zeta \vec{\lambda}\right) \exp\left(\frac{1}{2}s\|\vec{\lambda}\|^2\right). \quad (5.5)$$

5.2 The Lyapunov Equation

Gaussian states are well known to be fully characterised by their first and second moments. In terms of the mode operators in Eq. (2.77), the first moments are the expectation values of these

$$\langle \hat{\zeta} \rangle = \left[\langle \hat{a}_1 \rangle, \langle \hat{a}_1^\dagger \rangle, \cdots, \langle \hat{a}_N \rangle, \langle \hat{a}_N^\dagger \rangle \right]^\top, \quad (5.6)$$

and the second moments are the expectation values of the product of any two mode operators. The second moments can then be arranged in matrix form, known as the covariance matrix (CM). To obtain explicit forms of the moments, equations of motion and their solutions are needed, which is the topic of Sec. 5.2.

5.2.1 The Covariance Matrix and the Equations of Motion

From the expectation value in Eq. (4.5), we can introduce the fluctuation operators [56]

$$\delta \hat{a}_i = \hat{a}_i - \langle \hat{a}_i \rangle, \quad \text{and} \quad \delta \hat{a}_i^\dagger = \hat{a}_i^\dagger - \langle \hat{a}_i^\dagger \rangle, \quad (5.7)$$

which we can use to define the elements of the CM as

$$\Theta_{ij} \equiv \frac{1}{2} \left\langle \left\{ \hat{\zeta}_i, \hat{\zeta}_j^\dagger \right\} \right\rangle - \langle \hat{\zeta}_i \rangle \langle \hat{\zeta}_j^\dagger \rangle = \frac{1}{2} \left\langle \left\{ \delta \hat{\zeta}_i, \delta \hat{\zeta}_j^\dagger \right\} \right\rangle. \quad (5.8)$$

This definition makes the CM Hermitian, $\Theta = \Theta^\dagger$. From the above definition in the case of two bosonic modes, we have that the explicit form of the CM is

$$\Theta = \begin{bmatrix} \langle \delta \hat{a}_1^\dagger \delta \hat{a}_1 \rangle & \langle \delta \hat{a}_1 \delta \hat{a}_1 \rangle & \langle \delta \hat{a}_1 \delta \hat{a}_2^\dagger \rangle & \langle \delta \hat{a}_1 \delta \hat{a}_2 \rangle \\ \langle \delta \hat{a}_1^\dagger \delta \hat{a}_1 \rangle & \langle \delta \hat{a}_1^\dagger \delta \hat{a}_1 \rangle & \langle \delta \hat{a}_1^\dagger \delta \hat{a}_2^\dagger \rangle & \langle \delta \hat{a}_1^\dagger \delta \hat{a}_2 \rangle \\ \langle \delta \hat{a}_1^\dagger \delta \hat{a}_2 \rangle & \langle \delta \hat{a}_1 \delta \hat{a}_2 \rangle & \langle \delta \hat{a}_2^\dagger \delta \hat{a}_2 \rangle & \langle \delta \hat{a}_2 \delta \hat{a}_2 \rangle \\ \langle \delta \hat{a}_1^\dagger \delta \hat{a}_2^\dagger \rangle & \langle \delta \hat{a}_1 \delta \hat{a}_2^\dagger \rangle & \langle \delta \hat{a}_2^\dagger \delta \hat{a}_2^\dagger \rangle & \langle \delta \hat{a}_2^\dagger \delta \hat{a}_2 \rangle \end{bmatrix} + \frac{1}{2} \mathbb{1}_4. \quad (5.9)$$

Written in block form

$$\Theta = \begin{bmatrix} \Theta_1 & \Theta_{12} \\ \Theta_{21} & \Theta_2 \end{bmatrix}, \quad (5.10)$$

it has the nice property

$$\text{tr}_{1(2)}(\Theta) = \Theta_{2(1)}. \quad (5.11)$$

The CM in terms of mode operators can be transformed into the CM for quadrature operators, σ , via Eq. (2.88) through $\sigma = \mathbf{T}\Theta\mathbf{T}^\dagger$.

The equation of motion for the CM is given by

$$\frac{d}{dt}\Theta = \mathbf{W}\Theta + \Theta\mathbf{W}^\dagger + \mathbf{F}, \quad (5.12)$$

where \mathbf{W} and \mathbf{F} parametrise the covariance matrix. Here, \mathbf{F} is a matrix depending only on the dissipators. Equation (5.12) is known as the continuous¹ differential Lyapunov equation² (CDLE) [91], and is commonly used in the field of control theory. Taking Θ to be independent of time gives $\frac{d}{dt}\Theta = 0$, reducing the CDLE to

$$\mathbf{W}\Theta + \Theta\mathbf{W}^\dagger = -\mathbf{F}, \quad (5.13)$$

known as the continuous algebraic Lyapunov equation (CALE) [91], describing the steady state of the system. The first moments instead evolve according to the first-order non-homogeneous ordinary differential equation

$$\frac{d}{dt}\langle \hat{\zeta} \rangle = \mathbf{W}\langle \hat{\zeta} \rangle + \mathbf{f}, \quad (5.14)$$

where $\langle \hat{\zeta} \rangle$ is the vector of first moments, and \mathbf{W} and \mathbf{f} parametrise the vector. We note the fact that the matrix \mathbf{W} appearing in both Eq. (5.12) and Eq. (5.14) are the same matrix. Since a Gaussian state is fully characterised by its first and second moments, Eqs. (5.12) and (5.14) are sufficient for fully describing the system. As alternatives to the above equations, one can use the expressions in Sec. 4.2 to find the first and second moments. In the next section, we give a derivation showing that these equations of motion are correct in describing the first and second moments for a Gaussian state.

¹We emphasise that it is the continuous version, since there also exists a discrete version of the form $\frac{d}{dt}X = AXA^\dagger - X + Q$, also commonly appearing in control theory.

²The Lyapunov equation is a special case of the Sylvester equation of the form $\frac{d}{dt}X = AX + XB + Q$, with $B = A^\dagger$.

5. Gaussian States

The matrices appearing in the equations of motion can be written in the symplectic form introduced in Sec. 2.5.3. The dissipator for a HO has the form [56]

$$\mathcal{D}_{\text{HO}}[\hat{a}_i](\hat{\rho}) = \gamma_i \bar{n}_i \left[\hat{a}_i^\dagger \hat{\rho} \hat{a}_i - \frac{1}{2} \left\{ \hat{a}_i \hat{a}_i^\dagger, \hat{\rho} \right\} \right] + \gamma_i (\bar{n}_i + 1) \left[\hat{a}_i \hat{\rho} \hat{a}_i^\dagger - \frac{1}{2} \left\{ \hat{a}_i^\dagger \hat{a}_i, \hat{\rho} \right\} \right], \quad (5.15)$$

where the \bar{n}_i are the Bose-Einstein distributions for each mode. This dissipator gives the contribution

$$\mathbf{\Gamma} = \frac{1}{2} \bigoplus_{i=1}^N \gamma_i \mathbb{1}_2, \quad (5.16)$$

which leads to the matrix \mathbf{F} , depending only on the dissipators, being

$$\mathbf{F} = \mathbf{\Gamma} \bigoplus_{i=1}^N (2\bar{n}_i + 1) \mathbb{1}_2. \quad (5.17)$$

The matrix \mathbf{W} can then be identified with the Hamiltonian matrix in Eq. (2.91) with an added contribution from the dissipator coefficients

$$\mathbf{W} = -i\mathbf{K}_\zeta \mathbf{H} - \mathbf{\Gamma}. \quad (5.18)$$

The first moments can also be written in symplectic form as

$$\frac{d}{dt} \langle \hat{\zeta} \rangle = \mathbf{W} \langle \hat{\zeta} \rangle + i\mathbf{K}_\zeta \vec{\lambda}_D, \quad (5.19)$$

where we made the replacement $\mathbf{f} = i\mathbf{K}_\zeta \vec{\lambda}_D$, and $\vec{\lambda}_D \in \mathbb{C}^{2N}$ are the drive coefficients. Furthermore, a general quadratic Hamiltonian can be written, using \mathbf{H} and $\vec{\lambda}_D$, as [40]

$$\hat{H} = \hat{\zeta}^\dagger \mathbf{H} \hat{\zeta} + \hat{\zeta}^\dagger \vec{\lambda}_D. \quad (5.20)$$

For a Gaussian state, a unitary transformation of a density matrix on the corresponding Hilbert space, can then be mapped to [40]

$$\hat{\rho}' = \hat{U}^\dagger \hat{\rho} \hat{U} \longleftrightarrow \begin{cases} \hat{\zeta}' = \mathbf{S} \hat{\zeta} \\ \Theta' = \mathbf{S}^\dagger \Theta \mathbf{S} \end{cases}, \quad (5.21)$$

as is also shown in the schematic overview in Tab. D.1 of Appendix D.

5.2.2 The Lyapunov Equation From the Characteristic Function

One method of deriving the CDLE involves transforming a master equation into a partial differential equation for the phase space distributions or their characteristic functions discussed in Sec. 5.1. Moving to a phase space description is desirable because the phase space is much smaller than the Hilbert space, as shown in Tab. D.1 of Appendix D. The resulting differential equation is usually called the Fokker-Planck equation, discovered by the physicists A.D. Fokker in 1914 [92] and M.

Planck in 1917 [93] respectively, or the Kolmogorov forward equation based on its independent discovery by the mathematician A. Kolmogorov in 1931 [94]. Here, we choose to consider the characteristic function in its s -ordered form, given in Eq. (5.3). Using the Kermack-McCrae identities to get the disentangled versions of the displacement operator in Eqs. (2.98) and (2.99), we get the normal and anti-normal ordered disentangled equations for the characteristic function

$$\chi_s(\hat{\varrho}, \vec{\lambda}) = \text{tr} \left\{ \hat{\varrho} \prod_{i=1}^N \exp(\alpha_i \hat{a}_i^\dagger) \exp(-\alpha_i^* \hat{a}_i) \right\} \exp\left(\frac{1}{2}(s-1)|\vec{\lambda}|^2\right) \quad (5.22a)$$

$$= \text{tr} \left\{ \hat{\varrho} \prod_{i=1}^N \exp(-\alpha_i^* \hat{a}_i) \exp(\alpha_i \hat{a}_i^\dagger) \right\} \exp\left(\frac{1}{2}(s+1)|\vec{\lambda}|^2\right). \quad (5.22b)$$

To move from the the master equation to the Fokker-Planck equation we need a way of rewriting the actions of \hat{a}_i and \hat{a}_i^\dagger on $\hat{\varrho}$ in terms of actions on the characteristic function. As an example, \hat{a}_i^\dagger acting from the right gives

$$\chi_s(\hat{\varrho} \hat{a}_i^\dagger, \vec{\lambda}) = \left(-\frac{1}{2}(s-1)\alpha_i^* + \partial_{\alpha_i} \right) \chi_s(\hat{\varrho}, \vec{\lambda}). \quad (5.23)$$

which is obtained from taking the partial derivative of the normal ordered characteristic function with respect to α_i

$$\begin{aligned} \partial_{\alpha_i} \chi_s(\hat{\varrho}, \vec{\lambda}) &= \text{tr} \left\{ \hat{\varrho} \hat{a}_i^\dagger \prod_{i=1}^N \exp(\alpha_i \hat{a}_i^\dagger) \exp(-\alpha_i^* \hat{a}_i) \right\} \exp\left(\frac{1}{2}(s-1)|\alpha|^2\right) \\ &+ \frac{1}{2}(s-1)\alpha_i^* \chi_s(\hat{\varrho}, \vec{\lambda}), \end{aligned} \quad (5.24)$$

and subsequently rearranging and factorising. In total, with two derivatives as well as normal and anti-normal ordered equations, we can apply the derivatives in four different ways, which leads to the actions of a single ladder operator on $\hat{\varrho}$ having the following rewritings

$$\chi_s(\hat{\varrho} \hat{a}_i^\dagger, \vec{\lambda}) = \left(-\frac{1}{2}(s-1)\alpha_i^* + \partial_{\alpha_i} \right) \chi_s(\hat{\varrho}, \vec{\lambda}), \quad (5.25a)$$

$$\chi_s(\hat{a}_i^\dagger \hat{\varrho}, \vec{\lambda}) = \left(-\frac{1}{2}(s+1)\alpha_i^* + \partial_{\alpha_i} \right) \chi_s(\hat{\varrho}, \vec{\lambda}), \quad (5.25b)$$

$$\chi_s(\hat{a}_i \hat{\varrho}, \vec{\lambda}) = \left(\frac{1}{2}(s-1)\alpha_i - \partial_{\alpha_i^*} \right) \chi_s(\hat{\varrho}, \vec{\lambda}), \quad (5.25c)$$

$$\chi_s(\hat{\varrho} \hat{a}_i, \vec{\lambda}) = \left(\frac{1}{2}(s+1)\alpha_i - \partial_{\alpha_i^*} \right) \chi_s(\hat{\varrho}, \vec{\lambda}). \quad (5.25d)$$

These relations bear resemblance to the relations in Eq. (2.97). Relations similar to the ones above also exist for the phase space distributions in Sec. 5.1, and can be found in Ref. [95]. Subsequently, one can find successive actions of two or more ladder operators on $\hat{\varrho}$ by repeated use of the rewritings

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in Eqs. (5.25a)-(5.25d). For example, the action of $\hat{a}_i^\dagger \hat{a}_j^\dagger$ on $\hat{\rho}$, acting from the right is

$$\chi_s(\hat{\rho} \hat{a}_i^\dagger \hat{a}_j^\dagger, \vec{\lambda}) = \left(-\frac{1}{2}(s-1)\alpha_j^* + \partial_{\alpha_j} \right) \left(-\frac{1}{2}(s-1)\alpha_i^* + \partial_{\alpha_i} \right) \chi_s(\hat{\rho}, \vec{\lambda}). \quad (5.26)$$

If the operators applied to $\hat{\rho}$ commute, then so will the corresponding differential operators.

The desired action on $\hat{\rho}$ is obtained from taking the time derivative of the characteristic function

$$\frac{d}{dt} \chi_s = \text{tr} \left\{ \left(\frac{d}{dt} \hat{\rho} \right) \hat{D}(\vec{\lambda}) \right\} \exp\left(\frac{1}{2}s\|\vec{\lambda}\|^2\right) = \text{tr} \left\{ \hat{\mathcal{L}}_{\text{GKSL}}(\hat{\rho}) \hat{D}(\vec{\lambda}) \right\} \exp\left(\frac{1}{2}s\|\vec{\lambda}\|^2\right), \quad (5.27)$$

and identifying that the time derivative of $\hat{\rho}$ can be replaced using the GKSL equation in Eq. (4.3). Inserting the general quadratic Hamiltonian in Eq. (5.20) into the GKSL equation in Eq. (4.3) gives

$$\hat{\mathcal{L}}_{\text{GKSL}}(\hat{\rho}) = -i \left(\hat{\zeta}^\dagger \mathbf{H} \hat{\zeta} \hat{\rho} + \hat{\zeta}^\dagger \vec{\lambda}_D \hat{\rho} - \hat{\rho} \hat{\zeta}^\dagger \mathbf{H} \hat{\zeta} - \hat{\rho} \hat{\zeta}^\dagger \vec{\lambda}_D \right) + \sum_{k=1}^{2N} \gamma_k \mathcal{D}[\hat{L}_k](\hat{\rho}). \quad (5.28)$$

Accounting for all the terms and replacing the actions of the operators according to Eqs. (5.25a)-(5.25d) will give a very complicated expression involving mixed partial derivatives, which is the Fokker-Planck equation we are after. Inserting the Gaussian state in Eq. (5.5) into the Fokker-Planck equation and carrying out the differentiation then gives

$$\frac{d}{dt} \chi_s = -\frac{1}{2} \vec{\lambda}^\dagger \mathbf{K}_\zeta \left(\mathbf{W} \Theta + \Theta \mathbf{W}^\dagger + \mathbf{F} \right) \mathbf{K}_\zeta \vec{\lambda} \chi_s + \left(\mathbf{W} \langle \hat{\zeta} \rangle + i \mathbf{K}_\zeta \vec{\lambda}_D \right)^\dagger \mathbf{K}_\zeta \vec{\lambda} \chi_s. \quad (5.29)$$

On the other hand, we have that the time derivative of the characteristic function is

$$\frac{d}{dt} \chi_s = -\frac{1}{2} \vec{\lambda}^\dagger \mathbf{K}_\zeta \frac{d\Theta}{dt} \mathbf{K}_\zeta \vec{\lambda} \chi_s + \frac{d\langle \hat{\zeta} \rangle^\dagger}{dt} \mathbf{K}_\zeta \vec{\lambda} \chi_s. \quad (5.30)$$

Thus, by setting these equal to each other, the result is

$$\frac{d}{dt} \Theta = \mathbf{W} \Theta + \Theta \mathbf{W}^\dagger + \mathbf{F}, \quad (5.31)$$

and

$$\frac{d}{dt} \langle \hat{\zeta} \rangle = \mathbf{W} \langle \hat{\zeta} \rangle + i \mathbf{K}_\zeta \vec{\lambda}_D. \quad (5.32)$$

Further methods of obtaining the CDLE exist, one of which is given in [Appendix C](#) without dissipation. Alternative derivations can be found in Ref. [96].

5.2.3 Solutions to the CALE, CDLE and First Moments

Using the vectorisation operator, vec , which stacks the columns of a matrix into a vector, and the Kronecker product, \otimes , we can write the CALE in the Kronecker formulation as [97]

$$\mathcal{W} \text{vec}(\Theta) = \mathbf{W} \underline{\oplus} \mathbf{W}^* \text{vec}(\Theta) = (\mathbf{1}_{2N} \otimes \mathbf{W} + \mathbf{W}^* \otimes \mathbf{1}_{2N}) \text{vec}(\Theta) = -\text{vec}(\mathbf{F}), \quad (5.33)$$

where $\mathbb{1}_{2N}$ is the identity matrix of order $2N$, the same size as \mathbf{W} , and $\underline{\oplus}$ represents the Kronecker sum, defined as³ [97]

$$\mathbf{A} \underline{\oplus} \mathbf{B} \equiv (\mathbb{1}_n \otimes \mathbf{A} + \mathbf{B} \otimes \mathbb{1}_m), \quad (5.34)$$

where $\mathbf{A} \in \mathbb{C}^{n \times n}$, $\mathbf{B} \in \mathbb{C}^{m \times m}$. The Kronecker sum is not to be confused with the direct sum \oplus , though they share the same symbol. The Kronecker sum is thus distinguished through the use of a line beneath the symbol. The vectorisation allows $\text{vec}(\Theta)$ to be solved for by finding the inverse of \mathcal{W}

$$\text{vec}(\Theta) = -\mathcal{W}^{-1} \text{vec}(\mathbf{F}). \quad (5.35)$$

The matrix Θ is then found by unvectorising the solution. A vectorised solution of the CDLE can also be found [98]. From the CDLE in Eq. (5.12), we obtain

$$\begin{aligned} \text{vec}\left(\frac{d}{dt}\Theta\right) &= \frac{d}{dt}\text{vec}(\Theta) = \text{vec}\left(\mathbf{W}\Theta + \Theta\mathbf{W}^\dagger + \mathbf{F}\right) \\ &= \text{vec}\left(\mathbf{W}\Theta\right) + \text{vec}\left(\Theta\mathbf{W}^\dagger\right) + \text{vec}(\mathbf{F}) \\ &= [(\mathbb{1}_{2N} \otimes \mathbf{W}) + (\mathbf{W}^* \otimes \mathbb{1}_{2N})]\text{vec}(\Theta) + \text{vec}(\mathbf{F}) \\ &= \mathcal{W}\text{vec}(\Theta) + \text{vec}(\mathbf{F}), \end{aligned} \quad (5.36)$$

where the identity [97]

$$\text{vec}(\mathbf{ABC}) = (\mathbf{C}^\top \otimes \mathbf{A})\text{vec}(\mathbf{B}), \quad (5.37)$$

was used. Equation (5.36) has the solution [97]

$$\text{vec}(\Theta) = \exp(\mathcal{W}t)\text{vec}(\Theta_0) + \int_0^t \exp(\mathcal{W}[t-\tau])\text{vec}(\mathbf{F}(\tau))d\tau. \quad (5.38)$$

Under the assumption that \mathbf{F} is independent of time, or $\frac{d}{dt}\mathbf{F}(t) = 0$, which is the case with our chosen dissipator, the integral above can be evaluated as follows [98]

$$\begin{aligned} \int_0^t \exp(\mathcal{W}[t-\tau])d\tau &= \exp(\mathcal{W}t) \int_0^t \exp(-\mathcal{W}\tau)d\tau \\ &= \exp(\mathcal{W}t)\mathcal{W}^{-1}(\mathbb{1}_{2N} - \exp(-\mathcal{W}t)) \\ &= \mathcal{W}^{-1}(\exp(\mathcal{W}t) - \mathbb{1}_{2N}), \end{aligned} \quad (5.39)$$

making the full solution

$$\text{vec}(\Theta) = \exp(\mathcal{W}t)\text{vec}(\Theta_0) + \mathcal{W}^{-1}(\exp(\mathcal{W}t) - \mathbb{1}_{2N})\text{vec}(\mathbf{F}). \quad (5.40)$$

Alternatively, by using the identity [97]

$$\exp(\mathbf{A} \underline{\oplus} \mathbf{B}) = \exp(\mathbf{A}) \otimes \exp(\mathbf{B}), \quad (5.41)$$

³It is also possible to define it as $\mathbf{A} \underline{\oplus} \mathbf{B} \equiv (\mathbf{A} \otimes \mathbb{1}_n + \mathbb{1}_m \otimes \mathbf{B})$, which from the definition in Eq. (5.34) corresponds to $\mathbf{B} \underline{\oplus} \mathbf{A}$.

one obtains

$$\begin{aligned} \text{vec}(\Theta) = & \left(\exp(\mathbf{W}t) \otimes \exp(\mathbf{W}^*t) \right) \text{vec}(\Theta_0) \\ & + \mathcal{W}^{-1} \left(\exp(\mathbf{W}t) \otimes \exp(\mathbf{W}^*t) - \mathbb{1}_{2N} \right) \text{vec}(\mathbf{F}). \end{aligned} \quad (5.42)$$

The solution of the CDLE in Eq. (5.12) on integral form is [99]

$$\Theta = \exp(\mathbf{W}t) \Theta_0 \exp(\mathbf{W}^\dagger t) - \int_0^t \exp(\mathbf{W}\tau) \mathbf{F} \exp(\mathbf{W}^\dagger \tau) d\tau, \quad (5.43)$$

where $\Theta_0 = \Theta(t=0)$. If \mathbf{W} is a stable matrix (Routh-Hurwitz matrix), meaning that the real parts of the eigenvalues of \mathbf{W} are strictly negative, i.e. $\text{Re}[\lambda_i] < 0$, then

$$\Theta = - \int_0^\infty \exp(\mathbf{W}\tau) \mathbf{F} \exp(\mathbf{W}^\dagger \tau) d\tau, \quad (5.44)$$

is the steady-state solution (in the long-time limit) [99].

The solution to the matrix equation for the first moments given in Eq. (5.14), that is the equation

$$\frac{d}{dt} \langle \hat{\zeta} \rangle = \mathbf{W} \langle \hat{\zeta} \rangle + i \mathbf{K}_\zeta \vec{\lambda}_D, \quad (5.45)$$

has the same form as that in Eq. (5.38), which now is [97]

$$\langle \hat{\zeta} \rangle(t) = \exp(\mathbf{W}t) \langle \hat{\zeta} \rangle(0) + \exp(\mathbf{W}t) \int_0^t \exp(-\mathbf{W}\tau) i \mathbf{K}_\zeta \vec{\lambda}_D(\tau) d\tau. \quad (5.46)$$

Making the same assumption of time-independence as before, but this time with respect to $\vec{\lambda}_D$, which is also true in our case, the integral can be evaluated in the same way as in Eq. (5.39), giving

$$\langle \hat{\zeta} \rangle(t) = \exp(\mathbf{W}t) \langle \hat{\zeta} \rangle(0) + \mathbf{W}^{-1} (\exp(\mathbf{W}t) - \mathbb{1}_{2N}) i \mathbf{K}_\zeta \vec{\lambda}_D. \quad (5.47)$$

The steady state solution can then be obtained from

$$\langle \hat{\zeta} \rangle = -i \mathbf{W}^{-1} \mathbf{K}_\zeta \vec{\lambda}_D. \quad (5.48)$$

5.3 Entanglement

Quantum entanglement is an important aspect in the theory of quantum information, and by extension quantum computation. In our case, we want to study the entanglement of two resonators coupled via a DQD. Here, we thus restrict ourselves to considering bipartite quantum states $\hat{\rho}_{12}$. Such

a state is separable if, and only if, it can be expressed as [100–102]

$$\hat{\rho}_{12} = \sum_i p_i \hat{\rho}_{i;1} \otimes \hat{\rho}_{i;2}, \quad (5.49)$$

where $p_i \geq 0$ satisfying $\sum_i p_i = 1$, and $\hat{\rho}_{i;1}$ and $\hat{\rho}_{i;2}$ are the density operators of the two modes 1 and 2. A number of separability criteria exist for determining when the above is true. These include entropic criteria based on the Rényi- α entropy discussed below, entanglement witnesses and several other [100]. A necessary condition for $\hat{\rho}_{12}$ to be separable is the Peres-Horodecki criterion, or positive partial transpose (PPT) criterion since it involves finding if the partial transpose of $\hat{\rho}_{12}$ is a density operator [100]. A generalisation of the Peres-Horodecki criterion to continuous variables was made independently by R. Simon [101] and L.-M. Duan et al. [102]. They found a necessary and sufficient condition, in the form of an inequality relation, for the separability of Gaussian states of two HO's constituting a bipartite system. In the next section, we consider the approach of Duan et al. to this condition, and expand on it to find a more useful expression for our case. For a review of entanglement, see Ref. [100].

5.3.1 The Duan Criterion (Duan-Giedke-Cirac-Zoller)

The general statement of the Duan criterion is that a quantum state of two modes which is separable satisfies the inequality [100, 102]

$$\langle (\delta \hat{X})^2 \rangle + \langle (\delta \hat{P})^2 \rangle \geq A^2 - \frac{1}{A^2}, \quad (5.50)$$

for the Einstein-Podolsky-Rosen (EPR)-like operators

$$\hat{X} = |A| \hat{x}_1 + \frac{1}{A} \hat{x}_2, \quad \text{and} \quad \hat{P} = |A| \hat{p}_1 - \frac{1}{A} \hat{p}_2, \quad (5.51)$$

subject to the constraint $[\hat{x}_j, \hat{p}_{j'}] = i\delta_{jj'}$, ($j, j' = 1, 2$). For our purposes, we will reduce it to the case of linear combinations of quadrature operators, which have the form of the above EPR-like operators with $A = 1$ and an overall pre-factor $\frac{1}{\sqrt{2}}$. The quadrature operators are

$$\hat{x}_1 = \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger + \hat{a}_1), \quad \text{and} \quad \hat{p}_1 = \frac{i}{\sqrt{2}}(\hat{a}_1^\dagger - \hat{a}_1), \quad (5.52)$$

for the first mode, as well as

$$\hat{x}_2 = \frac{1}{\sqrt{2}}(\exp(i\phi)\hat{a}_2^\dagger + \exp(-i\phi)\hat{a}_2), \quad \text{and} \quad \hat{p}_2 = \frac{i}{\sqrt{2}}(\exp(i\phi)\hat{a}_2^\dagger - \exp(-i\phi)\hat{a}_2), \quad (5.53)$$

for the second, with phases added so that they are arbitrarily rotated compared to the first mode. These can both be shown to satisfy the commutation relation above. We thus define our EPR-like operators to be

$$\hat{X}_+ = \frac{1}{\sqrt{2}}(\hat{x}_1 + \hat{x}_2), \quad \text{and} \quad \hat{P}_- = \frac{1}{\sqrt{2}}(\hat{p}_1 - \hat{p}_2), \quad (5.54)$$

which in Eq. (5.50) gives

$$2\langle(\delta\hat{X}_+)^2\rangle + 2\langle(\delta\hat{P}_-)^2\rangle \geq 2 \implies \langle(\delta\hat{X}_+)^2\rangle + \langle(\delta\hat{P}_-)^2\rangle \geq 1. \quad (5.55)$$

The Duan criterion, expanded in terms of the ladder operators in Eq. (5.52) is then

$$\langle\delta\hat{a}_1^\dagger\delta\hat{a}_1\rangle + \langle\delta\hat{a}_2^\dagger\delta\hat{a}_2\rangle + \exp(i\phi)\langle\delta\hat{a}_1^\dagger\delta\hat{a}_2^\dagger\rangle + \exp(-i\phi)\langle\delta\hat{a}_1\delta\hat{a}_2\rangle \geq 0, \quad (5.56)$$

which in turn can be rewritten from the elements of the covariance matrix for the ladder operators

$$\Theta_{11} + \Theta_{33} + \exp(i\phi)\Theta_{41} + \exp(-i\phi)\Theta_{14} \geq 1. \quad (5.57)$$

The optimal choice of ϕ is the opposite angle to the phase of Θ_{14} (Θ_{41}), giving

$$\mathfrak{D} = \frac{1}{2} \text{tr}\{\Theta\} - 2|\Theta_{14}| \geq 1, \quad (5.58)$$

where we have named the expression on the left side of the inequality \mathfrak{D} .

For the case in which there is no two-mode squeezing, the elements Θ_{14} and Θ_{41} are going to be zero, meaning that the contribution to the variance is purely thermal

$$\frac{1}{2} \text{tr}\{\Theta\} = \langle\delta\hat{a}_1^\dagger\delta\hat{a}_1\rangle + \langle\delta\hat{a}_2^\dagger\delta\hat{a}_2\rangle + 1 \geq 1, \quad (5.59)$$

where $\langle\delta\hat{a}_{1(2)}^\dagger\delta\hat{a}_{1(2)}\rangle \geq 0$, meaning that the state is never entangled.

5.3.2 Rényi-2 Entropy

Rényi- α entropies [103] constitute a family of additive entropies, which for a density operator, $\hat{\rho}$, is given by [104]

$$S_R^{(\alpha)}(\hat{\rho}) = \frac{1}{1-\alpha} \ln(\text{tr}\{\hat{\rho}^\alpha\}). \quad (5.60)$$

In the limit of $\alpha \rightarrow 1$, the Rényi-2 entropy converges to the von Neumann entropy [100, 104, 105]

$$\lim_{\alpha \rightarrow 1} S_R^{(\alpha)} = -\text{tr}\{\hat{\rho} \ln(\hat{\rho})\}. \quad (5.61)$$

For Gaussian states, the most interesting entropy of the Rényi- α entropies is the Rényi-2 entropy⁴

$$S_R^{(2)} = -\ln(\text{tr}\{\hat{\rho}^2\}). \quad (5.62)$$

The quantity appearing in the logarithm of the Rényi-2 entropy in Eq. (5.62) is the purity, which can be computed from the Wigner-Ville function as [40, 56, 104]

$$\text{tr}\{\hat{\rho}^2\} = (2\pi)^N \int_{\mathbb{R}^{2N}} W^2(\vec{z}) d^{2N}\vec{z} = \frac{1}{\sqrt{\det(\Theta)}}. \quad (5.63)$$

⁴An important property of the Rényi-2 entropy is that it is strongly subadditive, like the von Neumann entropy. This means that it obeys the inequality $S(\rho_{A,B}) + S(\rho_{A,B}) \geq S(\rho_{A,B,C}) + S(\rho_B)$, for tripartite systems [104].

With the purity inserted back into Eq. (5.62), we have

$$S_R^{(2)} = \frac{1}{2} \ln(\det(\Theta)). \quad (5.64)$$

For a CM with only two-mode squeezing as the interaction, it further reduces to

$$S_R^{(2)} = \ln(|\Theta_{11}\Theta_{44} - \Theta_{14}\Theta_{41}|), \quad (5.65)$$

or since $\Theta_{14} = \Theta_{41}^*$ to

$$S_R^{(2)} = \ln(|\Theta_{11}\Theta_{44} - |\Theta_{14}|^2|). \quad (5.66)$$

6. Mean-Field Equations

6.1 Mean-Field Method

Making a MF approximation for the coupling between the resonators and the DQD allows for the use of the CDLE in describing the system [106, 107]. This means that we assume the correlations between the states of the DQD and the two resonator to be sufficiently small as to be neglected. In turn, this allows for the density matrix to be written as

$$\hat{\rho}(t) = \hat{\rho}_{\text{DQD}}(t) \otimes \hat{\rho}_{12}(t), \quad (6.1)$$

at all times, where $\hat{\rho}_{\text{DQD}}$ is the state of the DQD, and $\hat{\rho}_{12}$ is the joint state of the resonators. We also assume that $\hat{\rho}_{12}$ is not a product state, since such a state cannot facilitate entanglement. Inserting Eq.(6.1) into the chosen form of Eq. (4.3) along with the derived effective Hamiltonian

$$\hat{H} = \hbar\omega_1 \hat{a}_1^\dagger \hat{a}_1 + \hbar\omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2} \hbar\Omega \hat{\sigma}_z + \Lambda \left(\hat{\sigma}_+ \hat{a}_1 \hat{a}_2 + \hat{\sigma}_- \hat{a}_1^\dagger \hat{a}_2^\dagger \right), \quad (6.2)$$

and tracing over the degrees of freedom of either the resonators or the DQD, we get master equations for the other, with dependencies on the other entering only through their expectation values.

6.2 Equation for the Double Quantum Dot

For the DQD, the resulting Hamiltonian is

$$\hat{H}_{\text{DQD}} = \frac{1}{2} \hbar\Omega \hat{\sigma}_z + \hbar\Lambda \left(\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle \hat{\sigma}_- + \langle \hat{a}_1 \hat{a}_2 \rangle \hat{\sigma}_+ \right), \quad (6.3)$$

where the Pauli operators include a third level, as described at the end of Sec. 2.3. To rewrite this in terms of elements of the covariance matrix, we substitute in $\langle \delta \hat{a}_1^\dagger \delta \hat{a}_2^\dagger \rangle = \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle - \langle \hat{a}_1^\dagger \rangle \langle \hat{a}_2^\dagger \rangle$ and its hermitian conjugate

$$\hat{H}_{\text{DQD}} = \frac{1}{2} \Omega \hat{\sigma}_z + \hbar\Lambda \left(\langle \delta \hat{a}_1^\dagger \delta \hat{a}_2^\dagger \rangle \hat{\sigma}_- + \langle \delta \hat{a}_1 \delta \hat{a}_2 \rangle \hat{\sigma}_+ + \langle \hat{a}_1^\dagger \rangle \langle \hat{a}_2^\dagger \rangle \hat{\sigma}_- + \langle \hat{a}_1 \rangle \langle \hat{a}_2 \rangle \hat{\sigma}_+ \right). \quad (6.4)$$

From this we identify the corresponding elements to get

$$\hat{H}_{\text{DQD}} = \frac{1}{2} \hbar\Omega \hat{\sigma}_z + \hbar\Lambda \left(\Theta_{41} \hat{\sigma}_- + \Theta_{14} \hat{\sigma}_+ + \langle \hat{a}_1^\dagger \rangle \langle \hat{a}_2^\dagger \rangle \hat{\sigma}_- + \langle \hat{a}_1 \rangle \langle \hat{a}_2 \rangle \hat{\sigma}_+ \right). \quad (6.5)$$

For the DQD, the dissipators used are

$$\mathcal{D}[\hat{s}_g^\dagger](\hat{\rho}) = \hat{s}_g^\dagger \hat{\rho} \hat{s}_g - \frac{1}{2} \left\{ \hat{s}_g \hat{s}_g^\dagger, \hat{\rho} \right\}, \quad \text{and} \quad \mathcal{D}[\hat{s}_e](\hat{\rho}) = \hat{s}_e \hat{\rho} \hat{s}_e^\dagger - \frac{1}{2} \left\{ \hat{s}_e^\dagger \hat{s}_e, \hat{\rho} \right\}, \quad (6.6)$$

making the GKSL equation for the DQD

$$\partial_t \hat{\rho}_{\text{DQD}} = -i \left[\hat{H}_{\text{DQD}}, \hat{\rho}_{\text{DQD}} \right] + \gamma_{sg} \mathcal{D}[\hat{s}_g^\dagger](\hat{\rho}_{\text{DQD}}) + \gamma_{se} \mathcal{D}[\hat{s}_e](\hat{\rho}_{\text{DQD}}). \quad (6.7)$$

The density matrix includes the populations $p_e \in \mathbb{R}$ and $p_g \in \mathbb{R}$, the populations of the excited and ground state respectively, as well as the coherence $c \in \mathbb{C}$. The last diagonal element is then given by $1 - p_e - p_g$, since $\text{tr}\{\hat{\rho}_{\text{DQD}}\} = 1$. There is also no coherence between the lead and the excited or ground state.

6.3 Equations for the Resonators

For the HO Hamiltonian, we again use Eq. (6.1), but now trace over the DQD to get

$$\begin{aligned} \hat{H}_{\text{HO}} = & \hbar\omega_1 \hat{a}_1^\dagger \hat{a}_1 + \hbar\omega_2 \hat{a}_2^\dagger \hat{a}_2 + \hbar\Lambda \langle \hat{\sigma}_- \rangle \hat{a}_1^\dagger \hat{a}_2^\dagger + \hbar\Lambda \langle \hat{\sigma}_+ \rangle \hat{a}_1 \hat{a}_2 \\ & + \hbar\alpha_1 \hat{a}_1^\dagger + \hbar\alpha_1^* \hat{a}_1 + \hbar\alpha_2 \hat{a}_2^\dagger + \hbar\alpha_2^* \hat{a}_2, \end{aligned} \quad (6.8)$$

with drive added. The total dissipators for the HO's are

$$\begin{aligned} \mathcal{D}_{\text{HO}}[\hat{a}_{1(2)}](\hat{\rho}) = & \gamma_{1(2)} \bar{n}_{1(2)} \left[\hat{a}_{1(2)}^\dagger \hat{\rho} \hat{a}_{1(2)} - \frac{1}{2} \left\{ \hat{a}_{1(2)} \hat{a}_{1(2)}^\dagger, \hat{\rho} \right\} \right] \\ & + \gamma_{1(2)} (\bar{n}_{1(2)} + 1) \left[\hat{a}_{1(2)} \hat{\rho} \hat{a}_{1(2)}^\dagger - \frac{1}{2} \left\{ \hat{a}_{1(2)}^\dagger \hat{a}_{1(2)}, \hat{\rho} \right\} \right], \end{aligned} \quad (6.9)$$

though these are accounted for in the matrices $\mathbf{\Gamma}$ and \mathbf{F} in Eqs. (5.16) and (5.17). The bosonic Hamiltonian matrix is

$$\mathbf{H} = \begin{bmatrix} \omega_1 & 0 & 0 & \Lambda \langle \hat{\sigma}_- \rangle \\ 0 & \omega_1 & \Lambda \langle \hat{\sigma}_+ \rangle & 0 \\ 0 & \Lambda \langle \hat{\sigma}_- \rangle & \omega_2 & 0 \\ \Lambda \langle \hat{\sigma}_+ \rangle & 0 & 0 & \omega_2 \end{bmatrix}, \quad (6.10)$$

allowing us to find \mathbf{W} through Eq. (5.18). The HO's are then described by the CALE

$$\mathbf{W}\mathbf{\Theta} + \mathbf{\Theta}\mathbf{W}^\dagger = -\mathbf{F}, \quad (6.11)$$

and the first moments in the steady state

$$\mathbf{W}\langle \hat{\zeta} \rangle + i\mathbf{K}_\zeta \vec{\lambda}_D = 0. \quad (6.12)$$

7. Numerical Results

The system described in the previous section is not analytically solvable in the general case (all parameters being non-zero), and thus requires a numerical treatment. In this section, we present the numerical findings and compare them to analytical expressions for the solution of the coupled oscillators without a DQD where possible. The Python package used for solving the system of equations numerically was SymPy's `nsolve`.

We first turn our attention to justification of removing the extra terms in Eq. (3.46). This can be seen from rewriting the Hamiltonian as

$$\hat{H} \approx \hbar\omega_1 \hat{a}_1^\dagger \hat{a}_1 + \hbar\omega_2 \hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2} \hbar\Omega \hat{\sigma}_z + \eta \hbar \Lambda_z \hat{\sigma}_z + \eta \hbar \Lambda \left(\hat{\sigma}_+ \hat{a}_1 \hat{a}_2 + \hat{\sigma}_- \hat{a}_1^\dagger \hat{a}_2^\dagger \right) + \hbar\alpha_1 \hat{a}_1^\dagger + \hbar\alpha_1^* \hat{a}_1 + \hbar\alpha_2 \hat{a}_2^\dagger + \hbar\alpha_2^* \hat{a}_2, \quad (7.1)$$

where

$$\Lambda_z = \frac{(g_1^x)^2}{\omega_2} + \frac{(g_2^x)^2}{\omega_1} + \frac{(g_1^x)^2}{\omega_2} \hat{a}_1^\dagger \hat{a}_1 + \frac{(g_2^x)^2}{\omega_1} \hat{a}_2^\dagger \hat{a}_2, \quad (7.2)$$

and we also introduced a shared parameter η for both Λ_z and Λ . Solving the MF equation with the DQD Hamiltonian from the steady state of the GKSL equation in Eq. (4.3), and expanding it to first order in η gives

$$\hat{\varrho} \approx \begin{bmatrix} 0 & -\frac{i\eta\Lambda\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle}{\gamma_{se}+2i\Omega} & 0 \\ \frac{i\eta\Lambda\langle\hat{a}_1\hat{a}_2\rangle}{\gamma_{se}-2i\Omega} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (7.3)$$

From this, we can see that there is no dependence on Λ_z to first order in η . Doing the same thing for the oscillators by solving the CALE with the bosonic MF Hamiltonian, and expanding in orders of η , gives that the zeroth order is $\text{diag}(\bar{n}_1, \bar{n}_1, \bar{n}_2, \bar{n}_2) + \mathbb{1}_4/2$, and the first order is

$$\Theta \approx \begin{bmatrix} \bar{n}_1 + \frac{1}{2} & 0 & 0 & \frac{2\eta\Lambda\langle\hat{\sigma}_-\rangle(\bar{n}_1+\bar{n}_2+1)}{i\gamma_1+i\gamma_2-2\omega_1-2\omega_2} \\ 0 & \bar{n}_1 + \frac{1}{2} & -\frac{2\eta\Lambda\langle\hat{\sigma}_+\rangle(\bar{n}_1+\bar{n}_2+1)}{i\gamma_1+i\gamma_2+2\omega_1+2\omega_2} & 0 \\ 0 & \frac{2\eta\Lambda\langle\hat{\sigma}_-\rangle(\bar{n}_1+\bar{n}_2+1)}{i\gamma_1+i\gamma_2-2\omega_1-2\omega_2} & \bar{n}_2 + \frac{1}{2} & 0 \\ -\frac{2\eta\Lambda\langle\hat{\sigma}_+\rangle(\bar{n}_1+\bar{n}_2+1)}{i\gamma_1+i\gamma_2+2\omega_1+2\omega_2} & 0 & 0 & \bar{n}_2 + \frac{1}{2} \end{bmatrix}, \quad (7.4)$$

which also has no dependence on Λ_z . This justifies the use of the Hamiltonian in this case for small values of Λ . We will however push the range of Λ and α , for better visualisation of different behaviours. Finally, the full steady state solutions for the first moments in the MF are

$$\langle\hat{a}_{1(2)}\rangle = \frac{-4\Lambda\langle\hat{\sigma}_-\rangle\alpha_{2(1)}^* + 2i\alpha_{1(2)}(\gamma_{2(1)} - 2i\omega_{2(1)})}{4\Lambda^2\langle\hat{\sigma}_+\rangle\langle\hat{\sigma}_-\rangle + (\gamma_{1(2)} + 2i\omega_{1(2)})(\gamma_{2(1)} - 2i\omega_{2(1)})}, \quad (7.5)$$

and $\langle\hat{a}_{1(2)}^\dagger\rangle = \langle\hat{a}_{1(2)}\rangle^*$, with $\langle\hat{a}_{1(2)}^{(\dagger)}\rangle = \text{tr}\{\hat{a}_{1(2)}^{(\dagger)}\hat{\varrho}\}$, where $\hat{\varrho}$ is given in Eq. (6.1).

7.1 First Moments

From the numerical solutions, we are interested in looking at how the first and second moments, as well as the Duan criterion depend on a number of quantities, mainly the HO drive strength, α , and the coupling strength, Λ , which we normalise by the DQD level splitting, Ω . First, we look at the first moments, through the expression $\sqrt{\langle \hat{a}_i^\dagger \rangle \langle \hat{a}_i \rangle}$, thus neglecting the phase, since it plays a minor role.

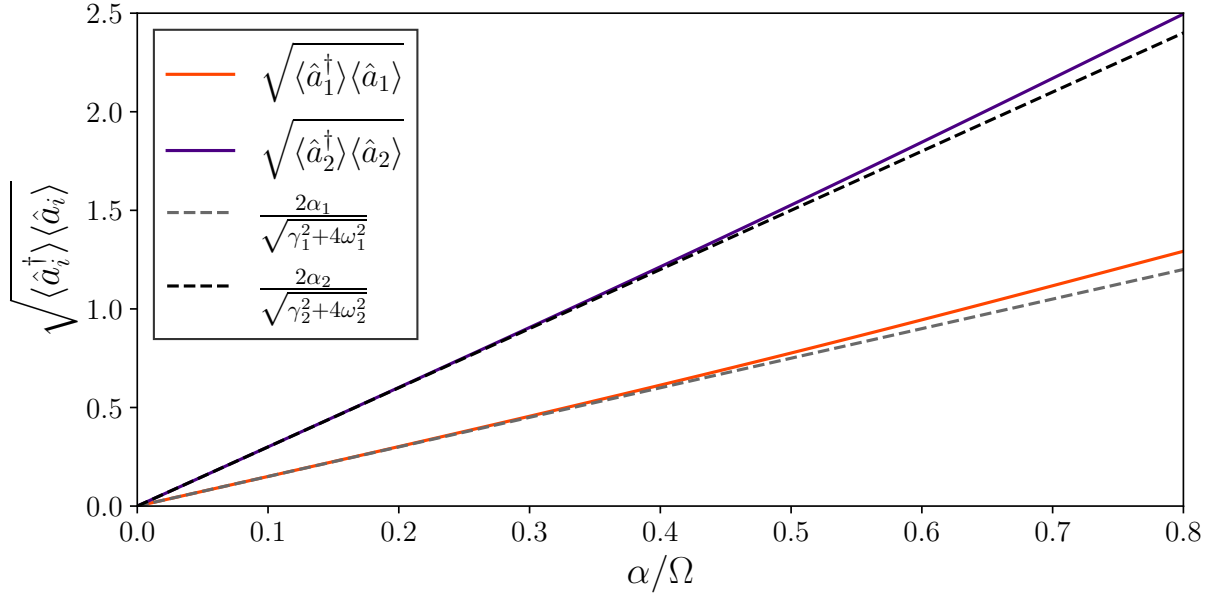


Figure 7.1: The figure shows the square root of the product of first moments for the two modes as a function of the drive strength on the HO's. The solid lines correspond to the numerical results, and the dashed lines are the analytic solutions for two coupled oscillators, to zeroth order in Λ . Parameter values used are $\Lambda = 0.1$, $\Omega = 1$, $\omega_1 = 2/3$, $\omega_2 = 1/3$, $\bar{n}_{1(2)} = 0$ and $\gamma_{1(2)} = \gamma_{s_e(s_g)} = 0.001$. Λ is required to be small in the MF. The frequencies are determined by the condition $\Omega = \omega_1 + \omega_2$, where $\omega_1 \neq \omega_2$. The dissipation rates, γ , are required to be small in comparison to Ω .

The dependence of the first moments on the HO drive strength, α , in the MF, shown as the solid lines in Fig. 7.1, is linear. By using the analytical solution of the first moments in Eq. (7.5), and inserting the numerical values of the coherence, c , a perfect match to the numerical solution is achieved. Expanding the analytical solutions to zeroth order in Λ , we get an approximation

$$\sqrt{\langle \hat{a}_{1(2)}^\dagger \rangle \langle \hat{a}_{1(2)} \rangle} \approx \frac{2\alpha_{1(2)}}{\sqrt{\gamma_{1(2)}^2 + 4\omega_{1(2)}^2}}, \quad (7.6)$$

which is not dependent on c or any other properties of the DQD. This is shown for the two modes as dashed lines in Fig. 7.1, and has a near perfect agreement for small values of α , while still exhibiting good

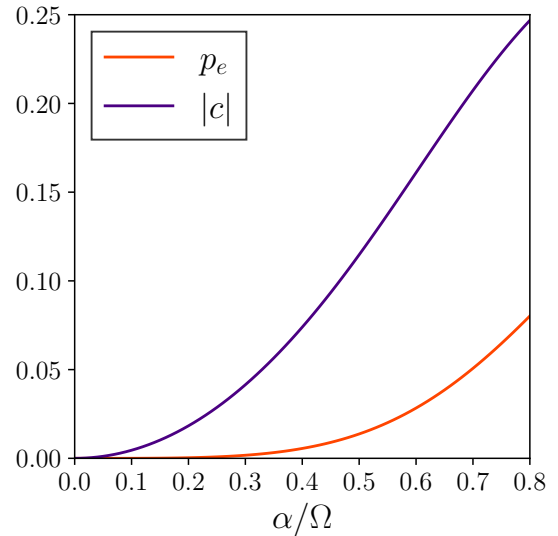


Figure 7.2: Dependence of the coherence of the DQD on HO drive. Also shown is the population of the excited state, p_e , of the DQD. Parameter values are $\Lambda = 0.1$, $\Omega = 1$, $\omega_1 = 2/3$, $\omega_2 = 1/3$, $\bar{n}_{1(2)} = 0$, $\gamma_{1(2)} = \gamma_{s_e(s_g)} = 0.001$.

7. Numerical Results

agreement for higher values. The order of the moment thus matches the order of its dependence on α .

The first moments exhibit dependence on the coupling strength Λ , as well as the oscillator dissipation rates, $\gamma_{1(2)}$. Though, in the range where they are physical, the first moments are constant and have a negligible dependence. There is also no dependence on the dissipation rates $\gamma_{s_e(s_g)}$. The DQD is not isolated from the environment, since at infinite time-scales, a small $\gamma_{s_e(s_g)}$ is still significant in order for the system to reach a steady state, even though its exact value is not.

Part of the reason for the discrepancy between the approximation in Eq. (7.6) and the full solution, is that Λ is non-zero, and that the coherence of the DQD has a non-linear dependence on α , which is plotted in Fig. 7.2. Also shown in Fig. 7.2 is the population of the excited state of the DQD, p_e , which is non-linear in the chosen regime.

7.2 Photon Population

Another property of the oscillators is the photon population, which is given by

$$\langle \hat{n}_{1(2)} \rangle = \langle \hat{a}_{1(2)}^\dagger \hat{a}_{1(2)} \rangle = \Theta_{11(33)} + \langle \hat{a}_{1(2)}^\dagger \rangle \langle \hat{a}_{1(2)} \rangle - \frac{1}{2}. \quad (7.7)$$

The numerical solutions for the two populations are shown as the solid lines in Fig. 7.3.

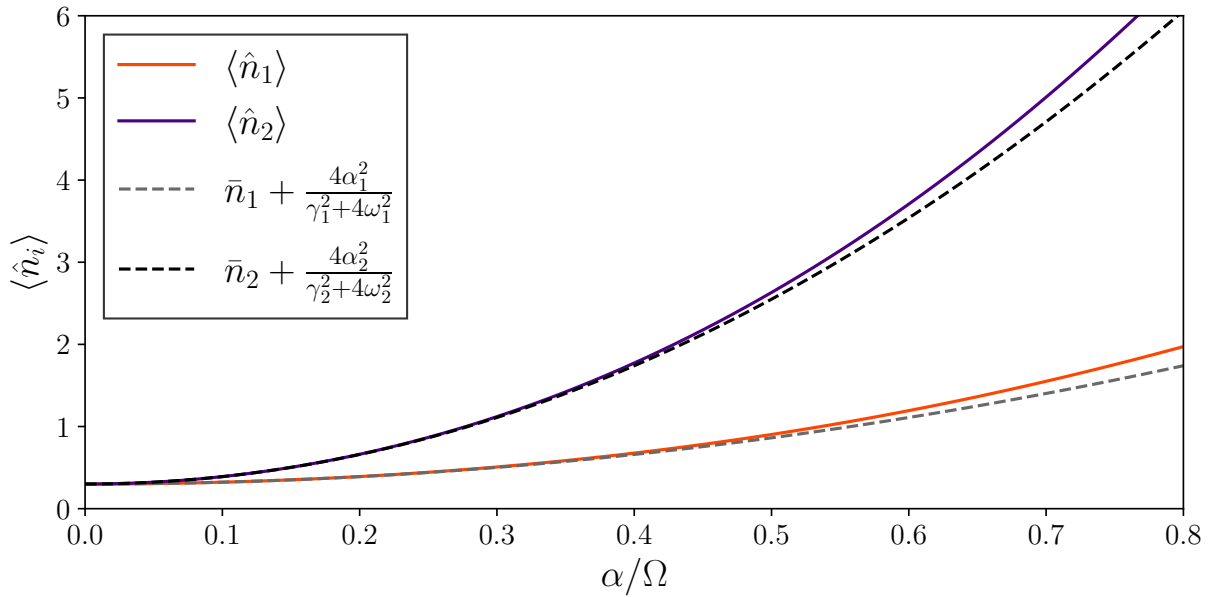


Figure 7.3: Photon population, $\langle \hat{n}_i \rangle$, as a function of HO drive strength. Parameter values used are $\Lambda = 0.1$, $\Omega = 1$, $\omega_1 = 2/3$, $\omega_2 = 1/3$, $\bar{n}_{1(2)} = 0.3$, $\gamma_{1(2)} = 0.001$ and $\gamma_{s_e(s_g)} = 0.001$.

Similar to the first moments, we can expand the analytical solution from Eq. (5.35) to first order in Λ to get

$$\langle \hat{n}_{1(2)} \rangle \approx \bar{n}_{1(2)} + \frac{4\alpha_{1(2)}}{\gamma_{1(2)} + 4\omega_{1(2)}}, \quad (7.8)$$

where $\bar{n}_{1(2)}$ is the Bose-Einstein distribution [69, 108]

$$\bar{n}_{1(2)} = \frac{1}{\exp\left(\frac{\hbar\omega_{1(2)}}{k_B T_{1(2)}}\right) - 1}, \quad (7.9)$$

with k_B being the Boltzmann constant, and $T_{1(2)}$ the temperature of the resonators. The total number of photons given in Eq. (7.8), $\langle \hat{n}_{1(2)} \rangle$, is thus constituted of both a thermal occupation at equilibrium from the Bose-Einstein distribution, and an additional occupation from the HO drive.

Equation (7.8) also has no dependence on coherence. The approximations of the populations of the two oscillators are also shown in Fig. 7.3, but as dashed lines. The dependence we see is close to quadratic, with excellent agreement of the approximation in Eq. (7.8) for small values of α , while also maintaining good agreement for higher values. The discrepancy is again caused by the coherence being dependent on α . The behaviour is quadratic because the second moments are quadratic in terms of operators, which are each linear with respect to α .

7.3 The Duan Criterion

In the MF, both the two-mode squeezing interaction strength, Λ , and the HO drive strength, α , affect \mathfrak{D} in the Duan criterion. The dependence on the drive stems from the expectation value of the Pauli operators, which with the density matrix used evaluates to

$$\langle \hat{\sigma}_- \rangle = c, \quad \text{and} \quad \langle \hat{\sigma}_+ \rangle = c^*. \quad (7.10)$$

The DQD coherences in turn depend on the drive as shown in Fig. 7.2. The dependence of \mathfrak{D} on α and Λ is shown as a contour plot in Fig. 7.4, where the oscillators are entangled if $\mathfrak{D} < 1$, and not entangled if $\mathfrak{D} \geq 1$. Thus, above the line indicating $\mathfrak{D} = 1$ in Fig. 7.4, the HO's are entangled, and below it, they are not. The dependence on α and Λ seems to be symmetric. Looking at cross-sections of the contour plot for specific values, shown in Fig. 7.5, we can see that this is the case in the range of the contour plot. We also note that the choice of oscillators to drive makes a very small difference, and thus we choose to have the same drive on both. Another point is that there needs to be non-zero drive on both HO's for it to affect \mathfrak{D} as can be seen from Eq. (6.5).

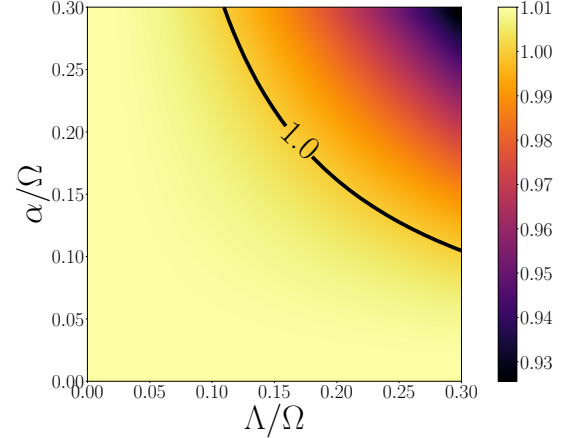


Figure 7.4: Contour plot of \mathfrak{D} in the Duan criterion, as a function of α and Λ . The Duan bound at 1 is indicated with a level line. Driving the HO's only contributes to entanglement in the MF. Parameter values are $\Omega = 1$, $\omega_1 = 2/3$, $\omega_2 = 1/3$, $\bar{n}_{1(2)} = 0.005$ and $\gamma_{1(2)} = \gamma_{s_e(s_g)} = 0.001$.

Without a DQD, that is, considering only two coupled cavities, \mathfrak{D} in the Duan criterion can be found analytically by solving the CALE. The Duan criterion in this case has no dependence on the oscillator drive, which can be seen from Eq. (7.4).

7. Numerical Results

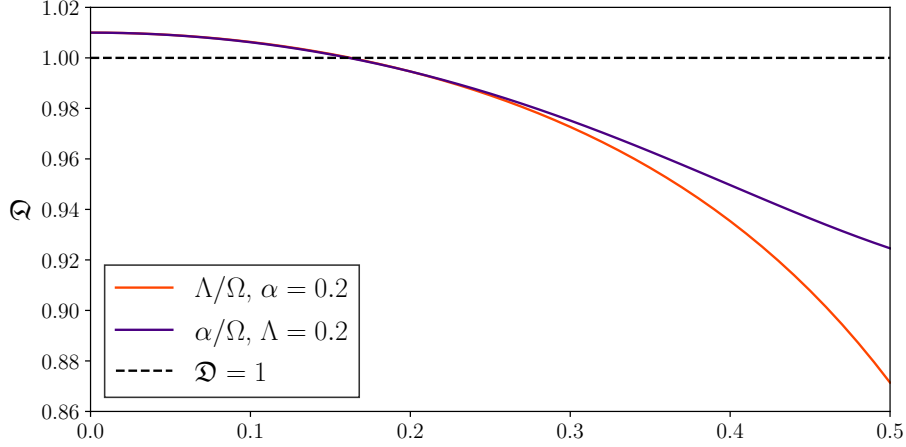


Figure 7.5: Cross-sections of Fig. 7.4, that is \mathfrak{D} as a function of Λ in one case when $\alpha = 0.2$, and in the other as a function of α when $\Lambda = 0.2$. The Duan bound, $\mathfrak{D} = 1$, is also indicated. The other parameter values used are $\Omega = 1$, $\omega_1 = 2/3$, $\omega_2 = 1/3$, $\bar{n}_{1(2)} = 0.005$, $\gamma_{1(2)} = 0.001$ and $\gamma_{s_e(s_g)} = 0.001$.

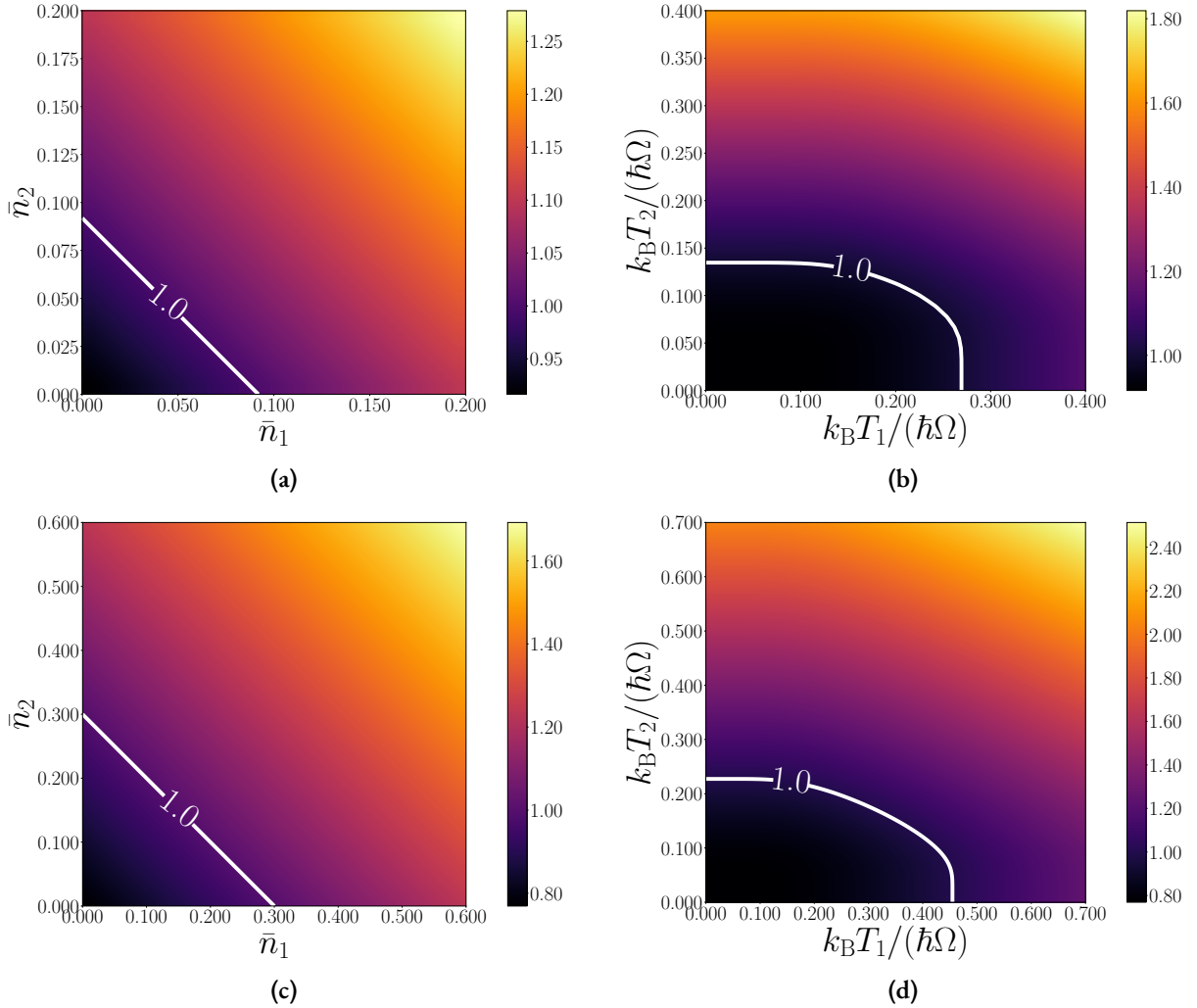


Figure 7.6: Contour plots for \mathfrak{D} in the Duan criterion. Panel (a) shows \mathfrak{D} as a function of the Bose-Einstein distributions of two oscillators from the numerical MF solution, while panel (b) shows the same thing, but as a function of their temperatures, both with $\Lambda = 0.3$. The panels (c) and (d) show \mathfrak{D} , but from the analytical solution for two coupled oscillators without the DQD, this time with $\Lambda = 0.15$. The other parameter values used are $\Omega = 1$, $\alpha_{1(2)} = 0.3$, $\omega_1 = 2/3$, $\omega_2 = 1/3$, $\gamma_{1(2)} = 0.001$ and $\gamma_{s_e(s_g)} = 0.001$.

The Duan criterion from the CALE for two coupled oscillators clearly has a dependence on $\bar{n}_{1(2)}$, from the first order solution in Eq. (7.4). This dependence will extend to the MF system, where we expect higher $\bar{n}_{1(2)}$ to break entanglement for the oscillators. \mathfrak{D} in the Duan criterion is thus shown as contour plots in the different cases in Fig. 7.6. The cases considered are: the MF solution as a function of $\bar{n}_{1(2)}$ in Fig. 7.6a, and as a function of temperature $T_{1(2)}$ in Fig. 7.6b, when $\bar{n}_{1(2)}$ is replaced by the explicit form in Eq. (7.9).

The corresponding cases for two coupled cavities without the DQD from their analytical solutions are shown in Figs. 7.6c and 7.6d. The analytical solution is not restricted by the value of the coherences, which has a maximum value of 1/2, and thus represents the best possible way to entangle the oscillators.

As a final aside in investigating the Duan criterion, we look at two analytical solutions for two coupled cavities. One is that from the CALE for local dissipators, and the other is the solution for a squeezed thermal state, which for zero temperature from Eq. (A.21) was found to be

$$\mathfrak{D} = \sqrt{\frac{\omega_1 + \omega_2 - 2|\Lambda|}{\omega_1 + \omega_2 + 2|\Lambda|}}, \quad (7.11)$$

for the diagonalised Hamiltonian in Eq. (A.13) of Appendix A. The two cases are shown in Fig. 7.7. It is interesting to note that our model does not predict the same result as the thermal solution, except for small Λ . This is because the dissipator we chose does not exactly thermalise the system, even if the temperatures in both HO's are the same. This is a problem in the field of open quantum systems, often described as *local dissipators* (the one we use) vs. *global dissipators* (which would have a thermal steady state). There is no consensus, though, on which approach is better, and both seem to fail in different situations [109–111].

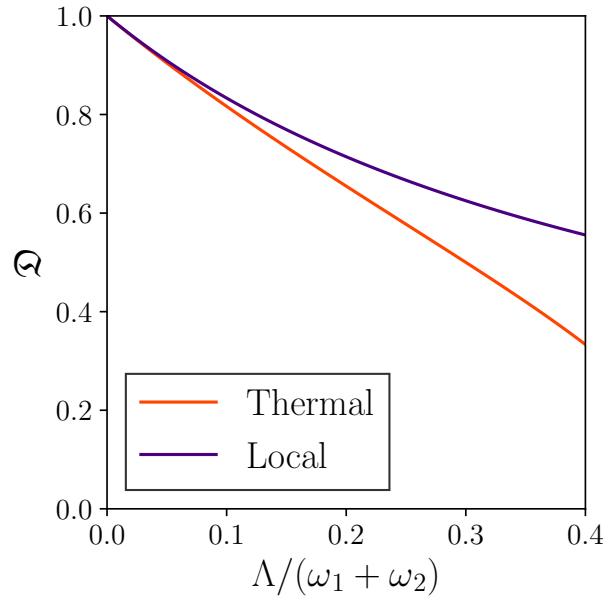


Figure 7.7: Plot of the Duan criterion \mathfrak{D} , comparing a state with local dissipators to a squeezed thermal state as a function of Λ . Parameter values used are $\omega_1 = 2/3$, $\omega_2 = 1/3$, $\bar{n}_{1(2)} = 0$ and $\gamma_{1(2)} = 0.001$.

8. Outlook

One of the inspirations for this work, Ref. [9], involved an experimental setup of a DQD coupled to a single resonator. The most exciting continuation of this work would be to do the same for the system we have considered, with coplanar waveguides and a semiconductor DQD, in order to obtain experimental data for comparison with our results.

For the purpose of simplifying calculations, a MF was introduced, shifting away from a Hilbert space description to a phase space description, where the phase space is much smaller than the Hilbert space. This makes the state of the oscillators Gaussian, and allows for the use of the CDLE. These calculations were not verified in any way, however, against theoretical results from a more exact model, such as one without the MF approximation, or against experimental data as discussed above. In Ref. [10] however, a similar MF was employed, giving accurate results in some regimes, including the weak coupling regime, compared to experimental data from Ref. [9]. A potential avenue to pursue is thus the verification of the validity of the MF approximation.

A possible theoretical extension to pursue would be the inclusion of more resonators. For larger systems of the type considered here, there exist similar criteria for entanglement [100]. More resonators represents but one of the possible models discussed in Appendix B, which could be considered. For purposes of computation, connecting several DQD-resonator systems in a network could also be explored, as mentioned in Sec. 1. The inclusion of more resonators can be handled effectively by the CDLE in the MF. In considering several connected DQD-resonator systems, the current flowing in the leads between the DQD's would need to be taken into account.

One of the obvious modifications that could be made to the system is replacing the DQD with another two-level system, such as a two-level atom, a single quantum dot with two levels, or a flux qubit as in Ref. [112]. Alternative extensions of the work could include a third or a fourth level, as well as to systems not in the Coulomb blockade regime. Moving away from the Coulomb blockade regime would introduce new physics stemming from electron-electron interactions, which are by no means trivial, and would have to be carefully considered.

In this work, entanglement of the system was analysed solely from application of the Duan criterion. This could be extended to other criteria, such as those based on entropy, or to measures of entanglement.

Similar to what was done in Ref. [10], applying the theory of counting statistics to this system would provide information on the transport statistics of the system. This involves modifying the dissipators of the DQD system as in Ref. [10] to look at electron transport, or alternatively modifying the dissipators of the oscillators to look at photon transport. More closely related to this work would be to analyse if the quantities one obtains from this framework can in any way be used to characterise entanglement.

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Appendix A: Coupled Cavities

A.1 Hamiltonian From cQED

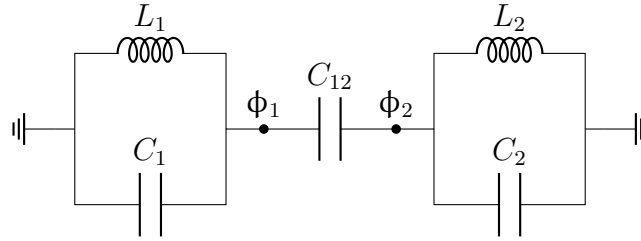


Figure 8.1: The circuit of two capacitively coupled LC -oscillators, corresponding to two coupled HO's.

Having quantised the circuit of a single LC -oscillator in Sec. 2.2.2, and the circuit for the full system considered in Sec. 3, we look at quantising two coupled LC -oscillators. The classical Lagrangian of the circuit in Fig. 8.1 is

$$\mathcal{L} = \frac{1}{2}C_1\dot{\phi}_1^2 - \frac{1}{2L_1}\phi_1^2 + \frac{1}{2}C_{12}(\dot{\phi}_1 - \dot{\phi}_2)^2 + \frac{1}{2}C_2\dot{\phi}_2^2 - \frac{1}{2L_2}\phi_2^2. \quad (\text{A.1})$$

From the matrix form in Eq. (2.9), the capacitance matrix is

$$\mathbf{C} = \begin{bmatrix} C_1 + C_{12} & -C_{12} \\ -C_{12} & C_{12} + C_2 \end{bmatrix}, \quad (\text{A.2})$$

and the inverse inductance matrix is¹

$$\mathbf{L}^{-1} = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \quad (\text{A.3})$$

We proceed to find the inverse capacitance matrix as being

$$\mathbf{C}^{-1} = \frac{1}{C_1C_2 + C_{12}C_1 + C_{12}C_2} \begin{bmatrix} C_2 + C_{12} & C_{12} \\ C_{12} & C_{12} + C_1 \end{bmatrix}. \quad (\text{A.4})$$

Introducing a new coupling constant [33]

$$C_0 \equiv \frac{C_{12}}{\sqrt{(C_1 + C_{12})(C_2 + C_{12})}}, \quad (\text{A.5})$$

¹In this case, the inverse inductance matrix is a true inverse matrix.

and using the first part of Eq. (2.19), we get

$$\mathbf{C}^{-1} = \begin{bmatrix} L_1\omega_1^2 & C_0\omega_1\omega_2\sqrt{L_1L_2} \\ C_0\omega_1\omega_2\sqrt{L_1L_2} & L_2\omega_2^2 \end{bmatrix}. \quad (\text{A.6})$$

Inserting this into Eq. (2.12) gives

$$\hat{H} = \frac{1}{2}L_1\omega_1^2q_1^2 + \frac{1}{2L_1}\phi_1^2 + C_0\omega_1\omega_2\sqrt{L_1L_2}q_1q_2 + \frac{1}{2}L_2\omega_2^2q_2^2 + \frac{1}{2L_2}\phi_2^2. \quad (\text{A.7})$$

Quantising this as before through Eq. (2.8) finally yields

$$\hat{H} = \hbar\omega_1\left(\hat{a}_1^\dagger\hat{a}_1 + \frac{1}{2}\right) + \hbar\omega_2\left(\hat{a}_2^\dagger\hat{a}_2 + \frac{1}{2}\right) + \hbar C_0\sqrt{\omega_1\omega_2}(\hat{a}_1^\dagger - \hat{a}_1)(\hat{a}_2^\dagger - \hat{a}_2). \quad (\text{A.8})$$

This can then be diagonalised with a Bogoliubov-Valatin transformation. An alternative approach to deriving the Hamiltonian is to diagonalise the Lagrangian first. More details on this can be found in Ref. [33].

A.2 Diagonalisation of the Two-Mode Squeezing Hamiltonian

If we restrict the Hamiltonian in the previous section to resonant emission and absorption, we get a Hamiltonian of the form

$$\hat{H} = \hbar\omega_1\left(\hat{a}_1^\dagger\hat{a}_1 + \frac{1}{2}\right) + \hbar\omega_2\left(\hat{a}_2^\dagger\hat{a}_2 + \frac{1}{2}\right) + \hbar(\Lambda\hat{a}_1^\dagger\hat{a}_2^\dagger + \Lambda^*\hat{a}_1\hat{a}_2). \quad (\text{A.9})$$

We can then diagonalise this Hamiltonian with a Bogoliubov-Valatin transformation from Eq. (2.65)

$$\hat{S}_2(\beta) = \exp\left(\beta\hat{a}_1^\dagger\hat{a}_2^\dagger - \beta^*\hat{a}_1\hat{a}_2\right), \quad (\text{A.10})$$

where the requirement on β becomes

$$\beta = \frac{1}{4}e^{i\arg(\Lambda)} \ln\left(\frac{\omega_1 + \omega_2 - 2|\Lambda|}{\omega_1 + \omega_2 + 2|\Lambda|}\right), \quad (\text{A.11})$$

or for $\omega_1 = \omega_2 = \omega_0$

$$\beta = \frac{1}{4}e^{i\arg(\Lambda)} \ln\left(\frac{\omega_0 - |\Lambda|}{\omega_0 + |\Lambda|}\right). \quad (\text{A.12})$$

With this, we get the Hamiltonian

$$\tilde{H} = S_2^\dagger\hat{H}S_2 = \Omega_+\left(\hat{a}_1^\dagger\hat{a}_1 + \frac{1}{2}\right) + \Omega_-\left(\hat{a}_2^\dagger\hat{a}_2 + \frac{1}{2}\right), \quad (\text{A.13})$$

with the new frequencies

$$\Omega_{\pm} = \pm \frac{1}{2} \hbar(\omega_1 - \omega_2) + \hbar \sqrt{\frac{(\omega_1 + \omega_2)^2}{4} - |\Lambda|^2}. \quad (\text{A.14})$$

A.3 Squeezed Thermal State

The covariance matrix of the two-mode squeezed vacuum can be found from the symplectic form of the two-mode squeezing operator in Eq. (2.68)

$$\Theta_{\text{STS}} = \mathbf{S}_2^\dagger \Theta_{\text{Th.}} \mathbf{S}_2, \quad (\text{A.15})$$

where the thermal state is $\Theta_{\text{Th.}} = \text{diag}(\bar{n}_1, \bar{n}_1, \bar{n}_2, \bar{n}_2) + \frac{1}{2} \mathbb{1}_4$. Evaluating this gives

$$\Theta_{\text{STS}} = \begin{bmatrix} A & 0 & 0 & C \\ 0 & A & C^* & 0 \\ 0 & C & B & 0 \\ C^* & 0 & 0 & B \end{bmatrix} + \frac{1}{2} \mathbb{1}_4, \quad (\text{A.16})$$

where

$$A = \bar{n}_1 \mu^2 + \bar{n}_2 |\nu|^2 + |\nu|^2, \quad B = \bar{n}_2 \mu^2 + \bar{n}_1 |\nu|^2 + |\nu|^2, \quad (\text{A.17})$$

and

$$C = (\bar{n}_1 + \bar{n}_2 + 1) \mu \nu. \quad (\text{A.18})$$

Using the Duan criterion in Eq. (5.50), we get

$$\mathfrak{D} = A + B - 2|C| = (\bar{n}_1 + \bar{n}_2) (\mu^2 + \nu^2) + 2|\nu|^2 - 2(\bar{n}_1 + \bar{n}_2 + 1) \mu |\nu|. \quad (\text{A.19})$$

Inserting the squeezing parameter β in Eq. (A.11) which diagonalises the Hamiltonian in Eq. (A.13), gives

$$\mathfrak{D} = (\bar{n}_1 + \bar{n}_2 + 1) \sqrt{\frac{\omega_1 + \omega_2 - 2|\Lambda|}{\omega_1 + \omega_2 + 2|\Lambda|}}, \quad (\text{A.20})$$

where the frequencies appearing in the Bose-Einstein distributions are modified according to Eq. (A.14). For zero temperature, \mathfrak{D} for a thermal state reduces to

$$\mathfrak{D} = \sqrt{\frac{\omega_1 + \omega_2 - 2|\Lambda|}{\omega_1 + \omega_2 + 2|\Lambda|}}. \quad (\text{A.21})$$

Appendix B: The Jaynes-Cummings Model and Its Extensions

Many theoretical models exist within quantum optics. The model used in the theoretical treatments in Refs. [9, 10] is the most common model, and the simplest, which is the Jaynes-Cummings model. The model considers one oscillator and one two-level system without counter-rotating terms. The interaction Hamiltonian is thus [26, 60, 113]

$$\hat{H}_{\text{JC}} = g\hat{a}^\dagger\hat{\sigma}_- + g^*\hat{a}\hat{\sigma}_+. \quad (\text{B.1})$$

This model can be extended to include more oscillators and more photons in the interaction with a single two-level system. The one-mode two-photon Jaynes-Cummings model would have the interaction Hamiltonian

$$\hat{H}_{\text{JC}}^{(1,2)} = g\hat{a}^\dagger\hat{a}^\dagger\hat{\sigma}_- + g^*\hat{a}\hat{a}\hat{\sigma}_+. \quad (\text{B.2})$$

If another oscillator is introduced, we have two modes and thus the two-mode two-photon Jaynes-Cummings Model has the interaction Hamiltonian [11]

$$\hat{H}_{\text{JC}}^{(2,2)} = g\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{\sigma}_- + g^*\hat{a}_1\hat{a}_2\hat{\sigma}_+. \quad (\text{B.3})$$

Model	Counter-rotating terms	Oscillators	Two-level systems
Jaynes-Cummings	No	$N = 1$	$N = 1$
Extended Jaynes-Cummings	No	$N > 1$	$N = 1$
Jaynes-Cummings-Hubbard	No	$N = M$	$M > 1$
Tavis-Cummings	No	$N = 1$	$N > 1$
Rabi	Yes	$N = 1$	$N = 1$
Dicke	Yes	$N = 1$	$N > 1$

Table B.1: Table giving an overview of theoretical models in quantum optics and some of their distinguishing features. This includes if the model contains counter-rotating terms and the number of oscillators and two-level systems.

The Rabi model is the Jaynes-Cummings model with the inclusion of counter-rotating terms. This modifies the interaction Hamiltonian to be [114]

$$\hat{H}_R = (g\hat{a}^\dagger + g^*\hat{a})\hat{\sigma}_x, \quad (\text{B.4})$$

which gives back the Jaynes-Cummings interaction when the RWA is applied. A number of these models are given in Tab. B.1 with some of their distinguishing features. Other variations on these models also exist.

Appendix C: The Lyapunov Equation From a Quadratic Function Subject to a 1D ODE

The simplest case of a linear matrix ODE is

$$\frac{d}{dt}\langle\hat{\zeta}\rangle = \mathbf{W}\langle\hat{\zeta}\rangle, \quad (\text{C.1})$$

where $\mathbf{W} \in \mathbb{C}^{2N \times 2N}$ and $\langle\hat{\zeta}\rangle \in \mathbb{C}^{2N \times 1}$. If we introduce a time-dependent quadratic function $Q(t)$, which can be written as

$$Q(t) = \langle\hat{\zeta}\rangle^\dagger(t) \mathbf{K}_\zeta \Theta(t) \mathbf{K}_\zeta \langle\hat{\zeta}\rangle(t), \quad (\text{C.2})$$

where the matrix formed by $\mathbf{K}_\zeta \Theta(t) \mathbf{K}_\zeta \in \mathbb{C}^{2N \times 2N}$ is a Hermitian matrix, i.e.

$$\mathbf{K}_\zeta \Theta(t) \mathbf{K}_\zeta = (\mathbf{K}_\zeta \Theta(t) \mathbf{K}_\zeta)^\dagger, \quad (\text{C.3})$$

we can take the time-derivative of this function, using the Leibniz product rule, to obtain

$$\begin{aligned} \dot{Q}(t) &= \langle\dot{\hat{\zeta}}\rangle^\dagger(t) \mathbf{K}_\zeta \Theta(t) \mathbf{K}_\zeta \langle\hat{\zeta}\rangle(t) + \langle\hat{\zeta}\rangle^\dagger(t) \mathbf{K}_\zeta \dot{\Theta}(t) \mathbf{K}_\zeta \langle\hat{\zeta}\rangle(t) \\ &\quad + \langle\hat{\zeta}\rangle^\dagger(t) \mathbf{K}_\zeta \Theta(t) \mathbf{K}_\zeta \langle\dot{\hat{\zeta}}\rangle(t), \end{aligned} \quad (\text{C.4})$$

and subsequently utilise Eq. (C.1) to further obtain

$$\begin{aligned} \dot{Q}(t) &= \left(\langle\hat{\zeta}\rangle^\dagger(t) \mathbf{W}^\dagger - i\vec{\lambda}^\dagger \mathbf{K}_\zeta \right) \mathbf{K}_\zeta \Theta(t) \mathbf{K}_\zeta \langle\hat{\zeta}\rangle(t) + \langle\hat{\zeta}\rangle^\dagger(t) \mathbf{K}_\zeta \dot{\Theta}(t) \mathbf{K}_\zeta \langle\hat{\zeta}\rangle(t) \\ &\quad + \langle\hat{\zeta}\rangle^\dagger(t) \mathbf{K}_\zeta \Theta(t) \mathbf{K}_\zeta \left(\mathbf{W} \langle\hat{\zeta}\rangle(t) + i \mathbf{K}_\zeta \vec{\lambda} \right) \\ &= \langle\hat{\zeta}\rangle^\dagger(t) \underbrace{\left(\mathbf{W}^\dagger \mathbf{K}_\zeta \Theta(t) \mathbf{K}_\zeta + \mathbf{K}_\zeta \dot{\Theta}(t) \mathbf{K}_\zeta + \mathbf{K}_\zeta \Theta(t) \mathbf{K}_\zeta \mathbf{W} \right)}_{= (*)} \langle\hat{\zeta}\rangle(t), \end{aligned} \quad (\text{C.5})$$

where we used $(\mathbf{W} \langle\hat{\zeta}\rangle(t))^\dagger = \langle\hat{\zeta}\rangle^\dagger(t) \mathbf{W}^\dagger$. Thus, a quadratic function subject only to Eq. (C.1) is governed by (*), which we can set equal to another matrix, $-\mathbf{K}_\zeta \mathbf{F} \mathbf{K}_\zeta$, as $(*) = -\mathbf{K}_\zeta \mathbf{F} \mathbf{K}_\zeta$ and express (*) as

$$\mathbf{K}_\zeta \left(\frac{d}{dt} \Theta \right) \mathbf{K}_\zeta = -\mathbf{W}^\dagger \mathbf{K}_\zeta \Theta \mathbf{K}_\zeta - \mathbf{K}_\zeta \Theta \mathbf{K}_\zeta \mathbf{W} + \mathbf{K}_\zeta \mathbf{F} \mathbf{K}_\zeta. \quad (\text{C.6})$$

We can then use the symplectic form of \mathbf{W} and its hermitian conjugate without the dissipative component Γ

$$\mathbf{W} = -i \mathbf{K}_\zeta \mathbf{H}, \quad \text{and} \quad \mathbf{W}^\dagger = i \mathbf{H} \mathbf{K}_\zeta, \quad (\text{C.7})$$

to get

$$\mathbf{W}^\dagger \mathbf{K}_\zeta = i \mathbf{H}, \quad \text{and} \quad \mathbf{K}_\zeta \mathbf{W} = -i \mathbf{H}. \quad (\text{C.8})$$

The latter can then be related to give

$$\mathbf{W}^\dagger \mathbf{K}_\zeta = -\mathbf{K}_\zeta \mathbf{W}. \quad (\text{C.9})$$

Using this in Eq. (C.6), we get

$$\mathbf{K}_\zeta \left(\frac{d}{dt} \boldsymbol{\Theta} \right) \mathbf{K}_\zeta = \mathbf{K}_\zeta \mathbf{W} \boldsymbol{\Theta} \mathbf{K}_\zeta + \mathbf{K}_\zeta \boldsymbol{\Theta} \mathbf{W}^\dagger \mathbf{K}_\zeta + \mathbf{K}_\zeta \mathbf{F} \mathbf{K}_\zeta, \quad (\text{C.10})$$

from which we can eliminate the matrices \mathbf{K}_ζ , since $\mathbf{K}_\zeta^2 = \mathbb{1}_{2N}$, to obtain

$$\frac{d}{dt} \boldsymbol{\Theta} = \mathbf{W} \boldsymbol{\Theta} + \boldsymbol{\Theta} \mathbf{W}^\dagger + \mathbf{F}. \quad (\text{C.11})$$

We can also relate the quadratic function Q to a Gaussian state through exponentiating it

$$\chi_Q(t) = e^{Q(t)}, \quad (\text{C.12})$$

and taking its derivative, giving

$$\dot{\chi}_Q(t) = \dot{Q}(t) e^{Q(t)}. \quad (\text{C.13})$$

Thus, $\dot{Q}(t)$ describes the evolution of a Gaussian state, which in turn is governed by Eq. (C.11). The omission of $\boldsymbol{\Gamma}$ from \mathbf{W} in this derivation seems to be necessary, although the CDLE still holds when it is included.

Appendix D: Overviews

D.1 Correspondence of a Microwave Cavity to an LC -Oscillator

A resonant cavity is physically, seemingly quite different compared to a resonant LC -circuit. They are however, both extreme cases of electromagnetic resonators [115]. By carefully altering the geometry and the number of inductors and capacitors of an LC -circuit, it is possible to obtain exactly a resonant cavity. This is explained and illustrated in full detail in Ref. [115]. This is also the reason as to why the HO Hamiltonian can be derived from an LC -circuit.

D.2 Correspondence Between Hilbert Space and Phase Space

In this work, we utilise two descriptions of quantum mechanics, one based on Hilbert spaces and the other on phase spaces. The correspondences between these descriptions are not immediately obvious, and as such, they are given as a schematic overview in Tab. D.1.

Property of the Space	Hilbert Space, \mathcal{H}	Phase Space, Γ
Dimension	∞	$2N$
Structure	\otimes	\oplus
Description	$\hat{\rho}$	$\langle \hat{\zeta} \rangle, \Theta$
<i>Bona Fide</i> Relation	$\hat{\rho} \geq 0$	$\Theta + \frac{1}{2}\mathbf{K}_\zeta \geq 0$
Unitary Operations	$\{\hat{U} : \hat{U}^\dagger \hat{U} = \mathbf{1}\}$ $\hat{\rho} \mapsto \hat{U}^\dagger \hat{\rho} \hat{U}$	$\{\mathbf{S} : \mathbf{S}^\dagger \mathbf{K}_\zeta \mathbf{S} = \mathbf{K}_\zeta\}$ $\hat{\zeta} \mapsto \mathbf{S} \hat{\zeta}, \Theta \mapsto \mathbf{S}^\dagger \Theta \mathbf{S}$
Spectra	$\hat{U} \hat{\rho} \hat{U}^\dagger = \text{diag}\{\lambda_i\}_{i=1}^\infty$ $0 \leq \lambda_i \leq 1$	$\mathbf{S} \Theta \mathbf{S}^\dagger = \text{diag}\{(\mu_k, \mu_k)\}_{k=1}^\infty$ $\frac{1}{2} \leq \mu_k < \infty$
Pure States	$\lambda_i = 1, \lambda_{j \neq i} = 0$	$\mu_k = 1, \forall k = 1, \dots, N$
Purity	$\text{tr}\{\hat{\rho}^2\} = \sum_i \lambda_i^2$	$(\sqrt{\det(\Theta)})^{-1} = \prod_k \frac{1}{\mu_k}$

Table D.1: Comparison of the properties in the Hilbert space and phase space descriptions of quantum mechanics for Gaussian states of N modes. The assumption made for the unitary operators \hat{U} is that they are quadratic. The table is adapted from Refs. [40, 69] with some of the modifications taken from Ref. [56]. For an almost equivalent table for quadrature operators, see Tab. 1 in Ref. [40].

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