# Constructing Subalgebras of $\mathbb{K}[x]$ Using the Minimal Polynomial

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**Abstract**

In this report we will be working with subalgebras $A$ of finite codimension in $\mathbb{K}[x]$. It is known that such subalgebras can be expressed using a set of linear conditions evaluated at a finite set of points called the spectrum elements of $A$. These conditions are of one of two types, equality conditions or $\alpha$-derivations, which in turn consists of the values and the values of the derivations of the elements in our algebra. From this representation we find a way to construct a polynomial, the zeros of which are exactly the spectrum element. This polynomial, called the minimal polynomial of $A$, has the property that its product with an arbitrary polynomial lies in our algebra. In order to find subalgebras of $A$ we can add an additional condition, namely an $\alpha$-derivation, where $\alpha$ lies in the spectrum of $A$. To find all such $\alpha$-derivations, which can be written as a linear combination of regular derivations, we find an upper limit to the order of the derivations involved. To fully determine the derivation we also construct a method of finding all the required restrictions on the coefficients of said linear combination.

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# Contents



# 1 Introduction

In the last couple of years researchers in Lund have worked on a new approach to understand polynomial algebras in one and several variables. The essential idea is to describe subalgebras using conditions a polynomial has to fulfill in order for it to be a member of the algebra. These conditions involves the polynomials themselves and the derivations of said polynomials evaluated at a finite set of points, called the spectrum of the algebra. This thesis is centered around the minimal polynomial which has these spectrum elements as its zeros. An additional property of this polynomial is that the product of the minimal polynomial and an arbitrary polynomial belongs to the algebra. We also want to find a way of finding subalgebras of the polynomial algebra, but first we need to define relevant concepts.

## 2 Background

Before introducing the polynomial algebra $\mathbb{K}[x]$ we will use [4] to introduce what a ring is

**Definition 1.** *A* ***ring*** *is a non-empty set $R$ with operations $+$ and $\cdot$ such that for all $a, b, c \in R$ we have*

- *If $a \in R$ and $b \in R$ then $a + b \in R$*
- $a + (b + c) = (a + b) + c$
- $a + b = b + a$
- $\exists\, 0 \in R$ *such that* $a + 0 = 0 + a = a$ *for all* $a \in R$
- *for each* $a \in R$ $\exists\, x \in R$ *such that* $a + x = 0$
- *If $a \in R$ and $b \in R$ then $a \cdot b \in R$*
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$

*A ring has an identity if there exists an element $1 \in R$ such that $a \cdot 1 = 1 \cdot a = a$ for every $a \in R$.*

From [1] we have

**Definition 2.** *A* $\mathbb{K}$***-algebra****, $A$, is an associative ring with an identity with a $\mathbb{K}$-vector space structure such that if $a, b \in A$ and $\lambda \in \mathbb{K}$ then we have*

$$a(\lambda b) = \lambda(ab) = (\lambda a)b$$

One such type of algebra is the polynomial algebra which we will now introduce

**Definition 3.** *A* ***polynomial algebra*** $\mathbb{K}[x]$ *can be defined as the algebra containing all polynomials*

$$p(x) = p_n x^n + p_{n-1} x^{n-1} + \ldots p_1 x + p_0, \quad p_i \in \mathbb{K}$$

*together with regular polynomial addition and multiplication.*

Note that this definition of $\mathbb{K}[x]$ satisfies the requirements of being a $\mathbb{K}$-algebra. On top of this it is also commutative when it comes to multiplication. We will also specify that an algebra $B$ such that $B \subseteq A$ is called a subalgebra of $A$. We will more specifically be working with subalgebras $A$ of finite codimension in the polynomial algebra $\mathbb{K}[x]$. This means that given a basis of $A$ we can add a finite amount of basis elements to obtain a basis of $\mathbb{K}[x]$. We will also assume that $\mathbb{K}$ is an algebraically closed field with $\text{char}(\mathbb{K}) = 0$. The following sections of the background, unless otherwise specified, are referenced from [3].

**Example 1.** *We will present two subalgebras of $\mathbb{K}[x]$. First is the algebra which is generated using the polynomial $x^2$. This algebra does not include any polynomials of odd degree and as such it is of infinite codimension in $\mathbb{K}[x]$.*

*The second algebra which is generated using the polynomials $x^2$ and $x^3$, consists of polynomials of degree two and above. This algebra is of codimension one in $\mathbb{K}[x]$, only missing polynomials of degree one.*

## 2.1 Representation Using a SAGBI Basis

As seen in the examples above subalgebras of $\mathbb{K}[x]$ are usually represented using a set of polynomials called generators. All polynomials in the algebra can then be created though addition and multiplication of these generators. The specific type of set of generators we will be using is called a SAGBI-basis, which allows us to check whether an element, $f$, belongs to the algebra or not. This is done by finding an element $g = \sum g_i^{k_i}$, where $g_i$ lies in our SAGBI basis, such $\deg(g) = \deg(f)$. Then we can create $f - \lambda g$, which has a lower degree than $f$. We then repeatedly lower the degree of the polynomial until we can continue no more. If the last difference is zero then $f$ is a member of $A$.

**Definition 4.** *A **SAGBI basis** is a set of generators $G$ such that $S = \{deg(g(x)) \mid g(x) \in G\}$ where $S$ is a generating set of the numerical semigroup corresponding to the set $\{deg(f(x)) \mid f(x) \in A\}$.*

A SAGBI basis is called minimal if no element of the basis can be removed and have the set still remain a SAGBI-basis.

**Definition 5.** *The **Type** of a subalgebra $A$ is defined as an ordered list of degrees in a minimal SAGBI basis.*

## 2.2 Representation Using Conditions

As mentioned previously, there is another way of representing these subalgebras using conditions evaluated at points, which we call the spectrum elements of A. For example the algebra generated by $x^2$ and $x^3$ can also be written as $\{f(x) \in \mathbb{K}[x] \mid f'(0) = 0\}$, this algebra has the spectrum element 0.

**Definition 6.** *The **spectrum** of a subalgebra $A$, denoted **Sp(A)**. Is elements $\alpha$ such that $f'(\alpha) = 0$, $\forall f \in A$ or if $\exists \beta \in Sp(A) \setminus \{\alpha\}$ such that $f(\alpha) = f(\beta)$, $\forall f \in A$.*

The definition of the spectrum gives way for a natural equivalence relation which we will define next.

**Definition 7.** *Given the spectrum of a subalgebra A we say two spectrum points,* $\alpha, \beta \in Sp(A)$*, belong to the same* ***cluster*** *if* $f(\alpha) = f(\beta)$*,* $\forall f \in A$*. We denote two elements being part of the same cluster as* $\alpha \sim \beta$*.*

Moving forward we will look at the previously mentioned conditions. The subalgebra A is constructed as the kernel of a set of conditions each of which can belong to one of two types. The first type we will look at is equality conditions.

**Definition 8.** *An* ***equivalence condition*** *for a subalgebra* $A \subset \mathbb{K}[x]$ *can be defined as a function* $E : A \to \mathbb{K}$

$$E(f) = f(\alpha) - f(\beta) \tag{1}$$

*where* $\alpha \neq \beta$ *are scalars belonging to* $\mathbb{K}$*.*

The second type of condition is called a derivation

**Definition 9.** *An* $\alpha$***-derivation*** *is defined as a linear function* $D : A \to \mathbb{K}$ *such that we have*

$$D(fg) = D(f)g(\alpha) + f(\alpha)D(g) \;\; \forall f, g \in A \tag{2}$$

We remark that an $\alpha$-derivation is not necessarily also a $\beta$-derivation. This definition however is not particularly easy to work with so we will introduce the following way of expressing a derivation

**Theorem 1.** *For* $\alpha \in Sp(A)$ *any* $\alpha$*-derivation,* $D$*, can be expressed as*

$$D(f) = \sum_{\alpha_i \sim \alpha} \sum_{k=1}^{n} c_k f^{(k)}(\alpha_i) \tag{3}$$

Going forward we will also be using the notation

$$D_k^{\alpha} = \frac{f^{(k)}(\alpha)}{k!}$$

Assuming that we have a function, how can we determine if it is a derivation in $A$?

**Theorem 2.** *Let* $D : A \to \mathbb{K}$ *be a linear map, then the following statements are equivalent*

- $D$ *is an* $\alpha$*-derivation in* $A$
- $D(1) = 0$ *and* $D(f^2) = 0$*,* $\forall f \in M_\alpha$

*where* $M_\alpha = \{f(x) \in A \mid f(\alpha) = 0\}$*.*

Now that we have introduced this way of expressing a subalgebra A using conditions we will move on to the next section, in which we will define the minimal polynomial, of a subalgebra $A$, which will be the main focus of this report.

## 2.3 Polynomials With Spectrum Elements as Zeros

It is becoming apparent that finding the spectrum elements of a subalgebra are of interest. If a subalgebra is generated by two polynomials then there is a way to construct a polynomial, the zeros of which is exactly the spectrum points. This polynomial is called the characteristic polynomial and will now be introduced.

**Definition 10.** *The* ***characteristic polynomial*** $\mathcal{X}_{p,q}$ *of subalgebra A generated by p(x) and q(x) is defined as*

$$\mathcal{X}_{p,q}(x) = Res\big(P(x,y), Q(x,y)\big) \tag{4}$$

*where* $P(x,y)=\frac{p(x)-p(y)}{x-y}$ *and* $Q(x,y)=\frac{q(x)-q(y)}{x-y}$.

From [2] we find that the resultant is defined as

**Definition 11.** *The* ***resultant*** *of the two polynomials* $p(x) = a_0 + a_1x + \cdots + a_nx^n$ *and* $q(x) = b_0 + b_1x + \cdots + b_mx^m$ *is the determinant of the following matrix*

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & a_1 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & b_1 & \dots & b_m \end{pmatrix}$$

**Example 2.** *We want to find the characteristic polynomial of the algebra A generated by the polynomials*

$$p(x) = x^2 - 1 \text{ and } q(x) = x^3 - x$$

*then we have*

$$P(x,y) = \frac{x^2 - 1 - y^2 + 1}{x - y} = y + x,\ Q(x,y) = \frac{x^3 - x - y^3 + y}{x - y} = y^2 + yx + x^2 - 1$$

*and so we get*

$$\mathcal{X}_{p,q}(x) = \left| \begin{pmatrix} x & 1 & 0 \\ 0 & x & 1 \\ x^2 - 1 & x & 1 \end{pmatrix} \right| = x \cdot (x - x) + (x^2 - 1) \cdot 1 = x^2 - 1$$

Let us see some more examples of characteristic polynomials in Table 1 where we among other things find algebras and their characteristic polynomials.

This polynomial is however difficult to find and is sometimes of a rather high degree. And so we will choose to look for another polynomial with the spectrum elements as its zeros. Although this is its most important property we will start our search looking at another property, namely that when it is multiplied with an arbitrary polynomial the product lies in our subalgebra. It is not trivial to see that such a polynomial exists and so we will start by introducing the following theorem

**Theorem 3.** *Let $A \subset \mathbb{K}[x]$ have $Sp(A) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and define $\pi_A(x) = \prod_{i=1}^{n}(x - \alpha_i)$. Then there exists $N > 1$ such that $\pi_A^N(x) \cdot x^k \in A$, $\forall k \geq 0$.*

And since such a polynomial exists then such a polynomial of minimal degree exists as well.

**Definition 12.** *The* ***minimal polynomial*** *$m(x)$ of $A$, which may be denoted $m_A(x)$ if confusion may occur, is the monic nonzero polynomial with smallest degree such that $m(x) \cdot x^k \in A, \ \forall k \in \mathbb{N}$.*

The minimal polynomial is unique. Assume it is not and we have the minimal polynomials $m_1(x)$ and $m_2(x)$, then the difference, $m_1(x) - m_2(x)$, is a polynomial of lower degree than our minimal polynomials. This is a contradiction if $m_1(x) - m_2(x) \neq 0$ and so we have $m_1(x) = m_2(x)$. In the following table we have some examples of algebras and their minimal polynomials

| $A$ | $p(x)$ | $q(x)$ | $m(x)$ | $\mathcal{X}_{p,q}$ |
|---|---|---|---|---|
| $\{f \mid f(\alpha) = f(\beta)\}$ | $(x-\alpha)(x-\beta)$ | $(x-\alpha)(x-\beta)x$ | $(x-\alpha)(x-\beta)$ | $(x-\alpha)(x-\beta)$ |
| $\{f \mid f'(\alpha) = 0\}$ | $(x-\alpha)^2$ | $(x-\alpha)^2 x$ | $(x-\alpha)^2$ | $(x-\alpha)^2$ |
| $\{f \mid f'(\alpha) = f''(\alpha) = 0\}$ | $(x-\alpha)^3$ | $(x-\alpha)^3 x$ | $(x-\alpha)^3$ | $(x-\alpha)^6$ |

Table 1: Some subalgebras generated by $\{p(x), q(x)\}$ together with their minimal polynomial, $m(x)$, and characteristic polynomial, $\mathcal{X}_{p,q}$.

Note that an algebra is not uniquely determined by its minimal polynomial, both $\{f \mid f(\alpha) = f(\beta), \ f'(\alpha) + f'(\beta) = 0\}$ and $\{f \mid f(\alpha) = f(\beta), \ f'(\alpha) = 0, \ f'(\beta) = 0\}$ have $m(x) = (x-\alpha)^2(x-\beta)^2$ as their minimal polynomial. This polynomial is of interest because similarly to the characteristic polynomial its zeros also consists of the spectrum elements.

**Theorem 4.** *Denote the minimal polynomial of $A$ as m(x) then the following is equivalent*

$$\alpha \in Sp(A) \iff m(\alpha) = 0$$

*Proof.* **(Theorem 4):**
We will start by showing that $\alpha \in \mathrm{Sp}(A) \implies m(\alpha) = 0$.

If there exists a $\beta \neq \alpha$ such that $\beta \in \mathrm{Sp}(A)$ and $f(\alpha) = f(\beta)$, $\forall f \in A$, we then use that $m(x) \in A$ to get

$$m(\alpha) = m(\beta)$$

and $m(x)x \in A$ to get

$$m(\alpha)\alpha = m(\beta)\beta \implies m(\alpha)(\alpha - \beta) = 0$$

where we use $\alpha \neq \beta$ to get $m(\alpha) = m(\beta) = 0$

Next take $\alpha \in \mathrm{Sp}(A)$ such that $f'(\alpha) = 0$ for all $f \in A$. Then $m(x) \in A$ gives

$$m'(\alpha) = 0$$

and $m(x)x \in A$ gives

$$\big(m(x) \cdot x\big)'(\alpha) = m'(\alpha) \cdot \alpha + m(\alpha) = m(\alpha) = 0$$

and so we can conclude that for any $\alpha \in \mathrm{Sp}(A)$ we have $m(\alpha) = 0$.

Next we will show that $m(\alpha) = 0 \implies \alpha \in \mathrm{Sp}(A)$. Since $m(x)$ is the minimal polynomial we know that it has a lower degree than the polynomial $\pi_A^N(x) = \prod_{i=1}^{n}(x - \alpha_i)^N$ defined in Theorem 3. Using polynomial division we can then write

$$\pi_A^N(x) = q(x) \cdot m(x) + r(x)$$

where $\deg\big(r(x)\big) < \deg\big(m(x)\big)$. We then get $r(x) \cdot x^k = \big(\pi_A^N(x) - q(x) \cdot m(x)\big) \cdot x^k \in A$, $\forall k \geq 0$. However since $m(x)$ is the minimal polynomial with this property we obtain that $r(x) = 0$. This means $m(x) \mid \pi_A^N(x)$ and so any zeros of $m(x)$ must also be zeros of $\pi_A^N(x)$ and as a result are in the spectrum of $A$. $\square$

Before moving on to find an explicit expression of the minimal polynomial we will show one final property of this polynomial.

**Theorem 5.** *Given a subalgebra $B \subseteq A \subseteq \mathbb{K}[x]$ of finite codimension we denote the minimal polynomials of $A$ and $B$ by $m_A(x)$ and $m_B(x)$ respectively. Then we have $m_A(x) \mid m_B(x)$*

*Proof.* **(Theorem 5):**
We have $m_B(x) \cdot x^k \in B \subseteq A$, $\forall k \geq 0$ and since $m_A(x)$ is the minimal polynomial with this property we have $\deg\big(m_A(x)\big) \leq \deg\big(m_B(x)\big)$. Using polynomial division we rewrite

$$m_B(x) = q(x) \cdot m_A(x) + r(x)$$

where $\deg\big(r(x)\big) < \deg\big(m_A(x)\big)$. Due to minimality we have $r(x) = 0$ and so $m_A(x) \mid m_B(x)$. $\square$

## 3 Constructing the Minimal Polynomial of A

In this section we will explain a simple way to obtain the minimal polynomial of a subalgebra A $\subseteq \mathbb{K}[x]$, if it is expressed using conditions. For example the algebra $\{f(\alpha) = f(\beta),\ f'(\alpha) = 0\}$ have the minimal polynomial $m(x) = (x - \alpha)^2(x - \beta)$ but how did we find this polynomial?

### 3.1 Effect of an Equality Condition

First we will start by looking at how the minimal polynomial is affected by an equality condition.

**Theorem 6.** *If one of the conditions of subalgebra $A \subseteq \mathbb{K}[x]$ is the equality condition $f(\alpha) = f(\beta)$, $\forall f \in A$ , where $\alpha, \beta$ are distinct spectrum elements, we have that*

$$(x - \alpha)(x - \beta) \mid m_A(x)$$

*where $m_A(x)$ is the minimal polynomial of $A$.*

*Proof.* **(Theorem 6):**
In Theorem 4 we found that since $\alpha$ and $\beta$ are in the spectrum of $A$ we have $m(\alpha) = m(\beta) = 0$. From the fact that $m(x)$ is nonzero, and $\alpha \neq \beta$, we use the factor theorem to obtain that $(x - \alpha)(x - \beta) \mid m_A(x)$. □

### 3.2 Effect of a Derivation

As the proof for the effect an arbitrary derivation is rather convoluted we will state the theorem and leave the proof for later once we have familiarized ourselves with the ideas required for the proof.

**Theorem 7.** *If one of the conditions of subalgebra is a derivation originating from a cluster of size M, $\alpha_i \sim \alpha_j$ $\forall i, j \in \{1, 2, \dots, M\}$, where we can write the derivation as*

$$D(f) = \sum_{j=1}^{M} \sum_{i=1}^{N_j} s_{j,i} f^{(i)}(\alpha_j)$$

*where $s_{j,N_j} \neq 0$. Then $\prod_{j=1}^{N} (x - \alpha_j)^{N_j+1}$ is a factor of the minimal polynomial.*

#### 3.2.1 Cluster of Size 1

We will start by looking at the case when the spectrum of our subalgebra consists of a single element, denoted $\alpha$.

**Theorem 8.** *Assume our subalgebra A, has spectrum $Sp(A) = \{\alpha\}$ and one of its conditions is the derivation $D(f) = \sum_{i=1}^{N} a_i f^{(i)}(\alpha)$, where $a_N \neq 0$. Then $(x - \alpha)^{N+1}$ is a factor of the minimal polynomial $m(x)$, of A.*

By the definition of the minimal polynomial, $m(x)$ which moving forward will be shortened to $m$ for convenience, we have $mx^k$ lies in $A$ for all $k \geq 0$, and so $D(mx^k) = 0$. This will be the foundation of our proofs moving forward. Starting off we will introduce the following notation

$$S(\alpha)_{k,n} = \sum_{i=k}^{N} a_i \frac{i!}{(i-k)!}(mx^n)^{(i-k)}(\alpha)$$

for $0 \le k \le N$, $n \ge 0$, and for $k = 0, n \ge 0$ we have

$$S(\alpha)_{0,n} = \sum_{i=1}^{N} a_i (mx^n)^{(i)}(\alpha) = D(mx^n)$$

We also define $S(\alpha)_{k,n} = 0$ for $k > N$. The product rule for derivation gives us

$$(mx^{n-1} \cdot x)^{(i)} = x(mx^{n-1})^{(i)} + i(mx^{n-1})^{(i-1)}$$

which we can rewrite using our new notations to obtain

$$S(\alpha)_{k,n} = \alpha \cdot S(\alpha)_{k,n-1} + S(\alpha)_{k+1,n-1} \text{ for } n \ge 1 \tag{5}$$

**Example 3.** *Assume we have a subalgebra $A$ with spectrum $Sp(A) = \{\alpha\}$. One of the conditions of $A$ can be expressed as following*

$$D(f) = a_1 f'(\alpha) + a_2 f''(\alpha) = 0, \quad \forall\, f \in A$$

*where $a_2 \neq 0$. Our goal is to show that $(x-\alpha)^3 \mid m(x)$. We will then rewrite $D(m) = 0$ as*

$$D(m) = a_1 m'(\alpha) + a_2 m''(\alpha) = \sum_{i=1}^{2} a_i m^{(i)}(\alpha) = S(\alpha)_{0,0}$$

*from which we get $S(\alpha)_{0,0} = 0$. Then we rewrite $D(mx) = 0$*

$$\begin{aligned} D(mx) &= a_1(mx)'(\alpha) + a_2(mx)''(\alpha) = \sum_{i=1}^{2} a_i (mx)^{(i)}(\alpha) = S(\alpha)_{0,1} = \\ &= \alpha\big(a_1 m'(\alpha) + a_2 m''(\alpha)\big) + \big(a_1 m(\alpha) + 2a_2 m'(\alpha)\big) = \\ &= \alpha \sum_{i=1}^{2} a_i m^{(i)}(\alpha) + \sum_{i=1}^{2} a_i i m^{(i-1)}(\alpha) = \alpha S(\alpha)_{0,0} + S(\alpha)_{1,0} \end{aligned}$$

*from which we get $S(\alpha)_{0,1} = S(\alpha)_{1,0} = 0$. Lastly we rewrite $D(mx^2) = 0$*

$$\begin{aligned} D(mx^2) &= a_1(mx^2)'(\alpha) + a_2(mx^2)''(\alpha) = \sum_{i=1}^{2} a_i (mx^2)^{(i)}(\alpha) = S(\alpha)_{0,2} = \\ &= \alpha\big(a_1(mx)'(\alpha) + a_2(mx)''(\alpha)\big) + \big(a_1(mx)(\alpha) + 2a_2(mx)'(\alpha)\big) = \\ &= \alpha \sum_{i=1}^{2} a_i (mx)^{(i)}(\alpha) + \sum_{i=1}^{2} a_i i (mx)^{(i-1)}(\alpha) = \alpha S(\alpha)_{0,1} + S(\alpha)_{1,1} = \\ &= \alpha^2\big(a_1 m'(\alpha) + a_2 m''(\alpha)\big) + 2\alpha\big(a_1 m(\alpha) + 2a_2 m'(\alpha)\big) + 2a_2 m(\alpha) = \\ &= \alpha^2 \sum_{i=1}^{2} a_i m^{(i)}(\alpha) + 2\alpha \sum_{i=1}^{2} a_i i m^{(i)}(\alpha) + \sum_{i=2}^{2} a_i i(i-1) m^{(i-2)}(\alpha) = \\ &= \alpha^2 S(\alpha)_{0,0} + 2\alpha S(\alpha)_{1,0} + S(\alpha)_{2,0} \end{aligned}$$

*from which we get* $S(\alpha)_{0,2} = S(\alpha)_{1,1} = S(\alpha)_{2,0} = 0$*. That* $(x-\alpha)^3 \mid m(x)$ *is equivalent to showing that* $m(\alpha) = m'(\alpha) = m''(\alpha)$*, and so we write*

$$\begin{array}{llll} S(\alpha)_{2,0} = 0 \implies & 2a_2 m(\alpha) = 0 \implies & & m(\alpha) = 0 \\ S(\alpha)_{1,0} = 0 \implies & a_1 m(\alpha) + 2a_2 m'(\alpha) = 0 & \implies & m'(\alpha) = 0 \\ S(\alpha)_{0,0} = 0 \implies & a_1 m'(\alpha) + a_2 m''(\alpha) = 0 & \implies & m''(\alpha) = 0 \end{array}$$

*where we have used* $a_2 \neq 0$ *to conclude our example.*

In Example 3 we begin to see patterns which we want to prove in general, one of which is

**Lemma 1.** *For any* $k \geq 0$*,* $n \geq 0$ *we have*

$$\sum_{i=0}^{k} \alpha^{k-i} \binom{k}{i} S(\alpha)_{i,n} = 0$$

*We will denote this equation as* $K_{k,n}$*.*

**Lemma 2.** *From* $D(mx^p) = 0$ *we obtain that*

$$\sum_{i=0}^{k} \alpha^{k-i} \binom{k}{i} S(\alpha)_{i,p-k} = 0$$

*holds true for* $0 \leq k \leq p$*. Which is equivalent to* $K_{k,p-k}$ *holding true for* $0 \leq k \leq p$*.*

*Proof.* **(Lemma 2):**
For our proof we will be using induction, first note that by definition we have

$$D(mx^n) = S(\alpha)_{0,n} = 0 \tag{6}$$

This means we have $D(mx^0) = S(\alpha)_{0,0} = 0$, and thus the lemma holds for $p = 0$. Then we assume the lemma to be true for up to and including an arbitrary $p \geq 0$. From $D(mx^{p+1}) = 0$ and Equation 6 we know that $K_{0,p+1}$ holds true. If we assume $K_{t,p+1-t}$ is true for an arbitrary $0 \leq t < p$ , then we have

$$\sum_{i=0}^{t} \alpha^{t-i} \binom{t}{i} S(\alpha)_{i,p+1-t} = 0$$

which together with equation (5) we can rewrite as

$$
\begin{aligned}
&\sum_{i=0}^{t} \alpha^{t-i}\binom{t}{i} S(\alpha)_{i,p+1-t} \\
&= \sum_{i=0}^{t} \alpha^{t+1-i}\binom{t}{i} S(\alpha)_{i,p+1-(t+1)} + \sum_{i=0}^{t} \alpha^{t-i}\binom{t}{i} S(\alpha)_{i+1,p+1-(t+1)} = \\
&= \sum_{i=0}^{t} \alpha^{t+1-i}\binom{t}{i} S(\alpha)_{i,p+1-(t+1)} + \sum_{i=1}^{t+1} \alpha^{t+1-i}\binom{t}{i-1} S(\alpha)_{i,p+1-(t+1)} = \\
&= \alpha^{t+1}\binom{t}{0} S(\alpha)_{0,p+1-(t+1)} + \alpha^{0}\binom{t}{t} S(\alpha)_{t+1,p+1-(t+1)} \\
&\quad + \sum_{i=1}^{t} \alpha^{t+1-i}\left(\binom{t}{i} + \binom{t}{i-1}\right) S(\alpha)_{i,p+1-(t+1)} = \\
&= \sum_{i=0}^{t+1} \alpha^{(t+1)-i}\binom{t+1}{i} S(\alpha)_{i,p+1-(t+1)} =
\end{aligned}
$$

and so we have shown that it also holds for $K_{t+1,p+1-(t+1)}$, and so by induction $K_{t,p+1-t}$ holds for all $t$. By induction our lemma holds for all $p$ and so we have reached the conclusion of the proof. □

*Proof.* **(Lemma 1):**
Take an arbitrary $K_{k,n}$ such that $k, n \geq 0$ then using Lemma 2 with $p = k + n$ we have our statement as $k \leq k + n = p$ and $n = (k + n) - k = p - k$. □

**Lemma 3.** *For all $k \geq 0$ we have*

$$S(\alpha)_{k,0} = 0$$

*Proof.* **(Lemma 3):**
We will be using induction, for $k = 0$ we have $D(m) = S(\alpha)_{0,0} = 0$. Next assume we have $S(\alpha)_{k,0} = 0$ for $0 \leq k \leq p$, then we have

$$0 = \sum_{i=0}^{p+1} \alpha^{p+1-i}\binom{p+1}{i} S(\alpha)_{i,0} = \alpha^{0}\binom{p+1}{p+1} S(\alpha)_{p+1,0} = S(\alpha)_{p+1,0}$$

□

*Proof.* **(Theorem 8):**
We will show that $m^{(n)}(\alpha) = 0$ for $n = 0, 1, \ldots, N$. For $n = 0$ we take

$$S(\alpha)_{N,0} = a_N \cdot N! \cdot m^{(0)}(\alpha) = 0 \implies m^{(0)}(\alpha) = 0 \text{ since } a_N \neq 0$$

Assume $m^{(n)}(\alpha) = 0$ for $n = 0, 1, \ldots, p$ for a $0 \leq p < N$ then from the definition of $S(\alpha)_{k,n}$ we have

$$S(\alpha)_{N-(p+1),0} = \sum_{i=N-(p+1)}^{N} a_i \frac{i!}{(i-N+p+1)!} m^{(i-N+(p+1))}(\alpha) = a_N \cdot \frac{N!}{(p+1)!} \cdot m^{(p+1)}(\alpha) = 0$$

and so we get $m^{(p+1)}(\alpha) = 0$. By induction we get $m^{(n)}(\alpha) = 0$ for $n = 0, 1, \ldots, N$ which is equivalent to $(x-\alpha)^{N+1} \mid m(x)$, and our proof has concluded. □

### 3.2.2 Cluster of Size 2

**Theorem 9.** *Assume one of the conditions of our subalgebra, A, is a derivation originating from a cluster of size two, $\alpha \sim \beta$, where we can write the derivation as*

$$D(f) = \sum_{i=1}^{N} a_i f^{(i)}(\alpha) + \sum_{i=1}^{M} b_i f^{(i)}(\beta), \text{ where } a_N \neq 0, b_M \neq 0$$

*Then $(x-\alpha)^{N+1}(x-\beta)^{M+1}$ is a factor of the minimal polynomial, $m(x)$, of A.*

The proof will be provided later. First we will extend our previous notations to also include

$$S(\beta)_{k,n} = \sum_{i=k}^{M} b_i \frac{i!}{(i-k)!}(mx^n)^{(i-k)}(\beta)$$

for $0 < k \leq M$ and $n \geq 0$. And for $k = 0$ and $n \geq 0$ we have

$$S(\beta)_{0,n} = \sum_{i=1}^{M} b_i (mx^n)^{(i)}(\beta)$$

We also define $S(\beta)_{k,n} = 0$ for $k > M$.

Similarly to before we also have

$$S(\beta)_{k,n} = \beta \cdot S(\beta)_{k,n-1} + S(\beta)_{k+1,n-1} \text{ for n} \geq 1 \tag{7}$$

since $(mx^n)^{(i)} = x(mx^{n-1})^{(i)} + i(mx^{n-1})^{(i-1)}$.

Before we work towards proving Theorem 9 we will look at the following example

**Example 4.** *Assume we have a subalgebra A with spectrum Sp(A)=$\{\alpha, \beta\}, \alpha \sim \beta$. One of the conditions of A can be expressed as follows*

$$D(f) = a_1 f'(\alpha) + a_2 f''(\alpha) + b_1 f'(\beta) = 0, \quad \forall\ f \in A$$

*where $a_2, b_1 \neq 0$. We will now show that $(x-\alpha)^3(x-\beta)^2 \mid m(x)$. Already in the case of one spectrum element the calculations were quite extensive and so in this example we will make use of the following table, which holds true for any $\alpha$-derivation.*

| | |
|---|---|
| $D(m)$ | $= S(\alpha)_{0,0} + S(\beta)_{0,0}$ |
| $D(mx)$ | $= S(\alpha)_{0,1} + S(\beta)_{0,1}$<br>$= S(\alpha)_{1,0} + (\beta-\alpha)S(\beta)_{0,0} + S(\beta)_{1,0}$ |
| $D(mx^2)$ | $= S(\alpha)_{0,2} + S(\beta)_{0,2}$<br>$= S(\alpha)_{1,1} + (\beta-\alpha)S(\beta)_{0,1} + S(\beta)_{1,1}$<br>$= (\alpha-\beta)S(\alpha)_{1,0} + S(\alpha)_{2,0} + (\beta-\alpha)S(\beta)_{1,0} + S(\beta)_{2,0}$ |
| $D(mx^3)$ | $= S(\alpha)_{0,3} + S(\beta)_{0,3}$<br>$= S(\alpha)_{1,2} + (\beta-\alpha)S(\beta)_{0,2} + S(\beta)_{1,2}$<br>$= (\alpha-\beta)S(\alpha)_{1,1} + S(\alpha)_{2,1} + (\beta-\alpha)S(\beta)_{1,1} + S(\beta)_{2,1}$<br>$= (\alpha-\beta)S(\alpha)_{2,0} + S(\alpha)_{3,0} + (\beta-\alpha)^2S(\beta)_{1,0} + 2(\beta-\alpha)S(\beta)_{2,0} + S(\beta)_{3,0}$ |
| $D(mx^4)$ | $= S(\alpha)_{0,4} + S(\beta)_{0,4}$<br>$= S(\alpha)_{1,3} + (\beta-\alpha)S(\beta)_{0,3} + S(\beta)_{1,3}$<br>$= (\alpha-\beta)S(\alpha)_{1,2} + S(\alpha)_{2,2} + (\beta-\alpha)S(\beta)_{1,2} + S(\beta)_{2,2}$<br>$= (\alpha-\beta)S(\alpha)_{2,1} + S(\alpha)_{3,1} + (\beta-\alpha)^2S(\beta)_{1,1} + 2(\beta-\alpha)S(\beta)_{2,1} + S(\beta)_{3,1}$<br>$= (\alpha-\beta)^2S(\alpha)_{2,0} + 2(\alpha-\beta)S(\alpha)_{3,0} + S(\alpha)_{4,0}$<br>$+(\beta-\alpha)^2S(\beta)_{2,0} + 2(\beta-\alpha)S(\beta)_{3,0} + S(\beta)_{4,0}$ |

*In our case we have $S(\alpha)_{k,n} = 0$ for $k > 2$ and $S(\beta)_{k,n} = 0$ for $k > 1$. And so looking only at $n = 0$ we can rewrite the table as*

| | |
|---|---|
| $D(m)$ | $= S(\alpha)_{0,0} + S(\beta)_{0,0}$ |
| $D(mx)$ | $= S(\alpha)_{1,0} + (\beta-\alpha)S(\beta)_{0,0} + S(\beta)_{1,0}$ |
| $D(mx^2)$ | $= (\alpha-\beta)S(\alpha)_{1,0} + S(\alpha)_{2,0} + (\beta-\alpha)S(\beta)_{1,0}$ |
| $D(mx^3)$ | $= (\alpha-\beta)S(\alpha)_{2,0} + (\beta-\alpha)^2S(\beta)_{1,0}$ |
| $D(mx^4)$ | $= (\alpha-\beta)^2S(\alpha)_{2,0}$ |

*Using $\alpha \neq \beta$ we get*

$$
\begin{array}{lll}
D(mx^4) = 0 \implies & (\alpha-\beta)^2S(\alpha)_{2,0} = 0 \implies & S(\alpha)_{2,0} = 0 \\
D(mx^3) = 0 \implies & (\beta-\alpha)^2S(\beta)_{1,0} = 0 \implies & S(\beta)_{1,0} = 0 \\
D(mx^2) = 0 \implies & (\alpha-\beta)S(\alpha)_{1,0} = 0 \implies & S(\alpha)_{1,0} = 0 \\
D(mx) = 0 \implies & (\beta-\alpha)S(\beta)_{0,0} = 0 \implies & S(\beta)_{0,0} = 0 \\
D(m) = 0 \implies & & S(\alpha)_{0,0} = 0
\end{array}
$$

*We then use the definition of $S(\alpha)_{k,n}$ to get*

$$
\begin{array}{lll}
S(\alpha)_{2,0} = 0 \implies & 2a_2m(\alpha) = 0 \implies & m(\alpha) = 0 \\
S(\alpha)_{1,0} = 0 \implies & 2a_2m'(\alpha) = 0 \implies & m'(\alpha) = 0 \\
S(\alpha)_{0,0} = 0 \implies & a_2m''(\alpha) = 0 \implies & m''(\alpha) = 0
\end{array}
$$

*where we have used $a_2 \neq 0$. Similarly we use the definition of $S(\beta)_{k,n}$ to get*

$$\begin{array}{lll} S(\beta)_{1,0} = 0 \implies & b_1 m(\beta) = 0 \implies & m(\beta) = 0 \\ S(\beta)_{0,0} = 0 \implies & b_1 m'(\beta) = 0 \implies & m'(\beta) = 0 \end{array}$$

*where we use $b_1 \neq 0$. Which together means we have $(x-\alpha)^3(x-\beta)^2 \mid m(x)$.*

**Lemma 4.** *The following statements are true for all $k, n \geq 0$.*

$$\sum_{i=0}^{k} \binom{k}{i} (\alpha-\beta)^{k-i} S(\alpha)_{k+i,n} + \sum_{i=0}^{k} \binom{k}{i} (\beta-\alpha)^{k-i} S(\beta)_{k+i,n} = 0 \tag{8}$$

$$\sum_{i=0}^{k} \binom{k}{i} (\alpha-\beta)^{k-i} S(\alpha)_{k+i+1,n} + \sum_{i=0}^{k+1} \binom{k+1}{i} (\beta-\alpha)^{k+1-i} S(\beta)_{k+i,n} = 0 \tag{9}$$

*which moving forward we will denote $K^0_{k,n}$ and $K^1_{k,n}$ respectively.*

From these equations we can trivially tell that we have $K^0_{0,n} = S(\alpha)_{0,n} + S(\beta)_{0,n} = D(mx^n) = 0$.

**Lemma 5.** *We have the following implications*

- $K^0_{k,n+1}$ *and* $K^0_{k,n}$ *holds* $\implies$ $K^1_{k,n}$ *holds*
- $K^1_{k,n+1}$ *and* $K^1_{k,n}$ *holds* $\implies$ $K^0_{k+1,n}$ *holds*

*Proof.* **(Lemma 5):**
Assume we have $K^0_{k,n+1}$ and $K^0_{k,n}$. We will rewrite $K^0_{k,n+1}$ using Equation 7 and then substitute using $K^0_{k,n}$

$$\begin{aligned}
&\sum_{i=0}^{k} \tbinom{k}{i} (\alpha-\beta)^{k-i} S(\alpha)_{k+i,n+1} + \sum_{i=0}^{k} \tbinom{k}{i} (\beta-\alpha)^{k-i} S(\beta)_{k+i,n+1} = \\
&\quad = \alpha \sum_{i=0}^{k} \tbinom{k}{i} (\alpha-\beta)^{k-i} S(\alpha)_{k+i,n} + \sum_{i=0}^{k} \tbinom{k}{i} (\alpha-\beta)^{k-i} S(\alpha)_{k+i+1,n} \\
&\qquad + \beta \sum_{i=0}^{k} \tbinom{k}{i} (\beta-\alpha)^{k-i} S(\beta)_{k+i,n} + \sum_{i=0}^{k} \tbinom{k}{i} (\beta-\alpha)^{k-i} S(\beta)_{k+i+1,n} = \\
&\quad = \sum_{i=0}^{k} \tbinom{k}{i} (\alpha-\beta)^{k-i} S(\alpha)_{k+i+1,n} + \sum_{i=0}^{k} \tbinom{k}{i} (\beta-\alpha)^{k-i+1} S(\beta)_{k+i,n} \\
&\qquad + \sum_{i=0}^{k} \tbinom{k}{i} (\beta-\alpha)^{k-i} S(\beta)_{k+i+1,n} = \\
&\quad = \sum_{i=0}^{k} \tbinom{k}{i} (\alpha-\beta)^{k-i} S(\alpha)_{k+i+1,n} + \tbinom{k}{0} (\beta-\alpha)^{k+1} S(\beta)_{k,n} \\
&\qquad + \sum_{i=1}^{k} \left( \tbinom{k}{i} + \tbinom{k}{i-1} \right) (\beta-\alpha)^{k-i+1} S(\beta)_{k+i,n} + \tbinom{k}{k} (\beta-\alpha)^0 S(\beta)_{2k+1,n} = \\
&\quad = \sum_{i=0}^{k} \tbinom{k}{i} (\alpha-\beta)^{k-i} S(\alpha)_{k+i+1,n} + \sum_{i=0}^{k+1} \tbinom{k+1}{i} (\beta-\alpha)^{k-i+1} S(\beta)_{k+i,n} = 0
\end{aligned}$$

where we have used $\binom{k}{i} + \binom{k}{i-1} = \binom{k+1}{i}$. And so we have shown the first statement of the lemma. Next we assume to have $K^1_{k,n+1}$ and $K^1_{k,n}$. Similarly we will rewrite $K^1_{k,n+1}$ using Equation 7 and then substitute using $K^1_{k,n}$

$$
\begin{aligned}
&\sum_{i=0}^{k} \binom{k}{i}(\alpha-\beta)^{k-i}S(\alpha)_{k+i+1,n+1} + \sum_{i=0}^{k+1} \binom{k+1}{i}(\beta-\alpha)^{k+1-i}S(\beta)_{k+i,n+1} = \\
&\quad = \alpha\sum_{i=0}^{k} \binom{k}{i}(\alpha-\beta)^{k-i}S(\alpha)_{k+i+1,n} + \sum_{i=0}^{k} \binom{k}{i}(\alpha-\beta)^{k-i}S(\alpha)_{k+i+2,n} \\
&\qquad +\beta\sum_{i=0}^{k+1} \binom{k+1}{i}(\beta-\alpha)^{k+1-i}S(\beta)_{k+i,n} + \sum_{i=0}^{k+1} \binom{k+1}{i}(\beta-\alpha)^{k+1-i}S(\beta)_{k+i+1,n} = \\
&\quad = \sum_{i=0}^{k} \binom{k}{i}(\alpha-\beta)^{k-i+1}S(\alpha)_{k+i+1,n} + \sum_{i=0}^{k} \binom{k}{i}(\alpha-\beta)^{k-i}S(\alpha)_{k+i+2,n} \\
&\qquad +\sum_{i=0}^{k+1} \binom{k+1}{i}(\beta-\alpha)^{k+1-i}S(\beta)_{k+i+1,n} = \\
&\quad = \binom{k}{0}(\alpha-\beta)^{k+1}S(\alpha)_{k+1,n} + \sum_{i=1}^{k} \left(\binom{k}{i} + \binom{k}{i-1}\right)(\alpha-\beta)^{k-i+1}S(\alpha)_{k+i+1,n} \\
&\qquad +\binom{k}{k}(\alpha-\beta)^{0}S(\alpha)_{2k+2,n} + \sum_{i=0}^{k+1} \binom{k+1}{i}(\beta-\alpha)^{k+1-i}S(\beta)_{k+i+1,n} = \\
&\quad = \sum_{i=0}^{(k+1)} \binom{(k+1)}{i}(\alpha-\beta)^{(k+1)-i}S(\alpha)_{(k+1)+i,n} + \sum_{i=0}^{(k+1)} \binom{(k+1)}{i}(\beta-\alpha)^{(k+1)-i}S(\beta)_{(k+1)+i,n} = 0
\end{aligned}
$$

□

**Lemma 6.** *From $D(mx^p) = 0$, where $p = 2q+r \geq 0$ and $r \in \{0,1\}$, we get $K^{\hat{r}}_{\hat{q},2(q-\hat{q})+(r-\hat{r})}$, for $0 \leq 2\hat{q} + \hat{r} \leq p$.*

*Proof.* **(Lemma 6):**
For $p = 0$ we need $K^0_{0,0}$ which can be written as $S(\alpha)_{0,0} + S(\beta)_{0,0} = D(m) = 0$ which we know since previously. Next we assume the lemma to be true for up to and including an arbitrary $p \geq 0$. We want to show that this statement also holds true for $p+1$. By definition we have $D(mx^{p+1}) = S(\alpha)_{p+1,0} + S(\beta)_{p+1,0} = 0$ which is equivalent to $K^0_{0,p+1}$. Assuming that we have $K^{\hat{r}}_{\hat{q},2(q-\hat{q})+(r-\hat{r})}$ for $0 \leq 2\hat{q} + \hat{r} \leq t$ for some $t = 2\tilde{q} + \tilde{r}$, where $\hat{r}, \tilde{r} \in \{0,1\}$. Then we will use Lemma 5.

If $\tilde{r} = 0$ we have $K^0_{\tilde{q},2(q-\tilde{q})+r}$ and $K^0_{\tilde{q},2(q-\tilde{q})+r-1}$ and so we also have $K^1_{\tilde{q},2(q-\tilde{q})+r-1}$.
If $\tilde{r} = 1$ we have $K^1_{\tilde{q},2(q-\tilde{q})+(r-1)}$ and $K^1_{\tilde{q},2(q-\tilde{q})+(r-1)-1}$ and so we also have
$K^0_{\tilde{q}+1,2(q-\tilde{q})+(r-1)-1} = K^0_{\tilde{q}+1,2(q-\tilde{q}-1)+r}$.

This is equivalent to $K^{\hat{r}}_{\hat{q},2(q-\hat{q})+(r-\hat{r})}$ being true for $0 \leq 2\hat{q}+\hat{r} \leq t+1$. We have now shown that our statement holds for $p+1$, so using induction we have finished our proof. □

*Proof.* **(Lemma 4):**

Pick arbitrary $a, b, c$ such that $a, b \geq 0$ and $c \in \{0, 1\}$, we want to show that $K_{a,b}^c$ holds. Then using Lemma (6) with $p = 2a+b+c = 2q+r$ where $r \in \{0,1\}$, $\hat{q} = a$, and $\hat{r} = c$ we have $2(q-\hat{q})+(r-\hat{r}) = p-2a-c = b$ which together with $2\hat{q}+\hat{r} = 2a+c \leq 2a+b+c = p$ concludes our proof.

□

**Lemma 7.** *For all $k \geq 0$ we have*

$$S(\alpha)_{k,0} = S(\beta)_{k,0} = 0$$

*Proof.* **(Lemma 7):**

By definition $S(\alpha)_{k,0} = 0$ for $k > N$ and $S(\beta)_{k,0} = 0$ for $k > M$. We will rewrite Equations 8 and 9 as

$$(\alpha-\beta)^k S(\alpha)_{k,n} = -\sum_{i=1}^{k} \binom{k}{i} (\alpha-\beta)^{k-i} S(\alpha)_{k+i,n} - \sum_{i=0}^{k} \binom{k}{i} (\beta-\alpha)^{k-i} S(\beta)_{k+i,n} \quad (10)$$

$$(\beta-\alpha)^{k+1} S(\beta)_{k,n} = -\sum_{i=0}^{k} \binom{k}{i} (\alpha-\beta)^{k-i} S(\alpha)_{k+i+1,n} - \sum_{i=1}^{k+1} \binom{k+1}{i} (\beta-\alpha)^{k+1-i} S(\beta)_{k+i,n} \quad (11)$$

In particular this is true for $n = 0$. Assume we have $S(\alpha)_{t,0} = S(\beta)_{t,0} = 0$ for all $t \geq p$ for some $p > 0$. Then from Equation 11 with $k = t-1$ we get

$$(\beta-\alpha)^{k+1} S(\beta)_{t-1,0} = 0 \implies S(\beta)_{t-1,0} = 0$$

and from Equation 10 we get

$$(\alpha-\beta)^k S(\alpha)_{t-1,0} = 0 \implies S(\alpha)_{t-1,0} = 0$$

By induction we finish our proof. □

*Proof.* **(Theorem 9):**
Starting with $S(\beta)_{M,0} = 0$ we get $b_M \cdot M! \cdot m(\beta) = 0 \implies m(\beta) = 0$, since $b_M \neq 0$. Assume $m^{(t)}(\beta) = 0$ for all $t$ such that $0 \leq t \leq p$, where $p$ is an arbitrary number $0 \leq p < M$. Then

$$S(\beta)_{M-p-1,0} = 0 \implies b_M \cdot \frac{M!}{(p+1)!} \cdot m^{(p+1)}(\beta) = 0$$

and so $m^{(p+1)}(\beta) = 0$. By induction we have $m(\beta) = m'(\beta) = \cdots = m^{(M)}(\beta) = 0$.

Similarly we will look at $S(\alpha)_{N,0} = 0$ to get $a_N \cdot N! \cdot m(\alpha) = 0 \implies m(\alpha) = 0$, since $a_N \neq 0$. Assume $m^{(t)}(\alpha) = 0$ for all $t$, such that $0 \leq t \leq p$, where $p$ is an arbitrary number $0 \leq p < N$. Then

$$S(\alpha)_{N-p-1,0} = 0 \implies b_N \cdot \frac{N!}{(p+1)!} \cdot m^{(p+1)}(\alpha) = 0$$

and so $m^{(p+1)}(\alpha) = 0$. By induction we have $m(\alpha) = m'(\alpha) = \cdots = m^{(N)}(\alpha) = 0$.

These two statements together gives us $(x-\alpha)^{N+1}(x-\beta)^{M+1} \mid m(x)$. □

#### 3.2.3 Cluster of Size M

In this section we will finally provide the proof for Theorem 7. We will start by introducing the following notations

$$S(\alpha_j)_{k,n} = \sum_{i=k}^{N_j} s_{j,i} \frac{i!}{(i-k)!} (mx^n)^{(i-k)}(\alpha_j)$$

for $0 < k \leq N_j$ and $n \geq 0$. And for $k = 0$ and $n \geq 0$ we have

$$S(\alpha_j)_{0,n} = \sum_{i=1}^{N_j} s_{j,i} (mx^n)^{(i)}(\alpha_j)$$

we also define $S(\alpha_j)_{k,n} = 0$ for $k > N_j$. Similar to before we also introduce the following equation

$$S(\alpha_j)_{k,n} = \alpha_j S(\alpha_j)_{k,n-1} + S(\alpha_j)_{k+1,n-1} \text{ for n} \geq 1 \tag{12}$$

which holds since $(mx^n)^{(i)} = x(mx^{n-1})^{(i)} + i(mx^{n-1})^{(i-1)}$.

When we are working with $M$ spectrum elements we will have some patterns which are similar to the ones in the 2 spectrum elements case. We will now look at the behavior of 3 spectrum elements to demonstrate the one behaviour which is not apparent in the case of 2 spectrum elements. Due to the length of these expressions we will introduce some temporary notations

$$A_k = S(\alpha)_{k,0} \quad B_k = S(\beta)_{k,0} \quad C_k = S(\gamma)_{k,0}$$

$$\begin{array}{ccc} \alpha_1 = 0 & \alpha_2 = \alpha - \beta & \alpha_3 = \alpha - \gamma \\ \beta_1 = \beta - \alpha & \beta_2 = 0 & \beta_3 = \beta - \gamma \\ \gamma_1 = \gamma - \alpha & \gamma_2 = \gamma - \beta & \gamma_3 = 0 \end{array}$$

As these notations hint at we will only look at the case when $n = 0$, but similar patterns appear for $n > 0$. As the definitions above might suggest $\alpha_1, \alpha_2, \alpha_3$ will not refer to the spectrum elements of $A$ whilst we are looking at the case of $|\text{Sp}(A)| = 3$.

$$
\begin{aligned}
D(m) &= A_0 + B_0 + C_0 \\
D(mx) &= \beta_1 B_0 + \gamma_1 C_0 + A_1 + B_1 + C_1 \\
D(mx^2) &= \gamma_1\gamma_2 C_0 + \alpha_2 A_1 + \beta_1 B_1 + (\gamma_1+\gamma_2)C_1 + A_2 + B_2 + C_2
\end{aligned}
$$

Looking at $D(mx^2)$ we find that $C_0$ has coefficient $\gamma_1\gamma_2$ then the coefficient in front of $C_1$ is $(\gamma_1+\gamma_2)$ which is a sum of all permutations of $\gamma_1\gamma_2$ where we have removed one of either $\gamma_1$ or $\gamma_2$, and similarly the coefficient in front of $C_2$ is the sum of all permutations of $\gamma_1\gamma_2$ where we have removed two of either $\gamma_1$ or $\gamma_2$, namely 1. Next we will look at the following three examples, where we will not only see this pattern reoccur but we will also take a look at what the initial coefficient looks like, which in the previous case we looked at was $\gamma_1\gamma_2$.

$$
\begin{aligned}
D(mx^3) = \;& \alpha_2\alpha_3 A_1 + (\alpha_2+\alpha_3)A_2 + A_3 \\
&+ \beta_1\beta_3 B_1 + (\beta_1+\beta_3)B_2 + B_3 \\
&+ \gamma_1\gamma_2 C_1 + (\gamma_1+\gamma_2)C_2 + C_3 \\
D(mx^4) = \;& \beta_1^2\beta_3 B_1 + (\beta_1^2 + 2\beta_1\beta_3)B_2 + (2\beta_1+\beta_3)B_3 + B_4 \\
&+ \gamma_1^2\gamma_2 C_1 + (\gamma_1^2 + 2\gamma_1\gamma_2)C_2 + (2\gamma_1+\gamma_2)C_3 + C_4 \\
&+ \alpha_2\alpha_3 A_2 + (\alpha_2+\alpha_3)A_3 + A_4 \\
D(mx^5) = \;& \gamma_1^2\gamma_2^2 C_1 + (2\gamma_1^2\gamma_2 + 2\gamma_1\gamma_2^2)C_2 + (\gamma_1^2 + 4\gamma_1\gamma_2 + \gamma_2^2)C_3 + (2\gamma_1+2\gamma_2)C_4 + C_5 \\
&+ \alpha_2^2\alpha_3 A_2 + (\alpha_2^2 + 2\alpha_2\alpha_3)A_3 + (2\alpha_2+\alpha_3)A_4 + A_5 \\
&+ \beta_1^2\beta_3 B_2 + (\beta_1^2 + 2\beta_1\beta_3)B_3 + (2\beta_1+\beta_3)B_4 + B_5
\end{aligned}
$$

Assume we are looking at $D(mx^p)$ where $p = 3q + r, r \in \{0,1,2\}$, then we find that the initial coefficient in front of $A$, $B$, and $C$ respectively is

| $A$ | $B$ | $C$ | |
|---|---|---|---|
| $\alpha_2^q \cdot \alpha_3^q$ | $\beta_1^q \cdot \beta_3^q$ | $\gamma_1^q \cdot \gamma_2^q$ | if $r = 0$ |
| $\alpha_2^q \cdot \alpha_3^q$ | $\beta_1^{q+1} \cdot \beta_3^q$ | $\gamma_1^{q+1} \cdot \gamma_2^q$ | if $r = 1$ |
| $\alpha_2^{q+1} \cdot \alpha_3^q$ | $\beta_1^{q+1} \cdot \beta_3^q$ | $\gamma_1^{q+1} \cdot \gamma_2^{q+1}$ | if $r = 2$ |

we can see a cyclical behavior with period 3, there is a similar cyclical behavior when we have M spectrum elements however then the period is M. Looking at only at the $C/\gamma$-part we get

$$
\gamma_1^q\gamma_2^q C_q + \left(\binom{q}{q}\binom{q}{q-1}\gamma_1^q\gamma_2^{q-1} + \binom{q}{q-1}\binom{q}{q}\gamma_1^{q-1}\gamma_2^q\right) C_{q+1} + \cdots + (q\,\gamma_1 + q\,\gamma_2)\,C_{2q-1} + C_{2q} \qquad \text{if } r = 0
$$

$$
\gamma_1^{q+1}\gamma_2^q C_q + \left(\binom{q+1}{q+1}\binom{q}{q-1}\gamma_1^{q+1}\gamma_2^{q-1} + \binom{q+1}{q}\binom{q}{q}\gamma_1^q\gamma_2^q\right) C_{q+1} + \cdots + ((q+1)\,\gamma_1 + q\,\gamma_2)\,C_{2q} + C_{2q+1} \qquad \text{if } r = 1
$$

$$
\gamma_1^{q+1}\gamma_2^{q+1} C_q + \left(\binom{q+1}{q+1}\binom{q+1}{q}\gamma_1^{q+1}\gamma_2^q + \binom{q+1}{q}\binom{q+1}{q+1}\gamma_1^q\gamma_2^{q+1}\right) C_{q+1} + \cdots + ((q+1)\,\gamma_1 + (q+1)\,\gamma_2)\,C_{2q+1} + C_{2q+2} \qquad \text{if } r = 2
$$

which combined into a single equation becomes

$$\sum_{\substack{a_1,a_2\dots,a_r\in\{0,1,\dots,q+1\}\\ a_{r+1},a_{r+2}\dots,a_3\in\{0,1,\dots,q\}}} \left(\prod_{i=1}^{r}\gamma_i^{a_i}\binom{q+1}{a_i}\right)\cdot\left(\prod_{i=r+1}^{3}\gamma_i^{a_i}\binom{q}{a_i}\right)C_\Delta$$

where $\Delta = a_3 + \sum_{i=1}^{r}(q+1-a_i) + \sum_{i=r+1}^{3}(q-a_i)$, note that in all nonzero terms we have $a_3 = 0$ since $a_3 \neq 0$ implies $\gamma_3 = 0$ is a factor of the term. In order to get the equations for $\alpha$ and $\beta$ we only need to change $\gamma_i$ to $\alpha_i$ or $\beta_i$ as well as change the $a_3$ in $\Delta$ to $a_1$ and $a_2$ respectively. Now that we hopefully have a better understanding of what our equations looks like we will go back to the general case where we have $M$ spectrum elements.

**Lemma 8.** *The following statement is true for all $k, n \geq 0$ and $0 \leq u \leq M-1$.*

$$\sum_{j=1}^{M}\left(\sum_{\substack{a_1,a_2\dots,a_u\in\{0,1,\dots,k+1\}\\ a_{u+1},a_{u+2}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u}(\alpha_j-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\left(\prod_{i=u+1}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_j)_{\Delta_j,n}\right)=0$$

*where $\Delta_j = a_j + \sum_{i=1}^{u}(k+1-a_i) + \sum_{i=u+1}^{M}(k-a_i)$. Going forward we will denote these this equation by $K_{k,n}^{u}$.*

**Lemma 9.** *We have the following implications*

- $K_{k,n+1}^{u}$ *and* $K_{k,n}^{u}$ *holds* $\implies K_{k,n}^{u+1}$ *holds*
- $K_{k,n+1}^{M-1}$ *and* $K_{k,n}^{M-1}$ *holds* $\implies K_{k+1,n}^{0}$ *holds*

*where $0 \leq u \leq M\text{-}2$ and $n \geq 0$.*

*Proof.* **(Lemma 9):**
We will rewrite $K_{k,n+1}^{u}$ using Equation 12 and then substitute using $K_{k,n}^{u}$ to get the following

$$\sum_{j=1}^{M}\left(\sum_{\substack{a_1,a_2\dots,a_u\in\{0,1,\dots,k+1\}\\ a_{u+1},a_{u+2}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u}(\alpha_j-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\left(\prod_{i=u+1}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_j)_{\Delta_j,n+1}\right)=$$

$$=\sum_{j=1}^{M}\alpha_j\left(\sum_{\substack{a_1,a_2\dots,a_u\in\{0,1,\dots,k+1\}\\ a_{u+1},a_{u+2}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u}(\alpha_j-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\left(\prod_{i=u+1}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_j)_{\Delta_j,n}\right)$$

$$+\sum_{j=1}^{M}\left(\sum_{\substack{a_1,a_2\dots,a_u\in\{0,1,\dots,k+1\}\\ a_{u+1},a_{u+2}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u}(\alpha_j-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\left(\prod_{i=u+1}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_j)_{\Delta_j+1,n}\right)=$$

$$=\sum_{\substack{j=1\\ j\neq u+1}}^{M}(\alpha_j-\alpha_{u+1})\left(\sum_{\substack{a_1,a_2\dots,a_u\in\{0,1,\dots,k+1\}\\ a_{u+1},a_{u+2}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u}(\alpha_j-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\left(\prod_{i=u+1}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_j)_{\Delta_j,n}\right)$$

$$+\sum_{j=1}^{M}\left(\sum_{\substack{a_1,a_2\dots,a_u\in\{0,1,\dots,k+1\}\\ a_{u+1},a_{u+2}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u}(\alpha_j-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\left(\prod_{i=u+1}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_j)_{\Delta_j+1,n}\right)=$$

$$=\sum_{\substack{j=1\\ j\neq u+1}}^{M}\left(\sum_{\substack{a_1,a_2\dots,a_u\in\{0,1,\dots,k+1\}\\ a_{u+1}\in\{1,2,\dots,k+1\}\\ a_{u+2},a_{u+3}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u}(\alpha_j-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\cdot\left((\alpha_j-\alpha_{u+1})^{a_{u+1}}\binom{k}{a_{u+1}-1}\right)\right.$$

$$\left.\cdot\left(\prod_{i=u+2}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_j)_{\Delta_j+1,n}\right)$$

$$+\sum_{j=1}^{M}\left(\sum_{\substack{a_1,a_2\dots,a_u\in\{0,1,\dots,k+1\}\\ a_{u+1},a_{u+2}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u}(\alpha_j-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\left(\prod_{i=u+1}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_j)_{\Delta_j+1,n}\right)=$$

$$=\sum_{\substack{j=1\\ j\neq u+1}}^{M}\left(\sum_{\substack{a_1,a_2\dots,a_u\in\{0,1,\dots,k+1\}\\ a_{u+1}\in\{1,2,\dots,k\}\\ a_{u+2},a_{u+3}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u}(\alpha_j-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\cdot\left((\alpha_j-\alpha_{u+1})^{a_{u+1}}\left(\binom{k}{a_{u+1}-1}+\binom{k}{a_{u+1}}\right)\right)\right.$$

$$\left.\cdot\left(\prod_{i=u+2}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_j)_{\Delta_j+1,n}\right)=$$

$$=\sum_{\substack{j=1\\ j\neq u+1}}^{M}\left(\sum_{\substack{a_1,a_2\dots,a_u\in\{0,1,\dots,k+1\}\\ a_{u+2},a_{u+3}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u}(\alpha_j-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\cdot\left((\alpha_j-\alpha_{u+1})^{k+1}\binom{k}{k}\right)\right.$$

$$\left.\cdot\left(\prod_{i=u+2}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_j)_{\Delta_j+1,n}\right)$$

$$+\sum_{j=1}^{M}\left(\sum_{\substack{a_1,a_2\dots,a_u\in\{0,1,\dots,k+1\}\\ a_{u+2},a_{u+3}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u}(\alpha_j-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\left(\prod_{i=u+2}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_j)_{\Delta_j+1,n}\right)$$

$$+\sum_{\substack{a_1,a_2\dots,a_u\in\{0,1,\dots,k+1\}\\ a_{u+1},a_{u+2}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u}(\alpha_{u+1}-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\left(\prod_{i=u+1}^{M}(\alpha_{u+1}-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_{u+1})_{\Delta_{u+1}+1,n}=$$

$$= \sum_{\substack{j=1\\ j\neq u+1}}^{M}\left(\sum_{\substack{a_1,a_2\dots,a_{u+1}\in\{0,1,\dots,k+1\}\\ a_{u+2},a_{u+3}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u+1}(\alpha_j-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\cdot\left(\prod_{i=u+2}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_j)_{\Delta_j+1,n}\right)$$

$$+\sum_{\substack{a_1,a_2\dots,a_u\in\{0,1,\dots,k+1\}\\ a_{u+1},a_{u+2}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u}(\alpha_{u+1}-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\left(\prod_{i=u+1}^{M}(\alpha_{u+1}-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_{u+1})_{\Delta_{u+1}+1,n} =$$

$$= \sum_{j=1}^{M}\left(\sum_{\substack{a_1,a_2\dots,a_{u+1}\in\{0,1,\dots,k+1\}\\ a_{u+2},a_{u+3}\dots,a_M\in\{0,1,\dots,k\}}}\left(\prod_{i=1}^{u+1}(\alpha_j-\alpha_i)^{a_i}\binom{k+1}{a_i}\right)\cdot\left(\prod_{i=u+2}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{k}{a_i}\right)S(\alpha_j)_{\Delta_j+1,n}\right)$$

Assuming that $0 \leq u \leq M-2$ this last statement is obviously $K^{u+1}_{k,n}$. However if $u = M-1$ it can be written as

$$\sum_{j=1}^{M}\left(\sum_{a_1,a_2\dots,a_M\in\{0,1,\dots,k+1\}}\left(\prod_{i=1}^{M}(\alpha_j-\alpha_i)^{a_i}\binom{(k+1)}{a_i}\right)S(\alpha_j)_{\Delta_j+1,n}\right)$$

which is the same as $K^0_{k+1,n}$ and thus we have finished the proof of Lemma 9. □

**Lemma 10.** *$D(mx^p) = 0$, $p = Mq + r$, $0 \leq r \leq M-1$, is equivalent to $K^{\hat{r}}_{\hat{q},M(q-\hat{q})+(r-\hat{r})}$ holding when $0 \leq M\hat{q} + \hat{r} \leq p$.*

*Proof.* **(Lemma 10):**
We will be using induction. First we make note that by definition we have

$$D(mx^n) = \sum_{j=1}^{M} S(\alpha_j)_{0,n} = 0 \tag{13}$$

Our lemma holding for $p = 0$ is equivalent to $K^0_{0,0}$ which we get from Equation 13 with $n = 0$. Next let us assume our lemma holds up to and including an arbitrary $p \geq 0$ and try to prove it holds for $p+1$. We get $K^0_{0,p+1}$ from Equation 13. If we then assume we have $K^{\hat{r}}_{\hat{q},M(q-\hat{q})+(r-\hat{r})}$ for all $0 \leq M\hat{q}+\hat{r} \leq t = M\tilde{q}+\tilde{r}$, for some $t \geq 0$. Then we have $K^{\tilde{r}}_{\tilde{q},M(q-\tilde{q})+(r-\tilde{r})}$ and $K^{\tilde{r}}_{\tilde{q},M(q-\tilde{q})+(r-\tilde{r})-1}$ which with Lemma 9 gives us $K^{\tilde{r}+1}_{\tilde{q},M(q-\tilde{q})+(r-\tilde{r})-1}$ at which point we have $K^{\hat{r}}_{\hat{q},M(q-\hat{q})+(r-\hat{r})}$ for all $0 \leq M\hat{q}+\hat{r} \leq t+1$. And so using induction we find that our lemma holds for $p+1$ and thus our proof has concluded. □

*Proof.* **(Lemma 8):**
Take arbitrary $a, b, c$ where $a, b \geq 0$ and $0 \leq c \leq M-1$, we want to show that $K^c_{a,b}$ holds. Then using Lemma 10 with $p = Ma+b+c = Mq+r$, $0 \leq r \leq M-1$. Then we take $\hat{q} = a$ and $\hat{r} = c$ and we get $M(q-\hat{q})+(r-\hat{r}) = p - Ma - c = b$ which is what we wanted and so our proof is concluded, since $Ma+c \leq Ma+b+c = p$. □

**Lemma 11.** *For all $k \geq 0$ and $j \in \{1, 2, \ldots, M\}$ we have*

$$S(\alpha_j)_{k,0} = 0$$

*Proof.* **(Lemma 11):**
By definition we know that $S(\alpha_j)_{k,n} = 0$ for $k > N_j$. So all that is left to prove it for $0 \leq k \leq N_j$. If we take $N = \max(N_j)$ we get a number for which all $S(\alpha_j)_{k,n} = 0$ for $k > N$. If we then take an arbitrary $p > 0$ and assume that $S(\alpha_j)_{k,n} = 0$ for $k > p$ then we want to that this is also true for $p - 1$. To start we will take a look at the following equation from Lemma 8, in the case when $n = 0$

$$\sum_{j=1}^{M} \left( \sum_{\substack{a_1,a_2\ldots,a_u \in \{0,1,\ldots,k+1\} \\ a_{u+1},a_{u+2}\ldots,a_M \in \{0,1,\ldots,k\}}} \left( \prod_{i=1}^{u} (\alpha_j - \alpha_i)^{a_i} \binom{k+1}{a_i} \right) \left( \prod_{i=u+1}^{M} (\alpha_j - \alpha_i)^{a_i} \binom{k}{a_i} \right) S(\alpha_j)_{\Delta_j,0} \right) = 0$$

From this equation we will specifically be looking at the term

$$\left( \prod_{i=1}^{u} (\alpha_{u+1} - \alpha_i)^{k+1} \cdot \prod_{i=u+2}^{M} (\alpha_{u+1} - \alpha_i)^{k} \right) S(\alpha_{u+1})_{k,0}$$

Under the assumption that

$$S(\alpha_j)_{t,0} = 0 \text{ for } \begin{cases} t > k & \text{if} \quad 0 \leq j \leq u+1 \\ t \geq k & \text{if} \quad u+2 \leq j \leq M-1 \end{cases}$$

this is the only term which is nonzero since if we look at $\Delta_j$ which is expressed as

$$\Delta_j = a_j + \sum_{i=1}^{u} (k + 1 - a_i) + \sum_{i=u+1}^{M} (k - a_i)$$

we have

$$\min(\Delta_j) = \begin{cases} \min\left(k + 1 + \sum\limits_{\substack{i=1 \\ i \neq j}}^{u} (k+1-a_i) + \sum\limits_{i=u+1}^{M} (k - a_i)\right) = k+1 & \text{if } 0 \leq j \leq u \\ \min\left(k + \sum\limits_{i=1}^{u} (k+1-a_i) + \sum\limits_{\substack{i=u+1 \\ i \neq j}}^{M} (k - a_i)\right) = k & \text{if } u+1 \leq j \leq M-1 \end{cases}$$

These assumptions are true in the case where we have $u = 0$ and $k = p$. Together with $\alpha_j \neq \alpha_1$ for all $j \neq 1$ this gives us $S(\alpha_1)_{p,0} = 0$. If we now assume that $S(\alpha_r)_{p,0} = 0$ for $1 \leq r \leq t$ for a $1 \leq t \leq M - 1$ then by the explanation above we get $S(\alpha_{t+1})_{p,0} = 0$ and by induction $S(\alpha_j)_{k,n} = 0$ for all $k > p - 1$ and all $j$. This completes our induction proof and so we have $S(\alpha_j)_{k,n} = 0$ for $k \geq 0$. □

*Proof.* **(Theorem 7):**
We will take $S(\alpha_j)_{N_j,0} = 0$ which implies $s_{j,N_j} \cdot N_j! \cdot m(\alpha_j) = 0$, since $s_{j,N_j} \neq 0$ we get $m(\alpha_j) = 0$. If we assume that $m^{(t)}(\alpha_j) = 0$ for all $0 \leq t \leq p$ for some $0 \leq p < N_j$. Then we take

$$S(\alpha_j)_{N_j-t-1,0} = 0 \implies s_{j,N_j} \cdot \frac{N_j!}{(p+1)!} \cdot m^{(t+1)}(\alpha_j) = 0 \implies m^{(t+1)}(\alpha_j) = 0$$

Using induction we end up with $m(\alpha_j) = m'(\alpha_j) = \cdots = m^{(N_j)}(\alpha_j) = 0$ for all $j$ which gives us the result of our theorem, namely $\prod_{j=1}^{N}(x-\alpha_j)^{N_j+1}$ is a factor of the minimal polynomial. □

## 3.3 Determining the Minimal Polynomial of A

Now that we know how equality conditions and derivations affect the minimal polynomial it is time to construct the minimal polynomial for a given algebra, $A$, with a set of arbitrary conditions. Looking at the previous discoveries we reach the conclusion that the minimal polynomial, of $A$, is a product $\prod_{i=1}^{N}(x-\alpha_i)^{N_i+1}$, where $\alpha_i \in \mathrm{Sp}(A)$ and $N_i$ is the largest order derivative, evaluated at $\alpha_i$, which occurs in our conditions. This is what we will express in the following theorem

**Theorem 10.** *Given a subalgebra $A$ of finite codimension in $\mathbb{K}[x]$, which is constructed using the conditions $\{L_i \mid i = 1, 2 \ldots, M\}$. We have the following minimal polynomial*

$$m(x) = \prod_{j=1}^{N}(x-\alpha_j)^{\max\limits_i (N_{i,j})+1}$$

*where Sp(A)=$\{\alpha_j \mid j = 1, 2, \ldots, N\}$ and we define $N_{i,j}$ as follows*

**If $L_i$ is an equality condition* $N_{i,j} = 0$ *if $\alpha_j$ is included and* $N_{i,j} = -1$ *if it is not included.*

**If $L_i$ is a derivation condition* $N_{i,j}$ *is the order of largest derivative included, which is evaluated at $\alpha_j$, if no such derivative is included* $N_{i,j} = -1$

*Proof.* **(Theorem 10):**
Let $m(x)$ be the minimal polynomial of $A$ and look at condition $L_i$, depending on what type of condition $L_i$ is we look at either Theorem 6 or Theorem 7 and from there we have

$$\prod_{j=1}^{N}(x-\alpha_j)^{N_{i,j}+1} \mid m(x) \tag{14}$$

Looking at a specific spectrum element $\alpha_k$ we have $(x-\alpha_k)^{N_{i,j}+1} \mid m(x)$ for all $i, j$. And so the largest power of $(x - \alpha_k)$ which divides the minimal polynomial is

$(x-\alpha_k)^{\max_i(N_{i,j})+1}$. Combined we get

$$S(x) = \prod_{j=1}^{N} (x-\alpha_j)^{\max_i(N_{i,j})+1} \mid m(x) \tag{15}$$

All that remains is to show that $S(x)$ satisfies the properties of a minimal polynomial i.e. $f(x) = S(x) \cdot x^k \in A$ for all $k$. We note that all equality conditions hold as $f(\alpha_j) = 0$ for any $\alpha_j$. Looking at derivations we find

$$f^{(t)}(\alpha_j) = \sum_{i=0}^{t} \binom{t}{i} \cdot S^{(i)}(\alpha_j) \cdot (x^k)^{(t-i)}(\alpha_j) = 0 \text{ for } 0 \leq t \leq \max_i(N_{i,j}) \tag{16}$$

since $m^{(i)}(\alpha_j) = 0$ for $0 \leq i \leq \max_i(N_{i,j})$. By the definition of $N_{i,j}$ we find that $L_i\big(S(x)x^k\big) = 0$ for all $k$ and so it satisfies the conditions necessary to be the minimal polynomial. From Equation 15 we also find that no polynomial of lower degree, which satisfies these conditions, can exist. □

Next we will look at an example where we will construct the minimal polynomial given the set of conditions of the algebra.

**Example 5.** *We want to construct the minimal polynomial $m(x)$ of the subalgebra, $A$, which is constructed using the conditions*

$$\begin{aligned}
L_1: &\quad f(\alpha) - f(\beta) = 0 \\
L_2: &\quad f(\beta) - f(\gamma) = 0 \\
L_3: &\quad f'(\alpha) = 0 \\
L_4: &\quad f'(\beta) = 0 \\
L_5: &\quad f'(\delta) = 0 \\
L_6: &\quad f''(\alpha) + 2f^{(3)}(\alpha) = 0
\end{aligned}$$

*We see that our spectrum is $\{\alpha, \beta, \gamma, \delta\}$, then we will look at what order of derivations occur. The highest derivative of $\alpha$ occurs in $L_6$ and is 3, the highest derivative of $\beta$ and $\delta$ is 1 and occurs in $L_4$ and $L_5$ respectively. $\gamma$ only occurs in the equality condition $L_2$ and as such its highest derivative is 0. Combined with Theorem 10 we obtain*

$$m(x) = (x-\alpha)^4(x-\beta)^2(x-\gamma)(x-\delta)^2$$

But if we have two subalgebras $A$ and $B$, what is the minimal polynomial of their intersection?

**Theorem 11.** *Assume we have two subalgebras $A$ and $B$ of finite codimension in $\mathbb{K}[x]$ with $Sp(A \cap B) = Sp(A) \cup Sp(B) = \{\alpha_i \mid i = 1, 2, \ldots, n\}$ and the minimal polynomials*

$m_A(x) = \prod_{i=1}^{n}(x-\alpha_i)^{a_i}$ *and* $m_B(x) = \prod_{i=1}^{n}(x-\alpha_i)^{b_i}$. *Then the intersection have the minimal polynomial*

$$m_{A\cap B}(x) = \prod_{i=1}^{n}(x-\alpha_i)^{\max\{a_i,b_i\}}$$

*Proof.* **(Theorem 11):**
The conditions of $A \cap B$ is the union of the conditions of $A$ and $B$, label these $L_k$. Then by Theorem 10 we get a minimal polynomial

$$m_{A\cap B}(x) = \prod_{i=1}^{n}(x-\alpha_i)^{c_i}$$

$c_i$, it is the largest order of a derivative, evaluated at $\alpha_i$, in $\{L_k\}$, let us say this occurs in condition $L_p$. This condition is then either in $A$ or $B$. Assume WLOG it is in $A$, then $a_i = c_i$. If not then there must exist a condition of $A$ with a larger order derivative, evaluated at $\alpha_i$ which contradicts how we created $c_i$. By a similar argument we find that $b_i \le c_i$ and so $c_i = a_i = \max\{a_i, b_i\}$, which concludes our proof. □

# 4 Adding an Additional Derivation to a Subalgebra

Given a subalgebra $A$ in $\mathbb{K}[x]$ we can create additional subalgebras by taking the kernel of an additional condition, for example a derivation of $A$. But given an algebra $A$ what does the non-trivial derivations look like? That is what we will look at in this section. We will start by explaining the method of finding derivations in $A$ before we show some examples.

## 4.1 Upper Limit for Additional Derivations

From Theorem 1 we find that there exists an upper limit to how high of an order of derivation we have in any given $\alpha$-derivation. We will dedicate this section to finding such a limit given a specific subalgebra $A$. First we will look at the case where we only use the minimal polynomial and the spectrum of $A$.

**Theorem 12.** *Assume we have a subalgebra $A \subset \mathbb{K}[x]$ of finite codimension with minimal polynomial $m(x) = \prod_{i=1}^{M}(x-\alpha_i)^{a_i}$. Assume we want to add a new $\alpha$-derivation, written*

$$D(f) = \sum_{\alpha_j\sim\alpha}\sum_{i=1}^{N_j} c_{i,j} f^{(i)}(\alpha_j)$$

*Then, for any $k$, we have $c_{b,k} = 0$ for $b \ge 2a_k$.*

**Example 6.** *Assume we have* $A = \{f(\alpha) = f(\beta),\ f'(\alpha) = f''(\alpha) = 0,\ \alpha \neq \beta\}$ *with minimal polynomial* $m(x) = (x-\alpha)^3(x-\beta)$*. Then any additional derivation can be written as*

$$D(f) = \sum_{i=1}^{N_1} a_i f^{(i)}(\alpha) + \sum_{i=1}^{N_2} b_i f^{(i)}(\beta)$$

*We are going to show that* $a_i = 0$ *for* $i \geq 6$ *and* $b_i = 0$ *for* $i \geq 2$*.*

*For this we want to use elements of* $A$ *on the form* $m^2(x) \cdot f(x)$*. Looking at the definition of an* $\alpha$*-derivation we have*

$$D\big(m(x) \cdot m(x)f(x)\big) = D\big(m(x)\big) \cdot m(\alpha)f(\alpha) + m(\alpha) \cdot D\big(m(x)f(x)\big) = 0$$

*where we have used* $m(\alpha) = 0$*. We will construct*

$$u_t(x) = m^2(x) \cdot (x-\beta)^{N_2+1} \cdot (x-\alpha)^t, \quad v_t(x) = m^2(x) \cdot (x-\alpha)^{N_1+1} \cdot (x-\beta)^t$$

*with* $t \geq 0$*, we can then write*

$$D(u_t) = \sum_{i=1}^{N_1} a_i u_t^{(i)}(\alpha) = 0 \quad \text{and} \quad D(v_t) = \sum_{i=1}^{N_2} b_i v_t^{(i)}(\beta) = 0$$

*since* $(x-\beta)^{N_2+1} \mid u_t$ *and* $(x-\alpha)^{N_1+1} \mid v_t$*.*

*We will use induction, to start we have*

$$D(u_{N_1-6}) = a_{N_1} \cdot N_1! \cdot (\alpha-\beta)^{N_2+3} = 0 \implies a_{N_1} = 0$$

*assuming* $a_r = 0$ *for* $r > R$ *for some* $R \geq 6$ *we have*

$$D(u_{R-6}) = a_R \cdot R! \cdot (\alpha-\beta)^{N_2+3} = 0 \implies a_R = 0$$

*and so by induction we get* $a_i = 0$ *for* $i \geq 6$ *similarly we have*

$$D(v_{N_2-2}) = b_{N_2} \cdot N_2! \cdot (\beta-\alpha)^{N_1+7} = 0 \implies b_{N_2} = 0$$

*assuming* $b_r = 0$ *for* $r > R$ *for some* $R \geq 2$ *we have*

$$D(v_{R-2}) = b_R \cdot R! \cdot (\beta-\alpha)^{N_1+7} = 0 \implies b_R = 0$$

*and so by induction we have* $b_i = 0$ *for* $i \geq 2$*, we conclude that, in this case, we can write any* $\alpha$*-derivation in* $A$ *as*

$$D(f) = \sum_{i=1}^{5} a_i f^{(i)}(\alpha) + b_1 f'(\beta)$$

and with this we we are ready to prove Theorem 12.

*Proof.* **(Theorem 12):**
Take an arbitrary $k$ and assume that $N_k \geq 2a_k$ and then define

$$S := \max\left\{\max_j\{N_j+1\}, \max_{1\leq i\leq M}\{2a_i\}\right\} \tag{17}$$

then we define

$$q_t(x) = (x-\alpha_k)^t \cdot \prod_{\substack{i=1\\ i\neq k}}^{M}(x-\alpha_i)^S \tag{18}$$

Since $S \geq N_j + 1$ for all $j$ we have

$$D(q_t) = \sum_{i=1}^{N_k} c_{i,k} q_t^{(i)}(\alpha_k)$$

for $t \geq 0$, and as long as $t \geq 2a_k$ we have $m(x)^2 \mid q_t(x)$, since $S \geq 2a_i$ $\forall i$, and so we can rewrite $q_t(x) = m(x) \cdot m(x) \cdot q(x)$, where $q(x) \not\equiv 0$. Both $m(x)$ and $m(x) \cdot q(x)$ are in $A$ and have $\alpha_k$ as a zero since $\alpha_k \in \mathrm{Sp}(A) \implies m(\alpha_k) = 0$. Since $\alpha_k \sim \alpha$ $D$ is an $\alpha_k$-derivation and we have

$$D\big(q_t(x)\big) = D\big(m(x)\big)m(\alpha_k)q(\alpha_k) + m(\alpha_k)D\big(m(x)q(x)\big) = 0 \tag{19}$$

Using this we get

$$D(q_{N_k}) = c_{N_k,k} \cdot N_k! \cdot \prod_{\substack{i=1\\ i\neq k}}^{M}(\alpha_k-\alpha_i)^S = 0 \implies c_{N_k,k} = 0$$

since $\alpha_i \neq \alpha_j$ if $i \neq j$. If we then assume $c_{b,k} = 0$ for all $b \geq s$ for some $2a_k < s \leq N_k$. Then we get

$$D(q_{s-1}) = c_{s-1,k} \cdot (s-1)! \cdot \prod_{\substack{i=1\\ i\neq k}}^{M}(\alpha_k-\alpha_i)^S = 0 \implies c_{s-1,k} = 0 \tag{20}$$

and so we have $c_{b,k} = 0$ for all $b \geq s-1$. Using induction we get $c_{b,k} = 0$ for all $b \geq 2a_k$, which concludes our proof. □

This theorem relies on $m(x)$ and $m(x) \cdot q(x)$ being elements in $A$ with $\alpha_k$ as a zero. So in order to improve our result we will exchange the first $m(x)$ with another element in $A$ with this same property.

**Theorem 13.** *Assume we have a subalgebra $A$ of finite codimension in $\mathbb{K}[x]$ with minimal polynomial $m(x) = \prod_{i=1}^{M} (x - \alpha_i)^{a_i}$. Assuming we want to add a new $\alpha$-derivation, written*

$$D(f) = \sum_{\alpha_j \sim \alpha} \sum_{i=1}^{N_j} c_{i,j} f^{(i)}(\alpha_j)$$

*and the conditions of $A$ can be rewritten such that they include $D_1^{\alpha_k} = D_2^{\alpha_k} = \cdots = D_p^{\alpha_k} = 0$ but not $D_{p+1}^{\alpha_k} = 0$, where $\alpha_k \sim \alpha$. Then we have $c_{b,k} = 0$ for $b \geq a_k + p + 1$.*

*Proof.* **(Theorem 13):**
Any element in $A$ can be written as $g(x) = (x - \alpha_k)^{p+1} r(x)$, for at least one element in $A$ we have $r(\alpha_k) \neq 0$, if this is not the case we would have $D_{p+1}^{\alpha_k} = 0$ for all elements in $A$, which leads to a contradiction since we would end up with $A \subseteq \ker(D_{p+1}^{\alpha_k})$. Next we will introduce the large number

$$S := \max \left\{ \max_j \{N_j + 1\}, \max_{1 \leq i \leq M} \{a_i\} \right\} \tag{21}$$

and then we will define

$$q_t(x) = r(x) \cdot (x - \alpha_k)^t \cdot \prod_{\substack{i=1 \\ i \neq k}}^{M} (x - \alpha_i)^S \tag{22}$$

Since $S \geq N_j + 1$ for all $j$ we have

$$D(q_t) = \sum_{i=1}^{N_k} c_{i,k} q_t^{(i)}(\alpha_k)$$

for $t \geq 0$, and as long as $t \geq a_k + p + 1$ we can rewrite $q_t(x) = g(x) \cdot m(x) \cdot q(x)$ where $q(x) \not\equiv 0$, since $S \geq a_i \; \forall i$. Both $g(x)$ and $m(x) \cdot q(x)$ are in $A$ and have $\alpha$ as a zero, since $g(\alpha) = g(\alpha_k) = 0$. $D$ is an $\alpha_k$-derivation and so we have

$$D\big(q_t(x)\big) = D\big(g(x)\big) m(\alpha_k) q(\alpha_k) + g(\alpha_k) D\big(m(x)q(x)\big) = 0 \tag{23}$$

and so for $t = N_k$ we get

$$D(q_{N_k}) = c_{N_k,k} \cdot N_k! \cdot r(\alpha_k) \cdot \prod_{\substack{i=1 \\ i \neq k}}^{M} (\alpha_k - \alpha_i)^S = 0 \implies c_{N_k,k} = 0$$

since $\alpha_i \neq \alpha_j$ if $i \neq j$, and $r(\alpha_k) \neq 0$. If we then assume $c_{b,k} = 0$ for all $b \geq s$ for some $a_k + p + 1 < s \leq N_k$. Then we get

$$D(q_{s-1}) = c_{s-1,k} \cdot (s-1)! \cdot r(\alpha_k) \cdot \prod_{\substack{i=1 \\ i \neq k}}^{M} (\alpha_k - \alpha_i)^S = 0 \implies c_{s-1,k} = 0 \tag{24}$$

and so we have $c_{b,k} = 0$ for all $b \geq s - 1$. Using induction we get $c_{b,k} = 0$ for all $b \geq a_k + p + 1$ □

## 4.2 Method for Finding Derivations

Now that we have found our upper limits we will explain our method for finding derivations which is based on Theorem 2. This Theorem says that if we have $D(f^2) = 0$ for every $f \in M_\alpha$ then $D$ is a $\alpha$-derivation in $A$. Logically this means we want to start by finding a way to express the elements in $M_\alpha$ and so we will look for a SAGBI basis, $\{v_i\}$ with the properties

- $v_i(\alpha) = 0, \ \forall v_i$
- if $\deg(v_i) \geq \deg(m(x))$ then $m(x) \mid v_i$

Such a basis can always be found. We know a finite SAGBI basis can always be found, so when creating our basis we start by taking an arbitrary SAGBI basis $\{\hat{u}_i(x)\}$. Then the first property can be achieved by creating the new basis $\{\tilde{u}_i(x) \mid \hat{u}_i(x) - \hat{u}_i(\alpha)\}$. In order to obtain the second property we write $\tilde{u}_i(x) = q_i(x) \cdot m(x) + r_i(x)$ where $\deg\big(r_i(x)\big) < \deg\big(m(x)\big)$ and so we can create our final SAGBI basis which consists of the sets $\{r_i(x)\}$ and $\{q_i(x) \cdot m(x)\}$, which we denote $V_1$ and $V_2$ respectively. Note that this means that $V_2$ consists of all basis elements such that $\deg(v_i) \geq \deg\big(m(x)\big)$ and $V_1$ of all for which $\deg(v_i) < \deg\big(m(x)\big)$.

We are going to show that $V_2$ is unnecessary and we need only work with the set $V_1$ when finding derivations. More specifically we will show that it is sufficient to ensure that $D(v_1^{k_1} v_2^{k_2} \dots v_{n-1}^{k_{n-1}} v_n^{k_n}) = 0$, for all $v_i \in V_1$ and $k_i \geq 0$, where $\sum_i k_i \geq 2$. It is trivial to see that

$$\begin{array}{lll} v_k^t \in M_\alpha & \Longrightarrow & \text{we need to check } D(v_k^{2t}) = 0 \text{ for all } k \text{ and } t > 0 \\ v_k^t + v_k \in M_\alpha & \Longrightarrow & \text{we need to check } D((v_k^t + v_k)^2) = 0 \text{ for all } k \text{ and } t > 0 \end{array}$$

Assuming we have $D(v_k^{2t}) = 0$ for all $t > 0$ and $k$ we can rewrite

$$D((v_k^t + v_k)^2) = D(v_k^{2t}) + 2D(v_k^{t+1}) + D(v_k^2) = 2D(v_k^{t+1}) = 0 \implies D(v_k^{t+1}) = 0$$

and so it is necessary to ensure that $D(v_i^t) = 0$ for all $t \geq 2$. Next we will look at the product of different elements in $V_1$.

$$v_1^{k_1} v_2^{k_2} \dots v_{n-1}^{k_{n-1}} v_n^{k_n} \in M_\alpha \implies D\left(v_1^{2k_1} v_2^{2k_2} \dots v_{n-1}^{2k_{n-1}} v_n^{2k_n}\right) = 0 \text{ for all } k_i \geq 0 \text{ and } \sum k_i \geq 1$$

So our original assertions hold true for all even powers. Next we will show that this is also true when we have odd powers. We have

$$v_k + v_j \in M_\alpha \quad \Longrightarrow \quad \text{we need to check } D((v_k + v_j)^2) = 0 \text{ for all } k \text{ and } j$$

which can be rewritten

$$D((v_k + v_j)^2) = D(v_k^2) + 2D(v_k v_j) + D(v_j^2) = 2D(v_k v_j) = 0 \implies D(v_k \cdot v_j) = 0$$

and through a similar argument, where we assume $k_r > 0$, we have

$$v_1^{k_1} v_2^{k_2} \dots v_r^{k_r - 1} \dots v_{n-1}^{k_{n-1}} v_n^{k_n} + v_r \in M_\alpha \quad \Longrightarrow$$
$$\text{we need to check } D\left(\left(v_1^{k_1} v_2^{k_2} \dots v_r^{k_r - 1} \dots v_{n-1}^{k_{n-1}} v_n^{k_n} + v_r\right)^2\right) = 0 \text{ for all } k_i \geq 0 \text{ and } v_i, 1 \leq i \leq n$$

which, assuming products with even powers have already been checked, can be rewritten as

$$D\left(\left(v_1^{k_1} v_2^{k_2} \dots v_r^{k_r - 1} \dots v_{n-1}^{k_{n-1}} v_n^{k_n} + v_r\right)^2\right) = \dots = 2D\left(v_1^{k_1} v_2^{k_2} \dots v_{n-1}^{k_{n-1}} v_n^{k_n}\right) = 0$$

for all $k_i \geq 0$, where $\sum_i k_i \geq 2$.

Next we will show that this is sufficient in order to determine that $D$ is a derivation in $A$. Take any $\alpha_k \in \mathrm{Sp}(A)$ and assume that the conditions of $A$ can be rewritten such that they include $D_1^{\alpha_k} = D_2^{\alpha_k} = \dots = D_{p_k}^{\alpha_k} = 0$ but not $D_{p_k+1}^{\alpha_k} = 0$. Then if we take an arbitrary $\hat{u} \in V_2$, and an arbitrary element $f(x) \in A$ we get

$$(x - \alpha_k)^{a_k + p_k + 1} | f(x) \cdot \hat{u} \implies (f(x) \cdot \hat{u})^{(t)}(\alpha_k) = 0 \text{ for } 0 \leq t \leq a_k + p_k$$

Combining this with the result of Theorem 13 we obtain $D\big(f(x) \cdot \hat{u}\big) = 0$ without any further restrictions on $D$. Denote $V_2 = \{u_i\}_{i \in I}$. Then we will take an arbitrary element $g(x) \in A$ which we then divide into two parts $g(x) = g_1(x) + g_2(x)$ where $g_1(x)$ can be written as

$$g_1(x) = \sum_d v_1^{k_{1,d}} v_2^{k_{2,d}} \dots v_n^{k_{n,d}}$$

and $g_2(x)$ can be written as a sum where every term is divisible by some element in $V_2$. Then we find that

$$D\big(g(x)^2\big) = D\big((g_1(x) + g_2(x))^2\big) = D(g_1(x)^2) + 2D(g_1(x)g_2(x)) + D(g_2(x)^2) = D(g_1(x)^2) = 0$$

where we have used that $D\big(f(x) \cdot \hat{u}\big) = 0$ for arbitrary $f(x) \in A, \hat{u} \in V_2$, without additional restrictions on $D$. Then all that is left to check is $D\big(g_1(x)^2\big) = 0$. Using the linearity of $D$ we find that this is a sum where each term can be written as $D\big(v_1^{k_1} v_2^{k_2} \dots v_{n-1}^{k_{n-1}} v_n^{k_n}\big)$ for $k_i \geq 0$, $\sum_i k_i \geq 2$, and $v_i \in V_1$, each of which we have determined is necessary to ensure is zero. And so we have shown that it is sufficient to ensure $D\big(v_1^{k_1} v_2^{k_2} \dots v_{n-1}^{k_{n-1}} v_n^{k_n}\big) = 0$, for all $k_i \geq 0$, where $\sum_i k_i \geq 2$, in order for $D$ to be an $\alpha$-derivation in $A$.

This will result in a finite amount of calculations since, by construction, we have $v_i(\alpha_j) = 0$ for all $\alpha_j \sim \alpha$. Which means that if $\sum_{i=1}^{n} k_i > \max\{a_j + p_j\}$ we have

$$\prod_{\alpha_j \sim \alpha} (x - \alpha_j)^{\max\{a_j + p_j\} + 1} \mid v_1^{k_1} v_2^{k_2} \dots v_{n-1}^{k_{n-1}} v_n^{k_n}$$

and so $D\left(v_1^{k_1}v_2^{k_2}\dots v_{n-1}^{k_{n-1}}v_n^{k_n}\right)=0$ according to Theorem 13.

Now that we have presented the method we will use we will show an example of where we find derivations before we present a list of algebras and their corresponding derivations.

**Example 7.** *We want to find all $\alpha$-derivations of the algebra*

$$A=\{\ f\mid f(\alpha)=f(\beta),\ D_1^{\alpha}(f)=0,\ D_1^{\beta}(f)-(\alpha-\beta)^2\cdot D_3^{\alpha}(f)=0\ \}$$

*which has minimal polynomial $m(x)=(x-\alpha)^4(x-\beta)^2$. First we will find $V_1$. From our first two conditions we know that any element in $A$ can be written as*

$$f(x)=(x-\alpha)^2(x-\beta)r(x)+c$$

*for some polynomial r(x) and constant c, however we will only be working with $f(x)$ where $c=0$. First we start by looking at whether $A$ has a polynomial of degree three or not. Such a polynomial would look like*

$$f(x)=(x-\alpha)^2(x-\beta).\ \text{Then}\ D_1^{\beta}(f)-(\alpha-\beta)^2\cdot D_3^{\alpha}(f)=0$$

*and so $(x-\alpha)^2(x-\beta)\in A$. Next will look for a polynomial of degree four, such a polynomial must be of the form*

$$f(x)=(x-\alpha)^2(x-\beta)(x-d),\ D_1^{\beta}(f)-(\alpha-\beta)^2\cdot D_3^{\alpha}(f)=-2(\alpha-\beta)^3\neq 0$$

*for some constant $d$. We find that since we have $\alpha\neq\beta$ it is not possible to have a polynomial of degree four in $A$. Lastly we look for a polynomial of degree five, we start of looking for a polynomial of the form*

$$f(x)=(x-\alpha)^3(x-\beta)(x-d),\ D_1^{\beta}(f)-(\alpha-\beta)^2\cdot D_3^{\alpha}(f)=-(\alpha-\beta)^3(\alpha+\beta-2d)=0$$

*and find that such a polynomial exists if we set $d=\frac{\alpha+\beta}{2}$ we get $(x-\alpha)^3(x-\beta)(x-\frac{\alpha+\beta}{2})\in A$. Next we use Theorem 13 to determine that the highest order of a derivation evaluated at $\alpha$ and $\beta$ is five and two respectively. We will use the following notation*

$$u(x)=(x-\alpha)^3(x-\beta)\left(x-\frac{\alpha+\beta}{2}\right),\ v(x)=(x-\alpha)^2(x-\beta)$$

*and write*

$$D(f)=b_1D_1^{\beta}(f)+b_2D_2^{\beta}(f)+a_1D_1^{\alpha}(f)+a_2D_2^{\alpha}(f)+a_3D_3^{\alpha}(f)+a_4D_4^{\alpha}(f)+a_5D_5^{\alpha}(f)$$

*Looking at $u(x)$ and $v(x)$ and find that it is sufficient to ensure that $D$ is zero evaluated at $u(x)^2, u(x)\cdot v(x)$, and $v(x)^2$ since any other product is divisible by $(x-\alpha)^6(x-\beta)^3$ and so $D(f)=0$ regardless of the constants $a_i$, $b_i$. After some calculations we obtain the following table*

| | $D_1^\beta$ | $D_2^\beta$ | $D_1^\alpha$ | $D_2^\alpha$ | $D_3^\alpha$ | $D_4^\alpha$ | $D_5^\alpha$ |
|---|---|---|---|---|---|---|---|
| $u(x)^2$ | 0 | $\frac{1}{4}(\alpha-\beta)^8$ | 0 | 0 | 0 | 0 | 0 |
| $u(x)\cdot v(x)$ | 0 | $\frac{1}{2}(\alpha-\beta)^6$ | 0 | 0 | 0 | 0 | $\frac{1}{2}(\alpha-\beta)^3$ |
| $v(x)^2$ | 0 | $(\alpha-\beta)^4$ | 0 | 0 | 0 | $(\alpha-\beta)^2$ | $2(\alpha-\beta)$ |

*which when used results in*

$$\begin{array}{lcl}
b_1\cdot 0 + b_2\cdot \frac{1}{4}(\alpha-\beta)^8 + a_1\cdot 0 + a_2\cdot 0 + a_3\cdot 0 + a_4\cdot 0 + a_5\cdot 0 = 0 & \implies & b_2 = 0 \\
b_1\cdot 0 + b_2\cdot \frac{1}{2}(\alpha-\beta)^6 + a_1\cdot 0 + a_2\cdot 0 + a_3\cdot 0 + a_4\cdot 0 + a_5\cdot \frac{1}{2}(\alpha-\beta)^3 = 0 & \implies & a_5 = 0 \\
b_1\cdot 0 + b_2\cdot (\alpha-\beta)^4 + a_1\cdot 0 + a_2\cdot 0 + a_3\cdot 0 + a_4\cdot (\alpha-\beta)^2 + a_5\cdot 2(\alpha-\beta) = 0 & \implies & a_4 = 0
\end{array}$$

*where we have used $\alpha \neq \beta$. And so we must have $b_2 = a_4 = a_5 = 0$, whilst we are free to pick the constants $b_1, a_1, a_2$, and $a_3$ arbitrarily. Taking the previous conditions of $A$ into account we end up with derivations which look like*

$$D(f) = b_1 D_1^\beta(f) + a_2 D_2^\alpha(f)$$

*where $b_1$ and $a_2$ are arbitrary constants, but not both zero.*

## 4.3 Examples of Derivations

This sections will consist of tables of algebras and all their derivations. We will restrict ourselves to subalgebras with a spectrum containing exactly two elements, $\alpha, \beta$, which are in the same cluster. For these algebras there exists a symmetry which we will make use of, for example $\{\ f \mid f(\alpha) = f(\beta),\ f'(\alpha) = 0\ \}$ and $\{\ f \mid f(\alpha) = f(\beta),\ f'(\beta) = 0\ \}$ can be considered the same, and so we will only include one of them in the following tables. This is because $\alpha$ and $\beta$ are simply notations of the two spectrum elements and can therefore be switched with each other. Below is an example of what the tables will look like

| Condition | Type | Derivations |
|---|---|---|
| $D_1^\alpha$ | (3,4,5) | $D_1^\beta, D_2^\alpha, D_3^\alpha$ |

In this example we have the algebra $\{\ f \mid f(\alpha) = f(\beta),\ D_1^\alpha(f) = 0\}$. $A$ consisting of two conditions means it is of codimension two in $\mathbb{K}[x]$ and since it has type (3,4,5) we can tell that the degrees missing are one and two. The derivations of $A$ can then be written as a linear combination of $D_1^\beta, D_2^\alpha$, and $D_3^\alpha$.

### 4.3.1 $f(\alpha) = f(\beta)$

In the tables in this section the algebras consists of the $f(\alpha) = f(\beta)$ as well as the condition in the column "Condition".

| Condition | Type | Derivations |
| --- | --- | --- |
| - | (2,3) | $D_1^\alpha, D_1^\beta$ |

| Condition | Type | Derivations |
| --- | --- | --- |
| $D_1^\alpha$ | (3,4,5) | $D_1^\beta, D_2^\alpha, D_3^\alpha$ |
| $D_1^\alpha + bD_1^\beta$, b=1 | (2,5) | $D_1^\beta, D_2^\alpha - b^2 D_2^\beta$ |
| $D_1^\alpha + bD_1^\beta$, b $\notin \{0,1\}$ | (3,4,5) | $D_1^\alpha, D_2^\alpha - b^2 D_2^\beta$ |

### 4.3.2 $f(\alpha) = f(\beta)$, $D_1^\alpha$

In this section the algebras in the table consists of the equality condition $f(\alpha) = f(\beta)$, the derivation $D_1^\alpha$, and condition in the column "Condition".

| Condition | Type | Derivations |
| --- | --- | --- |
| $D_1^\beta$ | (4,5,6,7) | $D_2^\alpha, D_2^\beta, D_3^\alpha, D_3^\beta$ |
| $D_2^\alpha$ | (4,5,6,7) | $D_1^\beta, D_3^\alpha, D_4^\alpha, D_5^\alpha$ |
| $D_3^\alpha$ | (4,5,6,7) | $D_1^\beta, D_2^\alpha, D_5^\alpha$ |
| $D_1^\beta + cD_2^\alpha$, $c = \beta - \alpha$ | (3,5,7) | $D_2^\alpha, D_3^\alpha, D_2^\beta - c^2 D_4^\alpha$ |
| $D_1^\beta + cD_2^\alpha$, $c \notin \{0, \beta - \alpha\}$ | (4,5,6,7) | $D_2^\alpha, D_3^\alpha, D_2^\beta - c^2 D_4^\alpha$ |
| $D_2^\alpha + cD_3^\alpha$, $c = \beta - \alpha$ | (3,5,7) | $D_1^\beta, D_2^\alpha, 2D_4^\alpha + cD_5^\alpha$ |
| $D_2^\alpha + cD_3^\alpha$, $c \notin \{0, \beta - \alpha\}$ | (4,5,6,7) | $D_1^\beta, D_2^\alpha, 2D_4^\alpha + cD_5^\alpha$ |
| $D_1^\beta + cD_3^\alpha$, $c = -(\alpha - \beta)^2$ | (3,5,7) | $D_1^\beta, D_2^\alpha$ |
| $D_1^\beta + cD_3^\alpha$, $c \notin \{0, -(\alpha - \beta)^2\}$ | (4,5,6,7) | $D_1^\beta, D_2^\alpha$ |
| $D_1^\beta + cD_2^\alpha + dD_3^\alpha$, $c = 2(\beta - \alpha), d = (\beta - \alpha)^2$ | (3,4) | $D_1^\beta, D_2^\alpha$ |
| $D_1^\beta + cD_2^\alpha + dD_3^\alpha$, $c \notin \{0, 2(\beta - \alpha)\}$, $d = c(\beta - \alpha) - (\beta - \alpha)^2 \neq 0$ | (3,5,7) | $D_1^\beta, D_2^\alpha$ |
| $D_1^\beta + cD_2^\alpha + dD_3^\alpha, c \neq 0$, $d \neq c(\beta - \alpha) - (\beta - \alpha)^2, d \neq 0$ | (4,5,6,7) | $D_1^\beta, D_2^\alpha$ |

### 4.3.3 $f(\alpha) = f(\beta)$, $D_1^\alpha$, $D_2^\alpha$

In this section the algebras in the table consists of the equality condition $f(\alpha) = f(\beta)$, the derivations $D_1^\alpha$ and $D_2^\alpha$, as well as the condition in the column "Condition".

| Condition | Type | Derivations |
|---|---|---|
| $D_1^\beta$ | (5,6,7,8,9) | $D_2^\beta, D_3^\beta, D_3^\alpha, D_4^\alpha, D_5^\alpha$ |
| $D_3^\alpha$ | (5,6,7,8,9) | $D_1^\beta, D_4^\alpha, D_5^\alpha, D_6^\alpha, D_7^\alpha$ |
| $D_4^\alpha$ | (5,6,7,8,9) | $D_1^\beta, D_3^\alpha, D_5^\alpha, D_7^\alpha$ |
| $D_5^\alpha$ | (4,6,7,9) | $D_1^\beta, D_3^\alpha, D_4^\alpha$ |
| $D_1^\beta + cD_3^\alpha, c = (\beta - \alpha)^2$ | (4,6,7,9) | $D_1^\beta, D_4^\alpha, D_5^\alpha, D_2^\beta - c^2 D_6^\alpha$ |
| $D_1^\beta + cD_3^\alpha, c \neq (\beta - \alpha)^2$ | (5,6,7,8,9) | $D_1^\beta, D_4^\alpha, D_5^\alpha, D_2^\beta - c^2 D_6^\alpha$ |
| $D_1^\beta + cD_4^\alpha,\ c = (\alpha - \beta)^3$ | (4,6,7,9) | $D_1^\beta, D_3^\alpha, D_5^\alpha$ |
| $D_1^\beta + cD_4^\alpha,\ c \notin \{0, (\alpha - \beta)^3\}$ | (5,6,7,8,9) | $D_1^\beta, D_3^\alpha, D_5^\alpha$ |
| $D_1^\beta + cD_5^\alpha, c \neq 0$ | (5,6,7,8,9) | $D_1^\beta, D_3^\alpha, D_4^\alpha$ |
| $D_3^\alpha + cD_4^\alpha$, c $= \beta - \alpha$ | (4,6,7,9) | $D_1^\beta, D_3^\alpha, D_5^\alpha, 2D_6^\alpha + cD_7^\alpha$ |
| $D_3^\alpha + cD_4^\alpha$, c $\notin \{0, \beta - \alpha\}$ | (5,6,7,8,9) | $D_1^\beta, D_3^\alpha, D_5^\alpha, 2D_6^\alpha + cD_7^\alpha$ |
| $D_3^\alpha + cD_5^\alpha, c \neq 0$ | (5,6,7,8,9) | $D_1^\beta, D_3^\alpha, D_4^\alpha$ |
| $D_4^\alpha + cD_5^\alpha, c \neq 0$ | (5,6,7,8,9) | $D_1^\beta, D_3^\alpha, D_4^\alpha$ |
| $D_3^\alpha + cD_4^\alpha + dD_5^\alpha,\ c = \beta - \alpha, d = (\beta - \alpha)^2$ | (4,5,7) | $D_1^\beta, D_3^\alpha, D_4^\alpha$ |
| $D_3^\alpha + cD_4^\alpha + dD_5^\alpha,\ c = \beta - \alpha, d \notin \{0, (\beta - \alpha)^2\}$ | (4,6,7,9) | $D_1^\beta, D_3^\alpha, D_4^\alpha$ |
| $D_3^\alpha + cD_4^\alpha + dD_5^\alpha,\ c \notin \{0, (\beta - \alpha)\}, d \neq 0$ | (5,6,7,8,9) | $D_1^\beta, D_3^\alpha, D_4^\alpha$ |
| $D_1^\beta + cD_3^\alpha + dD_4^\alpha, d = (\beta - \alpha)^3, c = 2(\beta - \alpha)^2$ | (4,5,7) | $D_1^\beta, D_3^\alpha, D_5^\alpha$ |
| $D_1^\beta + cD_3^\alpha + dD_4^\alpha$ $d = -(\beta - \alpha)^3 - (\beta - \alpha)c, c \notin \{0, 2(\beta - \alpha)^2\}$ | (4,6,7,9) | $D_1^\beta, D_3^\alpha, D_5^\alpha$ |
| $D_1^\beta + cD_3^\alpha + dD_4^\alpha, d \notin \{0, -(\beta - \alpha)^3 - (\beta - \alpha)c\} c \neq 0$ | (5,6,7,8,9) | $D_1^\beta, D_3^\alpha, D_5^\alpha$ |
| $D_1^\beta + cD_4^\alpha + dD_5^\alpha, d = -2(\beta - \alpha)^4, c = (\alpha - \beta)^3$ | (4,5,6) | $D_1^\beta, D_3^\alpha, D_4^\alpha$ |
| $D_1^\beta + cD_4^\alpha + dD_5^\alpha, d \notin \{0, -2(\alpha - \beta)^4\}, c = (\alpha - \beta)^3$ | (4,6,7,9) | $D_1^\beta, D_3^\alpha, D_4^\alpha$ |
| $D_1^\beta + cD_4^\alpha + dD_5^\alpha, d \notin \{0, -(\beta - \alpha)^3 - (\beta - \alpha)c\} c \neq 0$ | (5,6,7,8,9) | $D_1^\beta, D_3^\alpha, D_4^\alpha *$ |

Table 2: * is mostly checked except for some special cases.

As can be seen Table 2 there are algebras which consists of the equality condition $f(\alpha) = f(\beta)$, the derivations $D_1^\alpha$ and $D_2^\alpha$, and one additional $\alpha$-derivation which is not included, due to lack of time as well as complicated computations. These are the algebras containing the derivations
$D_1^\alpha, D_2^\alpha, D_1^\beta + cD_3^\alpha + dD_5^\alpha$ and
$D_1^\alpha, D_2^\alpha, D_1^\beta + cD_3^\alpha + dD_4^\alpha + eD_5^\alpha$.

#### 4.3.4 $f(\alpha) = f(\beta)$, Extras

This section contains some additional cases which might be of interest for the reader, but does not fit in the previous sections. These algebras only consists of the equality condition $f(\alpha) = f(\beta)$ and the conditions in the column "Conditions".

| Conditions | Type | Derivations |
|---|---|---|
| $D_1^\alpha, D_3^\alpha, D_1^\beta$ | (5,6,7,8,9) | $D_2^\beta, D_3^\beta, D_2^\alpha, D_5^\alpha$ |
| $D_1^\alpha, D_2^\alpha, D_3^\alpha, D_4^\alpha$ | (6,7,8,9,10,11) | $D_1^\beta, D_5^\alpha, D_6^\alpha, D_7^\alpha, D_8^\alpha, D_9^\alpha$ |
| $D_1^\beta, D_1^\alpha, D_2^\alpha, D_3^\alpha$ | (6,7,8,9,10,11) | $D_2^\beta, D_3^\beta, D_4^\alpha, D_5^\alpha, D_6^\alpha, D_7^\alpha$ |
| $D_1^\alpha, D_3^\alpha, D_1^\beta + cD_2^\alpha,\ c \notin \{0, 2(\beta-\alpha)\}$ | (5,6,7,8,9) | $D_1^\beta, D_5^\alpha, g(D_2^\beta, D_4^\alpha)$ |
| $D_1^\alpha, D_3^\alpha, D_1^\beta + cD_2^\alpha, c = 2(\beta-\alpha)$<br>$D_1^\alpha, D_3^\alpha, D_1^\beta + cD_2^\alpha, c \notin \{0, 2(\beta-\alpha)\}$ | (4,6,7,9)<br>(5,6,7,8,9) | $D_1^\beta, D_5^\alpha, D_2^\beta - c^2 D_4^\alpha$<br>$D_1^\beta, D_5^\alpha, D_2^\beta - c^2 D_4^\alpha$ |
| $D_1^\beta, D_2^\beta, D_1^\alpha, D_2^\alpha, D_3^\alpha$<br>$D_4^\alpha, D_3^\beta + cD_5^\alpha, c = (\beta-\alpha)^2$<br>$D_1^\beta, D_2^\beta, D_1^\alpha, D_2^\alpha, D_3^\alpha, D_4^\alpha$<br>$D_3^\beta + cD_5^\alpha, c \notin \{0, (\beta-\alpha)^2\}$ | (8,10,11,12,<br>13,14,15,17)<br>(9,10,11,12,13,<br>14,15,16,17) | $D_3^\beta, D_4^\beta, D_5^\beta, D_6^\alpha$<br>$D_7^\alpha, D_8^\alpha, D_9^\alpha, D_6^\beta - c^2 D_{10}^\alpha$<br>$D_3^\beta, D_4^\beta, D_5^\beta, D_6^\alpha$<br>$D_7^\alpha, D_8^\alpha, D_9^\alpha, D_6^\beta - c^2 D_{10}^\alpha$ |
| $D_1^\alpha + bD_1^\beta, D_2^\alpha - b^2 D_2^\beta,\ b^2 - b + 1 = 0$<br>$D_1^\alpha + bD_1^\beta, D_2^\alpha - b^2 D_2^\beta,\ b \neq 0,\ b^2 - b + 1 \neq 0$ | (3,5,7)<br>(4,5,6,7) | $D_1^\beta, D_3^\alpha + b^3 D_3^\beta$<br>$D_1^\beta, D_3^\alpha + b^3 D_3^\beta$ |
| $D_1^\alpha + D_1^\beta, D_1^\alpha + c(D_2^\alpha - D_2^\beta)$ | (4,5,6,7) | $D_1^\alpha,\ c_1 D_2^\alpha + c_2 D_2^\beta + c\frac{c_1+c_2}{2}(D_3^\alpha + D_3^\beta)$ |
| $D_1^\alpha, D_1^\beta + cD_2^\alpha, D_2^\alpha + dD_3^\alpha$<br>$c = d = \beta - \alpha$ | (3,7,8) | $D_1^\beta, c_1 D_2^\beta + c_2 D_4^\alpha$<br>$-\frac{1}{2}\left((\alpha-\beta)^3 c_1 + (\alpha-\beta)c_2\right) D_5^\alpha$ |
| $D_1^\beta, D_1^\alpha, D_2^\alpha + cD_3^\alpha, c \neq 0$<br>c=$-\left(\frac{\alpha^2+4\alpha\beta+\beta^2-12\alpha-12\beta+24}{2(\alpha-\beta)}\right)$ | (4,6,7,9) | $D_2^\beta, D_3^\beta, D_2^\alpha, 4D_4^\alpha + (\beta-\alpha)D_5^\alpha$ |

Table 3: Where g is some function.

Looking at all these tables we find some interesting patterns. First we take note of the fact that when our previous conditions includes a non-zero constant the structure of the derivations in $A$ stays the same regardless of the value of the constant, unlike the type which can change for specific values.

If we look at Table 2 we find that if our new condition is a single derivation then the higher the order of this derivation the less options we have to pick from for our next derivation.

Similarly if our new condition is a linear combination of derivations then we will, in most cases end up with the same set of derivations as in our previous algebra. There are however two conditions which keep appearing, namely $D_{2M+2}^\beta - c^2 D_{2N+2}^\alpha$ and $2D_{2N+2}^\alpha + cD_{2N+3}^\alpha$ for some positive integers $N$ and $M$. Although these are the two most commonly occurring conditions in the algebras we have looked at above it can be seen in Table 3 that this is likely not true in general. We will now take a closer look at the algebras where these two conditions appear.

## 4.4 Specific Algebras

### 4.4.1 Algebra: $D_i^{\alpha}, i = 1, 2, \ldots, N,\ D_j^{\beta}, j = 1, 2, \ldots, M,\ D_{N+1}^{\alpha} + cD_{M+1}^{\beta}$

Let us look at the subalgebra $A$ which consists, exclusively, of the following conditions

$$f(\alpha) = f(\beta),\ D_i^{\alpha}, i = 1, 2, \ldots, N,\ D_j^{\beta}, j = 1, 2, \ldots, M,\ D_{N+1}^{\alpha} + cD_{M+1}^{\beta},\ \text{where } c \neq 0$$

This subalgebra has the minimal polynomial $m(x) = (x-\alpha)^{N+2}(x-\beta)^{M+2}$. By Theorem 13 we have that the largest order of a derivative, in our $\alpha$-derivative, evaluated at $\alpha$ and $\beta$, are $2N + 2$ and $2M + 2$ respectively.

Looking at the conditions of $A$ we know that any $f(x) \in A$ can be written as

$$f(x) = (x - \alpha)^{N+1} \cdot (x - \beta)^{M+1} \cdot p(x)$$

for some polynomial $p(x)$. This means that when creating a SAGBI basis we will exclude the degrees $1, 2, 3, \ldots, M + N + 1$. The codimension of $A$ in $\mathbb{K}[x]$ is $N + M + 2$, and so we only need to find one basis vector of degree lower than the degree of the minimal polynomial. Going forward we will assume $n \geq m$. We will divide this problem in two cases depending on the value of $c$.

In the first case we have $c = (-1)^N(\alpha - \beta)^{N-M}$. In this case the basis element we are looking for is

$$q(x) = (x - \alpha)^{N+1}(x - \beta)^{M+1}$$

And so we end up with $A$ having the type $(M + N + 2, M + N + 4, M + N + 5, \ldots, 2M + 2N + 2, 2M + 2N + 3, 2M + 2N + 5)$.

To determine what derivations, $D$, we have in $A$ we only need to ensure that $D(q^2) = 0$. And so we have

$$\begin{cases} D_{2N+2}^{\alpha}(q^2) = (\alpha - \beta)^{2(M+1)} \\ D_{2M+2}^{\beta}(q^2) = (\beta - \alpha)^{2(N+1)} \end{cases}$$

all other derivations are either 0 or irrelevant. We end up with the requirement $a_{2N+2} = -(\alpha - \beta)^{2(N-M)} b_{2M+2}$. As a conclusion all $\alpha$-derivations in $A$ can be written as a linear combination of the following derivations

$$D_{2M+2}^{\beta} - c^2 D_{2N+2}^{\alpha}\ D_i^{\alpha},\ i = N + 2, N + 3, \ldots, 2N + 1,\ D_j^{\beta},\ j = M + 1, M + 2, \ldots, 2M + 1$$

In the second case we have $c \neq (-1)^N(\alpha - \beta)^{N-M}$. In this case the basis element can be written

$$q(x) = (x - \alpha)^{N+1}(x - \beta)^{M+1}(x - d),\ d = \frac{(-1)^{N+1}(\alpha - \beta)^{N-M}\beta + c\alpha}{(-1)^{N+1}(\alpha - \beta)^{N-M} + c}$$

In this case we have type $(M + N + 3, M + N + 4, \ldots, 2M + 2N + 4, 2M + 2N + 5)$. Same as before we only need to ensure that $D(q^2) = 0$. We have

$$\begin{cases} D_{2N+2}^{\alpha}(q^2) = (\alpha - \beta)^{2(M+1)} \cdot (\alpha - d)^2 \\ D_{2M+2}^{\beta}(q^2) = (\beta - \alpha)^{2(N+1)} \cdot (\beta - d)^2 \end{cases}$$

and all other derivations are either 0 or irrelevant. We end up with the requirement $a_{2N+2} = -c^2 b_{2M+2}$. As a conclusion any $\alpha$-derivation in $A$ can be written as a linear combination of the following derivations

$$D^{\beta}_{2M+2} - c^2 D^{\alpha}_{2N+2}\; D^{\alpha}_{i},\; i = N+2, N+3, \ldots, 2N+1,\; D^{\beta}_{j},\; j = M+1, M+2, \ldots, 2M+1$$

Note that the derivations in both cases look the same.

### 4.4.2 Algebra: $D^{\alpha}_{i}, i = 1, 2, \ldots, N,\; D^{\beta}_{j}, j = 1, 2, \ldots, M,\; D^{\alpha}_{N+1} + cD^{\alpha}_{N+2}$

Next we will look at the subalgebra $A$ which consists, exclusively, of the following conditions

$$f(\alpha) = f(\beta),\; D^{\alpha}_{i}, i = 1, 2, \ldots, N,\; D^{\beta}_{j}, j = 1, 2, \ldots, M,\; D^{\alpha}_{N+1} + cD^{\alpha}_{N+2},\text{ where } c \neq 0$$

This subalgebra has the minimal polynomial $m(x) = (x-\alpha)^{N+3}(x-\beta)^{M+1}$ Using Theorem 13, we can tell that the highest order of a derivative, in our $\alpha$-derivative, are $2N+3$ and $2M+1$ evaluated at $\alpha$ and $\beta$ respectively.

We know that any $f(x) \in A$ can be written as

$$f(x) = (x-\alpha)^{N+1} \cdot (x-\beta)^{M+1} \cdot p(x)$$

for some polynomial $p(x)$. $A$ is missing the degrees $1, 2, 3, \ldots, N+M+1$ and together with the codimension being $N+M+2$ we find that we only need one basis element, the degree of which is less than the degree of the minimal polynomial. Similarly to before we will divide this problem into two cases. We start by looking at the special case where we have $c = \frac{\beta-\alpha}{M+1}$. Then we find our basis element to be

$$q(x) = (x-\alpha)^{N+1}(x-\beta)^{M+1}$$

and so our algebra is of type $(N+M+2, N+M+4, N+M+5, \ldots, 2N+2M+2, 2N+2M+3, 2N+2M+5)$. In order for $D$ to be a derivation in $A$ we only need to ensure that $D(q^2) = 0$ and so we have

$$\begin{cases} D^{\alpha}_{2N+2}(q^2) = (\alpha-\beta)^{2M+2} \\ D^{\alpha}_{2N+3}(q^2) = 2(M+1)(\alpha-\beta)^{2M+1} \end{cases}$$

all other derivations are either 0 or irrelevant and so we end up with the requirement $a_{2N+2}(\alpha-\beta) + 2(M+1)a_{2N+3} = 0$. All $\alpha$-derivations in $A$ can therefor be written as a linear combination of the following derivations

$$2D^{\alpha}_{2N+2} + cD^{\alpha}_{2N+3}\; D^{\alpha}_{i},\; i = N+2, N+3, \ldots, 2N+1,\; D^{\beta}_{j},\; j = M+1, M+2, \ldots, 2M+1$$

In the second case where $c \notin \{0, \frac{\beta-\alpha}{M+1}\}$ our basis element looks like

$$q(x) = (x-\alpha)^{N+1}(x-\beta)^{M+1}(x-d)$$

where we have

$$d = \frac{(\alpha-\beta)\alpha + c(M+1)\alpha + c(\alpha-\beta)}{\alpha-\beta+(M+1)c}$$

This algebra is of type $(M+N+3, M+N+4, \ldots, 2M+2N+4, 2M+2N+5)$. As before we only need to ensure that $D(q^2) = 0$ and so we look at

$$\begin{cases} D^{\alpha}_{2N+2}(q^2) = (\alpha-\beta)^{2(M+1)}(\alpha-d)^2 \\ D^{\alpha}_{2N+3}(q^2) = (\alpha-\beta)^{2M+1}\Big(2(M+1)(\alpha-d)^2 + 2(\alpha-\beta)(\alpha-d)\Big) \end{cases}$$

which gives us the requirement $-ca_{2N+2} + 2a_{2N+3} = 0$ and all $\alpha$-derivations in $A$ can be written as a linear combination of

$$2D^{\alpha}_{2N+2} + cD^{\alpha}_{2N+3}\ D^{\alpha}_{i},\ i = N+2, N+3, \ldots, 2N+1,\ D^{\beta}_{j},\ j = M+1, M+2, \ldots, 2M+1$$

Note that as in the previous example the derivations in both cases look the same.

# 5 Future Areas of Study

This is the end of the report where we have managed to constructed the minimal polynomial and found a method to determine the $\alpha$-derivations of our algebra $A$. Lastly we have taken a look at some examples and determined that there seems to be some interesting patterns in what $\alpha$-derivations we can find in a given algebra. In the future it would also be of great interest to find a method to construct the minimal polynomial of an algebra which is not given on condition form. This because it would allow us to find the spectrum elements of $A$.

# 6 References


[1] Flávio U. Coelho Ibrahim Assem. *Basic Representation Theory of Algebras.* Springer Cham, 1 edition, 2020.

[2] Alexander M. Kytmanov Yaroslav M. Naprienko. An approach to define the resultant of two entire functions. *Complex Variables and Elliptic Equations*, 62(2):269–286, 2017.

[3] Anna Torstensson Victor Ufnarovski Rode Grönkvist, Erik Lefer. Subalgebras in k[x] of small codimension. volume 33 of *Applicable Algebra in Engineering, Communication and Computing*, 2022.

[4] Thomas W.Hungerford. *Abstract Algebra An Introduction.* Brooks/Cole, 3 edition, 2018.