Time-dependent light-matter coupling and quantum dynamics

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Thesis submitted for the degree of Bachelor of Science
Project duration: 2 months
May 2023

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Acknowledgement

My deepest thanks to Kalle Kansanen and Peter Samuelsson for the opportunity to work with their research group. I want to acknowledge their support throughout the project and their pedagogical way of leading me into new concepts and subjects.

A special thanks to my family: my parents, brother and grandmother. I have had their unconditional support throughout my studies and beyond.

Finally, I want to thank my fellow desk neighbors. Emil, Francesco and Zakaria made hard days easier and good days better.

Popular Science Summary

Electrons and photons are well-known entities in physics. The former are particles in atoms: they are fundamental particles with mass. The latter are particles of light. When these two are close to each other they can start to interact. Then, the interaction can be seen as a coupling. This coupling depends on the magnitude of their mutual *coupling constant*. In general, a coupling constant is just that: a constant. In the thesis, however, it will be investigated what happens when the coupling constant is time-dependent and how the coupling will look like then. But let us take a step back and investigate what properties of the particles are considered.

To explain this coupling, light-matter coupling, quantum mechanics must be taken into account. Consider the energy of an electron. In contrast to classical mechanics, energies in quantum mechanics are *discrete*: only specific values of energies are allowed called energy levels. To understand discrete energies, imagine throwing an electron in the air like a ball. You would see the electron magically disappear from your hand, appear in the air above you and then teleport back to your hand. The electrons energy can, for example, take a value of 1 or 2 but never 1.5!

Like electrons, light has discrete energies as well. One "unit" of light with a specific energy is called a photon: a particle without mass. The concept of photons are vital for light matter interaction. As famous physicist Max Planck said: only light considered as photons can interact with electrons. For light-matter interaction to occur, a photons energy must match, or be greater than, the energy difference between two energy levels of an electron.

Now, light-matter coupling can be explained. Consider a photon trapped in a box where it bounces around. Introduce an electron which has two energy levels and assume that the difference in energy between these levels correspond to the energy of the photon. When the electron, in its lower energy level, enters the box, it will absorb the photon and gain its energy. This means that the electron jumps up to its higher energy level. Then, the electron will proceed to emit the photon and go down to its original energy. This process will repeat itself indefinitely inside the box: the particles are coupled and the electron oscillates between its states.

So what would happen if the coupling changed over time? This question is to be answered in the thesis. It is known that the coupling is dependent on other physical properties, like the position of the electron, but not time. However, the position of the electron can change over time, which means that dynamical properties of the electron could affect the coupling. Therefore, the thesis will investigate a time-dependent coupling in terms of the dynamics of the system.

Abstract

In this thesis, two quantum dynamical system's of one and two qubits coupled to a field mode confined in an optical cavity have been studied. Within the Jaynes- and Tavis-Cummings model, the dynamics of the corresponding light-matter interaction was defined in terms of the system's coupling constants. From the setup of the system, the qubits flying through the cavity, the final state of the system could be adjusted by changing the dynamics via the time-dependent coupling constants. In the two qubit system, it was found that the entanglement between the qubits could be modified by the dynamics to obtain a maximal entanglement in the final state. These findings provides a starting point of how entanglement can be achieved via quantum dynamical light-matter interaction.

1 Introduction

Light-matter coupling arises when charged elementary particles interact with electromagnetic radiation. Light and a particle, like an electron, exchange energy and momentum in two processes. First, an electron can absorb light in which the electron jumps to a higher energy level and possibly increases its momentum. Secondly, an electron can, spontaneous or by stimulation, emit a photon which has gained some energy and momentum from the electron. These processes are vital for many fields of study, such as spectroscopy. In spectroscopy, the light emitted from matter is studied, which can thereafter be analyzed and reveal the properties of the material.

Investigating light-matter interaction, it can be seen that a classical model is not enough to explain the processes. As Albert Einstein discovered, only quantized quantities of light, photons, can interact with matter [2]. This means that in explaining light-matter coupling, quantum mechanics must be used. Furthermore, this thesis is not only interested in a single emittance and absorption of a photon, but a system that repeats these processes. For this to be possible one must confine light in a volume of space, so the photons does not simply fly away.

Light can be confined in an apparatus called an optical cavity or optical resonator. Generally, it consists of a configuration of mirrors, aligned such that it acts as a resonator for electromagnetic radiation [8]. Then, light confined in the cavity, given that its waveform does not change, result in a contained field mode of radiation [8]. A qubit[i] inside such a cavity can continuously interact with the confined radiation. The qubit will periodically absorb and emit a photon to the confined mode, the criteria being that the frequency of the atomic transition is in close proximity of the field mode frequency. Therefore, the electron will continuously change states, giving rise to Rabi oscillations between its states [12]. The frequency of these oscillations are proportional to the coupling strength between the electron and the photons, determined by their mutual interaction coupling constant.

The coupling strength is usually compared to the frequency of the radiation. By dividing the coupling constant with the frequency of the radiation, normalization of the coupling constant is achieved. Experiments have shown that the normalized coupling strength is in the order of 10^{-8} to 10^{-6} for atoms confined in optical cavities and microwave cavities [4]. In contrast, ultrastrong coupling regimes have shown to yield a normalized coupling constant close to unity in electron cyclotron resonance, intersubband polaritons and superconducting qubits [4]. However, this thesis will not consider ultrastrong coupling: the models used here for treating light-matter coupling breaks down at strong couplings.

The light-matter coupling strength affects the physics of such hybrid systems and varies considerably from one experimental realization to the next. In general, the magnitude of the coupling constant depends on the physical properties of the system: the frequency of the field, the frequency of the atomic transition and the position of the qubit [12].

[i] In this thesis, a qubit will exclusively refer to an atom or molecule with two relevant states.

No time dependence is apparent. However, this thesis is investigating systems where time-dependent effects will affect the coupling of light and matter in quantum dynamical systems. The corresponding time-dependence of the coupling constant will later be motivated by the dynamics of the specific systems considered.

In this thesis, two simple quantum dynamical systems with time-dependent coupling constants will be considered. As shown in Fig. 1.1 a), the first system is that of one qubit coupled to a single field mode in an optical cavity. The second system, Fig. 1.1 b), is that of two identical qubits coupled to one field mode in an optical cavity. In both cases, it will be investigated how time-dependent effects affect the coupling.

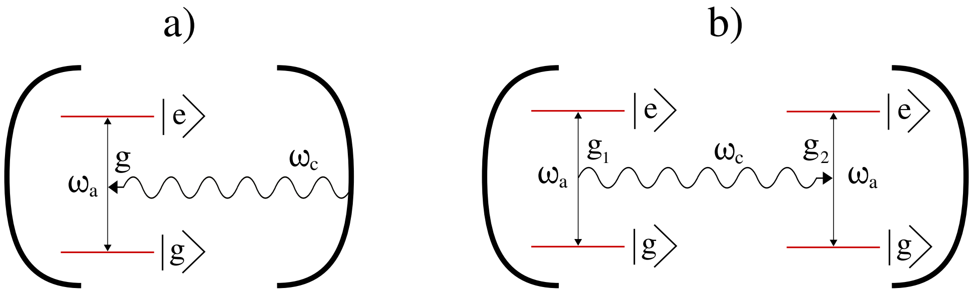

Figure 1.1: Schematics of the systems considered in the thesis. a) A qubit coupled to one field mode in an optical cavity. The atomic transition frequency ω_a between the ground state $|g\rangle$ and the excited state $|e\rangle$ of the qubit is in close proximity to the frequency of the cavity ω_c. The coupling constant g determines the coupling strength between the two. b) Two identical qubits, with different coupling constants g_1 and g_2, coupled to one field mode ω_c.

In the system with one qubit (Fig. 1.1 a)) a well-known quantum mechanical model applies, called the Jaynes-Cummings model, which treats the interaction between a qubit and one field mode [6]. This model is within the *rotating wave approximation* (RWA), meaning that atomic transition frequency and the frequency of the cavity is in close proximity of one another. The model will not hold for ultrastrong coupling. The qubit will be considered to fly through the cavity, as shown in Fig. 1.2, which will give rise to a time-dependence of the light-matter coupling. This is seen as an initial task to investigate how the system will behave for the given dynamics, as well as to determine to what extent an analytical solution of the total wavefunction is possible.

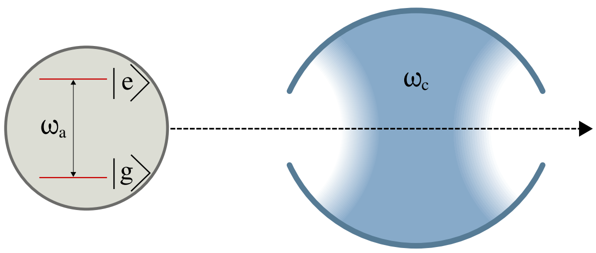

Figure 1.2: Schematic displaying the dynamics of the qubits. Here, the qubit is initially outside the cavity and flies through it in a straight path. The dynamics gives rise to a time-dependent light-matter interaction, characterized by the coupling constant.

In the system where two qubits are coupled to one field mode (Fig. 1.1 b)) an extension of the Jaynes-Cummings model is appropriate called the Tavis-Cummings model, which can treat an arbitrary number of qubits coupled to one field mode [11]. Like the single qubit system, the qubits are considered to fly through the cavity as shown in Fig. 1.2. Due to the increase of complexity in the system compared to the single qubit case, an analytical solution to the wavefunction may not be possible. Investigating the dynamics of this two qubit system, the point of interest will lie in how the dynamics will affect a possible entanglement of the two qubits.

The thesis will begin by presenting the relevant theory, namely the Jaynes-Cummings model, the Tavis-Cummings model and a relevant model of concurrence. Here, the assumptions and limitations of the models are discussed. The theory is followed by a method section, where the appropriate models are applied to each system and evaluated. In the following result section, the defined wave-functions of each system will be evaluated, in terms of their corresponding probabilities, where a time-dependence of their coupling constants are considered. In the two qubit system, entanglement will be investigated in terms of the system´s concurrence, as well as how the dynamics affects the entanglement.

2 Theory

In this section the necessary theory for the thesis is presented. It includes two quantum mechanical models which treats the interaction, and therefore coupling, between qubits and electromagnetic radiation. Since the thesis is treating purely quantum mechanical systems, the starting point for the theory will be of a quantum mechanical nature as well. Beside the relevant models, density matrices and concurrence will be introduced for the investigation of the entanglement in the two qubit system.

2.1 The Jaynes-Cummings model

The first system that is considered consists of one qubit coupled to a single electromagnetic field confined in an optical cavity. The full quantum mechanical model describing such a system takes a more general form where an arbitrary number of electronic energy levels are considered. Since the general case is not relevant to this thesis, the model will be restricted to one 2-level electron coupled to one mode electromagnetic field from the beginning.

The total Hamiltonian $\mathcal{H}$ for such a system includes the free Hamiltonian of the electronic system and the radiation field, denoted $\mathcal{H}_{el}$ and $\mathcal{H}_{field}$ respectively, as well as the interaction between them described by the interaction Hamiltonian, $\mathcal{H}_I$. The following Hamiltonians is presented in their occupation number representation: the operators only act on the number of occupations for a given state. Starting from the quantum mechanical model [12], the Hamiltonian for a 2-level electron is

$$\mathcal{H}_{\text{el}} = \sum_{i=1}^{2} \hbar\omega_i a_i^\dagger a_i \ . \tag{2.1}$$

In Eq. 2.1, the indices 1 and 2 denote the ground and excited state of the electron, respectively. $\hbar\omega_i$ corresponds to the energy of the electron in a state i with a_i and $a_i^\dagger$ being the electron operators. They respectively lower and raise the number of occupation for a given electron energy level i [12]. Due to the Pauli exclusion principle, the raising (or lowering) operator acting on an arbitrary electronic state $|\psi\rangle$ twice, would yield

$$(a_i^\dagger)^2 |\psi\rangle = 0,$$

since otherwise, two identical electrons would then occupy the same state.

As for the electron, the free Hamiltonian for the radiation field is presented in a quantum mechanical form. The free Hamiltonian for a single quantized field mode is

$$\mathcal{H}_{\text{field}} = \hbar\omega_c b^\dagger b. \tag{2.2}$$

Here, $b^\dagger$ and b are the creation and annihilation operators, respectively, for a photon with frequency ω_c [12]. Eq. 2.2 corresponds to the Hamiltonian of a harmonic oscillator with one mode frequency, without considering the zero-point energy of the mode [12].

Now, the interaction Hamiltonian $\mathcal{H}_{\mathrm{I}}$, for one qubit and one field mode, can be presented in terms of the electron and photon operators:

$$\mathcal{H}_{\mathrm{I}} = \hbar a_1^{\dagger} a_2 (gb + g^* b^{\dagger}) + \hbar a_2^{\dagger} a_1 (gb + g^* b^{\dagger}). \tag{2.3}$$

In Eq. 2.3, g and its complex conjugate g^* denotes the coupling constant between the electron and the field mode [12].

The coupling constant determines the strength of the coupling and is in essence, with its time-dependence, the coefficient this thesis aims to investigate. The electromagnetic field is assumed to have a wavelength greater than the spatial size of the qubit [12]. As a result, deviations in the interaction between field and different parts of the atomic wavefunction are neglected. Therefore, the coupling constant and the Hamiltonian are within the dipole approximation, and the coupling constant for an ideal cavity is proportional to

$$e^{i\mathbf{k}\cdot\mathbf{x}_0}, \tag{2.4}$$

where $\mathbf{k}$ is the wavevector of the field and $\mathbf{x}_0$ is the central position of the qubit [12]. Hence, the interaction considered in the model is only at the qubit's position $\mathbf{x}_0$. Note that Eq. 2.4 refers to a plane wave field mode. However, the cavity does not necessarily contain a plane wave. Therefore, the wavefunction can take a different form depending on the system, which would reflect on the proportionality to the coupling constant.

The time-dependence of the coupling constant arises from the assumption that the qubit flies through the cavity. When the qubit is inside the cavity it interacts with the confined field mode. When it is outside, no interaction occurs. Mathematically, this onset and offset of the interaction can be described by a time-dependent coupling constant, now denoted by $g(t)$.

Considering a qubit in motion, additional time-dependence of $g(t)$ arises from the qubits spatial position relative to the field it interacts with. Although the dipole approximation neglects interaction variations between the field and different parts of the atomic wavefunction, this approximation is limited to a stationary qubit. When the qubit now moves through an optical cavity, variations of the field due to the movement of the qubit can not be neglected. Therefore, the proportionality in Eq. 2.4 attains a time-dependence, motivated by the dynamical nature of the qubit:

$$g(t) \propto e^{i\mathbf{k}\cdot\mathbf{x}_0(t)}. \tag{2.5}$$

Depending on the field mode interacting with the qubit, $g(t)$ attains an additional time-dependent phase factor.

Now, the total Hamiltonian of the system can be described by Eqs. 2.1 to 2.3:

$$\mathcal{H} = \hbar\omega_c b^\dagger b - \frac{\hbar\omega_a}{2} a_1^\dagger a_1 + \frac{\hbar\omega_a}{2} a_2^\dagger a_2 + \hbar a_1^\dagger a_2 [g(t)b + g^*(t)b^\dagger] + \hbar a_2^\dagger a_1 [g(t)b + g^*(t)b^\dagger] \quad (2.6)$$

Here, the ground and excited energy levels for the electron have been set to $-\frac{\hbar\omega_a}{2}$ and $\frac{\hbar\omega_a}{2}$, respectively. The electron operators in the Hamiltonian obey the same commutation relations as the Pauli pseudo-spin operators [12]. Therefore, these operators can be exchanged with the Pauli pseudo-spin operators [12]:

$$a_2^\dagger a_1 \to \sigma^+$$

$$a_1^\dagger a_2 \to \sigma^-$$

$$a_2^\dagger a_2 - a_1^\dagger a_1 \to \sigma_z.$$

The total Hamiltonian, Eq. 2.6, can then be written as

$$\mathcal{H} = \hbar\omega_c b^\dagger b + \frac{\hbar\omega_a}{2}\sigma_z + \hbar(\sigma^- + \sigma^+)[g(t)b + g^*(t)b^\dagger]. \quad (2.7)$$

This general Hamiltonian for the system is called the Dicke model [3]. Although Eq. 2.7 refers to one qubit coupled to one field mode, the Dicke model can be generalized to an arbitrary amount of 2-level systems coupled to a single field mode.

Now, from a special case of the Dicke model, the Jaynes-Cummings model can be derived. If one moves Eq. 2.7 to the interaction picture using a unitary transformation [12]

$$U = e^{-i\frac{\mathcal{H}_{el}+\mathcal{H}_{field}}{\hbar}t}$$

it can be shown that the photon operators, and the raising and lowering operators, evolve in time as [5][12]

$$b(t) = b(0)e^{-i\omega_c t},\ b^\dagger(t) = b^\dagger(0)e^{i\omega_c t}$$

and

$$\sigma^-(t) = \sigma^-(0)e^{-i\omega_a t},\ \sigma^+(t) = \sigma^+(0)e^{i\omega_a t}.$$

Considering this time evolution in Eq. 2.7, the non-interacting part of the Hamiltonian remains unchanged, while the interacting terms are changed to [5][12]

$$\hbar g(t)\left[\sigma^-(0)b(0)e^{-i(\omega_a+\omega_c)t} + \sigma^+(0)b(0)e^{i(\omega_a-\omega_c)t}\right]$$

$$+\hbar g^*(t)\left[\sigma^-(0)b^\dagger(0)e^{i(\omega_c-\omega_a)t} + \sigma^+(0)b^\dagger(0)e^{i(\omega_a+\omega_c)t}\right]. \quad (2.8)$$

When ω_c is in close proximity or equal to ω_a, the terms in Eq. 2.8 that does not conserve the total number of excitations rapidly oscillates compared to the terms that does. Over time, these oscillations will average out to zero. The non-conserving terms are therefore neglected in the *rotating wave approximation* (RWA) [12]. Thus, in the case where $\omega_c = \omega_a = \omega$, Eq. 2.7 reduces to

$$\mathcal{H}_{\mathrm{JC}} = \hbar\omega b^\dagger b + \hbar\omega\sigma_z + \hbar(g(t)b\sigma^+ + g^*(t)b^\dagger\sigma^-), \quad (2.9)$$

where $\mathcal{H}_{\mathrm{JC}}$ is the Hamiltonian for the Jaynes-Cummings model [6][12]. Since this Hamiltonian commutes with the total number of excitations as

$$\left[\mathcal{H}_{\mathrm{JC}}, b^\dagger b + \sigma_z\right] = 0,$$

the number of excitations are now conserved as a constant of motion.

Generally, the Jaynes-Cummings Hamiltonian is used with constant g in the weak coupling regime, which is within the RWA. For strong couplings, the Rabi oscillations would have a corresponding frequency on the same scale as the terms neglected in the RWA. Therefore, the model does not hold for strong couplings. A similar argument can be made for the time-dependence of $g(t)$: if $g(t)$ oscillates in the same time scale as the neglected terms in the RWA, the model is not applicable.

2.2 Tavis-Cummings model

The second system consists of two qubits coupled to one field mode contained in an optical cavity. As an extension of the Jaynes-Cummings model, the Tavis-Cummings model can be derived from the Dicke model [3]. This can be done with the for an arbitrary amount of identical 2-level systems, but for the relevance of the thesis, it will be done for 2 qubits.

Consider the Hamiltonian for the given system. The 2-level systems are identical, meaning that they have identical atomic transition frequencies ω_a, and thus identical energy levels. In the model, it is assumed that there is no direct interaction between the qubits: their wave functions does not overlap and the qubits only interacts with the field mode [11]. The Dicke model Hamiltonian for two qubits reads as [3]

$$\mathcal{H} = \hbar \left[\omega_c b^\dagger b + \sum_{i=1}^{2} \omega_a \sigma_i^\dagger \sigma_i + (\sigma_i + \sigma_i^\dagger)(g_i^* b + g_i b^\dagger)\right], \tag{2.10}$$

where index i denotes a given qubit. $\sigma_i^\dagger$ (σ_i) is the raising (lowering) operator for a specific qubit and g_i denotes the coupling constant between a single qubit and the field mode. Like the Jaynes-Cummings Hamiltonian, $b^\dagger$ and b are the photon creation and annihilation operators, respectively.

From here, the RWA is applied to Eq. 2.10 which allows for a simplification of the Hamiltonian: operations that do not conserve energy, or excitations, are neglected. With this, and the limitation $\omega_c = \omega_a = \omega$, Eq. 2.10 becomes

$$\mathcal{H}_{\mathrm{TC}} = \hbar \left[\omega b^\dagger b + \sum_{i=1}^{2} \omega \sigma_i^\dagger \sigma_i + g_i \sigma_i^\dagger b + g_i^* b^\dagger \sigma_i\right]. \tag{2.11}$$

This is the Hamiltonian in the Tavis-Cummings model for 2 qubits [11].

Like the system with one qubit, a time-dependence of g_i is desired, relating to the dynamics of the qubits. The same arguments holds in this model. First, the qubits are considered to move through the cavity which result in a time-dependent interaction, which is described by a Fermi function included in each of the coupling constants. Second, this model is within the dipole approximation: the same proportionality as in Eq. 2.5 holds for each coupling constant, meaning that they will attain a time-dependent phase factor.

2.3 Density matrices and entanglement

For the previous model, the Tavis-Cummings model, the thesis aims to investigate a two-qubit system. When the wavefunction is studied after the qubits has left the cavity, the entanglement of the qubits is desired. To describe the entanglement, density matrices must be introduced. This allows defining the concurrence of the system: a measure that characterizes how much the qubits are entangled.

Generally, any matrix that is hermitian, a positive semi-definite and has a trace of one can be a density matrix. Consider the density matrix ρ of some wavefunction Ψ describing the states of a system. For example, the joint wavefunction for the system of two qubits coupled to a field mode. The density matrix can be written as [10]

$$\rho = |\Psi\rangle\langle\Psi| \,, \tag{2.12}$$

where the operation performed on the right-hand side is called the outer product.

Eq. 2.12 is not the only density matrix of a given system. In fact, any state in the Hilbert space of the system can be traced out, forming another density matrix which excludes states that are not of interest. For example, to find the concurrence of two qubits, a density matrix containing only the eigenstates of the qubits is required. If the qubits live in a larger, composed system, all states that are not an eigenstate of the qubits are traced out. The resulting density matrix is said to be reduced, defined as [7]

$$\rho_r = \sum_i \langle i|\Psi\rangle \langle\Psi|i\rangle \,. \tag{2.13}$$

Here, indices i denotes the states which are traced out. For two qubits coupled to one field mode, indices i would correspond to the states of the field mode, if a two qubit density matrix is desired.

To find the concurrence from a density matrix, another matrix must be introduced. This non-Hermitian matrix can be defined as [13]

$$P = \rho_r \tilde{\rho}_r. \tag{2.14}$$

Note that ρ_r in Eq. 2.14 must be a two qubit density matrix. $\tilde{\rho}_r$ is the matrix

$$\tilde{\rho}_r = (\sigma_y \otimes \sigma_y)\, \rho_r^* \,(\sigma_y \otimes \sigma_y)\,,$$

where σ_y is the Pauli y matrix and ρ_r^* is the complex conjugate of the density matrix [13]. Given that the density matrix is constructed from the states of two qubits, the concurrence $C(\rho_r)$ of those qubits can be defined. Concurrence is an entanglement monotone: a function that determines the entanglement of two qubits. In this case, the concurrence function is defined as

$$C(\rho_r) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{2.15}$$

where λ denotes the square root of the eigenvalues of P defined in Eq. 2.14, in decreasing order, and every λ is real and positive [13]. For any two-qubit system, the concurrence takes a value between zero and one. At zero, the system has no entanglement. At one, the entanglement is at its maximum [13].

3 Methods

In this section, the quantum mechanical models presented will be applied on a single qubit and double qubit system. Within the models, the derivation of the time-dependent differential equations describing the quantum dynamics of each system will be presented. The resulting equations for the wave function coefficients is solved numerically in both systems and analytically in the single qubit system. They are thereafter used to calculate the corresponding probabilities of the states, as the qubits interact with the cavity. In the system with two qubits, the density matrix and the corresponding concurrence is derived.

3.1 Single qubit coupled to one field mode

Here, the quantum dynamics of one qubit flying through an optical cavity will be derived. An analytical solution to the wavefunction coefficients will be derived in the case where the coupling constant is real. With a real coupling constant, the position-dependent phase, motivated by the couplings proportionality to the field in Eq. 2.5, is neglected. The system is assumed to move sufficiently slow such that deviations in the electromagnetic field is insignificant.

3.1.1 Derivation of the wavefunction coefficients differential equations

Since the light-matter coupling is the point of interest, some assumptions can be made for the wavefunction of the system. The qubit can only interact with a single photon of a given bosonic state at a time. For a convenient system still relevant to the aim of investigation, the field mode is limited to the low energy situation where the state can have a maximum occupancy of one photon. Within the defined Jaynes-Cummings model, the field mode energy then have a maximum value corresponding to the transition energy of the qubit.

Within this restriction, the wavefunction of the system can be defined. First consider the individual wave functions for the qubit, $|\Psi_q\rangle$, and mode frequency, $|\Psi_m\rangle$:

$$|\Psi_q\rangle = a_1(t)\,|g\rangle + a_2(t)\,|e\rangle \tag{3.1}$$

and

$$|\Psi_m\rangle = b_1(t)\,|0\rangle + b_2(t)\,|1\rangle\,. \tag{3.2}$$

Here, $|g\rangle$ and $|e\rangle$ denotes the ground and excited state of the qubit, respectively. $|0\rangle$ and $|1\rangle$ are the states describing the photon occupation of the field mode. $a_n(t)$ and $b_n(t)$ are the time-dependent amplitudes.

From here, the total wavefunction of the system can be described as a joint wavefunction of Eq. 3.1 and 3.2. Since the system is limited to always contain one excitation, the states $|e1\rangle$ and $|g0\rangle$ appearing in the total wavefunction is neglected: these states will not affect the light-matter coupling. It can be shown that including these states in the time-dependent Schrödinger equation would yield differential equations for the corresponding coefficients $a_1(t)b_1(t)$ and $a_2(t)b_2(t)$. The state $|g0\rangle$ will not affect the light-matter interaction and is therefore neglected. As for $|e1\rangle$, the system is restricted to one excitation in total: this state is neglected as well. The total wavefunction $|\Psi\rangle$ of the system is therefore defined as:

$$|\Psi\rangle = A(t)\,|g1\rangle + B(t)\,|e0\rangle\,, \tag{3.3}$$

where $|g1\rangle$ and $|e0\rangle$ forms an orthogonal basis for the system, and $|A(t)|^2 + |B(t)|^2 = 1$.

The wavefunction of the system, Eq. 3.3, together with the Jaynes-Cummings Hamiltonian, Eq. 2.9, can be used in the time-dependent Schrödinger equation to obtain a set of ordinary differential equations (ODEs) for the coefficients $A(t)$ and $B(t)$, where the orthogonality condition between the states has been used:

$$\Rightarrow \begin{pmatrix} \dot{A}(t) \\ \dot{B}(t) \end{pmatrix} = -i \begin{pmatrix} \frac{w}{2} & g(t) \\ g^*(t) & \frac{w}{2} \end{pmatrix} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = -iC(t) \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}. \tag{3.4}$$

Eq. 3.4 is the main set of equations for the system. From these equations, a numerical solution was found in the two cases where $g(t) \in \mathbb{R}$ and $g(t) \in \mathbb{C}$, using the classical Runge-Kutta method (RK4) [9]. In the case where the coupling constant was purely real, an analytical solution was found for the wavefunction coefficients, derived in the following section.

3.1.2 Analytical solution to the time-dependent ordinary differential equations

For any system of first order ODEs with a variable coefficient matrix $k(t)$, a general solution

$$x(t) = e^{\int_{t_i}^{t} k(t)\ dt} x(t_i)$$

can be obtained, where t_i is the initial time, if the commutation relation

$$[k(t), k(t')] = 0 \tag{3.5}$$

holds for arbitrary different times t and t' [1]. Applying Eq. 3.5 on the coefficient matrix $C(t)$ in Eq. 3.4, it can be seen that the commutation relation holds if the coupling constant is purely real at all times. For this reason, the following derivation is within that restriction.

The solution to Eq. 3.4 takes the form

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = e^{-i\int_{t_i}^{t} C(t)\ dt} \begin{pmatrix} A(t_i) \\ B(t_i) \end{pmatrix} \tag{3.6}$$

for some continuous and real $g(t)$ on the evaluated time interval. Integrating the fundamental matrix component-wise, the resulting coefficient matrix can be written in terms of Pauli matrices.

To simplify the expression in Eq. 3.6, its matrix exponential can be rewritten. For any matrix M in which its square is the identity matrix I, an exponential of that matrix can be rewritten as

$$e^{\pm i\theta M} = \cos(\theta)I \pm i\sin(\theta)M, \tag{3.7}$$

where θ is real. Applying Eq. 3.7 on the matrix exponential in Eq. 3.6, $A(t)$ and $B(t)$ are found to to be

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = e^{-i\frac{\omega}{2}(t-t_i)} \begin{pmatrix} \cos(G(t) - G(t_i)) \\ -i\sin(G(t) - G(t_i)) \end{pmatrix}, \tag{3.8}$$

where

$$G(t) - G(t_i) = \int_{t_i}^{t} g(t')\, dt'.$$

Note that this analytical result of the amplitudes was derived in the case where the qubit starts in an excited state. In other words, $A(t_i) = 1$. In Eq. 3.8, $e^{-i\frac{\omega}{2}(t-t_i)}$ acts as a phase factor, which will not affect the probabilities of the states. This means that the specific frequency of the cavity and the qubit is irrelevant for the solution, assuming that the condition of the RWA and the magnitude of the mode frequency are met.

3.2 Two qubits coupled to one field mode

Here, a quantum dynamical system of two qubits interacting with a single mode field contained in an optical cavity will be analyzed, within the Tavis-Cummings model. As for the single qubit system, the interpretation is that of the qubits flying through the cavity, either at the same or different times. Although the dynamics of each qubit may vary from one another, the maximum coupling strength is the same for each qubit.

3.2.1 Derivation of the wavefunction coefficients ODEs

As the Tavis-Cummings model is an extension of the Jaynes-Cummings model, similar assumption of the system is made as for the one qubit system. First, the field mode of interaction contains a maximum of one excitation: the photon wavefunction is that of Eq. 3.2. Second, the system is limited to always contain one excitation, which neglects irrelevant states of the joint photon-qubit wavefunction.

With the definition of the field mode wavefunction as in Eq. 3.2, consider the wavefunction of the individual qubits. Although the qubits are assumed to be identical in terms of their ground and excited states, their time-dependent amplitudes may vary, depending on the dynamics of the system. Then, consider each qubits wavefunction

$$|\Psi_1\rangle = a_1(t)\,|g\rangle + a_2(t)\,|e\rangle \tag{3.9}$$

and

$$|\Psi_2\rangle = a_3(t)\,|g\rangle + a_4(t)\,|e\rangle\,, \tag{3.10}$$

where $a_n(t)$ is the amplitude of each state.

In defining the joint wavefunction of the system (with Eq. 3.2, 3.9 and 3.10), all states corresponding to two or more excitations are neglected, as well as the vacuum state. As for the one qubit system, the vacuum state is not relevant in describing the light-matter coupling and the system is restricted to one excitation. Excluding these states, the total wavefunction of the system is defined as

$$|\Psi\rangle = A(t)\,|eg0\rangle + B(t)\,|ge0\rangle + C(t)\,|gg1\rangle\,, \tag{3.11}$$

where $|eg0\rangle$, $|ge0\rangle$ and $|gg1\rangle$ form an orthogonal basis for the system and $|A(t)|^2+|B(t)|^2+|C(t)|^2 = 1$.

Now the Tavis-Cummings Hamiltonian, Eq. 2.11, can be applied to the total wavefunction, Eq. 3.11, in the time-dependent Schrödinger equation. Then, a set of differential equations for the wavefunction amplitudes is obtained, using the orthogonality condition between the states:

$$\begin{pmatrix} \dot{A}(t) \\ \dot{B}(t) \\ \dot{C}(t) \end{pmatrix} = -i \begin{pmatrix} 0 & 0 & g_1 \\ 0 & 0 & g_2 \\ g_1^* & g_2^* & 0 \end{pmatrix} \begin{pmatrix} A(t) \\ B(t) \\ C(t) \end{pmatrix}. \tag{3.12}$$

In contrast to the one qubit system ODEs, Eq. 3.4, no analytical solution to Eq. 3.12 were found. Instead, the corresponding probabilities were found using the numerical RK4 method on the differential equations [9].

The ODEs of the amplitudes in Eq. 3.12, and therefore the dynamics of the system, are only dependent on the time-dependent coupling constants g_1 and g_2. This means that as long as the frequencies of the system is within the RWA, the time-evolution is only dependent on the light-matter coupling.

3.2.2 Reduced density matrix and entanglement for the two qubit system

To obtain the concurrence of the qubits, a density matrix only containing the states of the qubits is required. Since the wavefunction of the system includes the bosonic states, a density matrix with these states traced out are desired. Such a density matrix is created by using the wavefunction of the system, Eq. 3.11, in Eq. 2.13, where indices i denotes the states of the field mode. The corresponding reduced density matrix of the system read as

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |A(t)|^2 & A(t)B^*(t) & 0 \\ 0 & A^*(t)B(t) & |B(t)|^2 & 0 \\ 0 & 0 & 0 & |C(t)|^2 \end{pmatrix}, \tag{3.13}$$

where $|A(t)|^2$, $|B(t)|^2$ and $|C(t)|^2$ are the probabilities of the system states $|eg0\rangle$, $|ge0\rangle$ and $|gg1\rangle$, respectively.

To find an expression for the concurrence, the P matrix of the reduced density matrix, Eq. 3.13, was derived using Eq. 2.14. This matrix has only one non-zero eigenvalue λ, namely

$$\lambda = 4|A(t)|^2|B(t)|^2.$$

Using λ in the definition of concurrence in Eq. 2.15, the concurrence for the system is defined as

$$C(\rho) = \max\{0, \sqrt{\lambda}.\} \tag{3.14}$$

The concurrence for this system can be seen to only depend on the amplitudes of the excited qubit states, meaning that, a concurrence of one is possible, corresponding to maximally entangled qubits. This is achieved when the two qubits contain the total excitation with equal probability.

4 Results

4.1 One qubit coupled to a field mode

4.1.1 Time-independent coupling constant

Before introducing a dynamical system, first consider a one qubit system where the coupling is constant. The interpretation of this stationary system is that of a qubit fixed inside an optical cavity. Since this is a well-known system described by the Jaynes-Cummings model, this section introduces the basic physical phenomena related to light-matter coupling.

Since the analytical solution of the wavefunction amplitudes in Eq. 3.8 is only limited to a real coupling constant, the corresponding functions are applicable here. The coupling constant is set to be real and constant as $g = g_0$. Then, the analytical solution in Eq. 3.6 yields corresponding probabilities

$$|A(t)|^2 = \cos^2(g_0 t),$$

for the state $|e0\rangle$, and

$$|B(t)|^2 = \sin^2(g_0 t)$$

for the state $|g1\rangle$ where the initial time t_i is set to be zero. These definitions of the probabilities hold if the excitation initially is with the qubit, otherwise the states would interchange probability functions. Then, the probabilities evolve in time as shown in Fig. 4.1.

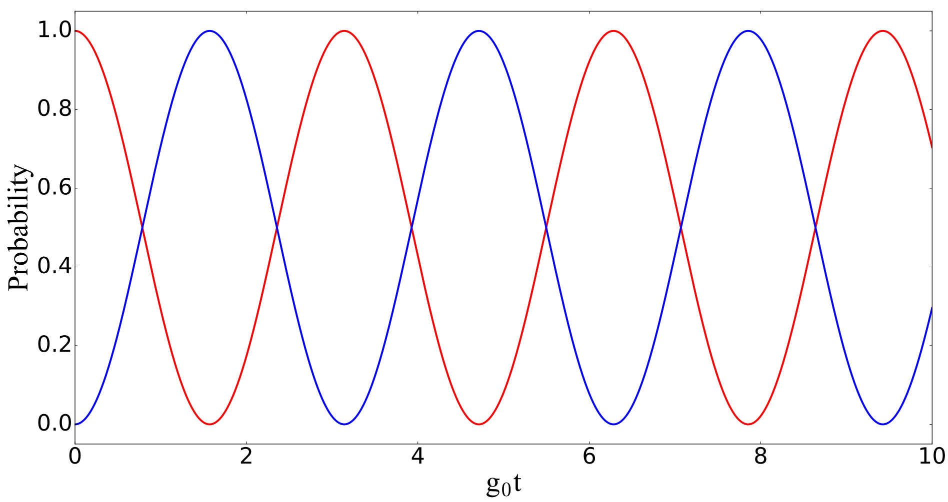

Figure 4.1: Probability of states evolving in time. Here, the coupling constant $g = g_0$ is taken to be real and constant. The excitation is initially in the qubit. Here, the qubit is stationary in an optical cavity, interacting with the confined field mode. Red: The probability of the state $|e0\rangle$. Blue: The probability of the state $|g1\rangle$.

In Fig. 4.1, the well-known Rabi oscillations are seen. The system oscillates between its states over time, corresponding to a continuous interaction between the electron and the photon in the cavity.

Since the system is described by the Jaynes-Cummings model, the corresponding assumptions must hold for the parameters of the system. The frequency of the Rabi oscillations shown in Fig. 4.1 must be within the RWA. If this frequency is in the range of 2ω, the oscillations between the states are too fast and the model is not applicable.

4.1.2 Time-dependent coupling constant

Now let us introduce the first dynamical property of the qubit. The qubit is initially outside the cavity and, with a constant speed, flies inside it. The coupling constant obtains time-dependence: no coupling is present when the qubit is outside the cavity but as it enters, the interaction to the mode field is initiated. This can be described in the coupling $g(t)$ with a time-dependent Fermi function:

$$g(t) = \frac{g_0}{1 + e^{-\frac{t-t_0}{\tau}}}. \tag{4.1}$$

Here, g_0 is the maximum amplitude of the coupling. t_0 is the time at which the qubit enters the cavity and τ determines to what extent the coupling strength will change as the qubit enters the cavity. If τ is small, $g(t)$ will be closer to a step function. Eq. 4.1 corresponds to a qubit initially outside a cavity at $t < t_0$, and thereafter entering it at $t = t_0$.

With the analytical result of the amplitudes in Eq. 3.8, the corresponding probabilities were plotted with a coupling constant as shown in Eq. 4.1. Beside the analytical solution, Fig. 4.2 contains the numerical RK4 solution of the probabilities, where Eq. 3.4 have been used.

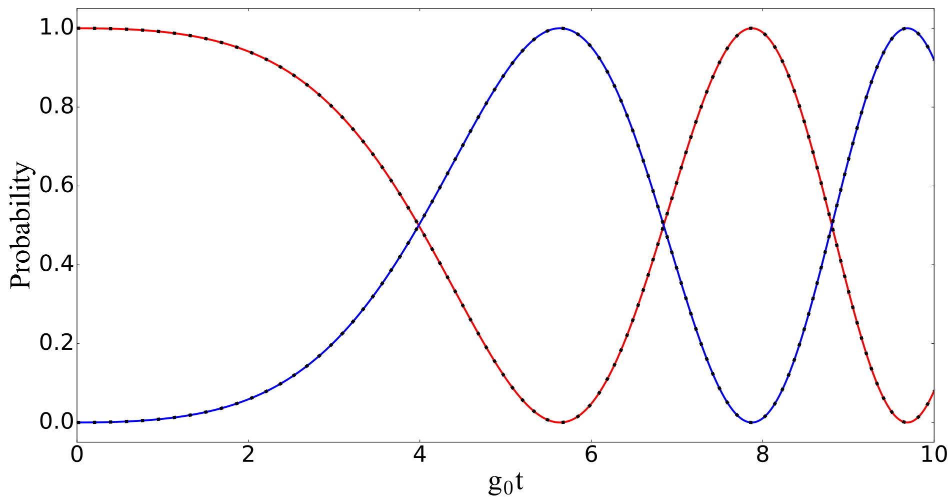

Figure 4.2: Probability of states evolving in time. Here, the coupling constant is time-dependent and takes the form of Eq. 4.1. The excitation is initially with the qubit and the qubit enters the cavity at $g_0 t = 5$ where $g_0\tau = 2$. Red: The probability of the state $|e0\rangle$ from the analytical result of the wavefunciton amplitudes. Blue: The probability of the state $|g1\rangle$ from analytical result of the wavefunciton amplitudes. Dotted black: Corresponding numerical result from a RK4 method applied to Eq. 3.4.

In Fig. 4.2, the excitation is initially in the qubit. In other words, the field mode in the cavity does not contain any photons when the qubit is outside the cavity. The interpretation is that of a qubit entering the cavity in an excited state and deexcite inside the cavity. Here, it can be seen that the defined time-dependence of $g(t)$ affects the Rabi oscillations as the qubit enters the cavity. When the qubit is fully inside, the oscillations between states again behave as Rabi oscillations, as those in Fig. 4.1.

As seen in Fig. 4.2, the analytical result is compared to the numerical approximation. Visually, there is no difference between them. However, the relative deviation is on the order of 10^{-15}, making it a good approximation. Furthermore, the conservation of probability allows for a reference point to which one can compare the approximation's accuracy. Here, the deviation in the total probability is again on the order of 10^{-15}, an accuracy that holds for all following numerical approximations.

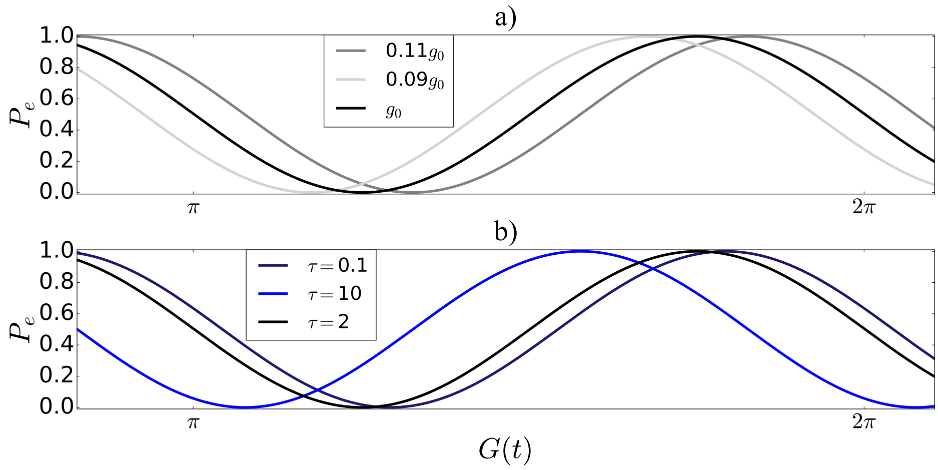

Figure 4.3: Probability P_e of the state $|e0\rangle$ over the average of $g(t)$, $G(t)$: shown in the analytical results of the wavefunction amplitudes in Eq. 3.8. $G(t)$ is plotted for different coupling strengths and τ. Black: The same timescale and parameters of the coupling constant were used as in Fig. 4.2 with a maximum coupling strength g_0. a): Plots with different maximum coupling strengths. b): Plots with different values of τ.

Looking at the analytical result of the amplitudes in Eq. 3.8, it can be seen that the Rabi oscillations are dependent on the integral of $g(t)$, or the time average of $g(t)$. The time-evolution of the coupling can be visualized by plotting the average of $g(t)$ as shown in Eq. 3.8. In Fig. 4.3, $G(t)$ has been plotted for different parameters, with the parameters used in Fig. 4.1 as a reference point (black). Beside the time evolution of the average of $g(t)$, its behavior is determined by the parameters of the coupling constant. The oscillations seen in Fig. 4.3 change phase when τ or g_0 is changed. Comparing the oscillations in Fig. 4.3 with the systems corresponding time evolution in Fig. 4.3, the oscillations are initiated at the moment the qubit is subjected to the coupling as it enters the cavity.

4.1.3 Time-dependent phase of the field mode

For a realistic mode field contained in the cavity, the coupling to the qubit changes depending on where the qubit is in the cavity. There would not only be an onset of the coupling as it enters the cavity, but the normal mode of the field would result in deviations in the coupling. Mathematically, this is represented as a phase factor included in the coupling function, reflecting on the proportionality in Eq. 2.5.

To describe the position dependent deviations in the coupling, first consider a finite cavity, in which the qubit can enter and leave the cavity. This is reflected on the coupling function which now, for a finite cavity, takes the form

$$g(t) = \frac{g_0}{1 + e^{\frac{-(t-t_0)}{\tau}} + e^{\frac{(t-t_f)}{\tau}}}. \tag{4.2}$$

Here, the qubit leaves the cavity at a time t_f. When τ is small, Eq. 4.2 corresponds to a step-function which returns g_0 in the time interval $[t_0, t_f]$ and zero otherwise.

For a cavity containing normal modes of an electromagnetic field, there would be deviations in the coupling depending on where to qubit is in the cavity. Consider the proportionality between the coupling constant and the field mode as shown in Eq. 2.5. For a finite cavity, this proportionality can be described by a time-dependent phase factor included in the coupling function as

$$g(t) = \frac{g_0}{1 + e^{\frac{-(t-t_0)}{\tau}} + e^{\frac{(t-t_f)}{\tau}}} e^{i\frac{t-t_0}{t_f-t_0}\phi}. \tag{4.3}$$

Here, the phase change is determined by the factor ϕ. The phase factor comes from the electric field of the normal mode which only matters when the qubit moves within the cavity. Furthermore, since the system is described by the Jaynes-Cummings model, the phase factor in the coupling must be within the RWA, meaning that its frequency must be much smaller than 2ω.

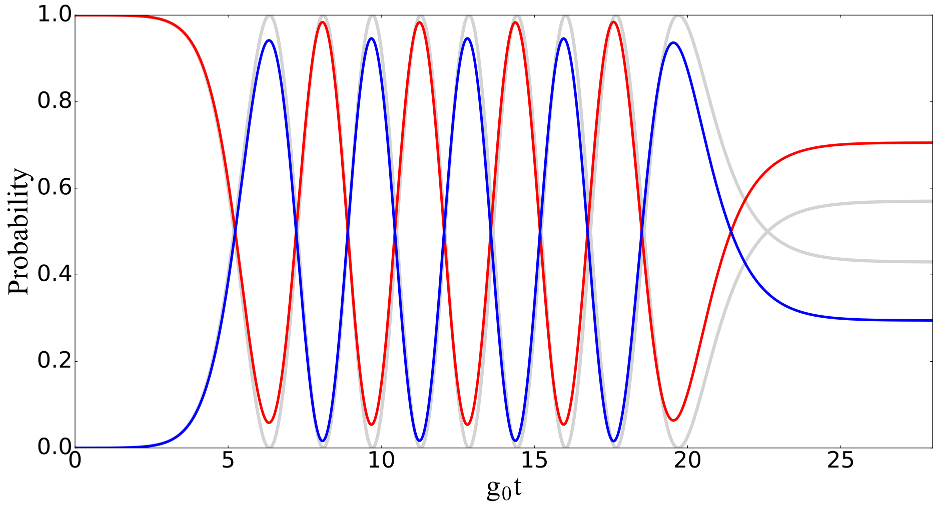

Figure 4.4: Probability of a qubit flying through an optical cavity with a phase included The coupling constant is that of Eq. 4.3 with $\phi = \pi$. The excitation is initially with the qubit. The qubit enters the cavity at $g_0 t = 5$ and leaves the cavity at $g_0 t = 20$, where $g_0 \tau = 1$. Blue: Probability of the state $|g1\rangle$. Red: Probability of the state $|e0\rangle$. Gray: Corresponding result with a coupling constant without phase as in Eq. 4.2.

An example of a qubit flying through a finite cavity with a phase factor, and thus with variations in the coupling, is shown in Fig. 4.4. In Fig. 4.4, the phase is seen to affect the Rabi oscillations. As the coupling strength is no longer uniform within the cavity due to the inclusion of the phase, the states now oscillates between each other with some distortion, reflecting on deviations in the coupling. Comparing the oscillations with phase to those without in Fig. 4.4, it can be seen that the phase decreases the amplitude of the oscillations. With an increasing phase, the Rabi oscillations would eventually break. Although the oscillations of the probabilities would still be present, the initial state with the excitation would be favored. In other words, the qubit and the cavity mode exchange excitations and energy less effectively due to the time-dependent phase arising from the coupling.

4.2 Two qubits coupled to a field mode

As an extension of the one qubit system, the two qubit system will be discussed here. The system considered has a corresponding wavefunction as defined in Eq. 3.11. It includes a system in which two qubits flies through the cavity in different cases. In light of the dynamics of the system, the concurrence and therefore entanglement between the qubits will be studied. In contrast to the single qubit system where a phase was introduced to the coupling, the focus is on the ideal case where the coupling is uniform when the qubit is within the cavity.

4.2.1 Real coupling constant without phase

Consider then two qubits coupled to a field mode with time-dependent couplings similar to those defined in the previous section (Eq. 4.2). With two qubits, an additional case can be considered: the dynamics of each qubit may differ from one another. Although the qubits are restricted to fly through the cavity in a straight path, different speeds of each qubit can be simulated by changing the time parameters in their corresponding coupling constants.

First, consider the qubits flying through the cavity, one at a time. This is described by the coupling functions, without phase, defined as in Eq. 4.2. Here, the onset and offset for each qubits coupling function is chosen such that the qubits fly through the cavity one at a time. With the same speeds, and the excitation initially being in the first qubit, the corresponding probabilities are shown in Fig. 4.5 as an example.

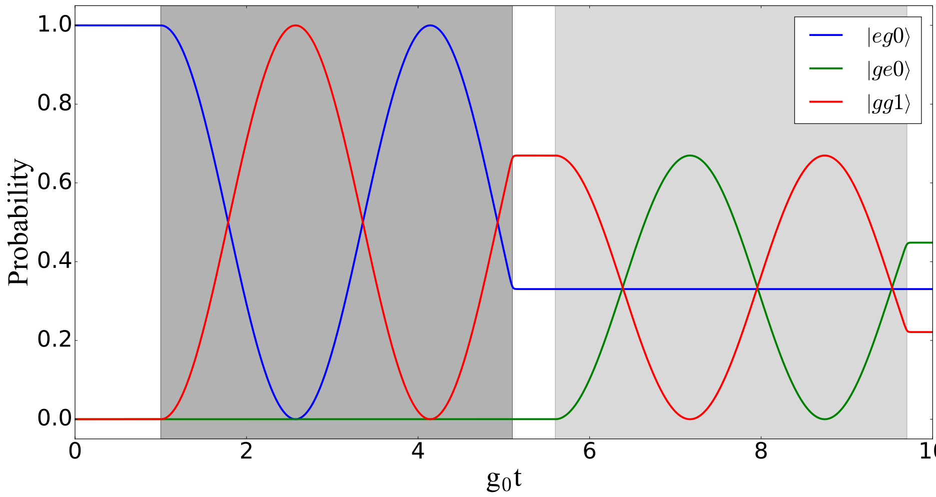

Figure 4.5: Probabilities of states, corresponding to two qubits flying through a cavity one at a time. The excitation is initially with the first qubit: the system starts in the state $|eg0\rangle$. The corresponding coupling constant for each qubit is that of Eq. 4.2, where the functions only differ in the time in which the qubits enters and leaves the cavity. Here, τ is chosen to be small such that the coupling functions acts like step functions. Blue: Probability of the state $|eg0\rangle$. Red: Probability of the state $|gg1\rangle$. Green: Probability of the state $|ge0\rangle$. Dark gray: The time in which the first qubit is inside the cavity. Light gray: The time in which the second qubit is inside the cavity.

kIn Fig. 4.5, each qubit is seen to produce Rabi oscillations within the cavity. Since the system as a whole is coupled, some probability is lost to the first qubit as it leaves the cavity. When both qubits have left the cavity, the system is left in a specific state which, depending on the initial conditions and the parameters of the coupling constants, can be tuned to any composed state.

As seen in Fig. 4.5, the qubits can interact remotely. Due to the light-matter coupling and the constraint of one excitation, the first qubit leaving the cavity determines to what extent the excitation can be with the second qubit. The amplitude of the Rabi oscillations of the second qubit is only conserved in the ideal case where the first qubit leaves the cavity in its ground state. As a result, this interaction between the qubits, besides the dynamics of the system, determines the final state of the system. If the qubits at some point were to be inside the cavity simultaneously, their respective light-matter interactions would overlap. Then, the excitation would be shared among the three states, leading to a breaking of the constituent Rabi oscillations and a potential non-sinusoidal behavior for them.

Looking at the state of the system after the qubits leave the cavity in Fig. 4.5, the entanglement of the qubits can be determined. In this case, with the definition in Eq. 3.14, the concurrence of the qubits is 0.77. Since the entanglement is determined by the state of the system, it is dependent on previously stated conditions.

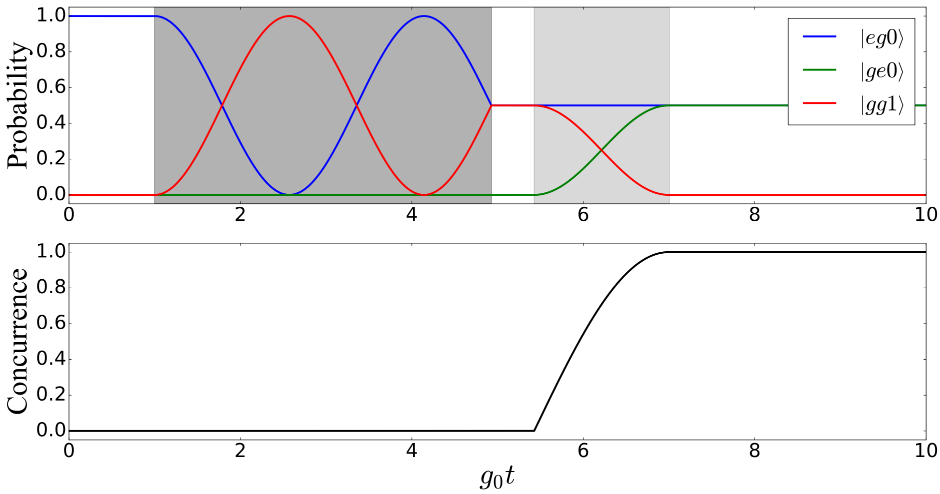

Figure 4.6: Probabilities of states, corresponding to two qubits flying through a cavity, one at a time. The excitation is initially with the first qubit. The qubits enters and leaves the cavity, one after the other, and the second qubit is within the cavity a shorter period of time. Blue: Probability of the state $|eg0\rangle$. Red: Probability of the state $|gg1\rangle$. Green: Probability of the state $|ge0\rangle$. Dark gray: The time in which the first qubit is inside the cavity. Light gray: The time in which the second qubit is inside the cavity. Black: Concurrence of the system as a function of g_0t.

A specific setup of interest is such that the qubits leave the cavity maximally entangled. Looking at the defined concurrence in Eq. 3.14, this is possible if no excitation is left in the cavity, and the qubits have an equal probability to contain the excitation. On way to obtain a concurrence of one is to increase the speed of the second qubit (or decrease the time it spends within the cavity). A faster qubit corresponds to the qubit reaching maximum coupling strength g_0 faster, corresponding to a smaller τ in the coupling function. However, this deviation is neglected, and the qubits increased speed is simulated simply by decreasing the time in which the qubit is inside the cavity, as shown in Fig. 4.6.

In Fig. 4.6, the system is tuned such that the qubits leaving the cavity is at a maximal entanglement. The speed of the second qubit is set such that its excitation exchanges probability with the cavity excitation. As a result, the concurrence of the system is seen to increase to the point where the second qubit leaves the cavity, maximally entangled.

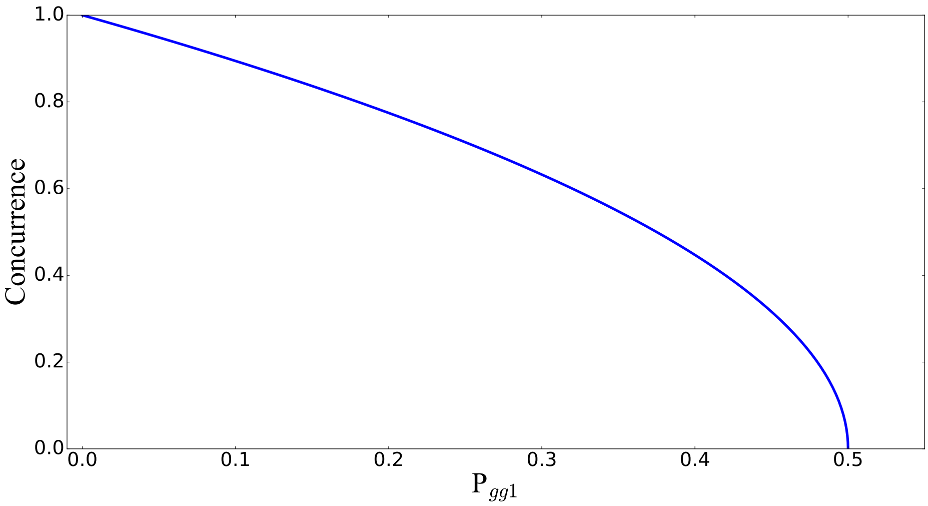

Figure 4.7: Probability to have one photon in the cavity (P_{gg1}) as a function of concurrence of the two excited qubit states. The concurrence is measured after the qubits leave the cavity and the maximal concurrence of one is obtained in Fig. 4.6. The decreasing concurrence with increasing ground state probability is obtained by decreasing the speed of the second qubit up to a point where no excitation is left with the second qubit. If the second qubit is within the cavity for a longer period of time, the concurrence will again increase.

One way to tune the system to any entanglement is to gradually change the speed of the second qubit. By changing the speed of the second qubit, the concurrence changes as a function of the ground state probability, as shown in Fig. 4.7. The concurrence in Fig. 4.7 is seen to decrease with increasing probability of the ground state. Then, any concurrence is obtainable after the qubits leave the cavity, depending on the speed of the second qubit.

In Fig. 4.7 it can be seen that the concurrence is zero when the probability to find the excitation in the cavity is 50%. This stems from the fact that as the first qubit leaves the cavity, the system is tuned such that the excitation has an equal probability to be found in the qubit and the cavity. With conserved probability, the total probability of the second qubit and cavity is now 50%. When the second qubit now leaves the cavity with no excitation, the concurrence is zero.

From Fig. 4.7 and the defined concurrence, a general statement can then be said relating the excitation and the entanglement. The probability to find the excitation in the cavity is inversely proportional to the maximum concurrence. This opens up to rather freely define the parameters of the coupling constant to obtain some entanglement.

5 Conclusion

In conclusion, two quantum mechanical models have been used to investigate simple quantum dynamical systems of one and two qubits coupled to a field mode confined in an optical cavity. With the dynamics of the system defined via the mutual coupling of qubit and field mode, the time evolution of the light-matter interaction was determined. For the double qubit system, it was found that the entanglement between the qubits could be determined by the nature of the time-dependent interaction, resulting in a possibility of maximum entanglement. These findings provide insight in quantum dynamical light-matter interaction for simple systems, within the restrictions of the Jaynes- and Tavis-Cummings model.

6 Outlook

This study has investigated light-matter interaction on a fundamental level for dynamical qubits. The research have been restricted by the limits of the models as well as the dynamical assumptions of the coupling constants, resulting in ideal time-evolution of the system. To further realize the impact of dynamical qubits on the light-matter interaction, different paths of the qubits through the cavity must be considered. Furthermore, the system could be expanded to a more realistic cavity by increasing the occupation limit of the field mode to contain more than one photon. The findings of the dynamical dependence of the concurrence for the double qubit system suggests a method to entangle qubits via light-matter interaction.

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