A short bit on copulas and alternative versions of Spearmans rho

Filip Carlsson

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Lund University

Faculty of Engineering
Centre for Mathematical Sciences
Mathematical Statistics

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Mathematical Statistics
Centre for Mathematical Sciences
Lund University
Box 118, SE-221 00 Lund, Sweden

http://www.maths.lu.se/

Supervisor: Tatyana Turova
Examiner: Stanislav Volkov
Lund University

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1 Introduction

The objective of this thesis is to better understand and clarify certain aspects of the paper *Multivariate conditional versions of Spearman's rho and related measures of tail dependence* [4]. The motivation of the alternative multivariate conditional versions of Spearman's rho proposed in the paper is that the commonly used, particularly in financial engineering, Pearson's correlation coefficient often is an unsuitable dependence measure as it measures linear dependence, is invariant to change of location and scale in the univariate margins, and is very sensitive to outliers. Schmidt has studied and obtained results in several publications on alternative correlation coefficients to Pearson's using copulas, among others *Dependence of Stock Returns in Bull and Bear Markets* [12] where it is applied to equity market data, as well as *Multivariate Extensions of Spearman's Rho and Related Statistics* [13] and *Measuring large co-movements in financial markets* [14] also co-written with Rafael Schmidt.

We study and discuss copula theory and the work of Sklar. We refer to and clarify certain statements and formulas in the Schmidt and Schmidt paper, and also explore how the conditional version of Spearman's rho could be applied to other parts of the distribution beyond the tail. Finally, we look at empirical versions of copulas and Spearman's rho, using a simple example with data from a bivariate normal distribution. Through this process, we gained a deeper understanding of copula theory and its use in measuring dependence. The results also support the usefulness of the theory and suggest further study using different types of copulas and other data sets.

2 Theoretical background

2.1 Copulas

The copula theory, introduced by Sklar in 1959, is an useful method for modelling joint distributions without making assumptions about the specific form of the distribution function. It allows for the decomposition of a d-dimensional joint distribution into d marginal distributions and a copula function. Copulas describe the connection between two distributions. When considering multiple sets of observations, a copula does not describe the relationship between the observations themselves, but rather the relationship between the order of the observations. The copula can be handy when we search for a multivariate model whose univariate distributions are rather well understood, but whose joint distribution is only partly understood [3]. For example, consider two random variables, X and Y, with the following joint distribution function:

$$F(X,Y) = P(X \leq x, Y \leq y) \tag{1}$$

Each of these variables will also have a marginal distribution function, $F_X(x) = P(X \leq x)$ and $F_Y(y) = P(Y \leq y)$. The joint distribution function can also be described as a function of the individual distribution functions:

$$F(X,Y) = C(F_X(x), F_Y(y)). \tag{2}$$

where the function $C(u,v)$ is termed the copula [3]. Or, in the more broad sense with $F_{X_i}(x) = P(X \leq x)$ for $x_i \in \mathbb{R}$ and $i = 1,...,d$ and the assumption that $F_i(x_i)$ are continuous functions, then according to Sklar's theorem, there exists a unique copula $C : [0,1]^d \rightarrow [0,1]$ such that

$$F(\mathbf{x}) = C(F_1(x_1),...,F_d(x_d)) \quad \text{for all } x_i \in \mathbb{R}^d. \tag{3}$$

That is, the copula is a mapping of d marginal distribution functions to their joint distribution function.

The theory makes use of the fact that the distribution function follows a uniform distribution from 0 to 1. That is $F(X) = U$, where $U \in U(0,1)$. This is, as mentioned, convenient as further assumptions on the selection of distribution function are not needed.

2.1.1 Sklar's theorem

Sklar proved the following results for copulas. The following theorem and definitions are a translation from the original article by Sklar from 1959 [11]:

Theorem 2.1. *Let G_d be a d-dimensional distribution function, having marginal distributions $F_1, F_2, \ldots, F_d$. Let $\mathbb{R}_k$ be the set of values of $F_k, k = 1,2,\ldots,d$. Then, there exists a unique function H_d, defined on the Cartesian product $\mathbb{R}_1 \times \mathbb{R}_2 \times \ldots \mathbb{R}_d$, such that*

$$G_d\left(x_1,\ldots,x_d\right) = H_d\left(F_1\left(x_1\right),\ldots,F_d\left(x_d\right)\right). \tag{4}$$

Definition 2.1. *We will call copula (having d dimensions) any continuous and non-decreasing function C_d, defined on $[0,1]^d$, satisfying the following conditions:*

(i) $C_d(0,\ldots,0)=0$ *, and*

(ii) $C_d(1,\ldots,1,\alpha,1,\ldots,1)=\alpha$

(iii) C_d *is d-non-decreasing, i.e., for each hyperrectangle*

$B=\prod_{i=1}^{d}[x_i,y_i]\subseteq[0,1]^d$ *the C-volume of B is non-negative:*

$$\int_B \mathrm{d}C(u)=\sum_{\mathbf{z}\in\prod_{i=1}^{d}\{x_i,y_i\}}(-1)^{N(\mathbf{z})}C(\mathbf{z})\geq 0,\ \text{where } N(\mathbf{z})=\#\left\{k:z_k=x_k\right\}. \tag{5}$$

Theorem 2.2. *The function H_d of Theorem 1 can be extended (in general, non-uniquely) to a copula C_d. Being an extension of H_d, the copula C_d satisfies the condition*

$$G_d\left(x_1,\ldots,x_d\right)=H_d\left(F_1\left(x_1\right),\ldots,F_d\left(x_d\right)\right). \tag{6}$$

Theorem 2.3. *Let one-dimensional distribution functions $F_1,\ldots,F_d$ be given. Let C_d be any d-dimensional copula. Then, the function*

$$G_d\left(x_1,\ldots,x_d\right)=C_d\left(F_1\left(x_1\right),\ldots,F_d\left(x_d\right)\right) \tag{7}$$

is an d-dimensional distribution function having marginals $F_1,F_2,\ldots,F_d$.

2.1.2 Frechet–Hoeffding bounds

Copulas satisfy a version of the Frechet–Hoeffding bounds inequality. Both the upper and the lower Frechet–Hoeffding bounds are copulas. Specifically; every copula C is bounded as follows where

$$W\left(u_1,\ldots,u_d\right)\leq C\left(u_1,\ldots,u_d\right)\leq M\left(u_1,\ldots,u_d\right), \tag{8}$$

the function W is called the lower Fréchet-Hoeffding bound and is defined as

$$W\left(u_1,\ldots,u_d\right)=\max\left\{1-d+\sum_{i=1}^{d}u_i,0\right\}, \tag{9}$$

and the function M is called the upper Fréchet-Hoeffding bound and is defined as

$$M(u_1, \ldots, u_d) = \min\{u_1, \ldots, u_d\}. \tag{10}$$

Proof for $W \leq C$:
$C(\mathbf{x})$ is a distribution function of, $(U_1, \ldots, U_n)$ random variables with $U(0,1)$ marginals, thus one-dimensional distribution functions $(F_1(x_1) \ldots F_d(x_d))$, follows uniform distributions. That said, for any $\mathbf{u} = (u_1, \ldots u_n) \in [0,1]^d$

$$C(\mathbf{u}) = \mathrm{P}(U_1 \leq u_1, \ldots, U_d \leq u_d) = P(A) = 1 - P(A^c)$$

where $A = \bigcap_{i=1}^{d} (A_i) = \bigcap_{i=1}^{d} (U_i \leq u_i)$ and A^c is the complement of A

and as the complement of an intercept is the union of the complements we get

$A^c = \bigcup_{i=1}^{d} A_i^c = \bigcup_{i=1}^{d} (U_i > u_i)$ also recall $\mathrm{P}\left(\bigcup_{i=1}^{d} A_i^c\right) \leq \sum_{i=1}^{d} \mathrm{P}(A_i^c)$ hence

$$1 - \mathrm{P}(A^c) \geq 1 - \sum_{i=1}^{d} \mathrm{P}(U_i > u_i)$$

$$= 1 - \sum_{i=1}^{d} (1 - \mathrm{P}(U_i \leq u_i) = 1 - \sum_{i=1}^{d} (1 - u_i) = 1 - d + \sum_{i=1}^{d} u_i. \tag{11}$$

Proof for $C \leq M$:
From the definition, a copula is non-decreasing, denoting the smallest u thus $min\{u_1, \ldots, u_d\} = u_s$. Then

$$C(u_1, ..., u_d) \leq C(u_1, ..., u_{d-1}, 1) \leq C(1, ..., u_s, ..., 1) = u_s \tag{12}$$

and as u_s is the smallest u

$$u_s = min\{u_1, \ldots, u_d\} = min\{u_1, \ldots, u_{d-1}, 1\} = min\{\ldots, 1, \ldots, u_s, 1\}. \tag{13}$$

In two dimensions, i.e. the bivariate case, the Fréchet-Hoeffding Theorem states

$$\max\{u + v - 1, 0\} \leq C(u, v) \leq \min\{u, v\}. \tag{14}$$

The lower bound, $\max\left\{1 - d + \sum_{i=1}^{d} u_i, 0\right\}$, can in two dimensions be written as $\max\{1 - 2 + F(Y_1) + F(Y_2), 0\}$, where Y_1 and Y_2 are random variables.

The lower bound becomes zero when the sum of the marginal probabilities is less or equal to one. For example, if A is an event and its complementary event is denoted by A^c, then Y_1 and Y_2 can be defined as $Y_1 = \mathbf{1}_A$ and $Y_2 = \mathbf{1}_{A^c}$, where $\mathbf{1}$ is the indicator function. We have the relationship $P(A) = 1 - P(A^c)$, thus as $F(Y_1) = P(A)$ and $F(Y_2) = P(A^c)$, we have $1 - 2 + F(Y_1) + F(Y_2) = 1 - 2 + P(A) + P(A^c) = 0$.

An example of the upper bound for two random variables X_1 and X_2, is when X_1 is a monotonic transformation of X_2. If $X_1 = aX_2 + b$, where a and b are constants. Thus X_1 and X_2 have a full dependency and the minimum copula is a lower bound. Thus if $F(X_1) = 0.3$ and $F(X_2) = 0.4$, $F(X_1, X_2) = 0.3 = min\{F(X_1), F(X_2)\} = M\{F(X_1), F(X_2)\}$.

Note that the upper bound is the Comonotonicity copula, which is a special copula characterising perfect positive dependence, i.e., it represents the copula of $X_1, ..., X_d$ is $F_{X_1} = ... = F_{X_d}$ with probability one, if there exists an almost surely strictly increasing functional relationship between X_i and X_j $(i \neq j)$ [4].

2.1.3 Copulas

In modern theory, Sklar's theorem is usually written as:

Theorem 2.4. *Sklar's Theorem [6]: Let $F(x_1, \ldots, x_d)$ be a multivariate cumulative distribution function, with marginal distributions $F_i(X_i)$ for $i \in \{1, \ldots, d\}$. Then there exists a d-copula C such that for all $\mathbf{x}$ in $\mathbb{R}_d$,*

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)) \tag{15}$$

If $F_i(X_i)$ for $i \in \{1, \ldots, d\}$ are all continuous, then C is unique; otherwise C is uniquely determined on $[0,1]^d$. Conversely if C is an d-copula and $F_1, F_2, ..., F_d$ are distribution functions, then the function F defined by (15) is an d-dimensional distribution function with margins, $F_1, F_2, ..., F_d$.

Here is the idea behind the proof: Using the definition of the multivariate distribution function in conjunction with the monotonic increasing property of distribution functions; for any multivariate random variable with continuous marginals and distribution F it holds that

$$
\begin{aligned}
F(x_1, \ldots, x_d) &= \mathbb{P}(X_1 \leq x_1, \ldots, X_d \leq x_d) \\
&\text{following that the distribution function is non-decreasing} \\
&= \mathbb{P}(F_1(X_1) \leq F_1(x_1), \ldots, F_d(X_d) \leq F_d(x_d)) \\
&= \mathbb{P}(U_1 \leq F_1(x_1), \ldots, U_d \leq F_d(x_d)) \\
&= C(F_1(x_1), \ldots, F_d(x_d)).
\end{aligned}
\tag{16}
$$

Definition 2.2. *Let $f : [a, b] \rightarrow [c, d]$ be a non-decreasing function. Then the quasi-inverse $f^{(-1)}$ of f is defined as follows:*

1. *if $t \in \operatorname{Ran} f$ then $f^{(-1)}(t) = x$ such that $f(x) = t$, that is*

$$f\left(f^{(-1)}(t)\right) = t.$$

2. *if $t \notin \operatorname{Ran} f$ then*

$$f^{(-1)}(t) = \inf\{x \mid f(x) > t\} = \sup\{x \mid f(x) < t\}$$

The range of f, $\operatorname{Ran} f$, is the set of all values that f takes. Note that the quasi-inverse of f will not necessarily be unique, as there might be multiple choices of x in 1.

Remark. If f is strictly increasing, we have that $f^{(-1)} = f^{-1}$, meaning that the regular inverse and the quasi-inverse of f coincide [7].

Let F be a d-dimensional distribution function and assume that its marginals $F_1, F_2, \ldots, F_d$ are continuous. Then the copula C satisfying (15) is determined, for all $\mathbf{u} \in [0, 1]^d$, by

$$C(\mathbf{u}) = F\left(F_1^{(-1)}(u_1), F_2^{(-1)}(u_2), \ldots, F_d^{(-1)}(u_d)\right) \tag{17}$$

were $F_i^{(-1)}$ is the quasi-inverse of F_i [7].

Despite Sklar's result that a copula function always exists, it is not always apparent to identify the copula function. Indeed, for many applications, the problem is that the joint distribution is not always given but can be assumed due to some stylised facts. For example, in financial problems, the relationships between different asset returns are given, then we usually make the assumption that the joint distribution follows a multivariate Gaussian or a log-normal distribution for calculation simplicity, even if these assumptions may not be accurate. To understand full multivariate outcomes, the modelling problem consists of two steps: *Identifying the marginal distributions* as well as *defining the appropriate copula function describing the dependence structure* [9].

Copulas are normally divided into three categories: fundamental copulas represent a number of important special dependence structures, this category include the ones showed above $M(\mathbf{u}), W(\mathbf{u})$ as well as the independence copula $\Pi(\mathbf{u}) := \prod_{i=1}^{d} u_i, \mathbf{u} \in [0,1]^d$; implicit copulas are extracted from well-known multivariate distributions using Sklar's Theorem, but do not necessarily possess simple closed form expressions; explicit copulas have simple closed-form expressions and follow general mathematical constructions known to yield copulas [8].

In this project, we will make use of the Gaussian copula, mentioned above, which is an example of an implicit copula, defined as:

Definition 2.3. *Gaussian copula*
The copula C_{Σ}^{Gauss} of a d-dimensional standard normal distribution, with linear correlation matrix Σ, is the distribution function of the random vector $(\Phi(X_1), ..., \Phi(X_d))$, where Φ is the univariate standard normal distribution function and $\mathbf{X}$ is $N_d(\mathbf{0}, \Sigma)$-distributed. Hence,

$$C_{\Sigma}^{Gauss}(\mathbf{u}) = P\left(\Phi(X_1) \leq u_1, \ldots, \Phi(X_d) \leq u_d\right) = \Phi_{\Sigma}^{d}\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\right) \tag{18}$$

where Φ_{Σ}^{d} is the distribution function of $\mathbf{X}$ [3].

We will also make use of the Farlie-Morgenstern copula, also often denoted as Farlie-Gumbel-Morgenstern copula, which is an example of an explicit copula, defined as:

Definition 2.4. *The Farlie-Gumbel-Morgenstern copula*
For some parameter $\theta \in [-1, 1]$ and $u, v \in [0, 1]$, the copula is defined as

$$C_{\theta}(u, v) = uv + \theta uv(1-u)(1-v). \tag{19}$$

Definition 2.5. *Farlie-Gumbel-Morgenstern d-copulas. The FGM family has the following extension to a* $\left(2^d - d - 1\right)$*-parameter family of d-copulas,* $d \geq 3$*:*

$$C(\mathbf{u}) = u_1 u_2 \cdots u_d \left[1 + \sum_{k=2}^{d} \sum_{1 \leq j_1 < \cdots < j_k \leq d} \theta_{j_1 j_2 \cdots j_k} \bar{u}_{j_1} \bar{u}_{j_2} \cdots \bar{u}_{j_k}\right] \tag{20}$$

(where $\bar{u} = 1 - u$ *). Each copula in this family is absolutely continuous with density*

$$\frac{\partial^d C(\mathbf{u})}{\partial u_1 \cdots \partial u_d} = 1 + \sum_{k=2}^{d} \sum_{1 \leq j_1 < \cdots < j_k \leq d} \theta_{j_1 j_2 \cdots j_k} \left(1 - 2u_{j_1}\right)\left(1 - 2u_{j_2}\right) \cdots \left(1 - 2u_{j_k}\right). \tag{21}$$

Due to its simple analytical form, the Farlie-Morgenstern Copula has been used in modelling, testing of association, and studying the efficiency of non-parametric procedures. However, the Farlie-Morgenstern Copula can only model relatively weak dependence [6].

2.2 Spearmans Rho - from 1904 paper to copula representation

Spearman's rho, denoted ρ_S, originally published by the psychologist C. Spearman in 1904, is in the original form the Pearson correlation coefficient applied to the ranks associated with a sample $\{(x_i, y_i)\}_{i=1}^n$. Let $R_i = \text{rank}(x_i)$ and $S_i = \text{rank}(y_i)$; then computing the sample (Pearson) correlation coefficient r for $\{(R_i, S_i)\}_{i=1}^n$ yields

$$\rho_S = \frac{\sum_{i=1}^n (R_i - \overline{R})(S_i - \overline{S})}{\sqrt{\sum_{i=1}^n (R_i - \overline{R})^2 \cdot \sum_{i=1}^n (S_i - \overline{S})^2}} = \tag{22}$$

$$= 1 - \frac{6 \sum_{i=1}^n (R_i - S_i)^2}{n(n^2 - 1)}, \tag{23}$$

where $\overline{R} = \sum_{i=1}^n R_i/n = (n+1)/2 = \sum_{i=1}^n S_i/n = \overline{S}$.
If X_1 and X_2 are random variables with respective distribution functions F_{X_1} and F_{X_2}, Spearmans rho is defined to be the Pearson correlation coefficient of the random variables $F_{X_1}(X_1)$ and $F_{X_2}(X_2)$:

Definition 2.6. *For random variables X_1 and X_2 with marginal distributions F_1 and F_2 Spearman's rho is given by $\rho_{\mathrm{S}}(X_1, X_2) = \rho(F_1(X_1), F_2(X_2))$, i.e.,*

$$\rho_S = \operatorname{corr}[F_1(X_1), F_2(X_2)] = \frac{\operatorname{cov}(F_1(X_1), F_2(X_2))}{\sqrt{\operatorname{var}(F_1(X_1))}\sqrt{\operatorname{var}(F_2(X_2))}}. \tag{24}$$

In other words, Spearman's rho is simply the linear correlation of the probability-transformed random variables, which for continuous random variables is the linear correlation of their unique copula. Since the copula C is the joint distribution function of the random variables $U_i = F_i(X_i)$, $i = 1, ..., d$. Spearmans rho can be defined in the sense of copulas as (recall for uniform $[0,1]$ distribution $\mathbb{E}[U] = \frac{1}{2}(1-0), \operatorname{var}[U] = \frac{1}{12}(1-0))$:

$$\begin{aligned}
\rho_{\mathrm{S}} &= \frac{\operatorname{cov}(F_1(X_1), F_2(X_2))}{\sqrt{\operatorname{var}(F_1(X_1))}\sqrt{\operatorname{var}(F_2(X_2))}} = \frac{\operatorname{cov}(U_1, U_2)}{\sqrt{\operatorname{var}(U_1)}\sqrt{\operatorname{var}(U_2)}} \\
&= \frac{\mathbb{E}[U_1U_2] - \mathbb{E}[U_1]\,\mathbb{E}[U_2]}{\sqrt{\frac{1}{12}}\sqrt{\frac{1}{12}}} = \frac{\int_0^1\int_0^1 uvc(u,v)dudv - \left(\frac{1}{2}\right)^2}{\sqrt{\frac{1}{12}}\sqrt{\frac{1}{12}}} \\
&= \frac{\int_0^1\int_0^1 uvdC(u,v) - \left(\frac{1}{2}\right)^2}{\sqrt{\frac{1}{12}}\sqrt{\frac{1}{12}}} = 12\int_0^1\int_0^1 C(u,v)dudv - 3,
\end{aligned} \tag{25}$$

where $c(u,v)$ is the joint density of U_1, U_2, thus $c(u,v) = \frac{\partial^2}{\partial u \partial v} C(u,v)$.

Further, we'd like to generalise (25) to d dimensions. Schmid and Schmidt [4] gives the following alternative representation, which is readily verified and plays a central role in the forthcoming definitions of conditional versions of Spearman's rho

$$\begin{aligned}
\rho_{\mathrm{S}_2} &= \frac{\int_0^1\int_0^1 C(u,v)dudv - \int_0^1\int_0^1 uvdudv}{\int_0^1\int_0^1 \min\{u,v\}dudv - \int_0^1\int_0^1 uvdudv} \\
&= \frac{\int_0^1\int_0^1 C(u,v)dudv - \int_0^1\int_0^1 \Pi(u,v)dudv}{\int_0^1\int_0^1 M(u,v)dudv - \int_0^1\int_0^1 \Pi(u,v)dudv}.
\end{aligned} \tag{26}$$

This version of ρ_{S_2} can be interpreted as the normalised average distance between the copula C and the independence copula $\Pi(u,v) = uv$. The numerator is derived as

$$\begin{aligned}\mathbb{E}\left[U_1U_2\right]-\mathbb{E}[U_1]\,\mathbb{E}[U_2] &= \int_0^1\int_0^1 uvc(u,v)dudv-\int_0^1 udu\int_0^1 vdv\\ &= \int_0^1\int_0^1 uvc(u,v)dudv-\int_0^1\int_0^1 uvdudv \qquad (27)\\ &= \int_0^1\int_0^1 C(u,v)dudv-\int_0^1\int_0^1 \Pi(u,v)dudv.\end{aligned}$$

Consider now the denominator

$$\int_{[0,1]^2} M(u,v)dudv = 1/3 \text{ and } \int_{[0,1]^2} \Pi(u,v)dudv = 1/4 \qquad (28)$$

thus

$$\sqrt{\operatorname{var}[U_1]\operatorname{var}[U_2]} = \frac{1}{12} = \frac{1}{3}-\frac{1}{4}. \qquad (29)$$

This shows why (26) corresponds to (25).

Wolff [16] introduces the following straightforward generalisation of Spearman's rho ρ_{S_2} to d dimensions;

$$\rho_{S_d} = \frac{\int_{[0,1]^d} C(\mathbf{u})d\mathbf{u}-\int_{[0,1]^d}\Pi(\mathbf{u})d\mathbf{u}}{\int_{[0,1]^d} M(\mathbf{u})d\mathbf{u}-\int_{[0,1]^d}\Pi(\mathbf{u})d\mathbf{u}} = \frac{d+1}{2^d-(d+1)}\left\{2^d\int_{[0,1]^d} C(\mathbf{u})d\mathbf{u}-1\right\}. \qquad (30)$$

2.3 Comparison with an alternative representation of Spearman's rho

The representation of Spearman's rho in (30) is not the only representation that has been studied. In this section, we compare it with an alternative version by Ruymgaart and van Zuijlen [15].

Ruymgaart and van Zuijlen [15] address the estimation of the alternative measure, $\tilde{\rho}_{S_d}$:

$$\begin{aligned}\tilde{\rho}_{S_d} &= \frac{\int_{[0,1]^d} \Pi(\mathbf{u})dC(\mathbf{u}) - \int_{[0,1]^d} \Pi(\mathbf{u})d\mathbf{u}}{\int_{[0,1]^d} M(\mathbf{u})d\mathbf{u} - \int_{[0,1]^d} \Pi(\mathbf{u})d\mathbf{u}} \\ &= \frac{d+1}{2^d-(d+1)}\left\{2^d \int_{[0,1]^d} \Pi(\mathbf{u})dC(\mathbf{u}) - 1\right\} \\ &= \frac{d+1}{2^d-(d+1)}\left\{2^d \int_{[0,1]^d} \Pi(\mathbf{u})c(\mathbf{u})d(\mathbf{u}) - 1\right\}.\end{aligned} \tag{31}$$

It is said by Schmidt and Schmidt [4] that both generalisations ρ_d and $\tilde{\rho}_d$ coincide with Spearman's rho if $d=2$. Here we show the general case when $d=2$, according to (25):

$$\begin{aligned}\tilde{\rho}_{S_2} &= 12\int_{[0,1]^2} \Pi(u,v)dC(u,v) - 3 \\ &= 12\int_{[0,1]^2} uvdC(u,v) - 3 \\ &= 12\int_{[0,1]^2} C(u,v)dudv - 3 = \rho_{S_2}.\end{aligned} \tag{32}$$

Further we look into a case with a specific copula. As an example, compare ρ_{S_d} and $\tilde{\rho}_{S_d}$ in the $d=2$ case with the bivariate Farlie-Morgenstern copula $C(u,v;\theta) = uv + \theta uv(1-u)(1-v), \theta \in [-1,1]$, with joint density function

$$\begin{aligned}c_d(u_1,\ldots,u_d;\boldsymbol{\theta}) &= \frac{\partial^d}{\partial u_1 \cdots \partial u_d} C(u_1,\ldots,u_d;\boldsymbol{\theta}) \\ &= 1 + \sum_{k=2}^{d} \sum_{1\le j_1<\cdots<j_k\le d} \theta_{j_1\cdots j_k}(1-2u_{j_1})\cdots(1-2u_{j_k}).\end{aligned} \tag{33}$$

Thus ρ_{S_2} and $\tilde{\rho}_{S_2}$ are denoted as;

$$\begin{aligned}\rho_{S_2} &= 12\int_{[0,1]^d} C(u,v)dudv - 3 \\ &= 12\int_{[0,1]^d} uv + \theta uv(1-u)(1-v)dudv - 3 \\ &= \frac{\theta+9}{3} - 3 = \frac{\theta}{3}\end{aligned} \tag{34}$$

and

$$\begin{aligned}\tilde{\rho}_{S_2} &= 12\int_{[0,1]^d} uv\cdot c(u,v)dudv - 3\\ &= 12\int_{[0,1]^d} uv\cdot(1+\theta(2u-1)(2v-1))\,dudv - 3 \\ &= \frac{\theta+9}{3} - 3 = \frac{\theta}{3},\end{aligned} \tag{35}$$

where we see that ρ_{S_d} corresponds with $\tilde{\rho}_{S_d}$ in the $d=2$ case for the Farlie-Morgenstern copula.

Further, we take a look at the $d=3$ case. By comparing the copula, it is noted that it becomes a bit more to calculate

$$\begin{aligned}C_2(u_1,u_2;\boldsymbol{\theta}) =& u_1u_2\left(1+\theta_{12}(1-u_1)(1-u_2)\right),\\ C_3(u_1,u_2,u_3;\boldsymbol{\theta}) =& u_1u_2u_3\left(1+\theta_{12}(1-u_1)(1-u_2)+\theta_{13}(1-u_1)(1-u_3)\right.\\ &\left.+\theta_{23}(1-u_2)(1-u_3)+\theta_{123}(1-u_1)(1-u_2)(1-u_3)\right),\end{aligned} \tag{36}$$

and

$$\begin{aligned}c_2(u_1,u_2;\boldsymbol{\theta}) =& 1+\theta_{12}(1-2u_1)(1-2u_2),\\ c_3(u_1,u_2,u_3;\boldsymbol{\theta}) =& \left(1+\theta_{12}(1-2u_1)(1-2u_2)+\theta_{13}(1-2u_1)(1-2u_3)\right.\\ &\left.+\theta_{23}(1-2u_2)(1-2u_3)+\theta_{123}(1-2u_1)(1-2u_2)(1-2u_3)\right).\end{aligned} \tag{37}$$

Thus ρ_{S_3} and $\tilde{\rho}_{S_2}$ are denoted as;

$$\begin{aligned}\rho_{S_3} &= 8\int_{[0,1]^3} C(u_1,u_2,u_3)du_1du_2du_3 - 1\\ &= 8\int_{[0,1]^3} u_1u_2u_3\left(1+\theta_{12}(1-u_1)(1-u_2)+\theta_{13}(1-u_1)(1-u_3)\right.\\ &\left.+\theta_{23}(1-u_2)(1-u_3)+\theta_{123}(1-u_1)(1-u_2)(1-u_3)\right)du_1du_2du_3 - 1\\ &= \frac{1}{9}\left(\theta_{12}+\theta_{13}+\theta_{23}+\frac{\theta_{123}}{3}\right)\end{aligned} \tag{38}$$

and

$$\begin{aligned}\tilde{\rho}_{S_3} &= 8\int_{[0,1]^3} u_1u_2u_3 \cdot c(u_1,u_2,u_3)du_1du_2du_3 - 1\\ &= 8\int_{[0,1]^3} u_1u_2u_3\left(1+\theta_{12}\left(1-2u_1\right)\left(1-2u_2\right)+\theta_{13}\left(1-2u_1\right)\left(1-2u_3\right)\right.\\ &\left.+\theta_{23}\left(1-2u_2\right)\left(1-2u_3\right)+\theta_{123}\left(1-2u_1\right)\left(1-2u_2\right)\left(1-2u_3\right)\right)du_1du_2du_3 - 1\\ &= \frac{1}{9}\left(\theta_{12}+\theta_{13}+\theta_{23}-\frac{\theta_{123}}{3}\right),\end{aligned} \tag{39}$$

where we see that, $\tilde{\rho}_{S_d} \neq \rho_{S_d}$, for $d = 3$. Thus we conclude that differences occur in more dimensions where the dependence structure gets more complex. The interested reader can look into more examples, to see if this accounts for all copulas in higher than two dimensions, or if there exist cases where the representations are equivalent also in higher dimensions. Also to be noted is that the Farlie-Morgenstern copula in $d = 2$ only depends on the chosen parameter θ. Thus the usefulness is rather limited but proves the point in this case. Different copulas are focused on different aspects of distributions, and there may be cases for simpler copulas where both generalisations of rho correspond for higher dimensions than two. However, as mentioned above, it is more likely that differences occur when the structure is more complex.

2.4 Conditional versions of Spearmans rho

In the interest of looking at the case of Spearmans rho for left tail events, i.e., low probability events in the left part of the distribution, we want to move towards conditional versions of Spearmans rho.
The following definition of the multivariate conditional version of Spearman's rho is motivated by (30):

$$\rho_{S_d}(g) := \frac{\int_{[0,1]^d} C(\mathbf{u})g(\mathbf{u})d\mathbf{u} - \int_{[0,1]^d} \Pi(\mathbf{u})g(\mathbf{u})d\mathbf{u}}{\int_{[0,1]^d} M(\mathbf{u})g(\mathbf{u})d\mathbf{u} - \int_{[0,1]^d} \Pi(\mathbf{u})g(\mathbf{u})d\mathbf{u}} \tag{40}$$

for some measurable function $g \geq 0$ such that the integrals exist.
The function of choice is $g(\mathbf{u}) = \mathbf{1}_{[0,p]^d}(\mathbf{u})$, $0 < p \leq 1$. Making the function concentrated to the lower part of the copula C. The resulting d dimensional

conditional version of Spearman's rho for $0 < p \leq 1$ is defined by

$$\rho_{S_d}(p) := \frac{\int_{[0,p]^d} C(\mathbf{u})d\mathbf{u} - \int_{[0,p]^d} \Pi(\mathbf{u})d\mathbf{u}}{\int_{[0,p]^d} M(\mathbf{u})d\mathbf{u} - \int_{[0,p]^d} \Pi(\mathbf{u})d\mathbf{u}} = \frac{\int_{[0,p]^d} C(\mathbf{u})d\mathbf{u} - \left(\frac{p^2}{2}\right)^d}{\frac{p^{d+1}}{d+1} - \left(\frac{p^2}{2}\right)^d}. \quad (41)$$

Another interesting case can be to introduce more degrees of freedom by splitting into other parts to be able to highlight different parts of the distribution. Thus we look at some other area, $[p_1, p_2]^d$, or several $[0, p_1]^d$ and $[p_2, p_3]^d$, by $g(\mathbf{u}) = \mathbf{1}_{[p_1,p_2]^d}(\mathbf{u})$, $0 < p_1 < p_2 < 1$, or $g(\mathbf{u}) = \mathbf{1}_{[0,p_1]^d}(\mathbf{u}) + \mathbf{1}_{[p_2,p_3]^d}(\mathbf{u})$, $0 < p_1 < p_2 < p_3 < 1$ giving

$$\rho_{S_d}(p) := \frac{\int_{[p_1,p_2]^d} C(\mathbf{u})d\mathbf{u} - \int_{[p_1,p_2]^d} \Pi(\mathbf{u})d\mathbf{u}}{\int_{[p_1,p_2]^d} M(\mathbf{u})d\mathbf{u} - \int_{[p_1,p_2]^d} \Pi(\mathbf{u})d\mathbf{u}} \quad (42)$$

or

$$\rho_{S_d}(p) := \frac{\int_{[0,p_1]^d} C(\mathbf{u})d\mathbf{u} + \int_{[p_2,p_3]^d} C(\mathbf{u})d\mathbf{u} - \int_{[0,p_1]^d} \Pi(\mathbf{u})d\mathbf{u} - \int_{[p_2,p_3]^d} \Pi(\mathbf{u})d\mathbf{u}}{\int_{[0,p_1]^d} M(\mathbf{u})d\mathbf{u} + \int_{[p_2,p_3]^d} M(\mathbf{u})d\mathbf{u} - \int_{[0,p_1]^d} \Pi(\mathbf{u})d\mathbf{u} - \int_{[p_2,p_3]^d} \Pi(\mathbf{u})d\mathbf{u}}. \quad (43)$$

Denote the smallest u thus $min\{u_1, \ldots, u_d\} = u_s$ and $M(\mathbf{u})$ and $\Pi(\mathbf{u})$ are given as

$$\int_{[p_1,p_2]^d} M(\mathbf{u})d\mathbf{u} = \int_{[p_1,p_2]^d} \min(\mathbf{u})d\mathbf{u} = \int_{p_1}^{p_2} u_s(\int_{p_1}^{u_s} 1du_1...du_{s-1})du_s = \int_{p_1}^{p_2} u_s(u_s - p_1)^{d-1}du_s = \frac{(p_2 - p_1)^d(dp_2 + p_1)}{d(d+1)} \quad (44)$$

and

$$\int_{[p_1,p_2]^d} \Pi(\mathbf{u})d\mathbf{u} = \left(\frac{p_2^2 - p_1^2}{2}\right)^d. \quad (45)$$

This gives us the ability to give weights to selected parts of the copula and we can study dependence in other parts than in the tail. For example, putting the upper percentile to one, $p_2 = 1$, yields the upper tail. It would for example be interesting to further look into if the upper tail corresponds to the lower tail plugged in with reversed data.

2.5 Estimation under unknown marginal distributions

2.5.1 Empirical copula

Consider a random sample $(\mathbf{X}_j)_{j=1,\ldots,n}$ from a d-dimensional random vector $\mathbf{X}$ with joint distribution function F and copula C. Assume that the univariate marginal distribution functions F_{X_i} of F are continuous but unknown. The marginal distribution functions F_{X_i} are estimated by their empirical counterparts

$$\hat{F}_{i,n}(x) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{\{X_{ij}\leq x\}}, \qquad \text{for } i = 1,\ldots,d \text{ and } x \in \mathbb{R}. \tag{46}$$

Further, set $\hat{U}_{ij,n} := \hat{F}_{i,n}(X_{ij})$ for $i = 1,\ldots,d, j = 1,\ldots,n$, and $\hat{\mathbf{U}}_{j,n} = \left(\hat{U}_{1j,n},\ldots,\hat{U}_{dj,n}\right)$. Note that

$$\hat{U}_{ij,n} = \frac{1}{n}(\text{ rank of } X_{ij} \text{ in } X_{i1},\ldots,X_{in}). \tag{47}$$

The estimation of the copula will therefore be based on ranks (and not on the original observations). In other words, we consider order statistics. The copula C is estimated by the empirical copula which is defined as the discrete function $\hat{C}_n$ given by

$$\hat{C}_n\left(\frac{i}{n},\frac{j}{n}\right) = \frac{\text{number of pairs } (x,y) \text{ in the sample with } x \leq x_{(i)}, y \leq y_{(j)}}{n}$$

where $x_{(i)}$ and $y_{(j)}, 1 \leq i,j \leq n$, denote order statistics from the sample. I.e., the share of elements fulfilling where both x and y are below order i,j respective. This can be written for d dimensions as

$$\hat{C}_n(\mathbf{u}) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d} \mathbf{1}_{\left\{\hat{U}_{ij,n}\leq u_i\right\}} \qquad \text{for} \qquad \mathbf{u} = (u_1,\ldots,u_d) \in [0,1]^d. \tag{48}$$

The empirical copula, being a particular multivariate empirical distribution function, often exhibits a large bias when the sample size is small. One way to counteract this is to use the empirical beta copula. The estimator is given by

$$\hat{C}_n^{\beta}(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{d} F_{n,R_{ij}}(u_j) \quad \mathbf{u} \in [0,1]^d \tag{49}$$

where $F_{n,r}$ represents a beta distribution function with parameters r and $n+1-r$ and where R_{ij} represents the rank of X_{ij} where $\mathbf{X}$ is the original data set used to "fit" the empirical copula.

2.5.2 Empirical rho

The empirical copula yields the following empirical version of Spearmans rho

$$\begin{aligned} \hat{\rho}_{S_{d},n} &:= \frac{\int_{[0,1]^d} \hat{C}_n(\mathbf{u})d\mathbf{u} - \int_{[0,1]^d} \Pi(\mathbf{u})d\mathbf{u}}{\int_{[0,1]^d} M(\mathbf{u})d\mathbf{u} - \int_{[0,1]^d} \Pi(\mathbf{u})d\mathbf{u}} \\ &= \frac{\frac{1}{n}\sum_{j=1}^n \int_{[0,1]^d} \prod_{i=1}^d \mathbf{1}_{\left\{\hat{U}_{ij,n}\leq u_i\right\}} d\mathbf{u} - \left(\frac{1}{2}\right)^d}{\frac{1}{d+1} - \left(\frac{1}{2}\right)^d} \\ &= \frac{\frac{1}{n}\sum_{j=1}^n \prod_{i=1}^d \left(1-\hat{U}_{ij,n}\right) - \left(\frac{1}{2}\right)^d}{\frac{1}{d+1} - \left(\frac{1}{2}\right)^d}, \end{aligned} \tag{50}$$

recall the approach for the conditional version of Spearmans rho, for $g(\mathbf{u}) = \mathbf{1}_{[0,p]^d}(\mathbf{u}), 0 < p \leq 1$, given in (40). For the estimated copula this gives us

$$\begin{aligned} \hat{\rho}_{S_{d},n}(g) &:= \frac{\int_{[0,1]^d} \hat{C}_n(\mathbf{u})g(\mathbf{u})d\mathbf{u} - \int_{[0,1]^d} \Pi(\mathbf{u})g(\mathbf{u})d\mathbf{u}}{\int_{[0,1]^d} M(\mathbf{u})g(\mathbf{u})d\mathbf{u} - \int_{[0,1]^d} \Pi(\mathbf{u})g(\mathbf{u})d\mathbf{u}} \\ &= \frac{\frac{1}{n}\sum_{j=1}^n \int_{\hat{\mathbf{U}}_{\mathbf{j},\mathbf{n}}\leq \mathbf{u}} g(\mathbf{u})d\mathbf{u} - \int_{[0,1]^d} \Pi(\mathbf{u})g(\mathbf{u})d\mathbf{u}}{\int_{[0,1]^d} M(\mathbf{u})g(\mathbf{u})d\mathbf{u} - \int_{[0,1]^d} \Pi(\mathbf{u})g(\mathbf{u})d\mathbf{u}} \end{aligned} \tag{51}$$

given g as earlier $g(\mathbf{u}) = \mathbf{1}_{[0,p]^d}(\mathbf{u}), 0 < p \leq 1$,

$$\begin{aligned} \hat{\rho}_{S_{2},n}(p) &= \frac{\frac{1}{n}\sum_{j=1}^n \int_{\hat{U}_{ij,n}\leq u_i} \mathbf{1}_{[0,p]^d}(\mathbf{u})d\mathbf{u} - \left(\frac{p^2}{2}\right)^d}{\frac{p^{d+1}}{d+1} - \left(\frac{p^2}{2}\right)^d} \\ &= \frac{\frac{1}{n}\sum_{j=1}^n \prod_{i=1}^d \left(p-\hat{U}_{ij,n}\right)^+ - \left(\frac{p^2}{2}\right)^d}{\frac{p^{d+1}}{d+1} - \left(\frac{p^2}{2}\right)^d}, \end{aligned} \tag{52}$$

where p is the chosen quantile and d the number of dimensions. Thus, for two dimensions ρ_S is estimated by

$$\hat{\rho}_{S_2,n}(p) = \frac{\frac{1}{n}\sum_{j=1}^{n}\left(p - \hat{U}_{1j,n}\right)^{+}\left(p - \hat{U}_{2j,n}\right)^{+} - \left(\frac{p^2}{2}\right)^2}{\frac{p^3}{3} - \left(\frac{p^2}{2}\right)^2}. \tag{53}$$

It is clear that the limiting laws for $\sqrt{n}\,\{\hat{\rho}_{S_d,n}(p) - \rho_{S_d}(p)\}$ depends on the asymptotic behaviour of the copula process $\sqrt{n}\left\{\hat{C}_n(\mathbf{u}) - C(\mathbf{u})\right\}$. I.e., if $\sqrt{n}\left\{\hat{C}_n(\mathbf{u}) - C(\mathbf{u})\right\}$ converges towards a Gaussian process, that is the estimate has asymptotic normality, then it is also the case for $\hat{\rho}_{S_d,n}$. The asymptotic behaviour of the ordinary empirical copula process is well studied. Schmidt and Schmidt [4] also shows that $\sqrt{n}\,\{\hat{\rho}_{S_d,n}(p) - \rho_{S_d}(p)\}$ converges weakly to a centred Gaussian process, i.e. a Gaussian process with mean 0. The weak convergence takes place in $\ell^\infty([\epsilon, 1])$ for arbitrary but fixed $0 < \epsilon < 1$, i.e.

$$\sqrt{n}\,\{\hat{\rho}_{S_d,n}(p) - \rho_{S_d}(p)\} \xrightarrow{w} N(0, \sigma^2). \tag{54}$$

3 Method

Here, we first check asymptotic normality for the empirical copula. We limit the scope to two dimensions. The dataset used is samples from a multivariate normal distribution. We then fit a Gaussian copula according to (18), and an empirical copula according to (48), using n samples from the multivariate normal distribution. To check the asymptotic behaviour of the copula process, we draw 4000 samples from each fitted copula and evaluate the distribution of $\sqrt{n}\left\{\hat{C}_n(\mathbf{u}) - C(\mathbf{u})\right\}$, to see if it approached a normal distribution when n increases.

To check the asymptotic behaviour of the copula process, we use synthetic data with samples from a multivariate normal distribution with two dimensions. We let the number of samples, n, increase and compare the empirical rho for $d = 2$ as in (50) and Spearmans rho as in (24) as n increases. To see if $\sqrt{n}\,\{\hat{\rho}_n(\mathbf{u}) - \rho_S(\mathbf{u})\}$ approaches a normal distribution as n increases.

4 Results

4.1 Empirical copula

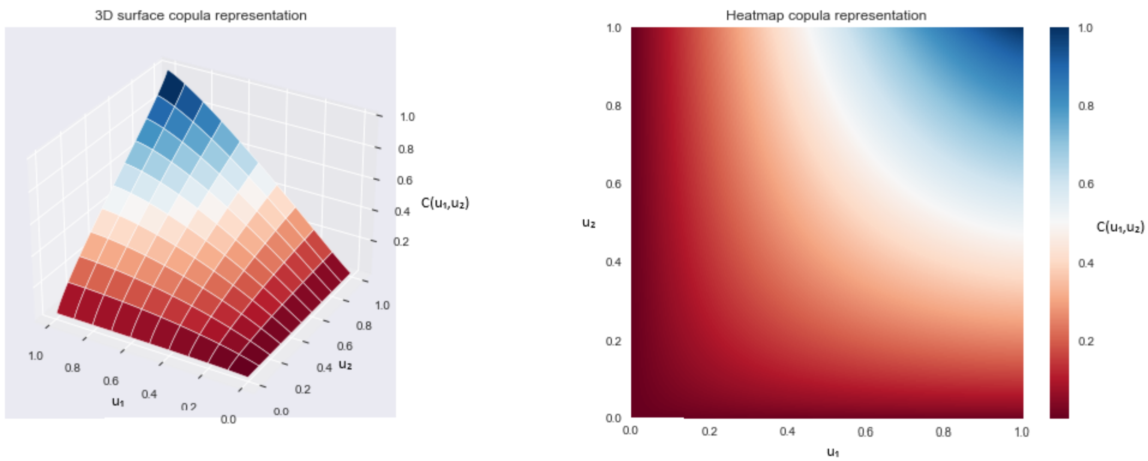

Figure 1: Gaussian copula for $n = 200$

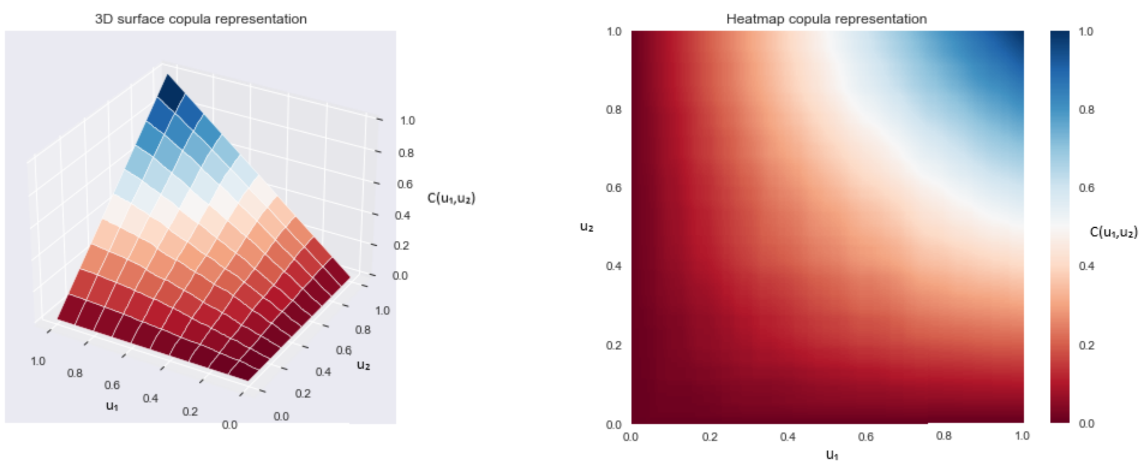

Figure 2: Empirical copula for $n = 200$

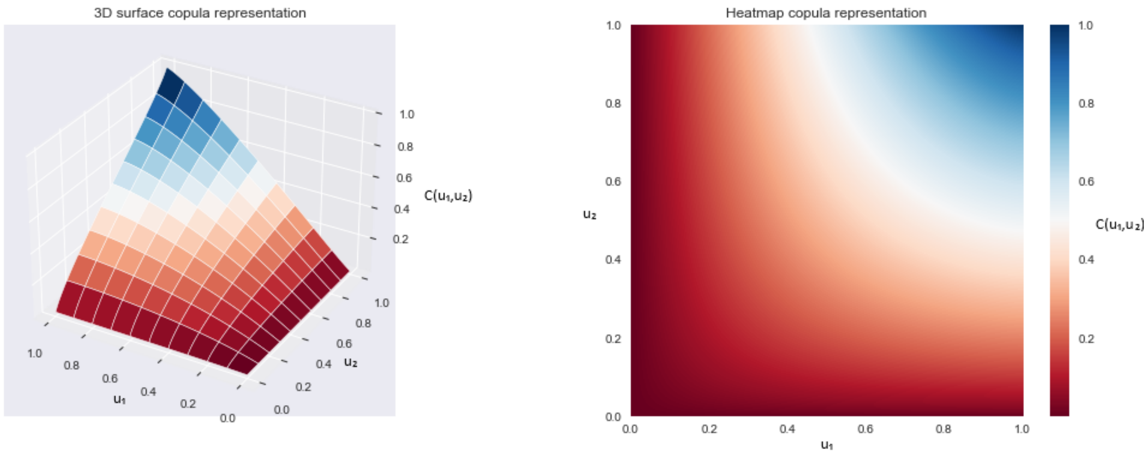

Figure 3: Gaussian copula for $n = 600$

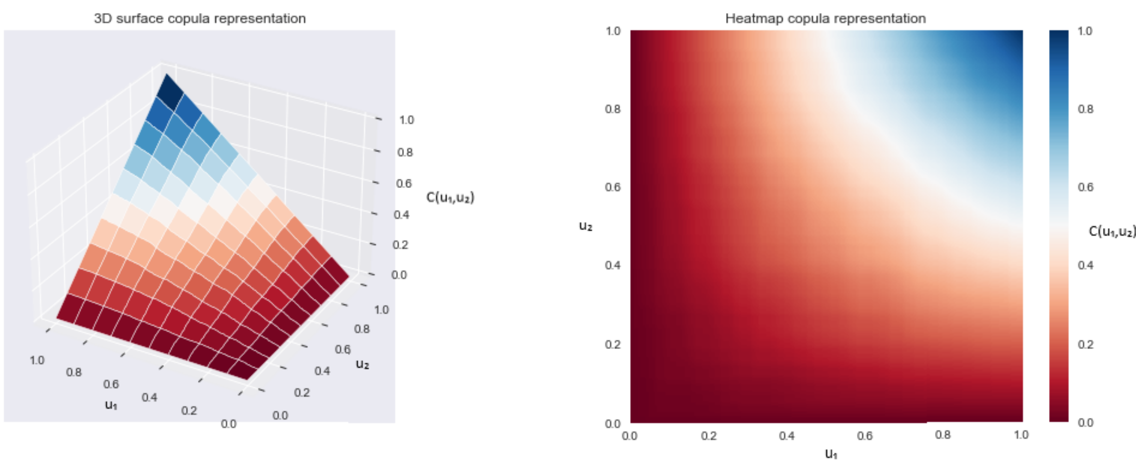

Figure 4: Empirical copula for $n = 600$

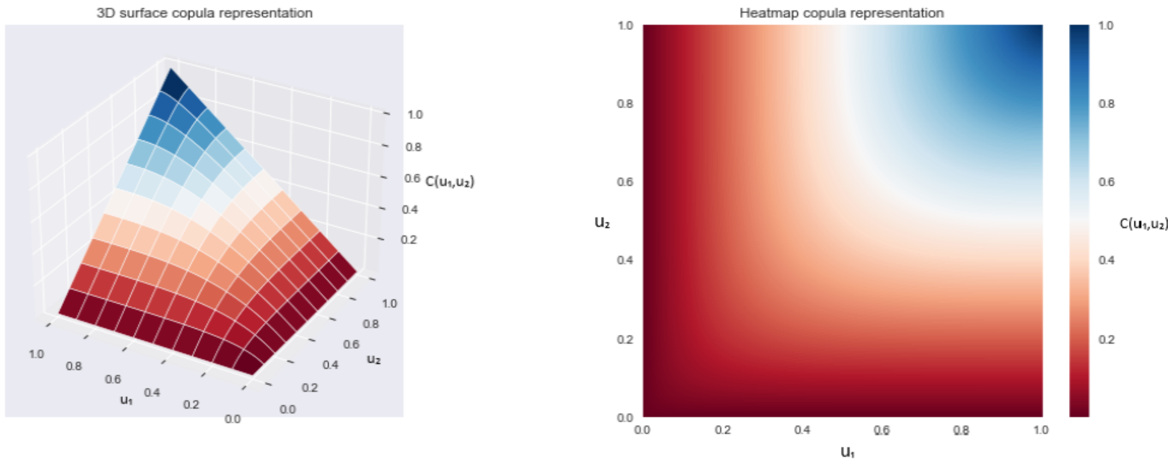

Figure 5: Gaussian copula for $n = 2000$

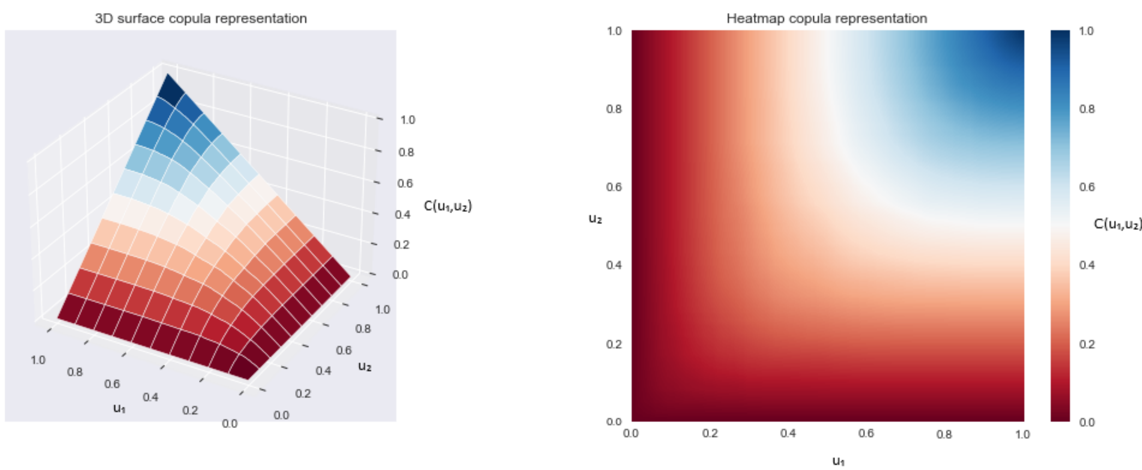

Figure 6: Empirical copula for $n = 2000$

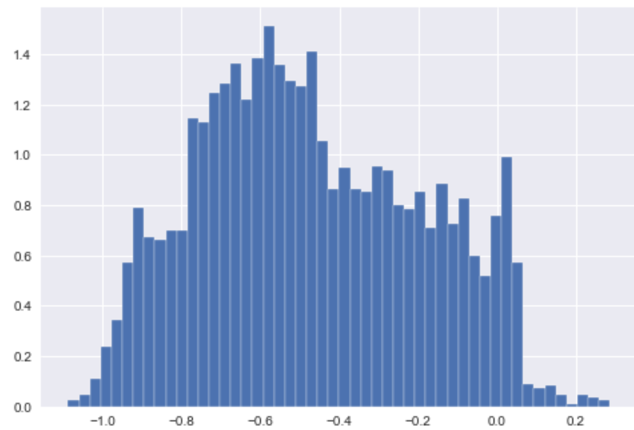

Figure 7: $n = 200$. The distribution of $\sqrt{n}\left\{\hat{C}_n(\mathbf{u}) - C(\mathbf{u})\right\}$ for 4000 draws from Empirical copula and 4000 draws from Gaussian copula

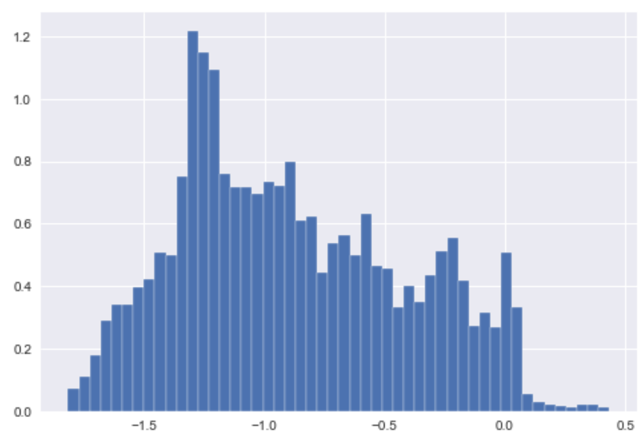

Figure 8: $n = 600$. The distribution of $\sqrt{n}\left\{\hat{C}_n(\mathbf{u}) - C(\mathbf{u})\right\}$ for 4000 draws from Empirical copula and 4000 draws from Gaussian copula

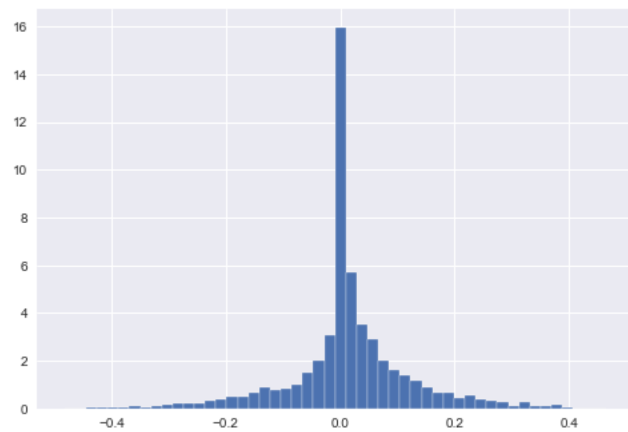

Figure 9: $n = 2000$. The distribution $\sqrt{n}\left\{\hat{C}_n(\mathbf{u}) - C(\mathbf{u})\right\}$ for 4000 draws from Empirical copula and 4000 draws from Gaussian copula

4.2 Empirical rho

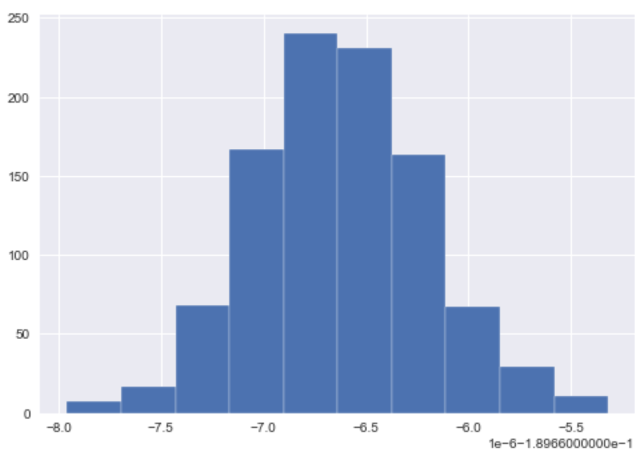

Figure 10: $n = 1000$. The distribution of $\sqrt{n}\,\{\hat{\rho}_{S_d,n}(p) - \rho_{S_d}(p)\}$ for 1000 draws from multivariate normal distribution. Mean $= -0.1897$, Variance $= 1.18476 \cdot 10^{-13}$.

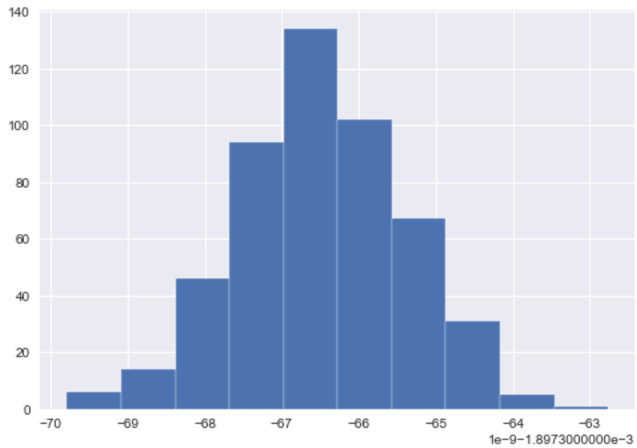

Figure 11: $n = 1000000$. The distribution of $\sqrt{n}\,\{\hat{\rho}_{S_d,n}(p) - \rho_{S_d}(p)\}$ for 1m draws from multivariate normal distribution. Mean $= -0.001897$, Variance $= 1.18476 \cdot 10^{-18}$.

5 Discussion

5.1 Empirical results

From the results, we note that for $d = 2$, $\sqrt{n}\left\{\hat{C}_n(\mathbf{u}) - C(\mathbf{u})\right\}$ tends towards a normal distribution as n increases and that we obtain a normal distribution when $n = 2000$, when the data is sampled from a bivariate normal distribution and $C(\mathbf{u})$ is a Gaussian copula. Also to be noted is that the copula gets smoother and more and more resembles a Gaussian copula as n increases.

For the empirical rho in $d = 2$, we note that $\sqrt{n}\,\{\hat{\rho}_{S_2,n}(\mathbf{u}) - \rho_S(\mathbf{u})\}$ tends towards a normal distribution quite fast, although the mean is slightly negative below zero and that the variance is very small. An interesting note is that when increasing n from 1000 to 1000000 is that the mean decreases by a factor 100, i,e, $\mu_{1000} = 100\mu_{1000000}$ and that the variance decreases by a factor 10^{-5}. Thus it seems that we have converged towards a distribution and that only a scale factor differs when we increase n.

For further research, it would be interesting to look into the results with other choices of the copula. It would also be interesting to investigate if

$\sqrt{n}\left\{\hat{C}_n(\mathbf{u}) - C(\mathbf{u})\right\}$ converges faster towards a normal distribution if the smoother empirical beta copula (49) is used.

5.2 Overall discussion and conclusion

The goal for this work was to understand and clarify some of the propositions and results in Schmidt and Schmidt's paper *Multivariate conditional versions of Spearman's rho and related measures of tail dependence* [4]. First, we gained a deeper understanding of the copula theory and started in Sklar's paper from [11] where copulas were introduced. We explained the Frechlet-Hoeffding bonds and discussed copulas further to gain a deeper understanding. We then analyse and discuss the work of Schmid and Schmidt's *Multivariate conditional versions of Spearman's rho and related measures of tail dependence*. We came up with some examples considering this and clarified how the conditional version was derived and also looked into how the conditional version of Sperman's rho could be applied to other areas of the distribution than to the tail. In the last part of the thesis, we investigated the empirical versions of the copula as well as the proposed Spearman's rho, where we modeled a simple case with samples from a bivariate normal distribution.

From this work we have gained a deeper knowledge of the copula theory and how it can be used for dependence measures. The results also indicate that the theory works and motivates further studies on its applicability with other types of copulas and other datasets.

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