Groups with Noetherian Group Rings

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Abstract

In this thesis we will attempt to classify groups and rings based on whether the associated group rings are Noetherian or not. As this is still an open problem, we present the current state of research. While there are several classes of groups proven not to have Noetherian group rings, such as non-amenable and non-Noetherian groups, so far, virtually polycyclic groups are the only class for which the group ring is known to be Noetherian. We will show some of these major results as well as explore the indications as to whether the unsolved cases might lean one way or another.

Popular Summary

In a sense, abstract algebra is a study of symmetry; rotations, translations, reflections and more. More importantly, it considers how those symmetries interact with each other.

Consider a group of knights sitting at a round table and imagine all the ways a king could move them around without changing who neighbours whom. The main two things he could do would be to have each knight move one seat to the right, or he could have each knight move to the seat across the table. Now, if the king was feeling bored, he might have them first move to the seat opposite and then one seat to the right. He could also have them move one seat to the left and then to the one opposite. A particularly astute king might realize that these two sets of movements end up with the same exact table configurations!

Letting the knights rest for a bit, one could similarly look at the behaviour of rotations and translations of an object in 3D space. Or one could look at ways to shuffle a deck of cards. Is a cut followed by a riffle shuffle the same as a riffle followed by a cut? Why is cutting a deck 17 times the same as cutting it once?

To answer these questions, we create 'groups' and 'rings', among other constructs. They are sets of things, be they ways to move a 3D object, shuffle cards or mess with increasingly rebellious knights. Crucially we place some constraints on how objects in these sets behave. For example, consider how a shuffled deck can always be rearranged to the original order, like how two perfectly even cuts do not shuffle the deck at all. Similarly, turning the knights a step to the right can be undone either by turning them a step to the left, or by turning them to the right until they complete a full circle. This property is called an inverse and it is one of the properties we require every element of a group to have.

The purpose of abstract algebra is to find the commonalities in these examples and find general, abstract results which can then be applied where appropriate. There is a balance to these constraints, the properties we demand the constructs to have. The more constraints there are the easier the constructs are to work with, the more tools we have available. On the other hand, the more constraints we place, the fewer applications there will be, the fewer situations we will find that fulfill all the constraints and let us actually apply our results.

Beyond the illustrative examples, the field has proper applications. They range from algorithms for solving the Rubik's cube to error correcting codes in cryptography. As it turns out, our knights of the round table are quite good at chemistry! Some molecular orbitals follow the same symmetries as the knights and group theory makes it possible to speed up calculations.

The concept of 'Noetherianity' puts a certain size constraint on particularly abstract groups and rings. If at a table of twelve knights we ignore every other knight, we notice that the remaining six behave exactly as they would if they were on a smaller table seating six. Similarly, the shufflings of a small deck of cards consisting only of spades are contained in the shufflings of a full deck. These are called subgroups and some truly enormous groups can have infinite chains of subgroups where the first one is contained in the second, which is

contained in the third and so on without stopping, all of them contained in the original group. Sort of like a Matryoshka doll where if you started in the innermost one, you would never get out. Groups without such an infinite chain are called Noetherian and for rings the definition is similar. Most common groups are, in fact, Noetherian. One particular non-Noetherian group could be represented by all the different ways of shuffling a deck of infinitely many cards! The purpose of this thesis is to survey the current research on the Noetherianity of a specific type of ring, a so-called 'group ring'.

Contents

1 Introduction

The goal of this thesis is to study the conditions on a group G and ring R with unity, under which their group ring will be Noetherian. It covers some more advanced topics in group and ring theory so the reader is expected to be familiar with them at a bachelor level. We show some smaller results and put emphasis on three larger ones. The first result states that the group must be Noetherian for the group ring to be Noetherian. The second one will cover virtually polycyclic groups for which we will prove that their group ring $R[G]$ is Noetherian for every Noetherian R. The last one is a newer result by Kropholler and Lorensen [17], who proved that for domains R, the group ring $R[G]$ is not Noetherian for any non-amenable group G.

The study of commutative and non-commutative rings developed separately. The commutative case emerged from the studies of algebraic number theory, algebraic geometry and invariant theory. As far as non-commutative rings are concerned, the first example comes from the invention of quaternions by Hamilton in 1843. The concept of quaternions was controversial at first. In his response to Hamilton, Graves is quoted as saying "I have not yet any clear view as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties". Despite the initial reluctance however, the mathematical community soon came around to the idea and the field of ring theory grew to be a rich area of inquiry [16].

While advances in the field were being made from the start, it took a while for the first axiomatic definition of a ring to be made by Fraenkel in 1914 [9]. Published by Hilbert in 1890, Hilbert's Basis Theorem [13], would become one of the earliest important results regarding Noetherian rings. The theorem is originally phrased in the context of binary forms as Noetherian rings were yet to be named. That would not occur until Noether studied the field in the 1920s. It was her work as well as that of Artin that finally established abstract ring theory as a branch beyond just rings as a concept.

In order to summarize the main results of the thesis, we proceed with some definitions.

Definition 1.1. A ring R is called *left (right) Noetherian* if it fulfills the *ascending chain condition (acc)* on its left (right) ideals, that is, if there is no infinite ascending chain of ideals I_k of R such that

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots .$$

A ring that is both left and right Noetherian is simply called *Noetherian.*

To clarify this property we show some examples and counterexamples.

Example 1.2. Any finite ring R will be Noetherian, for obvious reasons.

Example 1.3. The ring of integers $\mathbb{Z}$ is Noetherian. Indeed, let there be some infinite ascending chain of ideals

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

of $\mathbb{Z}$. Then take I to be the infinite union of these ideals. Since $\mathbb{Z}$ is a principal ideal domain, the ideal I must be generated by some element $r \in \mathbb{Z}$, that is $I = \langle r \rangle$. There must be some integer k such that $r \in I_k$ but then for every $j \geq k$ we have both $I_j \subseteq I$ and $r \in I_j \Rightarrow I = \langle r \rangle \subseteq I_j$ which gives us $I_j = I$, so the chain is not ascending and $\mathbb{Z}$ is Noetherian.

The above proof is not specific to the ring of integers, indeed it holds for any principal ideal domain.

Many common examples of rings will be Noetherian. It is somewhat difficult to construct counterexamples, as in order to contain an infinite ascending chain, the ring must in a certain sense be very large, like in the following example.

Example 1.4. Let R be the ring of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with addition and multiplication defined as addition and multiplication of functions. Then R is not Noetherian, as seen below.

We can construct a chain of ideals $I_k,\ k = 0, 1, 2, \dots$ as

$$I_k := \{f \in R \mid f(x) = 0 \text{ for } k \leq x \leq \infty\}.$$

These are easily verified to be ideals and clearly $I_k \subsetneq I_{k+1}$, so this is an infinite ascending chain of ideals and R is not Noetherian.

All the examples so far have been of commutative rings so there was no sense in differentiating between left and right Noetherian rings but it is possible for a ring to be left Noetherian without being right Noetherian and vice versa.

Example 1.5. Let R be a subset of 2×2 rational matrices,

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a \in \mathbb{Z}, b, c \in \mathbb{Q} \right\}.$$

Then we will see that R is left but not right Noetherian.

To show that it is left Noetherian define, for some prime p, the sets

$$I_k := \left\{ r \in R \mid r = \begin{bmatrix} 0 & \frac{n}{p^k} \\ 0 & 0 \end{bmatrix}, n \in \mathbb{Z} \right\}.$$

The sets I_k are closed under subtraction, and left multiplication yields

$$ri = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & \frac{n}{p^k} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{an}{p^k} \\ 0 & 0 \end{bmatrix} \in I_k,$$

for $r \in R,\ a \in \mathbb{Z},\ b, c \in \mathbb{Q}$ and $i \in I_k$, so they are left ideals. The sequence of ideals I_k is then infinitely ascending. The proof that R is right Noetherian is long but straightforward. The outline is that for any $R \ni r = \left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right]$, where $a, b, c \neq 0$, multiplication on the right by $\left[\begin{smallmatrix} 0 & 0 \\ 0 & c^{-1} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & a^{-1} \\ 0 & 0 \end{smallmatrix}\right]$ generates $\left[\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{smallmatrix}\right]$, so the proof simplifies to that of Example 1.3.

In this thesis we will mainly focus on left ideals and left Noetherian rings and so, unless specified otherwise, ideals and Noetherian rings will be assumed to be left ideals and left Noetherian rings. All the proofs apply symmetrically to right ideals.

The second concept important to the premise of the thesis is that of a group ring. Informally, a group ring is a module over the elements of the group where the elements of the ring serve as scalars. It must also have a product mapping. More stringently:

Definition 1.6. For a given ring R and multiplicative group G the *group ring* $R[G]$ is a free R-module with elements of the group as a basis. It forms a ring whose elements are finite sums of the form:

$$R[G] \ni \alpha := \sum_{g \in G'} r_g g \text{ for } r_g \in R,$$

where G' is some finite subset of G. Ring addition and multiplication are defined term-wise, i.e. for $\alpha, \beta \in R[G]$

$$\alpha + \beta = \sum_{g \in G'} r_g g + \sum_{g \in G'} l_g g = \sum_{g \in G'} (r_g + l_g) g$$

and

$$\alpha\beta = \sum_{g \in G_1} r_g g \sum_{h \in G_2} l_h h = \sum_{g \in G_3} \left(\sum_{h \in G_4} r_h l_{h^{-1}g} \right) g,$$

where G_i are finite subsets of G.

This construction is closed under addition and multiplication, both operations are associative and addition is commutative. It has a zero element, the empty sum and if R has an identity, $R[G]$ will have a multiplicative identity $1_R 1_G$, therefore the group ring is itself indeed a ring. By abuse of notation, we will usually omit ring or group identities and write group ring elements of the form $\alpha = r1_G$ and $\alpha = 1_R g$ as $\alpha = r$ and $\alpha = g$ respectively. The group ring identity $\alpha = 1_R 1_G$ will therefore be written as $\alpha = 1$.

As one might expect, the group rings will be diverse and their properties will depend on the groups and rings that define them. For small rings and groups, we can write their group ring explicitly.

Example 1.7. Let R be the ring $R = \mathbb{Z}/3\mathbb{Z}$ and $G = \{1, a\}$ the group of two elements. Then the elements of the group ring $R[G]$ are $\{0, 1, 2, a, 1 + a, 2 + a, 2a, 1 + 2a, 2 + 2a\}$. In fact, it can be shown that $R[G]$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$[1].

Example 1.8. Let $R = \mathbb{R}$ be the ring of real numbers and $G = \langle x \rangle$ an infinite cyclic group generated by one element. Then the group ring $R[G]$ will be the Laurent polynomial $\mathbb{R}[X, X^{-1}]$.

[1] The only groups of order 9, up to isomorphism, are $\mathbb{Z}/9\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, and none of these elements have order 9.

Finally, there are three types of groups which will be relevant in this thesis, Noetherian, virtually polycyclic groups and amenable groups. We give their definitions here but we will look into them more closely in Section 2.

Definition 1.9. A group is considered *Noetherian* if every one of its subgroups is finitely generated.

Example 1.10. Any finite group is Noetherian.

Example 1.11. The infinite cyclic group $G = \langle g \rangle$, is finitely generated and every one of its subgroups is isomorphic to itself. Therefore it is Noetherian.

Example 1.12. The lamplighter group L_2 is not Noetherian, as we will see in Subsection 2.8.

Definition 1.13. A group G is *polycyclic* if it contains a series of subgroups,

$$\mathbf{1} = H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_{n-1} \trianglelefteq H_n = G,$$

where a subgroup H_i is not necessarily normal in G but is normal in H_{i+1} and where the quotient groups H_k/H_{k-1} are cyclic for $k \in [2, n]$. Here, $\mathbf{1}$ stands for the trivial group $\{1\}$.

Example 1.14. The group $\mathbb{Z} \times \mathbb{Z}$ is polycyclic with the series $\mathbf{1} \trianglelefteq \mathbb{Z} \trianglelefteq \mathbb{Z} \times \mathbb{Z}$.

Definition 1.15. A group is called *virtually polycyclic* or *polycyclic-by-finite* if it has a subgroup of finite index that is polycyclic.

Example 1.16. The group $\mathbb{Z} \times A_5$, where A_5 is the alternating group of 5 elements is virtually polycyclic, as is shown in Subsection 2.4.

Amenable groups have several equivalent definitions but the one most appropriate for this paper is the following. We cover them more thoroughly and with examples in Subsection 2.5.

Definition 1.17. A group G is *amenable* if it fulfills the *Følner condition*, where for any finite subset $A \subseteq G$ and real $\epsilon > 0$, there is a finite non-empty subset $F \subseteq G$ such that for each $a \in A$

$$\frac{|aF \cup F|}{|F|} < 1 + \epsilon.$$

With these terms in mind we can state the main results on Noetherian group rings, which will be covered in Section 3.

Theorem 3.2. *For any group G and ring R with unity, if the group ring $R[G]$ is Noetherian, then so is G.*

Theorem 3.4. *For any non-zero Noetherian ring R and polycyclic-by-finite group G, the group ring $R[G]$ will be Noetherian.*

Theorem 3.7. *Let R be a ring which admits an ideal whose quotient ideal is a domain and G a group such that the group ring $R[G]$ is Noetherian. Then the group G is amenable.*

2 Preliminaries

2.1 Basic Group and Ring Properties

We show some common concepts from group and ring theory that are brought up later on in the thesis.

Following the terminology of Lam [18], we distinguish between commutative and non-commutative domains.

Definition 2.1. A ring R is a *domain* if it has no zero divisors.

Definition 2.2. A ring R is an *integral domain* if it is commutative and a domain.

An example of the later would be the ring of integers $\mathbb{Z}$, after which the integral domain gets its name. An example of a domain that is not an integral domain would be any division ring. This is because an element cannot be a zero divisor and have an inverse.

Example 2.3. The ring of real quaternions can be defined as a real vector space over vectors $1,\ i,\ j$ and k together with a distributive multiplication function governed by the relation $i^2 = j^2 = k^2 = ijk = -1$, as well as the intuitive $1i = i,\ 1j = j,\ 1k = k$. Explicitly:

$$R := \{a_0 + a_i i + a_j j + a_k k \mid a_0, a_i, a_j, a_k \in \mathbb{R}\}$$

One can see that $ij = k$ but $ji = -k$, so multiplication is not commutative. Each element of the quaternion ring can be shown to have an inverse and therefore the ring of real quaternions is a non-commutative division ring.

The following is a commonly used result in ring theory.

Theorem 2.4 (Correspondence Theorem)**.** *For any ideal I of a ring R there is a bijection between ideals of R which contain I and ideals of the quotient ring R/I.*

Proof. Let $A \supseteq I$ be an ideal of R and $J := \{k + I \mid k \in K\}$ an ideal of R/I for some subset K of R. Then we can set up functions $f(A) = \{a+I \mid a \in A\} \subseteq R/J$ and $g(J) = \{k + i \mid k \in K, i \in I\}$.

As A is closed under multiplication by elements of R, so is $f(A)$ closed under multiplication by $r + I$. Similarly for any $r \in R$ and $k + i \in g(J)$, we have $r(k + i) = rk + ri = rk + i'$ for some $i' \in I$ and since $rk + I \in J$, we have $rk + i' \in g(J)$. Both $f(A)$ and $g(J)$ will be closed under subtraction and therefore they are ideals of R/I and R, respectively, so the outputs of both functions are ideals.

We can now see that

$$g \circ f(A) = g\left(\{a + I \mid a \in A\}\right) = A$$

and

$$f \circ g(J) = f\left(\{k + i \mid k \in K, i \in I\}\right) = J,$$

so f and g are inverses of each other and therefore bijective. $\square$

Another important thing that can be gleamed from the proof of the correspondence theorem is that for two ideals I_1 and I_2 of R/I where $I_1 \subseteq I_2$, their corresponding ideals of R will also contain one another, $g(I_1) \subseteq g(I_2)$.

2.2 Noetherian Rings and Groups

There are several results concerning Noetherian rings we will need in order to more easily work with them. We start with some equivalent definitions.

Theorem 2.5. *For any ring R, the following are equivalent.*

(i) *The ring R is Noetherian.*

(ii) *Every ideal of R is finitely generated.*

(iii) *Every non-empty set of ideals of R has a maximal element.*

Proof. (i) $\Rightarrow$ (ii): If there is an ideal I of R that is not finitely generated, that is $I = \langle r_1, r_2, r_3, \dots \rangle$, then we can form a chain of ideals $I_1 = \langle r_1 \rangle,\ I_2 = \langle r_1, r_2 \rangle, \dots$, which can never stabilize since that would imply that I is finitely generated, so R is not Noetherian.

(ii) $\Rightarrow$ (iii): Let S be some set of ideals of R without a maximal element. Then each ideal in S is properly contained in some other element of S and we can form an infinite ascending chain of ideals $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$. Let $I = \bigcup_{k=1}^{\infty} I_k$. We now prove that I is an ideal. For any $i \in I$, i must be an element of I_k for some k, and since I_k is an ideal, $ri \in I_k \subseteq I$ for any $r \in R$. Similarly for any $i_1, i_2 \in I$, there are some I_{k_1}, I_{k_2} such that $i_1 \in I_{k_1}, i_2 \in I_{k_2}$ and so $i_1 - i_2 \in I_{\max(i_1,i_2)} \subseteq I$. Now since I is an ideal in R, there is some set $\{i_1, i_2, \dots, i_n\} \subseteq I$ which generates it and each of those elements must be contained in some I_k. The largest of these I_k then contains all the generators of I and therefore contains all subsequent ideals in the chain leading to a contradiction.

(iii) $\Rightarrow$ (i): If there were some infinite ascending chain of ideals which would make R not Noetherian then the set of those ideals would have no maximal element, leading to a contradiction. $\square$

Of these, the most relevant property will the second one. It provides a useful alternative way to demonstrate Noetherianity or lack thereof. It also motivates Definition 1.9, of Noetherian groups.

While intuitively it might seem that a group ring should be Noetherian if and only if the group itself is, as will be shown in Section 3, the implication is only one-directional.

To better manipulate them, it is also important to know how Noetherianity of a ring relates to its ideals and quotient rings.

Theorem 2.6. *If a ring R is Noetherian then for any ideal I of R, the quotient ring R/I is Noetherian.*

Proof. If R/I is not Noetherian then we can find some infinite ascending chain of ideals $I_1 \subsetneq I_2 \subsetneq \cdots$ of R/I. Then we use the correspondence theorem (Theorem 2.4) to construct an infinite ascending chain of ideals $g(I_1) \subsetneq g(I_2) \subsetneq \cdots$ of R and therefore R is not Noetherian. $\square$

2.3 Group Rings

There are a couple of useful tools and terms in regards to group rings so we define them here. Often when discussing an element α of a group ring we need to refer to the specific group elements that make up its sum.

Definition 2.7. For a given element α of a group ring $R[G]$, the *support* refers to elements of G which appear in the expression of α. That is,

$$\mathrm{Supp}(\alpha) = \{g \in G : x_g \neq 0\}.$$

If we take the group ring in Example 1.7 the supports of its elements $2, a$ and $1 + 2a$ will be $1, a$ and the whole of group G, respectively. In Example 1.8 we have, for example $\mathrm{Supp}(1 + 2.5x^3 - \pi x^{-7}) = \{1, x^3, x^{-7}\}$. It is not usually possible for the support of an element of a group ring $R[G]$ to be the entire group G. In fact as the elements of $R[G]$ must be finite sums, this is only possible if the order of the group is finite.

The term comes naturally from an alternative interpretation of group rings. The group ring can be seen as the set of all functions $f : G \to R$ with finite support. So $\alpha = 1 + 2.5x^3 - \pi x^{-7}$ can be interpreted as a function $\alpha : G \to \mathbb{R}$, explicitly:

$$\alpha(g) = \begin{cases} 1 & \text{for } g = 1 \\ 2.5 & \text{for } g = x^3 \\ -\pi & \text{for } g = x^{-7} \\ 0 & \text{otherwise} \end{cases}$$

The term 'support' then comes from the fact that $\mathrm{Supp}(\alpha)$ is the support of the function defined by α.

We also define an augmentation map, a common tool in the study of group rings. It will not see too much use in this thesis but it does simplify the proof of Theorem 3.2.

Definition 2.8. For a given group ring $R[G]$, we define the *augmentation map* $\rho : R[G] \to R$ as $\rho(\sum_{g \in G'} x_g g) = \sum_{g \in G'} x_g$.

The augumentation map sums up the ring scalars of a given element, ignoring their associated group elements. So, continuing Examples 1.7 and 1.8, we might have $\rho(2a) = 2$, $\rho(1 + 2a) = 0$ and $\rho(1 + 2.5x^3 - \pi x^{-7}) = 3.5 - \pi$.

2.4 Virtually Polycyclic Groups

We present virtually polycyclic groups more thoroughly here. First let us define a subnormal series.

Definition 2.9. For a group G, a subnormal series is a series of the form

$$\mathbf{1} = H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_{n-1} \trianglelefteq H_n = G,$$

where each subgroup is normal in the next one but not necessarily in G.

A polycyclic group then is a group with a subnormal series whose quotient groups H_{i+1}/H_i are cyclic, cf. Definition 1.13. This series is usually not unique. They are a special case of *solvable groups* - groups with a subnormal series with abelian quotient groups. Similarly to the polycyclic case, a *virtually solvable group* is a group with a solvable subgroup of finite index.

Example 2.10. Let G be the group $G = S_3 \times \mathbb{Z}$, where S_3 is the symmetric group of three elements. Then G is polycyclic. Some of its possible subnormal series are

$$\mathbf{1} \trianglelefteq \{(1), (1,2,3), (1,3,2)\} \trianglelefteq S_3 \trianglelefteq S_3 \times \mathbb{Z} = G, \tag{2.1}$$

$$\mathbf{1} \trianglelefteq 8\mathbb{Z} \trianglelefteq 2\mathbb{Z} \trianglelefteq \mathbb{Z} \trianglelefteq \{(1), (1,2,3), (1,3,2)\} \times \mathbb{Z} \trianglelefteq S_3 \times \mathbb{Z} = G. \tag{2.2}$$

Example 2.11. An alternating group A_n for $n \geq 5$ is not polycyclic. This is due to the fact that alternating groups are simple for $n \geq 5$. Since they themselves are not cyclic the subnormal series $1 \trianglelefteq A_n$ does not have a cyclic quotient group.

Example 2.12. Using Example 2.11, we can construct a virtually polycyclic group $\mathbb{Z} \times A_5$, which will have a polycyclic subgroup $\mathbb{Z}$ of index $|\mathbb{Z} \times A_5 : \mathbb{Z}| = |A_5| = 60 < \infty$. In fact for any polycyclic group G, the group $G \times A_5$ will be virtually polycyclic.

As can be seen in Example 2.10, a polycyclic group can have different subnormal series, even series of different lengths. On a closer look however, one might notice that these series are only superficially different, the series $\mathbf{1} \trianglelefteq 8\mathbb{Z} \trianglelefteq 2\mathbb{Z} \trianglelefteq \mathbb{Z}$ seems particularly artificial. In fact we can expand series (2.1) to

$$\mathbf{1} \trianglelefteq \{(1), (1,2,3), (1,3,2)\} \trianglelefteq S_3 \trianglelefteq S_3 \times 8\mathbb{Z} \trianglelefteq S_3 \times 2\mathbb{Z} \trianglelefteq S_3 \times \mathbb{Z} = G,$$

at which point we can notice that this is very similar to the series (2.2). The quotient groups are the same, only rearranged. This is a consequence of Schreier's refinement theorem [2]. The theorem is not directly relevant to the thesis but we bring it up for its significance to polycylcic and solvable groups. Loosely speaking, it says that any two subnormal series of the same group can be made equivalent in the same manner as those of Example 2.10.

2.5 Amenable Groups

Amenable groups first appear implicitly in the context of the Banach-Tarski paradox from 1924 [1]. It famously states that a three-dimensional ball in $\mathbb{R}^3$ can be split into a finite number of pieces, which upon rearrangement can be made to form two balls equal to the original one, apparently doubling it. Von Neumann was the first to define amenable groups and connect them to the Banach-Tarski result. They would not actually be named amenable until Day came up with the name in the 1940s [7], apparently as a pun. Indeed, Runde claims [23]:

> The first to use the adjective "amenable" was M. M. Day in [Day], apparently with a pun in mind: These groups G are called amenable because they have an invariant mean on $L^\infty(G)$, but also since they are particularly pleasant to deal with and thus are truly amenable - just in the sense of that adjective in colloquial English.

Amenable groups have several different equivalent definitions. We only concern ourselves with the case of discrete groups as they are considered in the Kropholler-Lorensen paper.

Definition 2.13. For a group G, the following are equivalent.

(i) The group G admits a finitely additive, left-invariant probability measure $\mu : \mathcal{P}(G) \to [0, 1]$.

(ii) There is a left-invariant mean on G.

(iii) The group G fulfills the Følner condition, where for any finite subset $A \subseteq G$ and real $\epsilon > 0$, there is a finite non-empty subset $F \subseteq G$ such that for each $a \in A$
$$\frac{|aF \cup F|}{|F|} < 1 + \epsilon.$$

A group fulfilling these conditions is called *amenable*.

In the above definition, *finitely additive* means that for any finite collection of n disjoint subsets I_i of G, the measure of their union equals the sum of their individual measures. That is,

$$\mu\left(\bigcup_{i=1}^{n} I_i\right) = \sum_{i=1}^{n} \mu(I_i).$$

The *left-invariant* property requires that for any subset I of G and any $g \in G$, the measures $\mu(I)$ and $\mu(gI)$ are equal. The *left-invariant mean* is a linear functional F on $\ell^\infty(G)$, the set of bounded sequences of G. It has the properties that $F(1_G) = 1$, where 1_G is the indicator function on G, it is positive for positive functions and $F(f) = F({}_g f)$, where the function ${}_g f$ is defined as ${}_g f(h) := f(gh)$, for $g, h \in G$. The existence of ℓ^∞ presumes some measure on G.

The first definition is the one given by von Neumann and is closely related to the Banach-Tarski paradox. The second was proven to be equivalent to the first one by Day [8]. Finally, the third one was proven by Følner [10] and it is the one we will be using. We mention the first two conditions for posterity and will not cover them in any more detail. We refer the interested reader to an introduction to amenable groups by Garrido [11], from where we extract the following few examples. The proofs we present are by the author. They are different than Garrido's and rely on the Følner condition, in order to avoid measure theory. Garrido's proof of Example 2.14, specifically, is much nicer than the one we present and Example 2.18 is particularly relevant later on for the case of lamplighter groups.

Example 2.14. If, for a group G with a normal subgroup N, groups $N \trianglelefteq G$ and G/N are amenable, then so is G.

Let $A \subseteq G$ be some finite subset of G and $\epsilon > 0$ some positive real number. We start by defining several different sets. Let T be a left transversal of N in G, that is, a set of left coset representatives of N in G. Then every element a of A has a unique representation tn, with $t \in T$ and $n \in N$. Let S be the finite set of cosets of N whose union contains A. As G/N is amenable, let S' be the finite set of cosets such that for any $s \in S$

$$\frac{|sS' \cup S'|}{|S'|} < 1 + \frac{\epsilon}{3}. \tag{2.3}$$

Let $R, R' \subseteq T$ be the sets of representatives of S and S', respectively. Now, let N_0 be the finite subset of N, consisting of elements which appear in the representation of some $a \in A$, that is,

$$N_0 := \{n_0 \in N \mid a = rn_0 \text{ for some } r \in R, a \in A\}.$$

Let $r \in R$, $r' \in R'$ and $n_0 \in N_0$ and consider the product rn_0r'. We can use the fact that N is a normal subgroup to rewrite the product as $rn_0r' = rr'n_1 = tn_2n_1 = tn^*$, for some $n_1, n_2, n^* \in N$ and $t \in T$. We remark that t is in R' if and only if $rNr'N \in S'$.

Now we define the finite set M as the set consisting of n^* for every possible $r \in R, r' \in R'$ and $n_0 \in N_0$. That is,

$$M := \{n^* \in N \mid rn_0r' = tn^* \text{ for some } r \in R, r' \in R', n_0 \in N_0 \text{ and } t \in T\}.$$

As M is a finite subset of the amenable group N, there is some finite subset M' of N such that for any $m \in M$

$$\frac{|mM' \cup M'|}{|M'|} < 1 + \frac{\epsilon}{3}. \tag{2.4}$$

With all the necessary sets defined we proceed with the technical part of the proof. Consider the product $R'M'$. For any $a = rn_0$, where $r \in R$ and $n_0 \in N_0$, we have

$$\begin{aligned} |rn_0R'M' \cup R'M'| &= \left| \left(\bigcup_{r' \in R'} rn_0r'M' \right) \cup R'M' \right| \\ &= \left| \left(\bigcup_{r' \in R'} t_{r'}m_{r'}M' \right) \cup R'M' \right|, \end{aligned}$$

where $t_{r'} \in T$ and $m_{r'} \in M$, such that $rn_0r' = t_{r'}m_{r'}$. Now let $M'_{r'} = m_{r'}M'$ and we continue:

$$\begin{aligned}
\text{LHS} &= \left| \left(\bigcup_{r' \in R'} t_{r'} M'_{r'} \right) \cup R'M' \right| \\
&= |R'M'| + \left| \left(\bigcup_{r' \in R'} t_{r'} M'_{r'} \backslash (R'M') \right) \right| \\
&\leq |R'M'| + \sum_{r' \in R'} |t_{r'} M'_{r'} \backslash (R'M')| \\
&= |R'M'| + \sum_{r' \in R',\, t_{r'} \in R'} |t_{r'} \left(M'_{r'} \cap M'\right) \backslash (R'M')| \\
&\qquad + \sum_{r' \in R',\, t_{r'} \in R'} |t_{r'} \left(M'_{r'} \backslash M'\right) \backslash (R'M')| \\
&\qquad + \sum_{r' \in R',\, t_{r'} \notin R'} |t_{r'} \left(M'_{r'} \cap M'\right) \backslash (R'M')| \\
&\qquad + \sum_{r' \in R',\, t_{r'} \notin R'} |t_{r'} \left(M'_{r'} \backslash M'\right) \backslash (R'M')|
\end{aligned} \tag{2.5}$$

Now we can put some upper bounds on the sums above. First, we have that $|\{r' \in R' \mid t_{r'} \in R'\}| \leq |R'|$ and $|M'_{r'} \cap M'| \leq |M'|$. Then, Equation 2.3 gives us

$$\frac{|sS' \cup S'|}{|S'|} = \frac{|sS' \backslash S'| + |S'|}{|S'|} = \frac{|sS' \backslash S'|}{|S'|} + 1 < 1 + \frac{\epsilon}{3}$$

and so $|sS' \backslash S'| < \frac{\epsilon}{3}||S'|$. Similarly, Equation 2.4 gives us $|M'_{r'} \backslash M'| < \frac{\epsilon}{3}|M'|$. Since $t_{r'}$ is in R' if and only if $rNr'N \in S'$, we get

$$|\{r' \in R' \mid t_{r'} \notin R'\}| = |\{r' \in R' \mid rNr'N \notin S'\}| = |sS' \backslash S'|,$$

where $s = rN$ is the coset containing r. This yields the final inequality

$$|\{r' \in R' \mid t_{r'} \notin R'\}| < \frac{\epsilon}{3}|R'|.$$

Now we can simplify the Inequality 2.5 to

$$\begin{aligned}
LHS < |R'M'| &+ \sum_{r' \in R',\, t_{r'} \in R'} 0 \quad + \sum_{r' \in R',\, t_{r'} \in R'} \frac{\epsilon}{3}|M'| \\
&+ \sum_{r' \in R',\, t_{r'} \notin R'} |M'| + \sum_{r' \in R',\, t_{r'} \notin R'} \frac{\epsilon}{3}|M'| \\
< |R'M'| &+ 0 + \frac{\epsilon}{3}|R'||M'| + \frac{\epsilon}{3}|R'||M'| + \frac{\epsilon^2}{9}|R'||M'|.
\end{aligned}$$

Finally, as R' is a subset of coset representatives of N and $M \subseteq N$, we have $|R'M'| = |R'||M'|$ and therefore for every $a \in A$ and $1 > \epsilon > 0$

$$\frac{|aR'M' \cup R'M'|}{|R'M'|} < 1 + \frac{2}{3}\epsilon + \frac{\epsilon^2}{9} < 1 + \epsilon,$$

so G is amenable.

The remaining examples are much more straightforward.

Example 2.15. Every finite group is amenable. This is a direct consequence of the Følner condition. For any subset A, one can simply choose the whole group as the finite subset F in the definition.

Example 2.16. The additive group of integers $\mathbb{Z}$ is amenable.

Let $A \subseteq \mathbb{Z}$ be some finite subset of $\mathbb{Z}$. Let m be the maximal absolute value of an element of A. For any positive integer n, let F_n be the finite set $F_n := \{x \in \mathbb{Z} \mid -n \leq x \leq n\}$. Then we have, for any $a \in A$:

$$|(a + F_n) \cup F_n| \leq 2n + 1 + m$$

and so

$$\frac{|(a + F_n) \cup F_n|}{|F_n|} \leq \frac{2n + 1 + m}{2n + 1} \leq 1 + \frac{m}{2n + 1}.$$

Since for any $\epsilon > 0$ there is some N such that $\frac{m}{2N+1} < \epsilon$, for every finite $A \subseteq \mathbb{Z}$ we can find a finite group F_N fulfilling the Følner condition.

Example 2.17. All abelian groups are amenable.

By the fundamental theorem of finitely generated abelian groups, every finitely generated abelian group is of the form $\mathbb{Z}^n \oplus H$, for some non-negative integer n and finite abelian group H. If we use Example 2.14 to repeatedly extend the previous two examples, we can see that this is amenable.

That leaves the case of infinitely generated abelian groups. Let be $G = \langle g_1, g_2, \ldots \rangle$ be such a group, with g_i its generators. Let $A \subseteq G$ be some finite subset of G. Then there must be some positive integer n for which the finitely generated abelian subgroup $G' = \langle g_1, \ldots, g_n \rangle$ contains all of A. As G' is amenable, there is some finite subset $F \subseteq G' \subsetneq G$ fulfilling the Følner condition for any $\epsilon > 0$.

Example 2.18. All virtually solvable groups are amenable. This follows directly from Examples 2.17 and 2.14.

Finally, we give an example of a broad category of non-amenable groups without proof, which can be found in the introduction by Garrido.

Example 2.19. Any group containing a subgroup isomorphic to F_2, the free group of rank two, is not amenable.

2.6 Modules and Strong Rank Condition

While we try to avoid them as much as possible, the result by Kropholler and Lorensen which leads to Theorem 3.7 requires an understanding of the strong rank condition and therefore modules. We describe modules here only as much as needed to define the concept of strong rank and prove Theorem 2.29. They are not used anywhere else, so the reader may skip this section and take the result of the theorem for granted. As many of the following concepts have close ring theory analogues, we omit some details.

Definition 2.20. For a given ring R, a *left R-module M* is an additive abelian group paired with a multiplication operation $\cdot : R \times M \to M$ with the following properties.

(i) $(r_1 + r_2) \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_1 \cdot m_2 + r_2 \cdot m_1 + r_2 \cdot m_2$

(ii) $r_1(r_2 \cdot m) = (r_1 r_2) \cdot m$

(iii) $1_R \cdot m = m$

for any $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$. By abuse of notation, we usually drop the multiplication symbol and simply write rm. A *right R-module* is defined similarly, with multiplication being defined as right-hand multiplication by elements of R, instead of left.

From now on, modules can be assumed to be left modules, the statements and proofs for right modules are analogous.

Example 2.21. In an n-dimensional vector space over the reals, the vectors can be added together and scaled by elements of $\mathbb{R}$, so the vector space is an $\mathbb{R}$-module. In general, a module can be seen as a vector space over a ring instead of a field and every vector space is, in fact, a module.

The following simple example of modules is the most relevant one to the thesis.

Example 2.22. Any ideal I of a ring R can be seen as an R-module over R and any ring R can be interpreted as a module over itself.

For the rest of this subsection, R refers to some non-zero ring. When talking about R-modules we will sometimes drop the R and simply call them 'modules', where it does not lead to confusion.

To work with modules we need some more definitions. These should feel familiar.

Definition 2.23. For R-modules A and B we can define their *cross product* $A \times B$ to be the Cartesian product of their underlining sets, with addition defined term-wise and multiplication by elements of R applied to each term individually.

Crucially, for any ring R, the expression R^n represents a cross product $R \times R \times \cdots \times R$ with n terms, with the ring R seen as an R-module.

Definition 2.24. For two R-modules M and N a function $f : M \to N$ is an *R-homomorphism* if it fulfills the following two conditions,

(i) $f(m_1 + m_2) = f(m_1) + f(m_2)$

(ii) $f(rm) = rf(m)$,

for all $m, m_1, m_2 \in M$ and $r \in R$.

If the function is a bijection, it is called an isomorphism and the two modules are said to be *R-isomorphic*, written $M \cong N$. This is easily verified to be an equivalence relation.

Definition 2.25. For some R-module M, an *R-submodule* of M is an abelian subgroup of M that is closed under action of R.

Lemma 2.26. *Let $f : M \to N$ be an R-homomorphism between R-modules M and N. For any submodule A of M, the image $f(A)$ is a submodule of N.*

Proof. Let r be an element of R and $f(x)$ and $f(y)$ any two elements of $f(A)$. Then by the definition of an R-homomorphism, we have $rf(x) = f(rx)$ and $f(x) + f(x + y) = f(x + y)$. As rx and $x + y$ are by definition elements of A, we get that $rf(x), f(x + y) \in f(A)$, so $f(A)$ is an R-submodule of N. $\square$

Now, as given by Lam [18, Section 1.1].

Definition 2.27. A ring R is said to fulfill the *strong rank condition (SRC)* if there is no injective R-homomorphism $f : R^{n+1} \to R^n$ as right R-modules, for any $n \in \mathbb{Z}^+$.

The equivalent definition on left R-modules is called a *left strong rank condition (LSRC)* and it will be the focus of the thesis as the results for SRC are analogous.

As is by now a pattern, a simple example is any ring which is finite and a counterexample is a ring which is very much not.

Example 2.28. [18, Example 1.31] Let R be some non-zero ring. The free algebra $P = R\langle a, b\rangle$ does not fulfill LSRC.

A free algebra is a polynomial ring where multiplication of the indeterminates is non-commutative. More strictly, we define a 'word' w as any finite combination of letters a and b, with order mattering. Multiplication of words is done by concatenating their letters in order, so if $w = a$ and $u = abb$ we have $wu = aabb \neq uw = abba$. Then our ring P can be written as the set of finite sums

$$P := \left\{ \sum_{i=0}^{n} r_i w_i \mid n \in \mathbb{N}_0, r_i \in R \right\},$$

where w_i is any word formed by letters a and b. The addition and multiplication of P are defined as for group rings. Practically, this is just a group ring except the words do not form a group but a semigroup, as they have no inverses.

To show that P does not follow LSRC, we need to find an injective homomorphism $f : P^2 \to P$. For $p_1, p_2 \in P$, we define f as

$$f(p_1, p_2) := (p_1 a, p_2 b),$$

where we add a to the end of every word in p_1 and b to the end of every word in p_2. This is injective, with the inverse function being the one which removes a from the ends of all the words in the first term in the ordered pair and b from the second. It is also a P-homomorphism as for $p, p_1, \ldots, p_4 \in P$ we have $pf(p_1, p_2) = p(p_1 a, p_2 b) = (pp_1 a, pp_2 b) = f(p(p_1, p_2))$ and $f((p_1, p_2)+(p_3, p_4)) = (p_1 a, p_2 b) + (p_3 a, p_4 b) = f(p_1, p_2) + f(p_3, p_4)$ and so P does not satisfy LSRC.

Now that SRC is properly defined, the following theorem is the only result we need to interpret Kropholler and Lorensen.

Theorem 2.29. *[18, Theorem. 1.35] Non-zero left Noetherian rings satisfy LSRC.*

In order to prove this we need to delve deeper into module theory. One can define Noetherianity of a module in an analogous manner to that of rings.

Definition 2.30. Let R be a ring. An R-module M is *Noetherian* if there it has no infinite ascending chain of submodules.

$$M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \cdots .$$

Similarly to rings this condition can also be equivalently stated in terms of finitely generated submodules.

Definition 2.31. Let M be an R-module and S some subset of M. We define the *submodule generated by* S as the set of all finite sums of the form

$$\sum_{s \in S'} r_s s,$$

where $r_s \in R$ and S' is some finite subset of S. This can straightforwardly be seen to indeed be a submodule.

A submodule that is generated by some finite subset we call *finitely generated*.

Any (non-)Noetherian R ring seen as an R-module will be a (non-)Noetherian R-module, as its ideals will be submodules. As always, a module with finitely many elements is Noetherian. There are, of course, more interesting examples.

Example 2.32. For any infinite set of indeterminates $S = \{x_1, x_2, \ldots\}$ the R-module generated by S will not be Noetherian, as it will have an infinite ascending chain of submodules $\langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \cdots$.

We can then frame Noetherianity of R-modules in three different ways. We omit the proofs as they are identical to those of Theorem 2.5.

Theorem 2.33. *For any R-module M, the following are equivalent.*

(i) *The module M is Noetherian.*

(ii) *Every submodule of M is finitely generated.*

(iii) *Every non-empty set of submodules of M has a maximal element.*

With these in hand, we can finally prove Theorem 2.29. We start with a lemma[2].

Lemma 2.34. *For any Noetherian ring R, the module R^n is Noetherian.*

Proof. We proceed by induction on n.

For $n = 1$ each submodule of R seen as an R-module is an ideal of R seen as a ring, so the module R^1 is Noetherian.

Now assume that R^n is Noetherian for some n. We can view R^{n+1} as a direct product $R^n \times R$. Let M be some submodule of R^{n+1}. We define the following two sets:

$$A := \{r_1 \in R^n \mid (r_1, 0) \in M\} \subseteq R^n$$
$$B := \{r_2 \in R \mid (r_1, r_2) \in M \text{ for some } r_1 \in R^n\} \subseteq R$$

Let $r_1, r_1' \in R^n$ and $r \in R$. We have $(r_1, 0) + (r_1', 0) = (r_1 + r_1', 0) \in M \Rightarrow r_1 + r_1' \in A$ and $r(r_1, 0) = (rr_1, 0) \in M \Rightarrow rr_1 \in A$, so A is an R-submodule of R^n. Similarly, we get that B is a submodule of R.

Since R^n and R are Noetherian R-modules, by the inductive assumption, sets A and B are finitely generated. Let $A = \langle a_1, \ldots, a_k \rangle$ and $B = \langle b_1, \ldots, b_l \rangle$. Then define the set A' of all the elements of the type $(a_i, 0)$ and the set B' which for each b_i contains an element (x_i, b_i), where x_i are some arbitrary elements of R^n such that (x_i, b_i) are in M. We will show that $S = A' \cup B'$ is the generating set of M.

Clearly, $\langle S \rangle \subseteq M$. For any element (r_1, r_2) of M, we can find an element (x, r_2) of $\langle B' \rangle$ for some x. Then we have that $(r_1, r_2) - (x, r_2) = (r_1 - x, 0) \in M$. The element $(r_1 - x, 0)$ is generated by A' and therefore the sum $(r_1, r_2) = (r_1 - x, 0) + (x, r_2)$ is in $\langle S \rangle$, so M is finitely generated and R^{n+1} is Noetherian. □

We are finally ready to prove Theorem 2.29.

Proof of Theorem 2.29. Let R be a Noetherian ring and let there be some $n \in \mathbb{Z}^+$ and injective R-homomorphism $f : R^{n+1} \to R^n$.

Then the R-module R^n contains a submodule which is isomorphic to R^{n+1}, call it A_1. We then have that $A_1 \cong R \times R^n$, so we can write $R^n \supseteq A_1 \cong R_1 \times B_1$, where $R_1 \cong R$ and $B_1 \cong R^n$. Then we can again find some submodule A_2 of B_1, again isomorphic to R^{n+1}, so that $R^n \supseteq R_1 \times A_2$.

[2]The proof is adjusted from the slightly more general result of Lemma 18.4.2 in [5].

Repeating this process we obtain the infinite cross product $R^n \supseteq R_1 \times R_2 \times R_3 \times \cdots$ where each R_i is an isomorphic copy of R. Then if we define $M_k := R_1 \times \cdots \times R_k$, the sequence

$$M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \cdots$$

is an infinite ascending sequence of submodules of R^n. Therefore R^n is not a Noetherian R-module which is in contradiction with Lemma 2.34. $\square$

2.7 Rings Graded by Groups

The result by Kropholler and Lorensen uses rings graded by groups, so we define them here.

Definition 2.35. A *ring R strongly graded by a group G*, is a ring which can be written as a direct sum of additive abelian subgroups $R = \bigoplus_{g \in G} R_g$ such that the product of any two subgroups $R_g R_h = R_{gh}$.

The example relevant to this thesis is that of group rings themselves. A group ring $R[G]$ is the direct sum $R = \bigoplus_{g \in G} Rg$ and clearly $RgRh = Rgh$. The subring $R_{1_G} = R1_G$ assigned to the identity element 1_G will be isomorphic to R itself. Other examples might be polynomial rings or ring algebras, for similar reasons.

2.8 Lamplighter Group L_2

Here we define and showcase the lamplighter group L_2, as it will serve as a useful example later on. The group has different uses and applications, for more we direct the interested reader to [6].

Definition 2.36. The *lamplighter group, L_2*, is constructed as follows.

Let there be an infinite line of lamps, labeled by integers, where each lamp can be turned on or off. There is a lamplighter, starting at lamp numbered 0, which can move a step to the right (t), a step to the left (t^{-1}) or toggle a lamp (a). An element of group L_2 is any finite sequence of moves and toggles by the lamplighter, where two elements are considered equivalent if they lead to the same configuration of lamps and leave the lamplighter at the same final position.

The group operation of two elements $g, h \in L_2$ is defined as simply a merger of the two sequences, the sequence of h is executed directly after that of g to obtain gh.

Essentially, in the product gh, the sequence of h is done as usual just with a new starting position, determined by where g leaves the lamplighter. This is clearly associative, as $g(hk)$ and $(gh)k$ yield the same sequence. There is a group identity element, the empty sequence. Every element also has an inverse $g^{-1} = t^k g t^{-k}$, where k is the integer such that t^k is the translation which returns the lamplighter to 0 after sequence g. This is because toggling each lamp an even number of times returns it to the original state. Therefore L_2 is indeed a group. The group L_n is defined the same way except each lamp has n possible states the lamplighter cycles through. We focus on L_2 from here on out.

Now that we have defined it, we will show that it is solvable and not Noetherian. For that we will need the commutator group.

Definition 2.37. For a group G, the *commutator subgroup* $[G, G]$ is the subgroup generated by all the *commutators* $[g, h] = ghg^{-1}h^{-1}$ for $g, h \in G$.

For S_3, the symmetric group of three elements, the commutator subgroup can be shown to be $[S_3, S_3] = \{(1), (1\,2\,3), (1\,3\,2)\}$. For any abelian group G, its commutator subgroup will be the trivial group $\{1\}$. A non-abelian group can equal its commutator subgroup, such a group is called perfect. For example the commutator subgroup of the alternating group of five elements, A_5, is not empty, so as A_5 is simple $[A_5, A_5]$ must equal A_5.

It is a commonly used subgroup in group theory because of its following useful property, among others.

Theorem 2.38. *For every group G, the commutator subgroup is normal and the quotient group $G/[G, G]$ is abelian.*

Proof. Let $N := [G, G]$. For any product gn where $g \in G, n \in N$, we can 'move' g to the right as follows $gn = gng^{-1}n^{-1}ng = n'ng = n''g$, where $n', n'' \in N$, with $n'' = n'n$. The middle equality follows from the fact that $n' := gng^{-1}n^{-1} = [g, n] \in N$.

Let $n \in N$ and $g \in G$. Then by the above we have $gng^{-1} = n''gg^{-1} \in N$, so N is normal.

Let $anbna^{-1}nb^{-1}n$ be a representative of the product of the cosets $aNbNa^{-1}Nb^{-1}N$. Then, same as above, we can move a, b, a^{-1} and b^{-1} one by one all the way to the right to obtain $anbna^{-1}nb^{-1}n = n''aba^{-1}b^{-1}$ for some $n'' \in N$. Again, $aba^{-1}b^{-1}$ is an element of N, so $anbna^{-1}nb^{-1}n \in N$ and $aNbNa^{-1}Nb^{-1}N = 1N$, so G/N is commutative. $\square$

Returning to the lamplighter group, we examine its commutator subgroup. We will call an element of G a 'resetting element' if, after executing its sequence, it returns the lamplighter to the same place where he started. It can be easily seen that every resetting element is its own inverse. Every element of $[L_2, L_2]$ is a resetting element. Indeed if $g \in L_2$ and $h \in L_2$ move the lamplighter by a and b, respectively, then $ghg^{-1}h^{-1}$ moves him by $a + b - a - b = 0$. Now we can show our two important results, the latter one being a proof of Example 1.12. These proofs are by the author although certainly not new, as the results are referred to matter-of-factly by others [4].

Theorem 2.39. *The lamplighter group L_2 is solvable.*

Proof. Its subnormal series is

$$\mathbf{1} \trianglelefteq [L_2, L_2] \trianglelefteq L_2.$$

Theorem 2.38 gives us the normality of $[L_2, L_2]$ and the abelian property of the quotient group $L_2/[L_2, L_2]$. All that is left is prove that the commutator subgroup is abelian.

This is straightforward once we notice that, since all the elements of $[L_2, L_2]$ are resetting, they essentially 'ignore' each other. For any resetting elements $g, h \in [L_2, L_2]$, in the sequence $ghg^{-1}h^{-1}$ the element h will return the lamplighter where he started, so g and $g^{-1} = g$ will toggle each lamp the same number of times. The same thing happens for elements h and h^{-1} with the resetting element g^{-1} between them. Therefore every lamp will be toggled an even number of times and stays unchanged, so $ghg^{-1}h^{-1} = 1_{[L_2,L_2]}$ and $[L_2, L_2]$ is abelian. □

Theorem 2.40. *The lamplighter group L_2 is not Noetherian.*

Proof. Its commutator subgroup is not finitely generated. Indeed, let $N := [L_1, L_2] = \langle g_1, \dots, g_n \rangle$ for some $n \in \mathbb{Z}^+$.

As every element of N is resetting, in any finite composition of elements $n_1 n_2 \cdots n_k,\ n_i \in N$, each n_i has the lamplighter start at zero, so each n_i will affect the same lamps as it would on its own. Now as each g_i toggles a finite number of lamps and n is finite, there exists a rightmost lamp that is toggled by the generators of N. Since each element of N is a finite composition of its generators, there is no element of N which toggles any lamp beyond that one.

This is of course a contradictions as if we let $g, h \in L_2$, where $g = t^k a$ and $h = t$, the commutator $[g, h] = t^k atat^{-k+1}$ will toggle lamps k and $k + 1$ for arbitrarily large k. □

3 Main Results

We start by imposing some basic limits on the type of ring that can form a Noetherian group ring. The first proof is by the author, although the result is well-known.

Theorem 3.1. *If, for a group G and ring R, the group ring $R[G]$ is Noetherian then so is the ring R.*

Proof. Let $R[G]$ be Noetherian and R not Noetherian. Then there is some infinite ascending chain of ideals $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots \subsetneq R$. Now for each ideal I_k we can construct a group ring $I_k[G]$. Then we prove that $I_k[G]$ are ideals of $R[G]$.

For any two $y_1, y_2 \in I_k[G]$, the elements y_1 and $-y_2$ can be written as finite sums of the elements of the form $i_g g$ where $i_g \in I_k$ and $g \in G$, therefore so can the difference $y_1 - y_2$. Now let G' and H' be finite subsets of G, such that

$$I_k[G] \ni y = \sum_{g \in G'} i_g g,$$

$$R[G] \ni x = \sum_{h \in H'} r_h h.$$

Consider the product xy

$$\begin{aligned} xy &= \left(\sum_{h \in H'} r_h h \right) \left(\sum_{g \in G'} i_g g \right) \\ &= \sum_{h \in H'} \sum_{g \in G'} r_h i_g h g \\ &= \sum_{h \in H'} \sum_{g \in G'} i_{h,g} h g \end{aligned}$$

where $i_{h,g} = r_h i_g \in I_k$ since I_k is an ideal of R.

Since they are closed under subtraction and multiplication by elements of $R[G]$, group rings $I_k[G]$ are its ideals. The infinite ascending chain $\{I_n\}$ then generates an infinite ascending chain of ideals $I_0[G] \subsetneq I_1[G] \subsetneq I_2[G] \subsetneq \cdots$ of $R[G]$ and therefore $R[G]$ is not Noetherian. □

With this restriction on R, we show our first important restriction on G. It is found concisely in [21]. The proof has been expanded for clarity.

Theorem 3.2. *[21, Lemma 10.2.2] For any group G and ring R with unity, if the group ring $R[G]$ is Noetherian, then so is G.*

Proof. Let G be a non-Noetherian group and R some non-zero ring. Then there is some subgroup $H \leq G$ that is not finitely generated. Since H is not finitely generated, we can construct some infinite sequence of elements $\{h_i\}$ such that the subgroups $\{H_i\}$ constructed by taking $H_i = \langle h_1, \ldots, h_i \rangle$ form an infinite sequence of proper subgroups $H_1 < H_2 < \cdots$ of H.

Now for each subgroup H_k let I_k be an ideal of $R[G]$ generated by the elements $(h_i - 1) \in R[G]$. If we can prove that these ideals are all different we will have obtained an infinitely ascending sequence of ideals of $R[G]$ so $R[G]$ will not be Noetherian either.

We will prove that for every positive integer k, we have $(h_{k+1} - 1) \notin I_k$. Let us assume otherwise for some k. Every element α of I_k is of the form

$$\alpha = \sum_{i=0}^{k} \alpha_i (h_i - 1), \quad \alpha_i \in R[G]$$

and we have

$$\sum_{i=0}^{k} \alpha_i (h_i - 1) = (h_{k+1} - 1).$$

Now when we write out each α_i as a separate sum, we get

$$(h_{k+1} - 1) = \sum_{i=0}^{k} \left(\sum_{g_i \in G_i} x_{g_i} g_i (h_i - 1) \right),$$

where $x_{g_i} \in R$ and G_i is some finite subset of G. If n is the total number of summands in the above sum, we can rewrite it as a sum over a finite sequence $\{a_i\}_n$ of elements of G

$$(h_{k+1} - 1) = \sum_{i=0}^{n} x_{a_i} a_i (h_{a_i} - 1), \tag{3.1}$$

where $x_{a_i} \in R$ and $h_{a_i} \in \{h_1, \ldots, h_k\}$ are the corresponding variables in each summand. Now we notice that on the left-hand side we have $h_{k+1} \notin H_k$ and $1 \in H_k$. On the other hand, on the right-hand side the support of the summand $x_{a_i} a_i (h_{a_i} - 1)$ will be in H_k if and only if $a_i \in H_k$. Therefore we can split the sequence $\{a_i\}_n$ into two sequences $\{b_i\}_{n_1}$ and $\{c_i\}_{n_2}$ where $b_i \in H_k$ and $c_i \notin H_k$ and $n_1 + n_2 = n$.

This lets us split Eq. (3.1) into two equations

$$\begin{aligned} -1 &= \sum_{i=0}^{n} x_{b_i} b_i (h_{b_i} - 1) \\ h_{k+1} &= \sum_{i=0}^{n} x_{c_i} c_i (h_{c_i} - 1). \end{aligned}$$

Finally, applying the augmentation map to either of the two equations gives us zero on the right-hand side and ± 1 on the left-hand side, yielding a contradiction, so there is an infinite ascending sequence of ideals $\{I_k\}_\infty$ and $R[G]$ is not Noetherian. $\square$

Now we move on to the proof of Theorem 3.4 on virtually polycyclic groups. It was first proven by Hall [12] in 1954. We present a version of the proof as presented by Putman [22], without using modules. It requires the following lemma, the proof of which is very similar to that of Lemma 2.34[3].

Lemma 3.3. *Let G be a group and R a ring. If there is some subgroup H of G such that $|G : H| < \infty$ and $R[H]$ is Noetherian, then $R[G]$ is Noetherian.*

Proof. Let $\{g_i\}_{i=1}^k$ be the set of representatives of right cosets of H in G, where $|G : H| = k$. Then we can write $R[G]$ as the direct sum

$$R[G] = R[H]g_1 \oplus \cdots \oplus R[H]g_k.$$

Any element $\alpha \in R[G]$ can be uniquely written as a sum $\alpha_1 g_1 + \cdots + \alpha_k g_k$, where $\alpha_i \in R[H]$ or in coordinate form $\alpha = (\alpha_1, \ldots, \alpha_k)$. Now let I be any ideal of $R[G]$, by Theorem 2.5 we need to show that I is finitely generated.

We define the following sets:

$$\begin{aligned}
&A_1 := \{\alpha_1 \in R[H] \mid (\alpha_1, 0, 0, \ldots, 0) \in I\} \\
&A_2 := \{\alpha_2 \in R[H] \mid (\alpha_1, \alpha_2, 0, \ldots, 0) \in I \text{ for some } \alpha_1 \in R[H]\} \\
&A_3 := \{\alpha_3 \in R[H] \mid (\alpha_1, \alpha_2, \alpha_3, 0, \ldots, 0) \in I \text{ for some } \alpha_1, \alpha_2 \in R[H]\} \\
&\quad\vdots \\
&A_k := \{\alpha_k \in R[H] \mid (\alpha_1, \alpha_2, \ldots, \alpha_k) \in I \text{ for some } \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \in R[H]\}
\end{aligned}$$

Now we will show that these sets are ideals of $R[H]$. Pick one of them, A_i. For any element α of A_i, let α' be an element of I which causes α to be included in A_i, that is an element with α at the i-th coordinate and zeros after it. Now let α, β be elements of A_i. Since I is an ideal of $R[G]$, we have $\alpha' - \beta' \in I$ and therefore $\alpha - \beta \in A_i$ with $(\alpha - \beta)' = \alpha' - \beta'$.

Similarly, let $\alpha \in A_i$ and $\gamma \in R[H]$. Then since I is an ideal, $\gamma\alpha'$ is in I. Writing out $\gamma\alpha'$, we get

$$\begin{aligned}
\gamma\alpha' &= \gamma(\alpha_1 g_1 + \cdots + \alpha g_i + 0 g_{i+1} + \cdots + 0 g_k) \\
&= (\gamma\alpha_1 g_1 + \cdots + \gamma\alpha g_i + 0 + \cdots + 0) \\
&= (\beta_1 g_1 + \cdots + \beta_{i-1} g_{i-1} + \gamma\alpha g_i + 0 + \cdots + 0) \\
&= (\beta_1, \ldots, \beta_{i-1}, \gamma\alpha, 0, \ldots, 0),
\end{aligned}$$

where $\beta_j = \gamma\alpha_j$ for $j < i$ is an element of $R[H]$.

So $\gamma\alpha$ is an element of A_i and A_i is an ideal of $R[H]$. As $R[H]$ is Noetherian this means that each A_i is finitely generated. Let $\alpha_{(i,j)}$ be the generators of A_i. Then for each A_i we can form a set B_i of elements of the form $(*, *, \ldots, *, \alpha_{(i,j)}, 0, \ldots, 0)$, for each $\alpha_{i,j}$ in the generator of A_i, where $*$ stands for some element of $R[H]$.

[3] Both proofs are very similar to that of Lemma 18.4.2 in [5].

We will show that I is generated by the union of all B_i. Let $\beta = (\beta_1, \dots, \beta_k)$ be some element of I. Using elements of B_k, we can construct an element $\gamma_k = (*, \dots, *, x, \beta_k)$, where $*$ and x stand for some elements of $R[H]$. Then since $\beta, \gamma_k \in I$, we have that $\beta - \gamma_k = (*, \dots, *, \beta_{k-1} - x, 0) \in I$ and therefore $(*, \dots, *, \beta_{k-1} - x, 0) \in \langle B_{k-1} \rangle$. Adding this element to γ_k, we obtain $\gamma_{k-1} = (*, \dots, *, x, \beta_{k-1}, \beta_k)$, with some new x. Repeating this process, descending to B_1, we eventually obtain $\gamma_1 = \beta$, so I is finitely generated by $\bigcup B_i$ and $R[G]$ is Noetherian. $\square$

Theorem 3.4. *[12, Theorem 1] For a Noetherian ring R with unity and a virtually polycyclic group G, the group ring $R[G]$ is Noetherian.*

Proof. We prove the case for the polycyclic subgroup H of G such that $|G : H|$ is finite. Lemma 3.3 then completes the proof.

Since H is polycyclic, let $\{H_k\}$

$$\mathbf{1} = H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_{n-1} \trianglelefteq H_n = H$$

be a subnormal series of H with quotient groups cyclic. We proceed by induction on the length of this series.

For the base case, $n = 1$, we have $H_n = \{1\}$, so $R[H_1] = R$ is Noetherian by the premise.

Now let $R[H_n]$ be Noetherian for some n and let $I \subseteq R[H_{n+1}]$ be an ideal of $R[H_{n+1}]$. We need to prove that I is finitely generated. The quotient group H_{n+1}/H_n is cyclic, so there is some coset xH_n which generates it, where $x \in H_{n+1}$ is some representative of that coset. Then for every $h_{n+1} \in H_{n+1}$ there is some $i \in \mathbb{Z}$ and some $h_n \in H_n$ such that $h_{n+1} = x^i h_n$. Then for every $\alpha \in R[H_{n+1}]$ we have $\alpha = \sum_{i \in C} x^i \beta_i$ for some finite set of integers C and $\beta_i \in R[H_n]$. These representations are not necessarily unique. We can call those representations that do not include any negative powers of x polynomials, the largest powers of x in a polynomial its degree and the coefficient $\beta_i \in R[H_n]$ in front of its largest power of x its leading coefficient.

Now let $L_j \subseteq R[H_n]$ be the union of $\{0\}$ and the set of all elements of $R[H_n]$ which appear as leading coefficients for some polynomial representation of degree j of some $\alpha \in I$. We prove that these L_j are ideals of $R[H_n]$.

Let $p, q \in I$ be two polynomials of degree j, with β_p and β_q their leading coefficients and β some element of $R[H_n]$. Then $p - q \in I$ and either the degree of $p - q$ is j or $\beta_p - \beta_q = 0$. In either case, $\beta_p - \beta_q \in L_j$.

Now consider $\beta \in R[H_n]$. It is of the form

$$\beta = \sum_{h \in H_n'} r_h h,$$

where H_n' is some finite subset of H_n and $r_h \in R$. Since H_n is a normal subgroup of H_{n+1} we can, for every i, write

$$x^{-i} \beta x^i = \sum_{h \in H_n'} x^{-i} r_h h x^i = \sum_{h \in H_n'} r_h h'$$

for some $h' \in H_n$. Alternatively, for any i, any $\beta \in R[H_n]$ can be written as $\beta = x^i \beta' x^{-i}$ for some $\beta' \in R[H_n]$. Now let $\beta = x^{-j}\beta' x^j$. Consider the product $\beta' p \in I$, where the polynomial p has the representation $\sum_{k=0}^{j} x^k \gamma_k$, with $\gamma_k \in R[H_k]$ and $\gamma_j = \beta_p$:

$$\begin{aligned}\beta' p &= \sum_{k=0}^{j} \beta' x^k \gamma_k \\ &= \sum_{k=0}^{j} x^k \beta'_k x^{-k} x^k \gamma_k \\ &= \sum_{k=0}^{j} x^k \beta'_k \gamma_k\end{aligned}$$

Here, β'_k are elements of $R[H_k]$ such that $\beta' = x^k \beta'_k x^{-k}$. The leading coefficient in this polynomial is of course $\beta\gamma_j = \beta\beta_p$, so finally we have that $\beta_p - \beta_q \in L_j$ and $\beta\beta_p \in L_j$, so L_j is a left ideal of $R[H_k]$.

Now as $1_R x \in R[H_{k+1}]$, multiplying an element of I by $1_R x$ gives a new element of I with the same leading coefficient but of one degree higher. Therefore $\beta \in L_j \Rightarrow \beta \in L_{j+1} \Rightarrow L_j \subseteq L_{j+1}$. This gives an ascending sequence of ideals of $R[H_k]$, $L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$. By the induction hypothesis, there must be some j such that $L_j = L_{j'}$ for all $j' > j$. The ideals L_1 through L_j must be finitely generated so we can construct a finite set P by putting in it all the polynomials $p_1, p_2, \ldots, p_{l_i}$ in I whose leading coefficients generate L_i for $1 \leq i \leq j$. Let I' be the ideal generated by P. We will prove that $I = I'$.

Clearly $I' \subseteq I$. Now let $\alpha \in I$. For any representation of α there is some i such that $x^i\alpha$ is a polynomial. Let its degree be n. Since polynomials in P generate all leading coefficients for polynomials of all degrees in I, there is some polynomial β_n in I' of degree n with the same leading coefficient as $x^i\alpha$, and if we subtract $x^i\alpha - \beta_n$ we get a new element of I of smaller degree. Continuing this process to degree zero we get that $x^i\alpha = \beta_n + \cdots + \beta_1 \in I' \Rightarrow \alpha \in I'$, which gives us $I \subseteq I' \Rightarrow I = I'$ and I is finitely generated which completes the proof. □

Finally, to prove Theorem 3.7, we arrive to the newest theorem regarding the topic, by Kropholler and Lorensen. The proof itself is beyond the scope of this thesis, so we merely present their result.

Theorem 3.5. *[17, Theorem A] Let G be a group and R a ring strongly graded by G such that R_1 is a domain. Then the following two statements are equivalent.*

(i) *R satisfies (L)SRC.*

(ii) *R_1 satisfies (L)SRC and G is amenable.*

It is the $(i) \Rightarrow (ii)$ implication that is relevant for Theorem 3.7. Combined with Theorem 2.29 and in the language of group rings it directly gives us the following lemma.

Lemma 3.6. *Let G be a group and R a domain. If the group ring $R[G]$ is Noetherian then the group G is amenable.*

We can expand this somewhat to obtain Theorem 3.7. The proof is by the author.

Theorem 3.7. *Let R be a ring which admits an ideal whose quotient ideal is a domain and G a group such that the group ring $R[G]$ is Noetherian. Then the group G is amenable.*

Proof. Let G be some non-amenable group. Let I be the ideal such that R/I is a domain. We then construct the ideal $I[G]$ of $R[G]$ and a group ring $(R/I)[G]$. It is straightforward to show that $R[G]/I[G] \cong (R/I)[G]$. Indeed let $f : (R/I)[G] \to R[G]/I[G]$, where

$$f\left(\sum_{g \in G'} (r_g + I)g\right) = \sum_{g \in G'} r_g g + I[G]$$

and the inverse of f is given by

$$f^{-1}\left(\left(\sum_{g \in G'} r_g g\right) + I[G]\right) = \sum_{g \in G'} (r_g + I)g.$$

This is a bijection and to prove the isomorphism we have to check that it is a homomorphism, which is straightforward although a bit unwieldy.

Let $\alpha, \beta \in (R/I)[G]$. If G' and H' are finite subsets of G, we have

$$\begin{aligned} f(\alpha + \beta) &= f\left(\sum_{g \in G'} (r_g + I)g + \sum_{h \in H'} (r_h + I)h\right) \\ &= \sum_{g \in G'} r_g g + \sum_{h \in H'} r_h h + I[G] \\ &= f(\alpha) + f(\beta), \end{aligned}$$

and

$$\begin{aligned}
f(\alpha\beta) &= f\left(\left(\sum_{g\in G'}(r_g+I)g\right)\left(\sum_{h\in h'}(r_h+I)h\right)\right)\\
&= f\left(\sum_{(g,h)\in G'\times H'}(r_g+I)(r_h+I)gh\right)\\
&= f\left(\sum_{(g,h)\in G'\times H'}(r_g r_h+I)gh\right)\\
&= \sum_{(g,h)\in G'\times H'} r_g r_h gh + I[G]\\
&= \left(\sum_{g\in G'} r_g g \ + I[G]\right)\left(\sum_{h\in H'} r_h h \ + I[G]\right)\\
&= f(\alpha)f(\beta).
\end{aligned}$$

Also,

$$\begin{aligned}
f(1_{(R/I)[G]}) &= f(1_{(R/I)}1_G) = f((1_R+I)1_g)\\
&= 1_R 1_G + I[G] = 1_{R[G]/R[I]}.
\end{aligned}$$

Now Theorem 2.6 tells us that if $R[G]$ is Noetherian, then the quotient ring $R[G]/I[G]$ must be Noetherian as well. This is then equivalent to showing that $(R/I)[G]$ is Noetherian. However since R/I is a domain, we can apply Lemma 3.6 to obtain that G is amenable, a contradiction. □

Now that we have two necessary conditions on G we can pit them against each other to show that neither is sufficient.

As previously stated, the implication of Theorem 3.2 does not apply in the other direction. While their construction is too involved, Tarski monster groups, constructed by Yu. Ol'shanskii [19] are Noetherian infinite groups with every subgroup cyclic. As Tarski monster groups are non-amenable [20], their group rings are not Noetherian as a consequence of Theorem 3.7. That Noetherian groups can have non-Noetherian group rings was, however, known even before the result by Kropholler and Lorensen. Ivanov provided a counterexample in 1989 [14].

On the other hand, the inverse of Theorem 3.7 has a counterexample too, lamplighter group L_2. Theorem 2.39 tells us it is solvable and then by Example 2.18 it is amenable, but as shown in Theorem 2.40, it is not Noetherian. With minor modifications, these proofs can also be broadened for more general L_n, where $n > 1$.

4 Summary

If we want to determine the Noetherianity of a group ring for a given group and ring with unity there are a range of conditions that narrow the answer down.

The results regarding the ring are simple. The main requirement is the straightforward one, that the ring itself must be Noetherian. The only other relevant condition on rings is that of Theorem 3.7, that the ring must have an ideal whose quotient ring is a domain. Neither of the conditions on G care about the properties of R beyond those two. In addition our proof of Theorem 3.7 would apply to all necessary conditions on G. If we limit ourselves to rings which admit such an ideal then any claim that for a certain type of group its group ring cannot be Noetherian only needs to be proven for (possibly non-commutative) domains.

The limitations on the group are more interesting. The sufficient condition, that it be virtually polycyclic was proven almost 70 years ago, yet no other example has been found so far. The Kourovka Notebook [15] is a collection tracking unsolved problems in group theory. In 1990, Ivanov added to it the following question.

> **11.39.** (Well-known problem). Does there exist a group which is not virtually polycyclic and whose integral group ring is Noetherian?

As of the 2022 edition, the proposition has not been removed from the notebook.

Neither of the two necessary conditions, amenability and Noetherianity is by itself sufficient. In addition, the few examples of Noetherian groups that are not virtually polycyclic are known not to be amenable. All of this would seem to suggest that the answer to Ivanov's question is a tentative no.

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