Axisymmetric capillary water waves on cylindrical fluid jets

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Axisymmetriska kapillära vattenvågor på cylindriska vätskestrålar

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Abstract

In their paper ([12]), Vanden-Broeck, Miloh and Spivack describe two limiting behaviours of irrotational, axisymmetric capillary water waves using numerical methods. They found a two-parameter family of nontrivial solutions. Some solutions of small amplitude approach a uniform stream, while others approach a static configuration. Our main interest is with the branch of static configurations, and most importantly, a bifurcating curve which occurs at each point of the static branch. The static branch begins at a cylinder-like solution and then smoothly varies with wave steepness until it terminates at a solution corresponding to spheres (droplets). In this work, we wish to show analytically that for each point of the static branch, there exists a connecting curve of non-static solutions. We do this by first identifying the static configuration, and then performing a continuation analysis at an arbitrary point of the static branch. Our methods include nonlinear functional analysis, as well as classical theory of existence and regularity of solutions through classical Schauder estimates for elliptic partial differential equations.

Populärvetenskaplig Sammanfattning

Inom fysiken och vissa delar av matematiken siktar man på att beskriva så mycket av naturen som möjligt. Det gör man genom att hitta modeller som passar bäst till varje situation. Naturligtvis är vissa system mycket svårare än andra att förstå matematiskt och vattenvågor är ett exempel på ett sådant system. Under 1800-talet formulerades **Euler**-ekvationerna, som beskriver rörelsen av ideala fluider, det vill säga inkompressibla fluider med konstant densitet. Kopplade till Euler-ekvationerna är de randvillkoren som kan avgöra vilket slags vågor formeras. Det finns *gravitationsvågor* och *kapillärvågor*. Gravitationsvågor påverkas främst av gravitation, vilket är deras återställande kraft. Åt andra sidan, kapillärvågornas dynamik domineras främst av ytspänning. I denna uppsatsen, studerar vi kapillärvågor som uppstår på en cylindrisk vattenstråle och som påverkas endast av ytspänning. Syftet är att vissa att icke-stationära lösningar existerar, vilket innebär att vi letar efter lösningar när våghastigheten är skild från noll.

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1 Introduction

This subsection is based on [6] and [12]. We study the periodic, capillary waves of an inviscid (frictionless), incompressible fluid in a simply connected three-dimensional jet-like domain $\Omega^\eta \subset \mathbb{R}^3$, bounded in the vertical direction and extending horizontally to the whole real line (see figure 1). The flow is described by the gradient of the potential function ϕ and is assumed to be irrotational, axisymmetric about the horizontal z-axis and periodic in z. The boundary $\partial\Omega^\eta$ is a free surface. It is a priori unknown and in cylindrical coordinates given by $r = \eta(z)$ where $r = \sqrt{x^2 + y^2}$. The domain is therefore given by $\Omega^\eta = \{(x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2} < \eta(z)\}$. Moreover, we assume there is no flow outwards from the boundary. Mathematically, this translates to

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega^\eta, \\ \partial_n \phi = 0 & \text{on } \partial\Omega^\eta \end{cases} \tag{1.1}$$

together with an additional boundary condition

$$\frac{1}{2}|\nabla\phi|^2 - \frac{T}{\rho}H(\eta) = B, \text{ on } \partial\Omega^\eta. \tag{1.2}$$

Equation (1.1) is known as the *Neumann problem* for the Laplace equation and (1.2) is known as *Bernoulli's boundary condition*. In the latter, T is the surface tension coefficient, ρ is the density of the fluid and $H(\eta)$ is the *mean curvature* of $\partial\Omega^\eta$, given by

$$H(\eta) = \frac{\eta_{zz}}{(1 + \eta_z^2)^{3/2}} - \frac{1}{\eta(1 + \eta_z^2)^{1/2}}.$$

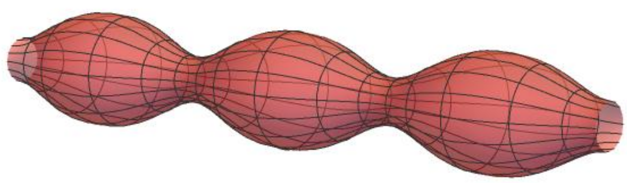

Figure 1: Water domain for 3 periods, following [9]

We now give some background to this model. The corresponding equations are derived form the so-called *Euler equations*. These describe the conservation of mass and the conservation of momentum. Let V be a volume containing water (assume it is a simply connected domain), and suppose $\mathbf{u}(\mathbf{x}, t)$ be the velocity of the fluid at position $\mathbf{x} = (x, y, z)$ and at time t. There are two types of forces we can consider.

1. Forces acting on the fluid when it is considered as a whole body, such as gravity.

2. Local forces acting as a normal force onto elements of the fluid, such as pressure.

Denote by $\mathbf{F} = \mathbf{F}(\mathbf{x}, t)$ the total force of type (1) per unit mass. Let $P(\mathbf{x}, t)$ be the pressure, and let $\rho = \rho(\mathbf{x}, t)$ be the density of the fluid. These are the force of type (2). First, the general formulae are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

which is the conservation of mass and

$$\rho\Big(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\Big) = \rho\mathbf{F} - \nabla P + \nu \Delta \mathbf{u}$$

the equation of conservation of momentum, with ν being the *dynamic viscosity*, a measurement of the amount of friction per density of the fluid. For our purposes, it is enough to consider a special case of these equations, namely Euler's equations, which model an ideal, incompressible fluid of constant density $\rho = \rho_0$ and no viscosity. Then, the Euler equations are

$$\nabla \cdot \mathbf{u} = 0 \tag{1.3}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla P + \mathbf{F}. \tag{1.4}$$

Vorticity, which measures the degree of local spin of the fluid, can also be left out from the model. Then, the fluid is called *irrotational*, and mathematically this corresponds to $\text{curl}(\mathbf{u}) = 0$. Consequently, for a simply connected domain, there exists some potential function $\phi(\mathbf{x}, t)$ such that we can write $\mathbf{u} = \nabla\phi$ and (1.3) becomes

$$\Delta\phi = 0.$$

Coupled to the equations of conservation of mass and momentum are the set of boundary conditions to the fluid problem. These are the *kinematic* and *dynamic* boundary conditions. They describe the surface of the fluid, which is not fixed. The surface is given by $r = \eta(z, t)$ in cylindrical coordinates. The kinematic boundary condition requires that the flow occurs only within the defined domain V, so that the volume of water always stays the same. Given that $\mathbf{n} = (\frac{x}{\eta}, \frac{y}{\eta}, -\eta_z)$ is the normal vector to the surface (not of unit length), the condition reads

$$\partial_n\phi - \partial_t\eta = 0. \tag{1.5}$$

If the problem assumes a bottom boundary (such as the bottom of the ocean), the condition there is defined analogously. The dynamic boundary condition accommodates for surface tension and is given by

$$P_i - P_o = TH(\eta) \tag{1.6}$$

where T is the surface tension coefficient, $H(\eta)$ denotes the mean curvature of the surface and P_i and P_o are the inside and outside fluid pressure respectively. For our purposes, $P_0 = 0$. Equation (1.6) can be combined with what is known as *Bernoulli's principle.* Assuming that the force $\mathbf{F}$ is a conservative vector field, Bernoulli's principle is derived from (1.4) and it is given by

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 - \frac{P + U}{\rho_0} = B$$

where $\nabla U = \mathbf{F}$ and B is known as the Bernoulli constant. This equation holds in the whole fluid domain. Since our problem involves capillary waves with no gravity, we assume that $\mathbf{F} = 0$, and the only force considered is the local one generated by surface tension, which we obtain by combining with (1.6). We then have *Bernoulli's boundary condition* given by

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 - \frac{T}{\rho_0}H(\eta) = B$$

with

$$H(\eta) = \frac{\eta_{zz}}{(1+\eta_z^2)^{3/2}} - \frac{1}{\eta(1+\eta_z^2)^{1/2}}$$

The equation for the mean curvature can easily be computed using the divergence of the unit normal. We have arrived at the following equations:

$$\Delta\phi = 0, \quad \text{in } V$$

$$\partial_n\phi - \partial_t\eta = 0, \quad \text{on } \partial V \tag{1.7}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 - \frac{T}{\rho_0}H(\eta) = B, \quad \text{on } \partial V \tag{1.8}$$

Since we are interested in steady, periodic capillary waves propagating in the z-direction at some positive speed c, we can get rid of the time variable in $\phi(x,y,z,t)$. This is done by first writing $\phi(x,y,z,t) = \widetilde{\phi}(x,y,z+ct)$. Now, we can define a new function $\overline{\phi}$ by

$$\overline{\phi}(x,y,\widetilde{z}) := \widetilde{\phi}(x,y,\widetilde{z}) + c\widetilde{z}$$

where $\widetilde{z} := z+ct$. Rewriting the first two terms of equation (1.8) in terms of $\widetilde{z}$ yields

$$\begin{aligned} c\frac{\partial\widetilde{\phi}}{\partial\widetilde{z}} + \frac{1}{2}|\nabla_{x,y,\widetilde{z}}\widetilde{\phi}|^2 &= c\left(\frac{\partial\overline{\phi}}{\partial\widetilde{z}} + c\right) + \frac{1}{2}|\nabla_{x,y,\widetilde{z}}\overline{\phi} - (0,0,c)|^2 \\ &= \frac{1}{2}|\nabla_{x,y,\widetilde{z}}\overline{\phi}|^2 - \frac{c^2}{2}. \end{aligned}$$

The Bernoulli condition then becomes

$$\frac{1}{2}|\nabla_{x,y,\widetilde{z}}\overline{\phi}|^2 - \frac{T}{\rho_0}H(\eta) = \widetilde{B}$$

where $\widetilde{B} = B - \frac{c^2}{2}$. Note that the equation for $H(\eta)$ is unchanged since it only concerns spatial variables. The same kind of argument applies to equation (1.7). If we redefine $\overline{\phi}$ as ϕ and $\widetilde{B}$ as B for simplicity, we finally have

$$\frac{1}{2}|\nabla\phi|^2 - \frac{T}{\rho_0}H(\eta) = B, \quad \text{on } \partial V$$

and

$$\partial_n\phi = 0, \quad \text{on } \partial V$$

and we get the functional setup for our problem.

Remark. As we shall see later, in the static solution, the term $|\nabla\phi|^2$ vanishes and we obtain $-\frac{T}{\rho_0}H(\eta) = B$. Hence, for a static solution, the mean curvature of the resulting surface is constant. We will then show that the surface is an unduloid.

2 Earlier result

In [12], Vanden-Broeck, Miloh and Spivack considered periodic, axisymmetric waves propagating at a constant velocity on a jet. As in our case, the model was given by (1.1) and (1.2), but the investigation was done numerically. They found that the waves were characterized by the thickness of the jet Q (the flux of the velocity field across a cross-section of the fluid) as well as their steepness S (the difference of the highest and lowest amplitude points of the free surface function). Two values of Q were investigated, Q_1 and Q_2. Our interest lies with Q_2. It was found that as S decreases, some of the solutions approach a uniform stream, while others reach a static configuration. Of these two results, we are interested in the static configuration, which exhibits two limiting behaviours. The static branch, illustrated in figure 2 as the solid curve, is our main point of interest.

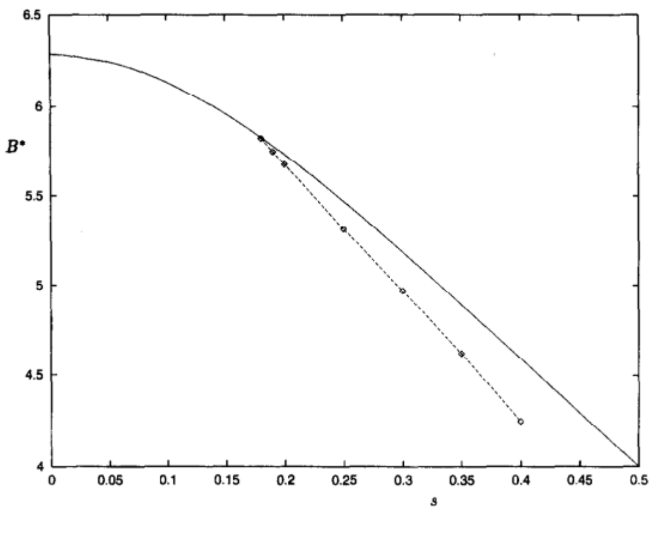

Figure 2: Branch of static solutions, picture from [12]

The vertical axis corresponds to a scaled value of the mean curvature of the profile curve of the unduloids, while the horizontal axis is for the steepness of the wave.

Upon inspecting this graph, one can see that the configuration begins at a cylinder-like shape and terminates at an array of spheres of a certain radius. The broken curve that bifurcates from one point on the solid branch is the curve corresponding to non-static solutions. This is the curve we aim to show exists in a rigorous way. While in [12] they directly (numerically) compute solutions for various values of the steepness, we work the other way around and establish first the static configuration. We then introduce perturbations and we linearize the problem in the direction of these perturbations. Applying the implicit function theorem, we get the non-static branch.

As introduced in section 1 and [12], the capillary wave problem is described by equations (1.1) and (1.2). We present here some details. First, we assume a moving frame of reference. Let (η, ϕ) be a solution to the steady problem given by the Neumann problem and the Bernoulli condition. This means that $\nabla\phi$ (and η) is periodic. Then, there is a constant $c > 0$ such that the function $\phi(x, y, z) = \varphi(x, y, z) + cz$ is periodic, with φ periodic in z and c being the wave speed (either to the left or right) chosen according to the definition in [12], that is,

$$c = \frac{1}{\lambda}\int_0^\lambda \partial_z\phi \, dz,$$

where λ is the wavelength of the wave (or spatial period). Note that the velocity potential ϕ is only unique up to a constant, but the decomposition $\phi = \varphi + cz$ makes it unique. As we can see from equation (1.2), ϕ depends on η in some way, but it also depends on c due to the decomposition we introduced above. Therefore, we write $(\eta, \phi_{\eta,c})$ to denote the solution to the full problem (equations (1.1) and (1.2)). Concerning the function describing the profile curve of the surface, we denote by $\eta_{s,k}(z)$ (for suitable parameters s, k which we introduce later) the one corresponding to the unduloid solutions, and by $\eta(z)$ the more general profile curve which occurs in the non-static case. Each point on the static branch is then given by $(\eta_{s,k}, 0)$, where the second component stands for the wave speed c.

The main outline of the thesis is as follows: in section 3 we go through the necessary preliminaries and we also define more rigorously the function describing the profile curve of the unduloids. Then, in section 4, we reformulate the problem by considering the left hand side of (1.2) as an operator of (η, c), where $\phi = \phi_{\eta,c}$ solves (1.1). We identify the static configuration (for a vanishing velocity field and $c = 0$) and present the main theorem we wish to prove. This involves taking the Fréchet derivative of the operator, and most importantly showing first that it exists, which we do in section 5. Finally, in subsection 5.6, we prove the main theorem.

We would like to mention that [12] also found that the limiting behaviour as the steepness S grows large indicates overhanging waves (in both cases Q_1 and Q_2). However, we will not be concerned with this case.

3 Background

3.1 Hölder spaces and nonlinear functional analysis

We introduce the implicit function theory for Hölder spaces, which we will use, as well as the framework of the problem.

Definition 3.1. Let $k \geq 0$ be a non-negative integer and let $\alpha \in (0, 1]$. Suppose Ω is an open subset of the Euclidean space $\mathbb{R}^n$ and let $u \in C^k(\Omega)$ be a k-times differentiable function from Ω to $\mathbb{R}$. The α^{th} *Hölder semi-norm* is defined by

$$[u]_{C^{0,\alpha}(\overline{\Omega})} := \sup_{x,y\in\Omega, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

and the α^{th} *Hölder norm* is given by

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} := \|u\|_{C(\overline{\Omega})} + [u]_{C^{0,\alpha}(\overline{\Omega})}$$

where $\|u\|_{C(\overline{\Omega})} = \sup_{x\in\Omega} |u(x)|$. The Hölder space $C^{k,\alpha}(\overline{\Omega})$ consists of all $u \in C^k(\overline{\Omega})$ for which the following norm is finite

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} := \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C(\overline{\Omega})} + \sum_{|\alpha|\leq k} [D^\alpha u]_{C^{0,\alpha}(\overline{\Omega})}$$

Note that we require u and its derivatives up to order k to be bounded on the closure of Ω.

Definition 3.2. The open unit ball in $\mathbb{R}^n$ is given by the set

$$B = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : |x| < 1\}.$$

We also define

$$B_+ = \{x \in B : x_n > 0\}, \ B_0 = \{x \in B : x_n = 0\}.$$

Definition 3.3. The domain Ω is said to be a $C^{k,\alpha}$-domain if, for every $p \in \partial\Omega$, there exists a neighbourhood U_p of p in $\mathbb{R}^n$ and a diffeomorphism $\kappa_p : B \to U_p$ such that

1. $\kappa_p \in C^{k,\alpha}(\overline{B})$ and $\kappa_p^{-1} \in C^{k,\alpha}(\overline{U_p})$;

2. $\kappa_p(B_+) = U_p \cap \Omega$;

3. $\kappa_p(B_0) = U_p \cap \partial\Omega$.

We present here a compactness result which is useful in showing regularity later on. It is a corollary of the Arzela-Ascoli theorem for functions in Hölder spaces.

Theorem 3.4. *Let Ω be as before, and additionally assume that it is a $C^{k,\alpha}$-domain (in fact we only need a $C^{1,\alpha}$ domain but we lose no information by assuming $C^{k,\alpha}$). Let $\{u_j\} \in C^{k,\alpha}(\overline{\Omega})$ be a sequence such that*

$$\sup_j \|u_j\|_{C^{k,\alpha}(\overline{\Omega})} < \infty.$$

Then, there is a subsequence $\{u_{j_h}\}$ of $\{u_j\}$ and a function $u \in C^{k,\alpha}$ such that

$$u_{j_h} \to u, \quad \text{as } h \to \infty$$

in $C^k(\overline{\Omega})$.

Note that the convergence occurs in the space of k-differentiable functions.

In the following part we assume that X and Y are Banach spaces and $\mathcal{L}(X, Y)$ is the space of bounded, linear operators equipped with the operator norm $\|T\| := \sup_{\|x\|=1} \|Tx\|$. Naturally, all results also hold for the previously defined Hölder spaces. The idea is to introduce calculus on infinite dimensional function spaces. We refer to [11] for the definitions and results.

Definition 3.5. Let $a \in U$, where $U \subset X$ is an open neighbourhood. Then, if there exists an operator $DT[a] \in \mathcal{L}(X, Y)$ such that

$$\lim_{\|h\|\to 0} \frac{\|T(a+h) - T(a) - DT[a]h\|_Y}{\|h\|_X} = 0$$

we call $DT[a]$ the Fréchet derivative of T at $a \in U$.

Remark. Fréchet differentiability can also be expressed as

$$T(a+h) - T(a) = DT[a]h + w(a,h)$$

where $w(a,h)$ is the remainder term for which it holds that

$$\lim_{\|h\|\to 0} \frac{\|w(a,h)\|}{\|h\|} = 0$$

and which can also be written in terms of little-O notation.

Theorem 3.6. *(Implicit function theorem for Banach spaces) Let X, Y and Z be Banach spaces, and let U, V be subset of X and Y respectively. Moreover, let $F \in C^k(U \times V, Z)$ for $k \geq 0$. Fix a point $(x_0, y_0) \in U \times V$ and suppose that $D_x F \in C(U \times V, \mathcal{L}(X, Z))$ exists and $D_x F[x_0, y_0] \in \mathcal{L}(X, Z)$ is an isomorphism. Then, there exists an open neighbourhood $U_1 \times V_1 \subseteq U \times V$ of (x_0, y_0) such that for each $y \in V_1$, there exists a unique point $(\nu(y), y) \in U_1 \times V_1$ satisfying $F(\nu(y), y) = F(x_0, y_0)$. Moreover, $\nu \in C^k(V_1, Z)$ and fulfills*

$$D\nu(y) = -\left(D_x F[\nu(y), y]\right)^{-1} \circ D_y F[\nu(y), y]$$

In other words, if the operator describing the static curve of solutions is regular enough, then locally, there exists a function expressing a curve of non-static solutions, at each point of the original curve.

3.2 Elliptic differential operators

Our focus is on second order operators. As before, Ω is an open subset of the Euclidean space $\mathbb{R}^n$.

Definition 3.7. A second order differential operator is an operator of the form

$$Lu = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u$$

with $(x_1, ..., x_N) \in \Omega \subset \mathbb{R}^n$ and with $C^{0,\alpha}(\overline{\Omega})$ coefficients $a_{ij}(x), b_i(x), c(x)$.

Definition 3.8. We say that L is *elliptic* at a point x if the coefficient matrix $[a_{ij}(x)]$ is symmetric and positive in the sense that

$$0 < \lambda(x)|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2$$

where $\lambda(x)$ and $\Lambda(x)$ are the smallest and largest eigenvalues of $[a_{ij}(x)]$ respectively, and $\xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N \setminus \{0\}$. If $\lambda > 0$ for all $x \in \Omega$, then L is *elliptic* in Ω, and *strictly elliptic* if $\lambda \geq \lambda_0 > 0$ for some constant λ_0. Finally, we call L *uniformly elliptic* in Ω if $\frac{\Lambda}{\lambda}$ is bounded in Ω, or equivalently, if the inequalities hold for constants λ, Λ independent of $x \in \Omega$. This definition is taken from [4].

3.3 Surfaces of revolution and constant mean curvature (CMC)

The following subsection is based on papers [2] and [3] and on the book [8]. We specifically look into surfaces of revolution and their construction since our setting involves an axisymmetric water jet stream. These surfaces were first studied by the French astronomer CH Delaunay in the 1840s. His main result is the following:

Theorem 3.9. *(Delaunay, [2]) The only surfaces of revolution with constant mean curvature are the cylinder, sphere, catenoid, undoloid and nodoid.*

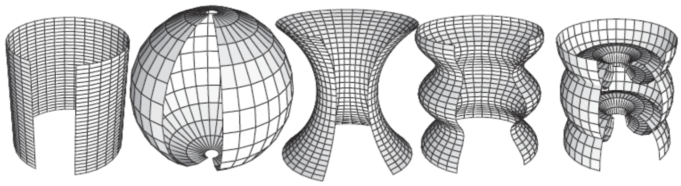

Figure 3: The five surfaces of Delaunay, Mladenov, Ivailo and Hadzhilazova, Mariana. (2012). Geometry of the Anisotropic Minimal Surfaces. An. St. Univ. Ovidius Constanta. 20. 79-88. 10.2478/v10309-012-0042-3.

Our goal is to isolate the surface we will work with, that is the unduloid, from the ones mentioned above. We begin by outlying briefly Delaunay's construction of these surfaces. First, a definition.

Definition 3.10. A *conic* is the intersection of a plane and a right circular cone. The equation of a conic is given by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Example 3.11. The non-degenerate conics are the circle, the ellipse, the parabola and the hyperbola. These are conics whose equations are non-reducible.

The construction of surfaces of revolution and constant mean curvature presented here is based on the one given in [3]. It consists in tracing out the the foci (or focus) of a conic as it rolls along a straight line. The resulting curve in the plane is then rotated to obtain a surface of revolution. For example, to obtain an unduloid, we trace out the foci of an ellipse, and to obtain a catenoid, we trace out the focus of a parabola. From the remark in the previous section, we know that $\frac{T}{\rho_0}H(\eta) = B$, with an arbitrary positive constant B, for vanishing velocity. We also seek a surface which is periodic. We can immediately rule out catenoids since they have a constant mean curvature of 0 and are not periodic ([8]). We also rule out nodoids, since they are self-intersecting and thus do not constitute a physical domain. This leaves us with unduloids. Note that cylinders appear as a special case of unduloids, and are in our case a limiting configuration. Similarly, the sphere is ruled out, but appears as a limiting configuration in the form of a family of parts of spheres repeated periodically.

3.4 Elliptic integrals

As the equations for an unduloid that we are going to use contain elliptic integrals, we give a short introduction to the subject, see [7]. Elliptic integrals originally arose in connection to the problem of finding the arc length of an ellipse, hence the name. We only consider elliptic integrals of the first and second kind, whose definition will be made clear shortly. Let an ellipse be given by

$$\frac{x^2}{c^2} + \frac{y^2}{a^2} = 1 \iff y = \pm\frac{a}{c}\sqrt{c^2 - x^2}$$

where a and c are the radii of the ellipse, assuming also that $a < c$. Using the formula for the arc length

$$s = \int_a^c \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

we get that the arc length of the ellipse (we call it W) is given by

$$W = 4\int_0^c \sqrt{\frac{c^4 + (a^2 - c^2)x^2}{c^2(c^2 - x^2)}} dx.$$

Using the substitutions $x = c\sin(u)$ and $dx = c\cos(u)du$, we obtain

$$W = 4c\int_0^{\frac{\pi}{2}} \sqrt{1 - m\sin^2(u)} du \tag{3.1}$$

with $m = 1 - \frac{c^2-a^2}{a^2}$. Taking $k = \sqrt{m}$ gives the eccentricity of the ellipse. From here, we can generalize and allow for an arbitrary angle ϕ in the integration limit. This directly gives us the definition of the *incomplete elliptic integral of the second kind.*

Definition 3.12. The *incomplete elliptic integral of the second kind* is defined as

$$P(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2(u)} du,$$

with $0 \leq k \leq 1$. Using an appropriate substitution, it can also be written as

$$\int_0^{\sin(\phi)} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} dt$$

for $0 < k < 1$ and $0 \leq \phi \leq \frac{\pi}{2}$. If $\phi = \frac{\pi}{2}$, we get the *complete* elliptic integral of the second kind

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2(u)} du = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} dt.$$

Remark. Equation (3.1) is a complete elliptic integral of the second kind given by $W = 4aE(\sqrt{m})$.

Definition 3.13. The *incomplete elliptic integral of the first kind* is defined as

$$F(\phi, k) = \int_0^{\phi} \frac{du}{\sqrt{1 - k^2 \sin^2(u)}} = \int_0^{\sin(\phi)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

and the *complete* elliptic integral of the first kind is given by

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1 - k^2 \sin^2(u)}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

with the same conditions on k and ϕ as before.

To finish this section, we present some formulae for derivatives and series of complete elliptic integrals which will be useful later. These identities are found in [7].

Lemma 3.14. *It holds that*

$$K'(k) = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k} \tag{3.2}$$

and that

$$E'(k) = \frac{E(k)}{k} - \frac{K(k)}{k}. \tag{3.3}$$

Lemma 3.15. *The power series expansions of the elliptic integrals of the first and second kind are*

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\Big(\frac{(2n)!}{2^{2n}(n!)^2}\Big)^2 k^{2n} \tag{3.4}$$

and

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\Big(\frac{(2n)!}{2^{2n}(n!)^2}\Big)^2 \frac{k^{2n}}{1-2n}. \tag{3.5}$$

Note that the power series have an infinite radius of convergence.

3.5 Parametrization of unduloids

The aim of this subsection is to first present a parametrization of unduloids given in [5], and then introduce a reparametrization which will be more convenient for the analysis of the problem. Let a and c be the radii of the profile curve of the unduloid as illustrated in the figure, with $c > a$. Note that these parameters are not necessarily the same as the radii of the underlying generating ellipse. Let $(z(u), r(u))$ denote the parametrization given by

$$z(u) = z_{a,c}(u) = aF\Big(\frac{\mu u}{2} - \frac{\pi}{4}, k\Big) + cE\Big(\frac{\mu u}{2} - \frac{\pi}{4}, k\Big) \tag{3.6}$$

$$r(u) = r_{a,c}(u) = \sqrt{m\sin(\mu u) + n} \tag{3.7}$$

where $u \in \mathbb{R}$, $c > a > 0$, F and E are incomplete elliptic integrals of the first and second kind respectively, and we have

$$\mu = \frac{2}{a+c},\ \ k^2 = \frac{c^2-a^2}{c^2},\ \ m = \frac{c^2-a^2}{2},\ \ n = \frac{c^2+a^2}{2}.$$

We underline the dependence of $(z(u), r(u))$ on the parameters a and c. Later this will be used to define new parameters.

Lemma 3.16. *The function z defined in* (3.6) *is invertible, that is, its inverse is given by some function $\mu_{a,c}u_{a,c}(z)$.*

Proof. Observe that the elliptic integrals F and E are given by

$$F\Big(\frac{\mu u}{2} - \frac{\pi}{4}, k\Big) = \int_0^{\frac{\mu u}{2}-\frac{\pi}{4}} \frac{d\nu}{\sqrt{1-k^2\sin^2(\nu)}} \tag{3.8}$$

and by

$$E\Big(\frac{\mu u}{2}-\frac{\pi}{4},k\Big)=\int_0^{\frac{\mu u}{2}-\frac{\pi}{4}}\sqrt{1-k^2\sin^2(\nu)}d\nu \tag{3.9}$$

Both integrands in (3.8) and (3.9) are strictly positive, continuous functions. The integration is taken in the interval $(0,\frac{\mu u}{2}-\frac{\pi}{4})$, which increases with u. Note that this also implies that the inverse function $u_{a,c}(z)$ is periodic. Thus, F and E are strictly increasing and $u\mapsto z(u)$ is strictly increasing. □

By inverting equation (3.6), and inserting it into (3.7), we can rewrite the profile curve in terms of a two-parameter function of z which we denote by $\rho_{a,c}(z)$:

$$r(z)=r(\mu_{a,c}u_{a,c}(z))=\rho_{a,c}(z)$$

The period of this function (in terms of the distance on the z-axis) is given by

$$L=2cE(k)+2aK(k) \tag{3.10}$$

where $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kind, respectively, and the mean curvature of the corresponding surface is given by

$$H=\frac{1}{a+c}. \tag{3.11}$$

Our aim now is to introduce another parametrization which will depend on different parameters than a and c.

Lemma 3.17. *It holds that* $\rho_{a,c}(z)=c\rho_{\frac{a}{c},1}(z/c)$.

Proof. We begin by rewriting (3.6) as

$$\frac{z(u)}{c}=\frac{a}{c}F\Big(\frac{\mu u}{2}-\frac{\pi}{4},k\Big)+E\Big(\frac{\mu u}{2}-\frac{\pi}{4},k\Big).$$

Now, note that since

$$k^2=\frac{c^2-a^2}{c^2}$$

we get that

$$\frac{a}{c}=\sqrt{1-k^2}.$$

If we let $\mathcal{G}(u,k)$ be the inverse of

$$\sqrt{1-k^2}F\Big(\frac{v}{2}-\frac{\pi}{4},k\Big)+E\Big(\frac{v}{2}-\frac{\pi}{4},k\Big)$$

where $v=\frac{2u/c}{\frac{a}{c}+1}$. Then, we have that

$$\mathcal{G}\Big(\frac{z}{c},k\Big)=\mu_{a,c}u_{a,c}(z).$$

Note that $\mathcal{G}$ is a continuous function we get from taking the inverse of the elliptic integrals. Since $\rho_{a,c}(z)$ is obtained by inserting $\mu_{a,c}u_{a,c}(z)$ into equation (3.7), we get

$$\rho_{a,c}(z)=\sqrt{m\sin\mathcal{G}\Big(\frac{z}{c},k\Big)+n}=\sqrt{\frac{c^2-a^2}{2}\sin\mathcal{G}\Big(\frac{z}{c},k\Big)+\frac{c^2+a^2}{2}}=$$

$$=c\sqrt{\frac{1}{2}\Big(\Big(1-\frac{a^2}{c^2}\Big)\sin\mathcal{G}\Big(\frac{z}{c},k\Big)+1+\frac{a^2}{c^2}\Big)}=c\rho_{\frac{a}{c},1}(z/c).$$

□

Since $\frac{a}{c}=\sqrt{1-k^2}$, we can write

$$\rho_{a,c}(z)=c\rho_{\frac{a}{c},1}(z/c)=c\rho_{\sqrt{1-k^2},1}(z/c).$$

Next, we fix the period. Using formula (3.10), we have

$$L=2c(E(k)+\sqrt{1-k^2}K(k))$$

Setting $L=2\pi$ and rearranging, we obtain a new parameter $c(k)$ which depends on k such that for $c=c(k)$, the period is 2π.

$$c(k)=\frac{\pi}{E(k)+\sqrt{1-k^2}K(k)} \tag{3.12}$$

and therefore

$$L(k)=2c(k)(E(k)+\sqrt{1-k^2}K(k))=2\pi$$

We can now define the reparametrized function describing the free surface by

$$\eta_{s,k}(z):=sc(k)\rho_{\sqrt{1-k^2},1}\left(\frac{z}{sc(k)}\right)$$

which now has a fixed period of $2\pi s$ and where $s > 0$ is a scaling factor. Thus, we have

$$L(s,k) = 2\pi \cdot s.$$

Additionally, taking

$$a = sc(k)\sqrt{1-k^2} \tag{3.13}$$

and

$$c = sc(k) \tag{3.14}$$

it follows that

$$\rho_{a,c}(z) = \rho_{sc(k),sc(k)\sqrt{1-k^2}}(z) = sc(k)\rho_{\sqrt{1-k^2},1}(z/sc(k)). \tag{3.15}$$

Using formula (3.12), we note that we have a bijection between (s,k) and (a,c) because for any $0 < a < c$, we can write

$$k = \sqrt{1-\frac{a^2}{c^2}} \quad \text{and} \quad s = \frac{c}{c(k)} = \frac{c}{c\sqrt{1-\frac{a^2}{c^2}}}$$

using equations (3.13) and (3.14). Therefore, also due to equation (3.15), we have a one-to-one correspondence between $\eta_{s,k}(z)$ and $\rho_{a,c}(z)$. The map $(s,k) \mapsto (a,c)$ is a bijection from $(0,\infty) \times (0,1)$ to $\{(a,c) \in \mathbb{R}^2 : 0 < a < c\}$. We can now reformulate equation (3.11) in terms of $c(k)$ to obtain

$$H_{s,k} = \frac{E(k) + \sqrt{1-k^2}K(k)}{\pi(1+\sqrt{1-k^2})s} \tag{3.16}$$

as the mean curvature of $\eta_{s,k}(z)$. This reparametrization allows us to, in a sense, have more control of the unduloids, and therefore of the free surface in the hydrodynamic problem. To illustrate, if we fix $s = 1$, we obtain a 2π-periodic surface which only depends on k, which we can vary. For $0 < k < 1$, $\eta_{1,k}(z)$ gives a one-parameter family of unduloids. Looking back at figure 2, it also makes sense to investigate the two limiting cases. The case $k = 0$ corresponds to taking $a = c$ which gives

$$r(u) = r_{a,c}(u) = \sqrt{n}$$

a constant. Therefore, the surface profile for $k = 0$ is a cylinder. Its radius can be found through equation (3.16):

$$\lim_{k\to 0} H_{s,k} = \frac{1}{2s}$$

where we see that the radius is s. On the other hand, letting k go to 1 gives

$$\lim_{k\to 1} H_{s,k} = \frac{1}{s\pi}$$

which we can recognize as the curvature of a sphere of radius $s\pi$. Equivalently, we can see this by letting $k \to 1$ in equations (3.6) and (3.7). We have

$$z(u) = s\pi E\Big(\frac{u}{s\pi} - \frac{\pi}{4}, 1\Big) = s\pi \int_0^{\frac{u}{s\pi}-\frac{\pi}{4}} \cos(x)dx = s\pi \sin\left(\frac{u}{s\pi} - \frac{\pi}{4}\right)$$

and

$$r(u) = \frac{s\pi}{\sqrt{2}}\sqrt{\sin\left(\frac{2u}{s\pi}\right) + 1} = \frac{s\pi}{\sqrt{2}}\sqrt{\cos\left(\frac{2u}{s\pi} - \frac{\pi}{2}\right) + 1} =$$

$$= \frac{s\pi}{\sqrt{2}}\sqrt{\cos\left(2\Big(\frac{u}{s\pi} - \frac{\pi}{4}\Big)\right) + 1} = s\pi\sqrt{\cos^2\left(\frac{u}{s\pi} - \frac{\pi}{4}\right)} = s\pi\cos\left(\frac{u}{s\pi} - \frac{\pi}{4}\right).$$

Remark. We show that the infinimum of the profile curve function $\eta_{s,k}(z)$ is strictly positive. Clearly, in the original parametrization, it follows that profile curve is bounded below by the constant $a > 0$. In the new parametrization, we have

$$\begin{aligned}\inf(\eta_{s,k}(z)) &= \inf\left(sc(k)\rho_{\sqrt{1-k^2},1}\left(\frac{z}{sc(k)}\right)\right)\\ &= \inf\left(sc(k)\sqrt{\frac{1}{2}\left(\left(1 - \frac{a^2}{(sc(k))^2}\right)\sin\mathcal{G}\left(\frac{z}{sc(k)}, k\right) + 1 + \frac{a^2}{(sc(k))^2}\right)}\right).\end{aligned}$$

Since the minimum of $\frac{z}{sc(k)}$ is achieved when $\sin\mathcal{G}\left(\frac{z}{sc(k)}\right) = -1$, we have that the above quantity is larger than

$$\inf\left(sc(k)\sqrt{-\frac{1}{2}\left(1 - \frac{a^2}{(sc(k))^2}\right) + 1 + \frac{a^2}{(sc(k))^2}}\right) = \inf\left(sc(k)\sqrt{\frac{1}{2} + \frac{a^2}{(sc(k))^2}\frac{3}{2}}\right) > 0,$$

since $s \in \mathbb{R}_+$, $\inf(c(k)) = 1$ and the expression inside the square root is strictly positive.

4 Continuation analysis

4.1 Reformulation of the problem

As in the beginning, the domain is given by $\Omega^\eta = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < \eta(z)^2\}$. Let $F : C^{2,\alpha}_{\text{per,e}}(\mathbb{R}) \times \mathbb{R} \to C^{0,\alpha}_{\text{per,e}}(\mathbb{R})$ be the second order operator given by

$$F(\eta, c) = \frac{1}{2}|\nabla\phi_{\eta,c}|^2 - \frac{T}{\rho}H(\eta) \tag{4.1}$$

where $\eta(z)$ is an, even, periodic function, $\phi_{\eta,c} = \varphi_{\eta,c} + cz$ ($\varphi_{\eta,c}$ 2π-periodic in z) is the solution to

$$\begin{cases} \Delta\phi = 0, & \text{in } \Omega^\eta \\ \partial_n\phi = 0, & \text{on } \partial\Omega^\eta \end{cases} \tag{4.2}$$

$c > 0$ is the wave speed and H is the mean curvature of $\partial\Omega^\eta$ given by

$$H(\eta) = \frac{\eta_{zz}}{(1+\eta_z^2)^{3/2}} - \frac{1}{\eta(1+\eta_z^2)^{1/2}}.$$

The function spaces are defined as $C^{k,\alpha}_{\text{per,e}} = \{f \in C^{k,\alpha}, k \in \mathbb{N} : f \textbf{ even and periodic}\}$. The selected definition of $\phi_{\eta,c} = \varphi_{\eta,c} + cz$ comes from our assumption of a moving frame of reference, with wave speed $c > 0$ as introduced in section 1. Our aim in this section is to first identify the static solutions, and then in section 5, we show that problem (4.2) has a solution which is unique up to a constant. This solution can be taken to be odd in z (if $\eta(z)$ is even), and thus be unique. More details on this will be provided in section 5.

4.2 Static case

In this subsection, we first investigate the static solutions for which the velocity field vanishes and the surface is a surface of revolution with constant mean curvature.

Proposition 4.1. *If $c = 0$, then $\phi = \varphi =$ constant.*

Proof. Recall that our domain is $\Omega^\eta = \{(x, y, z) \in \mathbb{R}^3 : r < \eta(z)\}$ and it is periodic. We restrict it to a single period. Denote this new domain by

$$\Omega^\eta_0 = \{(x, y, z) \in \mathbb{R}^3 : r = \sqrt{x^2 + y^2} < \eta(z), 0 < z < L\}$$

The boundary of Ω_0^η is made up of three components: $\partial D_1^\eta = \{r = \eta(z)\}$, i.e., this is the horizontal free surface and $\partial D_2^\eta = \{(x, y, 0) \in \mathbb{R}^3, r < \eta(z)\}$ and $\partial D_3^\eta = \{(x, y, L) \in \mathbb{R}^3, r < \eta(z)\}$ are the sides of the "cut off" surface. Since $c = 0$ and φ is periodic in z, we have periodic boundary conditions: $\varphi|_{\partial D_2^\eta} = \varphi|_{\partial D_3^\eta}$ and $\nabla\varphi|_{\partial D_2^\eta} = \nabla\varphi|_{\partial D_3^\eta}$. Therefore,

$$0 = \int_{\Omega_0^\eta} \varphi\Delta\varphi \; d(x, y, z) = -\int_{\Omega_0^\eta} |\nabla\varphi|^2 \; d(x, y, z) + \int_{\partial D_1^\eta} \varphi\partial_n\varphi \; dS + \int_{\partial D_2^\eta} \varphi(-\partial_z\varphi) \; dS + \int_{\partial D_3^\eta} \varphi\partial_z\varphi \; dS$$

and because $\partial_n\varphi = 0$ on ∂D_1^η, we get

$$0 = \int_{\Omega_0^\eta} |\nabla\varphi|^2 \; d(x, y, z)$$

Since $|\nabla\varphi|^2$ is non-negative, $\nabla\varphi = 0$ and since Ω_0^η is connected, $\varphi = constant$. □

Given the previous proposition, indeed if $c = 0$, it follows that

$$\frac{1}{2}|\nabla\phi_{\eta,c}|^2 - \frac{T}{\rho}H(\eta)\bigg|_{c=0} = -\frac{T}{\rho}H(\eta) = B$$

implying that in this case the mean curvature is constant, and therefore, we get a two-parameter family of static, unduloid solutions $\eta_{s,k}(z)$, as we have defined them in section 3.5. Note that they have period $2\pi s$, so for any given fixed period (that is any given fixed s), we get a one-parameter family instead.

4.3 Main result

Now that we have identified the static solutions in the form of $\eta_{s,k}(z)$, we want to perform a continuation analysis at one such solution in order to establish the existence of non-static solutions. We do this by applying the implicit function theorem to the operator F. We ultimately want to prove the following theorem.

Theorem 4.2. *The equation $F(\eta, c)$ given by*

$$\frac{1}{2}|\nabla\phi_{\eta,c}|^2 - \frac{T}{\rho}H(\eta) = B$$

for B chosen to be the one corresponding to a given η_{1,k_0}, where $H(\eta)$ is the mean curvature of $\partial\Omega^\eta$ and where $\phi_{\eta,c} = \varphi_{\eta,c} + cz$ with $\varphi_{\eta,c}$ periodic in z satisfies

$$\begin{cases} \Delta\phi_{\eta,c} = 0, & \text{in } \Omega^\eta \\ \partial_n\phi_{\eta,c} = 0, & \text{on } \partial\Omega^\eta \end{cases}$$

has a unique curve of non-static solutions near each η_{1,k_0}, that is, there exists a neighbourhood $U \times (-\delta,\delta) \subset C^{2,\alpha}_{per,e}(\mathbb{R}) \times \mathbb{R}$, for some $\delta > 0$, of the point $(\eta_{1,k_0}, 0)$ such that for each $c \in (-\delta,\delta)$, there exists a unique curve (η, c) in $U \times (-\delta,\delta)$, with $F(\eta_{1,k_0}, 0) = F(\eta, c)$.

The proof of this theorem constitutes the remaining sections.

5 Proof of the main result

We pick an arbitrary point on the static branch, corresponding to a static solution given by $(\eta_{1,k_0}, 0)$ for some fixed k_0, unit scale $s = 1$ and wave speed $c = 0$. This solution represents an unduloid of period 2π, and therefore it is also 2π periodic. It is possible to consider more general solutions, that is solutions which are L-periodic for some arbitrary period L. This is because we can always define a pair $(\widetilde{\eta}(z'), \widetilde{\phi}(x', y', z'))$ where

$$\begin{cases} \widetilde{\eta}(z') = L^{-1}\eta(Lz') \\ \widetilde{\phi}(x', y', z') = L^{-1}\phi(Lx', Ly', Lz') \end{cases}$$

and which is also a solution to the system of equations 4.1 and 4.2. Therefore, we can make the choice to restrict to 2π-periodic functions without loss of generality. Now, in order to use Theorem 3.6, we must check that the operator F is regular, that is, $D_\eta F \in C\left(C^{2,\alpha}_{\text{per,e}}(\mathbb{R}) \times \mathbb{R}, \mathcal{L}(C^{2,\alpha}_{\text{per,e}}(\mathbb{R}), C^{0,\alpha}_{\text{per,e}}(\mathbb{R}))\right)$ exists and $D_\eta F[\eta_{1,k_0}, 0]$ is an isomorphism between the spaces $C^{2,\alpha}_{\text{per,e}}(\mathbb{R})$ and $C^{0,\alpha}_{\text{per,e}}(\mathbb{R})$. Since the operator is of the form

$$F(\eta, c) = \frac{1}{2}|\nabla\phi_{\eta,c}|^2 - \frac{T}{\rho}H(\eta),$$

the first step is an investigation of the dependence between (η, c) and $\phi_{\eta,c}$. As opposed to the static case, we now assume $c \neq 0$, so $\phi_{\eta,c} = \varphi_{\eta,c} + cz$ and (4.2) is equivalent to

$$\begin{cases} \Delta\varphi_{\eta,c} = 0, & \text{in } \Omega^\eta \\ \partial_n\varphi_{\eta,c} = -c\partial_n z, & \text{on } \partial\Omega^\eta \end{cases} \tag{5.1}$$

The main difficulty we face lies in that η is unknown a priori. For this reason, we flatten the domain Ω^η through a diffeomorphism. Next, we introduce the corresponding velocity potential in the new domain, $\hat{\varphi}_{\eta,c}$. This naturally modifies the equations (4.2) for the capillary wave problem. Our goal is then to show regularity of F using the new equations for $\hat{\varphi}_{\eta,c}$. This we achieve by proving that the map $(\eta, c) \mapsto \hat{\varphi}_{\eta,c}$ is Lipschitz and Fréchet differentiable. The use of the diffeomorphism allows us to do work in the flat domain without changing the operator $F(\eta, c)$. The process involves obtaining a priori estimates for $\hat{\varphi}_{\eta,c}$ which are then also independent of (η, c) in some way which will be clarified later. It is helpful to think of the operator as a composition between the map $(\eta, c) \mapsto \hat{\varphi}_{\eta,c}$ and

$$G(\hat{\varphi}, \eta) = \frac{1}{2}|\hat{\nabla}\hat{\varphi} + (0, 0, c)|^2 - \frac{T}{\rho}H(\eta)$$

such that

$$F(\eta, c) = G(\hat{\varphi}_{\eta,c}, \eta)$$

where we denote by $\hat{\nabla}$ the gradient operator in terms of $\hat{\varphi}$ in the flat domain. Naturally, if both components of the composition are regular, so is their composition.

5.1 Flattening of the domain and equivalence of the free boundary problem to a problem with fixed domain

There are several ways of flattening and they involve finding a diffeomorphism between the physical domain and a corresponding cylindrical domain. Here we use a simple one where the x and y coordinates are scaled by $\eta(z)$. To avoid singularities, we omit cylindrical coordinates. The map J given by

$$J(x, y, z) = \left(\frac{x}{\eta}, \frac{y}{\eta}, z\right)$$

with inverse

$$J^{-1}(s,t,z) = (s\eta, t\eta, z),$$

where $s^2 + t^2 < 1$, can be taken as our flattening map. The velocity potential $\phi_{\eta,c} = \varphi_{\eta,c} + cz$ for $c \neq 0$ for the new flat domain, is then given by

$$\hat{\phi}_{\eta,c} = (\varphi_{\eta,c} + cz) \circ J^{-1}$$

Moreover, we have

$$J' = \begin{pmatrix} \frac{1}{\eta} & 0 & -\frac{x\eta_z}{\eta^2} \\ 0 & \frac{1}{\eta} & -\frac{y\eta_z}{\eta^2} \\ 0 & 0 & 1 \end{pmatrix}$$

and thus

$$(J^{-1})' = \begin{pmatrix} \eta & 0 & s\eta_z \\ 0 & \eta & t\eta_z \\ 0 & 0 & 1 \end{pmatrix}$$

Proposition 5.1. *Given that $\eta > 0$ and $\eta \in C^{2,\alpha}_{per,e}$, the map $J : (x,y,z) \mapsto \left(\frac{x}{\eta}, \frac{y}{\eta}, z\right)$ is a $C^{2,\alpha}$ diffeomorphism between the sets $\{(x,y,z) \in \mathbb{R}^3 : \sqrt{x^2+y^2} \leq \eta(z)\}$ and $\{(s,t,z) \in \mathbb{R}^3 : \sqrt{s^2+t^2} \leq 1\}$.*

Proof. Note that

$$J \circ J^{-1}(s,t,z) = \left(\frac{s\eta}{\eta}, \frac{t\eta}{\eta}, z\right) = (s,t,z)$$

and thus $J \circ J^{-1} = id$. Conversely, we have

$$J^{-1} \circ J(x,y,z) = (s\eta, t\eta, z) = (x,y,z)$$

and thus $J^{-1} \circ J = id$. Additionally, since $\eta > 0$ for all z, it follows that J is as smooth as η. □

We now rewrite the problem in terms of the new coordinates. With the transformation J, our new domain is given by $\Omega^1 = \{(s,t,z) \in \mathbb{R}^3 : s^2 + t^2 < 1\}$ and is now fixed. The surface profile is given by $\partial\Omega^1 = \{(s,t,z) \in \mathbb{R}^3 : s^2 + t^2 = 1\}$. Then, the problem (5.1) equates to

$$\begin{aligned}\Delta\varphi = \left(\frac{1+s^2\eta_z^2}{\eta^2}\right)\hat{\varphi}_{ss} + \left(\frac{1+t^2\eta_z^2}{\eta^2}\right)\hat{\varphi}_{tt} + \hat{\varphi}_{zz} + 2\frac{st\eta_z^2}{\eta^2}\hat{\varphi}_{st} - \\ 2\frac{s\eta_z}{\eta}\hat{\varphi}_{sz} - 2\frac{t\eta_z}{\eta}\hat{\varphi}_{tz} + \left(\frac{2s\eta_z^2 - s\eta\eta_{zz}}{\eta^2}\right)\hat{\varphi}_s + \left(\frac{2t\eta_z^2 - t\eta\eta_{zz}}{\eta^2}\right)\hat{\varphi}_t = 0, \text{ in } \Omega^1\end{aligned} \tag{5.2}$$

and

$$\partial_n\varphi = \left(\frac{s}{\eta}\partial_s + \frac{t}{\eta}\partial_t + \left(\frac{s\eta_z^2}{\eta}\partial_s + \frac{t\eta_z^2}{\eta}\partial_t - \eta_z\partial_z\right)\right)\hat{\varphi} = c\eta_z, \text{ on } \partial\Omega^1$$

We introduce a simplified notation for the new problem.

$$\begin{cases} L^\eta\hat{\varphi} = G, & \text{in } \Omega^1 \\ B^\eta\hat{\varphi} = H, & \text{in } \partial\Omega^1 \end{cases} \tag{5.3}$$

where $G = 0$ and $H = c\eta_z$. Now, the operator F is written as

$$F(\eta, c) = \frac{1}{2}|\nabla\varphi_{\eta,c}|^2 - \frac{T}{\rho}H(\eta)$$

with

$$|\nabla\varphi|^2 = \frac{1}{\eta^2}\hat{\varphi}_s^2 + \frac{1}{\eta^2}\hat{\varphi}_t^2 + \left(\left(-\frac{s\eta_z}{\eta}\partial_s - \frac{t\eta_z}{\eta}\partial_t + \partial_z\right)\hat{\varphi}\right)^2 \tag{5.4}$$

In order to adapt the methods we use to our periodic setting, we make one more change, where we repeat the steps used to make the set Ω_0^1, i.e., the endpoints of Ω^1 are identified with each other. Clearly, Ω_0^1 is a $C^{2,\alpha}$ domain. Some remarks follow.

Remark. The coefficients of $L^\eta\hat{\varphi} = 0$ are $C^{0,\alpha}(\overline{\Omega^1})$ for $\eta \in C^{2,\alpha}(\overline{\Omega^1})$. This follows from the fact that $\eta > A > 0$ for some constant A and each coefficient can be bounded by the finite quantity $\|\eta\|_{C^{2,\alpha}}$ which depends on A.

Remark. The problem (5.3) is equivalent to the one given by (4.2). Direct computation shows equivalence. Since composition with a $C^{2,\alpha}$ diffeomorphism preserves Hölder continuity, it follows that $\phi_{\eta,c}$ is Hölder continuous if and only if $\hat{\varphi}_{\eta,c}$ is Hölder continuous.

Lemma 5.2. *Assume that $\eta \geq C_1 > 0$, $\|\eta\|_{C^{2,\alpha}} \leq C_2$. Then, the modified Laplacian operator in* (5.2) *is uniformly elliptic with λ, Λ only depending on C_1 and C_2.*

Proof. We must check that

$$\lambda|\xi|^2 \leq \sum_{i,j} a_{ij}(z)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for

$$\sum_{i,j} a_{ij}(z)\xi_i\xi_j = \left(\frac{1+s^2\eta_z^2}{\eta^2}\right)\xi_s^2 + \left(\frac{1+t^2\eta_z^2}{\eta^2}\right)\xi_t^2 + \xi_z^2 + \\ + \left(\frac{2st\eta_z^2}{\eta^2}\right)\xi_s\xi_t - \left(\frac{2s\eta_z}{\eta}\right)\xi_s\xi_z - \left(\frac{2t\eta_z}{\eta}\right)\xi_t\xi_z$$

This can be rewritten by completing the square:

$$\frac{1}{\eta^2}\left(\xi_s^2+\xi_t^2\right) + \left(-\frac{s\eta_z}{\eta}\xi_s - \frac{t\eta_z}{\eta}\xi_t + \xi_z\right)^2 \tag{5.5}$$

We wish to bound (5.5) from below first, by an expression of the form

$$\lambda\left(\xi_s^2+\xi_t^2+\xi_z^2\right)$$

for some number $\lambda > 0$ which is independent of $z \in \Omega^1$. The term that poses problems is exactly

$$\left(-\frac{s\eta_z}{\eta}\xi_s - \frac{t\eta_z}{\eta}\xi_t + \xi_z\right)^2$$

since it is not clear how it behaves when it is close to zero. We can identify two cases.

1. $\left|-\frac{s\eta_z}{\eta}\xi_s - \frac{t\eta_z}{\eta}\xi_t + \xi_z\right|^2 \geq \frac{1}{4}\xi_z^2$. Then,

   $$\frac{1}{\eta^2}\left(\xi_s^2+\xi_t^2\right) + \left(-\frac{s\eta_z}{\eta}\xi_s - \frac{t\eta_z}{\eta}\xi_t + \xi_z\right)^2 \geq \frac{1}{\eta^2}\left(\xi_s^2+\xi_t^2\right) + \frac{1}{4}\xi_z^2 \geq \lambda|\xi|^2$$

   where $\lambda = \min\{\frac{1}{C_2^2}, \frac{1}{4}\}$.

2. $\left|-\frac{s\eta_z}{\eta}\xi_s - \frac{t\eta_z}{\eta}\xi_t + \xi_z\right|^2 < \frac{1}{4}\xi_z^2$. We have

   $$\left(-\frac{s\eta_z}{\eta}\xi_s - \frac{t\eta_z}{\eta}\xi_t + \xi_z\right)^2 < \frac{1}{4}\xi_z^2 \iff$$

   $$-\frac{1}{2}|\xi_z| < -\frac{s\eta_z}{\eta}\xi_s - \frac{t\eta_z}{\eta}\xi_t + \xi_z < \frac{1}{2}|\xi_z|.$$

Suppose first that $\xi_z \geq 0$. Then, we have

$$\frac{1}{2}\xi_z < \frac{s\eta_z}{\eta}\xi_s + \frac{t\eta_z}{\eta}\xi_t \iff \frac{1}{4}\xi_z^2 < \left(\frac{s\eta_z}{\eta}\xi_s + \frac{t\eta_z}{\eta}\xi_t\right)^2.$$

Therefore,

$$\begin{aligned}
\lambda(\xi_s^2 + \xi_t^2 + \xi_z^2) &< \lambda\left(\xi_s^2 + \xi_t^2 + \frac{\left(\frac{s\eta_z}{\eta}\xi_s + \frac{t\eta_z}{\eta}\xi_t\right)^2}{\frac{1}{4}}\right) \\
&= 4\lambda\left(\frac{1}{4}\xi_s^2 + \frac{1}{4}\xi_t^2 + \left(\frac{s\eta_z}{\eta}\xi_s + \frac{t\eta_z}{\eta}\xi_t\right)^2\right) \\
&\leq 4\lambda\left(\frac{1}{4}\xi_s^2 + \frac{1}{4}\xi_t^2 + 2\frac{s^2\eta_z^2}{\eta^2}\xi_s^2 + 2\frac{t^2\eta_z^2}{\eta^2}\xi_t^2\right) \\
&= 4\lambda\left(\left(\frac{1}{4} + 2\frac{s^2\eta_z^2}{\eta^2}\right)\xi_s^2 + \left(\frac{1}{4} + 2\frac{t^2\eta_z^2}{\eta^2}\right)\xi_t^2\right) \\
&\leq 4\lambda\left(\left(\frac{1}{4} + 2\frac{s^2C_2^2}{C_1^2}\right)\xi_s^2 + \left(\frac{1}{4} + 2\frac{t^2C_2^2}{C_1^2}\right)\xi_t^2\right) \\
&\leq 4\lambda\left(\left(\frac{1}{4} + 2\frac{C_2^2}{C_1^2}\right)\xi_s^2 + \left(\frac{1}{4} + 2\frac{C_2^2}{C_1^2}\right)\xi_t^2\right) \\
&= 4\lambda\left((\xi_s^2 + \xi_t^2)\left(\frac{1}{4} + 2\frac{C_2^2}{C_1^2}\right)\right) \\
&\leq \frac{1}{C_2^2}(\xi_s^2 + \xi_t^2)
\end{aligned}$$

provided that

$$4\lambda \leq \frac{\frac{1}{C_2^2}}{\frac{1}{4} + 2\frac{C_2^2}{C_1^2}}.$$

Thus, we conclude that

$$\begin{aligned}
\frac{1}{C_2^2}(\xi_s^2 + \xi_t^2) &< \frac{1}{\eta^2}(\xi_s^2 + \xi_t^2) \\
&< \frac{1}{\eta^2}(\xi_s^2 + \xi_t^2) + \left(-\frac{s\eta_z}{\eta}\xi_s - \frac{t\eta_z}{\eta}\xi_t + \xi_z\right)^2.
\end{aligned}$$

The case where $\xi_z \leq 0$ is analogous so we repeat the same process on

$$\frac{1}{2}|\xi_z| < -\frac{s\eta_z}{\eta}\xi_s - \frac{t\eta_z}{\eta}\xi_t.$$

This concludes the second case, as we have shown that

$$\begin{aligned}
\frac{1}{\eta^2}&\left(\xi_s^2 + \xi_t^2\right) + \left(-\frac{s\eta_z}{\eta}\xi_s - \frac{t\eta_z}{\eta}\xi_t + \xi_z\right)^2 \\
&\geq \frac{1}{C_2^2}(\xi_s^2 + \xi_t^2) \geq \lambda|\xi|^2
\end{aligned}$$

provided λ is chosen as above.

It remains to show that we also have a bound from above, i.e., that there is some constant $\Lambda > 0$ such that

$$a_{ij}(z)\xi_i\xi_j \le \Lambda|\xi|^2$$

But this follows first from

$$\left|-\frac{s\eta_z}{\eta}\xi_s - \frac{t\eta_z}{\eta}\xi_t + \xi_z\right|^2 \le 3\frac{C_2^2}{C_1^2}|\xi|^2$$

Hence,

$$\frac{1}{\eta^2}\left(\xi_s^2 + \xi_t^2\right) + \left(-\frac{s\eta_z}{\eta}\xi_s - \frac{t\eta_z}{\eta}\xi_t + \xi_z\right)^2 \le \Lambda|\xi|^2$$

where $\Lambda = \frac{1+3C_2^2}{C_1^2}$. It is important to note that these estimates (in particular the constants λ, Λ) are uniform in η and do not depend on $z \in \mathbb{R}$. □

5.2 Schauder estimates and existence results

The following results are based on the paper [10] and on [4]. In this subsection, we assume that $\Omega \subset \mathbb{R}^N$ is an open, bounded, connected domain. Let the operator L be given by

$$Lu = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u.$$

We refer to Theorem 6.31, p.128 in [4] for the proof of the following theorem.

Theorem 5.3. *Let Ω be a connected, $C^{2,\alpha}$-domain. Suppose L is a uniformly elliptic operator in Ω with $c \le 0$ and with coefficients in $C^{0,\alpha}(\overline{\Omega})$. Let the boundary operator on $\partial\Omega$ be given by $Nu \equiv \gamma u + \beta \cdot Du$ such that $\gamma(\beta \cdot \nu) > 0$, with ν the unit normal on $\partial\Omega$, and with $\beta, \gamma \in C^{1,\alpha}(\overline{\Omega})$. Then, the problem*

$$\begin{cases} Lu = f, & \text{in } \Omega \\ Nu = g, & \text{on } \partial\Omega \end{cases} \tag{5.6}$$

has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ for all $f \in C^{0,\alpha}(\overline{\Omega})$ and $g \in C^{1,\alpha}(\overline{\Omega})$.

Now that we have the appropriate conditions for the existence of the solution u, we turn to some a priori estimates.

Theorem 5.4. *(Schauder boundary estimate, Theorem 6.30, p.127 in [4]) Let* Ω *be a* $C^{2,\alpha}$*-domain and let* $u \in C^{2,\alpha}(\overline{\Omega})$ *be a solution to* (5.6)*. As before,* $f \in C^{0,\alpha}(\overline{\Omega}), g \in C^{1,\alpha}(\overline{\Omega}), a_{ij}, b_i, c \in C^{0,\alpha}(\overline{\Omega})$*. Then, if the normal component* β_ν *of the vector* β *is non-zero and satisfies*

$$|\beta_\nu| \geq \kappa > 0, \text{ on } \partial\Omega$$

we have that

$$\|u\|_{C^{2,\alpha}} \leq C\left(\|u\|_{C^{0,\alpha}} + \|g\|_{C^{1,\alpha}} + \|f\|_{C^{0,\alpha}}\right)$$

where $C = C(\lambda, \Lambda, \kappa, \Omega, \alpha, N)$*, with* λ, Λ *as in definition 3.8.*

In other words, u is of class $C^{2,\alpha}$ up to the boundary of the domain Ω. We'd like to make use of a theorem from [10], which is an application of theorem 5.3 for the Neumann problem. We need this so that we can show existence of a solution to our problem in the physical domain before we make the flattening. This is enough since the diffeomorphism carries over the solution to the flat domain.

Theorem 5.5. *Let* Ω *be a connected,* $C^{2,\alpha}$*-domain and* $f \in C^{0,\alpha}(\overline{\Omega})$*,* $g \in C^{1,\alpha}(\overline{\Omega})$ *be such that*

$$\int_\Omega f = \int_{\partial\Omega} g.$$

Then, the problem

$$\begin{cases} \Delta u = f, & \text{in } \Omega \\ \partial_n u = g, & \text{on } \partial\Omega \end{cases}$$

has a unique solution in the class

$$\mathcal{C} = \left\{ u \in C^{2,\alpha}(\overline{\Omega}) : \frac{1}{|\Omega|}\int_\Omega u = 0 \right\}.$$

Remark. Note that the solution to the Neumann problem is unique only up to a constant, hence the additional assumption on null average.

5.3 Application to our problem

In order to make use of these results and adapt them to a periodic setting, the fluid domain must be fixed and "corner-free", i.e., it must be a $C^{2,\alpha}$-domain. Thus, for a given $\eta(z)$, cut a slice from Ω^η given by

$$\Omega_0^\eta = \{(x,y,z) \in \Omega^\eta : 0 < z < 2\pi\}.$$

We view this slice as having its endpoints identified with each other. We will refer to this domain as the *physical* domain.

Lemma 5.6. *The domain Ω_0^η is a $C^{2,\alpha}$-domain.*

Proof. This follows directly from how the domain is defined. □

We now solve the problem for a general ϕ, meaning that as before $\phi = \varphi + cz$ and $c \neq 0$. Then, we have

$$\begin{cases} \Delta\varphi = 0, & \text{in } \Omega^\eta \\ \partial_n \varphi = -c\partial_n z, & \text{on } \partial\Omega^\eta \end{cases} \tag{5.7}$$

with φ 2π-periodic in z.

Lemma 5.7. *For the problem* (5.7)*, it holds that*

$$\int_{\partial\Omega_0^\eta} \partial_n z \, dS = 0$$

where $\partial\Omega_0^\eta$ is the boundary excluding the lateral parts.

Proof. This follows from the fact that

$$\int_{\partial\Omega_0^\eta} g \, dS = \int_{\partial\Omega_0^\eta} (-c\partial_n z) \, dS = -c \int_{\partial\Omega_0^\eta} \partial_n z \, dS = -c \int_{\Omega_0^\eta} \Delta z \, d(x,y,z) = 0$$

The two lateral parts of the boundary cancel out since their normal vectors point in opposite directions. This holds even though z is not periodic in z, as ∇z is periodic. □

Theorem 5.8. *The boundary value problem* (5.7) *admits a unique solution in the class*

$$\mathcal{C} = \left\{ \varphi \in C^{2,\alpha}_{per}(\overline{\Omega}_0^\eta) : \frac{1}{|\Omega_0^\eta|} \int_{\Omega_0^\eta} \varphi \, d(x,y,z) = 0 \right\}$$

Proof. The result follows by applying a modified version of the proof of theorem 5.3 to (5.7). The main difference is that the theorem must be adapted to our periodic setting, but this is done by considering Ω_0^η instead of Ω^η. $\square$

Remark. It holds that φ (and thus also ϕ) is odd in z, assuming $\eta(z)$ is even. In fact, if $\varphi(x, y, z)$ is a solution to the Neumann problem, then $-\varphi(x, y, -z)$ is also a solution since

$$\Delta(-\varphi(x, y, -z)) = -\Delta(\varphi(x, y, -z)) = 0$$

for any value (x, y, z). We know that the solution to the Neumann problem is unique up to a constant and so we must have that

$$-\varphi(x, y, -z) = \varphi(x, y, z) + C$$

for some constant C. Furthermore,

$$\begin{aligned}\int_{\Omega^\eta} -\varphi(x, y, -z)\ d(x, y, z) &= -\int_{\Omega^\eta} \varphi(x, y, -z)\ d(x, y, z) \\ &= -\int_{\Omega^\eta} \varphi(x, y, z)\ d(x, y, -z) = 0.\end{aligned}$$

But, as remarked earlier, a solution with 0 average is unique, and therefore $\varphi(x, y, z) = -\varphi(x, y, -z)$. This extends easily to ϕ since $\phi = \varphi + cz$ and cz is odd in z.

5.4 Smoothness of the map $(\eta, c) \mapsto \hat{\varphi}_{\eta,c}$

We can finally turn to the aforementioned Schauder estimates for $\hat{\varphi}_{\eta,c}$ which we will then use in the second result where we show that the map $(\eta, c) \mapsto \hat{\varphi}_{\eta,c}$ is Lipschitz and Fréchet differentiable. Observe again that in Theorem 5.4 the existence of a constant C is established but it is not clear how exactly that constant depends on the domain Ω. We aim to prove the following result:

Theorem 5.9. *Let $\hat{\varphi} \in C^{2,\alpha}_{2\pi per,e}\left(\overline{\Omega^1}\right)$ be a solution to the problem* (5.3)*. Then, there is an estimate for $\hat{\varphi}$ in the $C^{2,\alpha}$ norm given by*

$$\left\| \hat{\varphi} - \frac{1}{|\Omega_0^\eta|} \int_{\Omega_0^1} \hat{\varphi}\eta^2\ d(s, t, z) \right\|_{C^{2,\alpha}} \leq B\left(\|G\|_{C^{0,\alpha}} + \|H\|_{C^{1,\alpha}}\right)$$

with $B = C(C_1 + C_2, \alpha)$, where $\eta \geq C_1 > 0$ and $\|\eta\|_{C^{2,\alpha}} \leq C_2$ as in the previous lemma, and with G and H as in (5.3)*.*

Note that before the flattening, the domain Ω^η depended on η, but not the Laplace operator Δ. The diffeomorphism J passed this dependence from the domain onto the operator, that is, now Ω^1 is independent of η, but the operator L^η is instead dependent on η. Thus, while the constant from Theorem 5.4 depends on Ω^1, the constant we get from this theorem will depend only on λ and Λ (from Lemma 5.2), which in turn uniformly depend on C_1 and C_2. Hence, the estimate is independent of η, whenever η is in a ball in $C^{2,\alpha}$ and bounded from below by a positive constant.

Proof. The first step is to check that we can use Theorem 5.4 in order to establish the existence of a constant $A(\Omega^1, \lambda, \Lambda, \kappa, \alpha)$ such that

$$\|\hat{\varphi}\|_{C^{2,\alpha}} \leq A(\|\hat{\varphi}\|_{C^{0,\alpha}} + \|G\|_{C^{0,\alpha}} + \|H\|_{C^{1,\alpha}}) \tag{5.8}$$

holds on Ω^1. It must therefore be checked that the normal component of the boundary condition is bounded from below by a positive constant. We have

$$\beta = \left(\frac{s(1+\eta_z^2)}{\eta}, \frac{t(1+\eta_z^2)}{\eta}, -\eta_z \right)$$

and

$$\nu = (s, t, 0)$$

which is the unit vector pointing radially from the cylinder axis. Therefore,

$$\beta_\nu = m \cdot \nu = \left(\frac{s(1+\eta_z^2)}{\eta}, \frac{t(1+\eta_z^2)}{\eta}, -\eta_z \right) \cdot (s, t, 0) = \left(\frac{s^2(1+\eta_z^2)}{\eta} + \frac{t^2(1+\eta_z^2)}{\eta} \right).$$

So,

$$\beta_\nu = \frac{1+\eta_z^2}{\eta}.$$

using that $s^2 + t^2 = 1$ on the boundary. By the estimates in lemma 5.2 and that $s^2 + t^2 = 1$, it follows that

$$|\beta_\nu| = \left| \frac{1+\eta_z^2}{\eta} \right| \geq \frac{1}{C_2} = \kappa > 0$$

which holds at all $z \in \mathbb{R}$. Thus, we have the estimate (5.8). We wish to obtain a new constant $B = B(C_1, C_2)$ such that we remove the dependence on $\|\hat{\varphi}\|_{C^{0,\alpha}}$, i.e., we want an estimate of the form

$$\|\hat{\varphi}\|_{C^{2,\alpha}} \leq B(C_1, C_2)(\|G\|_{C^{0,\alpha}} + \|H\|_{C^{1,\alpha}}) \tag{5.9}$$

This constant only depends on the bounds C_1 and C_2 since the quantities λ and Λ are uniform with respect to η. Hence, suppose the inequality (5.9) is false. Then, for each $k \in \mathbb{N}$, there exist $\hat{\varphi}_k, \eta_k \in C^{2,\alpha}$ and $H_k \in C^{1,\alpha}$, $G_k \in C^{0,\alpha}$ with $\|\eta_k\|_{C^{2,\alpha}} \leq C_2$ and $\eta_k \geq C_1 > 0$ such that

$$\begin{cases} L^{\eta_k}\hat{\varphi}_k = G_k, & \text{in } \Omega^1 \\ B^{\eta_k}\hat{\varphi}_k = H_k, & \text{on } \partial\Omega^1 \\ \frac{1}{|\Omega^1|}\int_{\Omega^1} \hat{\varphi}_k \eta_k^2 \, d(s,t,z) = 0 \end{cases} \tag{5.10}$$

$$\begin{cases} \|\hat{\varphi}_k\|_{C^{2,\alpha}} = 1 \\ \|\hat{\varphi}_k\|_{C^{2,\alpha}} > k(\|G_k\|_{C^{0,\alpha}} + \|H_k\|_{C^{1,\alpha}}) \end{cases}$$

This means we must have $H_k, G_k \to 0$ as $k \to \infty$ in $C^{1,\alpha}(\overline{\Omega^1})$ and $C^{0,\alpha}(\overline{\Omega^1})$ respectively. Because

$$\|\hat{\varphi}_k\|_{C^{2,\alpha}} = 1$$

using the definition of this norm, it follows that for every multi-index $\beta, |\beta| = 0, 1, 2,$ the family $\{D^\beta \hat{\varphi}_k\}$ is uniformly bounded since

$$\sup_{z\in\Omega^1} |D^\beta \hat{\varphi}| \leq \sup_k \|\hat{\varphi}_k\|_{C^{2,\alpha}} = 1$$

The sequence $\{D^\beta \hat{\varphi}_k\}$ is equicontinuous, that is

$$\left|D^\beta \hat{\varphi}_k(x) - D^\beta \hat{\varphi}_k(y)\right| \leq C|x-y|^\alpha, \ \forall x, y \in \Omega^1, |\beta| = 0, 1$$

for some constant $C > 0$. By theorem 3.4 used on $\{D^\beta \hat{\varphi}_k\}$ for $|\beta| = 0, 1, 2$, we get a subsequence $\{\hat{\varphi}_{k_h}\}$ of $\{\hat{\varphi}_k\}$ and a function $\hat{\varphi} \in C^{2,\alpha}$ such that

$$\hat{\varphi}_{k_h} \to \hat{\varphi}_0, \text{ in } C^2(\overline{\Omega^1}).$$

Similarly, since $\eta_k \in C^{2,\alpha}$, we have

$$|\eta_k(x) - \eta_k(y)| \leq C|x-y|^\alpha$$

for some $x, y \in \mathbb{R}$ such that $x \neq y$ and some constant $C > 0$, so $\{\eta_k\}$ is also equicontinuous. By Arzelà-Ascoli, there exists a subsequence $\{\eta_{k_h}\}$ and a function $\eta_0 \in C^{2,\alpha}$ such that

$$\eta_{k_h} \to \eta_0, \text{ in } C^2(\overline{\Omega^1})$$

with $\eta_0 \geq C_1 > 0$ and $\|\eta_0\|_{C^{2,\alpha}} \leq C_2$, by uniform convergence. The next step involves taking a limit as $h \to \infty$, but this must be justified since the operator L depends on h at the same time as the sequence $\hat{\varphi}_{k_h}$ does. We consider only one term here since the rest can be handled in the same way. We have a product of the form

$$\frac{(\eta_z)^2_{k_h}}{\eta^2_{k_h}} \partial_s^2 \hat{\varphi}_{k_h}$$

By the product and quotient rules for sequences, it holds that

$$\lim_{h\to\infty} \left(\frac{(\eta_z)^2_{k_h}}{\eta^2_{k_h}} \partial_s^2 \hat{\varphi}_{k_h} \right) = \frac{\lim_{h\to\infty} (\eta_z)^2_{k_h}}{\lim_{h\to\infty} \eta^2_{k_h}} \lim_{h\to\infty} \partial_s^2 \hat{\varphi}_{k_h}$$

Therefore, we can take the limit outside and write

$$L^{\eta_0} \hat{\varphi}_0 = \lim_{h\to\infty} L^{\eta_{k_h}} \hat{\varphi}_{k_h} = \lim_{h\to\infty} G_{k_h} = 0$$

and

$$B^{\eta_0} \hat{\varphi}_0 = \lim_{h\to\infty} B^{\eta_{k_h}} \hat{\varphi}_{k_h} = \lim_{h\to\infty} (H_{k_h}) = 0.$$

Moreover

$$\frac{1}{|\Omega^1|} \int_{\Omega^1} \hat{\varphi}_0 \eta_0^2 \, d(s,t,z) = \lim_{h\to\infty} \frac{1}{|\Omega^1|} \int_{\Omega^1} \hat{\varphi}_{k_h} (\eta_{k_h})^2 \, d(s,t,z) = 0$$

by assumption. Passing to the limit in (5.10) yields

$$\begin{cases} L^{\eta_0} \hat{\varphi}_0 = 0, & \text{in } \Omega^1 \\ B^{\eta_0} \hat{\varphi}_0 = 0, & \text{on } \partial\Omega^1 \\ \frac{1}{|\Omega^1|} \int_{\Omega^1} \hat{\varphi}_0 \eta_0^2 \, d(s,t,z) = 0. \end{cases}$$

Since we can return to the physical domain bijectively using J, we have

$$\begin{cases} \Delta \varphi_0 = 0, & \text{in } \Omega^{\eta_0} \\ \partial_n \varphi_0 = 0, & \text{on } \partial\Omega^{\eta_0} \\ \frac{1}{|\Omega^{\eta_0}|} \int_{\Omega^{\eta_0}} \varphi_0 \, d(x,y,z) = 0 \end{cases}$$

But this problem is only solved by the trivial solution $\varphi_0 = 0$, by theorem 5.8. Therefore, $\hat{\varphi}_0 = 0$ as well. But this is a contradiction because

$$1 = \|\hat{\varphi}_{k_h}\|_{C^{2,\alpha}} \le A(\|\hat{\varphi}_{k_h}\|_{C^0} + \|G_{k_h}\|_{C^{0,\alpha}} + \|H_{k_h}\|_{C^{1,\alpha}}) \to 0$$

as $h \to \infty$. So, (5.9) holds. □

The previous result implies that $\hat{\varphi}$ can be bounded by a constant depending uniformly on η. We use this in the next theorem. From now on, we assume that $\hat{\varphi}_{\eta,c}$ is the solution to (5.3) with the integral condition $\frac{1}{|\Omega^1|}\int_{\Omega^1} \hat{\varphi}_{\eta,c}\eta^2 \ d(s,t,z) = 0$.

Theorem 5.10. *The function $(\eta, c) \mapsto \hat{\varphi}_{\eta,c}$ is globally Lipschitz continuous in the set $\{\eta \in C^{2,\alpha} : \eta \ge C_1, \ \|\eta\|_{C^{2,\alpha}} \le C_2\}$.*

Proof. Let (η, c) and $(\widetilde{\eta}, \widetilde{c})$ be two instances of the profile curve and consider the two corresponding solutions, namely $\hat{\varphi}_{\eta,c}$ and $\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}$. Set up the following equations:

$$\begin{cases} \int_{\Omega^1} \hat{\varphi}_{\eta,c}\eta^2 \ d(s,t,z) = 0 \\ \int_{\Omega^1} \hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\widetilde{\eta}^2 \ d(s,t,z) = 0 \end{cases}$$

Taking the difference and rewriting gives

$$\int_{\Omega^1} \Big(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c}\Big)\eta^2 \ d(s,t,z) + \int_{\Omega^1} \hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\Big(\widetilde{\eta}^2 - \eta^2\Big) \ d(s,t,z) = 0$$

$$\iff$$

$$\int_{\Omega^1} \Big(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c}\Big)\eta^2 \ d(s,t,z) = \int_{\Omega^1} \hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\Big(\eta^2 - \widetilde{\eta}^2\Big) \ d(s,t,z).$$

Bounding the right hand side, we have

$$|I| = \left|\int_{\Omega^1} \hat{\varphi}_{\widetilde{\eta},\widetilde{c}}(\eta^2 - \widetilde{\eta}^2) d(s,t,z)\right| \le \int_{\Omega^1} |\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}(\eta + \widetilde{\eta})(\eta - \widetilde{\eta})| d(s,t,z)$$

Now, using the estimate from theorem 5.4, we obtain

$$|I| \le A\|\eta - \widetilde{\eta}\|_{C^{2,\alpha}}$$

where the constant A depends on the quantities C_1 and C_2. What we have obtained is needed in order to use theorem 5.9, as the Schauder estimate there is proven for a solution with null average. Now, we use the elliptic equation $L^\eta \hat{\varphi}_{\eta,c} = 0$. We have

$$\begin{cases} L^\eta \hat{\varphi}_{\eta,c} = 0 \\ L^{\widetilde{\eta}} \hat{\varphi}_{\widetilde{\eta},\widetilde{c}} = 0. \end{cases}$$

Combining these yields

$$L^\eta(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c}) + \left(L^{\widetilde{\eta}} - L^\eta\right) \hat{\varphi}_{\widetilde{\eta},\widetilde{c}} = 0$$

$$\iff$$

$$L^\eta(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c}) = -\left(L^{\widetilde{\eta}} - L^\eta\right) \hat{\varphi}_{\widetilde{\eta},\widetilde{c}}.$$

Similarly, using the boundary condition $B^\eta \hat{\varphi}_{\eta,c}$,

$$\begin{cases} B^\eta \hat{\varphi}_{\eta,c} = c\eta_z \\ B^{\widetilde{\eta}} \hat{\varphi}_{\widetilde{\eta},\widetilde{c}} = \widetilde{c}\widetilde{\eta}_z \end{cases}$$

gives

$$B^\eta(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c}) = -\left(B^{\widetilde{\eta}} - B^\eta\right) \hat{\varphi}_{\widetilde{\eta},\widetilde{c}} + \eta_z(\widetilde{c} - c) + \widetilde{c}(\widetilde{\eta}_z - \eta_z).$$

The term $\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c}$ is a solution to a certain elliptic boundary value problem, namely

$$\begin{cases} L^\eta(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c}) = G, & \text{in } \Omega^1 \\ B^\eta(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c}) = H, & \text{on } \partial\Omega^1 \end{cases}$$

where $G = -\left(L^{\widetilde{\eta}} - L^\eta\right) \hat{\varphi}_{\widetilde{\eta},\widetilde{c}}$ and $H = -\left(B^{\widetilde{\eta}} - B^\eta\right) \hat{\varphi}_{\widetilde{\eta},\widetilde{c}} + \eta_z(\widetilde{c} - c) + \widetilde{c}(\widetilde{\eta}_z - \eta_z)$. By theorem 5.9, we have that for every η,

$$\|\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c} - U\|_{C^{2,\alpha}} \leq B(C_1, C_2)(\|G\|_{C^{0,\alpha}} + \|H\|_{C^{1,\alpha}})$$

with $U = \frac{1}{|\Omega^1|}\int_{\Omega^1}(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c})\eta^2 \, d(x, y, z)$. From this, we aim to move the integral condition to the right hand side and show boundedness for both G and H in terms of $\|\eta - \widetilde{\eta}\|_{C^{2,\alpha}}$ in order to establish Lipschitz continuity. Therefore, we have

$$\|\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c}\|_{C^{2,\alpha}} \leq B(C_1, C_2)\Bigg(\left\|-\left(L^{\widetilde{\eta}} - L^\eta\right) \hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\right\|_{C^{0,\alpha}} +$$

$$+ \left\|-\left(B^{\widetilde{\eta}} - B^\eta\right) \hat{\varphi}_{\widetilde{\eta},\widetilde{c}} + \eta_z(c - \widetilde{c}) + \widetilde{c}(\eta_z - \widetilde{\eta}_z)\right\|_{C^{1,\alpha}} \Bigg) + \|U\|_{C^{2,\alpha}}.$$

We begin by looking at the terms in the sum separately. We already know from our previous calculations that

$$\|U\|_{C^{2,\alpha}} = |U| \le A\|\eta - \widetilde{\eta}\|_{C^{2,\alpha}}.$$

For the term coming from the elliptic equation, we have

$$\left\|-\left(L^{\widetilde{\eta}} - L^{\eta}\right)\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\right\|_{C^{0,\alpha}} = \left\|-\left[\frac{1+s^2\widetilde{\eta}_z^2}{\widetilde{\eta}^2} - \frac{(1+s^2\eta_z^2)}{\eta^2}\right]\partial_s^2\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} + \cdots\right\|_{C^{0,\alpha}}.$$

It suffices to show the bound for only one of the coefficient terms since all of them are dealt with similarly. Thus, we have for instance

$$\left\|\left(\frac{s^2\widetilde{\eta}_z^2}{\widetilde{\eta}^2} - \frac{s^2\eta_z^2}{\eta^2}\right)\partial_s^2\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\right\|_{C^{0,\alpha}} = \left\|\frac{s^2(\eta\widetilde{\eta}_z + \widetilde{\eta}\eta_z)(\eta\widetilde{\eta}_z - \widetilde{\eta}\eta_z)}{\eta^2\widetilde{\eta}^2}\partial_s^2\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\right\|_{C^{0,\alpha}} =$$

$$= \left\|\frac{s^2(\eta\widetilde{\eta}_z + \widetilde{\eta}\eta_z)(\eta(\widetilde{\eta}_z - \eta_z) + \eta_z(\eta - \widetilde{\eta}))}{\eta^2\widetilde{\eta}^2}\partial_s^2\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\right\|_{C^{0,\alpha}}$$

$$\le \left\|\frac{s^2(\eta\widetilde{\eta}_z + \widetilde{\eta}\eta_z)\eta(\widetilde{\eta}_z - \eta_z)}{\eta^2\widetilde{\eta}^2}\partial_s^2\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\right\|_{C^{0,\alpha}} + \left\|\frac{s^2(\eta\widetilde{\eta}_z + \widetilde{\eta}\eta_z)\eta_z(\eta - \widetilde{\eta})}{\eta^2\widetilde{\eta}^2}\partial_s^2\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\right\|_{C^{0,\alpha}}$$

$$\le \left(\left\|\frac{1}{\eta^2\widetilde{\eta}^2}\right\|_{C^{0,\alpha}}(\|\eta\|_{C^{0,\alpha}} + \|\eta_z\|_{C^{1,\alpha}})\|\eta\widetilde{\eta}_z + \widetilde{\eta}\eta_z\|_{C^{0,\alpha}}\right)$$
$$\Big(\|\widetilde{\eta}_z - \eta_z\|_{C^{0,\alpha}} + \|\eta - \widetilde{\eta}\|_{C^{0,\alpha}}\Big)\|\partial_s^2\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\|_{C^{0,\alpha}}$$

$$\le 2\left(\left\|\frac{1}{\eta^2\widetilde{\eta}^2}\right\|_{C^{0,\alpha}}(\|\eta\|_{C^{0,\alpha}} + \|\eta_z\|_{C^{1,\alpha}})\|\eta\widetilde{\eta}_z + \widetilde{\eta}\eta_z\|_{C^{0,\alpha}}\right)\|\eta - \widetilde{\eta}\|_{C^{2,\alpha}}\|\partial_s^2\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\|_{C^{0,\alpha}}$$

where the last inequality comes from the fact that $C^{2,\alpha} \subset C^{0,\alpha}$ and

$$\|\widetilde{\eta}_z - \eta_z\|_{C^{0,\alpha}} \le \|\eta - \widetilde{\eta}\|_{C^{2,\alpha}}.$$

Thus, one such term in the sum can be bounded by the quantity

$$\Gamma\|\eta - \widetilde{\eta}\|_{C^{2,\alpha}}$$

where Γ is a constant which depends on the quantities $\|\eta_z\|_{C^{0,\alpha}}$, $\|\widetilde{\eta}_z\|_{C^{0,\alpha}}$, $\|\eta\|_{C^{0,\alpha}}$, $\|\widetilde{\eta}\|_{C^{0,\alpha}}$ and the positive lower bounds on $\eta, \widetilde{\eta}$. We get this because we can use the previous boundary Schauder estimate from theorem 5.9 to obtain

$$\|\partial_s^2\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\|_{C^{0,\alpha}} \le B(C_1, C_2)\|c\widetilde{\eta}_z\|_{C^{1,\alpha}}$$

Similarly, we have

$$\left\|- \left(B^{\widetilde{\eta}} - B^{\eta}\right) \hat{\varphi}_{\widetilde{\eta},\widetilde{c}} + \eta_z(c - \widetilde{c}) + \widetilde{c}(\eta_z - \widetilde{\eta}_z)\right\|_{C^{1,\alpha}} =$$

$$= \left\|- \left[\left(\frac{s(1+\widetilde{\eta}_z^2)}{\widetilde{\eta}} - \frac{s(1+\eta_z^2)}{\eta}\right) \partial_s \hat{\varphi}_{\widetilde{\eta},\widetilde{c}} + \cdots\right] + \eta_z(c - \widetilde{c}) + \widetilde{c}(\eta_z - \widetilde{\eta}_z)\right\|_{C^{1,\alpha}}$$

Again, we show the bound for one term in the sum.

$$\left\|\left(\frac{s}{\widetilde{\eta}} - \frac{s}{\eta}\right) \partial_s \hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\right\|_{C^{1,\alpha}} = \left\|\left(\frac{s(\eta - \widetilde{\eta})}{\eta\widetilde{\eta}}\right) \partial_s \hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\right\|_{C^{1,\alpha}} \leq$$

$$\leq s \left\|\frac{1}{\eta\widetilde{\eta}}\right\|_{C^{1,\alpha}} \|\eta - \widetilde{\eta}\|_{C^{1,\alpha}} \|\partial_s \hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\|_{C^{1,\alpha}} \leq$$

$$\leq B(C_1, C_2) \left\|\frac{1}{\eta\widetilde{\eta}}\right\|_{C^{1,\alpha}} \|\eta - \widetilde{\eta}\|_{C^{1,\alpha}} \|\widetilde{\eta}\|_{C^{2,\alpha}} \leq \beta \|\eta - \widetilde{\eta}\|_{C^{2,\alpha}}$$

where the constant β depends on the same quantities as the constant Γ. Adding the other two terms, we can incorporate the quantities $(c - \widetilde{c})$ and $\|\eta_z - \widetilde{\eta}_z\|_{C^{1,\alpha}}$ into the bound such that we obtain

$$\|\hat{\varphi}_{\widetilde{\eta},c} - \hat{\varphi}_{\eta,c}\|_{C^{2,\alpha}} \leq M(\|\widetilde{\eta} - \eta\|_{C^{2,\alpha}} + |c - \widetilde{c}|)$$

for some constant $M > 0$ which depends on the $C^{1,\alpha}$ norm of η, $\widetilde{\eta}$ (and the norms of their first and second derivatives respectively). From this it follows that $(\eta, c) \mapsto \hat{\varphi}_{\eta,c}$ is Lipschitz. □

Theorem 5.11. *The function $(\eta, c) \mapsto \hat{\varphi}_{\eta,c}$ is Fréchet differentiable.*

Proof. As before, $(\widetilde{\eta}, \widetilde{c})$ denotes an instance of the profile curve of $\partial\Omega^{\eta}$. We seek to differentiate with respect to (η, c) in the direction $(\widetilde{\eta} - \eta), (\widetilde{c} - c)$. We first find a candidate for the Fréchet derivative of $(\eta, c) \mapsto \hat{\varphi}_{\eta,c}$ and then show that it exists. Note that L^{η} does not depend on c in any way, and therefore we only vary in η. If $D_{\eta,c}\hat{\varphi}_{\eta,c}(\widetilde{\eta} - \eta, \widetilde{c} - c)$ were the Fréchet derivative of $\hat{\varphi}_{\eta,c}$ in the direction $(\widetilde{\eta} - \eta, \widetilde{c} - c)$, we could use the product rule to obtain

$$D_{\eta,c}L^{\eta}[\eta, c](\widetilde{\eta} - \eta, \widetilde{c} - c)\hat{\varphi}_{\eta,c} + L^{\eta}D_{\eta,c}\hat{\varphi}_{\eta,c}(\widetilde{\eta} - \eta, \widetilde{c} - c) = 0.$$

Similarly, from the boundary conditions we have

$$D_{\eta,c}B^{\eta}[\eta,c](\widetilde{\eta}-\eta,\widetilde{c}-c)\hat{\varphi}_{\eta,c}+B^{\eta}D_{\eta,c}\hat{\varphi}_{\eta,c}(\widetilde{\eta}-\eta,\widetilde{c}-c)=c(\widetilde{\eta}_z-\eta_z)+\eta_z(\widetilde{c}-c).$$

We also need the integral condition, which becomes

$$\int_{\Omega^1} D_{\eta,c}\hat{\varphi}_{\eta,c}(\widetilde{\eta}-\eta,\widetilde{c}-c)\eta^2+\hat{\varphi}_{\eta,c}2\eta(\widetilde{\eta}-\eta)\ d(s,t,z)=0 \tag{5.11}$$

or equivalently

$$\int_{\Omega^1} D_{\eta,c}\hat{\varphi}_{\eta,c}(\widetilde{\eta}-\eta,\widetilde{c}-c)\eta^2\ d(s,t,z)=-2\int_{\Omega^1}\hat{\varphi}_{\eta,c}\eta(\widetilde{\eta}-\eta)\ d(s,t,z).$$

This condition is fulfilled since we can modify $D_\eta\hat{\varphi}_{\eta,c}$ through a constant which will not change the equations, as they depend on the derivatives of $D_\eta\hat{\varphi}_{\eta,c}$. Now, these three equations define the linearization of $\hat{\varphi}_{\eta,c}$. The goal is to show that

$$\lim_{(\widetilde{\eta},\widetilde{c})\to(\eta,c)}\frac{\|\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}-\hat{\varphi}_{\eta,c}-D_{\eta,c}\hat{\varphi}_{\eta,c}(\widetilde{\eta}-\eta,\widetilde{c}-c)\|_{C^{2,\alpha}}}{\|\widetilde{\eta}-\eta\|_{C^{2,\alpha}}+|\widetilde{c}-c|}=0 \tag{5.12}$$

which we do by taking the limit of an equivalent expression in term of the operators L^η and B^η and by using the Schauder estimate from theorem 5.9. It must also be shown that the equations defining the Fréchet derivative of $\hat{\varphi}_{\eta,c}$ have a solution in the first place, in order for the derivative to be well-defined. This means that we must check the compatibility condition from theorem 5.5

$$\int_{\Omega^1} f=\int_{\partial\Omega^1} g$$

for the appropriate functions f and g. Using the transformation J, this can be done in the physical domain in which case we'd simply work with the Laplace operator with Neumann boundary conditions. Since we are already working in the flat domain, we show this condition directly in a separate lemma after this proof.

We have

$$\begin{cases} L^{\eta}\hat{\varphi}_{\eta,c}=0, & B^{\eta}\hat{\varphi}_{\eta,c}=c\eta_z\\ L^{\widetilde{\eta}}\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}=0, & B^{\widetilde{\eta}}\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}=\widetilde{c}\widetilde{\eta}_z\end{cases}$$

Looking only at the elliptic operator L^η first, we introduce some simpler notation.

$$L^{\eta}D_{\eta,c}\hat{\varphi}_{\eta,c}(\widetilde{\eta}-\eta,\widetilde{c}-c)=:L^{\eta}\delta\hat{\varphi}_{\eta,c}$$

which gives

$$D_{\eta,c}\hat{\varphi}_{\eta,c}(\widetilde{\eta}-\eta,\widetilde{c}-c)=:\delta\hat{\varphi}_{\eta,c}.$$

We also have

$$D_{\eta,c}L^{\eta}[\eta,c](\widetilde{\eta}-\eta,\widetilde{c}-c)\hat{\varphi}_{\eta,c}=:L^{\eta}_{\eta}\hat{\varphi}_{\eta,c}.$$

From equation (5.12), we have

$$L^{\widetilde{\eta}}\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}-L^{\eta}\hat{\varphi}_{\eta,c}-L^{\eta}\delta\hat{\varphi}_{\eta,c}=L^{\eta}_{\eta}\hat{\varphi}_{\eta,c}.$$

Adding and subtracting a certain term gives

$$L^{\widetilde{\eta}}\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}-L^{\eta}\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}+L^{\eta}\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}-L^{\eta}\hat{\varphi}_{\eta,c}-L^{\eta}\delta\hat{\varphi}_{\eta,c}=L^{\eta}_{\eta}\hat{\varphi}_{\eta,c}$$

which can be written as

$$L^{\eta}(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}-\hat{\varphi}_{\eta,c}-\delta\hat{\varphi}_{\eta,c})=-\left(L^{\widetilde{\eta}}-L^{\eta}\right)\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}+L^{\eta}_{\eta}\hat{\varphi}_{\eta,c}.$$

Repeating the previous steps, we finally get

$$L^{\eta}(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}-\hat{\varphi}_{\eta,c}-\delta\hat{\varphi}_{\eta,c})=-\left(L^{\widetilde{\eta}}-L^{\eta}-L^{\eta}_{\eta}\right)\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}-L^{\eta}_{\eta}(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}-\hat{\varphi}_{\eta,c}).$$

Therefore, to estimate (5.12), we must control

$$\left\|-\left(L^{\widetilde{\eta}}-L^{\eta}-L^{\eta}_{\eta}\right)\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}-L^{\eta}_{\eta}(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}-\hat{\varphi}_{\eta,c})\right\|_{C^{2,\alpha}}. \tag{5.13}$$

With a similar argument as the one used to show Lipschitz continuity, we investigate the growth rate. Again, it is enough to check for one term in the sum coming from applying the operator. First, we have $\left(L^{\widetilde{\eta}}-L^{\eta}-L^{\eta}_{\eta}\right)\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}$. Taking the first term, we get

$$\left(\frac{1}{\widetilde{\eta}^2}-\frac{1}{\eta^2}-\left(\frac{-2(\widetilde{\eta}-\eta)}{\eta^3}\right)\right)\partial_s^2\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}$$

because $L^{\eta}_{c}=0$. This is equal to

$$\left(\frac{\eta^3-3\eta\widetilde{\eta}^2+2\widetilde{\eta}^3}{\widetilde{\eta}^2\eta^3}\right)\partial_s^2\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}=\left(\frac{(\widetilde{\eta}-\eta)^2(\eta+2\widetilde{\eta})}{\widetilde{\eta}^2\eta^3}\right)\partial_s^2\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}.$$

From this, we have

$$\left\| \left(\frac{(\widetilde{\eta}-\eta)^2(\eta+2\widetilde{\eta})}{\widetilde{\eta}^2\eta^3} \right) \partial_s^2 \hat{\varphi}_{\widetilde{\eta},\widetilde{c}} \right\|_{C^{2,\alpha}} \leq \left\| \partial_s^2 \hat{\varphi}_{\widetilde{\eta},\widetilde{c}} \right\|_{C^{2,\alpha}} \left\| \frac{1}{\widetilde{\eta}^2\eta^3} \right\|_{C^{2,\alpha}} \|(\eta+2\widetilde{\eta})\|_{C^{2,\alpha}} \|(\widetilde{\eta}-\eta)\|^2_{C^{2,\alpha}}$$

where the right hand side is controlled by the quantity $\widetilde{C}\|\widetilde{\eta}-\eta\|^2_{C^{2,\alpha}}$ with the constant $\widetilde{C}$ depending on the Schauder estimate for $\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}$ and the other norms. Therefore,

$$\left\| - \left(L^{\widetilde{\eta}} - L^{\eta} - L^{\eta}_{\eta} - L^{\eta}_{c} \right) \hat{\varphi}_{\widetilde{\eta},\widetilde{c}} \right\|_{C^{2,\alpha}} = \mathcal{O}(\|\widetilde{\eta}-\eta\|^2_{C^{2,\alpha}})$$

Next, from the Lipschitz continuity of the map $(\eta, c) \mapsto \hat{\varphi}_{\eta,c}$, we have

$$\|\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c}\|_{C^{2,\alpha}} = \mathcal{O}(\|\widetilde{\eta}-\eta\|_{C^{2,\alpha}}).$$

Finally, since the derivative L^{η}_{η} at the point (η, c) is linear in the perturbation $(\widetilde{\eta}-\eta)$ it follows that

$$L^{\eta}_{\eta} = \mathcal{O}(\|\widetilde{\eta}-\eta\|_{C^{2,\alpha}})$$

and all the terms in (5.13) are of order $\mathcal{O}(\|\widetilde{\eta}-\eta\|^2_{C^{2,\alpha}})$. Thus,

$$L^{\eta}(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c} - \delta_{\eta}\hat{\varphi}_{\eta,c}) = \mathcal{O}(\|\widetilde{\eta}-\eta\|^2_{C^{2,\alpha}})$$

We repeat the same procedure with the boundary conditions operator, that is, we want to show

$$\lim_{(\widetilde{\eta},\widetilde{c})\to(\eta,c)} \frac{\left\| B^{\widetilde{\eta}}\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - B^{\eta}\hat{\varphi}_{\eta,c} - B^{\eta}\delta_{\eta}\hat{\varphi}_{\eta,c} \right\|_{C^{2,\alpha}}}{\|\widetilde{\eta}-\eta\|_{C^{2,\alpha}} + |\widetilde{c}-c|} = 0.$$

Using the notation

$$B^{\eta} D_{\eta,c}\hat{\varphi}_{\eta,c}(\widetilde{\eta}-\eta, \widetilde{c}-c) =: B^{\eta}\delta_{\eta,c}\hat{\varphi}_{\eta,c}$$

$$D_{\eta,c}B^{\eta}[\eta, c](\widetilde{\eta}-\eta, \widetilde{c}-c)\hat{\varphi}_{\eta,c} =: B^{\eta}_{\eta,c}\hat{\varphi}_{\eta,c}$$

As before, we have

$$B^{\widetilde{\eta}}\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - B^{\eta}\hat{\varphi}_{\eta,c} - B^{\eta}\delta_{\eta,c}\hat{\varphi}_{\eta,c} = B^{\eta}_{\eta,c}\hat{\varphi}_{\eta,c} + \widetilde{c}\widetilde{\eta}_z - c\eta_z - c(\widetilde{\eta}_z - \eta_z) - \eta_z(\widetilde{c}-c)$$

which can be written as

$$B^{\widetilde{\eta}}\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - B^{\eta}\hat{\varphi}_{\eta,c} - B^{\eta}\delta_{\eta,c}\hat{\varphi}_{\eta,c} = B^{\eta}_{\eta,c}\hat{\varphi}_{\eta,c} + (\widetilde{\eta}_z - \eta_z)(\widetilde{c}-c)$$

Similar computations as before give

$$B^{\eta}(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c} - \delta_{\eta,c}\hat{\varphi}_{\eta,c}) = -\left(B^{\widetilde{\eta}} - B^{\eta} - B^{\eta}_{\eta,c}\right)\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - B^{\eta}_{\eta,c}(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c}) + (\widetilde{\eta}_z - \eta_z)(\widetilde{c} - c). \tag{5.14}$$

Again, we check only one term, so we have

$$\left(\frac{s}{\widetilde{\eta}} - \frac{s}{\eta} - \left(\frac{-s(\widetilde{\eta}-\eta)}{\eta^2}\right)\right)\partial_s\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} = \left(-\frac{s(\widetilde{\eta}-\eta)^2}{\widetilde{\eta}\eta}\right)\partial_s\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}$$

and so

$$\left\|\frac{s(\widetilde{\eta}-\eta)^2}{\widetilde{\eta}\eta^2}\partial_s\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\right\|_{C^{2,\alpha}} \leq \left\|\frac{s}{\widetilde{\eta}\eta^2}\right\|_{C^{2,\alpha}} \|\partial_s\hat{\varphi}_{\widetilde{\eta},\widetilde{c}}\|_{C^{2,\alpha}} \|\widetilde{\eta}-\eta\|^2_{C^{2,\alpha}}$$

By the same arguments as before, we have that the terms B^{η}_{η}, B^{η}_{c} and $(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c})$ are $\mathcal{O}(\|\widetilde{\eta}-\eta\|_{C^{\in,\alpha}})$. The same applies to $(\widetilde{\eta}_z - \eta_z)$ and $(\widetilde{c} - c)$. Therefore, the right hand side in (5.14) is $\mathcal{O}(\|\widetilde{\eta}-\eta\|^2_{C^{2,\alpha}})$. We can now apply the Schauder estimate. Let

$$P = \hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c} - \delta\hat{\varphi}_{\eta,c}.$$

Then, we apply theorem 5.9 to

$$\left\|P - \int_{\Omega^1} D\eta^2 \; d(s,t,z)\right\|_{C^{2,\alpha}}.$$

Since we have already estimated the terms G and H, it remains to take care of the integral condition.

$$\left|\int_{\Omega^1} D\eta^2 \; d(s,t,z)\right| = \left|\int_{\Omega^1} (\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c} - \delta\hat{\varphi}_{\eta,c})\eta^2 \; d(s,t,z)\right|$$

Note that the middle term vanishes, that is $\int_{\Omega^1} \hat{\varphi}_{\eta,c}\eta^2 \; d(s,t,z) = 0$. We can write the integrand as

$$\int_{\Omega^1} |(\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c})(\eta^2 - \widetilde{\eta}^2)| \; d(s,t,z) + \int_{\Omega^1} |\hat{\varphi}_{\eta,c}\left((\eta^2 - \widetilde{\eta}^2) + 2\eta\widetilde{\eta}\right)| \; d(s,t,z)$$

by adding and substracting zero, and using the linearized integral condition (5.11). Finally, we write

$$\int_{\Omega^1} |\hat{\varphi}_{\eta,c}\left((\eta^2 - \widetilde{\eta}^2) + 2\eta\widetilde{\eta}\right)| \; d(s,t,z) = \int_{\Omega^1} \hat{\varphi}_{\eta,c}(-\eta^2 - \widetilde{\eta}^2 + 2\eta\widetilde{\eta}) \; d(s,t,z)$$

which then gives

$$\int_{\Omega^1} \hat{\varphi}_{\eta,c}(\eta - \widetilde{\eta})^2 \; d(s,t,z)$$

The integral condition is therefore also of order $\mathcal{O}(\|\widetilde{\eta}-\eta\|_{C^{2,\alpha}})$. Since we have shown that the numerator is of order $\mathcal{O}(\|\widetilde{\eta}-\eta\|^2_{C^{2,\alpha}})$, it follows that

$$\lim_{(\widetilde{\eta},\widetilde{c})\to(\eta,c)} \frac{\|\hat{\varphi}_{\widetilde{\eta},\widetilde{c}} - \hat{\varphi}_{\eta,c} - D_{\eta,c}\hat{\varphi}_{\eta,c}(\widetilde{\eta}-\eta,\widetilde{c}-c)\|_{C^{2,\alpha}}}{\|\widetilde{\eta}-\eta\|_{C^{2,\alpha}} + |\widetilde{c}-c|} = 0$$

and hence the map $(\eta, c) \mapsto \hat{\varphi}_{\eta,c}$ is Fréchet differentiable. □

It remains to show the compatibility condition. First, we identify the functions corresponding to f and g in theorem 5.5. From the definition of $D_{\eta,c}\hat{\varphi}_{\eta,c}(\widetilde{\eta}-\eta,\widetilde{c}-c)$, we know that it must satisfy

$$\begin{cases} L^\eta D_{\eta,c}\hat{\varphi}_{\eta,c}(\widetilde{\eta}-\eta,\widetilde{c}-c) = -D_{\eta,c}L^\eta[\eta,c](\widetilde{\eta}-\eta,\widetilde{c}-c)\hat{\varphi}_{\eta,c} \\ B^\eta D_{\eta,c}\hat{\varphi}_{\eta,c}(\widetilde{\eta}-\eta,\widetilde{c}-c) = -D_{\eta,c}B^\eta[\eta,c](\widetilde{\eta}-\eta,\widetilde{c}-c)\hat{\varphi}_{\eta,c} + c(\widetilde{\eta}_z-\eta_z) + \eta_z(\widetilde{c}-c). \end{cases}$$

where the first equation holds in Ω^1 and the second holds on $\partial\Omega^1$. In order for this problem to have a solution, we must show that the right hand side terms satisfy the compatibility condition.

Lemma 5.12. *It holds that*

$$\begin{aligned} &-\int_{\Omega^1} D_\eta L^\eta[\eta](\widetilde{\eta}-\eta)\hat{\varphi}_{\eta,c}\eta^2 \; d(s,t,z) \\ &= \int_{\partial\Omega^1} (-D_\eta B^\eta[\eta](\widetilde{\eta}-\eta)\hat{\varphi}_{\eta,c} + c(\widetilde{\eta}_z-\eta_z) + \eta_z(\widetilde{c}-c))\eta \; dS. \end{aligned}$$

Proof. We first write D_η as a difference quotient, that is,

$$D_\eta L^\eta[\eta](\widetilde{\eta}-\eta) = \lim_{t\to 0} \frac{L^{\eta+t(\widetilde{\eta}-\eta)} - L^\eta}{t}$$

and

$$D_\eta B^\eta[\eta](\widetilde{\eta}-\eta) = \lim_{t\to 0} \frac{B^{\eta+t(\widetilde{\eta}-\eta)} - B^\eta}{t}.$$

By theorems 5.3 and 5.5, we know that for $t \neq 0$ and for $\eta + t(\widetilde{\eta}-\eta)$ and $c + t(\widetilde{c}-c)$

$$\begin{aligned} &-\int_{\Omega^1} \frac{L^{\eta+t(\widetilde{\eta}-\eta)} - L^\eta}{t} \hat{\varphi}_{\eta+t(\widetilde{\eta}-\eta),c+t(\widetilde{c}-c)}\eta^2 \; d(s,t,z) \\ &= \int_{\partial\Omega^1} \frac{B^{\eta+t(\widetilde{\eta}-\eta)} - B^\eta}{t} \hat{\varphi}_{\eta+t(\widetilde{\eta}-\eta),c+t(\widetilde{c}-c)} + c(\widetilde{\eta}_z-\eta_z) + \eta_z(\widetilde{c}-c))\eta \; dS \end{aligned}$$

Consider the difference

$$
\begin{aligned}
&-\int_{\Omega^1} D_\eta L^\eta[\eta](\widetilde{\eta}-\eta)\hat{\varphi}_{\eta,c}\eta^2\ d(s,t,z) \\
&-\int_{\partial\Omega^1} (-D_\eta B^\eta[\eta](\widetilde{\eta}-\eta)\hat{\varphi}_{\eta,c} + c(\widetilde{\eta}_z-\eta_z)+\eta_z(\widetilde{c}-c))\eta\ dS \\
&= \lim_{t\to 0}\Bigg[-\int_{\Omega^1}\frac{L^{\eta+t(\widetilde{\eta}-\eta)}-L^\eta}{t}\hat{\varphi}_{\eta,c}\eta^2\ d(s,t,z) \\
&-\int_{\partial\Omega^1}\left(-\frac{B^{\eta+t(\widetilde{\eta}-\eta)}-B^\eta}{t}\hat{\varphi}_{\eta,c}+c(\widetilde{\eta}_z-\eta_z)+\eta_z(\widetilde{c}-c)\right)\eta\ dS\Bigg].
\end{aligned}
$$

Adding and subtracting 0, we obtain the long expression

$$
\begin{aligned}
&\lim_{t\to 0}\Bigg[-\int_{\Omega^1}\frac{L^{\eta+t(\widetilde{\eta}-\eta)}-L^\eta}{t}\hat{\varphi}_{\eta,c}\eta^2\ d(s,t,z) \\
&-\int_{\partial\Omega^1}\left(-\frac{B^{\eta+t(\widetilde{\eta}-\eta)}-B^\eta}{t}\hat{\varphi}_{\eta,c}+c(\widetilde{\eta}_z-\eta_z)+\eta_z(\widetilde{c}-c)\right)\eta\ dS \\
&+\int_{\Omega^1}\frac{L^{\eta+t(\widetilde{\eta}-\eta)}-L^\eta}{t}\hat{\varphi}_{\eta+t(\widetilde{\eta}-\eta),c+t(\widetilde{c}-c)}\eta^2\ d(s,t,z) \\
&+\int_{\partial\Omega^1}\left(-\frac{B^{\eta+t(\widetilde{\eta}-\eta)}-B^\eta}{t}\hat{\varphi}_{\eta+t(\widetilde{\eta}-\eta),c+t(\widetilde{c}-c)}+(c+t(\widetilde{c}-c))(\widetilde{\eta}_z-\eta_z)+\eta_z(\widetilde{c}-c)\right)\eta\ dS\Bigg].
\end{aligned}
$$

Simplifying the expression in the square brackets, we obtain

$$
\begin{aligned}
&\lim_{t\to 0}\Bigg[\int_{\Omega^1}\left(\frac{L^{\eta+t(\widetilde{\eta}-\eta)}-L^\eta}{t}\right)\left(\hat{\varphi}_{\eta+t(\widetilde{\eta}-\eta),c+t(\widetilde{c}-c)}-\hat{\varphi}_{\eta,c}\right)\eta^2\ d(s,t,z) \\
&+\int_{\partial\Omega^1}\left(-\left(\frac{B^{\eta+t(\widetilde{\eta}-\eta)}-B^\eta}{t}\right)\left(\hat{\varphi}_{\eta+t(\widetilde{\eta}-\eta),c+t(\widetilde{c}-c)}-\hat{\varphi}_{\eta,c}\right)+(\widetilde{\eta}_z-\eta_z)(t(\widetilde{c}-c))\right)\eta\ dS\Bigg].
\end{aligned}
$$

We begin with the term involving the elliptic operator L^η. Observe that by theorem 5.10, we have

$$
\left\|\left(\frac{L^{\eta+t(\widetilde{\eta}-\eta)}-L^\eta}{t}\right)\left(\hat{\varphi}_{\eta+t(\widetilde{\eta}-\eta),c+t(\widetilde{c}-c)}-\hat{\varphi}_{\eta,c}\right)\right\|_{C^{0,\alpha}} \leq B(C_1,C_2)\left\|\hat{\varphi}_{\eta+t(\widetilde{\eta}-\eta),c+t(\widetilde{c}-c)}-\hat{\varphi}_{\eta,c}\right\|_{C^{2,\alpha}}
$$

and by the same result, we can calculate that

$$
\left\|\hat{\varphi}_{\eta+t(\widetilde{\eta}-\eta),c+t(\widetilde{c}-c)}-\hat{\varphi}_{\eta,c}\right\|_{C^{2,\alpha}} \leq \Gamma(\|t(\widetilde{\eta}-\eta)\|_{C^{2,\alpha}}+|t(\widetilde{c}-c)|)
$$

which is of order $\mathcal{O}(t)$. The constant Γ depends on the $C^{0,\alpha}$ norms of η and $\widetilde{\eta}$ as well as their derivatives. Passing to the limit as $t\to 0$, this term becomes 0. We can do the same on the term involving the boundary operator B^η. We have

$$
\left\|\left(\frac{B^{\eta+t(\widetilde{\eta}-\eta)}-B^\eta}{t}\right)\left(\hat{\varphi}_{\eta+t(\widetilde{\eta}-\eta),c+t(\widetilde{c}-c)}-\hat{\varphi}_{\eta,c}\right)\right\|_{C^{0,\alpha}} \leq B(C_1,C_2)\left\|\hat{\varphi}_{\eta+t(\widetilde{\eta}-\eta),c+t(\widetilde{c}-c)}-\hat{\varphi}_{\eta,c}\right\|_{C^{2,\alpha}}
$$

and similarly,

$$\left\|\hat{\varphi}_{\eta+t(\widetilde{\eta}-\eta),c+t(\widetilde{c}-c)} - \hat{\varphi}_{\eta,c}\right\|_{C^{2,\alpha}} \leq M(\|t(\widetilde{\eta}-\eta)\|_{C^{2,\alpha}} + |t(\widetilde{c}-c)|)$$

which is also of order $\mathcal{O}(t)$. Finally, we have the term $(\widetilde{\eta}_z - \eta_z)(t(\widetilde{c}-c))$ of order $\mathcal{O}(t)$ as well. Combining these results, we conclude that the limit as $t \to 0$ is 0 and we are done. □

5.5 Invertibility of the Fréchet derivative

Now that we have established that the map $(\eta, c) \mapsto \hat{\varphi}_{\eta,c}$ is regular, so that $\hat{\varphi}_{\eta,c}$ is differentiable with respect to both η and c, we can find the Fréchet derivative of F with respect to (η, c) evaluated at the point $(\eta_{1,k}, 0)$, which we denote by $D_\eta F[\eta_{1,k_0}, 0]$. In the following, $\widetilde{\eta} \in C^{2,\alpha}_{\text{per,e}}(\mathbb{R})$ denotes a perturbation of the static solutions, that is a 2π periodic function with small $C^{2,\alpha}$ norm.

The Fréchet derivative is given by

$$\begin{aligned} D_\eta F[\eta_{1,k_0}, 0](\widetilde{\eta}, \widetilde{c}) = & \frac{\widetilde{\eta}_{zz}(1+\eta_z^2)^{3/2} - \eta_{zz}\frac{3}{2}(1+\eta_z^2)^{1/2} 2\eta_z\widetilde{\eta}_z}{(1+\eta_z^2)^3} + \\ & \frac{\widetilde{\eta}}{\eta^2}\Big(\frac{1}{(1+\eta_z^2)^{1/2}}\Big) - \frac{1}{\eta}\Big(\frac{-\eta_z\widetilde{\eta}_z(1+\eta_z^2)^{-1/2}}{(1+\eta_z^2)}\Big) \end{aligned} \tag{5.15}$$

Note that the expression coming from the term $|\nabla\phi_{\eta,c}|^2$ (and thus the derivative with respect to c) vanishes because

$$\frac{d}{d\tau}\left(\frac{1}{2}|\hat{\nabla}(\hat{\varphi}_{\eta_{1,k_0}+\tau\widetilde{\eta},0+\tau\widetilde{c}} + (0,0,\tau\widetilde{c}))|^2\right)\Bigg|_{\tau=0} =$$

$$= \left(\hat{\nabla}(\hat{\varphi}_{\eta_{1,k_0},0} + (0,0,0))\right) \cdot \left(\frac{d}{d\tau}\hat{\nabla}(\hat{\varphi}_{\eta_{1,k_0}+\tau\widetilde{\eta},\tau\widetilde{c}} + (0,0,\widetilde{c}))\right)\Bigg|_{\tau=0} = 0$$

where $\hat{\nabla}$ denotes the gradient in terms of $\hat{\varphi}$ in the flat domain (equation (5.4)) for simplicity. Since we differentiate at the point $(\eta_{s,k}, 0)$, and by proposition 4.1, $\nabla\hat{\varphi} = 0$. Moreover, note that since $\hat{\varphi}_{\eta,c}$ is regular with respect to (η, c), it follows that $\hat{\nabla}\hat{\varphi}_{\eta,c}$ is also regular and hence differentiable with respect to (η, c). Thus, our previous calculations are justified. Now, because all the partial derivatives in (5.15) are taken with respect to the variable z only, we can treat this as a second order ordinary linear differential equation. We first rewrite it in the following way:

$$D_\eta F[\eta_{s,k},0](\widetilde{\eta}) = \frac{1}{(1+\eta_z^2)^{3/2}}\widetilde{\eta}_{zz} - \Big(\frac{\eta_z(3\eta_{zz}\eta-(1+\eta_z^2))}{\eta(1+\eta_z^2)^{5/2}}\Big)\widetilde{\eta}_z + \Big(\frac{1}{\eta^2(1+\eta_z^2)^{1/2}}\Big)\widetilde{\eta}$$

with periodic boundary conditions. It is useful to first consider this linear operator on a larger spaces than the ones F is defined on. Thus, we omit the extra assumption of periodicity and evenness and consider $D_\eta F[\eta_{s,k},0]$ as an operator on $C^{2,\alpha}(\mathbb{R})$.

Lemma 5.13. *The solutions $\widetilde{\eta}$ are $C^{2,\alpha}(\mathbb{R})$ given that the coefficients of $D_\eta F$ are in $C^{0,\alpha}(\mathbb{R})$.*

Proof. Since $D_\eta F[\eta_{s,k},0]$ is Hölder continuous (in $C^{0,\alpha}(\mathbb{R})$), and it is an elliptic equation, we can apply an interior Schauder estimate which gives that $\widetilde{\eta}\in C^{2,\alpha}(\mathbb{R})$. □

Proposition 5.14. *For each $\eta_{s,k}$, the kernel of the linear differential operator $D_\eta F$ on the space $C^{2,\alpha}(\mathbb{R})$ is two-dimensional.*

Proof. This follows from Lemma 5.13 and the fact that the solution space of an n'th order linear ODE has dimension n. □

Proposition 5.15. *$D_\eta F[\eta_{1,k_0},0]$ has trivial kernel if the space $C^{2,\alpha}(\mathbb{R})$ is restricted to even and 2π periodic functions.*

Proof. Given the restriction above, we consider periodic boundary conditions for $D_\eta F[\eta_{s,k},0](\widetilde{\eta})$, namely $\widetilde{\eta}(0)=\widetilde{\eta}(2\pi)$ and $\widetilde{\eta}_z(0)=\widetilde{\eta}_z(2\pi)$. By proposition 5.14, the space $\{\widetilde{\eta}\in C^{2,\alpha}(\mathbb{R}) : D_\eta F[\eta_{1,k_0},0](\widetilde{\eta})=0\}$ has dimension 2. We can look for the basis elements of this space. Consider first

$$\partial_z F(\eta_{1,k_0},0)(z) = D_\eta F[\eta_{1,k_0},0]\eta'_{1,k_0}(z) = 0$$

We get that the first basis element is η'_{1,k_0}, which is an odd function. Next, consider

$$\partial_s F(\eta_{1,k_0},0)(z) = D_\eta F[\eta_{1,k_0},0]\partial_s\eta_{1,k_0} = \partial_s B_{1,k_0}$$

and

$$\partial_k F(\eta_{1,k_0},0)(z) = D_\eta F[\eta_{1,k_0},0]\partial_k\eta_{1,k_0} = \partial_k B_{1,k_0} \tag{5.16}$$

where B_{1,k_0} is the B corresponding to the static unduloids solutions parametrized by s and k. We combine these two equations in order to set $D_\eta F[\eta_{1,k_0}, 0] = 0$ and find another kernel element. For this we need to check that $\partial_k B_{1,k_0} \neq 0$, which we do at the end. Multiplying both sides of (5.16) by $\delta = \frac{\partial_s B_{1,k_0}}{\partial_k B_{1,k_0}}$, we obtain

$$D_\eta F[\eta_{1,k_0}, 0]\partial_k \eta_{1,k_0}\delta = \partial_s B_{1,k_0}.$$

Then, by subtracting we get

$$D_\eta F[\eta_{1,k_0}, 0]\Big(\partial_s \eta_{1,k_0} - \delta\partial_k \eta_{1,k_0}\Big) = 0.$$

Hence, the second basis element is given by $\partial_s \eta_{1,k_0} - \delta\partial_k \eta_{1,k_0}$. Observe that

$$\partial_s \eta_{1,k_0}(z) = \partial_s(s\eta_{1,k_0}(z/s))\Big|_{s=1} = \eta_{1,k_0}(z) - z\eta'_{1,k_0}(z)$$

is not periodic because of the second term. By writing

$$\partial_k \eta_{1,k_0}(z) = \lim_{h\to 0} \frac{sc(k_0+h)\eta_{1,k_0}(z/sc(k_0+h)) - sc(k_0)\eta_{1,k_0}(z/sc(k_0))}{h}$$

we see that both terms are 2π-periodic. The periodicity of the first term follows from the fact that k is a period preserving parameter, and so any change to it keeps the fixed period of 2π. This means the basis element $\partial_s \eta_{1,k_0} - \delta\partial_k \eta_{1,k_0}$ is not periodic. On the other hand, the basis element η'_{1,k_0} is odd. If we restrict the differential operator $D_\eta F[\eta_{1,k_0}]$ to 2π-periodic and even functions, it follows that the kernel of this restricted operator contains none of the basis elements and is trivial. Finally, we check the assumption that $\partial_k Q_{s,k} \neq 0$. This is equivalent to checking that the mean curvature is strictly monotone in k. Let $s = 1$ in equation (3.16), and differentiate with respect to k, for $0 < k < 1$, using formulae (3.2) and (3.3) to obtain

$$H'(k) = \frac{2(E(k) - K(k)) + k^2 K(k)}{(1+\sqrt{1-k^2})\pi\sqrt{1-k^2}\cdot k}$$

We want to show that

$$2\cdot(E(k) - K(k)) + k^2 K(k) < 0 \tag{5.17}$$

Using the series expansion formulae (3.4),(3.5) and making some simplifications, we write the left hand side of (5.17) as

$$\pi\sum_{n=2}^{\infty}\Big(\frac{(2n!)}{2^{2n}(n!)^2}\Big)^2 k^{2n}\Big(\frac{-2n(n-1)}{(2n-1)^2}\Big)$$

which is negative since

$$-2n(n-1) < 0$$

for all $n = 2, 3, 4, \ldots$. □

Theorem 5.16. *$D_\eta F[\eta_{1,k_0}, 0]$ is invertible when considered as an operator between the spaces $C^{2,\alpha}_{per,e}(\mathbb{R})$ and $C^{0,\alpha}_{per,e}(\mathbb{R})$ of 2π periodic, even functions.*

Proof. By proposition 5.15, $ker(D_\eta F[\eta_{1,k_0}, 0]) = \{0\}$ in the space $C^{2,\alpha}_{\mathrm{per,e}}(\mathbb{R})$. Then, the operator is injective. We now consider the differential equation $D_\eta F[\eta_{1,k_0}, 0](\widetilde{\eta}) = 0$ on the interval $(0, \pi)$ with homogenous boundary conditions $B(\widetilde{\eta}) = 0$ given by $\widetilde{\eta}'(0) = \widetilde{\eta}'(\pi) = 0$. From here, we can "build" the space $C^{2,\alpha}_{\mathrm{per,e}}(\mathbb{R})$ by first extending the function $\widetilde{\eta}$ evenly to $(-\pi, \pi)$ and then periodically to $\mathbb{R}$. By theorem 1 (b) from [1] (p.178), the differential operator $DF[\eta_{1,k_0}, 0]$ is a surjection between the spaces $C^{2,\alpha}(0, \pi)$ and $C^{0,\alpha}(0, \pi)$, since the homogenous problem $DF[\eta_{1,k_0}, 0](\widetilde{\eta}) = 0$ only has the trivial solution $\widetilde{\eta}$. Following the extension presented above, it then holds that the operator $DF[\eta_{1,k_0}, 0]$ between the spaces $C^{2,\alpha}_{\mathrm{per,e}}(\mathbb{R})$ and $C^{0,\alpha}_{\mathrm{per,e}}(\mathbb{R})$ is also surjective. We conclude that $DF[\eta_{1,k_0}, 0]$ is bijective. □

5.6 Conclusion

We now have all the necessary tools to give a proof of the theorem we set out to establish. We summarize the result here. Using the operator formulation (4.1) of the problem, we know that $F(\eta, c)$ is regular by theorem 5.10. Moreover, the Fréchet derivative $D_\eta F[\eta_{1,k_0}, 0]$ at the (static) point $(\eta_{1,k_0}, 0)$ is an isomorphism between the spaces $C^{2,\alpha}_{\mathrm{per,e}}(\mathbb{R})$ and $C^{0,\alpha}_{\mathrm{per,e}}(\mathbb{R})$ by theorem 5.16. Therefore, by the implicit function theorem for Banach spaces, in a neighbourhood of the static solution $(\eta_{1,k_0}, 0)$, there exists a unique one-parameter family of non-static solutions (η, c) in the space $U \times (-\delta, \delta) \subset C^{2,\alpha}_{\mathrm{per,e}}(\mathbb{R}) \times \mathbb{R}$.

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