Low Temperature Localization of the 2D Coulomb Gas

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Master's thesis
2023:E39

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CENTRUM SCIENTIARUM MATHEMATICARUM

Abstract

We fix a minor glitch in a proof recently stated by Ameur. The result shows that the Coulomb gas is well localized to the vicinity of the "droplet", created by the external field, which is described in terms of classical potential theory ("Frostman's theorem"). For this reason, the result was termed "Localization theorem" in [1].

Throughout this work it has been my firm intention to give reference to the stated results and credit to the work of others. All theorems, propositions, lemmas and examples left unmarked are assumed to be too well known for a reference to be given.

Acknowledgments

I would like to thank my supervisor Yacin Ameur for introducing me to the 2D Coulomb gas as well as providing me with help and insights throughout my work on this thesis.

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Chapter 1

Introduction

1.1 The Coulomb Gas in the Plane

Given a lower semi-continuous extended real valued function (**external potential**) $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ satisfying

$$\liminf_{|\zeta| \to \infty} \frac{Q(\zeta)}{2 \log |\zeta|} > 1, \tag{1.1}$$

we define the **Hamiltonian** $H_n : \mathbb{C}^n \to \mathbb{R} \cup \{+\infty\}$ as

$$H_n = \sum_{i \neq j} \log \frac{1}{|\zeta_i - \zeta_j|} + n \sum_{j=1}^{n} Q(\zeta_j). \tag{1.2}$$

The **2D Coulomb gas** at inverse temperature β is then defined as the random point configuration $\{\zeta_j\}_{j=1}^n$ with law given by the **Gibbs measure**,

$$dP_n^\beta(\boldsymbol{\zeta}) = \frac{1}{Z_n^\beta} e^{-\beta H_n(\boldsymbol{\zeta})} dA_n(\boldsymbol{\zeta}), \tag{1.3}$$

where $dA_n(\boldsymbol{\zeta}) = dA(\zeta_1) \cdots dA(\zeta_n)$ is n dimensional Lebesque volume measure on $\mathbb{C}^n$ divided by π^n, and the **partition function**

$$Z_n^\beta = \int_{\mathbb{C}^n} e^{-\beta H_n(\boldsymbol{\zeta})} dA_n(\boldsymbol{\zeta}), \tag{1.4}$$

is a normalizing constant. We interpret $\{\zeta_j\}_{j=1}^n$ as identical and equally charged particles, and the Hamiltonian (1.2) represents the energy of the configuration $\{\zeta_j\}_{j=1}^n$. The particles repel each other by a force proportional to the 2D Coulomb potential $-\log |z_i - z_j|$, but are kept away from dispersing to infinity by the external potential Q. The condition (1.1) precisely means that that the external potential is large enough at infinity in order to confine the particles, and moreover ensures that (1.4) is finite.

A natural question is the behaviour of the particles $\{\zeta_j\}_{j=1}^n$ as $n \to +\infty$. In this thesis we shall analyze this by looking at the **one-point function** (or more generally, the k-point function, see below),

$$R_n^\beta(z) = \lim_{\varepsilon \to 0} \frac{E_n^\beta[\{\text{number of particles in } D(z,\varepsilon)\}]}{\varepsilon^2}. \tag{1.5}$$

where

$$E_n^\beta[f] = \int f dP_n^\beta$$

is the usual expectation with respect to the Gibbs measure (1.3). We see that R_n^β is the expected number of particles per unit area, and thus provides a measure of the expected particle intensity at a given point. It was shown in [1] that outside of a certain compact set $S \subset \mathbb{C}$ known as the **droplet**, we have for sufficiently large n and β the bound

$$R_n^\beta(z) \leq C^\beta n^2 e^{-c\beta n\delta(z)^2}, \tag{1.6}$$

with $C, c > 0$ constants, and $\delta(z) = \inf\{|z - w|; w \in S\}$ is the distance from z to S. The goal of this thesis is to go through the proof of (1.6), which we do in Chapter 3. At its core, the proof uses weighted potential theory, which we develop partly in Chapter 2, in combination with an identity (Lemma 3.2) involving certain weighted polynomials.

The droplet S has the following interesting interpretation: for any compactly supported Borel probability measure, we define the **weighted logarithmic energy**:

$$I_Q(\mu) = \int \int \log \frac{1}{|z - w|} d\mu(z)d\mu(w) + \int Q\, d\mu. \tag{1.7}$$

Frostman's theorem [14] states that that, under some mild conditions on Q, there exists a unique compactly supported probability measure σ that minimizes (1.7). The droplet S is precisely the support of this measure. In Chapter 2 we provide a proof of Frostman's theorem.

It is a well-known result by Johansson [10] that as $n \to \infty$, we have

$$E_n^\beta \left[\frac{1}{n} \sum_{j=1}^n f(\zeta_j) \right] \to \int f\, d\sigma,$$

for all bounded continuous functions f on $\mathbb{C}$, which is a result much along the lines as the bound (1.6), however without any quantitative statements for large but finite n. The measure σ also has connections to so-called quadrature domains [12].

Finally, we shall briefly touch upon a subtlety in the definition of the point the particles $\{\zeta_j\}_{j=1}^n$. The configuration $\{\zeta_j\}_{j=1}^n$ is to be understood as the equivalence class of all vectors $(\zeta_j)_{j=1}^n$ modulo permutations of the elements, and any of these vectors are distributed according to (1.3). Since (1.3) is invariant under permutations, this is well-defined. When performing calculations with the Coulomb gas it is often convenient to work with a representative vector $\boldsymbol{\zeta} = (\zeta_j)_{j=1}^n \in \mathbb{C}^n$ instead.

1.2 The k-point Function

The **marginal probability measure** of (1.3) is defined as

$$P_{n,k}^\beta(B) = P_n^\beta(B \times \mathbb{C}^{n-k}),$$

for Borel sets $B \subseteq \mathbb{C}^k$. The Gibbs measure (1.3) gives the probability distribution of an ordered sample $(\zeta_j)_1^n$. Suppose we pick out the k first particles from this ordered sample. Then, the probability of these k particles being in $B \subseteq \mathbb{C}^k$ is $P_n^\beta(B \times \mathbb{C}^{n-k})$.

This is not quite what we want, as we are only interested in the configuration $\{\zeta_j\}_1^n$ since the particles are indistinguishable. There are $\frac{n!}{(n-k)!}$ ways to pick out k particles from the ordered sample $(\zeta_j)_1^n$, and since P_n^β is invariant under permutations we thus have

$$
\begin{aligned}
P_n^\beta(\{\zeta_j\}_{j=1}^n \in B \times \mathbb{C}^{n-k}) &:= P_n^\beta\left((\zeta_{\pi(j)})_1^n \in B \times \mathbb{C}^{n-k} \text{ for any permutation } \pi\right) \\
&= \frac{n!}{(n-k)!} P_n^\beta(B \times \mathbb{C}^{n-k}),
\end{aligned}
$$

where π is any permutation of the indices $1, \ldots, n$. We now define the **correlation measure** $\mu_{n,k}^\beta$ as.

$$
\mu_{n,k}^\beta = \frac{n!}{(n-k)!} P_{n,k}^\beta. \tag{1.8}
$$

With the preceding argument as motivation, $\mu_{n,k}$ is more relevant to the quantities we want to compute.

We can now generalize the one-point function (1.5) to higher dimensions. Let $(z_1, \ldots, z_k) \in \mathbb{C}^k$, and define the **k-point function** as

$$
R_{n,k}^\beta((z_1, \ldots, z_k)) = \lim_{\varepsilon \to 0^+} \frac{\mu_{n,k}^\beta(D(z_1, \varepsilon) \times \cdots D(z_k, \varepsilon))}{\varepsilon^{2k}}. \tag{1.9}
$$

Much as the one-point function, the k-point function is a multivariate measure of the expected intensity of particles at the points $z_1, \ldots, z_k$ simultaneously.

1.3 A Word on Notation

By a domain D we mean an open connected subset of the complex plane $\mathbb{C}$. We refer to a number $x \in [-\infty, +\infty]$ as an extended real number. If $x \in \mathbb{R}$ is merely a real number, we mean that it is finite and thus does not take on the values $-\infty$ or $+\infty$. The complement of a set $A \subset \mathbb{C}$ is denoted A^c.

We shall write $dA = \pi^{-1} dxdy$ for the normalized Lebesgue area measure, so that the unit disk has area 1. We also use a normalized the Laplacian $\Delta := \partial\bar{\partial}$, which is 1/4 times the usual Laplacian $(\partial_x^2 + \partial_y^2)$. Here $\partial = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ are the Wirtinger differential operators.

By P_n^β we shall always mean the Gibbs measure (1.3), and E_n^β will denote expectation with respect to P_n^β.

Chapter 2

Weighted Potential Theory

2.1 Subharmonic Functions

C^2-functions h on a domain (open, connected set) $D \subseteq \mathbb{C}$ satisfying **Laplace's equation**

$$\Delta h = 0, \tag{2.1}$$

are commonly known as harmonic functions and satisfy a number of interesting properties, one of the most important ones being the maximum principle: both the minimum and maximum of u on any compact set $K \subset D$ is attained at the boundary ∂K. There are of course also many different versions and formulations of this principle.

Harmonic functions possess continuous derivatives of all orders, and are unique in the sense that if they vanish on any open set, they must vanish on their entire domain [13]. This form of "rigidness" severely limits the construction of arbitrary harmonic functions, and it is therefore of interest to instead consider subharmonic and superharmonic functions which, as we shall see in this chapter, can be very non-smooth while still satisfying a maximum principle. Their properties will later aid us in proving the bound on the one-point function (1.6) in Chapter 3.

We begin by discussing a "one-sided" notion of continuity. Recall that an extended real valued function f on a domain D is said to be continuous at a point $z_0 \in D$ with $f(z_0) = +\infty$ if for all $\omega > 0$, there is a δ such that $|z - z_0| < \delta$ implies $f(z) > \omega$.

Definition 2.1. Let D be a domain. An extended real-valued function $u : D \to [-\infty, +\infty)$ is said to be **upper semi-continuous** at the point $z_0 \in D$ if either of the following hold:

- $u(z_0) > -\infty$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $|z - z_0| < \delta$ we have $u(z) < u(z_0) + \varepsilon$,
- $u(z_0) = -\infty$ and u is continuous at z_0.

If u is upper semi-continuous at every point of D, it is said to be upper semi-continuous on D.

We can also define lower semi-continuous functions analogously.

Definition 2.2. Let D be a domain. An extended real-valued function $u : D \to (\infty, +\infty]$ is said to be **lower semi-continuous** at the point $z_0 \in D$ if $-u$ is upper semi-continuous there. Similarly, u is said to be lower semi-continuous on D if $-u$ is upper semi-continuous on D.

Some simple examples of upper semi-continuous functions are the characteristic function $\mathbf{1}_V$ of any closed set V, any continuous function f and any positive linear combination of functions of these forms.

While continuity of a function f can be understood as the property that if z is "close" to z_0, then $f(z)$ is "close" to $f(z_0)$, upper semi-continuity of u only requires that for z close to z_0, the neighbouring values $u(z)$ are not much higher than $u(z_0)$. From the above definitions we see that a real-valued function is continuous if and only if it is both upper and lower semi-continuous.

As with continuity, there are a number of equivalent conditions of semi-continuity.

Theorem 2.3. *Let D be a domain, $u : D \to [-\infty, +\infty)$. The following are equivalent:*

(i) u is upper semi-continuous on D,

(ii) $\{z \in D; u(z) < x\} = u^{-1}([-\infty, x))$ is open for all $x \in \mathbb{R}$,

(iii) For every sequence (z_n) converging to z_0, we have $\limsup_{n\to+\infty} u(z_n) \le u(z_0)$.

Proof. (i) $\Longrightarrow$ (ii) Suppose u is upper semi-continuous in D, and let $z_0 \in \{u(z) < x\}$. Assume first that $u(z_0) > -\infty$. Given positive $\varepsilon < x - u(z_0)$ we pick a δ such that $|z - z_0| < \delta$ implies $u(z) \le u(z_0) + \varepsilon < x$. Thus, the disk $D(z_0, \delta)$ is contained in $\{u(z) < x\}$. Now, if $u(z_0) = -\infty$ we may by the continuity of u at z_0 directly pick a $\delta > 0$ such that $|z - z_0| < \delta$ implies $u(z) < x$. Since z_0 was arbitrary, $\{u(z) < x\}$ is open.

(ii) $\Longrightarrow$ (iii) Suppose $\{u(z) < x\}$ is open for all $x \in \mathbb{R}$, and let (z_n) in D be a sequence converging to $z_0 \in D$. Then, z_0 is contained in the open set $\{u(z) < u(z_0) + \varepsilon\}$ for all $\varepsilon > 0$. Thus, there is a $\delta = \delta(\varepsilon) > 0$ such that $D(z_0) \subseteq \{u(z) < u(z_0) + \varepsilon\}$. But this implies that $\limsup_{n\to+\infty} u(z_n) \le u(z) + \varepsilon$, and since this holds for all $\varepsilon > 0$, we must have $\limsup_{n\to+\infty} u(z_n) \le u(z)$.

(iii) $\Longrightarrow$ (i) Let $z_0 \in D$ with $u(z_0) > -\infty$ and assume u is not upper semi-continuous at z_0. Then there is an ε such that for each $n \in \mathbb{Z}_+$ there is a $z_n \in D$ with $|z - z_n| < 1/n$ and $u(z_n) \ge u(z_0) + \varepsilon$. Thus, $\limsup_{n\to\infty} u(z_n) \ge u(z_0) + \varepsilon > u(z_0)$, which contradicts (iii). Now let $z_0 \in D$ with $u(z_0) = -\infty$ and assume u is not continuous at z_0. Then there is an $\omega > 0$ such that for every $n \in \mathbb{Z}_+$ there is a $z_n \in D$ with $|z - z_n| < 1/n$ and $u(z_n) \ge -\omega$. But then $\limsup_{n\to\infty} u(z_n) \ge -\omega > -\infty = u(z_0)$ and we contradict (iii) also in this case. $\square$

A family of upper semi-continuous functions $\{u_v, v \in I\}$ have the nice property that their pointwise infimum

$$u(z) = \inf_{v \in I} u_v(z), \tag{2.2}$$

is also upper semi-continuous. Indeed, this follows easily from condition (ii) in the previous theorem since $\{u(z) < x\} = \cup_{v\in I}\{u_v(z) < x\}$ is open for all $x \in \mathbb{R}$. In the other direction, we have the following theorem.

Theorem 2.4. *Let D be a domain, $u : D \to [-\infty, +\infty)$ an upper semi-continuous function bounded above, $u < C$, $C \in \mathbb{R}$. Then there is a monotone sequence of real-valued continuous functions (f_n) on D satisfying $f_1 \ge f_2 \ge \ldots \ge u$ that converges pointwise to u.*

Proof. If $u = -\infty$ everywhere we make simply take $f_n = -n$ and we are done. In the general case, define f_n by

$$f_n(z) = \sup_{w \in D} \left(u(w) - n|z - w| \right).$$

Then f_n are monotonically non-increasing, satisfy $u \leq f_n \leq C$ for all n and $f_n(z) > -\infty$ everywhere. We show that each f_n is also continuous. Fix $n \geq 1$, , $z \in D$ let $\varepsilon > 0$, and let $\tilde{z} \in D$ be a point such that

$$f_n(z) \leq u(\tilde{z}) - n|z - \tilde{z}| + \varepsilon.$$

Note that this implies $u(\tilde{z}) > -\infty$. Then, for all $z' \in D$,

$$\begin{aligned} f_n(z) - f_n(z') &\leq (u(\tilde{z}) - n|z - \tilde{z}| + \varepsilon) - (u(\tilde{z}) - n|z' - \tilde{z}|) \\ &= n\left(|z - \tilde{z}| - |\tilde{z} - z'|\right) + \varepsilon \\ &\leq n|z - z'| + \varepsilon, \end{aligned}$$

where the last inequality is the reverse triangle inequality. Since this holds for all $\varepsilon > 0$, we must have

$$f_n(z) - f_n(z') \leq n|z - z'|.$$

By switching the roles of z, z' we conclude that in fact

$$|f_n(z) - f_n(z')| \leq n|z - z'|,$$

so that each f_n is Lipschitz continuous, and in particular continuous.

Now, since the (f_n) is a monotone sequence, it has a limit satisfying $\lim_{n\to+\infty} f_n(z) \geq u(z)$ everywhere. To show the reverse inequality, we observe that for any fixed $r > 0$ such that $D(z, r) \subseteq D$ it holds that

$$\begin{aligned} f_n(z) &\leq \max\left(\sup_{w \in D(z,r)} u(w), \sup_{w \in D} (u(w) - nr) \right) \\ &\leq \max\left(\sup_{w \in D(z,r)} u(w), C - nr \right) \end{aligned}$$

In the limit as $n \to +\infty$ we thus have,

$$\lim_{n\to+\infty} f_n(z) \leq \sup_{w \in D(z,r)} u(w).$$

Let $\varepsilon' > 0$. The upper semi-continuity of u implies that for all sufficiently small $r > 0$,

$$\sup_{w \in D(z,r)} u(w) < u(z) + \varepsilon'.$$

Hence, $\lim_{n\to+\infty} f_n(z) < u(z) + \varepsilon'$ and since this holds for all $\varepsilon' > 0$ we must have $\lim_{n\to+\infty} f_n(z) \leq u(z)$.

□

Theorem 2.4 allows us to approximate upper semi-continuous functions u from above with continuous functions. We illustrate a situation when this can be useful:

suppose (μ_m) is a sequence of positive Borel measures on some bounded domain D, such that in the weak-star sense of convergence we have

$$\mu_m \to \mu,$$

for some positive Borel measures μ, that is,

$$\lim_{m\to+\infty} \int f d\mu_m = \int f d\mu,$$

for all continuous bounded functions f on D. Let u be an upper semi-continuous function on D and (f_n) an approximating sequence as in Theorem 2.4. Then,

$$\lim_{m\to+\infty} \int f_n d\mu_m = \int f_n d\mu,$$

for all $n > 0$. Since we always have $f_n \geq u$, it then holds that

$$\limsup_{m\to+\infty} \int u\, d\mu_m \leq \int f_n d\mu,$$

for all $n > 0$. Letting $n \to +\infty$ and applying monotone convergence theorem, we obtain

$$\limsup_{m\to+\infty} \int u\, d\mu_m \leq \int u d\mu. \tag{2.3}$$

Thus, while we generally cannot conclude that $\int u d\mu_m \to \int u d\mu$, we at least always have the one-sided bound (2.3).

Upper semi-continuity also implies upper boundedness on compact sets.

Theorem 2.5. *Let D be a domain, $u : D \to [-\infty, +\infty)$ an upper semi-continuous function. If $K \subset D$ is compact, there exists a point $z_0 \in K$ with $u(z_0) = \sup_{z\in K} u(z)$. In particular, $\sup_{z\in K} u(z) < \infty$.*

Proof. Let (z_n) be a sequence in K such that

$$\lim_{n\to+\infty} u(z_n) = \sup_{z\in K} u(z). \tag{2.4}$$

By passing to a subsequence we may assume that (z_n) converges to some point $z_0 \in K$. In view of condition (iii) in theorem 2.3 this means that

$$\sup_{z\in K} u(z) = \limsup_{n\to+\infty} u(z_n) \leq u(z_0). \tag{2.5}$$

Since we trivially also have $u(z_0) \leq \sup_{z\in K} u(z)$, it must hold that $u(z_0) = \sup_{z\in K} u(z)$. □

The advantage of semi-continuity is that it allows us to consider less well behaved functions while still retaining many of the important properties of continuous functions, although typically such properties are now “one-sided” as in Theorem 2.3 and 2.5.

We are now ready to give our definition of subharmonic functions.

Definition 2.6. An upper semi-continuous real-valued function u on a domain D is said to be subharmonic in D if for each $\zeta_0 \in D$ there is a $\rho > 0$ such that

$$u(z_0) \leq \int_{-\pi}^{\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi}, \tag{2.6}$$

holds for all $0 < r < \rho$.

One of the most important example of a subharmonic function on $\mathbb{C}$ is $\log|z|$. For $z \neq 0$, we have $\log|z| = \Re(\log z)$ locally for some branch of $\log z$, and thus $\log|z|$ is harmonic and satisfies the mean value equality there. Moreover, the mean value inequality (2.6) holds trivially at the origin since $\log|0| = -\infty$.

If u is a C^2-function we can do a second order Taylor expansion around z_0, which we express in terms of the Wirtinger differential operators $\partial = \frac{1}{2}(\partial_x - i\partial_y), \overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ [7],

$$\begin{aligned} u(z_0 + w) &= u(z_0) + w\partial u(z_0) + \overline{w}\overline{\partial}u(z_0) \\ &\quad + w^2\frac{1}{2}\partial^2 u(z_0) + \overline{w}^2\frac{1}{2}\overline{\partial}^2 u(z_0) + w\overline{w}\partial\overline{\partial}u(z_0) + o(|w|^2). \end{aligned}$$

Switching to polar coordinates, using that $w^n = r^n e^{in\theta}$ and $\int_{-\pi}^{\pi} e^{in\theta} d\theta = 0$ for all nonzero integers n, we then see that

$$\int_{-\pi}^{\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} = u(z_0) + r^2 \Delta u(z_0) + o(r^2), \tag{2.7}$$

where we used that $\partial\overline{\partial} = \Delta$. Thus, u is subharmonic if and only if $\Delta u \geq 0$ everywhere. The above calculation also gives the intuitive meaning of the Laplacian Δ as the limiting difference of the mean value around z_0, $\int u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi}$ and the value there, $u(z_0)$.

If u is not C^2 but $\Delta u \geq 0$ is still satisfied in the distributional sense, namely that for all infinitely differentiable nonnegative functions $\phi : \mathbb{C} \to [0, +\infty)$ with compact support we have

$$\int u \Delta\phi dA \geq 0,$$

it is still true that u is a subharmonic function. In this case, a famous theorem from distribution theory [5] tells us that any distribution v satisfying $v \geq 0$ everywhere is a nonnegative Borel measure. Thus, there is a nonnegative Borel measure μ on $\mathbb{C}$ such that

$$\Delta u = \mu, \tag{2.8}$$

which, as before, should be interpreted in the sense of distributions. Equation (2.8) is known as **Poisson's equation**, and is the inhomogenous version of Laplace's equation. Its fundamental solution on $\mathbb{C}$ is well known [13] to be $2\log|z|$, and thus (2.8) implies that on any open disk $D(w_0, \rho) \subseteq D$ we can express u as the convolution of $2\log|z|$ with μ plus some solution of Laplace's equation,

$$u(z) = 2\int_{D(w_0,\rho)} \log|z - w| d\mu(w) + h(z),$$

with h harmonic, $z \in D(w_0, \rho)$. The upper semi-continuity follows from Theorem 2.4 and the representation

$$\int_{D(w_0,\rho)} \log |z - w| d\mu(w) = \lim_{M\to+\infty} \int_{D(w_0,\rho)} \max\left(-M, \log |z - w|\right) d\mu(w),$$

which holds by the monotone convergence theorem. The distributional subharmonicity of $\log |z|$ together with Fubini also implies that

$$\begin{aligned} u(z_0) &= 2 \int_{D(w_0,\rho)} \log |z_0 - w| d\mu(w) + h(z_0) \\ &\leq 2 \int_{D(w_0,\rho)} \int_{-\pi}^{\pi} \log |z_0 + re^{i\theta} - w| \frac{d\theta}{2\pi} d\mu(w) + h(z_0) \\ &= \int_{-\pi}^{\pi} 2 \int_{D(w_0,\rho)} \log |z_0 - w| d\mu(w) \frac{d\theta}{2\pi} + \int_{-\pi}^{\pi} h(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} \\ &= \int_{-\pi}^{\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi}, \end{aligned}$$

for $0 < r < \rho$, and thus the mean value inequality (2.6) is satisfied.

Subharmonicity is preserved under non-decreasing convex maps, a simple consequence of Jensen's inequality.

Theorem 2.7. *Let u be subharmonic in a domain D and $\varphi : [-\infty, +\infty) \to [-\infty, +\infty)$ be a non-decreasing convex function. Then $\varphi \circ u$ is subharmonic.*

Proof. Let $z_0 \in D$ and take $r > 0$ small enough so that $D(\zeta, r)$ is contained in D and that the submean inequality (2.6) holds. Then,

$$\varphi(u(z_0)) \leq \varphi\left(\int_{-\pi}^{\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi}\right) \leq \int_{-\pi}^{\pi} \varphi\left(u(z_0 + re^{i\theta})\right) \frac{d\theta}{2\pi}. \tag{2.9}$$

The first inequality is due to φ being non-decreasing, the second inequality is Jensen's inequality. $\square$

An important case is when $\varphi(x) = e^x$ is the exponential function. Suppose f is some complicated nonnegative function (for example, a large product of other nonnegative functions), but $\log f$ is easier to analyze. Then, if we can show that $\log f$ is subharmonic, Theorem 2.7 tells us that $f = e^{\log f} = \varphi(\log f)$ is subharmonic.

The following version of the maximum principle follows fairly easily from the definition: if u attains its global maximum at z_0, the submean inequality (2.6) forces u to be constant in a whole disk around z_0. Since D is connected, we can iterate this process to exhaust all of D.

Theorem 2.8. *Let u be subharmonic in a domain D. If u attains a global maximum in D, then u is constant.*

Proof. We may exclude the case when $u = -\infty$ identically since the theorem is trivial in that case. Let $M > -\infty$ be the global maximum and consider the sets

$$A = \{u(z) < M\} \qquad B = \{u(z) = M\}. \tag{2.10}$$

Assume that A is nonempty. By the upper semi-continuity of u, A is open. We shall now show that also B is open; this will contradict the connectedness of D.

Let $z_0 \in B$, such a point exists by assumption. The subharmonicity of u then implies the existance of a $\rho > 0$ such that

$$\int_{-\pi}^{\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi} = M, \tag{2.11}$$

for all $0 < r < \rho$. Suppose there is a $w_0 \in D(z_0, \rho)$ with $u(w_0) < M$. Then there is a small ε such that $u(w_0) < M - \varepsilon$, and the upper semi-continuity of u implies that $u(w) < u(w_0) + \frac{\varepsilon}{2} < M - \frac{\varepsilon}{2}$ for all w in some disk $D(w_0, \delta)$. Thus, letting $r = |w_0 - z_0|$ we have

$$\int_{-\pi}^{\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi} \leq \alpha(M - \frac{\varepsilon}{2}) + (1-\alpha)M < M, \tag{2.12}$$

where $0 < \alpha < 1$ is the proportion of the part of the circle $\partial D(z_0, r)$ lying in $D(w_0, \delta)$. But this contradicts (2.11), so we must have $u(w) = M$ for all $w \in D(z_0, \rho)$, that is, $D(z_0, \rho) \subseteq B$. This shows that B is open. $\square$

Using the previous theorem we may also prove the following maximum principle variant.

Theorem 2.9. *Let u be subharmonic in a bounded domain D. If*

$$\limsup_{z \to w} u(z) \leq C, \qquad C \in \mathbb{R}, \tag{2.13}$$

for all $w \in \partial D$, then $u \leq C$ on D.

Proof. Extend u to ∂D by setting $u(w) = \limsup_{z \to w} u(z)$ for $w \in \partial D$. Then u is upper semi-continuous on the compact set $\overline{D}$, so by Theorem 2.5 it attains its maximum at some point $z_0 \in \overline{D}$. If $z_0 \in \partial D$ we automatically have $u \leq C$ on D, and if $z_0 \in D$ then by the previous theorem u must be constant on D. The condition (2.13) then ensures that $u \leq C$ everywhere. $\square$

We can extend the previous theorem to unbounded domains if we in addition we assume that

$$\limsup_{|z| \to +\infty} u(z) \leq C. \tag{2.14}$$

Indeed if we pick a large enough radius R that $u \leq C$ outside $|z| > R$, we get the result by applying the maximum principle to u on the bounded domain $D \cap D(0, 2R)$.

Corollary 2.10. *Let u be subharmonic in an unbounded domain D. If*

$$\limsup_{z \to w} u(z) \leq C, \qquad C \in \mathbb{R}, \tag{2.15}$$

for all $w \in \partial D \cup \{\infty\}$, then $u \leq C$ on D.

For a complete treatment of subharmonic functions, we refer to [13].

2.2 The Equilibrium Measure

Suppose we have a fixed external electric field over $\mathbb{R}^n$, and we insert an arbitrary (positive) charge distribution of total charge 1. The external electric field will then cause said charge to move until a stable configuration is established. This "equilibrium configuration" will be such that the energy of the total electric field, given as the superposition of the external field and the field generated by the point charges, is minimized. It is a natural mathematical question weather or not the "equilibrium configuration" is unique, and what conditions on the electric field are needed in order to guarantee existence of such a configuration. In this section we shall provide answers to both of those questions in the case of dimension $n = 2$, which is the case most relevant to us as we shall use theory from this section to analyze the 2D Coulomb Gas in Chapter 3. We shall however begin by briefly looking at a general dimension n, in order to get a feel for the problem as a whole and better understand the contrast between dimension $n = 2$ and $n \geq 3$.

Consider a compactly supported Borel probability[1] measure μ on $\mathbb{R}^n$, which we shall think of as a distribution of charge over $\mathbb{R}^n$. Maxwell's famous equations of electromagnetism [8] state that in a static charge configuration the electric field may be expressed as the (negative) gradient of a **potential** U^μ which satisfies Poisson's equation,

$$-2\Delta U^\mu = \mu, \tag{2.16}$$

which for a general μ (not absolutely continuous with respect to Lebesgue measure) should be interpreted in the sense of distributions [5]. It is well-know [5] that the fundamental solution $\tilde{g}_n$ of (2.16) in dimension n is

$$\tilde{g}_n(\boldsymbol{x}) = \begin{cases} \log \frac{1}{|\boldsymbol{x}|} & n = 2 \\ C_n \frac{1}{|\boldsymbol{x}|^{n-2}} & n \geq 3 \end{cases}, \tag{2.17}$$

for constants $C_n > 0$, where $|\boldsymbol{x}| = \sqrt{x_1^2 + \cdots x_n^2}$, $\boldsymbol{x} \in \mathbb{R}^n$. The function $\tilde{g}_n$ is also known as the **Coulomb potential** in dimension n. The general solution of (2.16) is thus given by convolving $\tilde{g}_n$ with μ [5]:

$$U^\mu(\boldsymbol{x}) = \int \tilde{g}_n(\boldsymbol{x} - \boldsymbol{y})d\mu(\boldsymbol{y}), \tag{2.18}$$

which is well-defined as an extended real-valued function. What is then the energy of the electric field generated by μ? Since the electric field is the (negative) gradient of U^μ, the difference $U^\mu(\boldsymbol{x}) - U^\mu(\tilde{\boldsymbol{x}})$ is precisely the energy it takes to move a unit charge from $\tilde{\boldsymbol{x}}$ to $\boldsymbol{x}$. In dimension $n \geq 3$ we see from (2.17) and (2.18) that $U^\mu \to 0$ as $|\boldsymbol{x}| \to \infty$, and thus the energy to put a unit charge at $\boldsymbol{x}$ in the first place (which is equivalent to move it in "from infinity") is simply $U^\mu(\boldsymbol{x})$. The energy of the charge configuration μ is thus given by[2]

$$\int U^\mu(\boldsymbol{x})d\mu(\boldsymbol{x}) = \iint \tilde{g}_n(\boldsymbol{x} - \boldsymbol{y})d\mu(\boldsymbol{x})d\mu(\boldsymbol{y}). \tag{2.19}$$

This is the energy of μ in a vacuum, that is, without the influence of any external electric field. If there is an external field present, it too can be expressed as the gradient of some **external potential** Q, and the energy contribution from this is simply $\int Q d\mu$, so that the total energy is given by

$$\iint \tilde{g}_n(\boldsymbol{x} - \boldsymbol{y})d\mu(\boldsymbol{x})d\mu(\boldsymbol{y}) + \int Q(\boldsymbol{x})d\mu(\boldsymbol{x}). \tag{2.20}$$

[1]Recall that a measure μ is a probability measure if it is nonnegative and of total mass 1.

[2]Actually, there is a factor $\frac{1}{2}$ missing in front of the integral (2.19) which accounts for the fact that there is no charge present when we start to bring in charges "from infinity". However, this will be of no importance for us.

Thus, to see how a positive charge distribution of unit mass would redistribute iteself under Q, we simply have to minimize (2.20) over all compactly supported Borel probability measures μ. Strictly speaking, (2.20) is not the total energy of the system but rather the energy increase we get by inserting the charge μ into it. However, it is still the only portion of the energy dependent on μ, so it is equivalent for minimization purposes.

From this point on we shall limit ourselves to the case of dimension $n = 2$, and identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. By (2.18), the potential U^μ is then

$$U^\mu(z) = \int \log \frac{1}{|z-w|} d\mu(w), \tag{2.21}$$

where μ is any compactly supported signed Borel measure on $\mathbb{C}$ of finite total variation $|\mu|(\mathbb{C}) = \sup\{\sum_{n=1}^N |\mu(E_n)|, E_n \text{ measurable partition of } \mathbb{C}\}$. We shall almost always take μ to be a probability measure but, as we shall see, there are also some certain scenarios where signed measures also are of interest, such as when μ is the difference of two probability measures.

While the argument leading up to the energy (2.19) is only valid for $n \geq 3$, we now simply take (2.19) as the definition for $n = 2$. Thus, the **logarithmic energy** of a compactly supported signed Borel measure μ of finite total variation is defined as

$$I(\mu) := \iint \log \frac{1}{|z-w|} d\mu(w) d\mu(z). \tag{2.22}$$

We now examine the potential U^μ slightly closer; outside of $\operatorname{supp} \mu$ we see that U^μ is harmonic, and we can make an expansion in z: for $|z| > |w|$ we have that

$$\begin{aligned}
\log \frac{1}{|z-w|} &= \log \frac{1}{|z|} + \log \frac{1}{|1 - \frac{z}{w}|} \\
&= \log \frac{1}{|z|} - \frac{1}{2} \log \left(1 - \frac{w}{z}\right) - \frac{1}{2} \log \left(1 - \frac{\overline{w}}{\overline{z}}\right) \\
&= \log \frac{1}{|z|} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \frac{w^k}{z^k} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} \frac{\overline{w}^k}{\overline{z}^k}.
\end{aligned}$$

We can pick z, R large enough that $|z| > R > |w|$ for all $w \in \operatorname{supp} \mu$ and observe that

$$\int \sum_{k=1}^{\infty} \frac{1}{k} \frac{|w|^k}{|z|^k} d\mu(w) \leq |\mu|(\mathbb{C}) \sum_{k=1}^{\infty} \frac{1}{k} \frac{R^k}{|z|^k},$$

where $|\mu|(E) = \sup\{\sum_{n=1}^N |\mu(E_n)|, E_n \text{ measurable partition of } E\}$ is the total variation of μ. Since this is finite, Fubini's theorem let's us change the order of integration and summation to conclude that

$$U^\mu(z) = \mu(\mathbb{C}) \log \frac{1}{|z|} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{kz^k} \int w^k d\mu(w) + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k\overline{z}^k} \int \overline{w}^k d\mu(w), \tag{2.23}$$

for all z outside some disk $D(0, R)$ containing $\operatorname{supp} \mu$. This is known as the **multipole expansion** of U^μ. In particular, we see that if the total mass $\mu(\mathbb{C})$ is positive, then we always have $U^\mu(z) \to -\infty$ as $|z| \to +\infty$. It is interesting to note that if μ is uniform over a disk, only the log-term shows up in the multipole expansion (2.23), as can be seen in the following lemma.

Lemma 2.11. *In the special case when* $d\mu = \frac{1}{\rho^2}\mathbf{1}_{D(z_0,\rho)}dA$ *is the uniform probability measure on the disk* $D(z_0,\rho)$*, we have*

$$U^\mu(z) = \begin{cases} \log \frac{1}{|z-z_0|}, & |z-z_0| > \rho \\ -\frac{|z-z_0|^2}{2\rho^2} + \log\frac{1}{\rho} + \frac{1}{2}, & |z-z_0| \le \rho \end{cases}. \tag{2.24}$$

Proof. Indeed, for $|z-z_0| > \rho$ the function $\log\frac{1}{|z-w|}$, seen as a function of $w \in \mathbb{C}$, is harmonic in a neighbourhood of $D(z_0,\rho)$. It follows from the mean value property that

$$U^\mu(z) = \frac{1}{\rho^2}\int_{D(z_0,\rho)} \log\frac{1}{|z-w|}dA(w) = \log\frac{1}{|z-z_0|}, \quad (|z-z_0| > \rho).$$

When $|z-z_0| < \rho$, U^μ satisfies $-2\Delta U^\mu = \frac{1}{\rho^2}$ and can thus be written as

$$U^\mu(z) = -\frac{1}{2\rho^2}|z-z_0|^2 + h(z), \quad (|z-z_0| < \rho)$$

with h harmonic in $D(z_0,\rho)$. The continuity of U^μ at $\partial D(z_0,\rho)$ forces h to be constant, so

$$h = \log\frac{1}{\rho} + \frac{1}{2}.$$

□

We now consider the situation with an external electric field, which is given by an **external potential** $Q : \mathbb{C} \to [0,+\infty]$. We shall assume that Q satisfies the following conditions:

1. $\liminf_{|z|\to+\infty} \frac{Q(z)}{2\log|z|} > 1$.
2. The set $\Sigma = \{Q(z) < \infty\}$ has nonempty interior.
3. Q is lower semi-continuous on $\mathbb{C}$.

The first condition is a growth condition to ensure that far away from the origin, the force generated by Q is greater than that of the Coulomb potential $\log|z|$, and thus that Q is "strong enough" to keep charges from escaping to infinity. Conditions 2 and 3 are mild regularity conditions.

In the influence of the external potential Q, we get the formula (2.20) for the total energy in dimension $n \ge 3$. As for the free energy (2.22), we again now take this as our definition in dimension 2, and define the **weighted logarithmic energy** as

$$\begin{aligned} I_Q(\mu) &:= \iint \log\frac{1}{|z-w|}d\mu(w)d\mu(z) + \int Q(z)d\mu(z) \\ &= \iint \left(\log\frac{1}{|z-w|} + \frac{1}{2}Q(z) + \frac{1}{2}Q(w)\right) d\mu(w)d\mu(z) \end{aligned} \tag{2.25}$$

again for compactly supported signed Borel measure on $\mathbb{C}$ of finite total variation. We now seek to minimize this energy over the set of all all compactly supported Borel *probability* measures μ. We denote the minimal energy by $V_Q = \inf_\mu I_Q(\mu)$.

Lemma 2.12. *V_Q is finite.*

Proof. Since Σ has non-empty interior it contains a disk $D(z_0, \rho)$. By Lemma 2.11, the logarithmic energy of the measure $\frac{1}{\rho^2}\mathbf{1}_{D(z_0,\rho)}dA$ is finite, so $V_Q < \infty$.

To show that $V_Q > -\infty$, we simply note that the growth condition on Q implies that

$$\log \frac{1}{|z-w|} + \frac{1}{2}Q(z) + \frac{1}{2}Q(w) \geq 0,$$

for all sufficiently large $z, w \in \mathbb{C}$.

□

We shall now show that there does indeed exist a compactly supported probability measure μ that minimizes (2.25) over all compactly supported Borel probability measures, and that this μ is unique. The main tool for proving existence is the following well-known result from functional analysis.

Lemma 2.13. *Let $K \subseteq \mathbb{C}$ be a compact set, $(\mu_n)_n$ a sequence of Borel probability measures with $\operatorname{supp} \mu_n \subset K$ for all n. Then there exists a Borel probability measure μ with $\operatorname{supp} \mu \subset K$ and a subsequence $(\mu_{n_k})_k$ such that $\mu_n \to \mu$ in the weak-star sense, that is,*

$$\int f d\mu_n \to \int f d\mu,$$

as $n \to +\infty$ for all complex-valued continuous functions f on K.

Proof. Let $C(K) = \{f : K \to \mathbb{C}, f \text{ continuous}\}$ be the vector space of continuous functions on K equipped with the L^∞ norm,

$$||f||_{L^\infty(K)} = \sup_{z\in K} |f(z)|.$$

From functional analysis [11], we know that $C(K)$ is a Banach space, and that its dual $\mathcal{M}(K) := C'(K)$ consists of all signed Borel measures on K of finite total mass, which certainly contains all Borel probability measures on K. The Banach–Alaoglu theorem [11] states that any closed ball in $\mathcal{M}(K)$ is compact in the weak-star topology. Since the sequence $(\mu_n)_n$ is contained in such a ball, the result follows. □

However, we are considering probability measures over all of $\mathbb{C}$, which is not a compact set. We thus first need to show the following lemma.

Lemma 2.14. *The minimization is of (2.25) over all compactly supported Borel probability measures is equal to the minimization over all Borel probability measures μ with $\operatorname{supp} \mu \subseteq D_R := D(0, R)$, for some fixed $R \geq 0$.*

Proof. Consider the function

$$\log \frac{1}{|z-w|} + \frac{1}{2}Q(z) + \frac{1}{2}Q(w).$$

The growth condition on Q ensures that

$$\log \frac{1}{|z-w|} + \frac{1}{2}Q(z) + \frac{1}{2}Q(w) \to +\infty,$$

as $\max\{|z|,|w|\} \to +\infty$. In particular, we can choose an $R > 0$ so large that

$$\log \frac{1}{|z-w|} + \frac{1}{2}Q(z) + \frac{1}{2}Q(w) > V_Q + 1, \tag{2.26}$$

whenever $\max\{|z|,|w|\} > R$. Writing $D_R = D(0,R)$, this is equivalent to $(z,w) \notin D_R \times D_R$.

Now, consider a compactly supported probability measure μ having positive mass outside of D_R, that is satisfying $\mu(D_R^c) > 0$, and with sufficiently small logarithmic energy $I_Q(\mu) < V_Q + 1$. We will show that we then can construct another compactly supported probability measure $\tilde{\mu}$ with $\operatorname{supp}\tilde{\mu} \subseteq D_R$ and $I_Q(\tilde{\mu}) < I_Q(\mu)$, which in turn proves the lemma.

Firstly, observe that the condition $I_Q(\mu) < V_Q + 1$ together with (2.26) implies that $\mu(D_R) > 0$. Indeed, otherwise we would have

$$\begin{aligned} I_Q(\mu) &= \int_{D_R^c}\int_{D_R^c} \left(\log \frac{1}{|z-w|} + \frac{1}{2}Q(z) + \frac{1}{2}Q(w)\right) d\mu(z)d\mu(w) \\ &\geq \iint (V_Q + 1)\, d\mu(z)d\mu(w) \\ &= V_Q + 1. \end{aligned}$$

The fact that $\mu(D_R) > 0$ then allows us to construct $\tilde{\mu}$ as

$$\tilde{\mu}(E) = (\mu(D_R))^{-1}\, \mu(E \cap D_R).$$

In other words, $\tilde{\mu}$ is simply μ conditioned on D_R. Clearly $\operatorname{supp}\tilde{\mu} \subseteq D_R$, and moreover,

$$\begin{aligned} I_Q(\mu) &= \left(\iint_{D_R \times D_R} + \iint_{(D_R \times D_R^c)}\right) \left(\log \frac{1}{|z-w|} + \frac{1}{2}Q(z) + \frac{1}{2}Q(w)\right) d\mu(z)d\mu(w) \\ &\geq \mu(D_R)^2 I_Q(\tilde{\mu}) + (1 - \mu(D_R)^2)(V_Q + 1) \end{aligned}$$

This then implies that $I_Q(\tilde{\mu}) < I_Q(\mu)$, because if we would have $I_Q(\tilde{\mu}) \geq I_Q(\mu)$ the above inequality would imply that $I_Q(\mu) \geq \mu(D_R)^2 I_Q(\mu) + (1 - \mu(D_R)^2)(V_Q + 1)$, or equivalently,

$$I_Q(\mu) \geq V_Q + 1,$$

which is a contradiction since $I_Q(\mu) < V_Q + 1$ by assumption. □

We shall also need a result of the type "if the energy (2.22) is zero then μ is zero", which is the following lemma.

Lemma 2.15. *Let μ be a compactly supported Borel measure on $\mathbb{C}$ of zero total mass. Then,*

$$\iint \log \frac{1}{|z-w|} d\mu(w)d\mu(z) \geq 0, \tag{2.27}$$

with equality if and only if $\mu = 0$.

This lemma is technical to prove, so we refer to [14] for a proof. However the main idea is somewhat elementary, so we sketch the proof in the special case when $d\mu(w) = f(w)dA(w)$ for some smooth compactly supported function f with zero total mass $\int f(z)dA(z) = 0$. Then, using the divergence theorem to integrate by parts,

$$\begin{aligned}\int U^\mu(z)f(z)dA(z) &= \int U^\mu(z)\left(-\Delta U^\mu(z)\right)dA(z)\\ &= \lim_{r\to\infty}\left(\int_{|z|<r} \nabla U^\mu(z)\cdot\nabla U^\mu(z)dA(z)\right.\\ &\left.-\ 2\pi r\int_{-\pi}^{\pi} U^\mu(re^{i\theta})\partial_r U^\mu(re^{i\theta})d\theta\right)\\ &= \int \nabla U^\mu(z)\cdot\nabla U^\mu(z)dA(z) + 0\\ &= \int |\nabla U^\mu(z)|^2 dA(z) \geq 0,\end{aligned}$$

where the boundary term in the integration by parts vanishes because the vanishing of $\mu(\mathbb{C})$ in the multipole expansion (2.23) implies that

$$U^\mu(z)\partial_r U^\mu(z) = \mathcal{O}\left(\frac{1}{|z|}\cdot\frac{1}{|z|^2}\right) = \mathcal{O}\left(\frac{1}{|z|^3}\right),$$

and thus decays sufficiently rapidly. Thus, the energy

$$\int U^\mu(z)f(z)dA(z) = \int\int \log\frac{1}{|z-w|}f(z)f(w)dA(z)dA(w)$$

is zero if and only if U^μ is zero, which implies that $f = -2\Delta U^\mu$ is zero.

We are now ready to prove our uniqueness and existence of the equilibrium measure.

Theorem 2.16. *There exists a unique Borel probability measure μ that minimizes (2.25) among all compactly supported Borel probability measures.*

Proof. Let (μ_n) be a sequence of compactly supported probability measures such that $I_Q(\mu_n) \to V_Q$. By Lemma 2.14 we may assume that this sequence is chosen so that $\operatorname{supp}\mu_n \subseteq \overline{D_R}$ with $R > 0$, for all n. By Lemma 2.13, since $\overline{D_R}$ is compact we can by passing to a subsequence also assume that (μ_n) converges to some compactly supported probability measure μ in the weak-star sense. This also implies that [13]

$$\mu_n \times \mu_n \to \mu \times \mu,$$

in weak-star sense, that is

$$\iint f\mu_n \times \mu_n \to \iint f\mu \times \mu$$

for all complex-valued continuous f on $\overline{D_R} \times \overline{D_R}$. Unfortunately, we cannot immediately conclude from this that $I_Q(\mu_n) \to I_Q(\mu)$, because $\log\frac{1}{|z-w|}$ and Q are not continuous, bounded functions. However, we can do a regularization to get a one-sided bound: for all $M > 0$ we have

$$\iint \min\left\{\log\frac{1}{|z-w|}, M\right\} d\mu_n(w)d\mu_n(z) \to \iint \min\left\{\log\frac{1}{|z-w|}, M\right\} d\mu(w)d\mu(z).$$

Hence,

$$\liminf_{n\to+\infty}\iint \log\frac{1}{|z-w|}d\mu_n(w)d\mu_n(z) \geq \iint \min\left\{\log\frac{1}{|z-w|}, M\right\} d\mu(w)d\mu(z),$$

for all $M > 0$. Letting $M \to +\infty$ we have by the monotone convergence theorem that

$$\liminf_{n\to+\infty}\iint \log\frac{1}{|z-w|}d\mu_n(w)d\mu_n(z) \geq \iint \log\frac{1}{|z-w|}d\mu(w)d\mu(z).$$

Now we just apply a similar approach to the Q-integral: since by assumption Q is lower semi-continuous, there is an increasing sequence of continuous functions $f_1 \leq f_2 \leq \ldots \leq Q$ such that $f_m \to Q$ pointwise. For each f_m we thus have

$$\int f_m d\mu_n \to \int f_m d\mu.$$

By a similar reasoning as before we get that

$$\liminf_{n\to+\infty}\int Q d\mu_n \geq \int Q d\mu.$$

In total we finally have

$$\begin{aligned}
V_Q &= \liminf_{n\to+\infty} I_Q(\mu_n)\\
&= \liminf_{n\to+\infty}\left(\iint \log\frac{1}{|z-w|}d\mu_n(w)d\mu_n(z) + \int Q d\mu_n\right)\\
&\geq \iint \log\frac{1}{|z-w|}d\mu(w)d\mu(z) + \int Q d\mu\\
&= I(\mu).
\end{aligned}$$

Since μ is a compactly supported probability measure, we have by definition $V_Q \leq I(\mu)$, which together with the above result shows that $I(\mu) = V_Q$. Moreover, Q is bounded on $\operatorname{supp}\mu$ because $V_Q < +\infty$, which shows that μ has finite (unweighted) logarithmic energy:

$$-\infty < \iint \log\frac{1}{|z-w|}d\mu(w)d\mu(z) < \infty.$$

We now turn to uniqueness. Suppose another measure $\tilde{\mu}$ satisfies $I(\tilde{\mu}) = V_Q$. Then by the same reasoning as for μ, also $\tilde{\mu}$ must have finite logarithmic energy and satisfy $\operatorname{supp}\tilde{\mu} \subseteq \overline{D_R}$. Since $(\mu - \tilde{\mu})(\mathbb{C}) = 0$, it follow by Lemma 2.15 that

$$J := \iint \log\frac{1}{|z-w|}d\frac{1}{2}(\mu-\tilde{\mu})(z)d\frac{1}{2}(\mu-\tilde{\mu})(w) \geq 0,$$

with equality if and only if $J = 0$. Moreover, we also see that by the monotone convergence theorem

$$\begin{aligned}
&I_Q(\frac{1}{2}(\mu+\tilde{\mu})) + J =\\
&= \iint \log\frac{1}{|z-w|}d\frac{1}{2}(\mu+\tilde{\mu})(z)d\frac{1}{2}(\mu+\tilde{\mu})(w) + \int Q(z)d\frac{1}{2}(\mu+\tilde{\mu})(z)
\end{aligned}$$

$$
\begin{aligned}
&+\iint \log\frac{1}{|z-w|}d\frac{1}{2}(\mu-\tilde{\mu})(z)d\frac{1}{2}(\mu-\tilde{\mu})(w)\\
&=\lim_{M\to+\infty}\Big(\iint \min\Big\{\log\frac{1}{|z-w|},M\Big\}\,d\frac{1}{2}(\mu+\tilde{\mu})(z)d\frac{1}{2}(\mu+\tilde{\mu})(w)+\int Qd\frac{1}{2}(\mu+\tilde{\mu})\\
&+\iint \min\Big\{\log\frac{1}{|z-w|},M\Big\}\,d\frac{1}{2}(\mu-\tilde{\mu})(z)d\frac{1}{2}(\mu-\tilde{\mu})(w)\Big)\\
&=\lim_{M\to+\infty}\Big(\frac{1}{2}\iint \min\Big\{\log\frac{1}{|z-w|},M\Big\}\,d\mu(z)d\mu(w)+\frac{1}{2}\int Qd\mu\\
&\frac{1}{2}\iint \min\Big\{\log\frac{1}{|z-w|},M\Big\}\,d\tilde{\mu}(z)d\tilde{\mu}(w)+\frac{1}{2}\int Qd\tilde{\mu}\Big)\\
&=\frac{1}{2}I_Q(\mu)+\frac{1}{2}I_Q(\tilde{\mu})\\
&=V_Q.
\end{aligned}
$$

However, by the definition of V_Q we must also have

$$
I_Q(\frac{1}{2}(\mu+\tilde{\mu}))+J\geq V_Q+J,
$$

so that in fact $J=0$, which shows that $\mu=\tilde{\mu}$. □

On account of Theorem 2.16, we can now define the **equilibrium measure** as the unique minimizer of (2.25), which we shall denote by σ, and its support by $S=\operatorname{supp}\sigma$. We can also consider the **equilibrium potential**:

$$
U^\sigma(z):=\int \log\frac{1}{|z-w|}d\sigma(w). \tag{2.28}
$$

Its properties will play a major role in the bound of the one-point function (1.6), to be proven in Section 3. For a thorough discussion on the equilibrium measure and potential, we refer to the standard reference [14].

Chapter 3

Low Temperature Localization

Recall that the droplet $S = \operatorname{supp} \sigma$ where σ is the unique measure minimizing the logarithmic energy (2.25). In this section we will go through the proof of Theorem 1 in [1], which states that near the outside of the droplet, the one-point function (1.9) of the Coulomb gas (1.3) satisfies

$$R_n^\beta(z) \leq C^\beta n^2 e^{-c\beta n\delta(z)^2}, \tag{3.1}$$

where $\delta(z) = \inf\{|z-w|; w \in S\}$ is the distance from z to S, $C, c > 0$ are constants and n, β sufficiently large (exact statement will be given as Theorem 3.8 at the end of this section). As R_n^β measures the amount of particles per unit area, (3.1) illustrates that the particle intensity just outside S will be very low for large n or β, and thus that the Coulomb gas (1.3) is localized around S as $n, \beta \to +\infty$. The bound (3.1) is believed to be nearly sharp, see the discussion at the end of this chapter.

For convenience, we restate the conditions on the external potential $Q : \mathbb{C} \to [0, +\infty]$ given in Section 2.2.

1. $\liminf_{|z|\to+\infty} \frac{Q(z)}{2\log|z|} > 1$.
2. The set $\Sigma = \{Q(z) < \infty\}$ has nonempty interior.
3. Q is lower semi-continuous.

Throughout this section, we also impose the following additional assumptions on Q:

(i) Q is strictly subharmonic in a neighbourhood of the boundary ∂S.

(ii) The boundary ∂S has finitely many components.

(iii) Each component of ∂S is an everywhere smooth C^1 Jordan curve.

(iv) $S^* = S$ where S^* is the coincidence set for the obstacle problem given in section 3.2.

(v) Q is C^2 on $\Sigma = \{Q < +\infty\}$.

3.1 Weighted Polynomials

By the definition of R_n^β (1.9), we have

$$R_n^\beta(\zeta) = \lim_{\varepsilon\to 0} \frac{E_n^\beta[\{\text{number of particles in } D(\zeta, \varepsilon)\}]}{\varepsilon^2}$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0} \sum_{k=1}^{n} k P_n^\beta(\#(D(\zeta,\varepsilon)\cap\{\zeta_j\}_1^n)=k),\\
&= \lim_{\varepsilon\to 0} \sum_{k=1}^{n} k \binom{n}{k} P_n^\beta(\zeta_1\in D(\zeta,\varepsilon),\ldots,\zeta_k\in D(\zeta,\varepsilon)),
\end{aligned}
$$

with $dP_n^\beta = \frac{1}{Z_n^\beta} e^{-\beta H_n} dA_n$ being the Gibbs measure of the Coulomb gas (1.3). Thus, in order to control R_n^β we simply need to control the probabilities

$$
P_n^\beta(\zeta_1\in D(\zeta,\varepsilon),\ldots,\zeta_k\in D(\zeta,\varepsilon)). \tag{3.2}
$$

It turns out that there exists a certain class of "weighted polynomials" that serve as good tools to deal with this.

Let $\mathcal{W}_n$ be the space of functions f of the form $f = q\cdot e^{-nQ/2}$, where q is a holomorphic polynomial of degree at most $n-1$, and Q as usual is the external potential. Given a vector $\boldsymbol{z} = (z_j)_1^n \in \mathbb{C}^n$, define the (weighted) **Lagrange polynomial** in z as

$$
l_j^{(\boldsymbol{z})}(z) = e^{-n(Q(z)-Q(z_j))/2} \prod_{k\neq j} \frac{z-z_k}{z_j-z_k}. \tag{3.3}
$$

We then have $l_j^{(\boldsymbol{z})}\in\mathcal{W}_n$, and these polynomials are intimately connected with the Coulomb gas density via the following remarkable identity:

$$
\begin{aligned}
|l_j^{(\boldsymbol{z})}(w)|^{2\beta} e^{-\beta H_n(\boldsymbol{z})} &= \left(\prod_{k\neq j}\left|\frac{w-z_k}{z_j-z_k}\right|^{2\beta}\right)\left(\prod_{1\le k<m\le n}|z_k-z_m|^{2\beta}\right)\\
&\quad\cdot e^{-n\beta(Q(w)-Q(z_j))-n\beta\sum_{k=1}^n Q(z_k)}\\
&= e^{-\beta H_n(\boldsymbol{z}|z_j=w)},
\end{aligned} \tag{3.4}
$$

where $\boldsymbol{z}|z_j = w$ is the $\boldsymbol{z}$-vector but with the j'th component replaced by w. We now show how this identity can aid in computing the probabilities (3.2).

Lemma 3.1. *[1, Lemma 2.5] Let $W\subseteq\mathbb{C}$ be a measurable set and define*

$$
X_j^W = \int_W |l_j^{(\boldsymbol{\zeta})}(w)|^{2\beta} dA(w). \tag{3.5}
$$

Then, for all measurable $U\subseteq\mathbb{C}$

$$
E_n^\beta[\mathbf{1}_U(\zeta_j)X_j^W] = |U| p_n^\beta(W), \tag{3.6}
$$

where $p_n^\beta(W) := P_n^\beta(\zeta_j\in W) = P_n^\beta(\zeta_1\in W)$.

Proof. It follows by Fubini's theorem that

$$
\begin{aligned}
E_n^\beta[X_j^W] &= \int_{\mathbb{C}^n} \mathbf{1}_U(\zeta_j)\int_W |l_j^{(\boldsymbol{\zeta})}(w)|^{2\beta}\frac{1}{Z_n^\beta} e^{-\beta H_n(\boldsymbol{\zeta})} dA(w) dA_n(\boldsymbol{\zeta})\\
&= \int_{\mathbb{C}^n}\int_{\mathbb{C}} \mathbf{1}_U(\zeta_j)\mathbf{1}_W(w)\frac{1}{Z_n^\beta} e^{-\beta H_n(\boldsymbol{\zeta}|\zeta_j=\eta)} dA(w) dA_n(\boldsymbol{\zeta})\\
&= \int_{\mathbb{C}}\int_{\mathbb{C}^n} \mathbf{1}_U(\zeta_j)\mathbf{1}_W(w)\frac{1}{Z_n^\beta} e^{-\beta H_n(\boldsymbol{\zeta}|\zeta_j=w)} dA_n(\boldsymbol{\zeta}) dA(w)
\end{aligned}
$$

$$
\begin{aligned}
&= |U| \int_{\mathbb{C}^n} \mathbf{1}_W(w) \frac{1}{Z_n^\beta} e^{-\beta H_n(\boldsymbol{\zeta}|\zeta_j = w)} dA_n(\boldsymbol{\zeta}|\zeta_j = w) \\
&= |U| p_n^\beta(W).
\end{aligned}
$$

□

The Lagrange polynomials together with (a version of) Lemma 3.1 were also used [2] to study the separation of particles for large values of β.

We can extend the identity (3.4) to the two variable case as

$$
\begin{aligned}
|l_k^{(\boldsymbol{z}|z_j=w_j)}(w_k)|^{2\beta} |l_j^{(\boldsymbol{z})}(w_j)|^{2\beta} e^{-\beta H_n(\boldsymbol{z})} &= |l_k^{(\boldsymbol{z}|z_j=w_j)}(w_k)|^{2\beta} e^{-\beta H_n(\boldsymbol{z}|z_j=w_j)} \\
&= e^{-\beta H_n(\boldsymbol{z}|z_k=w_k, z_j=w_j))},
\end{aligned}
$$

when $j \neq k$. This process may be iterated to obtain a generalization of Lemma 3.1.

Lemma 3.2. *[1, Lemma 2.6] Let k be an integer, $1 \leq k \leq n$. It then holds that*

$$
\left(\prod_{j=1}^{k} |l_j^{(\boldsymbol{\zeta}|\zeta_1=w_1,\ldots,\zeta_{j-1}=w_{j-1})}(w_j)|^{2\beta} \right) e^{-\beta H_n(\boldsymbol{z})} = e^{-\beta H_n(\boldsymbol{z}|\zeta_1=w_1,\ldots,\zeta_k=w_k)}. \tag{3.7}
$$

Furthermore, if $W_1, \ldots, W_k, U_1, \ldots, U_k$ are measurable subsets of $\mathbb{C}$ and we define the random variable

$$
X_{1,\ldots,k}^{W_1,\ldots,W_k} = \int_{W_1} \cdots \int_{W_k} \prod_{j=1}^{k} |l_j^{(\boldsymbol{\zeta}|\zeta_1=w_1,\ldots,\zeta_{j-1}=w_{j-1})}(w_j)|^{2\beta} dA(w_1) \cdots dA(w_k),
$$

we have that

$$
E_n^\beta \left[\left(\prod_{j=1}^{k} \mathbf{1}_{U_j}(\zeta_j) \right) X_{1,\ldots,k}^{W_1,\ldots,W_k} \right] = |U_1| \cdots |U_k| p_{n,k}(W_1, \ldots, W_k). \tag{3.8}
$$

Moreover, the identities (3.7) and (3.8) remain valid under any permutation of the indices $1, \ldots, k$. Here of course $p_{n,k}(W_1, \ldots, W_k) = P_n^\beta(\{\zeta_1 \in W_1, \ldots, \zeta_k \in W_k\})$.

Proof. By Lemma 3.1, (3.7) and (3.8) holds whenever $k = 1$. Suppose (3.7) is valid up to some $k = p$, $p < n$. Then,

$$
\begin{aligned}
&\left(\prod_{j=1}^{p+1} |l_j^{(\boldsymbol{\zeta}|\zeta_1=w_1,\ldots,\zeta_{j-1}=w_{j-1})}(w_j)|^{2\beta} \right) e^{-\beta H_n(\boldsymbol{\zeta})} = \\
&= |l_{p+1}^{(\boldsymbol{\zeta}|\zeta_1=w_1,\ldots,\zeta_p=w_p)}(w_{p+1})|^{2\beta} e^{-\beta H_n(\boldsymbol{\zeta}|\zeta_1=w_1,\ldots,\zeta_p=w_p)} \\
&= e^{-\beta H_n(\boldsymbol{\zeta}|\zeta_1=w_1,\ldots,\zeta_p=w_p,\zeta_{p+1}=w_{p+1})},
\end{aligned}
$$

and thus (3.7) holds also for $k = p + 1$. By (finite) induction we have thus proven (3.7) for all $1 \leq k \leq n$.

To prove (3.8), we note that by (3.7) and Fubini we have

$$
E_n^\beta \left[\left(\prod_{j=1}^{k} \mathbf{1}_{U_j}(\zeta_j) \right) X_{1,\ldots,k}^{W_1,\ldots,W_k} \right] =
$$

$$= |U_1| \cdots |U_k| \int_{\mathbb{C}^n} \left(\prod_{j=1}^{k} \mathbf{1}_{W_j}(w_j) \right) \frac{1}{Z_n^\beta} e^{-\beta H_n(\boldsymbol{\zeta}|\zeta_1 = w_1, \ldots, \zeta_k = w_k)} d\tilde{A}_{n,k}$$
$$= |U_1| \cdots |U_k| p_{n,k}(W_1, \ldots, W_m),$$

where $d\tilde{A}_{n,k} = dA(w_1) \cdots dA(w_k) dA(z_{k+1}) \cdots dA(z_n)$.

The fact that (3.7) and (3.8) are invariant under permutations of the indices follows from the invariance of P_n^β under permutations. □

In the following we'll write

$$p_{n,k}(W) = p_{n,k}(W, W, \ldots, W).$$

3.2 The Effective Potential

Lemma 3.1 and 3.1 provide the connection to the probabilities (3.2) in the one-point function that we need. In order to get the bound on R_n^β (3.1) we thus only need a bound on $l_j^{\boldsymbol{\zeta}}$ near the droplet S. The key to this will be to analyze a special function known as the "effective potential", which is closely related to our external potential Q.

Let $\mathcal{F}_Q$ denote the set of subharmonic functions u on $\mathbb{C}$ which satisfy $u \leq Q$ and $u(\zeta) \leq 2 \log |\zeta| + \mathcal{O}(1)$ as $z \to \infty$, and define $\check{Q}(z)$ as the pointwise supremum:

$$\check{Q}(z) = \sup\{u(z), u \in \mathcal{F}_Q\} \tag{3.9}$$

The function $\check{Q}$ is known as the **obstacle function** corresponding to the obstacle Q, the external potential. As we shall soon see, $\check{Q}$ is the solution to a certain free boundary value problem, which is a partial differential equation where both the function and the boundary of the domain is unknown. More precisely, consider the problem of determining a function $v : \mathbb{C} \to \mathbb{R}$ and a **coincidence set** S^* (which we for the moment just think of as an arbitrary set, not related to the droplet S) such that

(a) $v \leq Q$ everywhere on $\mathbb{C}$.

(b) $v = Q$ on S^* and harmonic otherwise; $\Delta v = 0$ on $\mathbb{C} \setminus S^*$.

(c) $v(\zeta) = 2 \log |\zeta| + \mathcal{O}(1)$ as $|z| \to +\infty$.

The above problem is also known as the **obstacle problem**. Similar free boundary value problems also show up in the pricing of American options in finance [4], and in the study of water waves [6], for example. We now show that $v = \check{Q}$ solves this problem. It follows from the definition that $\check{Q} \leq Q$, but to prove that the two other constraints are also satisfied we shall need the following theorem.

Theorem 3.3. *Suppose Q is C^2 in a neighbourhood of the droplet $S = \operatorname{supp} \sigma$, with σ being the equilibrium measure which minimizes the weighted logarithmic energy $I_Q(\cdot)$ in (2.25), and $V_Q = I_Q(\sigma)$ is the minimal energy. Define the (modified) Robin's constant*

$$\gamma := 2V_Q - \int Q \, d\sigma, \tag{3.10}$$

Then, the potential U^σ of the equilibrium measure σ,

$$U^\sigma(z) = \int \log \frac{1}{|z-w|} d\sigma(w) \tag{3.11}$$

satisfies

$$2U^\sigma(z) + Q(z) \geq \gamma, \tag{3.12}$$

everywhere on $\mathbb{C}$, and

$$2U^\sigma(z) + Q(z) = \gamma, \tag{3.13}$$

on S.

The proof of Theorem 3.3 above involves theory of capacities of sets, which would require a slight detour, and we therefore choose to skip it and refer to [14, Theorem 1.3] for a full proof. For a general Q, Theorem 3.3 is only true "quasi-everywhere", that is, up to a set of infinite logarithmic energy. In particular, quasi-everywhere implies dA-almost everywhere. For details we again refer to [14].

We can now show the following theorem.

Theorem 3.4. *Let $\check{Q}$ be the obstacle function (3.9). Then*

$$\check{Q} = \gamma - 2U^\sigma, \tag{3.14}$$

where U^σ, γ are as in Theorem 3.3.

Proof. Let

$$\tilde{Q} = \gamma - 2U^\sigma,$$

It follows from Theorem 3.3 that $\tilde{Q} \in \mathcal{F}$, and that $\tilde{Q} = Q$ on S (since Q is C^2 there). Moreover, the multipole expansion (2.23) of U^σ shows that $\tilde{Q} = \log|z|^2 + \mathcal{O}(1)$ as $\zeta \to \infty$.

Now, consider an arbitrary $u \in \mathcal{F}_Q$ and form the function

$$\tilde{u}(z) = \max(u(z), \tilde{Q}(z)) \in \mathcal{F}_Q.$$

It follows that $\tilde{u}(z) = \log|z|^2 + \mathcal{O}(1)$ and $\tilde{u} = \tilde{Q}$ on S. The function $\tilde{u} - \tilde{Q}$ thus vanishes on S and is subharmonic and bounded outside S, since $\tilde{Q}$ is harmonic there. By the maximum principle for subharmonic functions, $\tilde{u} - \tilde{Q}$ is constant; since $\tilde{u} - \tilde{Q} = 0$ on S this constant must be 0. We conclude that $\tilde{u} = \tilde{Q}$ everywhere. Since $u \in \mathcal{F}_Q$ was arbitrary, this shows that

$$\tilde{Q}(z) = \sup\{u(z), u \in \mathcal{F}_Q\} = \check{Q}(z).$$

□

The coincidence set S^* in the obstacle problem can be expressed as

$$S^* = \{(\check{Q} - Q)(z) = 0\}. \tag{3.15}$$

In general it holds that $S \subseteq S^*$ with possible strict inclusion [9], but our assumption (iv) eliminates this case. Since $S = S^*$ by (iv), combining Theorem 3.3 and 3.4 shows that $\check{Q}$ satisfies $\check{Q} = Q$ on S, and also that $\check{Q} = 2\log|\zeta| + \mathcal{O}(1)$ as $|\zeta| \to +\infty$ since we by the multipole expansion (2.23) of a potential of a probability measure have $U^\sigma = -\log|\zeta| + \mathcal{O}(1)$, as $|\zeta| \to +\infty$. Thus, $(\check{Q}, S)$ solves the obstacle problem.

We now define the **effective potential** as

$$Q^{\text{eff}} := Q - \check{Q}. \tag{3.16}$$

We note that by definition, $Q^{\text{eff}} = 0$ on the droplet S and furthermore that $Q^{\text{eff}}(z) \to +\infty$ as $|z| \to +\infty$ since the growth condition on Q (1.1) means that Q outgrows $\check{Q}(z) = 2\log|z| + \mathcal{O}(1)$ at infinity. Furthermore, since $\check{Q}$ is harmonic outside of S, we have

$$\Delta Q^{\text{eff}} = \Delta Q - \Delta\check{Q} = \Delta Q,$$

outside of S. It can be shown [14] that the potential U^σ is $C^{1,1}$-smooth, meaning that its gradient satisfies a Lipschitz condition. By Theorem 3.4 this also applies to $\check{Q}$ and, since Q is C^2 on $\Sigma = \{Q < +\infty\}$, we also have that Q^{eff} is $C^{1,1}$-smooth.

The effective potential Q^{eff} may be used to study the decay of the Lagrange polynomials $l_j^{(z)}$ from Section 3.1 near the droplet with the help of the following theorem, which is sometimes called the "weighted maximum principle" for its close resemblance to the usual maximum principle for subharmonic functions, Theorem 2.9.

Lemma 3.5. *[1, Lemma 2.3] For $f = q \cdot e^{-nQ/2} \in \mathcal{W}_n$ we have the bound*

$$|f(z)| \leq ||f||_{L^\infty(S)} \cdot e^{-nQ^{\text{eff}}(z)/2}, \tag{3.17}$$

where

$$||f||_{L^\infty(S)} = \sup_{z\in S} |f(z)|,$$

is the supremum norm of f over S.

Proof. Without loss of generality, we may assume $||f||_{L^\infty(S)} = 1$. Taking the logarithm yields that

$$\frac{1}{n}\log|q(z)|^2 = \frac{1}{n}\log|f(z)|^2 + Q(z). \tag{3.18}$$

The function $u := \frac{1}{n}\log|q|^2$ is harmonic everywhere except at a finite number of points (the zero set of q) where it equals $-\infty$, and is thus subharmonic on $\mathbb{C}$. Furthermore, it satisfies $u \leq Q$ on S since $f \leq 1$ there. Then, since $Q = \check{Q}$ on S we also have $u \leq \check{Q}$ on S and the harmonicity of $\check{Q}$ outside S shows that $u - \check{Q}$ is subharmonic in S^c. By the maximum principle for subharmonic functions $u \leq \check{Q}$ everywhere, which means that

$$\frac{1}{n}\log|f(z)|^2 + Q(z) \leq \check{Q}(z), \tag{3.19}$$

or

$$|f(z)| \leq e^{n(\check{Q}(z)-Q(z))/2}, \tag{3.20}$$

which is (3.17). ☐

Theorem 3.5 is precisely the tool we need to bound the Lagrange polynomials outside of S, but in order to utilise it we first need one bound on the supremum norm $||f||_{L^\infty(S)}$, and one more on the effective potential Q^{eff}. We begin with Q^{eff}.

Lemma 3.6. *[1, Lemma 2.1] There is a constant $a_0 > 0$ such that*

$$Q^{\textit{eff}}(z) \geq 2\min\{c\delta(z)^2, a_0\}, \tag{3.21}$$

holds for any $c < c_0$, where

$$c_0 = \min\{\Delta Q(\eta); \eta \in \partial S\}. \tag{3.22}$$

Proof. Consider a boundary point $p \in \partial S$ of S, and let N, T be the outwards unit normal and (positively oriented) unit tangent at p. Our aim is to Taylor expand Q^{eff} a small distance outside S. Since Q^{eff} is only smooth strictly outside of S, we let V be a C^2 extension of $Q|_{S^c}$ to a neighbourhood of ∂S, and expand V instead. We let $\partial_N V(p), \partial_T V(p)$ denote the corresponding directional derivatives at p.

By the $C^{1,1}$-continuity of Q^{eff} together with the fact that $Q^{\text{eff}} = 0$ on S, we have that $\partial_N V(p) = 0$. For the same reason we also have $V = 0$ identically on ∂S, so that in particular $\partial_T V(p) = \partial_T^2 V(p) = 0$. However, by assumption $(\partial_N^2 + \partial_T^2)Q = 4\Delta Q > 0$ in a neighbourhood of ∂S and $\Delta \check{Q}$ is harmonic outside S, so

$$\partial_N^2 V(p) = 4\Delta V(p) = 4\Delta Q(p) \geq 4c_0 > 0. \tag{3.23}$$

We are now ready to invoke Taylor's theorem; consider a small positive number δ, then

$$\begin{aligned} Q^{\text{eff}}(p+\delta N) &= V(p+\delta N) \\ &= V(p) + \frac{1}{2}\partial_N V(p)\delta + \partial_N^2 V(p)\delta^2 + o(\delta^2) \\ &= \frac{1}{2}\partial_N^2 V(p)\delta^2 + o(\delta^2) > 2c\delta^2 > 0 \quad (\delta \to 0^+), \end{aligned} \tag{3.24}$$

holds for all positive δ smaller than some number δ_0. Therefore, $Q^{\text{eff}}(z) > 2c\delta(z$ on the set $\{\delta(z) < \delta_0\}$.

Now, by the growth assumption (1.1) and Theorem 3.4 we see that $Q^{\text{eff}}(z) \to \infty$ as $|z| \to \infty$. Since Q is lower semi-continuous and $\check{Q}$ is $C^{1,1}$, the effective potential Q^{eff} is lower semi-continuous and thus attains a strict minimum $2a_0$ over the set $\{\delta(z) \geq \delta_0\}$. In conclusion,

$$Q^{\text{eff}}(z) \geq \begin{cases} 2c\delta(z) & \delta(z) < \delta_0 \\ 2a_0 & \delta(z) \geq \delta_0, \end{cases} \tag{3.25}$$

from which (3.21) follows. □

We now turn to analyze the supremum norm $||f||_{L^\infty(S)}$. To this end, let V be a neighbourhood of S, small enough so that ΔQ is continuous and positive in a neighbourhood V_1 of its closure $\overline{V}$. Such a neighbourhood exists because of assumption (i). We then have the following pointwise bound.

Lemma 3.7. *[1, Lemma 2.4] Let $f \in \mathcal{W}_n$, let s be any number satisfying*

$$s > \max\{\Delta Q(z); \zeta \in \overline{V}\}. \tag{3.26}$$

Then for $z_0 \in V$,

$$|f(z_0)|^{2\beta} \leq ne^{s\beta} \int_{D(z_0, 1/\sqrt{n})} |f|^{2\beta} dA, \tag{3.27}$$

holds for all n bigger than some $n_0 > 0$.

Proof. Consider the auxiliary function

$$F(z) = |f(z)|^{2\beta} e^{sn\beta|z_0 - z|^2}. \tag{3.28}$$

Since $f = q \cdot e^{-nQ/2}$ we have

$$\begin{aligned}\Delta \log F(z) &= 2\beta\Delta \log |q(z)| - n\beta Q(z) + sn\beta\Delta|z - z_0|^2 \\ &\geq -n\beta Q(z) + sn\beta \geq 0, \end{aligned} \tag{3.29}$$

for $z \in V$, where the first inequality is due to $\log|q|$ being subharmonic and $\Delta|z - z_0|^2 = \partial\overline{\partial}(z - z_0)\overline{(z - z_0)} = 1$. Thus, $\log F$ is subharmonic, and by Theorem 2.7 F is thus itself subharmonic. Now, we apply the mean value inequality for subharmonic functions to conclude that

$$\begin{aligned}|f(z)|^{2\beta} &= F(z_0) \\ &\leq n \int_{D(z_0,1/\sqrt{n})} F dA \\ &= n \int_{D(z_0,1/\sqrt{n})} |f(w)|^{2\beta} e^{sn\beta|w-z_0|^2} dA(w) \\ &\leq ne^{s\beta} \int_{D(z_0,1/\sqrt{n})} |f|^{2\beta} dA, \end{aligned} \tag{3.30}$$

which holds for any n large enough so that $D(z_0, 1/\sqrt{n})$ is strictly contained in V. ☐

Using Lemma 3.7, we can now obtain the following uniform bound of the Lagrange polynomials

$$|l_j^{(\boldsymbol{z})}(z)|^{2\beta} \leq ne^{s\beta} \int_{D(z_0,1/\sqrt{n})} |l_j^{(\boldsymbol{z})}|^{2\beta} dA \leq ne^{s\beta} \int_{\mathbb{C}} |l_j^{(\boldsymbol{z})}|^{2\beta} dA,$$

and thus

$$||l_j^{(\boldsymbol{z})}||^{2\beta}_{L^\infty(S)} \leq ne^{s\beta} \int_{\mathbb{C}} |l_j^{(\boldsymbol{z})}|^{2\beta} dA.$$

The last integral is familiar to us as it shows up in Lemma 3.1.

3.3 Bounding the One-point Function

We are now in position to prove the bound on the one-point function R_n^β (3.1). The idea is to to use Lemma 3.1 and 3.2 to bound the probabilities in the expression of R_n^β (3.2) with expectations of Lagrange polynomials $l_j^{(\boldsymbol{z})}$ (3.3), and applying the weighted maximum principle (3.5).

Theorem 3.8. *[1, Theorem 1] Write $\delta(z) = d(z, S)$ for the distance to the droplet. There are then constants C and $c > 0$ such that*

$$R_n^\beta(z) \leq C^\beta n^2 e^{-c\beta n\delta(z)^2},$$

for all n greater than some $n_0 > 0$.

Proof. Let $\varepsilon > 0$, small enough so that the disk $W = D(z, \varepsilon)$ is contained in $\overline{V} \setminus S$. Define the random variables

$$X_j^W = \int_W |l_j^{(\boldsymbol{\zeta})}(w)|^{2\beta} dA(w) \tag{3.31}$$

$$Z_j = \int_{\mathbb{C}} |l_j^{(\zeta)}(w)|^{2\beta} dA(w). \tag{3.32}$$

We would like to provide an estimate for $E_n^\beta[\mathbf{1}_U(\zeta_j)X_j]$ and we do so by the following: by Lemma 3.5 and 3.6 and we have the bound

$$|l_j^{(\zeta)}(z)| \leq ||l_j^{(\zeta)}||_{L^\infty(S)} e^{-nQ^{\mathrm{eff}}(z)/2} \leq ||l_j^{(\zeta)}||_{L^\infty(S)} e^{-nc\delta(z)^2}. \tag{3.33}$$

Now, to get a bound for $||l_j^{(\zeta)}||_{L^\infty(S)}$ we first note that by Lemma 3.7,

$$|l_j^{(\zeta)}(z)|^{2\beta} \leq ne^{s\beta} \int_{D(z,1/\sqrt{n})} |l_j^{(\zeta)}|^{2\beta} dA \leq nC^\beta \int_{\mathbb{C}} |l_j^{(\zeta)}|^{2\beta} dA = nC^\beta Z_j, \tag{3.34}$$

with $C = e^s$. This bound is uniform in z, so in particular we have $||l_j^{(\zeta)}||_{L^\infty(S)}^{2\beta} \leq ne^{s\beta} Z_j$. In total we thus have

$$|l_j^{(\zeta)}(z)|^{2\beta} \leq nC^\beta e^{-2\beta nc\delta(z)^2} Z_j. \tag{3.35}$$

Integrating over W,

$$\begin{aligned} X_j^W &= \int_W |l_j^{(\zeta)}(w)|^{2\beta} dA(w) \\ &\leq Z_j nC^\beta \int_W e^{-2\beta nc\delta(w)^2} dA(w) \\ &\leq \varepsilon^2 nC^\beta e^{-2\beta nc\delta^2} Z_j, \end{aligned}$$

where $\delta = \inf\{|z-w|; w \in W, z \in S\}$ is the distance between W and S. Now we can multiply by $\mathbf{1}_U(\zeta_j)$, take expectations and use Lemma 3.1 to conclude that

$$|U| p_n^\beta(W) \leq |U| \varepsilon^2 nC^\beta e^{-2\beta nc\delta^2}, \tag{3.36}$$

from which it follows that

$$p_n^\beta(W) \leq \varepsilon^2 nC^\beta e^{-2\beta nc\delta^2}. \tag{3.37}$$

What is left is to use Lemma 3.2 to obtain a similar bound for $p_{n,k}^\beta(W)$, $1 < k \leq n$. We mimic the idea above: first we define the random variables

$$\begin{aligned} \tilde{X}_j^W &= \int_W |l_j^{(\zeta|\zeta_1=w_1,\ldots,\zeta_{j-1}=w_{j-1})}(w_j)|^{2\beta} dA(w_j) \\ \tilde{Z}_j &= \int_{\mathbb{C}} |l_j^{(\zeta|\zeta_1=w_1,\ldots,\zeta_{j-1}=w_{j-1})}(w_j)|^{2\beta} dA(w_j), \end{aligned}$$

for $1 \leq j \leq k$. By the same reasoning as before, we then have the bound

$$\tilde{X}_j^W \leq \varepsilon^2 nC^\beta e^{-2\beta nc\delta^2} \tilde{Z}_j. \tag{3.38}$$

Since $\tilde{X}_j^W, \tilde{Z}_j$ are non-negative this implies

$$\prod_{j=1}^k \tilde{X}_j^W \leq \varepsilon^{2k} q^k \prod_{j=1}^k \tilde{Z}_j, \tag{3.39}$$

where $q = nC^\beta e^{-2\beta cn\delta}$. We note that, with notation from Lemma 3.2,

$$\begin{aligned}\prod_{j=1}^{k}\tilde{X}_j^W &= \int_W\cdots\int_W\prod_{j=1}^{k}|l_j^{(\zeta|\zeta_1=w_1,\dots,\zeta_{j-1}=w_{j-1})}(w_j)|^{2\beta}dA(w_1)\cdots dA(w_k)\\ &= X_{1,\dots,k}^{W,\dots,W}\\ \prod_{j=1}^{k}\tilde{Z}_j &= \int_{\mathbb{C}}\cdots\int_{\mathbb{C}}\prod_{j=1}^{k}|l_j^{(\zeta|\zeta_1=w_1,\dots,\zeta_{j-1}=w_{j-1})}(w_j)|^{2\beta}dA(w_1)\cdots dA(w_k)\\ &= Z_{1,\dots,k}\end{aligned}$$

Thus, (3.39) together with Lemma 3.2 gives

$$p_{n,k}^\beta(W) \le \varepsilon^{2k}q^k. \tag{3.40}$$

It follows that

$$\begin{aligned}E_n^\beta(\#(W\cap\{\zeta_j\}_1^n)) &= \sum_{k=1}^{n}kP_n^\beta(\#(W\cap\{\zeta_j\}_1^n = k))\\ &\le \sum_{k=1}^{n}k\binom{n}{k}q^k\varepsilon^{2k}.\end{aligned} \tag{3.41}$$

Sending $\varepsilon \to 0$ we now conclude that

$$R_n^\beta(w_0) = \lim_{\varepsilon\to 0}\frac{E_n^\beta(\#(W\cap\{\zeta_j\}_1^n))}{\varepsilon^2} \le C^\beta n^2 e^{-cn\beta\delta(w_0)^2}. \tag{3.42}$$

□

Theorem 3.8 above illustrates again the fact that the configuration $\{\zeta_j\}_{j=1}^n$ tends to concentrate at the droplet S as $n \to +\infty$ and also at low temperatures ($\beta \to +\infty$).

It was recently shown [16, Theorem 2] that for the specialized case when $Q(z) = |z|^2$ (and other similar potentials), one has the bound $R_n^\beta(z) \le C^\beta n$ for all $z \in \mathbb{C}$, including in the droplet. It is therefore reasonable to conjecture that

$$R_n^\beta(z) \le C^\beta n e^{-c\beta n\delta(z)^2}, \tag{3.43}$$

outside S, for all potentials obeying the standard conditions 1, 2, 3 (compare to the n^2-factor in Theorem 3.8). Indeed, in the case $\beta = 2$, (3.43) has been known to hold in many situations [3].

As a final remark, we shall briefly discuss an alternative formulation of Lemma 3.1. Recall that the **marginal density** of ζ_1 is given by

$$f_1(\zeta_1) := \frac{dP_{n,1}^\beta}{dA} = \int_{\mathbb{C}^{n-1}}\frac{e^{-\beta H_n(\zeta_1,\zeta_2,\dots,\zeta_n)}}{Z_n^\beta}dA(\zeta_2)\cdots dA(\zeta_n).$$

If Q is continuous everywhere, which we shall assume from now on, f_1 is an everywhere continuous function. For $E \subseteq \mathbb{C}^{n-1}$ Borel we define the Gibbs probability measure **conditioned on** ζ_1 as

$$P_n^\beta(E|\zeta_1 = z) = \frac{1}{f_1(z)}\int_E\frac{e^{-\beta H_n(z,\zeta_2,\dots,\zeta_n)}}{Z_n^\beta}dA(\zeta_2)\cdots dA(\zeta_n), \tag{3.44}$$

for all $z \in \{f(z) > 0\}$, and $P_n^\beta(E|\zeta_1 = z) = 0$ otherwise. Then, mirroring the proof of Lemma 3.1, we see that

$$\begin{aligned}
E_n^\beta\left[|l_1^{(\boldsymbol{\zeta})}(w)|^{2\beta}|\zeta_1 = z\right] &= \frac{1}{f_1(z)}\int_{\mathbb{C}^{n-1}} |l_1^{(\boldsymbol{\zeta}|\zeta=z)}(w)|^{2\beta}\frac{e^{-\beta H_n(z,\zeta_2,\ldots,\zeta_n)}}{Z_n^\beta}dA(\zeta_2)\cdots dA(\zeta_n) \\
&= \frac{1}{f_1(z)}\int_{\mathbb{C}^{n-1}} \frac{e^{-\beta H_n(w,\zeta_2,\ldots,\zeta_n)}}{Z_n^\beta}dA(\zeta_2)\cdots dA(\zeta_n) \\
&= \frac{1}{f_1(z)}f_1(w).
\end{aligned}$$

By (1.9) we have the simple relation $nf_1 = R_n^\beta$, so we end up with the following proposition.

Proposition 3.9. *Denote by $E_n^\beta[\,\cdot\,|\zeta_1 = z]$ expectation with respect to the conditional Gibbs measure (3.44). Let R_n^β be the one-point function (1.5) and $l_1^{(\boldsymbol{\zeta})}$ the Lagrange polynomial (3.3). Then we have*

$$E_n^\beta\left[|l_1^{(\boldsymbol{\zeta})}(w)|^{2\beta}|\zeta_1 = z\right] = \frac{R_n^\beta(w)}{R_n^\beta(z)}, \tag{3.45}$$

whenever $z \neq w$ and z is such that $R_n^\beta(z) > 0$.

While (3.45) conveys the same type of relation as Lemma 3.1, it does so with a more direct link to the one-point function.

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Master's Theses in Mathematical Sciences 2023:E39
ISSN 1404-6342

LUNFMS-3118-2023

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