Gravity Water Waves over Constant Vorticity Flows

From Laminar Flows to Touching Waves

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Abstract

In a recent paper [31], Hur and Wheeler proved the existence of periodic steady water waves over an infinitely deep, two-dimensional and constant vorticity flow and subject to gravity whose profile overhangs, among which, waves whose surface touches at a point, enclosing a bubble of air. We take this further, proving the existence of a continuous curve of water waves from a laminar flow up to a touching wave for fixed non-zero gravity. This implies the existence of a wave profile that is vertical at a point but not overhanging, which is referred to as a breaking wave. This allows us to study the behaviour of critical layers, which are points where the horizontal velocity vanishes, at locations where the wave profile is vertical. This applies to both overhanging and breaking waves. We also extend our results regarding the continuous curve of water waves from a laminar flow up to a touching wave to finite but very large depth. We formulate our problem as a modified version of the Babenko equation. We then use methods from local bifurcation theory to construct solutions near the laminar flow and use a compactness argument to ensure the maps obtained from the different Implicit Function Theorems coincide. In the last Section, we extend our results to the finite depth case. To do this, we formulate the problem utilising the periodic Hilbert transform on a finite strip. Properties of this operator discovered by Constantin, Strauss and Varvaruca [12] turn out instrumental for our purposes.

Popular Abstract

I denna uppsats studerar vi en matematisk modell av vattenvågor över djupt vatten. I synnerhet studerar vi vågor som färdas med konstant hastighet i en horisontell riktning och är likformiga i den vinkelräta horisontella riktningen. På grund av den matematiska svårigheten med den allmänna modellen är det vanligt att utgå från förenklade versioner av problemet. I denna uppsats, gör vi antaganden om att vätskan har konstant densitet, att det inte finns någon luftrörelse utanför vattnet, att det inte finns någon viskositet, att vorticiteten är konstant och att vätskan är oändligt djup.

Specifikt fokuserar vi på vågor som utvecklar en överhängande profil. Vårt huvudresultat visar att det finns en kontinuerlig kurva av lösningar mellan ett laminärt flöde, det vill säga en vattenvåg vars yta är platt, och en vidrörande våg, det vill säga en vattenvåg vars yta vidrör sig själv i en punkt och som omsluter en luftbubbla i vätskan. Vi studerar också egenskaperna hos de punkter där flödet är rent vertikalt. I det sista avsnittet utvidgar vi våra resultat från fallet med oändligt djup till fallet med begränsat djup.

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Contents

1 Introduction

The mathematical theory of water waves models a body of water lying underneath a layer of air. Mathematically speaking, it is a free boundary problem, a problem to be solved in an unknown domain that must be determined as part of the problem. The boundary conditions being non-linear together with the boundary being unknown makes the problem challenging, and therefore there are still a lot of unanswered questions. In this thesis, we consider two-dimensional periodic water waves that are travelling with constant velocity over an incompressible inviscid fluid. Particularly we focus on water waves whose surface develops an overhanging profile.

The equations that describe the motion of flow in fluids were originally derived by Euler [25]. In 1776, Laplace [40] was the first to study water waves as an initial value problem. Around the same time, Lagrange [38] formulated the water wave problem with the particles as the variables instead of the spatial coordinates. Extensive work on the initial value problem was done by Cauchy [10] and Poisson [48]. In 1802, Gerstner [27] found explicit solutions for infinite depth and nonzero vorticity. In Gerstner's solutions, the particles move in circles, whose radii decrease with depth. Famously, Stokes [52] conjectured that the gravity wave of the greatest possible height has a one hundred and twenty-degree angle at the crest. Moreover, Stokes noticed that for gravity waves, as the amplitude increases, crests become sharper, and throughs become flatter, among other contributions.

A lot of progress was achieved for the irrotational case. Using power series expansion, Nekrasov [43] and Levi-Civita [41] were able to show the existence of small amplitude solutions. The existence of solutions with large amplitude was first proven by Krasovskii [37]. Using bifurcation theory for positive operators [19], Keady & Norbury [32] constructed a smooth curve of solutions with large amplitude. Finally, Amick, Fraenkel & Toland [3] and Plotnikov [46, 47] established Stokes' conjecture regarding the wave of greatest height.

Throughout most of the history of water wave theory, research focused mainly on irrotational flow. Shifting the research focus beyond irrotational flow, Dubreil-Jacotin [22] made significant contributions in 1934 by constructing small amplitude solutions for a general vorticity flow. Using global bifurcation theory, Constantin & Strauss [14] were able to prove the existence of large amplitude solutions over a general vorticity flow. In this thesis, we will assume the vorticity to be constant. This is primarily due to it being possible to adapt methods from the irrotational case to the constant vorticity case. Recently, Constantin, Strauss & Vărvărucă [12] constructed a curve of solutions under the assumption of constant vorticity, which possibly allows for overhanging profiles.

In 1957 Crapper [17] discovered an explicit, smooth curve of exact solutions for the case of capillary waves over an irrotational flow without the effects of gravity. As the amplitude increases, the crests of Crapper's solutions become flatter, and their throughs become sharper, the opposite of what happens for gravity waves. While proving the non-existence of a bifurcation from Crapper's solutions, Okamoto & Shōji [44, 45] produced closed-form recurrence relations between the Fourier coefficients of the Fréchet derivative of an operator about Crapper's waves. Using this relation, Akers, Ambrose & Wright [2] and de Boeck [21] were able to construct small gravity solutions close to Crapper's solutions. Córdoba, Enciso and Grubic [15] took matters further and constructed a touching wave.

Recently, Hur & Wheeler [30] discovered a family of explicit solutions for the case without surface tension and gravity over a constant vorticity flow. The geometry of the free surface of these solutions is the same as Crapper's solutions, but the flow under the surface is vastly different. Recently, Crowdy [18] showed that Hur & Wheeler's solutions can be seen as part of a larger framework of explicit solutions for the rotational case under the assumption of zero surface tension and gravity.

Under the assumption of no capillarity, Spielvogel [51] showed that gravity waves over an irrotational flow can not be overhanging. Using an Implicit Function Theorem argument, Hur & Wheeler [31] were able to construct overhanging and touching waves for small gravity over a constant vorticity flow. In this thesis, we take these matters further, using local bifurcation theory and compactness to improve the results of Hur & Wheeler by making the argument uniform. This leads to a smooth curve of solutions from the laminar flow up to a touching wave for a fixed small nonzero gravity and implies the existence of waves whose profile is vertical at a point but never overhanging, that is, breaking waves. We then analyse the local behaviour of critical layers (points where the horizontal velocity becomes zero) at the surface for breaking and overhanging waves. Finally, we use a formulation of Constantin, Strauss & Vărvărucă to extend our results to large finite depth.

For a more detailed overview of the history of water wave theory, we refer the reader to [16, 20, 28, 53].

We start by introducing the necessary notation and the mathematical machinery used throughout the thesis in Section 2. Then, in Section 3, we formulate the water wave problem. Afterwards, we give a brief overview of the solutions found by Crapper and Hur & Wheeler in Section 4. We construct a continuous curve of solutions starting from the laminar flow until the touching wave in Section 5. In Section 6, we analyze the local behaviour of critical layers at points where the wave has a vertical tangent. Finally, we extend the results from Section 5 to the finite depth case, in Section 7.

2 Mathematical Background

In this section, we aim to introduce notation and present results that may be unfamiliar to the reader. We start by going over some of the notation used throughout the Thesis in the first subsection. The second subsection discusses Bifurcation theory, which is used to prove our main result. In the last subsection, we state some results concerning the periodic Hilbert transform.

2.1 Notation

Let X and Y be Banach spaces, x_0 be a point in X and $F : X \to Y$ be a function defined in a neighbourhood of x_0. Then we write

$$F(x) = \mathcal{O}(x - x_0) \text{ as } x \to x_0 \qquad \text{if} \qquad \overline{\lim_{x \to x_0}} \frac{||F(x)||_Y}{||x - x_0||_X} < \infty$$

and

$$F(x) = o(x - x_0) \text{ as } x \to x_0 \qquad \text{if} \qquad \lim_{x \to x_0} \frac{||F(x)||_Y}{||x - x_0||_X} = 0,$$

where $||F(x)||_Y$ and $||x - x_0||_X$ are the corresponding Banach space norms.

We denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y.

The range and the kernel of a linear operator F are denoted by $\operatorname{ran} F$ and $\ker F$, respectively.

We use $\mathcal{F}^n(X, Y)$, to denote the space of bounded multilinear operators from X to Y. Let $m \in \mathcal{F}^n(X, Y)$, then we use $m(x^n)$ to denote $m(x, ..., x)$.

Let z be a complex number, then we write $\Re z$ and $\Im z$ to denote the real and imaginary parts of z, respectively.

We use $\mathbb{D}$ to denote the open complex unit disc.

2.2 Bifurcation theory

We start by introducing the Fréchet derivative, a generalization of the derivative from single variable calculus to Banach spaces.

Definition 1 (Fréchet derivative)**.** Let X and Y be Banach spaces, $\mathcal{U}$ an open subset of X and $F : \mathcal{U} \to Y$ an operator. Then, F is said to be Fréchet differentiable at $x_0 \in \mathcal{U}$ if there exists

$DF[x_0] \in \mathcal{L}(X, Y)$ satisfying

$$F(x_0 + u) - F(x_0) - DF[x_0](u) = o(u)$$

as $u \to 0$. The linear operator $DF[x_0]$ is called the Fréchet derivative of F at x_0. Moreover, F is said to be Fréchet differentiable on $\mathcal{U}$ if it is Fréchet differentiable at x for all $x \in \mathcal{U}$.

We also introduce the notion of partial Fréchet derivatives.

Definition 2 (Partial Fréchet derivative)**.** Let X, Y and Z be Banach spaces, $\mathcal{U} \subset X \times Y$ an open set, (x_0, y_0) a point in $\mathcal{U}$ and $F : \mathcal{U} \to Z$ an operator. If $F(\cdot, y_0)$ is Fréchet differentiable at x_0 then F is said to be partially Fréchet differentiable with respect to x at (x_0, y_0). Moreover, the partial Fréchet derivative with respect to x at the point (x_0, y_0) is denoted $D_x F[x_0, y_0]$.

Next, we state some properties of the Fréchet derivative. We refer to chapter 9 of [54] for a more detailed discussion.

Proposition 1 (Properties of Fréchet derivative)**.**

(i) If the Fréchet derivative exists, then it is unique.

(ii) When $F : \mathbb{R}^n \to \mathbb{R}^m$, the Fréchet derivative corresponds to the Jacobian matrix.

Afterwards, we give the definition of a real-analytic operator.

Definition 3 (Real-analytic operator)**.** Let X and Y be Banach spaces over $\mathbb{R}$, $\mathcal{U}$ an open subset of X, x_0 an arbitrary point in $\mathcal{U}$ and $F : \mathcal{U} \to Y$ an operator. Then F is said to be real-analytic at x_0 if there exist $r > 0$ and bounded symmetric multilinear operators $m_k \in \mathcal{F}^k(X, Y)$ such that

$$F(x_0 + h) - F(x_0) = \sum_{k=1}^{\infty} m_k(h^k)$$

and the series converges uniformly for all $||h||_X < r$. A sufficient and necessary condition for the above to hold is that

$$\sup_{k \geq 0} r^k ||m_k|| = M < \infty,$$

where

$$||m_k|| := \sup_{\substack{||x_1||\leq 1,\\ \dots,\\ ||x_k||\leq 1}} ||m(x_1, ..., x_k)||$$

denotes the norm of the multilinear operator m_k. F is said to be real-analytic on $\mathcal{U}$ if F is real-analytic at x for all $x \in \mathcal{U}$.

Throughout this thesis, we sometimes omit the real when referring to real-analytic operators.

Next, we state the analytic version of the Implicit Function Theorem. For proof, we refer to Theorem 4.5.4 in [8].

Theorem 2 (Analytic Implicit Function Theorem)**.** *Let X, Y and Z be Banach spaces, $\mathcal{U}$ an open subset of $X \times Y$, and $F : \mathcal{U} \to Z$ an analytic operator. Assume (x_0, y_0) is a point in $\mathcal{U}$ satisfying the following two properties.*

(i) $F(x_0, y_0) = 0$*;*

(ii) $D_x F[x_0, y_0] : X \to Z$ *is an isomorphism.*

Then there exist open sets $V \subset X$ and $W \subset Y$ containing x_0 and y_0, respectively, and an analytic operator $\chi : W \to V$ such that $F(x, y) = 0$ for $(x, y) \in V \times W$ if and only if $x = \chi(y)$ for $(x, y) \in V \times W$.

It is interesting to note that a weaker result holds when the operator is not analytic. However, the Fréchet derivative needs to be continuous. Moreover, in such a case, the operators F and X above have the same level of regularity (see Theorem 3.5.4 in [8]).

We will now introduce Fredholm operators. For more information on Fredholm operators, we refer to Chapter 7.2 in [54].

Definition 4 (Fredholm operator)**.** Let X and Y be Banach spaces and $F : X \to Y$ a continuous linear operator. Then F is said to be a Fredholm operator if it satisfies the following three properties.

(i) $\ker F$ is finite-dimensional,

(ii) $\mathrm{ran}(F)$ is closed in Y,

(iii) $Y/\mathrm{ran}(F)$ is finite-dimensional.

The index of F is defined as $\dim(\ker F) - \dim(Y/\operatorname{ran}(F))$.

When the Fréchet derivative is not an isomorphism but is Fredholm, one can use projections and the Implicit Function Theorem to transform an infinite-dimensional problem into a finite-dimensional one. This process is known as Lyapunov-Schmidt reduction. For a proof, we refer to Theorem 8.2.1 in [8].

Theorem 3 (Analytic version of Lyapunov-Schmidt reduction)**.** *Let X and Y be Banach spaces over $\mathbb{R}$, $\mathcal{U}$ an open set of $X \times \mathbb{R}^n$, $(0, \mu_0) \in \mathcal{U}$ and $F : \mathcal{U} \to Y$ an analytic operator, satisfying*

(i) $F(0, \mu) = 0$, *for all* $\mu \in \mathbb{R}$;

(ii) $L \coloneqq D_x F[0, \mu_0]$ *is a Fredholm operator with kernel* X_0 *and range* Y_0;

(iii) $X_0 \neq \{0\}$ *and* $m \in \mathbb{N}$ *is the codimension of* Y_0.

Then there exist open sets $\mathcal{V} \subset \mathcal{U}$ *and* $\mathcal{W} \subset X_0 \times \mathbb{R}^n$, *and two analytic operators* $\chi : \mathcal{W} \to X$ *and* $h : \mathcal{W} \to \mathbb{R}^m$, *such that* $(0, \mu_0) \in \mathcal{V}$, $(0, \mu_0) \in \mathcal{W}$ *and* $\chi(0, \mu_0) = 0$. *Moreover,* $F(x, \mu) = 0$ *if and only if* $\chi(x_0, \mu) = x$ *for some* $(x_0, \mu) \in \mathcal{V}$ *with* $h(x_0, \mu) = 0$.

For convenience, we give an outline of the proof in the following remark.

Remark. Let X_1 and Y_1 denote complements of X_0 and Y_0, respectively, in the sense that $X_0 \oplus X_1 = X$ and $Y_0 \oplus Y_1 = Y$, where $\oplus$ denotes the topological direct sum operator. Additionally, define $P : X \to X_0$, $Q : Y \to Y_0$, $(I - P) : X \to X_1$ and $(I - Q) : Y \to Y_1$ be projection mappings. Define $x_0 = Px$ and $x_1 = (I - P)x$, this implies $x = x_0 + x_1$. Then the problem $F(x, \mu) = 0$ is equivalent to

$$QL(x_1) + QN(x_0 + x_1, \mu) = 0, \tag{1a}$$

$$(I - Q)N(x_0 + x_1, \mu) = 0, \tag{1b}$$

where $N \coloneqq F - L$. Since QL is an isomorphism, we apply Theorem 2 to (1a), yielding that $F(x, \mu) = 0$ is equivalent to the finite-dimensional problem

$$(I - Q)\left(N\left(x_0 + \tilde{X}(x_0, \mu), \mu\right)\right) = 0, \tag{2}$$

where $\tilde{X} : \mathcal{W} \to X_1$. Moreover, the analytic map h is defined as the left-hand side of (2) and the analytic map χ is given by $x_0 + \tilde{X}(x_0, \mu)$.

Next, we state an analytic version of the Crandall-Rabinowitz theorem. It can be proved using Lyapunov-Schmidt's reduction as above. See Theorem 8.3.1 in [8] for more details.

Theorem 4 (Analytic version of the Crandall-Rabinowitz theorem)**.** *Let X and Y be Banach spaces, $\mu_0 \in \mathbb{R}$ and $F : X \times \mathbb{R} \to Y$ an analytic operator satisfying the following three properties.*

(i) $F(0, \mu) = 0$, *for all* $\mu \in \mathbb{R}$,

(ii) $D_x F[0, \mu_0]$ *is a Fredholm operator of index zero with a one-dimensional kernel spanned by some* $x_0 \in X$,

(iii) $D_{x\mu} F[0, \mu_0](x_0) \notin \mathrm{ran}(D_x F[0, \mu_0])$ *(transversality condition).*

Then there exist $\epsilon > 0$ and a curve $(\chi(s), \mu(s))$ where $\chi(s) = sx_0 + \mathcal{O}(s^2)$ and $\mu(s) = \mu_0 + \mathcal{O}(s)$ such that $F(\chi(s), \mu(s)) = 0$ for $s \in (-\epsilon, \epsilon)$. In addition, there exists a neighbourhood of $(0, \mu_0)$ where these are the only non-trivial solutions.

Moreover, $\chi(s)$ and $\mu(s)$ are analytic functions from $(-\epsilon, \epsilon)$ to X and $\mathbb{R}$, respectively.

The next Proposition states formulas for the first and second derivative of the parameter map under the same assumptions as the Crandall-Rabinowitz Theorem. For a proof, we refer to Section I.6 of [33].

Proposition 5. *Under the assumptions of Theorem 4, we have that*

$$\frac{\partial}{\partial s}\mu(0) = -\frac{1}{2}\frac{(I-Q)D^2_{xx}F[0,\mu_0](x_0)^2}{(I-Q)D_{x\mu}F[0,\mu_0](x_0)}$$

and

$$\frac{\partial^2}{\partial s^2}\mu(0) = -\frac{1}{3}\frac{(I-Q)D^3_{xxx}F[0,\mu_0](x_0)^3}{(I-Q)D_{x\mu}F[0,\mu_0](x_0)},$$

where $(I - Q)$ denotes a projection onto the complement of the range of $D_x F[0, \mu_0]$.

The next Proposition is a modified version of Theorem 4 for the case when there exists one extra parameter, the proof is very similar to the proof of the Crandall-Rabinowitz Theorem.

Proposition 6. *Let X and Y be Banach spaces, $\mu_0, \lambda_0 \in \mathbb{R}$ and $F : X \times \mathbb{R} \times \mathbb{R} \to Y$ an analytic operator satisfying the following three properties.*

(i) $F(0, \mu, \lambda) = 0$, *for all* $\mu, \lambda \in \mathbb{R}$;

(ii) $D_x F[0, \mu_0, \lambda_0]$ *is a Fredholm operator of index zero with a one-dimensional kernel spanned by* $x_0 \in X$;

(iii) $D_{x\mu} F[0, \mu_0, \lambda_0](x_0) \notin \mathrm{ran}(D_x F[0, \mu_0, \lambda_0])$ *(transversality condition).*

Then there exist $\epsilon > 0$ *and a curve* $(\chi(s, \lambda), \mu(s, \lambda), \lambda)$, *where* $\chi(s, \lambda) = sx_0 + \mathcal{O}\left(s^2 + s(\lambda - \lambda_0)\right)$ *and* $\mu(s, \lambda) = \mu_0 + \mathcal{O}\left(s + (\lambda - \lambda_0)\right)$ *such that* $F(\chi(s, \lambda); \mu(s, \lambda), \lambda) = 0$ *for* $s, (\lambda - \lambda_0) \in (-\epsilon, \epsilon)$. *In addition, there exists a neighbourhood of* $(0, \mu_0, \lambda_0)$ *where these are the only non-trivial solutions.*

Moreover, $\chi(s, \lambda)$ *and* $\mu(s, \lambda)$ *are analytic functions from* $(-\epsilon, \epsilon)^2$ *to* X *and* $\mathbb{R}$, *respectively.*

Proof. We apply Lyapunov-Schmidt reduction to the problem and obtain $\mathcal{V}, \mathcal{W}, \chi$ and h as in the Theorem 3. Since $F(0, \mu, \lambda) = 0$ for all μ and λ, it follows $\chi(0, \lambda, \mu) = 0$. We then define $g : \mathcal{V} \to \mathbb{R}$ by

$$g(\xi, \mu, \lambda) = \int_0^1 D_\xi h[t\xi, \mu, \lambda] x_0 dt.$$

Since $g(0, \mu_0, \lambda_0) = 0$ and $D_\mu g(0, \mu_0, \lambda_0) = D^2_{\xi,\mu} h[0, \mu_0, \lambda_0](x_0) \neq 0$, we can apply the Implicit Function Theorem to $g(\xi, \mu, \lambda) = 0$. Therefore, there exists an $\epsilon > 0$ and a real-analytic map $\mu : (-\epsilon, \epsilon) \times (\lambda_0 - \epsilon, \lambda_0 + \epsilon) \to \mathbb{R}$ such that $\mu(0, \lambda_0) = \mu_0$ and $g(sx_0, \mu(s, \lambda), \lambda) = 0$ for all $s \in (-\epsilon, \epsilon)$ and $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. Moreover, note that

$$g(sx_0, \mu, \lambda) = \begin{cases} \dfrac{h(sx_o, \mu, \lambda)}{s} & \text{if } s \neq 0, \\ D_\xi h(0, \mu, \lambda) & \text{if } s = 0, \end{cases}$$

and therefore $h(sx_0, \mu(s, \lambda), \lambda) = 0$ for $(s, \lambda) \in (-\epsilon, \epsilon) \times (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. Furthermore, it follows that

$$F(\chi(s, \lambda), \mu(s, \lambda), \lambda) = 0,$$

where $\chi(s, \lambda) \coloneqq \chi(s, \mu(\lambda), \lambda)$. The formulas for χ and μ stated in the proposition follow from the Taylor expansions of both instances where the Implicit Function Theorem is used. $\square$

Corollary 7. *Assuming the same conditions as stated in Proposition 6, and further, that there exists a continuous curve of parameters* $S := \{(\mu(t), \lambda(t)) : t \in (-\delta, \delta)\}$*, satisfying the following four properties.*

(i) $(\mu(0), \lambda(0)) = (\mu_0, \lambda_0)$*;*

(ii) For all $t \in (-\delta, \delta)$*,* $D_x F[0, \mu(t), \lambda(t)]$ *is a Fredholm operator of index zero with one-dimensional kernel spanned by* $x_0 \in X$*;*

(iii) For all $t \in (-\delta, \delta)$*,* $D_{x\mu} F[0, \mu(t), \lambda(t)](x_0) \notin \operatorname{ran}(D_x F[0, \mu_0, \lambda_0])$ *(transversality condition)*

(iv) The mapping $t \to D_{x\mu} F[0, \mu(t), \lambda(t)](x_0)$ *is continuous.*

Then, there exists $\epsilon > 0$ *and a curve* $(\chi(s, \lambda), \mu(s, \lambda), \lambda)$ *where* $\chi(s, \lambda) = sx_0 + \mathcal{O}\left(s^2 + s(\lambda - \lambda_0)\right)$ *and* $\mu(s, \lambda) = \mu_0 + \mathcal{O}\left(s + (\lambda - \lambda_0)\right)$ *such that* $F(\chi(s, \lambda); \mu(s, \lambda), \lambda) = 0$ *for* $s, (\lambda - \lambda_0) \in (-\epsilon, \epsilon)$*.*

Moreover, for every fixed $\lambda(t) \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$*, the curve* $(\chi(s, \lambda(t)), \mu(s, \lambda(t))$ *is the same as the curve obtained by applying Crandall-Rabinowitz at the point* $(0, \mu(t), \lambda(t))$*. In addition, there exists a neighbourhood of* $(0, \mu_0, \lambda_0)$ *where these are the only non-trivial solutions not contained in* S*.*

This statement follows from applying Theorem 4 at every point along the parameter curve while keeping λ fixed, and then shrinking the neighbourhoods so that the uniqueness from Proposition 6 implies that the new solutions are the same as those obtained from Proposition 6.

2.3 Hilbert transform

In water wave theory, it is common to use the Hilbert transform to restate the problem. We will state the results used throughout this thesis without proof. For a detailed discussion, we refer to [35].

Before introducing the Hilbert transform, we recall the notion of a Hölder space.

Definition 5 (Hölder space)**.** Let $\mathcal{U} \subset \mathbb{R}^n$ be an open set, $\alpha \in (0, 1)$ and $f : \mathcal{U} \to \mathbb{R}$. Then the Hölder seminorm of f is defined by

$$[f]_{C^\alpha(\mathcal{U})} := \sup_{\substack{x,y \in \mathcal{U} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Furthermore, if k is a non-negative integer, then the Hölder space $C^{k+\alpha}(\mathcal{U})$ is the space of functions $f \in C^k(\mathcal{U})$, satisfying

$$||f||_{C^{k+\alpha}(\mathcal{U})} \coloneqq \sum_{|\alpha|\leq k} ||D^\alpha f||_{C(\mathcal{U})} + \sum_{|\alpha|=k} [D^\alpha f]_{C^\alpha(\mathcal{U})} < \infty.$$

For more on Hölder spaces, we refer to Section 5.1 in [26]. Next, we define the Hilbert transform on the span of $\{\exp(inx)\}_{n\in\mathbb{Z}}$ by

$$\mathcal{H}(\exp(inx)) = \begin{cases} -i\,\mathrm{sign}\,(inx)\exp(inx), & \text{if } n \neq 0 \\ 0, & \text{if } n = 0 \end{cases}$$

Let f be a 2π periodic, real-valued Hölder function, then

$$\begin{aligned} f(x) &= \sum_{n\in\mathbb{Z}} \hat{f}(n)\exp(inx) = \hat{f}(0) + \sum_{n=1}^{\infty} \hat{f}(n)\exp(inx) + \sum_{n=1}^{\infty} \overline{\hat{f}(n)\exp(inx)} \\ &= \hat{f}(0) + 2\Re\left(\sum_{n=1}^{\infty} \hat{f}(n)\exp(inx)\right). \end{aligned}$$

Notice f can be seen as the real part of the boundary value of a holomorphic function $h_{\mathbb{D}} \to \mathbb{C}$, given by

$$h(z) = \hat{f}(0) + 2\sum_{n=1}^{\infty} \hat{f}(n)z^n.$$

Moreover, a short computation reveals

$$\Im\left(h(\exp(ix))\right) = \mathcal{H}f(x).$$

Next, we note the Periodic Hilbert transform has a singular integral representation of the form

$$\mathcal{H}(f)(t) = \frac{1}{\pi}\text{p.v.}\int_{-\pi}^{\pi} \frac{f(x)}{\tan((t-x)/2)}dx,$$

where p.v. denote the Cauchy principal value. This can be shown by writing out the Fourier series coefficients and switching the order of summation and integration.

Next, we state Privalov's theorem, originally proven in [50]. We refer to Chapter 8, Section 13 of [5] for a proof in English.

Theorem 8 (Privalov's theorem)**.** *Let* $\alpha \in (0, 1)$ *and*

$$C^{\alpha}_{2\pi}(\mathbb{R}) \coloneqq \{f \in C^{\alpha}(\mathbb{R}) : f \text{ is } 2\pi \text{ periodic}\}.$$

Then $\mathcal{H} : C^{\alpha}_{2\pi}(\mathbb{R}) \to C^{\alpha}_{2\pi}(\mathbb{R})$ *is a bounded linear operator.*

3 Water wave problem

In this section, we formulate the water wave problem. We introduce Euler's equations of motion and two-dimensional steady waves. Afterwards, in the second subsection, we formulate the free boundary problem we will consider throughout the thesis. In the third subsection, we prove the existence of a conformal map between the half-plane and the water domain. Finally, in the fourth subsection, we derive a modified version of the Babenko equation for the case of constant vorticity and no capillarity.

3.1 Two-dimensional steady water waves

Euler's equations. Let D be a simply connected, time-dependent domain in $\mathbb{R}^3$. Define the velocity vector $\vec{u} : D \to \mathbb{R}^3$, the pressure $P : D \to \mathbb{R}$, the density ρ and the gravitational constant g. Then the Euler equations are given by

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + \frac{\nabla P}{\rho} - g\vec{e_y} = 0 \qquad \text{in } D, \tag{3a}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \qquad \text{in } D, \tag{3b}$$

$$\nabla \cdot \vec{u} = 0 \qquad \text{in } D, \tag{3c}$$

where ∇ denotes the spatial derivatives. We assume the density constant. In Cartesian coordinates, we consider gravity acting along the y-axis in the negative direction.

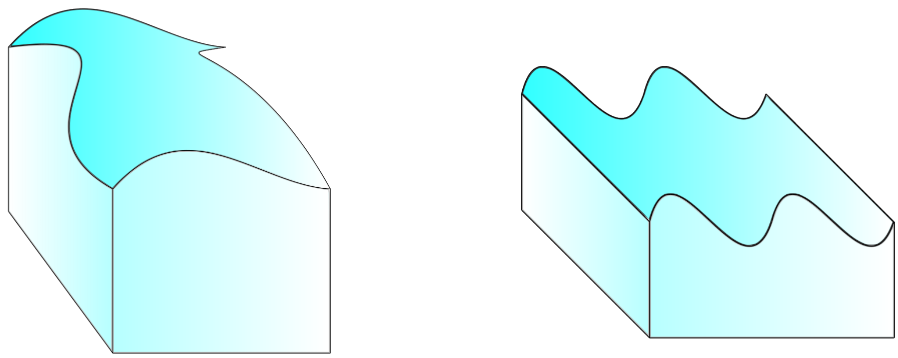
Two-dimensional water waves. We assume there is no fluid motion in the z-axis, that is

$$\vec{u} = (u_1, u_2, 0) \qquad \text{in } D$$

and that flow is constant in the z-direction, as in

$$\vec{u}(x, y, z_1) = \vec{u}(x, y, z_2),$$

for all z_1, z_2 such that (x, y, z_1) and (x, y, z_2) are in D. Similarly, we assume that the pressure remains constant in the z-direction. Under these assumptions, our problem is independent of z and



(a) Part of a three-dimensional domain. (b) Part of a two-dimensional domain.

Figure 1: Examples of three and two-dimensional domains.

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we can consider a two-dimensional problem. See Figure 1 for examples of two and three-dimensional domains.

Steady water waves. We assume the wave is travelling with constant speed c along the x-axis in the positive direction. We perform the change of variables

$$(x, y; t) \to (\tilde{x}, y; t),$$

where $\tilde{x} = x - ct$ and define $\tilde{u}(\tilde{x}, y; t) \coloneqq \vec{u}(x - ct, y; t)$. Furthermore, we assume $\tilde{u}$ can be expressed as a function of $(\tilde{x}, y)$, meaning there exists v such that

$$\tilde{u}(\tilde{x}, y, t) = \vec{v}(\tilde{x}, y),$$

for all $t \in (0, \infty)$. Similarly, we assume the pressure can be expressed as a function of $(\tilde{x}, y)$. Then

$$(\vec{v} \cdot \nabla_{x,y})\vec{v} - \frac{\partial \vec{v}}{\partial t} = (\vec{v} \cdot \nabla_{\tilde{x},y} - c\vec{e}_{\tilde{x}})\vec{v}.$$

Defining $\vec{w} = \vec{v} - c\vec{e}_{\tilde{x}}$, it follows that

$$(\vec{w} \cdot \nabla_{\tilde{x},y})\vec{w} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla_{x,y})\vec{u}.$$

Now, we can simplify the Euler equations (3) into the time-independent problem

$$(\vec{w} \cdot \nabla)\vec{w} + \frac{\nabla P}{\rho} - g\vec{e_y} = 0 \qquad \text{in } D, \tag{4a}$$

$$\nabla \cdot \vec{w} = 0 \qquad \text{in } D, \tag{4b}$$

where the derivatives are with respect to the spatial coordinates $\tilde{x}$ and y.

3.2 Free boundary problem

Let $D \subset \mathbb{R}^2$ be a simply connected domain bounded from above by a free surface S, periodic in the x direction with period $2\pi/k$ and symmetric with respect to the vertical lines below the crest and the trough. Mathematically, we can write this as

$$S \coloneqq \{(x(t), y(t)) : t \in \mathbb{R}\} \tag{5}$$

with $x(t)$ and $y(t)$ satisfying

$$-x(-t) = x(t) = x(t - \frac{2\pi}{k}) + \frac{2\pi}{k} \quad \text{and} \quad y(-t) = y(t) = y(t + \frac{2\pi}{k}).$$

If $\vec{w} = (u, v, 0)$, then (4) becomes

$$\rho(uu_x + vu_y) = -P_x \qquad \text{in } D, \tag{6a}$$

$$\rho(uv_x + vv_y) = -P_y - g\rho \qquad \text{in } D, \tag{6b}$$

$$u_x + v_y = 0 \qquad \text{in } D. \tag{6c}$$

Since we are interested in periodic and symmetric solutions, we also make the assumptions

$$u(-x, y) = u(x, y) = u(x + \frac{2\pi}{k}, y), \qquad -v(-x, y) = v(x, y) = v(x + \frac{2\pi}{k}, y)$$

By using Equation (6c), we can define a stream function ψ, which satisfies $\psi_y = u$ and $-\psi_x = v$. Notice that ψ is not unique and is defined up to an additive constant.

The vorticity describes how much a fluid particle rotates around a point. It is defined as the curl of the velocity vector, $\vec{\omega} = \nabla \times \vec{w}$. In the two-dimensional case, we can express it as $(0, 0, \omega)$, where

$$\omega = v_x - u_y = -\triangle\psi \qquad \text{in } D. \tag{7}$$

Throughout this thesis, we will assume ω to be constant.

Kinematic boundary condition. The kinematic boundary condition states that each fluid particle that is on the boundary remains on the boundary. On the surface S, it can be expressed in any of the following manners

$$uy_t - vx_t = 0 \qquad \text{on } S, \tag{8a}$$

$$\psi_y y_t + \psi_x x_t = 0 \qquad \text{on } S, \tag{8b}$$

where the subscript t denotes the derivative with respect to t.

Equation (8) implies ψ is constant on S. We choose the constant mentioned in the discussion after (6) so that ψ is equal to zero on S.

There are two cases commonly considered in the literature, finitely deep and infinitely deep water waves. These cases lead to two alternative kinematic boundary conditions,

$$v = 0 \qquad \text{at } y = -h, \tag{9a}$$

$$(u, v) - (-\omega y - c, 0) \to (0, 0) \qquad \text{as } y \to -\infty. \tag{9b}$$

The case (9a) corresponds to D having a flat bottom at $y = -h$. The infinite depth case (9b) is a normalization of (9a) for large h and states that at great depths, the fluid motion is only horizontal. We note that for every constant C, the pair $(u, v) = (-\omega y - C, 0)$ is a solution to the Euler equations. We choose to set the constant equal to the wave speed since when $\omega = 0$, this corresponds to there being no fluid motion at the infinite bottom. Unless otherwise stated, we will consider (9b).

Dynamic boundary condition. Assuming no air motion, the dynamic boundary condition states that pressure must remain constant across the fluid's boundary and can be expressed as follows

$$P = \text{const} - T\kappa \qquad \text{on } S, \tag{10}$$

where T denotes the coefficient of surface tension and

$$\kappa = \frac{x_t y_{tt} - y_t x_{tt}}{(x_t^2 + y_t^2)^{\frac{3}{2}}} \qquad \text{on } S$$

stands for the curvature. Using the vector calculus identity

$$(u \cdot \nabla)u = \frac{1}{2}\nabla|u|^2 + (\nabla \times u) \times u = 0,$$

we can express the first two equations of (6) as

$$\frac{\partial}{\partial x}\left(\frac{1}{2}|\nabla\psi|^2 + \omega\psi + \frac{P}{\rho} + gy\right) = 0 \qquad \text{in } D, \tag{11a}$$
$$\frac{\partial}{\partial y}\left(\frac{1}{2}|\nabla\psi|^2 + \omega\psi + \frac{P}{\rho} + gy\right) = 0 \qquad \text{in } D. \tag{11b}$$

Integrating (11) yields

$$\frac{1}{2}|\nabla\psi|^2 + \omega\psi + gy + \frac{P}{\rho} = \text{const} \qquad \text{in } D. \tag{12}$$

Now (10) and (12) give

$$\frac{1}{2}|\nabla\psi|^2 + \omega\psi + gy - T\kappa = b \qquad \text{on } S, \tag{13}$$

where b represents the Bernoulli constant, which is a fixed value encompassing both the pressure and the integration constant.

Nondimensional variables. We now make the change of variables

$$x \to kx, \qquad y \to ky, \qquad \psi \to \frac{k}{c}\psi.$$

and introduce dimensionless parameters

$$\Omega = \frac{\omega}{ck}, \qquad \tau = \frac{Tk}{\rho c^2}, \qquad G = \frac{g}{kc^2}, \qquad B = \frac{b}{c^2\rho}.$$

In the non-dimensional variables, equations (7), (8), (9b) and (13) become

$$\begin{aligned}
&\triangle\psi = -\Omega && \text{in } D, && \text{(14a)}\\
&\psi = 0 && \text{on } S, && \text{(14b)}\\
&\nabla\psi - (0, -\Omega y - 1) \to (0,0) && \text{as } y \to -\infty, && \text{(14c)}\\
&\frac{1}{2}|\nabla\psi|^2 + Gy - \tau\kappa = B && \text{on } S. && \text{(14d)}
\end{aligned}$$

The problem (14) can be seen illustrated on the right-hand side of Figure 3. By setting the pressure such that the expression within the parentheses of (11) becomes zero, we can transition from a solution of (14) to a solution of the Euler equations.

3.3 Conformal map from the lower half plane

We recall that a map between two open subsets of $\mathbb{C}$ is conformal if and only if it is biholomorphic. Define the lower complex half-plane by

$$\mathbb{H}^- := \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}; \beta < 0\}.$$

Lemma 9. *Let S be as in (5), and D be a simply connected, 2π periodic domain that's bounded from above by S. Then there exists a conformal map z between the lower complex half-plane $\mathbb{H}^-$ and D satisfying the following properties.*

(i) z extends continuously to $\{\beta = 0\}$ and maps $\{\beta = 0\}$ to S;

(ii) $x(\alpha + i\beta) - \alpha$ and $y(\alpha + i\beta)$ are 2π periodic functions of α;

(iii) $z(\alpha + i\beta) - (\alpha + i\beta) \to 0$ as $\beta \to -\infty$.

where $x(\alpha + i\beta)$ and $y(\alpha + i\beta)$ are the real and imaginary parts of $z(\alpha + i\beta)$, respectively.

We note that our argument is inspired by a proof in the appendix of [13]. Moreover, we note Theorem 2.6 and Corollary 2.7 of [49], originally proven by Carathéodory [9], ensure the existence, uniqueness and extension to the boundary of the map Φ_3 below.

Proof. Define

$$\begin{aligned}\mathbb{H}_c &:= \{x + iy : x, y \in \mathbb{R}; y < c\},\\ \mathbb{H}_c^{\mathrm{per}} &:= \{x + iy : x, y \in \mathbb{R}; -\pi \leq x \leq \pi, y < c\},\\ D^{\mathrm{per}} &:= \{x + iy : x, y \in D; -\pi \leq x \leq \pi\}.\end{aligned}$$

Now, fix d to be a real constant large enough such that $D \cup S \subset \mathbb{H}_d$. Afterwards, define the conformal maps $\Phi_1 : \mathbb{D} \to \mathbb{H}_0^{\mathrm{per}}$ and $\Phi_2 : \mathbb{D} \to \mathbb{H}_d^{\mathrm{per}}$ by

$$\begin{aligned}\Phi_1(\zeta) &:= i \log \zeta;\\ \Phi_2(\zeta) &:= i \log(\zeta) + id,\end{aligned}$$

where log denotes the principal branch of the logarithm. Note that the boundary of $\Phi_2^{-1}(D^{\mathrm{per}})$ is contained in $\mathbb{D}$, is symmetric over the real-axis and intersects it at a positive number a and a negative number b. Moreover, note that both Φ_1 and Φ_2 map the negative real axis to the set where $x = -\pi$ and the positive real axis to the set where $x = \pi$. Let $\Phi_3 : \mathbb{D} \to \Phi_2^{-1}(D^{\mathrm{per}})$ be the conformal map with homeomorphic extension to the boundary, satisfying $\Phi_3(0) = 0$, $\Phi_3(1) = a$ and $\Phi_3(-1) = b$. The uniqueness, coupled with $\Phi_2^{-1}(D^{\mathrm{per}})$ being symmetric imply $\Phi_3(\overline{z}) = \overline{\Phi_3(z)}$ and therefore Φ_3 maps $[-1, 1]$ to a subset of $[-1, 1]$. Define the conformal map $z_0 : \mathbb{H}_0^{\mathrm{per}} \to D^{\mathrm{per}}$, which has a homeomorphic extension to the boundary, by

$$z_0(\alpha + i\beta) := \Phi_2 \circ \Phi_3^{-1} \circ \Phi_1^{-1}(\alpha + i\beta).$$

We note there is an illustration of this map in Figure 2. Moreover, the construction ensures that the subset of $\mathbb{H}_0^{\mathrm{per}}$ where $x = \pi$ gets mapped to the subset of D^{per} where $x = \pi$ and similarly for the subsets where $x = -\pi$. By a similar argument, for all $k \in \mathbb{Z}$, we can construct conformal maps

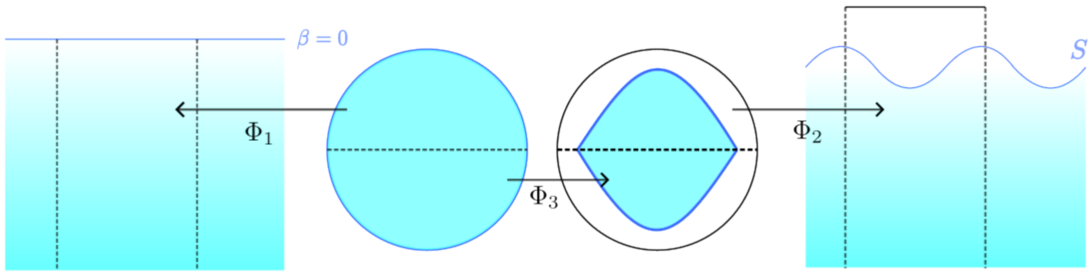

Figure 2: Illustration of the proof of Lemma 9.

$z_k : \mathbb{H}_0^{\mathrm{per}} + 2k\pi \to D^{\mathrm{per}} + 2k\pi$. We then define $z : \mathbb{H}^- \to D$ by

$$z(\alpha + i\beta) := z_k(\alpha + i\beta), \qquad \text{for } \alpha \in \Big((2k-1)\pi, (2k+1)\pi\Big].$$

It follows that (ii) is satisfied, since for all $k \in \mathbb{Z}$, the subsets $\{(2k+1)\pi + i\beta : \beta < 0\}$ get mapped to $\{(2k+1)\pi + i\beta :\} \cap D$. It remains to show z is holomorphic on the sets $\{(2k+1)\pi + i\beta : \beta < 0\}$ for all $k \in \mathbb{Z}$. Let ℓ be one of these lines, and consider a domain D_ℓ satisfying $\ell \subset D_\ell \subset D$. Let T be a triangle contained in D_ℓ, then there are three cases to consider.

- If T does not intersect with ℓ, it follows that $\oint_T z d(\alpha + i\beta) = 0$;
- If T has one edge on ℓ, then we can approximate T with triangles that do not intersect ℓ and use uniform continuity on compact subsets to obtain $\oint_T z d(\alpha + i\beta) = 0$;
- If T intersects ℓ not at an edge, we can add integrals along line segments in opposite directions and split the integral over T into the integral over two or three triangles of the previous two cases. The number of triangles depends on whether T intersects ℓ at a vertex or not.

By Morera's theorem, it follows that z is holomorphic on ℓ. Since ℓ was an arbitrary line of the form $\{(2k+1)\pi + i\beta : \beta < 0\}$, it follows z is holomorphic on D. $\square$

See Figure 3 for an illustration of this map. We also note that z is holomorphic on $\mathbb{H}^-$ and therefore x and y satisfy the Cauchy-Riemann equations

$$x_\alpha = y_\beta \qquad \text{and} \qquad x_\beta = -y_\alpha.$$

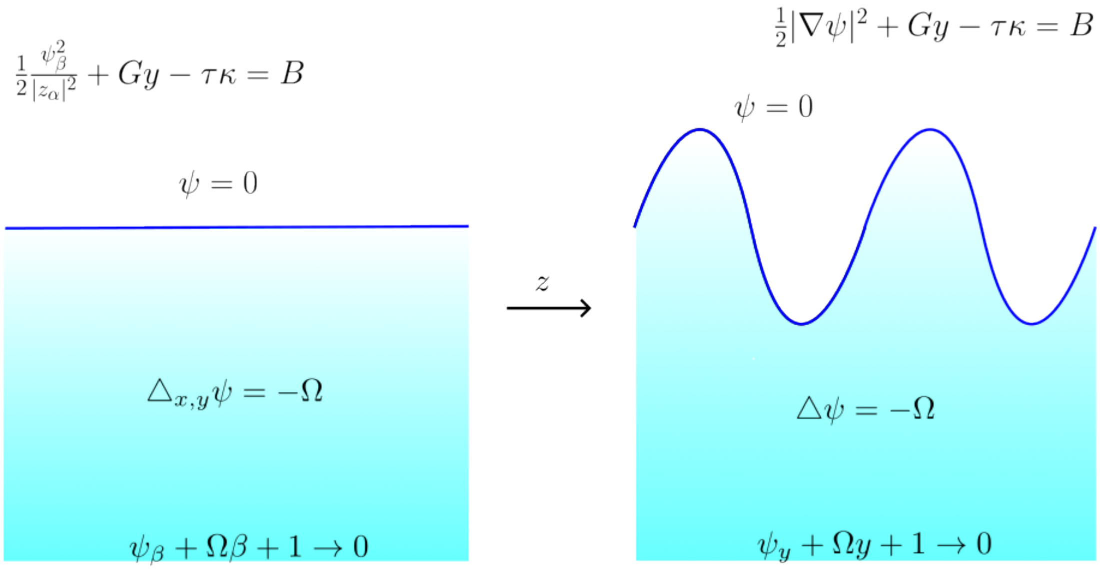

Figure 3: Conformal map between the half plane and D.

In the new coordinates, we can rewrite (14) as

$$|z_\alpha|^{-2}\,\triangle_{\alpha,\beta}\psi = -\Omega \qquad \text{in } \mathbb{H}^-, \tag{15a}$$

$$\psi = 0 \qquad \text{on } \beta = 0, \tag{15b}$$

$$\nabla\psi - (0, -\Omega\beta - 1) \to (0,0) \qquad \text{as } \beta \to -\infty, \tag{15c}$$

$$\frac{1}{2}\frac{\psi_\beta^2}{|z_\alpha|^2} + Gy - \tau\kappa = B \qquad \text{on } \beta = 0. \tag{15d}$$

This problem can be seen illustrated on the left-hand side of Figure 3.

Next, we will state a result concerning the analyticity of the free surface S and analytic extensions of the stream function ψ. Constantin and Escher [11] derived the primary result on the analyticity of the free surface, assuming a fluid devoid of stagnation points and with a surface that can be represented as a graph. Notably, their analysis permits the vorticity to be a real analytic function. With the additional assumption of the vorticity being affine, Aasen and Varholm (Theorem 2.5 in [1]) were able to prove to lower one of the assumptions to the fluid having no stagnation points on the boundary. Our proof is a modification of that in [1], using a local argument to allow for the possibility of the surface S not being the graph of a function.

Proposition 10. *Let z be the conformal map obtained in Lemma 9 and ψ be a solution of (15). Assume the free surface S is a C^1 curve and that z_α has no zeros on the set $\{\beta = 0\}$. Then the following three properties hold.*

(i) The free surface S is real analytic;

(ii) The map z has a complex analytic extension to an open neighbourhood of $\mathbb{H}^- \cup \{\beta = 0\}$;

(iii) The stream function ψ has a real analytic extension that satisfies $\triangle\psi = -\Omega$ to an open neighbourhood of $\mathbb{H}^- \cup \{\beta = 0\}$.

Proof. (i) Let p be an arbitrary point on S. Then there exists an open set U contained in D, such that $p \in \partial U \cap S$ and $S \cap U$ can be represented as either $(x, \eta(x))$ or $(\eta(y), y)$ for some function η. Applying Theorem 2 of [34] with u as the stream function and g as the dynamic boundary condition yields that $\partial U \cap S$ is analytic. Since p was an arbitrary point, it follows that S is real analytic.

(ii) Follows from applying Theorem 2.2 on page 299 in [39].

(iii) Since $\triangle\psi + \Omega = 0$ in D and $\psi = 0$ on the real analytic curve S, we can apply Theorem A in [42] to yield the claim.

□

3.4 A modified Babenko equation

In this subsection, we will derive a modified version of the Babenko equation, as in [23] or [24]. Assume $\tau = 0$ and let z be the conformal map from Lemma 9. Consider the holomorphic function $(x + iy)(\alpha + i\beta) - (\alpha + i\beta)$, it follows from the discussion in Section 2.3 that

$$x(\alpha + i0) - \alpha = \mathcal{H}(y)(\alpha + i0), \tag{16}$$

which implies

$$z(\alpha + i0) = \alpha + (i + \mathcal{H})(y)(\alpha + i0). \tag{17}$$

Similarly, we define ϕ as $\phi = \psi + \Omega\frac{y^2}{2} + y$. Then ϕ is a harmonic function. Let φ be the harmonic conjugate of $-\phi$, in the sense that $\varphi + i\phi$ is a holomorphic function on $\mathbb{H}^-$. Then it follows from the discussion in Section 2.3 that

$$(\varphi + i\phi)(\alpha + 0i) = (\mathcal{H} + i)\phi(\alpha + 0i). \tag{18}$$

Notice that (15b) implies

$$\phi_\alpha = \Omega y y_\alpha + y_\alpha \qquad \text{on } \beta = 0. \tag{19}$$

By taking Hilbert transforms of both sides of (19) and by (18), we obtain

$$\varphi_\alpha = \Omega\mathcal{H}(y y_\alpha) + \mathcal{H}y_\alpha. \tag{20}$$

Provided $z_\alpha \neq 0$ on $\{\beta = 0\}$, we can rewrite (15d) as

$$\frac{1}{2}\psi_\beta^2 = (B - Gy)\Big((1 + \mathcal{H}y_\alpha)^2 + y_\alpha^2\Big) \qquad \text{on } \beta = 0. \tag{21}$$

Next, we use the Cauchy-Riemann equations to compute

$$\begin{aligned} \psi_\beta &= \phi_\beta - \Omega y y_\beta - y_\beta \\ &= \varphi_\alpha - \Omega y x_\alpha - x_\alpha \qquad \text{in } \mathbb{H}^-. \end{aligned} \tag{22}$$

Therefore, we can plug the expression obtained for ψ_β in (22) in (21), yielding

$$\frac{1}{2}(\varphi_\alpha - \Omega y x_\alpha - x_\alpha)^2 = (B - Gy)\Big((1 + \mathcal{H}y_\alpha)^2 + y_\alpha^2\Big) \qquad \text{on } \beta = 0. \tag{23}$$

Finally, differentiating (20) and (16) with respect to α and plugging the obtained expressions for φ_α and x_α in (23) yields

$$\frac{1}{2}(1 + \Omega(y + y\mathcal{H}y_\alpha - \mathcal{H}(y y_\alpha)))^2 = (B - Gy)((1 + \mathcal{H}y_\alpha)^2 + y_\alpha^2) \qquad \text{on } \beta = 0. \tag{24}$$

A smooth solution of (24) generates a solution of (14), provided that $z(\alpha + i0)$ is injective and satisfies $z_\alpha(\alpha + i0) \neq 0$ for all $\alpha \in \mathbb{R}$.

This formulation using the Hilbert transform comes originally from [4], in which Babenko derived a similar equation for the zero vorticity case, which was studied in [6] and [7], among others. We also note a similar equation was derived in [12] for the finite depth case.

4 Previous work

In this section, we present some of the earlier results that were mentioned in the introduction. In Section 4.1, we discuss solutions found by Crapper [17] for the case of pure capillary waves (no gravity) over an irrotational flow. In Section 4.2, we provide a concise overview of the solutions discovered by Hur & Wheeler [30] for the rotational case in the absence of gravity and surface tension. Finally, in Section 4.3, we give a summary of how Hur & Wheeler [31] used an implicit function theorem to find nearby solutions for small but nonzero gravity.

4.1 Pure capillary waves over an irrotational flow

In this subsection, we will assume zero gravity and vorticity. The solutions discussed here were initially derived by Crapper [17], although the following exposition follows [30].

Let z, x, y, α and β be as in Lemma 9 and the preceding discussion. Define

$$\psi = -\beta. \tag{25}$$

It is a straightforward computation to verify this choice of ψ satisfies the first three equations of (15). We note that

$$|\nabla_{x,y}\psi|^2 = \frac{\psi_\beta^2}{|z_\alpha|^2} \qquad \text{on } \beta = 0 \tag{26}$$

and thus (14d) becomes

$$\frac{1}{2|z_\alpha|^2} - \tau\kappa = B \qquad \text{on } \beta = 0. \tag{27}$$

It is now a matter of a direct computation to verify that for each non-negative constant A, the map

$$z(\alpha + i0; A) := \alpha - 4iA\frac{\exp(-i\alpha)}{1 + A\exp(-i\alpha)} \tag{28}$$

is a solution of (27). The corresponding surface tension and Bernoulli constant are given by

$$\tau = \frac{1 + A^2}{1 - A^2}, \qquad B = \frac{1}{2}. \tag{29}$$

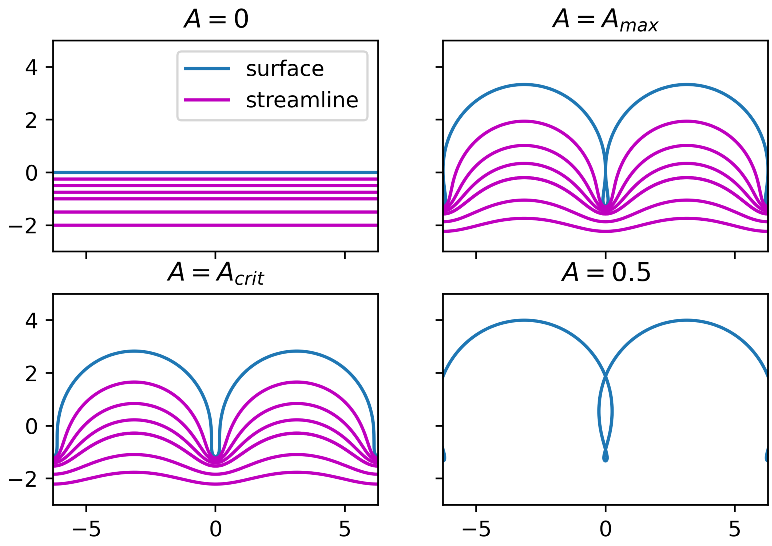

Figure 4: Streamlines of Crapper waves for laminar flow, touching waves, and breaking waves. Along with a plot of the surface of a non-physical solution.

Note that for $A > A_{\max} \approx 0.4546700164520109$, the solutions intersect themselves, and thus they are not physically realistic. At the point $A_{\max}$, the surface of the wave touches without the fluid being double valued. This is illustrated in Figure 4.

We also note the conformal extension of (28) for $A \in [0, A_{\max}]$ is given by

$$z(\alpha + i\beta; A) := \alpha + i\beta - 4iA\frac{\exp(-i(\alpha + i\beta))}{1 + A\exp(-i(\alpha + i\beta))}. \tag{30}$$

It is worth noting that Kinnersley [36] discovered finite depth analogues of these solutions.

4.2 Rotational flows without gravity

In this subsection, we will assume surface tension and gravitational forces to be zero. Under these assumptions, Dyachenko & Hur [24] and [23], as well as Hur & Vanden-Broeck [29], found numerical evidence of solutions with the same wave profile as Crapper's solutions. This was then rigorously justified by Hur & Wheeler [30]. We will give an overview of how to derive these solutions.

We will assume the conformal mapping z to the half plane is the same as in (30) and try to identify the corresponding stream function. Notice that we can express the stream function in the form $\psi = -\frac{1}{2}\Omega y^2 - y - f$, where f satisfies

$$\begin{aligned} &\triangle f = 0 && \text{in } D && (31a)\\ &f = -\frac{1}{2}\Omega y^2 - y && \text{on } S && (31b)\\ &\nabla f \to (0,0) && \text{as } y \to -\infty. && (31c) \end{aligned}$$

Then we express f as a function of $\zeta = \exp(-i(\alpha + i\beta))$, taking values in the complex unit disc and mapping S to the boundary of the complex unit disc. After that, we apply Poisson integral formula for a function satisfying (31b) and evaluate the integral using residue calculus. This will lead to the stream function

$$\psi(\alpha, \beta; A) := -\beta - \frac{1}{2}\Omega y^2 - \frac{4\Omega}{A^2 - 1}\Re\left(\frac{(1 - 2A^2)\zeta^2 + 1}{(\zeta + \frac{1}{A})^2}\right). \tag{32}$$

The corresponding values of the vorticity and the Bernoulli constant are given by

$$\Omega(A) := \frac{1 - A^2}{1 - 3A^2}, \qquad B(A) := \frac{1}{2}\left(\frac{1 + A^2}{1 - 3A^2}\right)^2. \tag{33}$$

We plot the flow generated by Hur & Wheeler's solutions in Figures 5 and 6.

4.3 Rotational flows with gravity

In this subsection, we will assume surface tension to be zero. Using an implicit function argument, Hur & Wheeler [31] were able to show the existence of overhanging and touching waves for small gravity.

Recall the equation derived in Section 3.4,

$$\frac{1}{2}(1 + \Omega(y + y\mathcal{H}y_\alpha - \mathcal{H}(yy_\alpha)))^2 = (B - Gy)((1 + \mathcal{H}y_\alpha)^2 + y_\alpha^2) \qquad \text{on } \beta = 0. \tag{34}$$

As done in [31], we introduce $\zeta : \mathbb{C} \to \mathbb{C}$

$$\zeta = \exp(-i(\alpha + i\beta)),$$

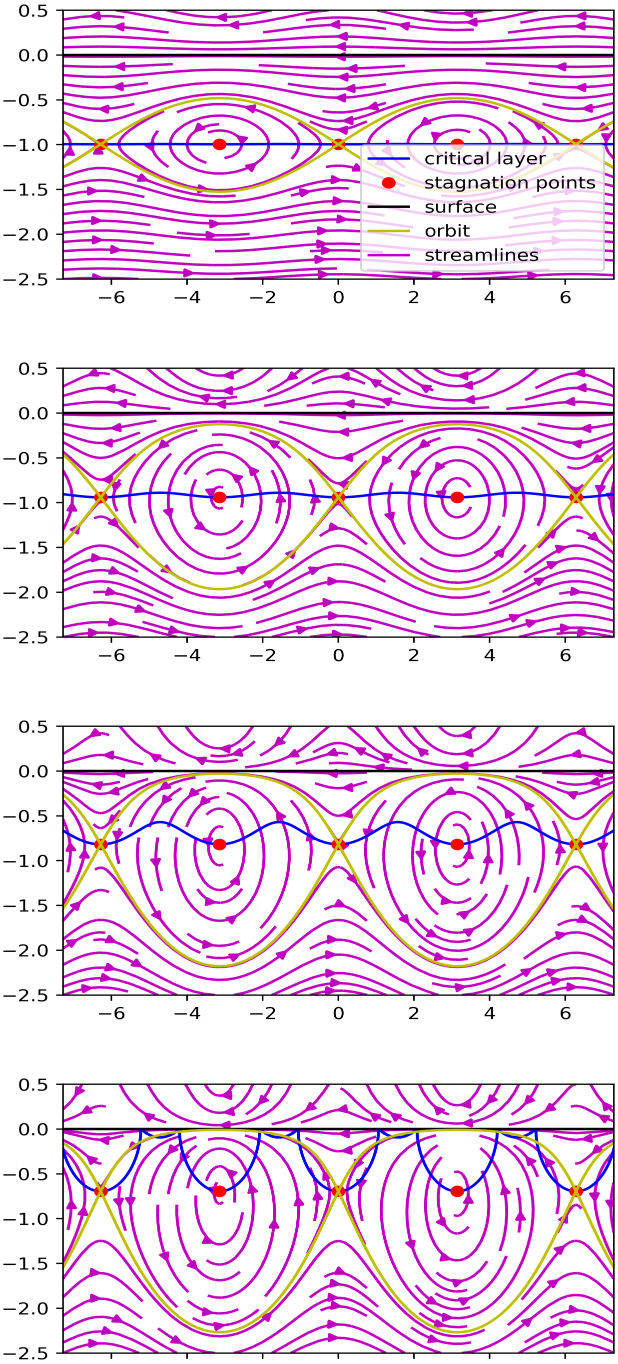

Figure 5: Streamplots in the α, β plane of Hur & Wheeler's explicit solution for the cases $A = 0.05, 0.2, 0.35, A_{\max}$. The critical layers correspond to the points where the horizontal velocity of the fluid vanishes. Points at which the velocity of the fluid vanishes are called stagnation points. The heteroclinic orbit denotes a path that joins two stagnation points.

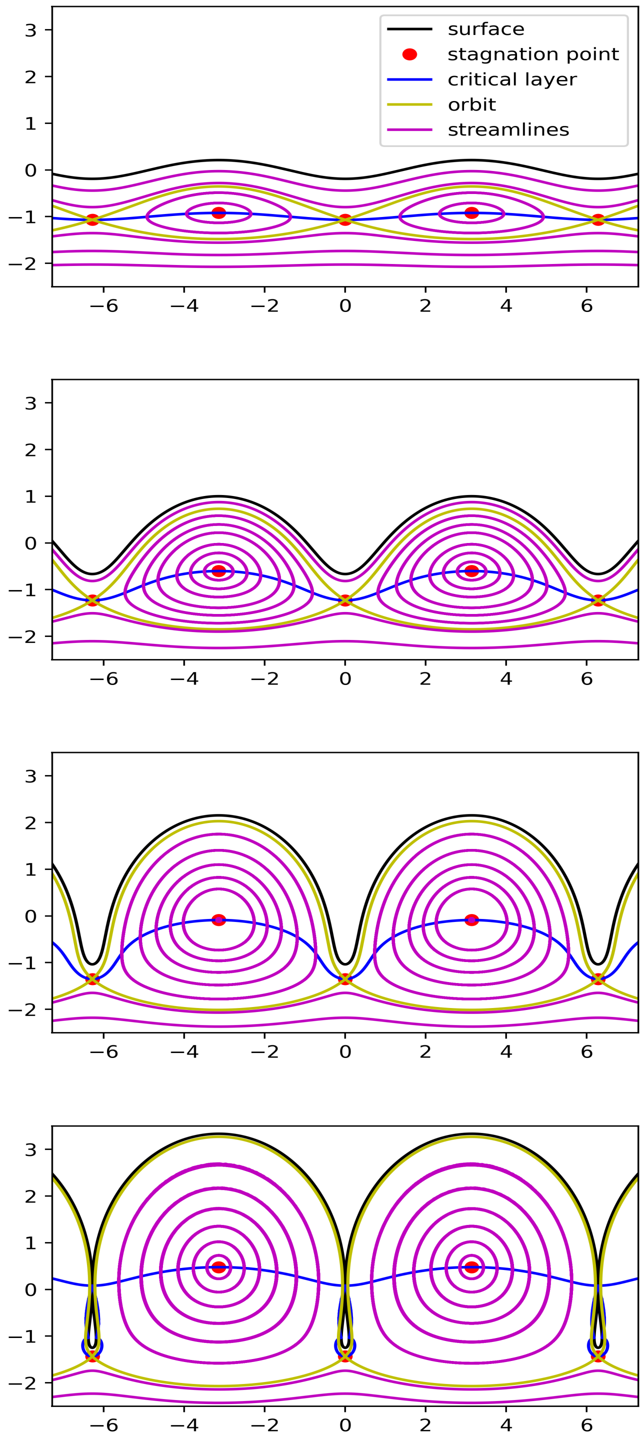

Figure 6: Streamplots in the x, y plane of Hur & Wheeler's explicit solution for the cases $A = 0.05, 0.2, 0.35, A_{\max}$.

which maps the lower half of the complex plane to the inside of the unit disc and the real axis to the complex unit circle. Now define $w : \mathbb{D} \to \mathbb{C}$ to be such that

$$z(\zeta) = i \log \zeta + w(\zeta). \tag{35}$$

Plugging this decomposition of z in (34) yields

$$B - Gy = \frac{1}{2} \frac{(1 + \Omega(\Im w + \mathcal{Q}(w)))^2}{|1 - i\zeta w_\zeta|^2} \qquad \text{for } |\zeta| = 1 \tag{36}$$

where

$$\mathcal{Q}(w(\zeta)) := -\frac{\zeta}{2\pi i} \int_{|\zeta'|=1} \left(\frac{\Im(w(\zeta) - w(\zeta'))}{\zeta - \zeta'} \right)^2 d\zeta' \qquad \text{for } |\zeta| = 1. \tag{37}$$

The commutator formula,

$$y\mathcal{H}y_\alpha - \mathcal{H}(yy_\alpha) = \frac{1}{8\pi} \int_0^{2\pi} \frac{(y(\alpha) - y(\alpha'))^2}{\sin^2((\alpha - \alpha')/2)} d\alpha' \tag{38}$$

is used to obtain (36). We refer to Section 3 of [6] for a proof of (38).

Now we are ready to formulate the operator form of the problem. Define the following spaces

$$X := \{w \in C^{3+a}(\mathbb{D}, \mathbb{C}) : w \text{ is holomorphic in } \mathbb{D} \text{ and } w(\overline{\zeta}) = -\overline{w(\zeta)}\} \tag{39}$$

$$Y := \{f \in C^{2+a}(\partial\mathbb{D}, \mathbb{R}) : f(\zeta) = f(\overline{\zeta})\} \tag{40}$$

and the subset

$$U := \{w \in X : 1 - i\zeta w_\zeta(\zeta) \neq 0 \text{ for } |\zeta| = 1\}. \tag{41}$$

Let $\mathcal{G} : U \times \mathbb{R}^2 \to Y$ be the operator defined by

$$\mathcal{G}(w; G, a) := \frac{1}{2} \frac{(1 + \Omega(a)(\Im w + \mathcal{Q}(w)))^2}{|1 - i\zeta w_\zeta|^2} + G\Im w - B(a) \qquad \text{for } |\zeta| = 1, \tag{42}$$

where

$$\Omega(a) := \frac{1 - a}{1 - 3a} \qquad \text{and} \qquad B(a) := \frac{1}{2} \left(\frac{1 + a}{1 - 3a} \right)^2.$$

We note Theorem 8 and Lemma 3.6 of [6] ensure the operator above maps U to Y.

Notice that $\mathcal{G}(w; G, a) = 0$ implies $w, G, \Omega(a)$ and $B(a)$ satisfy (36), which means

$$z(\alpha + i\beta) = i \log \zeta + w(\zeta), \qquad \text{where } \zeta = \exp(-i(\alpha + i\beta))$$

is the conformal map described in Section 3.3 that solves the water wave problem. We now define

$$w(a)(\zeta) := -\frac{4i\sqrt{a}\zeta}{1 + \sqrt{a}\zeta}. \tag{43}$$

In fact, (43) is the decomposition of (30) in the form (35), meaning the family (43) has overhanging waves. It is a long but straightforward calculation, using residue calculus to verify

$$\mathcal{G}(w(a); 0, a) = 0, \qquad \text{for } a > 0.$$

Using the Implicit Function Theorem, Hur & Wheeler were able to construct solutions for nonzero gravity. We note that the version stated here differs slightly from Theorem 3 of [31]. This is due to which formulation of the Implicit Function Theorem is used.

Theorem 11 (Hur & Wheeler,[31])**.** *For each $a_0 \in (0, \frac{1}{4})$ there exists $\epsilon > 0$ and a real-analytic operator*

$$W : (-\epsilon, \epsilon) \times (a_0 - \epsilon, a_0 + \epsilon) \to U$$

such that $W(0, a) = w(a)$ and

$$\mathcal{G}(W(G, a); G, a) = 0.$$

Moreover, there exists $\delta > 0$ such that for all $(a, G) \in (-\epsilon, \epsilon) \times (a_0 - \epsilon, a_0 + \epsilon)$ the following statements are equivalent.

(i) $\mathcal{G}(w; G, a) = 0$ and $||w - w(a)||_X < \delta$;

(ii) $w = W(G, a)$.

We refer to Theorem 3 of [31] for a proof. Note that Theorem 11 does not hold at $a = 0$ due to the Fréchet derivative of (42) not being invertible.

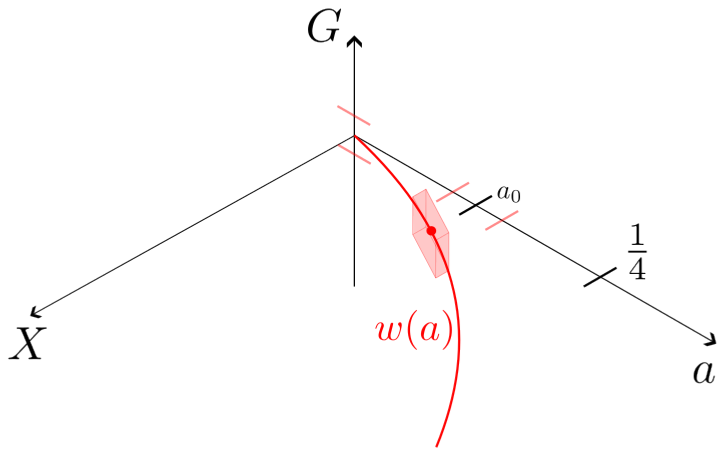

Figure 7: Illustration of Theorem 11. The neighbourhood where there exists uniqueness of solutions is depicted in faded red.

Hur & Wheeler employ Theorem 11 to prove the existence of overhanging waves and of touching waves for small but nonzero gravity.

5 Continuous curve of solutions with fixed gravity

This section aims to demonstrate the existence of a continuous curve of gravity waves connecting laminar flow and touching waves. We start by stating our results in the first subsection. Subsequently, in the second subsection, we establish a uniform version of Theorem 11.

Throughout this section, we assume there are no effects from surface tension. Additionally, we define $X, Y, U, \mathcal{G}$ and $w(a)$ to be as in (39),(41), (42) and (43), respectively.

5.1 Statement of results

We will prove a uniform version of Theorem 11 in the second subsection, but before that, let us state the theorem.

Theorem 12 (Main Theorem)**.** *For every $\gamma \in (0, \frac{1}{8})$, there exists $\epsilon > 0$ and a continuous operator $W : S \to U$, where S is the set defined as*

$$S = \left\{ (G, a) \in (-\epsilon, \epsilon) \times (-\infty, \frac{1}{4} - \gamma] : a \geq \tilde{G}^{-1}(G) \right\}.$$

Here, $\tilde{G}^{-1}$ is a continuous bijective map between two one-dimensional neighbourhoods of 0 such that $\tilde{G}(0) = 0$. The operator satisfies the following three conditions.

(i) $W(0, a) = w(a)$;

(ii) $W(G, \tilde{G}^{-1}(G))$ *is a constant function of* ζ;

(iii) $\mathcal{G}(W(G, a); G, a) = 0$.

Moreover, there exists a constant $\delta > 0$ such that for any $(G, a) \in S$ and any non-constant function w of ζ, the following are equivalent:

(i) $\mathcal{G}(w; G, a) = 0$ *and* $||w - w(a)||_X < \delta$;

(ii) $w = W(G, a)$.

Finally, the operator W is real-analytic on the set

$$\left\{ (G, a) \in (-\epsilon, \epsilon) \times (-\infty, \frac{1}{4} - \gamma] : a > \tilde{G}^{-1}(G) \right\}.$$

Theorem 12 implies the following two results.

Theorem 13. *For every sufficiently small $\epsilon > 0$ and $G \in (-\epsilon, \epsilon)$ fixed, there exists a continuous curve of solutions of (14) between a laminar flow and a touching wave.*

Corollary 14 (Breaking Waves)**.** *For every sufficiently small $\epsilon > 0$ and $G \in (-\epsilon, \epsilon)$ fixed, there exists a solution of (14) whose profile is vertical at a point but never overhanging.*

We note Theorem 13 is the first to establish the existence of such a curve of solutions and Corollary 14 is the first to rigorously establish the existence of breaking waves with gravity.

5.2 Proof of Main Theorem

In the following section, we provide a summary of the proof. Firstly, we use compactness to establish a version of Theorem 12 for compact subsets of $(0, \frac{1}{4})$. Then, we compute constant solutions and redefine the operator to obtain trivial solutions for the operator problem for any choice of parameters. Subsequently, we find a curve of parameters such that the Fréchet Derivative of the operator at every point along the curve has the same kernel as at the point $(0, 0)$. Next, we use a local bifurcation result to construct solutions near the point $(0, 0)$. Finally, we connect the different curves of solutions.

Lemma 15. *For all $\lambda \in (0, \frac{1}{8})$ there exist $\epsilon > 0$ and a real-analytic operator*

$$W : (-\epsilon, \epsilon) \times [\lambda, \tfrac{1}{4} - \lambda] \to U$$

such that $W(0, a) = w(a)$ and

$$\mathcal{G}(W(G, a); G, a) = 0.$$

Moreover, there exists $\delta > 0$ such that for all $(G, a) \in (-\epsilon, \epsilon) \times [\lambda, \frac{1}{4} - \lambda]$ the following statements are equivalent.

(i) $\mathcal{G}(w; G, a) = 0$ *and* $||w - w(a)||_X < \delta$;

(ii) $w = W(G, a)$.

Proof. We start by applying Theorem 11 to each $a \in [\lambda, \frac{1}{4} - \lambda]$, yielding an ϵ_a and a δ_a satisfying the conditions in the theorem. Afterwards, we cover $[\lambda, \frac{1}{4} - \lambda]$ with open sets, as below.

$$\left[\lambda, \tfrac{1}{4} - \lambda\right] \subset \bigcup_{a \in [\lambda, \frac{1}{4} - \lambda]} \Big(a - \epsilon_a, a + \epsilon_a\Big).$$

By the Heine-Borel Theorem, we can take a finite subcover, meaning

$$\left[\lambda, \tfrac{1}{4} - \lambda\right] \subset \bigcup_{k=1}^{n} \Big(a_k - \epsilon_{a_k}, a_k + \epsilon_{a_k}\Big).$$

Afterwards, we define

$$\delta \coloneqq \min_{k=1,..,n} \delta_{a_k}.$$

Let

$$W^k : (-\epsilon_{a_k}, \epsilon_{a_k}) \times (a_k - \epsilon_{a_k}, a_k + \epsilon_{a_k}) \to U \tag{44}$$

be the operator obtained from applying Theorem 11 to a_k. The operator (44) satisfies

$$||W^k(G, a) - w(a)||_X < \delta_{a_k} \qquad \text{for } |a - a_k| < \epsilon_{a_k} \text{ and } |G| < \epsilon_{a_k}.$$

Since W^k is continuous, we can choose $\tilde{\epsilon}_{a_k}$ such that

$$||W^k(G, a) - w(a)||_X < \delta, \qquad \text{for } |a - a_k| < \tilde{\epsilon}_{a_k} \text{ and } |G| < \epsilon_{a_k}.$$

Then, we define

$$\epsilon \coloneqq \min_{k=1,..,n} \tilde{\epsilon}_{a_k}.$$

Next, we define the operator $W : (-\epsilon, \epsilon) \times [\lambda, \frac{1}{4} - \lambda] \to U$ by

$$W(G, a) \coloneqq W^k(G, a) \qquad \text{if } |a - a_k| < \epsilon_{a_k}, k = 1, ..., n. \tag{45}$$

Let a satisfy $|a - a_j| < \epsilon_{a_j}$ and $|a - a_k| < \epsilon_{a_k}$ for some j and k and G satisfy $|G| < \epsilon$. Then (G, a)

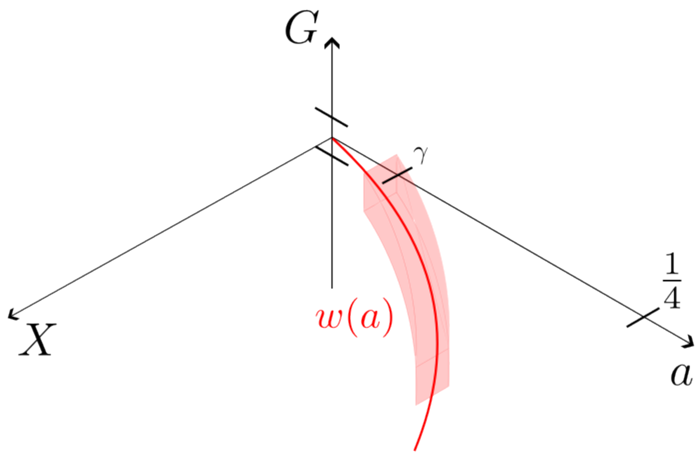

Figure 8: Illustration of Lemma 15.The neighbourhood where there exists uniqueness of solutions is depicted in faded red.

is in the domain of both W^j and W^k. Since

$$||W^i(G,a) - w(a)||_X < \delta \qquad \text{for } i = j, k,$$

it follows from the uniqueness of Theorem 11 that $W^j(G,a) = W^k(G,a)$, and therefore W is well defined. $\square$

We turn our attention now to the bifurcation point, starting by computing the partial Fréchet derivative at $(w(0), 0, 0)$,

$$D_w\mathcal{G}[w(0);0,0]v = \Im(v - \zeta v_\zeta) = -\Im\Big(\sum_{n=0}^{\infty}(n-1)v_n\zeta^n\Big), \tag{46}$$

which is not invertible. Note that due to the symmetry assumption on X, it follows that the coefficients v_n are purely imaginary. In fact, it follows from (46) that the kernel and the complement of the range of $D_w\mathcal{G}[w(0);0,0]$ are both one-dimensional. The kernel is spanned by $i\zeta$, and the complement of its range is spanned by $\Re(\zeta)$.

The next step is to look for constant solutions of $\mathcal{G}(w;G,a) = 0$, which correspond to laminar flow solutions of (14).

Lemma 16. *There exists a real-analytic operator* $d : \mathbb{R} \times (-\infty, 1/3) \to \{iR : R \in \mathbb{R}\}$ *such that*

$$\mathcal{G}(d(G,a); G, a) = 0 \qquad \text{and} \qquad d(0,0) = 0$$

for $(G, a) \in \mathbb{R} \times (-\infty, 1/3)$. *Moreover, there exist open neighbourhoods of zero* $V, T \subset \mathbb{R}$ *and a bijective analytic map* $\tilde{G} : V \to T$, *with the property that*

$$\ker D_x \mathcal{G}\Big[d(\tilde{G}(a), a); \tilde{G}(a), a\Big] = \langle i\zeta \rangle_X$$

for $a \in V$ *and* $\tilde{G}(0) = 0$

Proof. Let c be a fixed constant. A computation shows that

$$\mathcal{G}(ic; G, a) = \frac{1}{2}(1 + \Omega(a)c)^2 + Gc - B(a).$$

Thus $\mathcal{G}(ic; G, a) = 0$ if and only if

$$c = -\frac{\Omega(a) + G}{\Omega(a)^2} \pm \frac{1}{\Omega(a)^2}\sqrt{G^2 + 2\Omega(a)G + 2B(a)\Omega(a)^2}. \tag{47}$$

Evaluating this expression at $a = G = 0$, we find that when the sign before the square root is positive, $c = 0$. We define the operator $d : \mathbb{R} \times (-\infty, 1/3) \to \{iR : R \in \mathbb{R}\}$ by

$$d(G, a) = i\bigg(-\frac{\Omega(a) + G}{\Omega(a)^2} + \frac{1}{\Omega(a)^2}\sqrt{G^2 + 2\Omega(a)G + 2B(a)\Omega(a)^2}\bigg). \tag{48}$$

Note that the operator $d(G, a)$ defined in (48) is real-analytic over the domain where it is defined.

We turn our focus to the second part of the lemma and compute the Fréchet derivative of (42) at $(d(G, a); G, a)$.

$$D_x\mathcal{G}[d(G,a); G, a]v = \Im\bigg(\Big((1 + \Omega(a)\tilde{d}(G,a))\Omega(a) + G\Big)v - (1 + \Omega(a)\tilde{d}(G,a))^2 \zeta v_\zeta\bigg), \tag{49}$$

where $\tilde{d}(G, a) = \Im(d(G, a))$. By plugging in $v = i\zeta$ in (49) and equating it to zero, we find that $i\zeta$

spans the kernel of $D_x\mathcal{G}[d(G,a);G,a]$ if and only if

$$(1+\Omega(a)\tilde{d}(G,a))\Omega(a)+G-(1+\Omega(a)\tilde{d}(G,a))^2=0. \tag{50}$$

We note that in a neighbourhood of $(0,0)$, (48) can be described in the following way,

$$\tilde{d}(G,a)=4a(1+\mathcal{O}(a+G)). \tag{51}$$

Therefore, we can compute the partial derivative with respect to G of the left-hand side of (50) evaluated at $(G,a)=(0,0)$ to be equal to 1. By the analytic version of the Implicit Function Theorem, there exist neighbourhoods $V\subset\mathbb{R}$ and $T\subset\mathbb{R}$ containing zero and a map $\tilde{G}:V\to T$ such that the pairing $(\tilde{G}(a),a)$ satisfies (50). $\square$

Now define $\mathcal{K}:U\times\mathbb{R}\times(-\infty,1/3)\to Y$ by

$$\mathcal{K}(v;G,a)=\mathcal{G}(id(G,a)+v;G,a). \tag{52}$$

Notice that $\mathcal{K}(0;G,a)=0$ for all G and a, and

$$\mathcal{K}(v,G,a)=0 \iff \mathcal{G}(v+id(G,a),G,a)=0.$$

Lemma 17. *There exist $\epsilon>0$ and a continuous operator $v:\tilde{S}\to U$, where $\tilde{S}$ is defined as*

$$\tilde{S}:=\left\{(G,a)\in(-\epsilon,\epsilon)\times\mathbb{R}:\tilde{G}^{-1}(G)\le a<\tilde{G}^{-1}(G+\epsilon)\right\},$$

such that $v(G,a)$ satisfies the equation

$$\mathcal{K}(v(G,a);G,a)=0,$$

for $(G,a)\in\tilde{S}$. Moreover, $v(G,a)$ is of the form

$$v(G,a)=C_G\left(a-\tilde{G}^{-1}(G)\right)^{\frac{1}{2}}i\zeta+\mathcal{O}(a-\tilde{G}^{-1}(G)), \tag{53}$$

where C_G is a positive constant that depends continuously on G.

Furthermore, there exists $\delta > 0$ such that $v(G,a)$ is the only nontrivial solution of $\mathcal{K}\big(v, G, a\big) = 0$ satisfying $||v||_X < \delta$.

Finally, the operator v is real-analytic on the set

$$\Big\{(G,a) \in (-\epsilon, \epsilon) \times \mathbb{R} : \tilde{G}^{-1}(G) < a < \tilde{G}^{-1}(G+\epsilon)\Big\}.$$

Proof. By Lemma 16 and considering a Fourier series expansion of v in (49), the following three properties hold.

(i) $\mathcal{K}(0; G, a) = 0$, for all $G, a \in \mathbb{R} \times (-\infty, \frac{1}{3})$;

(ii) $\ker D_v\mathcal{K}\Big(0; \tilde{G}(a_0), a_0)\Big) = \langle i\zeta\rangle_X$, for all $a_0 \in V$;

(iii) $\operatorname{ran} D_v\mathcal{K}\Big(0; \tilde{G}(a_0), a_0\Big) = Y \ominus \langle\Re(\zeta)\rangle_Y$, for all $a_0 \in V$;

where $Y \ominus \langle\Re(\zeta)\rangle_Y$ denotes the orthogonal complement of $\langle\Re(\zeta)\rangle_Y$ in Y. Next, we compute the mixed derivative

$$D^2_{va}\mathcal{K}\Big(0; \tilde{G}(a_0), a_0\Big)(i\zeta) = \Big(-2 + \mathcal{O}(a_0)\Big)\Re\zeta. \tag{54}$$

This expression needs to be nonzero in order to satisfy the transversality condition. To achieve this, we redefine V if necessary. Then, we can apply Corollary 7 to obtain real analytic operators

$$\tilde{v} : (-\tilde{\epsilon}, \tilde{\epsilon})^2 \to U \qquad \text{and} \qquad \tilde{a} : (-\tilde{\epsilon}, \tilde{\epsilon})^2 \to \mathbb{R}, \tag{55}$$

such that

$$\tilde{K}\Big(\tilde{v}(s,G), G, \tilde{a}(s,G)\Big) = 0, \text{for all } |s|, |G| < \tilde{\epsilon},$$

$\tilde{v}(s,G) = si\zeta + \mathcal{O}(s^2 + sG)$ and $\tilde{a}(0,G) = \tilde{G}^{-1}(G)$. Moreover, the maps agree with the maps obtained by applying Theorem 4 at the point $(0, \tilde{G}^{-1}(G))$ with $G \in (-\epsilon, \epsilon)$ fixed, and therefore we can use

Proposition 5. We proceed with computing the second-order linearization in v at $(0, \tilde{G}(a_0), a_0)$,

$$D^2_{vv}\mathcal{K}(0, \tilde{G}(a_0), a_0)(u)^2 = \Omega(a_0)\mathcal{Q}_{ww}(d(G,a))(u)^2 - 4\Omega(a_0)\Im(u)\Im(\zeta u_\zeta) + 4\Big(\Im(\zeta u_\zeta)\Big)^2 - \Im(i|u_\zeta|^2), \tag{56}$$

where we use $\mathcal{Q}_w(d(G,a))(u) = 0$. Evaluating (56) at $u = i\zeta$ reveals

$$D^2_{vv}\mathcal{K}(0, \tilde{G}(a_0), a_0)(i\zeta)^2 = \Omega(a_0) - 4\Omega(a_0)\Re(\zeta)^2 + 4\Re(\zeta)^2 - 1, \tag{57}$$

where we used $\mathcal{Q}_w(d(G,a))(v) = 0$ and $\mathcal{Q}_{ww}(d(G,a))(i\zeta)^2 = 1$. Note that (57) has zero projection on $\langle\Re(\zeta)\rangle_Y$ and therefore, by Proposition 5, $\frac{\partial}{\partial s}\tilde{a}(0, \tilde{G}(a_0)) = 0$. Subsequently, we compute the third-order linearization in v at $(0, \tilde{G}(a_0), a_0)$,

$$\begin{aligned} D^3_{vvv}\mathcal{K}(0, \tilde{G}(a_0), a_0)(u)^3 &= 3\Omega(a_0)^2(\Im u)\mathcal{Q}_{ww}(u)^2 - 6\Omega(a_0)\mathcal{Q}_{ww}(u)^2\Im(\zeta u_\zeta) - 6\Omega(a_0)^2(\Im u)^2\Im(\zeta u_\zeta) \\ &\quad + 3\Omega(a_0)\Im(u)\left(8\Big(\Im(\zeta u_\zeta)\Big)^2 - 2\Im(i|u_\zeta|^2)\right) \\ &\quad - 24\Big(\Im(\zeta u_\zeta)\Big)^3 + 12\Im(\zeta u_\zeta)\Im(i|u_\zeta|^2). \end{aligned} \tag{58}$$

Evaluating (58) at $u = i\zeta$ reveals

$$D^3_{vvv}\mathcal{K}(0, \tilde{G}(a_0), a_0)(i\zeta)^3 = 3\big(\Omega(a_0) - 2\big)^2\Re\zeta - 6(\Omega(a_0) - 2)^2(\Re\zeta)^3,$$

which has nonzero projection on $\langle\Re(\zeta)\rangle_Y$. Applying Proposition 5 yields $\frac{\partial^2}{\partial s^2}\tilde{a}(0, \tilde{G}(a_0)) = \frac{(\Omega(a_0)-2)^2}{2+\mathcal{O}(a_0)}$. We, again, possibly, redefine V to ensure $\frac{\partial^2}{\partial s^2}\tilde{a}(0, \tilde{G}(a_0)) > 0$, for all $a_0 \in V$ (recall $\Omega(0) = 1$). Since $\frac{\partial^2}{\partial s^2}\tilde{a}(0, \tilde{G}(a_0)) > 0$, it follows that we can choose an ϵ such that the real analytic operator $\tilde{a}(\cdot, \tilde{G}(a_0))$ is injective on the set $[0, \epsilon)$. We define $v : (-\epsilon, \epsilon) \times [a_0, a_0 + \epsilon) \to U$ by

$$v(G, a) \coloneqq \tilde{v}\left(\tilde{a}^{-1}(s, G)\right).$$

The formula in (53) follows directly from the definition above. $\square$

We are now ready to prove Theorem 12. The proof is illustrated in Figure 9.

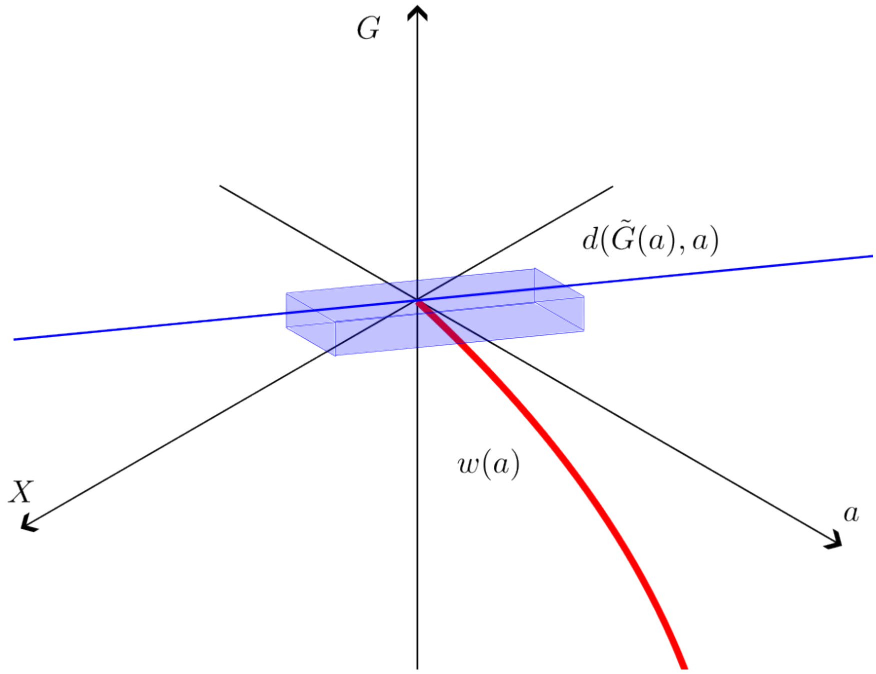

Figure 9: Connection of solution curves. The faded blue illustrates the neighbourhood where there exists uniqueness of non-trivial solutions, which was obtained in Lemma 17.

Proof of Theorem 12. Let $\gamma > 0$ be given. Let v, ϵ_1, δ_1 and $\tilde{S}$ be as described in Lemma 17. Next, we note that for small and positive a, (43) can be expressed as

$$w(a) = -\frac{4i\sqrt{a}\zeta}{1+\sqrt{a}\zeta} = -4i\sqrt{a}\zeta + \mathcal{O}(a). \tag{59}$$

Therefore, we can choose a positive and small enough λ such that $||w(\lambda)||_X < \frac{\delta_1}{2}$, $||d(0,\lambda)||_X < \frac{\delta_1}{2}$, $\lambda < \epsilon_1$ and $\lambda < \gamma$. We then apply Lemma 15 for such a λ, which yields ϵ_2, δ_2 and W^λ as in the Lemma. We then define

$$\delta \coloneqq \min(\delta_1, \delta_2).$$

We choose $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$ to be small enough such that

$$||v(G,a) - d(G,a)||_X < \delta \qquad \text{for } |G| < \tilde{\epsilon}_1$$

and

$$||W^{\lambda}(G,a) - w(a)||_X < \delta \qquad \text{for } |G| < \tilde{\epsilon}_2.$$

We then define

$$\epsilon := \min(\tilde{\epsilon}_1, \tilde{\epsilon}_2)$$

and

$$S := \Big\{(G,a) \in (-\epsilon,\epsilon) \times (-\infty, \tfrac{1}{\sqrt{2}} - \gamma] : a \geq \tilde{G}^{-1}(G)\Big\}.$$

Let $W : S \to U$ be defined by

$$W(G,a) := \begin{cases} W^{\lambda}(G,a) & \text{if } a > \lambda \\ v(G,a) + d(G,a) & \text{if } (G,a) \in \tilde{S} \cap S. \end{cases} \tag{60}$$

Next, we show (60) is well-defined. Let $(G,a) \in \tilde{S}$ and $a > \lambda$, then

$$\mathcal{K}\Big(W_{\lambda}(G,a) - d(G,a), G, a\Big) = 0$$

and

$$||W_{\lambda}(G,a) - d(G,a)||_X < \frac{\delta(\mu)}{2} + \frac{\delta(\mu)}{2} \leq \delta(\mu).$$

By Lemma 17, the solutions coincide, meaning (60) is well-defined. □

6 Critical Layers

In this section, we analyze the behaviour of critical layers at points where the wave is vertical. We consider both the cases of overhanging and breaking waves. This analysis is done using Taylor expansions at the point where the wave is vertical.

Definition 6. Critical layers are defined as the set of points in the (x, y) plane where the horizontal velocity

$$u = \psi_y = \frac{\psi_\alpha y_\alpha + \psi_\beta x_\alpha}{|z_\alpha|^2} \tag{61}$$

vanishes.

Even though the critical layers are defined in the (x, y) plane, the existence of the conformal map proved in Lemma 9 implies that angles are preserved under the map. A numerical computation of the critical layers can be seen plotted in Figures 10 and 11.

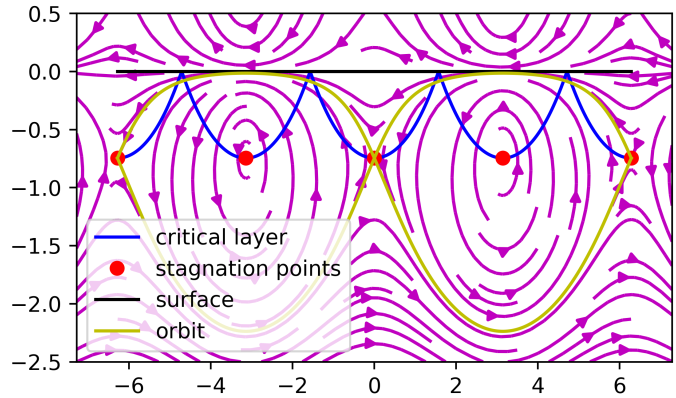

Figure 10: Streamlines in the (α, β) plane for Hur & Wheeler's breaking wave. We refer to the Caption of Figure 5 for an explanation of the terms in the Figure.

6.1 General results

In the following Proposition, we state our results regarding the behaviour of a critical layer at a point on the surface where the tangent is vertical but which is not a stagnation point. Notably, the results hold for a general water wave and not just the ones we constructed.

Proposition 18. *Let z denote the conformal map to the half plane of a solution to (15) with $\tau = 0$ and assume the same conditions as in Proposition 10. Let z denote the conformal map to the half plane of a solution to (15) with $\tau = 0$ and assume the same conditions as in Proposition 10. Additionally, assume α_{crit} to be a point in the (α, β) plane at which the wave is vertical.*

(i) If the function $x(\alpha + i0)$ has a local extremum at α_{crit} and $G > 0$, there exist no critical layers from the water touching the point α_{crit} (but they exist from outside the water);

(ii) If the function $x(\alpha + i0)$ has a local extremum at α_{crit} and $G < 0$, there exist critical layers from the water touching the point α_{crit} (but they do not exist from outside the water);

(iii) If the function $x(\alpha + i0)$ does not have a local extremum at α_{crit} and $G \neq 0$, there exist critical layers from the water touching the point α_{crit} (and from outside the water as well).

Proof. Without loss of generality, assume the wave is travelling in the positive x direction. Let α_{crit} denote the point at which the wave is vertical. It follows that

$$x_\alpha(\alpha_{\text{crit}}, 0) = 0$$

and the smallest positive integer k, such that $\frac{\partial^k}{\partial \alpha^k} x(\alpha_{\text{crit}}, 0)$ does not vanish, must be odd if the wave is a breaking wave and even if the wave overhangs at the point. Moreover

$$\frac{\partial^k}{\partial \alpha^k} x(\alpha_{\text{crit}}, 0) > 0 \tag{62}$$

if k is odd. It follows from (61) that

$$\psi_y = 0 \iff \psi_\alpha y_\alpha + \psi_\beta x_\alpha = 0. \tag{63}$$

Let U be the open set from Proposition 10 where ψ has a real-analytic extension and define $F : U \to \mathbb{R}$ by

$$F(\alpha, \beta) = \psi_\alpha(\alpha, \beta) y_\alpha(\alpha, \beta) + \psi_\beta(\alpha, \beta) x_\alpha(\alpha, \beta). \tag{64}$$

Equation (63) implies (α, β) belongs to the critical layer if and only if $F(\alpha, \beta) = 0$. Since $F(\alpha_{\text{crit}}, 0) =$

0, we want to use Taylor expansions of (64) near the point $(\alpha_{\text{crit}}, 0)$ to figure out the local behaviour of the critical layers.

Notice that ψ is constant on $\{\beta = 0\}$, and therefore

$$\frac{\partial^j}{\partial\alpha^j}\psi(\alpha_{\text{crit}}, 0) = 0, \text{ for all } j \in \mathbb{Z}^+. \tag{65}$$

Next, we rewrite (15d) as

$$\psi_\beta = \pm\sqrt{2(x_\alpha^2 + y_\alpha^2)(B - Gy)} \qquad \text{on } \{\beta = 0\} \tag{66}$$

for later use. Given all the alpha derivatives of ψ vanish at the critical point, exploring alternative branches of (66) results in the partial derivatives of F at the critical point differing only in sign. Consequently, we can consider the positive branch of equation (66) without any loss of generality.

We proceed to compute the partial derivatives of F at the point $(\alpha_{\text{crit}}, 0)$. Starting by

$$\frac{\partial^j}{\partial\alpha^j}F(\alpha, \beta) = \sum_{l=0}^{j}\binom{j}{l}\left(\frac{\partial^l\psi_\alpha}{\partial\alpha^l}\frac{\partial^{j-l}y_\alpha}{\partial\alpha^{j-l}} + \frac{\partial^l\psi_\beta}{\partial\alpha^l}\frac{\partial^{j-l}x_\alpha}{\partial\alpha^{j-l}}\right), \text{ for all } j \in \mathbb{Z}^+.$$

Using (65) and $\frac{\partial^j}{\partial\alpha^j}x = 0$ for $j < k$ to evaluate the expression above at $(\alpha_{\text{crit}}, 0)$ yields

$$\frac{\partial^j}{\partial\alpha^j}F(\alpha_{\text{crit}}, 0) = \begin{cases} 0, & \text{if } j < k - 1, \\ \psi_\beta \frac{\partial^k x}{\partial\alpha^k}, & \text{if } j = k - 1. \end{cases} \tag{67}$$

Next, we compute the derivatives with respect to β

$$\frac{\partial^j}{\partial\beta^j}F(\alpha, \beta) = \sum_{l=0}^{j}\binom{j}{l}\left(\frac{\partial^l\psi_\alpha}{\partial\beta^l}\frac{\partial^{j-l}y_\alpha}{\partial\beta^{j-l}} + \frac{\partial^l\psi_\beta}{\partial\beta^l}\frac{\partial^{j-l}x_\alpha}{\partial\beta^{j-l}}\right). \tag{68}$$

Plugging in $(\alpha_{\text{crit}}, 0)$ for $j = 1$ above yields

$$\begin{aligned}
\frac{\partial}{\partial \beta} F(\alpha_{\text{crit}}, 0) &= \psi_{\alpha\beta} y_\alpha - \psi_\beta y_{\alpha\alpha} \\
&= \left(\sqrt{2(B - Gy)} y_{\alpha\alpha} - \frac{G y_\alpha^2}{\sqrt{2(B - Gy)}} \right) y_\alpha - \sqrt{2(B - Gy)} y_\alpha y_{\alpha\alpha} \\
&= -\frac{G y_\alpha^3}{\sqrt{2(B - Gy)}}.
\end{aligned} \tag{69}$$

We utilized equation (66) above to calculate the derivatives of ψ_β. Additionally, we assume $\sqrt{y_\alpha^2} = y_\alpha$ because symmetry implies the existence of a critical point where y_α is positive and another where it is negative. The Taylor expansion of F at the point $(\alpha_{\text{crit}}, 0)$ is then

$$\begin{aligned}
&\left(-\frac{G y_\alpha^3}{\sqrt{2(B - Gy)}} \right) \Bigg|_{\alpha = \alpha_{\text{crit}}} \beta + \left(\sqrt{2(B - Gy)} y_\alpha \frac{\partial^k x}{\partial \alpha^k} \right) \Bigg|_{\alpha = \alpha_{\text{crit}}} (\alpha - \alpha_{\text{crit}})^{k-1} \\
=& \mathcal{O}\Big(|\alpha - \alpha_{\text{crit}}|^k + |\beta|^2 + |(\alpha - \alpha_{\text{crit}})\beta| \Big).
\end{aligned} \tag{70}$$

When k is odd, (70) has the form

$$-C_1 G \beta + C_2 (\alpha - \alpha_{\text{crit}})^{k-1} = \mathcal{O}\Big(|\alpha - \alpha_{\text{crit}}|^k + |\beta|^2 + |(\alpha - \alpha_{\text{crit}})\beta| \Big), \tag{71}$$

where C_1 and C_2 denote positive constants that depend on G. It follows from applying the Implicit Function Theorem to (71) that solutions of $F(\alpha, \beta) = 0$ in a neighbourhood of $(\alpha_{\text{crit}}, 0)$ have the form

$$\beta = \frac{C_2}{C_1 G} (\alpha - \alpha_{\text{crit}})^{k-1} + \mathcal{O}(|\alpha - \alpha_{\text{crit}}|^k). \tag{72}$$

Since the exponent $k - 1$ is even, statements (i) and (ii) follow. When k is even, (70) has the form

$$C_3 G \beta + C_4 (\alpha - \alpha_{\text{crit}})^{k-1} = \mathcal{O}\Big(|\alpha - \alpha_{\text{crit}}|^k + |\beta|^2 + |(\alpha - \alpha_{\text{crit}})\beta| \Big), \tag{73}$$

where C_3 and C_4 denote constants that depend on G. It follows from applying the Implicit Function

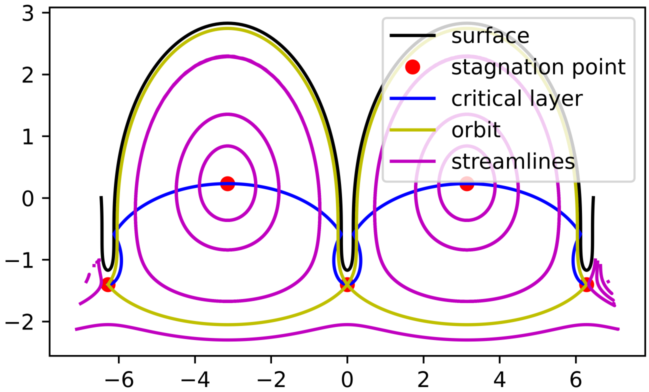

Figure 11: Streamlines in the (x, y) plane for Hur & Wheeler's breaking wave. We refer to the Caption of Figure 5 for an explanation of the terms in the Figure.

Theorem to (73) that solutions of $F(\alpha, \beta) = 0$ in a neighbourhood of $(\alpha_{\text{crit}}, 0)$ have the form

$$\beta = \frac{C_4}{C_3 G}(\alpha - \alpha_{\text{crit}})^{k-1} + \mathcal{O}(|\alpha - \alpha_{\text{crit}}|^k). \tag{74}$$

Since the exponent $k-1$ is odd, statement (iii) follows. $\square$

6.2 Hur & Wheeler's zero gravity solutions

Throughout this subsection, we assume $G = 0$. Recall the zero gravity solutions from Section 4.2. For these solutions, we can compute the parameters for which the solution is a breaking or overhanging wave explicitly. Since $x_\alpha(\alpha, 0, A) = 0$ corresponds to the wave profile being vertical, the computation is as follows,

$$x_\alpha(\alpha, 0; A) = \Re\left(\frac{(1 - A\exp(-i\alpha))^2}{(1 + A\exp(-i\alpha))^2}\right) = \frac{(A^2 - 1 - 2A\sin\alpha)(A^2 - 1 + 2A\sin\alpha)}{(A^2 + 2A\cos\alpha + 1)^2}. \tag{75}$$

Recall that $A_{\max}$ denotes the value of a for which the surface profile of the solutions touches at a point without the fluid being double valued. For $A \in (\sqrt{2}-1, A_{\max}]$, the expression above undergoes a sign change. When $A = \sqrt{2} - 1$, the expression vanishes at $\alpha = (2n+1)\frac{\pi}{2}$ for $n \in \mathbb{Z}$, but there is no sign change. This means that the solution corresponding to $A = \sqrt{2} - 1$ is a breaking wave.

Furthermore, notice that

$$x_\alpha(\alpha,0;a)=0 \iff \alpha=\arcsin\big((A^2-1)/(2A)\big) \text{ or } \pi-\arcsin\big((A^2-1)/(2A)\big),$$

for $A\in[\sqrt{2}-1,A_{\max}]$. We define

$$\alpha_{\mathrm{crit},1}(A)=\arcsin\big((A^2-1)/(2A)\big) \text{ for } A\in[\sqrt{2}-1,A_{\max}] \tag{76}$$

and

$$\alpha_{\mathrm{crit},2}(A)=\pi-\arcsin\big((A^2-1)/(2A)\big) \text{ for } A\in[\sqrt{2}-1,A_{\max}],$$

where we consider the principal branch of the arcsin function. We pause to remark that the symmetry and periodicity assumptions on the fluid and surface imply that different branches of the arcsin function correspond to points that are either related by symmetry and/or periodicity. Below we write α_{crit} when a computation would be true for both $\alpha_{\mathrm{crit},1}$ and $\alpha_{\mathrm{crit},2}$.

We now investigate the behaviour of the critical layers in Hur & Wheeler's zero gravity solutions. Starting by noting that there exists a neighbourhood of the critical point at which we can express the surface curve as $(\mathcal{F}(y),y)$. Moreover, we note

$$\mathcal{F}\big(y(\alpha)\big)=x(\alpha), \tag{77}$$

for α in a neighbourhood of α_{crit}. Differentiating both sides of (77) and evaluating at $\alpha=\alpha_{\mathrm{crit}}$ yields

$$\mathcal{F}'\Big(y\big(\alpha_{\mathrm{crit}}(A)\big)\Big)y_\alpha\big(\alpha_{\mathrm{crit}}(A)\big)=x_\alpha\big(\alpha_{\mathrm{crit}}(A)\big). \tag{78}$$

It follows from (78) that $\mathcal{F}'\Big(y\big(\alpha_{\mathrm{crit}}(A)\big)\Big)=0$, since $x_\alpha\big(\alpha_{\mathrm{crit}}(A)\big)=0$ and $y_\alpha\big(\alpha_{\mathrm{crit}}(A)\big)\neq 0$ for all $A\in[\sqrt{2}-1,A_{\max}]$. Taking the derivatives on both sides of (78) and evaluating at $\alpha=\alpha_{\mathrm{crit}}$ gives

$$\mathcal{F}''\Big(y\big(\alpha_{\mathrm{crit}}(A)\big)\Big)y_\alpha\big(\alpha_{\mathrm{crit}}(A)\big)^2+\mathcal{F}'\Big(y\big(\alpha_{\mathrm{crit}}(A)\big)\Big)y_{\alpha\alpha}\big(\alpha_{\mathrm{crit}}(A)\big)=x_{\alpha\alpha}\big(\alpha_{\mathrm{crit}}(A)\big). \tag{79}$$

Since the first derivative of $\mathcal{F}$ at the point vanishes, it follows that

$$\mathcal{F}''\Big(y\big(\alpha_{\text{crit}}(A)\big)\Big) = \frac{x_{\alpha\alpha}\big(\alpha_{\text{crit}}(A)\big)}{y_\alpha\big(\alpha_{\text{crit}}(A)\big)^2}.$$

We note that

$$\begin{aligned} x_{\alpha\alpha}(\alpha) = & -4A\Bigg(\frac{A\cos(\alpha)\,\big(1+A^2+2A\cos(\alpha)\big)\cos(\alpha)\big(A^2-1+2A\sin(\alpha)\big)}{(1+A^2+2A\cos(\alpha))^3} \\ & - \frac{\sin(\alpha)\,\big(A^2-1-2A\sin(\alpha)\big(A^2-1+2A\sin(\alpha)\big)\big)}{(1+A^2+2A\cos(\alpha))^3}\Bigg), \end{aligned}$$

which evaluated at $\alpha_{\text{crit},1}(A)$ becomes

$$\frac{\big(-A^5+A\big)\sqrt{\frac{-A^4+6A^2-1}{A^2}} + A^6 - 7A^4 + 7A^2 - 1}{(-A^5-6A^3-A)\sqrt{\frac{-A^4+6A^2-1}{A^2}} + A^6 - 9A^4 - 9A^2 + 1},$$

which is nonzero for $A \in (\sqrt{2}-1, A_{\max}]$ and zero for $A = \sqrt{2}-1$. Note that a similar computation can be done for $\alpha_{\text{crit},2}(A)$. This implies that the same is true for (79). We proceed by differentiating both sides of (79) and evaluating at $A = \sqrt{2}-1$ and $\alpha = \alpha_{\text{crit}}$, yielding

$$\mathcal{F}'''\Big(y\big(\alpha_{\text{crit}}(A)\big)\Big)y_\alpha\big(\alpha_{\text{crit}}(A)\big)^3 = x_{\alpha\alpha\alpha}\big(\alpha_{\text{crit}}(A)\big), \tag{80}$$

where above, we used that $\mathcal{F}'\Big(y\big(\alpha_{\text{crit}}(A)\big)\Big) = 0 = \mathcal{F}''\Big(y\big(\alpha_{\text{crit}}(A)\big)\Big)$. It follows from (80) that

$$\mathcal{F}'''\Big(y\big(\alpha_{\text{crit}}(A)\big)\Big) = \frac{x_{\alpha\alpha\alpha}\big(\alpha_{\text{crit}}(A)\big)}{y_\alpha\big(\alpha_{\text{crit}}(A)\big)^3},$$

when $A = \sqrt{2} - 1$. Moreover, note that

$$\begin{aligned}
x_{\alpha\alpha\alpha}(\alpha) =& \frac{8A}{(1 + A^2 + 2A\cos(\alpha))^4}\Bigg(\Big(A^5\cos(\alpha)^2 + 4A^4\cos(\alpha)^3 + \Big(4\cos(\alpha)^4 + 2\cos(\alpha)^2\Big)A^3 \\
&+ \Big(4\cos(\alpha)^3 - \cos(\alpha)\Big)A^2 + \Big(5\cos(\alpha)^2 - 6\Big)A - \cos(\alpha)\Big)A\sin(\alpha)\big(A^2 - 1 + 2A\sin(\alpha)\big) \\
&+ \frac{A\sin(\alpha)\left(1 + A^2 + 2A\cos(\alpha)\right)\left(A^2 - 6A\cos(\alpha) + 1\right)\cos(\alpha)\left(A^2 - 1 + 2A\sin(\alpha)\right)}{2} \\
&+ \frac{A^4\cos(\alpha)}{2} - 2A^3\cos(\alpha)^2 + 3A^3 + 2A\cos(\alpha)^2 - 3A - \frac{\cos(\alpha)}{2}\Bigg),
\end{aligned}$$

which evaluated at $A = \sqrt{2} - 1$ and $\alpha_{\text{crit},1}(\sqrt{2} - 1) = \alpha_{\text{crit},2}(\sqrt{2-1}) = \frac{\pi}{2}$ becomes

$$-\frac{6\left(\sqrt{2} - 1\right)^3}{\left(\sqrt{2} - 2\right)^4},$$

which is nonzero.

Next, we state a result concerning the local behaviour of critical layers of a certain class of breaking and overhanging waves, which include the solutions constructed by Hur & Wheeler.

Proposition 19. *Let z denote the conformal map to the half plane of a solution to (15) with $\tau = G = 0$ and assume the same conditions as in Proposition 10. Let z denote the conformal map to the half plane of a solution to (15) with $\tau = 0$ and assume the same conditions as in Proposition 10. Additionally, assume there exists a point $z_{crit} = (x_{crit}, y_{crit})$ at which the wave is vertical and that there exists a neighbourhood of said point where the surface profile of the wave can be represented as $(F(y), y)$.*

(i) If $F''(y_{crit}) = 0$ and $F'''(y_{crit}) \neq 0$, then the critical layers intersect the surface and form a corner from both inside and outside the water.

(ii) If $F''(y_{crit}) \neq 0$, then the critical layers intersect the surface from both inside and outside the water.

Proof. Differentiating the kinematic boundary condition (14b) twice and evaluating it at $y = y_{\text{crit}}$

yields

$$\psi_{yy}(z_{\text{crit}}) + 2\psi_{yx}(z_{\text{crit}})F'(y_{\text{crit}}) + \psi_{xx}(z_{\text{crit}})F'(y_{\text{crit}}))^2 + \psi_x(z_{\text{crit}})F''(y_{\text{crit}}) = 0, \tag{81}$$

since $F'(y_{\text{crit}}) = 0$, it follows that

$$\psi_{yy}(z_{\text{crit}}) = -\psi_x(z_{\text{crit}})F''(y_{\text{crit}})). \tag{82}$$

Notice that since $\psi_x(z_{\text{crit}}) \neq 0$, it follows the right-hand side of the equation above does not vanish in case (ii) but vanishes in case (i). We proceed with squaring and then differentiating the dynamic boundary condition (14d), and evaluating it at $y = y_{\text{crit}}$, resulting in

$$\begin{aligned} 2\Bigg(&\psi_x(z_{\text{crit}})\Big(\psi_{xy}(z_{\text{crit}}) + \psi_{xx}(z_{\text{crit}})F'(y_{\text{crit}})\Big) \\ &+ \psi_y(z_{\text{crit}})\Big(\psi_{yy}(z_{\text{crit}}) + \psi_{yx}(z_{\text{crit}})F'(y_{\text{crit}})\Big)\Bigg) = 0. \end{aligned} \tag{83}$$

Since $\psi_y(z_{\text{crit}}) = 0 = F'(y_{\text{crit}})$, it follows that $\psi_{xy}(z_{\text{crit}}) = 0$. Subsequently, we differentiate (81) with respect to y and evaluate it at $y = y_{\text{crit}}$, yielding

$$\psi_{yyy}(z_{\text{crit}}) + \psi_x(z_{\text{crit}})F'''(y_{\text{crit}}) = 0.$$

We note that above, we used that $F'(y_{\text{crit}}) = 0 = \psi_{yx}(z_{\text{crit}})$. It then follows $\psi_{yyy}(z_{\text{crit}}) = -\psi_x(z_{\text{crit}})F'''(y_{\text{crit}})$. Afterwards, we differentiate the Poisson equation (14a) with respect to y and evaluate it at $y = y_{\text{crit}}$, resulting in

$$\psi_{yyy}(z_{\text{crit}}) + \psi_{xxy}(z_{\text{crit}}) = 0,$$

which implies $\psi_{xxy}(z_{\text{crit}}) = -\psi_{yyy}(z_{\text{crit}})$.

In case (i), the Taylor expansion of ψ_y at $y = y_{\text{crit}}$ is

$$\begin{aligned}&\psi_{yyy}\big(z_{\text{crit}}\big)\left(\left((y-y_{\text{crit}})^2\right)-\big(x-F\left(y_{\text{crit}}\right)\big)^2\right)\\=&\mathcal{O}\left(|(x-F(y_{\text{crit}}),y-y_{\text{crit}}(a_{\text{crit}}))|^3\right).\end{aligned}$$

Afterwards, we apply the non-analytic version of the Implicit Function Theorem to the equation above with respect to the variable $(y - y_{\text{crit}})^2$, yielding

$$(y-y_{\text{crit}})^2 = \big(x - F\left(y_{\text{crit}}\right)\big)^2 + \mathcal{O}\left(|(x-F(y_{\text{crit}}),y-y_{\text{crit}}(a_{\text{crit}}))|^3\right).$$

The last equation implies (i). On the other hand, in case (ii), the Taylor expansion of ψ_y at $y = y_{\text{crit}}$ is

$$-\psi_x\big(z_{\text{crit}}\big)F''\big(y_{\text{crit}}\big))\left(y-y_{\text{crit}}(a_{\text{crit}})\right) = \quad \mathcal{O}\left(|(x-F(y_{\text{crit}}),y-y_{\text{crit}}(a_{\text{crit}}))|^2\right).$$

We can then apply the analytic version of the Implicit Function Theorem to the equation above, yielding that there exists a neighbourhood of z_{crit} and an analytic function ι, such that the pair (x, y) satisfying $\psi_y = 0$ is given by $(x, \iota(x))$ in said neighbourhood. Statement (ii) now follows. $\square$

7 Finite depth

In this Section, we consider the case of a fluid bounded from below by a flat bottom. In the first Subsection, we go over the formulation and derive an analogous equation to (34) for the finite depth case. In the second Subsection, we discuss Hur & Wheeler's argument [31] for the existence of overhanging and touching waves for large finite depth. Finally, in the third Subsection, we extend our argument from Section 5 to construct a continuous curve of solutions from laminar flow to a touching wave for the finite depth case.

7.1 Formulation

In this section, we consider (9a) instead of (9b). Under this assumption, the same argument as in Section 3.2 leads to

$$\triangle\psi = -\Omega \qquad \text{in } D, \tag{84a}$$

$$\psi = 0 \qquad \text{on } S, \tag{84b}$$

$$\psi = k \qquad \text{on } y = -h, \tag{84c}$$

$$\frac{1}{2}|\nabla\psi|^2 + Gy - \tau\kappa = B \qquad \text{on } S, \tag{84d}$$

where k denotes an a priori unknown constant and D is 2π periodic domain that is bounded from above by S and from below by $\{y = -h\}$.

For any $d > 0$, we define the strip

$$\mathbb{S}_d \coloneqq \{\alpha + i\beta \in \mathbb{C}; 0 > \beta > -d\}.$$

A similar result to Lemma 9 holds for a finitely deep domain. Namely, that for a given domain D as above, there exists a unique constant d and conformal map z between $\mathcal{S}_d$ and D satisfying the following three conditions.

(i) The continuous extension of z maps $\mathbb{S}_d \cup \{\beta = 0\}$ to $D \cup S$ continuously;

(ii) $x(\alpha + i0) - \alpha$ and $y(\alpha + i0)$ are 2π periodic;

(iii) The continuous extension of z maps $\mathbb{S}_d \cup \{\beta = -d\}$ to $D \cup \{y = -h\}$ continuously;

where $x(\alpha + i\beta)$ and $y(\alpha + i\beta)$ denote the real and imaginary part of $z(\alpha + i\beta)$. For a proof, we refer to Appendix A of [13].

Next, we define the periodic Hilbert transform on a strip.

Definition 7 (Periodic Hilbert transform on a strip)**.** We define the periodic Hilbert transform on $\mathbb{S}_d$, $\mathcal{H}_d : L^2_{2\pi} \to L^2_{2\pi}$ by its action on the orthonormal basis $\{\exp(inx)\}_{n\in\mathbb{Z}}$, as follows

$$\mathcal{H}_d(\exp(inx)) = \begin{cases} -i\coth(nd)\exp(inx), & \text{if } n \neq 0 \\ 0, & \text{if } n = 0 \end{cases}$$

where coth denotes the hyperbolic cotangent.

Remark. A singular integral formula for the periodic Hilbert transform on a strip was recently derived in the Appendix of [12] under a stronger regularity assumption. Let $f \in C^{1+\alpha}_{2\pi}$, then

$$(\mathcal{H}_d(f))(x) = \frac{1}{2\pi}\text{p.v.}\int_{-\pi}^{\pi} g_d(x-y)f(y)dy,$$

where $g_d : \mathbb{R} \setminus 2\pi\mathbb{Z}$ is given by

$$g_d(x) = -\frac{x}{d} + \frac{\pi}{d}\coth\frac{\pi x}{2d} + \frac{\pi}{d}\sum_{k\in\mathbb{Z}} \frac{2\sinh(\pi x/d)}{\cosh(\pi x/d) - \cosh(2\pi^2 k/d)}.$$

Moreover, note that

$$||\mathcal{H}_d(f) - \mathcal{H}(f)||_{C^{1+\alpha}_{2\pi,\circ}} \leq 2\pi \sup_{y\in[-\pi,\pi]\setminus 0} |g_d(y) - \cot(\frac{y}{2})| \cdot ||f||_{C^{1+\alpha}_{2\pi,\circ}}. \tag{85}$$

Using equations (A.11) and (A.12) of [12], we obtain the following estimate.

$$\begin{aligned} |g_d(y) - \cot(\frac{y}{2})| &= |\sum_{n=1}^{\infty} \frac{4}{\exp(2nd) - 1}\sin(ny)| \\ &\leq 4\sum_{n=1}^{\infty} \frac{1}{\exp(2nd) - 1} \to 0 \end{aligned}$$

as $d \to \infty$ independently of y for all $y \in [-\pi, \pi] \setminus 0$. It follows that $\mathcal{H}_d$ converges to $\mathcal{H}$ as $d \to \infty$ in the strong operator topology.

A similar argument as in Section 3.4 leads to a finite version of a modified Babenko equation,

$$\frac{1}{2}(1 + \Omega(y + y\mathcal{H}_d y_\alpha - \mathcal{H}_d(yy_\alpha)))^2 = (B - Gy)((1 + \mathcal{H}_d y_\alpha)^2 + y_\alpha^2) \qquad \text{on } \beta = 0. \tag{86}$$

We refer to [24] for a thorough derivation of (86). Next, we introduce the annulus

$$\mathbb{A}_d := \{\zeta \in \mathbb{D} : |\zeta| > \exp(-d)\}$$

and define $w : \mathbb{A}_d \to \mathbb{C}$ to be such that

$$z(\zeta) = i \log \zeta + w(\zeta). \tag{87}$$

This decomposition transforms (86) into

$$(B - G\Im(w)) = \frac{1}{2} \frac{\left(1 + \Omega(\Im w + \mathcal{Q}^d(w)\right)^2}{|1 - i\zeta w_\zeta|^2} \qquad \text{for } |\zeta| = 1,$$

where

$$\mathcal{Q}^d(w(\zeta)) := \Im(w)\mathcal{H}_d\left(\Im(-i\zeta w_\zeta)\right) - \mathcal{H}_d\left(\Im w \Im(-i\zeta w_\zeta)\right) \qquad \text{for } |\zeta| = 1,$$

is the commutator operator for the periodic Hilbert transform on a strip. Note that when f is defined on the unit circle, we perform the following slight abuse of notation

$$\mathcal{H}_d(f) = \mathcal{H}_d\left(f \circ \exp(i\cdot)\right).$$

In this case, the function f can be seen as the real part outer boundary value of a function holomorphic on an annulus.

7.2 Overhanging and Touching waves

In this subsection, we provide a more detailed explanation of Hur & Wheeler's proof, which establishes the existence of finite depth overhanging and touching waves. This proof is discussed in Section 5 of [31].

We define the Banach spaces

$$X := \{w \in C^{3+a}(\partial\mathbb{D}, \mathbb{R}) : w(\overline{\zeta}) = w(\zeta)\},$$
$$Y := \{w \in C^{2+a}(\partial\mathbb{D}, \mathbb{R}) : w(\overline{\zeta}) = w(\zeta)\}.$$

We pause to note that in the infinite depth case, the counterpart of X (39) is defined in terms of holomorphic functions on the disc. However, in the finite depth case, the functions would be holomorphic on an annulus that would change as the parameter d changes. To handle this issue, we instead consider a space defined in terms of real-valued functions on the unit circle. For a given $w \in X$ and constant d, we can extend w to a function $\tilde{w}$ that is harmonic on $\mathbb{A}_d$ and satisfies the following conditions

(i) $\tilde{w} = w$ on $\partial\mathbb{D}$;

(ii) $\tilde{w}$ is constant on the inner boundary of $\mathbb{A}_d$. Moreover, the constant is chosen so that the mean over the inner circle of $\mathbb{A}_d$ is equal to the mean over the outer circle of $\mathbb{A}_d$.

We can then use $\tilde{w}$ and its harmonic conjugate to define a holomorphic function on $\mathbb{A}_d$. We then introduce the map $E : X \times (0, \infty] \to Y$ that maps a pair (w, d) to the function that's holomorphic on $\mathbb{A}_d$, whose imaginary part satisfies (i) and (ii) when $d \in (0, \infty)$ or to the function that's holomorphic on $\mathbb{D}$, whose imaginary part satisfies (i) when $d = \infty$. We then define the subset

$$V := \{w \in X : 1 - i\zeta \frac{\partial E(w, \infty)}{\partial \zeta} \neq 0\}$$

and the operator $\mathcal{F} : V \times \mathbb{R} \times (-\infty, 1/3) \times \mathbb{R} \to Y$

$$\mathcal{F}(w, G, a, l) = \frac{1}{2} \frac{\left(1 + \Omega(a)\left(w + \mathcal{Q}^{\frac{1}{l^2}}\left(E(w, \tfrac{1}{l^2})\right)\right)\right)^2}{|1 - i\zeta\partial_\zeta E(w, \frac{1}{l^2})|^2} - (B(a) - Gw) \qquad \text{for } |\zeta| = 1 \tag{88}$$

where

$$\mathcal{Q}^{\frac{1}{l^2}} = \begin{cases} \mathcal{Q}^{\frac{1}{l^2}} & \text{if } l \neq 0, \\ \mathcal{Q} & \text{if } l = 0 \end{cases}$$

and $\mathcal{Q}$ denotes the operator defined in (37).

The operator $\mathcal{F}$ is real-analytic in the first three variables. Recall (85) and the discussion following it, we can then differentiate the Fourier series of $g_d(y) - \cot(\frac{y}{2})$ termwise with respect to d, yielding

$$\sum_{n=1}^{\infty} \frac{16}{(\exp(2n/l^2) - 1)^2} \frac{n}{l^3} \exp(2n/l^2) \sin(ny).$$

Note that for fixed l, the series above converges uniformly, meaning $\mathcal{H}_{\frac{1}{l^2}}$ is partial Fréchet differentiable with respect to l (note that $\mathcal{H}$ does not depend on l). Moreover, observe that as $l \to 0$, the series above converges to zero. Similarly, we can take higher-order derivatives of the Fourier series of $g_d(y) - \cot(\frac{y}{2})$ term-wise with respect to l and the series obtained will converge uniformly due to the exponent on the denominator. Furthermore, as $l \to 0$, the series obtained will converge to zero. From this argument, it follows that $\mathcal{H}_{\frac{1}{l^2}}$ is smooth with respect to l. In turn, this implies $\mathcal{F}$ is a smooth operator.

Let U and $\mathcal{G}$ be as in (41) and (42), respectively. Then for a function $v \in U$, $\mathcal{G}(v, G, a)$ coincides with $\mathcal{F}(\Im v, G, a, 0)$, except the latter one is written in terms of the boundary data. In particular, this implies

$$\mathcal{F}(\Im(w(a)), 0, a, 0) = 0,$$

where $w(a)$ is defined as in (43) but in terms of boundary values. The partial Fréchet derivative of $\mathcal{F}$ can be computed to be

$$\begin{aligned} D_x\mathcal{F}[w, G, a, 0]v = & \frac{1 + \Omega(a)(w + \mathcal{Q}(E(w, \infty))}{|1 - i\zeta\partial_\zeta E(w, \infty)|^2} \Omega(a)(v + \mathcal{Q}_w(E(w, \infty))E(v, \infty) \\ & - \frac{(1 + \Omega(a)(w + \mathcal{Q}(E(w, \infty)))^2}{|1 - i\zeta\partial_\zeta E(w, \infty)|^4} \Im\left(\overline{(1 - i\zeta\partial_\zeta E(w, \infty))}\zeta\partial_\zeta E(v, \infty)\right) + Gv. \end{aligned}$$

Which resembles the Fréchet derivative of the operator for the infinite depth case. See the discussion following Theorem 3 in [31]. In fact, we have that $D_x\mathcal{F}[\Im(w(a)), 0, a, 0]$ coincides with $D_x\mathcal{G}[w(a), 0, a]$ written in terms of the imaginary part of $w(a)$ restricted to the unit circle. This implies that

$D_x\mathcal{F}[\Im(w(a)), 0, a, 0]$ is an isomorphism. By the non-analytic version of Theorem 2, the following result now holds.

Theorem 20. *For each $a_0 \in (0, \frac{1}{4})$ there exists $\epsilon > 0$ and a smooth operator*

$$W : (-\epsilon, \epsilon) \times (a_0 - \epsilon, a_0 + \epsilon) \times (-\epsilon, \epsilon) \to V$$

such that $W(0, a, 0) = \Im(w(a))$ and

$$\mathcal{F}(W(G, a, l); G, a, l) = 0.$$

Moreover, there exists $\delta > 0$ such that for all $(G, a, l) \in (-\epsilon, \epsilon) \times (a_0 - \epsilon, a_0 + \epsilon) \times (-\epsilon, \epsilon)$, the following statements are equivalent.

(i) $\mathcal{F}(w; G, a, l) = 0$ and $||w - \Im(w(a))||_X < \delta$;

(ii) $w = W(G, a, l)$.

Theorem 20 implies the following results.

Theorem 21. *For all $a \in (a_{crit}, a_{max})$, there exists an $\epsilon > 0$, such that for all $G \in (-\epsilon, \epsilon)$ and some finite h, there exists a solution of (84) with $\Omega = \Omega(a)$ and $B = B(a)$ whose profile is overhanging.*

Theorem 22. *There exists an $\epsilon > 0$, such that for all $G \in (-\epsilon, \epsilon)$ and some finite h, there exists a solution of (84) whose profile intersects itself tangentially, enclosing a small bubble of air.*

7.3 Continuous curve of solutions for finite depth

In this subsection, we extend our results from Section 5 to the finite depth case. We will highlight the differences and refer to Section 5 when the details are similar.

It follows from the discussion preceding Theorem 20 that a similar argument to the one in Lemma 15 yields the following Lemma.

Lemma 23. *For all $\lambda \in (0, \frac{1}{8})$ there exist $\epsilon > 0$ and a real-analytic operator*

$$W : (-\epsilon, \epsilon) \times [\lambda, \tfrac{1}{4} - \lambda] \times (-\epsilon, \epsilon) \to V$$

such that $W(0,a,0) = \Im(w(a))$ *and*

$$\mathcal{F}(W(G,a,l);G,a,l) = 0.$$

Moreover, there exists $\delta > 0$ *such that for all* $(G,a,l) \in (-\epsilon,\epsilon) \times [\lambda, \frac{1}{4} - \lambda] \times (-\epsilon,\epsilon)$ *the following statements are equivalent.*

(i) $\mathcal{G}(w;G,a,l) = 0$ *and* $||w - \Im(w(a))||_X < \delta$*;*

(ii) $w = W(G,a,l)$.

Let $\tilde{d}$ denote a constant. Then

$$\mathcal{F}(\tilde{d},G,a,l) = \frac{1}{2}(1+\Omega(a)\tilde{d})^2 + G\tilde{d} - B(a) \tag{89}$$

and

$$D_x\mathcal{F}[\tilde{d},G,a,l](v) = \Im\left(\left((1+\Omega(a)\tilde{d})\Omega(a)+G\right)E(v,\frac{1}{l^2}) - (1+\Omega(a)\tilde{d})^2\zeta\partial_\zeta E(v,\frac{1}{l^2})\right) \qquad \text{for } |\zeta| = 1. \tag{90}$$

The symmetry assumption placed on X implies that the coefficients of the Fourier series of $E(v,\frac{1}{l^2})$ are imaginary. We can then employ a method analogous to the proof of Lemma 16 to demonstrate a corresponding result. The first statement of the Lemma below follows from choosing $\tilde{d}$ so that the right-hand side of (89) is equal to zero. The second statement follows from inserting $\tilde{d}$ in (90) and choosing $\tilde{G}$ so that the resulting expression has its kernel spanned by $\Re(\zeta)$. The choice of $\tilde{d}$ does not depend on l, however the map $\tilde{G}$ depends smoothly on l. This can be seen in the Fourier series expansion of $E(v,\frac{1}{l^2})$ (see (A.2) and the discussion following it in [12]).

Lemma 24. *There exists a smooth operator* $\tilde{d} : \mathbb{R} \times (-\infty, 1/3) \to \{t : t \in \mathbb{R}\}$ *such that*

$$\mathcal{F}(\tilde{d}(G,a);G,a,l) = 0 \qquad \text{and} \qquad \tilde{d}(0,0) = 0$$

for $(G,a) \in \mathbb{R} \times (-\infty, 1/3)$*. Moreover, there exist open neighbourhoods* $V \subset \mathbb{R}^2, T \subset \mathbb{R}$ *containing* *zero and a smooth map $\tilde{G} : V \to T$, such that*

$$\ker D_x \mathcal{G}\Big[\tilde{d}(\tilde{G}(a,l),a); \tilde{G}(a,l), a, l\Big] = \langle \Re(\zeta)\rangle_X$$

for $a \in V$ and $\tilde{G}(0,0) = 0$.

We note that

$$\mathcal{Q}_w^{\frac{1}{l^2}}[\tilde{d}, l](\Re(\zeta)) = 0$$

and

$$\mathcal{Q}_{ww}^{\frac{1}{l^2}}[\tilde{d}, l](\Re(\zeta))^2 = \coth\tfrac{2}{l^2} + (1 - \Re(\zeta^2))(\coth\tfrac{1}{l^2} - \coth\tfrac{2}{l^2}).$$

Using non-analytic analogues of the bifurcation theorems used in the proof of Lemma 17, the following result can be obtained.

Lemma 25. *There exist $\epsilon > 0$ and a smooth operator $v : \tilde{S} \to U$, where $\tilde{S}$ is defined as*

$$\tilde{S} := \Big\{(G,a,l) \in \mathbb{R} \times (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) : \tilde{G}(a,l) \le G < \tilde{G}(a+\epsilon, l+\epsilon)\Big\}$$

such that $v(G,a,l)$ satisfies the equation

$$\mathcal{F}\left(v(G,a,l) + \tilde{d}(G,a); G, a, l\right) = 0$$

for $(G,a,l) \in \tilde{S}$. Moreover, $v(G,a,l)$ is of the form

$$v(G,a,l) = C_{G,l}(a - \tilde{G}^{-1}(G))^{\frac{1}{2}}\Re\zeta + \mathcal{O}((a - \tilde{G}^{-1}(G)) + l),$$

where $C_{G,l}$ is a positive constant that depends continuously on G and l. Furthermore, there exists $\delta > 0$ such that $v(G,a,l) + \tilde{d}(G,a)$ is the only nontrivial solution of $\mathcal{F}\big(v, G, a, l\big) = 0$, satisfying $||v||_X < \delta$.

We note that in a proof of the Lemma above, one has to consider a version of Proposition 6 and Corollary 7 with an additional parameter.

We can repeat the same argument as in the latter part of Section 5.2 to yield a generalization of Theorem 12.

Theorem 26. *For all $\gamma \in (0, \frac{1}{8})$ there exists $\epsilon > 0$ and a smooth operator*

$$W : S \to U,$$

where

$$S := \left\{ (G, a, l) \in (-\epsilon, \epsilon) \times (-\epsilon, \tfrac{1}{4} - \gamma] \times (-\epsilon, \epsilon) : G \geq \tilde{G}(a, l) \right\}.$$

Here, $\tilde{G}$ is as in Lemma 24. The operator satisfies the following three conditions.

1. $W(0, a, l) = \Im(w(a))$*;*

2. $W(\tilde{G}(a, l), a, l)$ *is a constant function of* ζ*;*

3. $\mathcal{F}(W(G, a, l); G, a, l) = 0$.

Moreover, there exists $\delta > 0$ such that for all $(G, a, l) \in \tilde{S}$, the following statements are equivalent.

(i) $\mathcal{G}(w; G, a, l) = 0$ *and* $||w - \Im(w(a))||_X < \delta$*;*

(ii) $w = W(G, a, l)$.

Theorem 26 implies the following two results.

Theorem 27. *For every sufficiently small $\epsilon > 0$, $G \in (-\epsilon, \epsilon)$ and some finite h, there exists a continuous curve of solutions of (84) between a laminar flow and a touching wave.*

Corollary 28 (Finite depth breaking waves)**.** *For every sufficiently small $\epsilon > 0$, $G \in (-\epsilon, \epsilon)$ and some finite h, there exists a solution of (84) whose profile is vertical at a point but never overhanging.*

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