Factor HJM Yield Curve Modelling for Pricing of Danish Callable Mortgage Bonds

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Factor HJM Yield Curve Modelling for Pricing of Danish Callable Mortgage Bonds - Popular Scientific Summary

Ivar Fagerfjäll and Erik Hu

The Danish mortgage market is unique due to its long tradition of funding loans through issuing bonds rather than using bank deposits. This thesis explores the use of a new interest rate model for pricing these mortgage bonds.

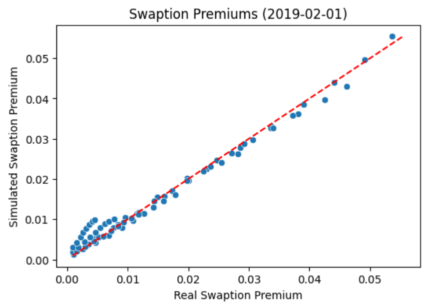
The Danish mortgage market works very differently than most others in the world. If you apply for a mortgage in Sweden, you go to a standard bank that funds the loan using deposits. In Denmark, these loans are instead issued by specialized mortgage banks and funded by pooling together many loans with the same structure and issuing a *Mortgage Backed Bond* (MBB), traded on the open market. The interest and amortization payments of the loan then go directly to the bondholder instead of the mortgage bank.

Historically this structure has proven very effective with the Danish housing market faring well even through tumultuous economic periods. One notable feature of this system is its built-in prepayment option, allowing borrowers to optionally prepay their loans every quarter. This gives loan takers added flexibility as they can refinance their loans if interest rates fall.

The drawback of the prepayment option is that it complicates pricing of the MBBs. Normal coupon bonds that pay out a fixed amount every month are priced by simply discounting their cash flows. The only uncertainty is thus the evolution of the market rates. The prepayment option of the MBBs adds another level of risk as they make the cash flows themselves stochastic. Pricing the bonds is generally done in two steps; modeling the interest rate and estimating future prepayments.

For the first step, this thesis explores the use of a new model, the *Factor HJM* (FHJM) model, which aims to bridge the gap between how interest rates are modelled in pricing and "real-world" applications. It does so by adapting the methodology and reasoning used in real-world modelling to the constraints of pricing applications. In financial markets, arbitrage is the practice of taking advantage of mispricings in the market that allow for riskless profit. Pricing models must be designed to prevent arbitrage opportunities to ensure fair and accurate pricing.

The figure below illustrates the calibrated model's estimates of swaption premiums plotted against the observed premiums in the market. Swaptions, which are options on interest rate swap contracts, are commonly used to calibrate interest rate models. Although the model appears to overestimate the lower premiums, it is evident that it captures the market behavior well.



The second step involved using a Probit model. Each quarter until the loan's maturity, the percentage of underlying loans that are prepaid was modeled by combining various explanatory variables. The most significant variable was the refinancing gain, which represents the amount of money borrowers would obtain from refinancing to a new interest rate. The refinancing gain was estimated by using the interest rate model to simulate future market rate evolutions and comparing them to the current loan interest rate.

Overall, the model performed relatively well. While the results could be further improved, they demonstrate that the FHJM model is a viable, albeit complex, method for grounding the pricing process of interest rate products in real world dynamics.

Abstract

A distinguishing component of the mortgage bond market in Denmark is the option to prepay a loan before its maturity date. From the debtors perspective, this is often done with the purpose of refinancing to a lower lending rate. Thus, interest rate modelling becomes a natural part of the pricing process of Danish callable mortgage bonds. Today, yield curves are modelled and thought of differently in real world versus pricing applications, creating a divide between the two. In this thesis, a factor modelling approach was combined with the Musiela HJM framework, as a means of bridging the divide.

The interest rate model was calibrated to forward rates and swaption premiums. Subsequently, yield curves were simulated and used in the calculation of refinance gains. Comparing with historical data, the factor HJM model showcased a good calibration to market volatility and rates.

The refinance gain, as well as a selection of other explanatory variables, were tested for significance in a probit model. The optimisation used a maximum likelihood estimation with historical prepayment rates as targets. The different model variations demonstrated predictive power albeit with some systematic biases and areas of improvement. Notably, the best model achieved an R^2 value of around 0.45, indicating a substantial portion of the variability in prepayment rates could be explained by the model. However, better calibration and further data exploration, including individual bond and debtor group data alongside macroeconomic factors, could enhance predictive accuracy.

While the factor HJM model exhibited strong calibration to the market, its utility for pricing callable mortgage bonds remains uncertain. The thesis highlights that a complex interest rate model may not significantly impact results when coupled with a relatively simple prepayment model. Nevertheless, the attempt at bridging the gap between real world and pricing applications represents a theoretical advancement, albeit one that may not yet be widely adopted in practice.

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Contents

1 Introduction

If you would be looking to buy a house in Denmark, you would normally go to a mortgage bank and ask for a mortgage loan. The bank would then pool your loan together with other loans of similar terms and structure, before issuing it on the market as a mortgage bond. In turn, the money from investors in the mortgage bond would be used to fund your loan. However, mortgage bonds in Denmark are often callable, meaning the borrower can choose to prepay their loan at any time until loan maturity. This makes the process more intricate for the mortgage bank and investors, as a prepayment would change the bonds cash flow, and would therefore need to be considered when pricing it.

As will be explained later, a major part of the prepayment behaviour of borrowers can be assumed to depend on interest rates and the yield curve. As such, a key part of the pricing process of Danish callable mortgage bonds is to forecast interest rates. Interest rates and the yield curve are however thought of and modelled differently in pricing applications versus real world applications. While no-arbitrage conditions lead to risk neutrality in pricing applications, the real world behaviour and actions of e.g the debtors of callable mortgage bonds cannot be assumed to be risk neutral.

The aim of this thesis is to explore a potential improvement in bridging the gap between yield curve modelling in real world versus pricing applications, by implementing a factor Heath-Jarrow-Morton (HJM) model, and explore its practical implications.

The factor HJM model will be implemented in Python, and calibrated to forward rates as well as swaption premiums, that are calculated from implied volatility using the Black-76 and Bachelier models. The calibrated model will be used to simulate yield curves, which in turn will be used in the calculations of refinance gain. Using refinance gain and some additional explanatory variables, a probit model will be introduced to calculate the prepayment rates. A total of 265 bonds, issued between 2004 and 2024 by two of the largest mortgage banks in Denmark, will be used and split into training as well as out-of-sample data. The former will be used to train the model and the latter to test its performance. All raw data will be provided by SEB, who will also be co-supervising this thesis. Finally, given the resulting prepayment rates, the general steps to pricing callable mortgage bonds will be presented.

2 Two Branches of Quantitative Finance

Quantitative finance is a multidisciplinary field that applies mathematical and statistical methods to financial markets, instruments, and models. This field can in turn be split into two general branches, namely "real-world applications" and "pricing applications", which are often associated with buy-side and sell-side practitioners respectively. These are usually considered to have different goals, use different types of tools, and operate under different measures and dimensions. In the following sections, an overview will be given to both of these branches.

2.1 Real-World Applications

The real-world side of quantitative finance is generally used for risk and portfolio management on the buy-side, and aims to "model the future" [1]. The measure under which real-world models operate is called the probability distribution $\mathbb{P}$. This probability distribution is assumed to be known when using real-world models and frameworks. As such, a large part of the work is focused on estimating the probability distribution, based on for example historical price movements and relationship to other financial variables. As Meucci describes, these are usually only monitored on specific time instances, which is why the dynamics are represented by discrete time series [1]. With dimensions usually being large, due to having to model the "joint distribution of all the securities" in the market, the main tools used involve multivariate analysis of discrete time series.

2.2 Pricing Applications

On the other side of the coin of quantitative finance, is the sell side and pricing applications. The aim here is to determine the fair price of a financial product in such a way that no arbitrage possibilities arise between products in the market. As summarised by Meucci, financial modelling in pricing applications is, therefore, a task of "extrapolating the present". This occurs under the risk-neutral probability measure $\mathbb{Q}$. In order for the discounted price of a financial derivative P_0 to be arbitrage-free, its future evolution needs to follow a stochastic martingale process, with constant expected value [2]

$$P_0 = \mathbb{E}\{P_t\},\ t \geq 0. \tag{2.1}$$

Furthermore, since equation 2.1 must be satisfied for all t, financial modelling in pricing applications mainly consists of continuous time martingales processes, using tools like stochastic calculus and partial differential equations. As financial derivatives are priced individually, the dimensionality in pricing applications is usually low.

2.3 Bridging the Divide in Yield Curve Modelling

While the two branches of quantitative finance are adapted and fit for their respective purposes, in some use cases this can lead to a divide and theoretical model inconsistency. One example of such an area is yield curve modelling.

As Lyashenko and Goncharov points out, the two branches lead to a divide in how yield curves are represented and generally thought of by practitioners [3]. In pricing applications, yield curves are usually modelled by first assuming short rate dynamics and then deducing the rest of the term points from the short rate. In contrast, real-world yield curve modelling is usually done by interpolating or bootstrapping a vector of observable rates with fixed tenors. As a result, pricing interest rate dependent derivatives such as callable mortgage bonds under risk-neutral assumptions clashes with actual borrower prepayment behaviour, which cannot be assumed to be risk-neutral. To bridge this divide between pricing applications and real-world applications, Lyashenko and Goncharov suggest a factor Heath-Jarrow-Morton (FHJM) model [3]. As opposed to the typical short rate model that infers the entire yield curve from just the short rate, the FHJM model constructs the yield curve by interpolating a set of observable rates, but still operates under the risk-neutral measure $\mathbb{Q}$. Theoretically, this should be an improvement in bridging the divide between modelling in real world and pricing applications.

3 The Danish Mortgage Bond Market

The Danish Mortgage market is unique worldwide due to the principle of matching loans with issued bonds. In most other countries, mortgage loans are issued by universal banks, as a part of their business. In Denmark, most mortgage loans are issued by specialised mortgage banks and funded by investors purchasing the issued bond rather than, e.g. bank deposits. The general system is as follows;

1. The loan taker applies for a loan at a mortgage bank.

2. The mortgage bank pools loans with similar structures, issues a mortgage bond, sells it on the open market and uses the revenue to fund the loans.

3. The loan taker repays amortisation plus interest to the mortgage bond holder through the mortgage bank.

In other words, the mortgage banks act as financial intermediaries, administrating the loan but not actively cashing in on the interest. The business model of the mortgage banks is instead to charge a margin on the loan for its services [4].

A fundamental part of the mortgage bank system is the match-funding principle, meaning the terms of the issued bond match the terms of the loan exactly. If the loan is issued with a five percent interest rate and a 30-year maturity, the corresponding bond will have a coupon rate of five percent and a maturity of 30 years. It should be noted that as of July 2007, following the implementation of Capital Requirements Directive from EU into Danish law, mortgage banks are not legally obliged to follow this principle but many have kept doing so [5].

There are also various repayment terms and types of bonds. For example, the Danish mortgage market includes Adjustable Rate Mortgage Securities (ARMs), floating-rate notes, capped floaters or callable bonds. The loan taker can choose to repay over maturities of up to 30 years, and might also be offered interest-only (IO) options where no amortisation is paid for part of the loan maturity and the remaining is paid as an annuity profile [4].

Due to its unique structure, the Danish mortgage market has historically been very liquid, transparent, and resilient to crises. With comparatively high yields, it has grown to be one of the largest mortgage bond markets in the world, at a market value of more than 400bn EUR in 2020 according to Jyske Capital.

3.1 Mortgage Backed Bonds (MBBs)

As previously mentioned, traded bonds are not backed by single mortgages but rather by pools of thousands of mortgages created by mortgage credit institutions. This report will focus on the most common type of bond, with fixed coupon payments and callable mortgages, henceforth referred to as mortgage-backed bonds, MBBs. The underlying loans in each MBB are standardised, sharing the same coupon rate, settlement dates, amortisation schedule, and time to maturity. [6]

The complexity in pricing these bonds comes from the prepayment option of the underlying loans which allows each loan taker to prepay their loan at par every quarter, throughout the life of the mortgage. As this risk changes the cash flows of the loans, it needs to be considered when pricing the mortgage bonds. Prepayments are often done to refinance the loan at a lower interest rate rather than to pay off mortgages [4]. As such, prepayment rates typically increase when market rates decrease. However, as with most financial products involving private individuals rather than financial institutions, behaviour cannot be assumed to be rational [6]. Modelling of the prepayment rate is further explored in chapter 8.

Note that in this report, default risk will not be taken into account. The structure of the market, with mortgage banks as intermediaries, function as protection for the investor from borrower default. This is a major contributing factor to the fact that historically, "not one Danish Mortgage bondholder has lost the investment or even a part of it" [4]. However, "black swan events" are always possible and should be considered to affect the level of risk involved, if the models discussed in this paper were to be used in a practical setting.

4 Interest Rate Theory

When pricing any interest rate derivative, a core aspect is to look at the dynamics and evolution of future rates using interest rate models. These can be explained as mathematical models that describe the probabilistic evolution of interest rates over time. In this chapter, we introduce key concepts in interest rate theory and take a look at different types and examples of interest rate models. Finally, we focus on and derive the HJM framework which constitutes the main class of interest rate models that will be used in the coming parts of this project.

4.1 The Yield Curve and Term Structure of Interest Rates

The yield curve does not have a singular definition but comprises a collection of interest rate curves for different interest rate products, connected by arbitrage arguments. In financial literature, the full set of yield curves and their connection is often referred to as the *Term Structure of Interest Rates*. The simplest interest rate product is the *Zero Coupon Bond* (ZCB) that gives a payment of one unit at some future time T. The value of a ZCB is defined as

$$P(t,T) = e^{-r(t,T)}$$

where $r(t,T)$ defines the spot rate, i.e. the interest rate from time t to T.

The spot rate from time t to time $\lim_{\Delta t \to 0} t + \Delta t$, in this report denoted as $r(t)$, is known as the short rate and acts as the basis for the risk-neutral pricing equation.

Theorem 1 (The risk-neutral pricing equation)**.** *The value of a ZCB, $P(t,T)$, under the risk-neutral measure, $\mathbb{Q}$, is given by*

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}(t)\right]$$

Proof. Omitted, see [7]. □

Using these we can define two more base rates used throughout this thesis. The first is the continuously compounded forward rated, in this thesis denoted $F(t,T_1,T_2)$, defined as the interest rate gained by buying a ZCB with maturity T_2 and selling a ZCB with maturity T_1. Mathematically this becomes

$$F(t,T_1,T_2) = -\frac{\log(P(t,T_2)) - \log(P(t,T_1))}{T_2 - T_1} = \frac{r(t,T_2) - r(t,T_1)}{T_2 - T_1} \tag{4.1}$$

The second is the instantaneous forward rate that can be seen as the forward version of the short rate and is defined as the forward at time t, from T to $\lim_{\Delta t \to 0} T + \Delta t$

$$f(t,T) = F(t,T,T) = \lim_{\Delta t \to 0} -\frac{\log(P(t,T+\Delta t)) - \log(P(t,T))}{\Delta t} = -\frac{\partial \log(P(t,T))}{\partial T} \tag{4.2}$$

As $P(t,t) = 1$ integrating gives

$$\int_t^T f(t,s)ds = -\int_t^T \frac{\partial}{\partial s} \log(P(t,T))ds = -\log(P(t,T)) + \log(P(t,t)) = -\log(P(t,T))$$

It thus holds that

$$P(t,T) = e^{-\int_t^T f(t,s)ds} \tag{4.3}$$

4.2 Interest Rate Modelling and Short Rates

There are two main categories of interest rate models, namely equilibrium models and arbitrage-free models, which will be explained further in the following sections. Many of the models within these categories are so called "short rate models". As opposed to HJM models which directly give the curve dynamics, the short rate models only model the short rate as some statistical process, and then assume that the rest of the rates in the term structure are driven by the short rate. Once the dynamics of the short rate are determined, the interest rate of all other maturities is derived by applying basic no-arbitrage conditions. Models that deduce the curve dynamics by only assuming short-rate dynamics are for obvious reasons referred to as one-factor models.

Multi-factor models such as the two-factor Hull-White [8] are also possible, and are often more precise in modelling the yield curve. However, as these are more complex, there is a trade-off between precision and analytical tractability and ease-of-use in practice.

Most interest rate models consist of stochastic differential equations and describe the movements of interest rates over time as some combination of drift, volatility, and mean-reversion. Drift can be explained as the expected rate of change of interest rates over time. It reflects tendency of interest rates to move in a particular direction in the longer term. Volatility describes the magnitude of random movements around the long-term mean and drift that cannot be predicted. Mean reversion refers to the tendency of the interest rate to revert back to a long-term mean over time.

4.2.1 Equilibrium interest rate models

Equilibrium models begin with defining fundamental macroeconomic factors that are assumed to inflict change in interest rates. These models describe interest rate change as a move towards equilibrium between the supply and demand for borrowing. A closed form solution is derived that describes the dynamics of the term structure of the interest rates. Some examples of equilibrium models are Vasicek and Cox-Ingersoll-Ross (CIR) [9] [10].

4.2.2 Arbitrage-free interest rate models

Arbitrage-free interest rate models are designed to be consistent with the prices of existing financial instruments, ensuring no arbitrage opportunities exist in the market. These models are fundamentally constructed to match current market data, thereby enabling accurate pricing of interest rate derivatives. Unlike equilibrium models that explain the underlying economic reasons for interest rate movements, arbitrage-free models focus on ensuring that their parameters can be calibrated to fit observed prices of a wide range of financial instruments such as bonds, interest rate swaps, and swaptions. Some examples of arbitrage-free models include the Ho-Lee model , the Hull-White model the LIBOR Market Model (LMM) [11] [12] [13].

4.3 Measure Theory

Measure theory plays a central role in interest theory. This section gives a short overview of the subject and the theorems used in the project based on the book *Statistics for Finance* by Lindström et. al. [7].

Measure theory is centred around three core concepts.

1. **Sets**: A well-defined collection or system of objects, as defined in basic set theory [14].
2. **Sigma Algebras:** A σ-algebra of a set X, is a collection of sets that includes the empty set and X itself, and fulfils the following criteria:
   (a) **Closed under complementation:** if the σ-algebra, Σ, contains a set A then it also contains its complements A^c, or formally: if $A \in \Sigma$ then $X \backslash A \in \Sigma$
   (b) **Closed under countable unions:** this means that if a countable number of sets, $A_1, A_2, ...$ are all in Σ, then their union is also in sigma, or formally: if $A_i \in \Sigma$ for $i \in \{1, 2, 3, ...\}$ then $\cup_{i=1}^{\infty} A_i \in \Sigma$
3. **Measures:** A measure is a function that assigns a non-negative real number to sets in a σ-algebra, satisfying the properties
   (a) **Non-negativity:** for every set A in the σ-algebra the measure $\mu(A) \geq 0$
   (b) **Null empty set:** the measures for the empty set is $\mu(\emptyset) = 0$
   (c) **Countable additivity (or σ-additivity):** if $A_1, A_2, ...$ are disjoint sets in the σ-algebra, then $\mu\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$

These lead to four more definitions.

1. **Measurable Space:** a set X equipped with a σ-algebra is called a measurable space and the sets of the σ-algebra are called measurable sets

2. **Measure space:** a measure space is defined by a set X, a σ-algebra Σ, and a measure μ, i.e. (X, Σ, μ)

3. **Probability space:** a probability space is a measure space with a total measure of 1, i.e. the sum of the measures of all disjoint subsets in the σ-algebra is equal to 1

4. **Filtration:** Given a probability space (Ω, Σ, μ), a filtration is a sequence of sigma-algebras $\{\Sigma_t\}_{t\geq 0}$ such that:

   (a) Each Σ_t is a sub-sigma-algebra of Σ, i.e., $\Sigma_t \subseteq \Sigma$ for all $t \geq 0$.

   (b) The sequence is increasing over time, i.e., if $s \leq t$ then $\Sigma_s \subseteq \Sigma_t$.

The final concepts needed before introducing the theorems used throughout this report are those of absolute continuous and equivalent measures.

Def 1 (Equivalent measures)**.** *Let (Ω, Σ) be a measurable space and μ and v be two measures on that space. If it holds that*

$$\textit{if } E \in \Sigma \textit{ and } \mu(E) = 0 \textit{ then } v(E) = 0$$

v is said to be absolute continuous with respect to μ, written $v \ll \mu$. If it further holds that $v \ll \mu$ and $\mu \ll v$, then v and μ are said to be equivalent measures, denoted $v \sim \mu$.

4.3.1 The Radon-Nikodym and Girsanov Theorems

Two important theorems in pricing application are the Radon-Nikodym and Girsanov theorems. These are used to change probability measures which allows for risk-neutral pricing and can often drastically simplify calculations.

Theorem 2 (Radon-Nikodym theorem)**.** *Let (X, Σ, μ) be a be a finite measurable space and v a finite measure on (X, Σ) s.t. $v \ll \mu$, then there exists a positive function λ_v^μ for which it holds that*

$$\begin{aligned} \lambda_v^\mu \textit{ is } \Sigma\textit{-measurable} \\ \int_X \lambda_v^\mu(x)dx < \infty \\ v(E) = \int_E \lambda_v^\mu(x)d\mu(x); \textit{ for all Borel sets } E \in \Sigma \end{aligned}$$

The function λ_v^μ is called the Radon-Nikodym derivative of v with respect to μ on the σ-algebra Σ and is denoted by

$$\lambda_v^\mu = \frac{dv}{d\mu} \textit{ or } dv(x) = \lambda_v^\mu(x)d\mu(x)$$

Proof. For proof, see [15]. □

Lemma 1. *Let $g(t)$ be an $\mathcal{F}(t)$ adapted process that satisfies*

$$\mathbb{P}\left[\int_0^T g^2(t)dt < \infty\right] = 1$$

Then the equation

$$d\lambda(t) = g(t)\lambda(t)dW^{\mathbb{P}}(t),\ L(0) = 1$$

Has the unique solution

$$\lambda(t) = \exp\left(\int_0^t g(s)dW^{\mathbb{P}}(s) - \frac{1}{2}\int_0^t g^2(s)ds\right)$$

Theorem 3 (The Girsanov Theorem)**.** *Let $W^{\mathbb{P}}$ be a $(\mathbb{P}, \mathcal{F})$-Wiener process and let $g(t)$ and $\lambda(t)$ be as in lemma 1. Further, assume that*

$$\mathbb{E}^{\mathbb{P}}[\lambda(T)] = 1$$

and define the probability measure $\mathbb{Q}$ by $d\mathbb{Q} = \lambda(t)d\mathbb{P}$ on $\mathcal{F}(t)$ then the process

$$W^{\mathbb{Q}}(t) = W^{\mathbb{P}}(t) - \int_0^t g(s)ds$$

becomes a $(\mathbb{Q}, \mathcal{F}(t))$-Wiener process.

Proof. For proof, see [15] □

4.3.2 Probability Measures in Finance

As discussed by Oosterlee and Grzelak in their book *Mathematical Modelling and Computation in Finance* [16], measures and measure changes play an important role in financial engineering. In finance, measure theory is closely tied to the concept of *numéraire*, the commodity or unit of measurement the price of a product is expressed in. As a simple example, assume that you have a store that sells apples for DKK 5 and bananas for DKK 10, you could then use apples as numéraire and express the price of a banana as 2 apples.

An important takeaway from the example is that the real price measure and the apple price measure for the bananas are equivalent by definition 1. As such we can use the Radon-Nikodym theorem for the measure change. Let $A(q) = 5q$ and $B(q) = 10q$ be the price of q apples and bananas respectively, under the real world measure $\mathbb{P}$. The Radon-Nikodym derivative to change from the real-world measure to the apple measure then becomes

$$\lambda_{\mathbb{P}}^{A} = \frac{1}{dA} = \frac{1}{5}$$

and we indeed get that the price of one banana is $\lambda^A B(1) = 2$ apples. The Radon-Nikodym theorem is obviously unnecessary for this example but the concept translates to more complicated financial measure changes that can sometimes drastically simplify calculations.

Mathematically, a numéraire is a strictly positive price process, adapted to the relevant filtration. Assuming the absence of arbitrage, it should further hold that a traded asset following an Ito-process under the numéraires measure is a martingale when divided by the numéraire. Let $\mathbb{A}$ be a measure with numéraire $a(t)$ and $X(t)$ be defined by the Ito process

$$dX(t) = \mu^{\mathbb{A}}(t)dt + \sigma(t)dW^{\mathbb{A}}(t)$$

Assuming an arbitrage free market, the process $X(t)/a(t)$ will then be a martingale under the $\mathbb{A}$ measure, i.e. [16].

$$\mathbb{E}^{\mathbb{A}}\left[\frac{X(t)}{a(t)}|\mathcal{F}(0)\right] = \frac{X(0)}{a(0)}$$

The $\mathbb{Q}$ Measure

The $\mathbb{Q}$, often referred to as the risk-neutral probability measure, is one the most important measures in mathematical finance and has the money market account $M(t)$ defined as

$$dM(t) = r(t)M(t)dt$$

as numéraire. The reason this measure is called risk-neutral is that the money market account is typically used as a discount factor meaning that the discounted value of a traded asset $X(t)$, defined under the $\mathbb{Q}$ measure, will be a martingale, or in mathematical terms

$$dX(t) = \mu^{\mathbb{Q}}(t)dt + \sigma(t)dW^{\mathbb{Q}}(t) \implies \mathbb{E}^{\mathbb{Q}}\left[\frac{X(t)}{M(t)}|\mathcal{F}(0)\right] = \frac{X(0)}{M(0)} \tag{4.4}$$

The $\mathbb{Q}^T$-Forward Measure

The $\mathbb{Q}^T$-forward measure uses a ZCB with maturity $\mathbb{T}$ as numéraire and is commonly used to price interest rate derivatives such as swaptions. As discussed by Oosterlee and Grzelak [16], the Radon-Nikodym derivative for changing from the risk-neutral measure to the $\mathbb{Q}^T$-forward measure is given by

$$\lambda_{\mathbb{Q}}^{\mathbb{Q}^T}(t) = \frac{P(0,T)}{P(t,T)}\frac{M(t)}{M(0)} = \frac{P(0,T)}{P(t,T)M(t)}$$

4.4 The HJM Framework

Originally developed by Heath, Jarrow and Morton the HJM modelling framework introduced a new way to model the evolution of the yield curve in pricing applications.

Instead of modelling the short rate using a Brownian motion and letting its evolution imply the yield curve, the HJM framework allows the modelling of the evolution of the entire yield curve simultaneously [17].

4.4.1 Deriving the HJM model

Following the approach outlined by Lindström et. al. [7] the HJM-framework can be derived in the following fashion. Assume the instantaneous forward rate follows the Ito-process

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)dW^{\mathbb{P}}(t) \tag{4.5}$$

As the price of a ZCB is given by (4.2) we can use the instantaneous forward rate to derive the entire yield curve. As previously discussed, any traded asset under $\mathbb{Q}$ divided by the money market account has to be a martingale under $\mathbb{Q}$. Finding an arbitrage-free model for the instantaneous forward curve is thus equivalent to finding dynamics that result in the process

$$Z(t,T) = \frac{P(t,T)}{M(t)} \tag{4.6}$$

being a martingale under $\mathbb{Q}$. To find the instantaneous forward rate dynamics that result in an arbitrage-free market we make use of the following lemma shown in [15]

Lemma 2 (Connection between f(t,T) and P(t,T))**.** *Assume the dynamics of the instantaneous forward rate to be given by* (4.5)*. The dynamics of a ZCB are then given by*

$$dP(t) = (r(t) + b(t,T))P(t,T)dt + a(t,T)P(t,T)dW^P(t) \tag{4.7}$$

$$a(t,T) = -\int_t^T \sigma(t,s)ds \tag{4.8}$$

$$b(t,T) = -\int_t^T \mu(t,s)ds + \frac{1}{2}a^2(t,T) \tag{4.9}$$

Proof. Omitted, can be found in [15]. □

Further following the approach in [7], we can now apply Ito's lemma to (4.6) to derive the dynamics of Z

$$\begin{aligned}
dZ(t,T) &= \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial P}dP(t,T) + \frac{\partial Z}{\partial M}dM(t) && (4.10)\\
&= \frac{1}{M(t)}(r(t,0) + b(t,T))P(t,T)dt + a(t,T)P(t,T)dW(t) \\
&\quad - \frac{P(t,T)}{M^2(t)}r(t,0)M(t)dt && (4.11)\\
&= \frac{P(t,T)}{M(t)}\left(b(t,T)dt + a(t,T)dW^{\mathbb{P}}(t)\right) && (4.12)\\
&= b(t,T)Z(t,T)dt + a(t,T)Z(t,T)dW^{\mathbb{P}}(t) && (4.13)
\end{aligned}$$

Note that all second-order terms involving $dM(t)$ will be 0 as $dM(t)$ lacks a diffusion term and the second-order derivative of $Z(t,T)$ with respect to $P(t,T)$ is 0. We now know the dynamics of $Z(t,T)$ under $\mathbb{P}$ and want to transform to the $\mathbb{Q}$ measure to find the arbitrage-free condition. Using the Radon-Nikodym theorem we identify that switching to the $\mathbb{Q}$ comes down to finding the Radon-Nikodym derivative, $\lambda_{\mathbb{P}}^{\mathbb{Q}}(t)$ that satisfies

$$d\mathbb{P} = \lambda_{\mathbb{P}}^{\mathbb{Q}}(t)d\mathbb{Q}$$

Girsanov's theorem gives that

$$\begin{aligned} d\lambda_{\mathbb{P}}^{\mathbb{Q}}(t) &= g(t)\lambda_{\mathbb{P}}^{\mathbb{Q}}(t)dW^{P}(t) \\ dW^{\mathbb{P}}(t) &= g(t)dt + dW^{\mathbb{Q}}(t) \end{aligned}$$

Plugging this into (4.13) gives

$$\begin{aligned} dZ(t,T) &= b(t,T)Z(t,T)dt + a(t,T)\left(g(t)dt + dW^{\mathbb{Q}}(t)\right) \\ &= (b(t,T) + a(t,T)g(t))dt + a(t,T)dW^{\mathbb{Q}}(t) \end{aligned}$$

It is now easy to identify that

$$g(t,T) = -\frac{b(t,T)}{a(t,T)} \tag{4.14}$$

results in $Z(t,T)$ being driftless under $\mathbb{Q}$. As we want the model to be arbitrage-free regardless of choice of T, it must further hold that

$$g(t,T) = g(t,S) \implies -\frac{b(t,T)}{a(t,T)} = -\frac{b(t,S)}{a(t,S)}$$

for all T and S such that $t < \min(T,S)$. Using the expression for a and b from lemma 2 we get that

$$\begin{aligned} g(t,S) &= \frac{\int_t^T \mu(t,u)du - \frac{1}{2}a^2(t,T)}{a(t,T)} \\ \implies \int_t^T \mu(t,u)du &= g(t,S)a(t,T) + \frac{1}{2}a^2(t,T) \\ &= -g(t,S)\int_t^T \sigma(t,s)ds + \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2 \\ \implies \mu(t,T) &= \sigma(t,T)\int_t^T \sigma(t,s)ds - g(t,S)\sigma(t,T) \end{aligned}$$

We now use this result to derive the arbitrage-free dynamics of $f(t,T)$. Using Girsanov's theorem we get

$$\begin{aligned} df(t,T) &= (\mu(t,T) + g(t)\sigma(t,T))dt + \sigma(t,T)dW^{\mathbb{Q}}(t) \\ &= \left(\sigma(t,T)\int_t^T \sigma(t,s) - g(t)\sigma(t,T) + g(t)\sigma(t,T)\right)dt \\ &\quad + \sigma(t,T)dW^{\mathbb{Q}}(t) \\ &= \left(\sigma(t,T)\int_t^T \sigma(t,s)\right)dt + \sigma(t,T)dW^{\mathbb{Q}}(t) \end{aligned}$$

This results in the following two theorems

Theorem 4 (HJM-conditions). *Assume the setup with $f(t,T)$, $M(t)$, $P(t,T)$, and $Z(t,T)$ described above. The four following conditions are then equivalent and sufficient for the market to be arbitrage-free*

1. *There exists a measure $\mathbb{Q}$ under which every $Z(t,T)$ process becomes a martingale.*

2. *For every choice of T and S s.t. $t < \min(T,S)$ we have*

$$\frac{b(t,T)}{a(t,T)} = \frac{b(t,S)}{a(t,S)}$$

3. *The process $g(\cdot,T)$ is independent of the choice of T*

4. *For every choice of S and T s.t. $t < \min(S,T)$ it holds that*

$$\mu(t,T) = -\sigma(t,T)\left(g(t,S) - \int_t^T \sigma(t,u)du\right) \tag{4.15}$$

Proof. Omitted, see Björk [15] □

Theorem 5 (The HJM framework). *Assume the forward dynamics under $\mathbb{P}$ are given by*

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)dW^{\mathbb{P}}(t)$$

and assume one of the conditions of theorem 4 is fulfilled. The arbitrage-free dynamics of $f(t,T)$ under the risk-free measure, $\mathbb{Q}$, are then given by

$$df(t,T) = \bar{\mu}(t,T)dt + \sigma(t,T)dW^{\mathbb{Q}}(t) \tag{4.16}$$

where

$$\bar{\mu}(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du$$

Note that this thesis leaves out several important steps in deriving the HJM framework. For a complete derivation, see the original paper written by Heath, Jarrow, and Morton [17] or the book *Arbitrage Theory in Continuous Time* by Björk [15].

4.4.2 Markovian HJM model

The Markov property of a stochastic process can be defined as "memorylessness". In other words, if a stochastic process satisfies the Markov property, the evolution of the future of the process is only dependent on its current state, and not on its past. Furthermore, another key assumption is that the dimension of the state-space generated by the process needs to be finite [18].

The original HJM formulation is not Markovian, since under the risk neutral measure, the short rates derived from it are are not a self contained diffusion. This non-Markovian nature of the original HJM can also be seen by further studying the integral of equation (4.16)

$$\int_0^t df(t,T) = f(t,T) - f(0,T) = \int_0^t (\sigma(s,T)\int_s^T \sigma(s,u)du)ds + \int_0^t \sigma(s,T)dW(s) \tag{4.17}$$

The first term on the right-hand side of (4.17) is deterministic, whereas the second term can be seen as normally distributed Brownian increments. In other words, only the volatility term will determine if the process is Markovian or not. Let us consider the increments of the stochastic term from t to T

$$D(t) = \int_0^t \sigma(s,t)dW(s) \text{ and } D(T) = \int_0^T \sigma(s,T)dW(s). \tag{4.18}$$

$$D(T) - D(t) = \int_0^T \sigma(s,T)dW(s) - \int_0^t \sigma(s,t)dW(s) \tag{4.19}$$

We can separate the first term on the right hand side of (4.19) into an integral from 0 to t, and an integral from t to T, and include the former into the second term of the right hand side as follows

$$D(T) - D(t) = \int_t^T \sigma(s,T)dW(s) + \int_0^t (\sigma(s,T) - \sigma(s,t))dW(s) \tag{4.20}$$

To see if the process is Markovian or not, we can check if the expected value of these increments require any past information of the process or only the information at t. The expected value of the first term in (4.20) is 0 since it is a deterministic function of a Brownian motion, which is a martingale. The best guess for the future value is therefore today's value, and the integral becomes 0. The second term however, has t as its upper integral boundary, and therefore is already realised and not equal to 0. Furthermore, the deterministic function is not dependent on only t, but also T. Hence, the process does not satisfy the Markov property.

In order to transform the general HJM into a Markov HJM, the process needs to be separable. If we define the stochastic variable as a multiplication of a function of only time and a function of only maturity as in (4.21), then we can show that the transformed process is Markovian.

$$\sigma(s,T) = g(s)h(T), \text{ where } h \text{ is deterministic.} \tag{4.21}$$

Note that if both g and h are deterministic, then the function becomes Gaussian. Inserting this into (4.18) we get

$$D(t) = \int_0^t \sigma(s,t)dW(s) = h(t)\int_0^t g(s)dW(s), \tag{4.22}$$

and

$$D(T) = \int_0^T \sigma(s,T)dW(s) = h(T)\int_0^T g(s)dW(s). \tag{4.23}$$

Substituting (4.22) and (4.23) into (4.20) we get

$$D(T) - D(t) = h(T)\int_t^T g(s)dW(s) + (h(T) - h(t))\int_0^t g(s)dW(s) \tag{4.24}$$

Note the resemblance between the second term of (4.24) and (4.22). If we multiply and divide with $h(t)$ in the second term, we get

$$D(T) - D(t) = h(T)\int_t^T g(s)dW(s) + \frac{h(T) - h(t)}{h(t)}h(t)\int_0^t g(s)dW(s) \tag{4.25}$$

$$D(T) - D(t) = h(T)\int_t^T g(s)dW(s) + \frac{h(T) - h(t)}{h(t)}D(t) \tag{4.26}$$

Following the same reasoning as before, the expected value of the first term in (4.26) is 0, but now the second term is only dependent on a deterministic function of the state at time t. As such, we have now proven that if the volatility is separable, the process can be transformed to be Markovian.

4.4.3 Musiela parametrisation of the HJM model

The Musiela parametrisation of the general HJM framework is a simple modification of the second parameter of the instantaneous forward rate $f(t,T)$, namely, the maturity T. In the original HJM framework, T is a constant and t is moving. Musiela redefines the second argument to be the remaining maturity, and denotes it as $\tau = T - t$. If we solve for T and substitute this in our instantaneous forward rate, we get $f(t,T) = f(t, t+\tau)$. The t in the second argument can however be seen as redundant, which means we can simply write $f(t,\tau)$.

The main idea with this representation of the function is that it makes it more obvious to the reader that there could be potentially infinite dimensions for the instantaneous forward rate. In the original HJM specification this can also be seen by considering one equation for each fixed maturity T. From now on, the Musiela HJM will be the choice of formulation of the HJM framework. Note that Musiela also changes the notation of the substituted function to $r(t,\tau)$. To prevent the reader from confusing this with the short rate $r(t)$ however, we will simply keep the same notation as before in the same way as Lyashenko and Goncharov do.

4.5 Pricing of Interest Rate Products

4.5.1 Pricing of Interest Rate Swaps

An interest rate swap is a financial contract where the *payer* receives payments according to the current simply compounded floating forward rate and the *receiver* receives payments according to some fixed, simply compounded, rate K. The payoff of a general swap contract is given by

$$\text{payoff}_{swap} = \sum_{i=m_F+1}^{n_F} N\left(e^{\tau_i F(T_{i-1},T_{i-1},T_i)} - 1\right) - \sum_{j=m_K+1}^{n_K} \tau_j NK$$

where N is the notional, $F(t, T_1, T_2)$ is as defined in (4.1), $\tau_k = T_k - T_{k-1}$, and $T_{m_F}, \dots, T_{n_F}$ and $T_{m_K}, \dots, T_{n_K}$ are the payoff grids for the fixed and floating rates respectively. [16] This thesis will use *AB6C* swaps that pay out the fixed rate every year and the floating Copenhagen Interbank Offered Rate (CIBOR) every 6 months.

The swap rate is the fixed rate K that results in a swap trading at par, i.e. the rate that results in the expected cash flow of the floating leg equaling the discounted cash flow of the fixed leg. If we, for simplicity, assume the notional to be 1, the value of the floating leg of the swap at time t is given by

$$V_F^{swap}(t) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=m_F+1}^{n_F} \frac{1}{M(T_i)}\left(e^{\tau_i F(T_{i-1},T_{i-1},T_i)} - 1\right)\middle|\mathcal{F}(t)\right]$$

Changing to the T_i forward measure gives

$$V_F^{swap}(t) = \sum_{i=m_F+1}^{n_F} P(t, T_i)\mathbb{E}^{\mathbb{Q}^{T_i}}\left[e^{\tau_i F(T_{i-1},T_{i-1},T_i)} - 1|\mathcal{F}(t)\right]$$

Using the definition of the forward rate from (4.1) the expression can be further simplified

$$\begin{aligned} V_F^{swap}(t) &= \sum_{i=m_F+1}^{n_F} P(t, T_i)\mathbb{E}^{\mathbb{Q}^{T_i}}\left[\exp\left(-\tau_i \frac{\log(P(T_{i-1}, T_i)) - \log(P(T_{i-1}, T_{i-1}))}{\tau_i}\right) - 1\middle|\mathcal{F}(t)\right] \\ &= \sum_{i=m_F+1}^{n_F} P(t, T_i)\mathbb{E}^{\mathbb{Q}^{T_i}}\left[\frac{P(T_{i-1}, T_{i-1})}{P(T_{i-1}, T_i)} - 1\middle|\mathcal{F}(t)\right] \end{aligned}$$

As $P(T_{i-1}, T_i)$ is numeraire under the T_i-forward measure, the fraction inside the expectation will be a martingale. We thus have

$$\begin{aligned} V_F^{swap}(t) &= \sum_{i=m_F+1}^{n_F} P(t, T_i)\left(\frac{P(t, T_{i-1})}{P(t, T_i)} - 1\right) \\ &= \sum_{i=m_F+1}^{n_F} P(t, T_{i-1}) - P(t, T_i) \\ &= P(t, T_{m_F}) - P(t, T_{n_F}), \end{aligned}$$

As the fixed and floating legs of the swap contract are assumed to be equal this gives

$$P(t, T_{m_F}) - P(t, T_{n_F}) = \sum_{j=m_K+1}^{n_K} \tau_j P(t, T_j) K.$$

This implies the following result holds [16].

Proposition 1. *A swap contract with payoff*

$$payoff_{swap} = \sum_{i=m_F+1}^{n_F} \left(e^{\tau_i F(T_{i-1}, T_{i-1}, T_i)} - 1 \right) - \sum_{j=m_K+1}^{n_K} \tau_j K$$

has the value

$$V^{swap}(t) = P(t, T_{m_F}) - P(t, T_{n_F}) - \sum_{j=m_K+1}^{n_K} \tau_j P(t, T_j) K$$

The swap rate for a swap trading at par, i.e. $V^{swap}(t) = 0$, *is thus given by*

$$K = \frac{P(t, T_{m_F}) - P(t, T_{n_F})}{\sum_{j=m_K+1}^{n_K} \tau_j P(t, T_j)}.$$

4.5.2 Pricing of Interest Rate Swaptions

An interest rate swaption is a European put or call option on an underlying swap contract. For simplicity, this report will only consider swaption contracts with maturity T_m where the underlying forward starting swap has an initial payoff date T_m and is trading at par. The value of the swaption at time T_m will thus be the same as its payoff. For a call option, this would be

$$\begin{aligned}
V^S_{call}(T_m) &= \max\left(V^{swap}_C(T_m), 0\right) \\
&= \max\left(P(T_{m_F}, T_{m_F}) - P(T_{m_F}, T_{n_F}) - \sum_{j=m_K+1}^{n_K} \tau_j P(T_m, T_j) K, 0 \right) \\
&= \max\left(1 - P(T_{m_F}, T_{n_F}) - \sum_{j=m_K+1}^{n_K} \tau_j P(T_m, T_j) K, 0 \right)
\end{aligned}$$

The arbitrage-free price of the swaption at time $t \leq T_m$ is then simply given as the expectation of the value at T_m discounted to t. [16]

Proposition 2. *The price of a call swaption with exercise date T_{m_F}, payoff grids $T_{m_F}, \ldots, T_{n_F}$ and $T_{m_K}, \ldots, T_{n_K}$ for the fixed and floating rates respectively, and strike price K at time t is given by*

$$V_{call}^S(t) = \frac{1}{M(T_m)} \mathbb{E}^{\mathbb{Q}} \left[\max \left(1 - P(T_{m_F}, T_{n_F}) - \sum_{j=m_K+1}^{n_K} \tau_j P(T_m, T_j) K, 0 \right) \middle| \mathcal{F}(t) \right] \tag{4.27}$$

Or equivalently

$$V_{call}^S(t) = P(t, T_m) \mathbb{E}^{\mathbb{Q}^{T_m}} \left[\max \left(1 - P(T_{m_F}, T_{n_F}) - \sum_{j=m_K+1}^{n_K} \tau_j P(T_m, T_j) K, 0 \right) \middle| \mathcal{F}(t) \right] \tag{4.28}$$

5 Factor HJM Model

The factor-HJM model, introduced by Lyashenko and Goncharov [3], aims to bridge the divide between how yield curves are modelled in real-world applications, under $\mathbb{P}$, compared to pricing applications, under $\mathbb{Q}$. It accomplishes this by adapting the standard real-world modelling approach to the HJM framework.

5.1 Yield Curve Factor Modelling in Real-World Applications

As discussed by Lyashenko and Goncharov [3] the standard method of modelling the yield curve under $\mathbb{P}$ is to decompose the spot rate curve, $r(t,\tau)$, into a part spanned by K linearly independent basis functions, $a_1(\tau),\dots,a_K(\tau)$, and a remainder, $\bar{r}(t,\tau)$. Defining the basis row-vector as

$$A(\tau) = (a_1(\tau),\dots,a_K(\tau))$$

results in the spot rate function

$$r(t,\tau) = A(\tau)X(t) + \bar{r}(t,\tau) \tag{5.1}$$

where $X(t)$ is a K-dimensional column vector process defining the spot rate evolution over time under the basis vector $A(\tau)$.

Assuming $r(t,\tau)$ and $a_k(\tau)$ are smooth, (5.1) can be rewritten to instantaneous forward rate form

$$f(t,\tau) = B(\tau)X(t) + \bar{f}(t,\tau) \tag{5.2}$$

where

$$B(\tau) = (b_1(\tau),\dots,b_K(\tau))$$

From the definition of the instantaneous forward rate (4.3) we have that

$$\tau r(t,\tau) = \int_0^\tau f(t,u)du$$

which gives

$$b_k(\tau) = (\tau a_k(\tau))',\ k = 1,\dots,K$$

Lyashenko and Goncharov go on to point out that two of the most commonly used bases are the Nelson-Siegel

$$B(\tau) = \left(1, e^{-\lambda\tau}, \tau e^{-\lambda\tau}\right)$$

and its Svensson extension

$$B(\tau) = \left(1, e^{-\lambda_1\tau}, \tau e^{-\lambda_1\tau}, \tau e^{-\lambda_2\tau}\right)$$

5.2 Yield Curve Factor Modelling in Pricing Applications

The direct translation from the spot rate to the instantaneous forward rate done under $\mathbb{P}$ does not produce an arbitrage-free market under $\mathbb{Q}$. Lyashenko and Goncharov instead use the multidimensional HJM-framework

$$df(t,\tau) = \left(f'(t,\tau) + \Sigma^*(t,\tau) \int_0^\tau \Sigma(t,u)du \right) + \Sigma^*(t,\tau)dW^{\mathbb{Q}}(t) \tag{5.3}$$

where $f'(t,\tau)$ is the derivative of $f(t,\tau)$ with respect to τ, $\Sigma(t,\tau)$ is an N-dimensional column vector process and * signifies the transpose of the vector or matrix. The factor decomposition from (5.2) now gives

$$\begin{aligned} df(t,\tau) &= B(\tau)dX(t) + d\bar{f}(t,\tau) \\ &= \left(B'(\tau)X(t) + \bar{f}'(t,\tau) + \Sigma^*(t,\tau) \int_0^\tau \Sigma(t,u)du \right) + \Sigma(t,\tau)dW^{\mathbb{Q}}(t) \end{aligned}$$

Lyashenko and Goncharov then eliminate more terms by assuming $\bar{f}(t,\tau)$ follows the evolution of the originally observed remainder, i.e.

$$\bar{f}(t,\tau) = \bar{f}(0, t+\tau)$$

In other words, they let $\bar{f}$ be given by the originally observed yield curve and $B(\tau)X(t)$ define the yield curve evolution. By further assuming the volatility process to be spanned by the functional basis $B(\tau)$, i.e.

$$\Sigma(t,\tau) = B(\tau)\mathbf{\Sigma}(t)$$

where $\mathbf{\Sigma}(t)$ is a $K \times N$-dimensional matrix process, Lyashenko and Goncharov arrive at the conclusion that

$$f(t,\tau) = B(\tau)X(t) + \bar{f}(t,\tau) \tag{5.4}$$

defines an arbitrage-free evolution for the forward curve if $\bar{f}$ is differentiable and $X(t)$ satisfy

$$B(\tau)dX(t) = \left(B'(\tau)X(t) + B(\tau)\mathbf{\Sigma}(t)\mathbf{\Sigma}^*(t) \int_0^\tau B^*(u)du \right) dt + B(\tau)\mathbf{\Sigma}(t)dW^{\mathbb{Q}}(t) \tag{5.5}$$

5.2.1 Exponential-Polynomial Bases

Lyashenko and Goncharov limits possible choices of basis functions by stipulating that (5.5) should admit a solution for the case of zero volatility. The condition then simplifies to

$$B(\tau)dX(t) = B'(\tau)X(t)dt$$

implying there must exist a matrix D such that

$$B'(\tau) = B(\tau)D$$

Solving the ordinary differential equation gives

$$B(\tau) = B(0)e^{D\tau} \tag{5.6}$$

The individual basis functions can thus be expressed as

$$b_k(\tau) = \sum_{i,j} b_{k,i,j}\tau^i e^{\lambda_j \tau}$$

where λ_i $i = 1, \dots, K$ are the eigenvalues of D. Following Björk and Christensen [19], Lyashenko and Goncharov call functions that can be represented in this form the Exponential Polynomial family. For this thesis, we limit ourselves to pure exponential basis functions on the form

$$b_k(\tau) = e^{\lambda_k \tau}$$

As argued by Lyashenko and Goncharov, due to

$$\tau^n e^{-\lambda\tau} = -\frac{\partial}{\partial\lambda}\left(\tau^{n-1}e^{-\lambda\tau}\right)$$

the space spanned by an exponential polynomial basis can be approximated by that spanned by a pure exponential basis. As such, using pure exponential basis functions should have a limited effect on performance compared to using exponential polynomial basis functions assuming a large enough parameter space. Pure exponential basis functions also have the advantage of leading to diagonal-generating matrices

$$D = \begin{pmatrix} -\lambda_1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda_K \end{pmatrix}$$

which significantly simplifies calculations in the case of, for example, inversions as the exponential matrix $e^{D\tau}$ also ends up being diagonal

$$e^{D\tau} = \begin{pmatrix} e^{-\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{-\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\lambda_K} \end{pmatrix}$$

In line with Lyashenko and Goncharov, we also limit λ to positive values to avoid unintuitive features like exploding yields for long tenors.

5.2.2 The FHJM-Framework

Using the definition of the exponential polynomial basis, (5.6), equation (5.5) becomes

$$\begin{aligned} B(\tau)dX(t) = (B(\tau)DX(t))\,dt + B(\tau)\mathbf{\Sigma}(t)dW^{\mathbb{Q}}(t) \\ + B(\tau)\mathbf{\Sigma}(t)\mathbf{\Sigma}^*(t)\left(\int_0^\tau B^*(u)du\right)dt \end{aligned}$$

The term $B(\tau)\mathbf{\Sigma}(t)\mathbf{\Sigma}^*(t)\left(\int_0^\tau B^*(u)du\right)$ is not spanned by the $B(\tau)$ basis functions. Instead, it is spanned by functions on the form $b_k(\tau)\int_0^\tau b_k(u)du$. This results in an expanded basis with generating matrix $\tilde{D}$ with eigenvalues $\tilde{\Lambda} = \Lambda \cup \{\lambda_i + \lambda_j | i, j = 1, \ldots, K\}$ where Λ denotes the set of eigenvalues of D. As an example, if D is

$$D = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$$

then $\tilde{D}$ is some diagonal matrix containing the values $\{-\lambda_1, -\lambda_2, -(\lambda_1 + \lambda_2)\}$. Lyashenko and Goncharov uses this basis to define a new variable, $\Omega(t)$, by the equation

$$\tilde{B}(\tau)\Omega(t) = B(\tau)\mathbf{\Sigma}(t)\mathbf{\Sigma}^*(t)\left(\int_0^\tau B^*(u)du\right)$$

To absorb the part of the equation spanned by the new basis they then add a locally deterministic variable, $Y(t)$, spanned by the $\tilde{B}(\tau)$ to the forward curve model, (5.4).

$$f(t,\tau) = B(\tau)X(t) + \tilde{B}(\tau)Y(t) + \bar{f}(0, t+\tau)$$

The HJM-equation, (5.3), then becomes

$$\begin{aligned} B(\tau)dX(t) + \tilde{B}(\tau)dY(t) &= \left(B(\tau)DX(t) + \tilde{B}(\tau)\tilde{D}Y(t) + \tilde{B}(\tau)\Omega(t)\right)dt \\ &\quad + B(\tau)\mathbf{\Sigma}(t)dW^{\mathbb{Q}}(t) \\ &= B(\tau)\left(DX(t)dt + \mathbf{\Sigma}(t)dW^{\mathbb{Q}}(t)\right) + \tilde{B}(t)\left(\tilde{D}Y(t) + \Omega(t)\right)dt \end{aligned}$$

which implies that

$$\begin{aligned} dX(t) &= DX(t)dt + \mathbf{\Sigma}(t)dW^{\mathbb{Q}}(t) \\ dY(t) &= \left(\tilde{D}Y(t) + \Omega(t)\right)dt \end{aligned}$$

This leads us to the FHJM model.

Proposition 3 (FHJM-Model)**.** *Let the instantaneous forward curve be defined by*

$$f(t,\tau) = B(\tau)X(t) + \tilde{B}(\tau)Y(t) + \bar{f}(0, t+\tau)$$

where

$$\begin{aligned} B(\tau) &= B(0)e^{\tau D} \\ \tilde{B}(\tau) &= \tilde{B}(0)e^{\tau \tilde{D}} \end{aligned}$$

D is $K \times K$ matrix and $\tilde{K}$ is a $\tilde{K} \times \tilde{K}$ matrix. The dynamics

$$\begin{aligned} dX(t) &= DX(t)dt + \mathbf{\Sigma}(t)dW^{\mathbb{Q}}(t) \\ dY(t) &= (\tilde{D}Y(t) + \Omega(t))dt \end{aligned}$$

where $W(t)$ is an N-dimensional Wiener process, $\mathbf{\Sigma}(t)$ is a $K \times N$ volatility matrix, and $\Omega(t)$ is given by

$$\tilde{B}(\tau)\Omega(t) = B(\tau)\mathbf{\Sigma}(t)\mathbf{\Sigma}^*(t)\left(\int_0^\tau B(u)du\right)^*$$

defines an arbitrage-free evolution of the forward rate.

5.2.3 Volatility Structure

Many possible volatility structures are applicable for the FHJM model framework. Sepp and Rakhmonov proposed a version using stochastic volatility processes, adding another layer of flexibility, but also complexity to the model [20]. For this thesis, we have instead chosen to use a constant volatility matrix, i.e.

$$\mathbf{\Sigma}(t) = \mathbf{\Sigma}$$

This leads to some further simplification as $\Omega(t)$ also becomes time-invariant.

Furthermore, while $\mathbf{\Sigma}$ in the original FHJM-model is a $K \times N$ matrix and N can be any value bigger than one, both Y and the covariance matrix of X will be dependent on $\mathbf{\Sigma\Sigma}^*$ which is a $K \times K$ matrix. For any value of $N > K$ the model can thus, in some sense, be considered over-determined with many equivalent values of $\mathbf{\Sigma}$. Similar issues arise even if $K \leq N$ as the matrix root of a square matrix is not uniquely defined. One possible solution to this problem is defining $\mathbf{\Sigma}$ as a lower triangular matrix with positive diagonal values. This leads to $\mathbf{\Sigma}$ being the Cholesky decomposition of $\mathbf{\Sigma\Sigma}^*$ making it uniquely defined for any covariance matrix for X.

Combining the above changes we arrive at the specified version of the FHJM model used in this report.

Proposition 4 (Specified FHJM-Model). *Let the instantaneous forward curve be defined by*

$$f(t, \tau) = B(\tau)X(t) + \tilde{B}(\tau)Y(t) + \bar{f}(0, t + \tau)$$

where

$$B(\tau) = B(0)e^{\tau D}$$
$$\tilde{B}(\tau) = \tilde{B}(0)e^{\tau \tilde{D}}$$

D is $K \times K$ matrix and $\tilde{K}$ is a $\tilde{K} \times \tilde{K}$ matrix. The dynamics

$$dX(t) = DX(t)dt + \mathbf{\Sigma} dW^{\mathbb{Q}}(t) \tag{5.7}$$
$$dY(t) = (\tilde{D}Y(t) + \Omega)dt \tag{5.8}$$

where $W(t)$ is an K-dimensional Wiener process, $\mathbf{\Sigma}$ is a $K \times K$ lower triangular volatility matrix with positive diagonal values, and Ω is given by

$$\tilde{B}(\tau)\Omega = B(\tau)\mathbf{\Sigma\Sigma}^* \left(\int_0^\tau B(u)du \right)^*$$

defines an arbitrage-free evolution of the forward rate.

5.2.4 Selection of Basis

Lyashenko and Goncharov discuss several schemes for selecting an appropriate set of basis functions for the FHJM model. One said scheme uses the fact that the pure exponential basis

$$PE_{\lambda_1,\ldots,\lambda_K} = \left(e^{-\lambda_1 \tau}, \ldots, e^{-\lambda_K \tau}\right)$$

is equivalent to the basis

$$\hat{PE}_{\lambda_1,\ldots,\lambda_K} = \left(e^{-\lambda_1\tau}, \frac{e^{-\lambda_1\tau} - e^{-\lambda_2\tau}}{c_2} \ldots, \frac{e^{-\lambda_{K-1}\tau} - e^{-\lambda_K\tau}}{c_K}\right)$$

in the sense that the first basis can be transformed into the second by multiplying it with a non-singular matrix M. Letting the constants $(c_2, \ldots, c_K)$ be defined s.t. each basis function in $\hat{PE}_{\lambda_1,\ldots,\lambda_K}$ has a maximum value of one now results in a set of humped functions, each reaching its maximum at

$$\tau_i = \frac{\ln(\lambda_i) - \ln(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}}$$

This can be used to define a set of key tenors and then solve for the corresponding λ values. As the maximum tenor of the mortgage bonds modelled in this thesis is 30-years, we decided to use the key tenors suggested by Lyashenko and Goncharov of 1, 5, 10, and 30 years, resulting in the lambda values,

$$\Lambda = (0.0001, 0.02, 0.06, 0.16, 0.25, 2.5)$$

5.3 Pricing and Dynamics of Market Instruments

5.3.1 Price of Money Market Account

As previously discussed, the money market account is defined as

$$M(T) = M(t)e^{\int_t^T r(s)ds}$$

It further holds that

$$\begin{aligned}
\int_t^T r(s)ds &= \int_t^T f(s,0)ds \\
&= \int_t^T B(0)X(s) + \tilde{B}(0)Y(s) + \bar{f}(0, t+s)ds \\
&= \mathbb{1}_K \int_t^T X(s)ds + \mathbb{1}_{\tilde{K}} \int_t^T Y(s)ds + \int_t^T \bar{f}(0, t+s)ds
\end{aligned}$$

where $\mathbb{1}_K$ is a vector of ones of size K. Using equation (5.7) we have that

$$\begin{aligned}
X(T) = X(t) + \int_t^T DX(s)ds + \int_t^T \mathbf{\Sigma} dW^{\mathbb{Q}}(s) \\
\implies \int_t^T X(s)ds = D^{-1}\left(X(T) - X(t) - \int_t^T \mathbf{\Sigma} dW^{\mathbb{Q}}(s)\right)
\end{aligned}$$

Similarly, using (5.8) we get

$$\begin{aligned}
Y(T) = Y(t) + \int_t^T (\tilde{D}Y(s) + \Omega)ds \\
\implies \int_t^T Y(s)ds = \tilde{D}^{-1}(Y(T) - Y(t) - \Omega(T-t))
\end{aligned}$$

Proposition 5. *The price of the money market account with tenor T is given by*

$$M(T) = M(t)\exp\bigg(\mathbb{1}_K D^{-1}(X(T)-X(t)) + \mathbb{1}_{\tilde{K}}\tilde{D}^{-1}(Y(T)-Y(t)+\Omega(T-t)) \\ + \int_t^T \bar{f}(0,t+s)ds - \mathbb{1}_K D^{-1}\int_t^T \mathbf{\Sigma} dW^{\mathbb{Q}}(s)\bigg)$$

5.3.2 Pricing and Dynamics of ZCBs

As shown by Sepp and Rakhmonov [20], the price of a ZCB under the FHJM model dynamics is given by

Theorem 6 (Price of a ZCB under the FHJM-model)**.** *Assume the FHJM given by proposition 4. The arbitrage-free price of a ZCB is then given by*

$$P(t,t+\tau) \equiv P(t,t+\tau;X(t),Y(t)) = \frac{P(0,t+\tau)}{P(0,t)}\exp\left(-\beta(\tau)X(t) - \tilde{\beta}(\tau)Y(t)\right)$$

Where

$$\beta(\tau) = \int_0^\tau B(u)du$$
$$\tilde{\beta}(\tau) = \int_0^\tau \tilde{B}(u)du$$

Proof. We have that

$$P(t,t+\tau) = \exp\left(-\int_0^\tau f(t,u)du\right) = \exp\left(-\int_0^\tau \left(B(u)X(t) + \tilde{B}(u)Y(t) + \bar{f}(t,u)\right)du\right)$$
$$= \exp\left(-\int_0^\tau f(0,t+u)du\right)\exp\left(-\int_0^\tau \left(B(u)X(t)+\tilde{B}(u)Y(t)\right)du\right)$$
$$= \frac{P(0,t+\tau)}{P(0,t)}\exp\left(-\left(\int_0^\tau B(u)du\right)X(t) - \left(\int_0^\tau \tilde{B}(u)du\right)Y(t)\right)$$

Giving the expression in the theorem. [20] □

Using this result we can derive the arbitrage-free evolution of the ZCB curve.

Theorem 7 (Dynamics of ZCB's under the FHJM-model)**.** *Assume the specified FHJM-model given by proposition 4. The dynamics of ZCB's are then given by*

$$\frac{dP(t,t+\tau)}{P(t,t+\tau)} = \bigg(-\beta(\tau)DX(t) - \tilde{\beta}(\tau)\left(\tilde{D}Y(t)+\Omega(t)\right) \\ + \frac{1}{2}tr\left(\mathbf{\Sigma}\mathbf{\Sigma}^*\beta(\tau)\beta^*(\tau)\right)\bigg)dt - \beta(\tau)\mathbf{\Sigma}dW^{\mathbb{Q}}(t) \quad (5.9)$$

where tr signifies the trace of a matrix

Proof. We start by applying Ito's lemma to the price given in theorem 6

$$\begin{aligned} dP(t,T) &= \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial X}dX + \frac{\partial P}{\partial Y}dY + \frac{1}{2}dX^* \frac{\partial^2 P}{\partial X^2}dX \\ &= 0 - \beta(\tau)P(t,T)dX - \tilde{\beta}(\tau)P(t,T)dY + \frac{1}{2}dX^*\beta(\tau)\beta^*(\tau)dX \end{aligned}$$

All second derivatives involving $Y(t)$ are 0 as $Y(t)$ only has a drift term. Inserting the dynamics from proposition 1 and dividing by $P(t, t+\tau)$ gives

$$\begin{aligned} \frac{dP(t,t+\tau)}{P(t,t+\tau)} &= -\beta(\tau)(DX(t)dt + \mathbf{\Sigma} dW^{\mathbb{Q}}(t)) - \tilde{\beta}(\tilde{D}Y(t) + \Omega(t))dt \\ &\quad + \frac{1}{2}(\mathbf{\Sigma} dW^{\mathbb{Q}}(t))^* \beta(\tau)\beta^*(\tau)(\mathbf{\Sigma} dW^{\mathbb{Q}}(t)) \\ &= \Big(- \beta(\tau)DX(t) - \tilde{\beta}(\tau)(\tilde{D}Y(t) + \Omega(t)) \\ &\quad + \frac{1}{2}\operatorname{tr}\left(\mathbf{\Sigma}\mathbf{\Sigma}^*\beta(\tau)\beta^*(\tau)\right) \Big) dt - \beta(\tau)\mathbf{\Sigma} dW^{\mathbb{Q}}(t) \end{aligned}$$

□

5.3.3 Pricing of Forward Rates

Theorem 8 (The Forward Rate under the FHJM-model). *Under the FHJM model, the forward rate is given by*

$$F(t, t+\tau_1, t+\tau_2) = F(0, t+\tau_1, t+\tau_2) + \beta_F(\tau_1,\tau_2)X(t) + \tilde{\beta}_F(\tau_1,\tau_2)Y(t)$$

where

$$\begin{aligned} \beta_F(\tau_1,\tau_2) &= \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} B(s)ds \\ \tilde{\beta}_F(\tau_1,\tau_2) &= \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \tilde{B}(s)ds \end{aligned}$$

Proof. From (4.2) we have that the forward rate is given by

$$F(t, t+\tau_1, t+\tau_2) = \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} f(t,s)ds$$

Proposition 4 further gives

$$\begin{aligned} F(t, t+\tau_1, t+\tau_2) &= \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \Big(\bar{f}(0,t+s) + B(s)X(t) + \tilde{B}(s)Y(t)\Big)\, ds \\ &= F(0, t+\tau_1, t+\tau_2) + \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \Big(B(s)X(t) + \tilde{B}(s)Y(t)\Big)\, ds \end{aligned}$$

Giving the expression in the theorem □

5.3.4 Model Dynamics under $\mathbb{Q}^T$-forward Measure

As previously discussed, the $\mathbb{Q}^T$-forward measure has a ZCB with maturity $\mathbb{Q}^T$ as numéraire and is commonly used for pricing forward-starting interest rate derivatives like swaps and swaptions. Under the $\mathbb{Q}^T$-forward measure, the dynamics of the FHJM-model are given by the following theorem.

Theorem 9 (FHJM Dynamics Under $\mathbb{Q}^T$-Forward Measure)**.** *The dynamics of X under the $\mathbb{Q}^T$-forward measure are given by*

$$dX^{\mathbb{Q}^T}(t) = (DX(t) - \mathbf{\Sigma}(\beta(\bar{\tau})\mathbf{\Sigma})^*)\, dt + \mathbf{\Sigma} dW^{\mathbb{Q}^T}(t)$$

where the numéraire is the ZCB $P(t,T)$ and $\bar{\tau} = T - t$.

Proof. We start by deriving the Radon-Nikodym derivative

$$\lambda_{\mathbb{Q}}^{\mathbb{Q}^T}(t) = \left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}(t)} = \frac{P(t,T)/P(0,T)}{M(t)}$$

Where $M(t)$ is the money market account used as numéraire under $\mathbb{Q}$ with dynamics

$$dM(t) = f(t,0)M(t)dt$$

Using the dynamics of a ZCB given by theorem 7 and the assumption that the Radon-Nikodym derivative will be a martingale under $\mathbb{Q}$, we can now derive its dynamics as

$$\begin{aligned}
d\lambda_{\mathbb{Q}}^{\mathbb{Q}^T}(t) &= \frac{\partial \lambda_{\mathbb{Q}}^{\mathbb{Q}^T}}{\partial t} dt + \frac{\partial \lambda_{\mathbb{Q}}^{\mathbb{Q}^T}}{\partial P} dP(t,T) + \frac{\partial \lambda_{\mathbb{Q}}^{\mathbb{Q}^T}}{\partial M} dM(t) + \frac{1}{2}\frac{\partial^2 \lambda_{\mathbb{Q}}^{\mathbb{Q}^T}}{\partial P^2} dP^2(t,T) \\
&= 0 + \frac{1}{P(0,T)}\left(\frac{1}{M(t)} dP(t,T) - \frac{P(t,T)}{M^2(t)} dM(t) + 0\right) \\
&= \frac{P(t,T)/P(0,T)}{M(t)}\left([\dots]dt - \beta(\bar{\tau})\mathbf{\Sigma} dW^{\mathbb{Q}}(t)\right) \\
&= -\lambda_{\mathbb{Q}}^{\mathbb{Q}^T}(t)\beta(t)\mathbf{\Sigma} dW^{\mathbb{Q}}(t)
\end{aligned}$$

Note that second derivative terms involving $M(t)$ equal 0 as $M(t)$ lacks a diffusion term. The Girsanov theorem now gives

$$dW^{\mathbb{Q}} = dW^{\mathbb{Q}^T} - (\beta(\bar{\tau})\mathbf{\Sigma})^* dt$$

Inserting this result in the dynamics for X given in proposition 4 gives

$$\begin{aligned}
dX^{\mathbb{Q}^T}(t) &= DX(t)dt + \mathbf{\Sigma}(dW^{\mathbb{Q}^T} - (\beta(\bar{\tau})\mathbf{\Sigma}^*)dt) \\
&= (DX(t) - \mathbf{\Sigma}(\beta(\bar{\tau})\mathbf{\Sigma})^*)\, dt + \mathbf{\Sigma} dW^{\mathbb{Q}^T}(t)
\end{aligned}$$

□

5.4 Monte Carlo Simulation of the FHJM-Model

As $\bar{f}(t,\tau) = \bar{f}(0, t+\tau)$, simulating the evolution of the yield curve comes down to simulating the $X(t)$ and $Y(t)$ processes.

5.4.1 Simulation of X

Continuous stochastic processes often have to be discretised before simulation but in this case, the distribution of $X(t+u)$ given $X(t)$ can be directly calculated. Proposition 4 gives

$$dX(t) = DX(t) + \mathbf{\Sigma} dW^{\mathbb{Q}}(t) \implies X(t) = e^{Dt}X(0) + \int_0^t e^{D(t-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s)$$

Note that $X(t+u)$ will be dependent on $X(t)$. To properly simulate X for a grid of times, $X(t+u)$ should thus be expressed as a function of $X(t)$. Inserting $t+u$ in the expression above gives

$$\begin{aligned}
X(t+u) &= e^{D(t+u)}X(0) + \int_0^{t+u} e^{D(t+u-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s) \\
&= e^{Du}e^{Dt}X(0) + \int_0^{t} e^{D(t+u-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s) + \int_t^{t+u} e^{D(t+u-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s) \\
&= e^{Du}\left(e^{Dt}X(0) + \int_0^{t} e^{D(t-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s)\right) + \int_t^{t+u} e^{D(t+u-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s) \\
&= e^{Du}X(t) + \int_t^{t+u} e^{D(t+u-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s)
\end{aligned}$$

$X(t+u)$ will thus be normally distributed and using $\mathbb{E}[dW^{\mathbb{Q}}(s)] = 0$ and the Ito isometry we further have

$$\begin{aligned}
\mathbb{E}[X(t+u)] &= \mathbb{E}\left[e^{Du}X(t) + \int_t^{t+u} e^{D(t+u-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s)\right] = e^{Du}X(t) \\
\mathbb{V}\left(X(t+u)\right) &= \mathbb{V}\left(\int_t^{t+s} e^{D(t+u-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s)\right) = \int_t^{t+u} e^{D(t+u-s)}\mathbf{\Sigma}\mathbf{\Sigma}^* e^{D(t+u-s)} ds
\end{aligned}$$

as D is a diagonal matrix which implies that $(e^{D(t+u-s)})^* = e^{D(t+u-s)}$. Setting $v = t+u-s$ gives

$$\mathbb{V}\left(X(t+u)\right) = \int_0^u e^{Dv}\mathbf{\Sigma}\mathbf{\Sigma}^* e^{Dv} dv$$

The expression can be further decomposed using vectorisation and the Kronecker product. Given matrices A, B and C of appropriate sizes, it holds that

$$\begin{aligned}
e^A \otimes e^B &= e^{A \oplus B} \\
\text{vec}(ABC) &= (C^* \otimes A)\text{vec}(B)
\end{aligned}$$

Where $\oplus$ denotes the *Kronecker sum* defined as

$$A \oplus B = A \otimes I + B \otimes I$$

Using these properties we have

$$\begin{aligned}
\text{vec}\left(\mathbb{V}\left(X(t+u)\right)\right) &= \text{vec}\left(\int_0^u e^{Dv}\mathbf{\Sigma}\mathbf{\Sigma}^* e^{Dv} dv\right) && (5.10) \\
&= \int_0^u e^{(D\oplus D)v}\text{vec}\left(\mathbf{\Sigma}\mathbf{\Sigma}^*\right) dv && (5.11) \\
&= \left[(D\oplus D)^{-1}e^{(D\oplus D)v}\right]_{v=0}^u \text{vec}\left(\mathbf{\Sigma}\mathbf{\Sigma}^*\right) && (5.12) \\
&= (D\oplus D)^{-1}\left(e^{(D\oplus D)u} - I\right)\text{vec}\left(\mathbf{\Sigma}\mathbf{\Sigma}^*\right) && (5.13)
\end{aligned}$$

Where $\mathbb{V}$ defines the variance and I is an identity matrix of the same dimensions as $D \oplus D$. As D is a diagonal matrix $D \oplus D$ will also be a diagonal matrix which means inverting it is straightforward. As such we can obtain the covariance matrix of $X(t+u)$ by simply reshaping the vector in (5.13).

Proposition 6 (Simulation of X)**.** *Given the dynamics in proposition 4, $X(t+u)$ given $X(t)$ will be normally distributed with mean and variance*

$$\begin{aligned}
\mathbb{E}[X(t+u)] &= e^{Du}X(t) \\
\mathbb{V}\left(X(t+u)\right) &= mat_{K\times K}\left((D\oplus D)^{-1}\left(e^{(D\oplus D)u}-I\right) vec\left(\mathbf{\Sigma}\mathbf{\Sigma}^*\right)\right)
\end{aligned}$$

Where $mat_{K\times K}$ is the reshape operator from a $K^2 \times 1$ vector to a $K \times K$ matrix. Simulating X thus comes down to sampling the above distribution in sequence.

Simulation of X under $\mathbb{Q}^T$-forward measure

From theorem 9 we have

$$\begin{aligned}
dX^{\mathbb{Q}^T}(t) &= \left(DX(t) - \mathbf{\Sigma}(\beta(\bar{\tau})\mathbf{\Sigma})^*\right)dt + \mathbf{\Sigma}dW^{\mathbb{Q}^T}(t) \\
\implies X^{\mathbb{Q}^T}(t) &= e^{Dt}C - \int_0^t e^{D(t-s)}\mathbf{\Sigma}(\beta(\bar{\tau})\mathbf{\Sigma})^* ds + \int_0^t e^{D(t-s)}\mathbf{\Sigma}dW^{\mathbb{Q}^T}(s)
\end{aligned}$$

We now express $X(t+u)$ as a function of $X(t)$

$$\begin{aligned}
X^{\mathbb{Q}^T}(t+u) &= e^{D(t+u)}C - \int_0^{t+u} e^{D(t+u-s)}\mathbf{\Sigma}(\beta(\bar{\tau})\mathbf{\Sigma})^* ds + \int_0^{t+u} e^{D(t+u-s)}\mathbf{\Sigma}dW^{\mathbb{Q}^T}(s) \\
&= e^{Du}\left(e^{Dt}C - \int_0^t e^{D(t-s)}\mathbf{\Sigma}(\beta(\bar{\tau})\mathbf{\Sigma})^* ds + \int_0^t e^{D(t-s)}\mathbf{\Sigma}dW^{\mathbb{Q}^T}(s)\right) \\
&\quad - \int_t^{t+u} e^{D(t+u-s)}\mathbf{\Sigma}(\beta(\bar{\tau})\mathbf{\Sigma})^* ds + \int_t^{t+u} e^{D(t+u-s)}\mathbf{\Sigma}dW^{\mathbb{Q}^T}(s) \\
&= e^{Du}X(t) - \int_t^{t+u} e^{D(t+u-s)}\mathbf{\Sigma}(\beta(\bar{\tau})\mathbf{\Sigma})^* ds + \int_t^{t+u} e^{D(t+u-s)}\mathbf{\Sigma}dW^{\mathbb{Q}^T}(s)
\end{aligned}$$

The stochastic integral has the same solution as under $\mathbb{Q}$. The deterministic integral has the solution

$$\begin{aligned}
-\int_t^{t+u} e^{D(t+u-s)}\mathbf{\Sigma}(\beta(\bar{\tau})\mathbf{\Sigma})^* ds &= \left[e^{D(t+u-s)}D^{-1}\mathbf{\Sigma}(\beta(\bar{\tau})\mathbf{\Sigma})^*\right]_t^{t+u} \\
&= \left(I - e^{Du}\right)D^{-1}\mathbf{\Sigma}(\beta(\bar{\tau})\mathbf{\Sigma})^*
\end{aligned}$$

Proposition 7. *Given the dynamics in theorem 9, $X(t+u)$ given $X(t)$ under the $\mathbb{Q}^T$-forward measure will be normally distributed with mean and variance*

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^T}\left[X^{\mathbb{Q}^T}(t+u)|X^{\mathbb{Q}^T}(t)\right] &= (I - e^{Du})D^{-1}\mathbf{\Sigma}(\beta(\bar{\tau})\mathbf{\Sigma})^* + e^{Du}X^{\mathbb{Q}^T}(t) \\
\mathbb{V}^{\mathbb{Q}^T}\left[X^{\mathbb{Q}^T}(t+s)|X^{\mathbb{Q}^T}(t)\right] &= mat_{KxK}\left((D\oplus D)^{-1}\left(e^{(D\oplus D)u}-I\right) vec\left(\mathbf{\Sigma}\mathbf{\Sigma}^*\right)\right)
\end{aligned}$$

Where $mat_{K\times K}$ is the reshape operator from a $K^2 \times 1$ vector to a $K \times K$ matrix. Simulating X under the $\mathbb{Q}^T$-forward measure thus comes down to sampling the above distribution in sequence.

5.4.2 Calculation of Y

Theorem 10. *The ordinary differential equation defining the dynamics of $Y(t)$ in proposition 4*

$$dY(t) = (\tilde{D}Y(t) + \Omega)dt$$

has the solution

$$Y(t) = (e^{\tilde{D}t} - I)\tilde{D}^{-1}\Omega$$

Proof. The differential equation can be solved by moving $\tilde{D}Y(t)$ to the left side and multiplying by the integrating factor $e^{-\tilde{D}t}$.

$$e^{-\tilde{D}t}\left(\frac{d}{dt}Y(t)\right) - e^{-\tilde{D}t}\tilde{D}Y(t) = e^{-\tilde{D}t}\Omega$$

Recognising that $\frac{d}{dt}e^{-\tilde{D}t}Y(t) - e^{-\tilde{D}t}\tilde{D}Y(t) = \frac{d}{dt}(e^{-\tilde{D}t}Y(t)$ and integrating both sides of the equation gives

$$e^{-\tilde{D}t}Y(t) = \int e^{-\tilde{D}t}\Omega dt = -e^{-\tilde{D}t}\tilde{D}^{-1}\Omega + C$$
$$\implies Y(t) = e^{\tilde{D}t}C - \tilde{D}^{-1}\Omega$$

where C is a constant K-dimensional column vector. As we initiate the model with X and Y set to zero, we have the starting condition

$$Y(0) = e^{\tilde{D}0}C - \tilde{D}^{-1}\Omega = 0 \implies C = \tilde{D}^{-1}\Omega$$

Inserting this in the expression for $Y(t)$ gives

$$Y(t) = e^{\tilde{D}t}\tilde{D}^{-1}\Omega - \tilde{D}^{-1}\Omega = (e^{\tilde{D}t} - I)\tilde{D}^{-1}\Omega$$

We now check the solution by first taking its derivative

$$\frac{d}{dt}Y(t) = \tilde{D}e^{\tilde{D}t}\tilde{D}^{-1}\Omega$$

Inserting the derived expression for $Y(t)$ and its derivative into the differential equation gives

$$\tilde{D}e^{\tilde{D}t}\tilde{D}^{-1}\Omega = \tilde{D}(e^{\tilde{D}t}\tilde{D}^{-1}\Omega - \tilde{D}^{-1}\Omega) + \Omega = \tilde{D}e^{\tilde{D}t}\tilde{D}^{-1}\Omega$$

Showing we have the correct solution. □

5.4.3 Simulation of the Money Market Account

From proposition 5 we have that the value of a money market account with tenor T is given by

$$M(t+u) = M(t)\exp\Bigg(\mathbb{1}_K D^{-1}(X(t+u) - X(t)) + \mathbb{1}_{\tilde{K}}\tilde{D}^{-1}(Y(t+u) - Y(t) + \Omega(T-t)) + \int_t^{t+u}\bar{f}(0,t+s)ds\Bigg) - \mathbb{1}_K D^{-1}\int_t^{t+u}\mathbf{\Sigma}dW^{\mathbb{Q}}(s)$$

To be able to simulate both a series of X and M values in unison, we thus need the expectation and variance of

$$\int_t^{t+u} \mathbf{\Sigma} dW^{\mathbb{Q}}(s)$$

given X, i.e. given

$$\int_t^{t+u} e^{D(t+u-s)} \mathbf{\Sigma} dW^{\mathbb{Q}}(s)$$

In general, for two zero mean Gaussian variables G and H, it holds that

$$\begin{aligned}\mathbb{E}[H|G] &= \mathbb{C}(H,G)\mathbb{V}^{-1}(G)G \\ \mathbb{V}[H|G] &= \mathbb{V}(H) - \mathbb{C}(H,G)\mathbb{V}^{-1}(G)\mathbb{C}(H,G)^*\end{aligned}$$

where $\mathbb{C}$ denotes the covariance. Now let

$$\begin{aligned}H &= D^{-1}\int_t^{t+u} \mathbf{\Sigma} dW^{\mathbb{Q}}(s) \\ G &= \int_t^{t+u} e^{D(t+u-s)} \mathbf{\Sigma} dW^{\mathbb{Q}}(s)\end{aligned}$$

From proposition 6 we have that

$$\mathbb{V}(G) = \text{mat}_{K\times K}\left((D\oplus D)^{-1}\left(e^{(D\oplus D)u} - I\right)\text{vec}\left(\mathbf{\Sigma\Sigma}^*\right)\right)$$

We further have that

$$\mathbb{V}(H) = D^{-1}\mathbf{\Sigma\Sigma}^* D^{-1} u$$

Finally, the covariance is given by

$$\begin{aligned}\mathbb{C}(H,G) &= \mathbb{E}^{\mathbb{Q}}[HG^*] - \mathbb{E}^{\mathbb{Q}}[H](\mathbb{E}^{\mathbb{Q}}[G])^* \\ &= D^{-1}\mathbb{E}^{\mathbb{Q}}\left[\int_t^{t+u} \mathbf{\Sigma} dW^{\mathbb{Q}}(s)\left(\int_t^{t+u} e^{D(t+u-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s)\right)^*\right] \\ &= D^{-1}\int_t^{t+u} \mathbf{\Sigma\Sigma}^* e^{D(t+u-s)} ds \\ &= D^{-1}\mathbf{\Sigma\Sigma}^* D^{-1}\left(e^{Du} - I\right)\end{aligned}$$

Proposition 8 (Simulation of money market account)**.** *Given simulated values $X(t)$ and $X(t+u)$ under the $\mathbb{Q}$-measure. The money market account at $t+u$ is given by*

$$\begin{aligned}M(t+u) = M(t)\exp\Bigg(&\mathbb{1}_K D^{-1}(X(t+u) - X(t)) + \mathbb{1}_{\tilde{K}}\tilde{D}^{-1}(Y(t+u) - Y(t) \\ &- \Omega u) + \int_t^{t+u} \bar{f}(0, t+s)ds - \mathbb{1}_K H\Bigg)\end{aligned}$$

where H is normally distributed with expectation and variance

$$\begin{aligned}\mathbb{E}^Q[H|X(t)] = {} & D^{-1}\mathbf{\Sigma\Sigma}^* D^{-1}\left(1 - e^{Du}\right)\left(mat_{K\times K}\left((D\oplus D)^{-1}\left(e^{(D\oplus D)u} - I\right) vec\left(\mathbf{\Sigma\Sigma}^*\right)\right)\right)^{-1} \\ & (X(t+u) - e^{Du}X(t)) \\ \mathbb{V}[H|G] = {} & D^{-1}\mathbf{\Sigma\Sigma}^* D^{-1}u - D^{-1}\mathbf{\Sigma\Sigma}^* D^{-1}\left(1 - e^{Du}\right) \\ & \left(mat_{K\times K}\left((D\oplus D)^{-1}\left(e^{(D\oplus D)u} - I\right) vec\left(\mathbf{\Sigma\Sigma}^*\right)\right)\right)^{-1} \\ & \left(D^{-1}\mathbf{\Sigma\Sigma}^* D^{-1}\left(1 - e^{Du}\right)\right)^*\end{aligned}$$

6 Black-76 and Bachelier Model

One asset type that can be used to calibrate the volatility matrix of the FHJM model is interest rate swaptions. This thesis uses Danish swaptions with a 6-month CIBOR swap as underlying. Swaptions are often quoted in their implied volatility, but are in this case converted to premiums before comparison. To convert the implied volatility to premiums, the Black-76 model and the Bachelier model are used. The main distinction between the two is that the latter allows for negative interest rates, whereas the former does not. The implementation of these models, especially when calculating the forward starting swap rate, can require interpolating or bootstrapping data. For this thesis, this is done through a cubic spline and using the QuantLib library in Python.

6.1 The Black-76 Formula

As presented by Black himself [21], the discounted Black-76 formula for a call option price can be expressed as

$$C_{BS}(K) = e^{-rT}(F_0 N(d_1) - K N(d_2)), \tag{6.1}$$

where

$$d_1 = \frac{ln(F_0/K) + (\sigma_{BS}^2/2)T}{\sigma_{BS}\sqrt{T}}$$

and

$$d_2 = \frac{ln(F_0/K) - (\sigma_{BS}^2/2)T}{\sigma_{BS}\sqrt{T}} = d_1 - \sigma_{BS}\sqrt{T}.$$

The Black-76 model assumes that the forward price F_t of the underlying asset follows a geometric Brownian motion,

$$\frac{dF_t}{F_t} = \sigma_{BS} dW_t.$$

Note that, as mentioned before, the model does not allow for negative interest rates. Since the formula contains a logarithm of the forward rates and strike price, negative rates would give imaginary results.

In the case of pricing swaptions on Danish interest rate swaps, the logarithm of the forward rate is assumed to follow a Brownian motion, and the variables are defined as follows.

F_0 = Fixed leg of a forward swap, starting at the expiry of the swaption.
K = Strike price of swaption.
r = Risk-free interest rate, set to Danish Overnight Index Swaps (OIS).
T = Time to maturity of swaption.
σ = Implied volatility as defined in the Black-Scholes model [22].

6.1.1 Forward Starting Swap Rate

The fixed leg F_0 of a forward starting swap can be calculated for a given swaption by equating the discounted fixed leg cash flows with the discounted floating leg cash flows of the underlying swap. To understand this better, assume the forward starting swap rate of a European-style swaption that expires in 1 month is to be calculated, where the underlying is a 1-year payer swap with semi-annual payments based on the 6-month CIBOR rate. If exercised, the underlying swap will have two cash flow payments, at months 7 and 13 respectively, equal to the difference between the determined fixed leg and the CIBOR rate. These are then discounted to the start of the swap using OIS rates. Note that OIS rates are used due to them being inherently less susceptible to credit risk and market manipulation, as opposed Interbank Offered Rates, and are therefore more accurately reflecting the true cost of funding. Denoting the CIBOR and OIS rate for month i as CIB_i and OIS_i respectively, the cash flows of the swap, assuming the swaption is exercised, will then be as follows.

$$\frac{CIB_7 - F_0}{(1 + OIS_7)^{0.5}} + \frac{CIB_{13} - F_0}{(1 + OIS_{13})^1} = 0 \tag{6.2}$$

Note that if the CIBOR and OIS rates are given on an annual basis, they first need to be converted to the respective period of the cash flow. The right-hand side of 6.2 is set to 0 since the discounted fixed and floating legs should be equated to avoid arbitrage opportunities. By solving for F_0, an expression for the forward starting swap rate is found.

$$F_0 = \left(\frac{CIB_7}{(1 + OIS_7)^{0.5}} + \frac{CIB_{13}}{(1 + OIS_{13})^1} \right) \Big/ \left(\frac{1}{(1 + OIS_7)^{0.5}} + \frac{1}{(1 + OIS_{13})^1} \right) \tag{6.3}$$

In the general case, for a swaption with expiry k, underlying swap tenor n and number of annual payments q, the formula can be expressed as

$$F_0 = \sum_{i=1}^{qn} \left(\frac{CIB_{12i/q+k}}{(1 + OIS_{12i/q+k})^{i/2}} \right) \Big/ \sum_{i=1}^{q} \left(\frac{1}{(1 + OIS_{12i/q+k})^{i/2}} \right) \tag{6.4}$$

6.2 The Bachelier Formula

The Bachelier model gives the discounted price of a call option through the following [23].

$$C_N(K) = e^{-rT}((F_0 - K)N(d_N) + \sigma_N \sqrt{T} n(d_N)), \tag{6.5}$$

where

$$d_N = \frac{F_0 - K}{\sigma_N \sqrt{T}}$$

and F_t is assumed to follow an arithmetic Brownian motion,

$$dF_t = \sigma_N dW_t.$$

In the case of pricing swaptions, it is the forward swap rate that is assumed to follow an arithmetic Brownian motion. Note that while all the other variables are defined in the same way, the volatility terms in 6.1 and 6.2 are not the same. While the Black volatility is expressed in relative terms, the Bachelier volatility measures the absolute change, which gives rise to the following relationship between the two.

$$\sigma_N = \sigma_{BS} F_0.$$

As can be seen, this model does not contain logarithms or any other mechanism preventing negative rates.

7 Calibrating the FHJM Model

7.1 Calibration Methods

As previously discussed, the λ variables defining the basis of the model are predefined. Calibrating the FHJM-model thus comes down to calibrating $\mathbf{\Sigma}$. Two different approaches are combined in this thesis. The first is fitting it to historical volatility using the past evolution of interest rates. The second is fitting the model to the markets implied belief about the future evolution of market volatility using interest rate options such as swaptions or caps or floors. In pricing applications it is typically better to fit the model to option data as it, in some sense, gives information about the future evolution of the yield curve rather than the past.

7.1.1 Calibrating to Forward Rates

Following Lyashenko and Goncharov [3] we can calibrate the volatility process of the model to a set of spanning forward rates. Let $\mathbf{F}(t)$ define a set of forward rates on the grid $\tau_1, \ldots, \tau_M$ s.t.

$$\mathbf{F}_m(t) = F(t, \tau_{m-1}, \tau_m)$$

Using theorem 8 we have that

$$\mathbf{F}(t) = \mathbf{F}(0) + \beta_{\mathbf{F}} X(t) + \tilde{\beta}_{\mathbf{F}} Y(t) + \epsilon(t) \tag{7.1}$$

where $\beta_{\mathbf{F}}$, $\tilde{\beta}_{\mathbf{F}}$ are $M \times K$ and $M \times \tilde{K}$ matrices respectively, defined by

$$\beta_{\mathbf{F}} = \begin{pmatrix} \frac{1}{\tau_1} \int_0^{\tau_1} B(s)ds \\ \vdots \\ \frac{1}{\tau_M - \tau_{M-1}} \int_{\tau_{M-1}}^{\tau_M} B(s)ds \end{pmatrix}$$

$$\tilde{\beta}_{\mathbf{F}} = \begin{pmatrix} \frac{1}{\tau_1} \int_0^{\tau_1} \tilde{B}(s)ds \\ \vdots \\ \frac{1}{\tau_M - \tau_{M-1}} \int_{\tau_{M-1}}^{\tau_M} \tilde{B}(s)ds \end{pmatrix}$$

and $\epsilon(t)$ some model error process. As proven by Lyashenko and Goncharov, $\beta_{\mathbf{F}}$ will be of full rank, if we define $\bar{\mathbf{F}}(t) = \mathbf{F}(t) - \mathbf{F}(0) - \tilde{\beta}_{\mathbf{F}} Y(t)$, we thus have

$$(\beta_{\mathbf{F}}^* \beta_{\mathbf{F}})^{-1} \beta_{\mathbf{F}}^* \bar{\mathbf{F}}(t) = X(t) + \tilde{\epsilon}(t)$$

where $\tilde{\epsilon}(t) = (\beta_{\mathbf{F}}^* \beta_{\mathbf{F}})^{-1} \beta_{\mathbf{F}}^* \epsilon(t)$. Assuming the model error is small we can use this to generate a set of $X(t_k)$ values for each of our data points, $\mathbf{F}(t_k)$. For each data point, it holds that

$$X(t_{k+1}) = e^{D(t_{k+1}-t_k)}X(t_k) + \int_{t_k}^{t_{k+1}} e^{D(t_{k+1}-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s) \tag{7.2}$$

$$\implies X(t_{k+1}) - e^{D(t_{k+1}-t_k)}X(t_k) = \int_{t_k}^{t_{k+1}} e^{D(t_{k+1}-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s) \tag{7.3}$$

From proposition 6 we know that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_{t_k}^{t_{k+1}} e^{D(t_{k+1}-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s)\right] = 0$$

$$\mathbb{V}\left[\int_{t_k}^{t_{k+1}} e^{D(t_{k+1}-s)}\mathbf{\Sigma} dW^{\mathbb{Q}}(s)\right] = \text{mat}_{K\times K}\left((D\oplus D)^{-1}\left(e^{(D\oplus D)(t_{k+1}-t_k)} - I\right)\text{vec}\left(\mathbf{\Sigma}\mathbf{\Sigma}^*\right)\right)$$

The matrix $(D\oplus D)^{-1}\left(e^{(D\oplus D)u} - I\right)$ will be a diagonal matrix with non-singular values making inverting it trivial. We can thus numerically estimate the covariance matrix for the process $X(t_{k+1}) - e^{D(t_{k+1}-t_k)}$ and solve for the corresponding $\text{vec}(\mathbf{\Sigma}\mathbf{\Sigma}^*)$ using

$$\text{vec}(\mathbf{\Sigma}\mathbf{\Sigma}^*) = \left((D\oplus D)^{-1}\left(e^{(D\oplus D)(t_{k+1}-t_k)} - I\right)\right)^{-1}\text{vec}(\tilde{V}) \tag{7.4}$$

where $\tilde{V}$ is the estimated covariance matrix. A value for $\mathbf{\Sigma}$ can then finally be obtained by taking the Cholesky decomposition of $\mathbf{\Sigma}\mathbf{\Sigma}^*$.

The challenge with this approach lies in the necessity of $\mathbf{\Sigma}$ to calculate the values of $Y(t)$. It is therefore not possible to directly calculate the correct value of $\mathbf{\Sigma}$. However, by defining an initial guess for $\mathbf{\Sigma}$ and iteratively refining it, using the estimated $\mathbf{\Sigma}$ from the previous iteration to calculate $Y(t)$, the values of $\mathbf{\Sigma}$ should converge.

7.1.2 Calibrating to Swaptions

Finding a closed-form solution for swaption prices under the FHJM model is difficult. As the model is multidimensional, using the method proposed by Jamshidian in [24] is not possible. Sepp and Rakhmonov did propose a swaption pricing formula for the FHJM model with stochastic volatility using moment-generating functions in [20]. Still, for this thesis, we decided to use Monte Carlo simulation and minimise the Mean Squared Error (MSE) between the simulations and the real prices.

There are two main ways of simulating swaption prices using the formulas from previous sections. The first is simultaneously simulating X under the $\mathbb{Q}$ measure using proposition 6 and the money market account using proposition 8. Swaption prices can then be calculated for each simulation using (4.27). The second is simulating X under different $\mathbb{Q}^T$-forward measures using proposition 7 and then pricing the simulated swaptions with (4.28). In both cases, pricing the ZCBs used in the swaption pricing formulas can be done with theorem 6. For this thesis, we decided to go with the latter approach.

The main issue with the $\mathbb{Q}^T$-forward approach is that it requires X to be simulated under a different forward measure for each option tenor of the swaption contracts. This makes it harder to simulate the X values in a unified time series as we can not use the X-value for the previous option tenor as the base for simulating the value for the next. As all forward measures have the same volatility, this problem can however be solved by simulating all X values under the forward measures for all option tenors simultaneously.

7.2 Implementation

The FHJM model was implemented in Python using an object-oriented approach with different classes for different aspects of the model, such as volatility, basis, and simulation. A UML diagram of the implementation can be found in appendix A.

The calibration of the model was done in two steps. First the volatility matrix, $\mathbf{\Sigma}$, was calibrated to one month of spanning forward rates as discussed in section 7.1.1. The implemented algorithm was as follows

Algorithm 1 Algorithm for Fitting $\mathbf{\Sigma}$ to Forwards

function FIT_TO_FORWARDS
    Calculate $\beta_{\mathbf{F}}$ and $\tilde{\beta}_{\mathbf{F}}$ parameters for the forward rate intervals.
    Calculate the inverse matrix used in (7.4)
    while not converged **do**
        Calculate new $\mathbf{\Sigma}$ using CALCULATE_NEW_SIGMA
        Update model volatility with the new $\mathbf{\Sigma}$
        if changes below tolerance threshold **then**
            Declare convergence and exit loop.
        end if
    end while
    return model parameters and final X and Y values
end function
function CALCULATE_NEW_SIGMA
    Compute Y-values for the current $\mathbf{\Sigma}$ for each forward
    Compute adjusted forward rates $\bar{\mathbf{F}}$ using Y-values and $\beta_{\mathbf{F}}$ and $\tilde{\beta}_{\mathbf{F}}$
    Calculate X-values using (7.2).
    Estimate covariance matrix $\tilde{V}$.
    Calculate the new covariance matrix for X using (7.4)
    Calculate the new $\mathbf{\Sigma}$ by taking the Cholesky decomposition of the new volatility matrix
    return the new $\mathbf{\Sigma}$ and calculated X and Y values
end function

The resulting $\mathbf{\Sigma}$ from the calibration to forwards was then used as a starting point for optimising the volatility to swaption data. The calibration was done using the optimiser from the the SciPy python package with the following algorithm for estimating the MSE.

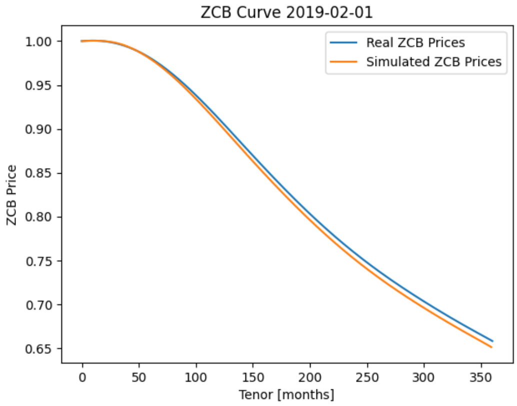
Algorithm 2 Mean Squared Error Calculation for Swaptions Pricing

```
function CALCULATE_MSE(new_Sigma, date, X_noise, nr_sims)
    Update the volatility matrix with new values from new_Sigma.
    Price the swaptions using the price_swaptions function.
    Calculate the MSE between the real and the simulated swaption prices.
    return MSE
end function
function PRICE_SWAPTIONS(date, nr_sims = 10000, X_noise, mean = True)
    Simulate paths for X under T-forward measures for every option maturity.
    Calculate the values for Y for every option maturity.
    Retrieve real swaption premiums and forward swap rates for the given date.
    Calculate the present values of ZCBs for all swap cash flow dates.
    Simulate swaption payoffs using the above data.
    Aggregate real and simulated swaption values
    return swaption premium data
end function
```

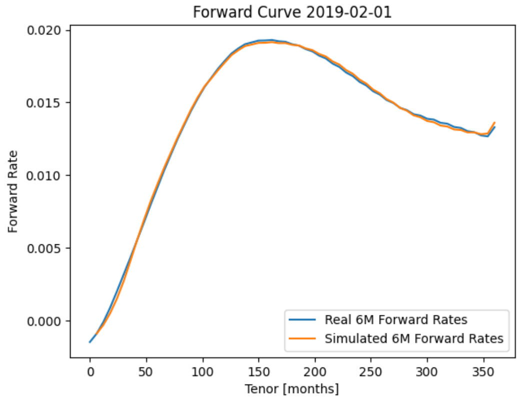
7.3 Calibration Results

After running algorithm 1, fitting the FHJM model to a set of spanning forward rates, the resulting X- and Y-values where used to calculate ZCB prices and 6m-forward and swap rates. Plots of the results can be seen in figure 7.1. While the ZCB and forward curves are similar to the observed values the swap curve deviates slightly. One possible explanation is that the simulated swap rates are calculated using proposition 1 accumulating the smaller errors observed for the ZCB prices. Another possibility is that the deviation stems from the initial bootstrapping of the yield curve. This was done using swap and OIS rates meaning the forward data used to fit the FHJM model, and plotted in 7.1 (b), were not observed in the market but calculated and interpolated from the bootstrap. Any error in the bootstrapping would thus follow to the calibration of the model and could cause a deviation when simulated swap rates are plotted against the original, observed ones.

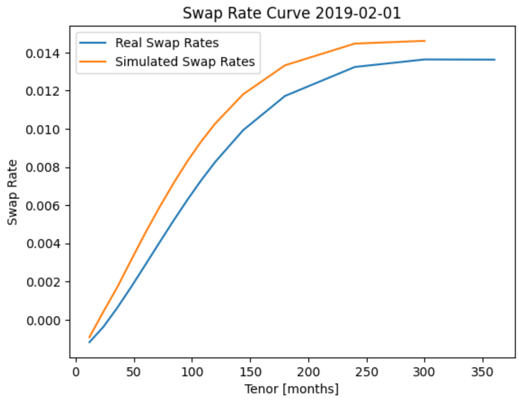
After calibrating the FHJM-model to forward rates, the resulting $\mathbf{\Sigma}$ was used to as a starting point to numerically estimate the optimal $\mathbf{\Sigma}$ for swaption premiums. From figure 7.2 it is clear that calibrating the model to swaptions is an important step and the calibration to forwards seems to have significantly underestimated the volatility for longer tenors. Figures 7.2 and 7.4 also show larger errors for smaller premiums, an observation that can explain the large errors for short tenors in figure 7.3 as short tenor swaptions typically have smaller premiums. This issue most likely stems from using MSE as an objective function when calibrating the model rather than some relative error or likelihood function. One solution to this problem could be calibrating the model directly to Black volatility rather than premiums. These are typically more uniform in size making MSE-based optimisation less likely to overfit to larger values. Due to time constraints, this was not explored. As there is no closed-form solution for calculating the black volatility from a swaption premium, this would also add another layer of complexity to an already computationally expensive calibration algorithm.



(a) Observed and simulated ZCB prices

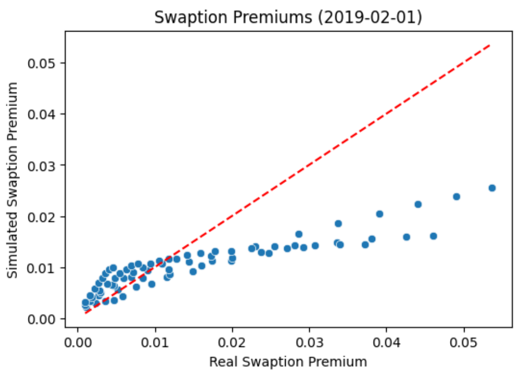


(b) Observed and simulated 6m-forward rates

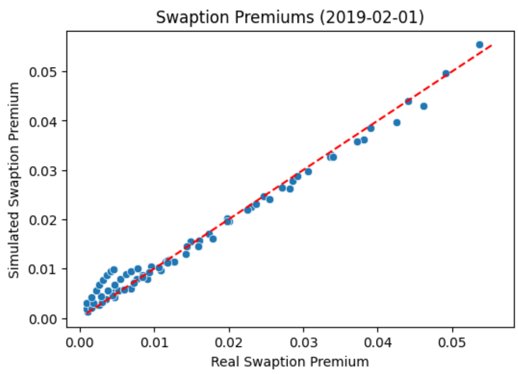


(c) Observed and simulated CIBOR-swap rates

Figure 7.1: Plots of the resulting interest rates from the final X-values outputted by algorithm 1.



(a) Simulated vs. real swaption-premiums before calibration to swaptions.



(b) Simulated vs. real swaption-premiums after calibration to swaptions.

Figure 7.2: Simulated vs. real swaption-premiums after calibrating the FHJM-model to swaption data using algorithm 2.

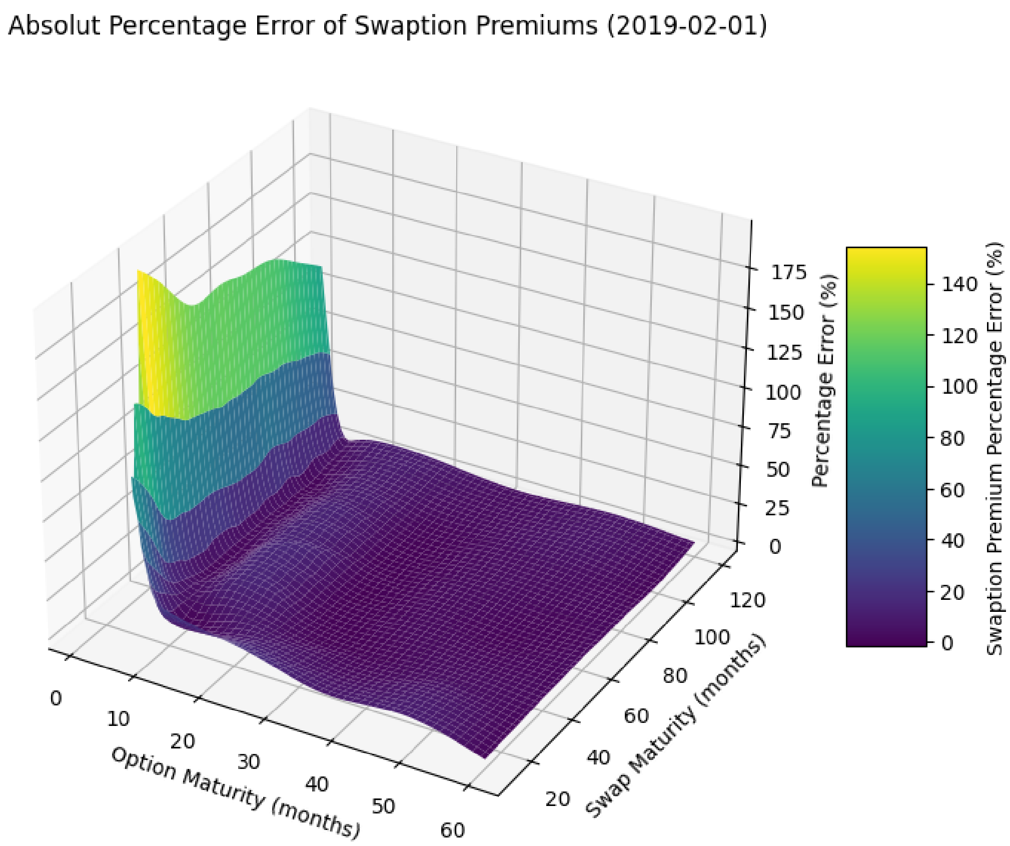
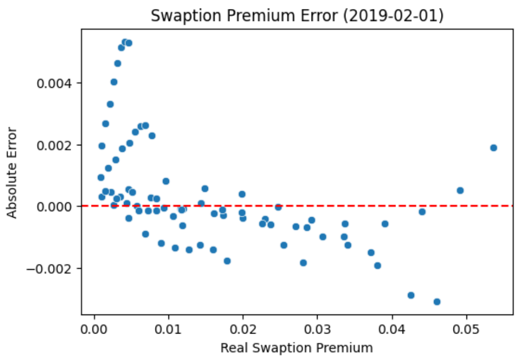


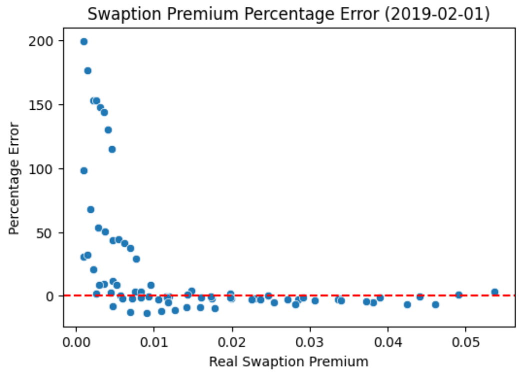
Figure 7.3: Percentage swaption-premium pricing errors against option- and swap-tenor after calibrating the FHJM-model to swaption data using algorithm 2.

7.4 Sensitivity Analysis

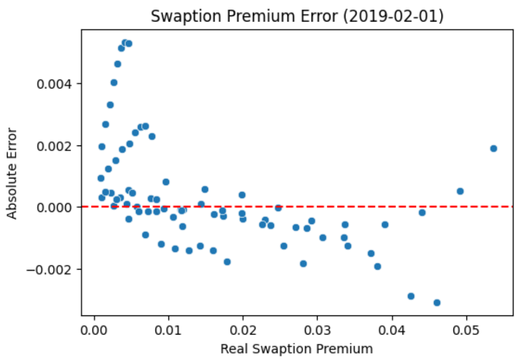
Two tests of the sensitivity of the calibrated $\mathbf{\Sigma}$ were conducted. First $\mathbf{\Sigma}$ was scaled by 1.1 and 0.9 and the resulting estimated swaption premiums were plotted, see figure 7.5. Secondly, the $\mathbf{\Sigma}$ calibrated to 2019-02-01 was used to estimate swaption premiums one month into the future, at 2019-03-01, and one year in the future, at 2020-02-03. The resulting plots can be seen in figure 7.6. The results are about as expected with the scaled $\mathbf{\Sigma}$'s either over or underestimating the swaption premiums slightly and the calibration to 2019-02-01 performing well for swaptions priced one month later but significantly poorer for swaptions priced a year later.



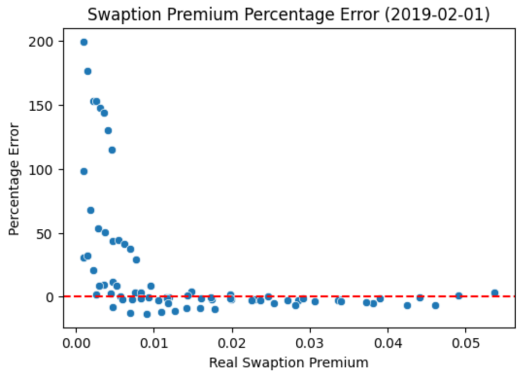
(a) Swaption-premium errors against observed swaption.



(b) Percentage swaption-premium errors against observed swaption premiums.

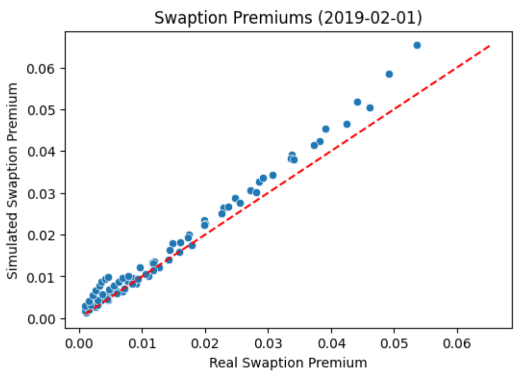


(c) Swaption-premium errors against observed swaption.

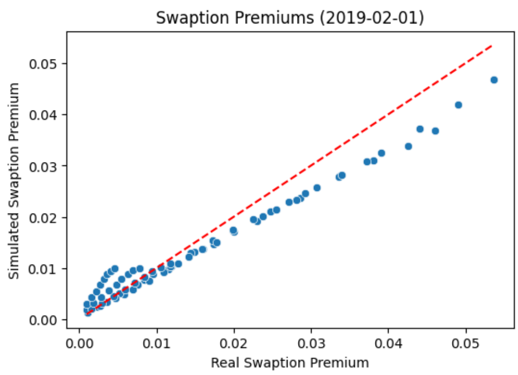


(d) Percentage swaption-premium errors against observed swaption premiums.

Figure 7.4: Percentage and real swaption-premium pricing errors against against observed swaption prices after calibrating the FHJM-model to swaption data using algorithm 2.

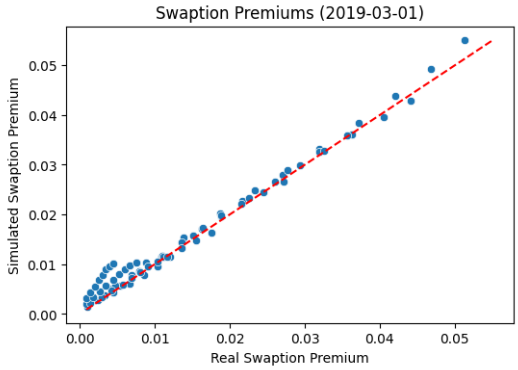


(a) Simulated vs. real swaption-premiums with $\boldsymbol{\Sigma}$ scaled by 1.1

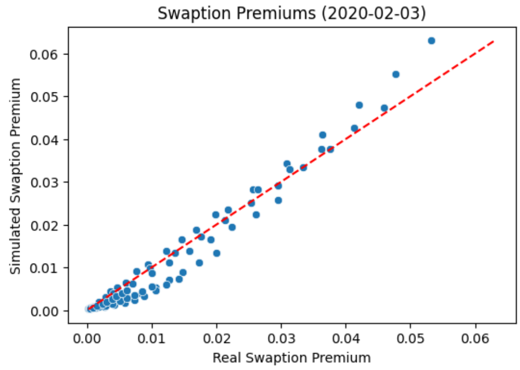


(b) Simulated vs. real swaption-premiums with $\boldsymbol{\Sigma}$ scaled by 0.9

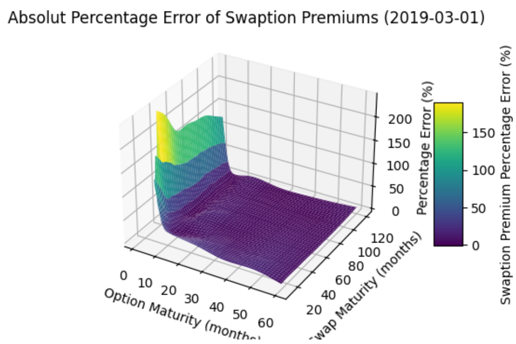
Figure 7.5: Simulated and real swaption premiums using the calibrated $\boldsymbol{\Sigma}$ scaled by a constant.



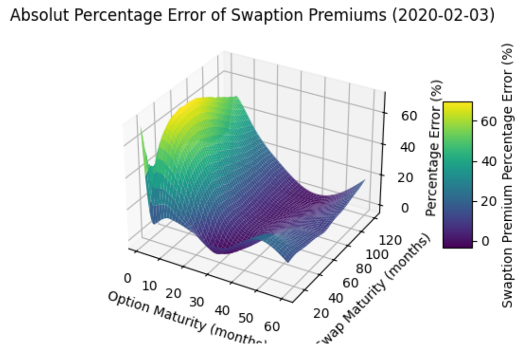
(a) Simulated vs. real swaption-premiums with $\mathbf{\Sigma}$ scaled by 1.1



(b) Simulated vs. real swaption-premiums with $\mathbf{\Sigma}$ scaled by 0.9



(c) Swaption pricing errors plotted against swap- and option tenors with $\mathbf{\Sigma}$ scaled by 1.1



(d) Swaption pricing errors plotted against swap- and option tenors with $\mathbf{\Sigma}$ scaled by 0.9

Figure 7.6: Simulated and real swaption premiums using the calibrated $\mathbf{\Sigma}$ scaled by a constant.

8 Forecasting Prepayment Rates

As previously discussed, the prepayment behaviour observed in the market is not optimal in the sense that it minimises the observable cost for the loan takers. There could be many factors influencing the deviation from optimal behaviour, ranging from a lack of knowledge to unobservable costs, such as the time investment needed to refinance the loan. It is however reasonable to assume financial gain is the main motivating factor behaviour behind prepayments [6]. Let us therefore begin by exploring what optimal prepayment behaviour would look like, and then review the explanatory variables used in the prepayment model.

8.1 Optimal Prepayment Behaviour

Assume, as done in [6], that the total cost of prepaying is given by

$$\tilde{W}_P(t,\gamma) = (1+\gamma)W_P(t) \tag{8.1}$$

where $W_P(t)$ is the value of the loan given prepayment and γ represents various refinancing costs, as a fixed percentage of $W_P(t)$. If we similarly let $\tilde{W}_N(t,\gamma)$ denote the value of the mortgage in the case of no prepayment, the value-minimising principle of American option theory gives that

$$\tilde{W}(t,\gamma) = \min\left(\tilde{W}_P(t,\gamma), \tilde{W}_N(t,\gamma)\right) \tag{8.2}$$

as rational behaviour would be to minimise the value of the loan. The rational prepayment rate would thus be

$$PR(t,\gamma) = \begin{cases} 1 & \text{if} \quad \tilde{W}_P(t,\gamma) < \tilde{W}_N(t,\gamma) \\ 0 & \text{otherwise} \end{cases} \tag{8.3}$$

To make a prepayment decision at t we can start at the maturity of the bond and then backtrack through the decision periods basing the valuation of W_N on the decisions that would be made at later periods [6].

8.2 Refinancing gain

Recall that it is assumed that the financial gain from prepaying and refinancing is the main explanatory variable behind the rate of prepayment in a pool of bonds. In this section, prepayment gain will be further explored, and the parameters necessary to calculate it will be presented. Note that in some literature, the refinancing gain is calculated as the ratio between the discounted old loan and the discounted new

[6]. However, many practitioners only focus on the after-tax first year ratio between the old and new loan, with the motivation that "today, borrowers focus primarily on liquidity savings in the form of lower net payments" [25]. The latter is what will be used in this thesis.

Assuming that the time to maturity on the new and old loan are the same, and denoting the after tax monthly payment for the first year as FYP, the refinancing gain can therefore be calculated as

$$\text{gain} = \frac{\text{FYP}_{\text{old loan}} - \text{FYP}_{\text{new loan}}}{\text{FYP}_{\text{old loan}}} \tag{8.4}$$

Note that the first year payments are calculated using the debt including estimated cost incurred for refinancing.

8.2.1 After Tax Monthly Payment

The after tax monthly payment can be divided into three parts, namely the first year interest-included payment $PMT_{IR,1}$, the first year amortisation payment $amort_1$ and the first year administration margin payment $admin_1$. The after tax monthly payment for the first year FYP, can then be calculated as

$$\frac{(PMT_{IR,1} + admin_1 - amort_1)(1 - tax) + amort_1}{12} \tag{8.5}$$

For simplicity and to stay within the scope of this thesis, the tax and admin margin are assumed to be constant over time, and are set to 25% and 0.7106% respectively.

In this case, since the loans are assumed to behave like straight annuity with quarterly payments, the standard annuity formula can be used to calculate the sum of the first four quarterly payments, to ultimately get the first annual payments. This can easily be achieved with NumPy financials "PMT" and "PPMT" functions.

The "PMT" function takes the arguments *(rate, nper, pv)*, where *rate* is the interest rate per period, *nper* is the number of compounding periods and *pv* is the present value. The function outputs the monthly payment amount, calculated by solving the following equation [26].

$$\frac{fv + pv \cdot (1 + rate)^{nper} + PMT \cdot (1 + rate \cdot when)}{rate \cdot ((1 + rate)^{nper} - 1)} = 0, \tag{8.6}$$

where "when" specifies if the payment is made in the beginning or end of each period.

The "PPMT" function takes the arguments *(rate, per, nper, pv)*, where "per" is the period of interest and all other arguments are defined similarly as in the "PMT" function. The "PPMT" function gives only payment against loan principal, without interest. It does this through the same calculation principle as in the "PMT" function less the interest rate payment [27].

First year interest-included payment

To calculate the first year payment including the interest rate payment, the "PMT" function is used, where *rate* is the annual coupon rate divided by four. Note that in the new loan, the prepayment rate will act as the coupon rate. The "nper" is set to the number of years to maturity of the loan, multiplied by four to get the number of quarters. The "pv" argument is set to the outstanding debt including estimated cost incurred of the loan, henceforth expressed as just *debt*. Finally, the resulting payment amount expressed in quarterly terms is multiplied by four to express the annual payment of the first year. This can be expressed as follows.

$$PMT(\frac{annual\ coupon}{4}, loan\ length \cdot 4, debt) \cdot 4 \tag{8.7}$$

First year amortisation payment

To find the first year amortisation payment, the sum of four "PPMT" functions are used where the "per" argument ranges from one to four, representing the quarters of the first year. As can be seen below, all other arguments are defined similarly as in the calculation of the first year interest-included payment. Note that the resulting amortisation payment is the same as taking the first year interest-included payment less the first year interest rate payment.

$$\sum_{i=1}^{4}(PPMT(\frac{annual\ coupon}{4}, i, loan\ length \cdot 4, debt)) \tag{8.8}$$

First year administration margin payment

From hereon, the above expression for each "PPMT" term is simplified to $PPMT(i)$, where "i" is the period of interest. Administration margin payment of the first year is calculated as an average of the quarterly payments of the first year, each of which is calculated as a margin on the remaining debt.

As previously mentioned, administration margin is set to a constant of 0.7106%. For the first quarter, the payment is simply the total debt times the margin.

$$Q1\ payment = (debt) \cdot administration\ margin \tag{8.9}$$

For the second quarter the remaining notional is the original debt less the amortisation in period 1.

$$Q1\ payment = (debt - PPMT(1)) \cdot administration\ margin \tag{8.10}$$

Following the same principle, the third quarter payment becomes

$$Q3\ payment = (debt - PPMT(1) - PPMT(2)) \cdot administration\ margin \tag{8.11}$$

Lastly, for the fourth quarter we get

$$Q4\ payment = (debt - PPMT(1) - PPMT(2) - PPMT(3)) \cdot administration\ margin \tag{8.12}$$

The yearly administration margin payment is found by taking the average of equation 8.9, 8.10, 8.11, and 8.12. This can be written as

$$\frac{(4 \cdot debt - 3 \cdot PPMT(1) - 2 \cdot PPMT(2) - 1 \cdot PPMT(3)) \cdot administration\ margin}{4} \tag{8.13}$$

8.2.2 prepayment rate

As previously mentioned, for all calculations involving the loan that the borrower refinances into, the annual coupon rate is set to the refinancing rate. The prepayment rate can be calculated by finding the weighted asset swap rate ASW for bond cash flow, and adding a refinance spread.

$$prepayment\ rate = ASW + refinance\ spread$$

Refinance spread in this case is defined as the spread over DKK swaps, where borrowers can refinance their loans. For consistency, the refinance spreads provided as data for this thesis were all calculated on a fixed bond price of 98, by weighting one bond above 98 and another one below. This creates a hybrid generic bond priced at 98, by interpolating the refinance spread on the two bonds. It is assumed that the prepayment rate is the same for all bonds with the same time to maturity, and that future prepayment rate is the same as today's.

The weighted asset swap rate ASW is found by first discounting all the cash flows of the bond of interest on the swap curve, thereby finding a price of the bond, and then solving for the yield to maturity needed to get the aforementioned price. In other words, the weighted asset swap rate is equal to the yield to maturity of the bond when using the price you get by discounting the cash flows on the swap curve.

In training data, the swap curve is constructed by simply interpolating observed swap rates of different tenors, whereas in the out-of-sample model the swap curve is constructed using simulated rates given from the factor HJM model, as described in proposition 1.

8.3 Burn-Out Rate

While it is intuitive to assume that the number of prepayments increase with the financial gain that borrowers receive from refinancing their loans, this relation might look different in different bond pools. Something that can affect this relationship is a so called "burn-out rate". Assume that each pool of bonds consists of borrowers with different traits, risk-profile, knowledge of the market, etc. Each pool would then also consist of some borrowers who are more likely to refinance, and some who are less likely. The burn-out effect is created when the borrowers who are more likely to refinance achieve their required refinancing gain, and therefore refinances. The pool now consists of a larger ratio of borrowers who, due to aforementioned reasons, are less likely to refinance, even if their respective required refinancing gain levels are met. Thus, the pool is now "burned out" and will not behave similarly to a pool with an even distribution of different types of borrowers.

This phenomenon is captured in the so called "pool-factor" variable used in the prepayment model of this thesis. For each mortgage bond, the mortgage institute provides some information about the structure and the composition of the loan pool. For example, loans are split into "residential loans" and "commercial loans", the size of the loans are split into buckets, and the current outstanding amount is divided with maximum of historical outstanding amount is calculated to give the pool factor for each debtor group. In this case, the latter is used to represent the historical prepayment rate of the bond, which should capture some of the aforementioned burn-out effect. The loans are categorised into residential and commercial because taxation schemes differ between private households and business debtors, and their respective prepayment behaviours are also assumed to be different. Unique taxation schemes and more detailed debtor information are however treated as beyond the scope of this thesis.

8.4 Maturity, Issuer, Deferred Amortisation, Historical Average Prepayment Rate

Aside from the refinancing gain and burn-out rate, which are often seen as some of the most relevant factors to consider when building a prepayment model, we will also explore some other potential explanatory variables.

As explained by basic option theory, the remaining time to maturity might also affect the prepayment rate [6]. With a longer remaining time to maturity, there is more time for the prepayment option to eventually run into the in-the-money zone, or deeper in-the-money than it already is. As a result, a borrower with a long remaining time to maturity on their loan will have a higher level of required refinancing gain, since they expect their potential gain in the future to be higher. The borrower with a shorter remaining time to maturity is more likely to prepay at lower required gain levels, since they do not expect their potential refinancing gain to increase much during the remaining loan length.

All bonds used in this thesis are issued by either *Nykredit* or *Realkredit Danmark*, two of the major mortgage banks in Denmark. The issuer of each bond is used as a categorical variable in the prepayment model, based on the hypothesis that typical borrowers of each mortgage bank might have different profiles and prepayment probabilities. This could be due to many different factors, such as how the bank market themselves, their size, reputation, fees and user-friendliness of the prepayment process. As such, each mortgage bank is given their own respective coefficients that are multiplied with the modelled prepayment probabilities.

The type of loan is also hypothesised to have an effect on the prepayment rate. The different types of loans "annuity" or "interest-only" are created through the option of having deferred amortisation of a specified length. In other words, the borrower can choose amortise "normally" and therefore have a loan of annuity-type, or choose to have an initial period of only interest, before amortising the remaining loan period like an annuity. The latter option is called an "interest-only" loan. These two types of loans are represented in the prepayment model as a categorical variable, with the idea that prepayment rates will vary depending on the category.

Furthermore, as prepayment behaviour in many cases can be irrational, a historical average prepayment rate is used as a variable to try to capture some of the behaviour that is not predicted by the aforementioned variables.

8.5 Probit Model for Prepayments

In [6], Jakobsen proposes using a probit type model for estimating prepayment percentages. Common in econometrics, probit models are typically used to estimate the likelihood of a binary outcome by taking a unit normal CDF of a linear combination of explanatory variables. If we let y be a binary outcome variable and $\mathbf{x}$ a column vector of explanatory variables, the standard probit model gives the likelihood of y being one as

$$P(y = 1|\mathbf{x}) = \Phi(\beta\mathbf{x}) \tag{8.14}$$

where Φ is a unit normal CDF and β a row vector of weights.

Jakobsen instead proposes using the probit framework to calculate the prepayment rate as

$$PR(t) = \Phi(\beta\mathbf{x}(t)) \tag{8.15}$$

This method ensures the estimated rate is between 0 and 1 while maintaining an easily interpretable relationship between variables [6].

8.5.1 Calibrating the Probit Model

The prepayment model is calibrated by first initialising a β vector, and then using a maximum likelihood estimation to find the β vector that maximises the probability of getting the actual historical prepayment rates. The log-likelihood function to be maximised for a probit model is [28].

$$\log(\mathcal{L}(\beta|Y,X) = \sum_{i=1}^{n}\left(y_i \log\left(\Phi(\mathbf{x}_i^T\beta)\right) + (1-y_i)\log\left(1-\Phi(\mathbf{x}_i^T\beta)\right)\right) \tag{8.16}$$

The initial β vector was chosen randomly from a $N(0,1)$ distribution. Note that the optimiser did not always converge. Depending on the randomly generated initial vector, the optimiser would sometimes fail to converge and give unreasonable values for both the parameters and the inverted Hessian matrix. Most of the times it did converge, it obtained approximately the same β values, which could be a sign that it found a global maximum. However, depending on the loss landscape, there is still a possibility that the resulting vector is just a local maximum. If that were to be the case, it would mean that the obtained parameter estimates are not optimal and should be further explored to find a better fit.

Different model variations were tested for significance and performance. The performance metrics used were mean-squared-error (MSE), R-squared and Akaike Information Criterion (AIC). An MSE value measures the average squared difference between actual observed values and estimated values, and is for obvious reasons better the lower it is. The R-squared value measures how much of the variability in the dependent variable that can be explained by the model, where the maximum value of 1 would indicate perfect accuracy of the model's predictions. It is however important to note that the R-squared value never decreases with added parameters, regardless of variable significance, and does not punish model size. As such, looking at only R-squared as a means of judging the model performance is generally not a good idea and would likely lead to overfitted models. The AIC on the other hand, does punish an increase in the number of parameters and measures the quality of models by estimating the relative amount of information lost. As such, a lower AIC value typically corresponds to a higher-quality model.

To test the significance of individual variables, t-ratios are computed. These act as indicators of how statistically different the relationship between the dependent variable and the independent variable of interest is from 0. In this case, on a 5% significance level, a t-distribution critical level of 1.96 is used. In other words, any variable with an absolute t-ratio greater than 1.96 is deemed significant on a 95% confidence level.

The calibration results of the probit model is presented in table 8.1, where the explanatory variables are listed in order of significance. As can be seen in model A, just the 3 first variables (adjusted gain, pool factor and historical average prepayment rates) give an R-squared value of around 41.9%, indicating some predicting power in the model. All the included variables show a statistical significance based on the t-tests. Note that the AIC in this model variation alone is not useful as it should only be interpreted in relation to the AIC of other model variations.

When adding remaining time to maturity in model B, the MSE and AIC decrease compared to model A, whereas the R-squared increased to 45.7%. At the same time, all variables are still shown to be significant. As such, it can be concluded that model B is clearly better than model A, and model A will therefore not be further explored.

Model:	A/*t-ratio*	B/*t-ratio*	C/*t-ratio*	D/*t-ratio*
CONST	-1.5881/*-37.990*	-1.7290/*-25.594*	-1.7216/*-26.608*	-1.7260/*-16.776*
GAINxPF	23.7780/*9.640*	21.4669/*11.942*	21.6227/*36.664*	21.8112/*9.991*
POOL-FAC	-0.9284/*-6.840*	-0.8806/*-7.333*	-0.8656/*-10.829*	-0.8691/*-9.115*
HIST-AVG-REFI	-2.0542/*12.526*	-4.4054/*-2.550*	-4.5859/*-2.195*	-4.6105/*-1.647*
MATURITY		0.0110/*3.837*	0.0093/*2.770*	0.0092/*2.212*
IO			0.0744/*1.195*	0.0765/*1.030*
Q1				0.0857/*1.578*
Q2				0.0350/*0.601*
Q3				-0.0294/*-0.439*
ISSUER				-0.0407/*-0.742*
MSE	0.0024	0.0022	0.0022	0.0022
R^2	0.4191	0.4577	0.4570	0.4690
AIC	-43079	-43559	-43551	-43710

Table 8.1: Parameter estimates for four prepayment models of increasing complexity.

In model C, the "interest only" categorical variable was added. Compared to model B, this variation showed a similar MSE but marginally worse R^2 and AIC. Note that in theory, R^2 should never decrease by just adding more parameters. In this case, the decrease can be attributed to the fact that the β vector was initialised randomly, and therefore led to somewhat inconsistent convergence. With a complex loss landscape, the R^2 can therefore vary between each calibration. The decrease in R^2 should instead be interpreted as a 0 gain in predicting power. Furthermore, the t-test of the interest only variable showed that it cannot be concluded to have a significant relationship with prepayment rates. As such, model C is discarded and model B is kept.

From hereon, the variables IO, quarters and issuer were added to model B and tested in all possible combinations. Note that only Q1, Q2 and Q3 variables are added since Q4 is reflected when all the former are 0. While some of the variations resulted in improved R-squared and AIC (see for example model D), the t-tests showed that these variables were not significant. A model with these variables included would therefore likely be more prone to overfitting and have unnecessarily large model complexity, while not significantly improving out-of-sample predicting power. Furthermore, analysis of the relations between independent variables showed multicollinearity in some cases. For example, figure 8.1 shows that there is some sort of dependency between the type of loan (annuity profile or interest only) and the remaining time to maturity. The interest only loans have a significantly higher proportion of high maturities compared to the annuity loans, where the distribution is more even. This can be explained by the fact that most interest only loans have a deferred amortisation length of at least 10 years, and a remaining annuity profile after that. As such, they usually have a very long maturity, beyond dates of which target values (observed prepayment rates) are available in the data of this thesis. As a result, the model will only be calibrated to training data where the interest only loans still have a long remaining time to maturity, leading to a dependence between the IO variable and maturity variable.

Considering the multicollinearity tendencies and insignificance of additional variables, it is concluded that model B is the best model variation, and will therefore be used for the rest of this thesis.

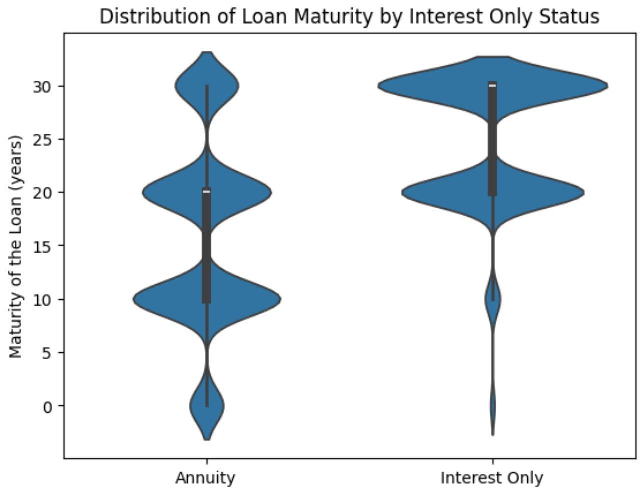


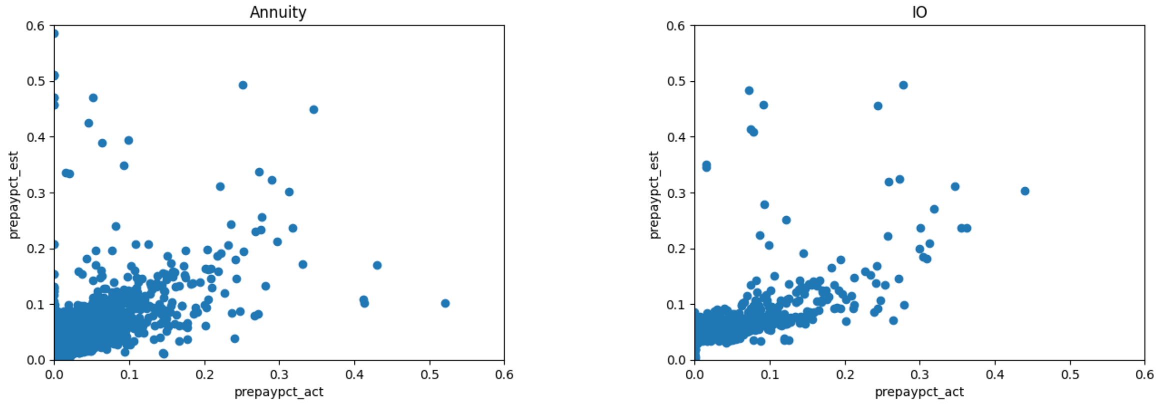
Figure 8.1: Violin plot showing distribution of remaining time to maturity for annuity type loans and interest-only type loans in training data.

8.6 Out of Sample Validation

All the prepayment data that was available for this thesis were split into a training part and an "out-of-sample" part, used for validating the model. For the model validation, model B and its respective coefficients shown in table 8.1, are used on the out-of-sample data. Note that unlike the in-sample model calibration, where the prepayment rate was calculated using bootstrapped observed swap rates, the out-of-sample uses simulated rates given from the factor HJM model. The resulting estimated prepayment rates are plotted against the observed actual prepayment rates in figure 8.2, for annuity and interest-only loans respectively. There seems to be a strong correlation between the estimated and actual prepayment rates. However, it is also obvious that systematic biases exist in the model. For both loan types, the model seems to overestimate low and underestimate high prepayment rates. The potential reasons leading to this are further explored in chapter 10.

8.7 Alternative Prepayment Models

There are of course many alternative prepayment models. In his master's thesis at the University of Copenhagen, Emil Rode used an extended Stanton model to directly construct a PDE for the evolution of the bond prices [29]. Another intuitive area



(a) Modelled prepayment rates against actual prepayment rates for annuity loans.

(b) Modelled prepayment rates against actual prepayment rates for interest-only loans.

Figure 8.2: Estimated versus actual prepayment rates on out of sample data for model B, where swap curve is simulated from factor HJM.

of model exploration is machine learning. There are many ways this could be implemented. One possible strategy is to modify the probit model presented above to use a machine learning model, such as random forest or gradient boosting, instead of linearly combining the explanatory variables. As a point of comparison, a Categorical Boosting model was implemented using the Python CatBoost library. Categorical boosting is a tree-based gradient-boosting algorithm introduced in 2018 to handle problems with previous, open-source gradient boost algorithms [30]. As the algorithm output is not limited to outputting values between 0 and 1 it was instead trained to output an input for the normal CDF function, i.e.:

$$PR(t) = \Phi(cat(x(t))) \tag{8.17}$$

where cat is the CatBoost algorithm and x the vector of features. To create a set of training targets for the model, the observed prepayment rates were converted using the inverse of the normal CDF function. With very limited hyperparameter tuning the CatBoost version achieved an in-sample R^2 of 64.5% and out-of-sample R^2 of 37.9% showing it has the potential to outperform the probit model.

9 Pricing of Danish Callable Bonds

After constructing the yield curve and prepayment models, pricing the callable bonds can be done through Monte Carlo simulation or by constructing a lattice grid, similar to [6]. Due to time constraints, the pricing model was never implemented but the theory is discussed below.

Let us start by introducing some notation similar to that used in [6]. Let $k = 1, \dots, M$ be an index of the fixed settlement days, t_k, of an MBB. $a = (a_1, \dots, a_M)$ is the vector of amortisation payments, $i = (i_1 \dots, i_M)$ the corresponding interest payments and $F_k = \sum_{j>k} a_j$ the remaining principal amount at t_k. Further, let $\kappa(t)$ be the resulting prepayment date index if the borrower decides to prepay at time t. For example, as prepayments occur every quarter, if a loan taker decides to prepay on the first of January, they will continue paying interest and amortisation as before until the end of March, when the remaining notional will be prepaid. With the introduced notation, if t corresponds to the first of January, $\kappa(t)$ would be the end of March.

Let $W_P(t)$ denote the value of an MBB if the prepayment decision is made at t, $W_N(t, h)$ the value if no prepayment decision is made in the period t to $t + h$, and $PR(t, h)$ the fraction of the underlying loans measured in notional value to decide to prepay in the period t to $t+h$. The value of the MBB at time t can then be expressed as [6]

$$W(t) = PR(t, h)W_P(t) + (1 - PR(t, h))W_N(t, h)$$

If the prepayment decision is made, the resulting cash flow becomes deterministic and the arbitrage-free price of $W_P(t)$ can thus be directly calculated as

$$W_P(t) = \sum_{k \leq \kappa(t)} \frac{M(t)}{M(t_k)}(a_k + i_k) + \frac{M(t)}{M(\kappa(t_k))} g_{\kappa(t)}$$

where $g_{\kappa(t)}$ is the outstanding amount at $\kappa(t)$. If the prepayment decision is not made, the resulting cash flows depend on future prepayment decisions and are thus not deterministic. However, at bond maturity, t_M, the cash flow will always equal the remaining notional amount, g_M. Given an appropriate model for the prepayment behaviour we can let h equal the time between prepayment dates and use the fact that

$$\begin{aligned} W_N(t, h) &= \sum_{k \leq \kappa(t)} \frac{M(t)}{M(t_k)}(a_k + i_k) + \frac{M(t)}{M(\kappa(t_k))} W(t_{\kappa(t)}) && (9.1) \\ &= \sum_{k \leq \kappa(t)} \frac{M(t)}{M(t_k)}(a_k + i_k) \\ &\quad + \frac{M(t)}{M(\kappa(t_k))} \big(PR(t_{\kappa(t)}, h)W_P(t_{\kappa(t)}) + (1 - PR(t_{\kappa(t)}, h))W_N(t_{\kappa(t)}, h)\big) && (9.2) \end{aligned}$$

to simulate the cash flow of an MBB until its maturity.

9.1 Monte Carlo Simulation for Pricing Bonds

As previously discussed, the default risk of the bonds is assumed to be negligible. This means that their returns should equal the risk-free rate under $\mathbb{Q}$ and simulating the price can be broken down into three steps.

1. Simulate future evolutions of the yield curve.
2. Estimate prepayments for each settlement day.
3. Price the bond using equation (9.2).

The first step can be done using the theory from section 5.4. Two types of rates are needed for the simulation. Swap rates used to calculate the ASW discussed in section 8.2.2 and the value of the money market account for discounting the cash flows back to the pricing date. As previously discussed, calculating the swap rates can be done using proposition 1 and the money market account using proposition 8. It should also be noted that the simulated rates are needed at different points in time. While prepayments, amortisation, and interest rate payments all happen at the end of each quarter the decision to prepay is made, at the latest, two months prior. As such, the swap rates need to be simulated for the end of the first month of each quarter while the money market account values need to be simulated for the end of each quarter.

The second step is estimating the prepayment rates for each settlement date using the prepayment model discussed in chapter 8. The third step can then be done by expanding the sum in equation (9.2) to maturity where it will hold that

$$W(\kappa(t_M)) = g_M$$

as the remaining outstanding amount will be paid. By combining multiple simulations, the distribution of the price of the bond can be estimated.

9.2 Lattice Grids for Pricing Bonds

The implementation of a lattice grid for pricing the bonds is done by estimating and discounting the cash flows on a set grid instead of generating random paths. At each grid point, pre-defined state evolutions are applied to reach the following points. This method is easy to implement for a one-factor short-rate model as each step could be given by a pre-defined value of the driving Brownian motion. Figure 9.1 shows a graphical representation of a trinomial lattice grid, where the downward-sloping arrows represent a decrease, the upward-sloping arrows represent an increase, and the straight arrows represent no change in the short rate. The prepayment model can then be used to calculate the prepayment at each point until maturity. After building the grid, pricing the bond can be done by back-propagating through the grid, at each point calculating the bond price by taking the discounted weighted average of connected points at the next time point [6].

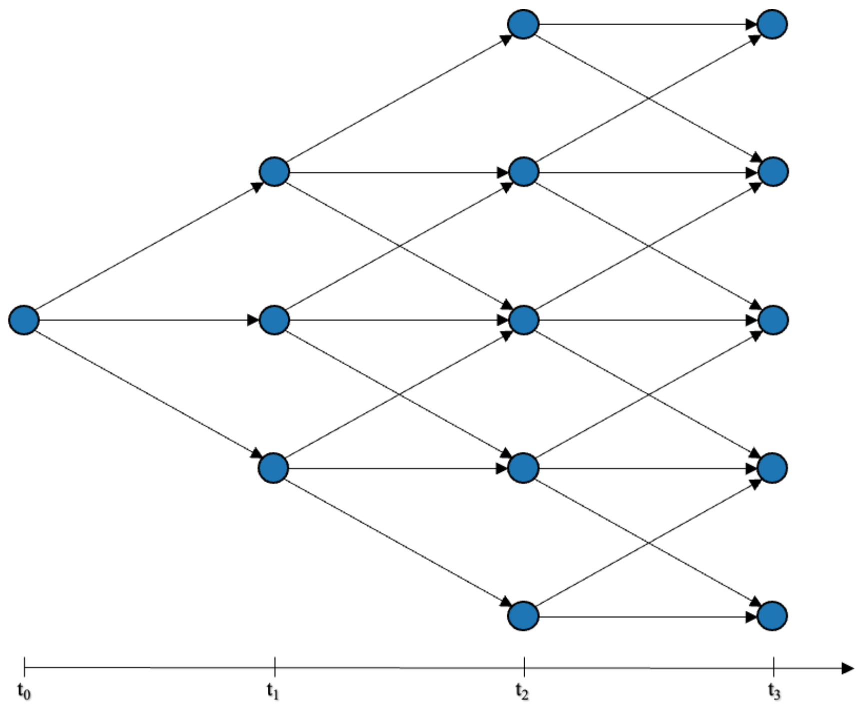


Figure 9.1: Diagram of a self recombining lattice grid of the sort used to price MBBs in [6].

For a multifactor model, such as the FHJM model, using a lattice grid becomes significantly more complicated. To capture the evolution of the yield curve, combinations of the possible evolutions of the Brownian motions would have to be defined for each point in the grid. For the 6-dimensional model used in this thesis, defining three different evolutions of each Brownian motion would lead to $3^6 = 729$ state changes at each point in the grid. As the bonds have tenors of up to 30-years and the grid would need to be evaluated for every quarter, the number of permutations makes the grid computationally unfeasible.

9.3 The Delivery Option

Apart from prepaying, the borrower also has the option to buy back the underlying bond at the current market price [4]. This is called the *delivery option* and complicates the prepayment and cash flow simulation by altering the total cost of prepaying from (8.1) to

$$\tilde{W}_P(t,\gamma) = \max((1+\gamma)W_P(t), W(t)) \tag{9.3}$$

This introduces two problems: the refinancing gain used in the prepayment model changes, and the bond value in the case of prepayment changes. Solving the first problem is challenging because the estimated prepayment rate is needed to calculate the bond price, creating a feedback loop if the bond price is used to estimate the prepayment rate. However, the second problem is relatively easy to solve when using a lattice grid. Since valuation is done through back-propagation, the bond value is known when evaluating each settlement date, allowing (9.3) to be used to calculate the true value of the prepayments.

A similar strategy can be applied to Monte Carlo simulations by first simulating the prepayment rates until maturity and then back-propagating through each simulation. This, however, increases computational complexity, as it necessitates looping through each settlement date in the simulation instead of using vector multiplication.

Overall, the delivery option is assumed to have a relatively limited effect on market prices and behaviour. Mortgage bonds typically trade below par if their interest rates are lower than overall market rates, which in turn implies that prepayment rates should be low. Since the delivery option only comes into play if a prepayment decision is made, exercising the option should be rare.

9.4 Option Adjusted Spread

The modelled prices of bonds with embedded options, like MBBs, typically do not match observed market prices. Even though prepayments are included in the pricing model introduced in this thesis, investors will typically require extra compensation for the increased volatility the option creates. The difference between the modelled and the observed price is called the *Option Adjusted Spread* (OAS). There is no clearly defined way to calculate or model the OAS, differences in model and observed prices could stem from market or model errors just as much as risk premiums [31]. Considering the uncertainty in the prepayment and the overall complexity of the MBB market, it is reasonable to assume the OAS of the bonds will be relatively big.

10 Discussion

10.1 FHJM-model

The results in section 7.3 are promising, indicating the FHJM-model specification can calibrate well to market volatility and rates. There are, however, many possible areas of exploration and improvement. As previously discussed, the calibration could most likely be improved by using Black or Bachilier volatilities rather than swaption premiums. This could increase performance for shorter tenor swaptions with lower premiums. On the other hand, as most MBB contracts have long tenors, up to 30 years, deviating volatility for short tenors might not harm performance for this particular task.

To fully evaluate the model performance, further testing is also needed. Most importantly, $\mathbf{\Sigma}$ should be calibrated to different points in time with different rates and volatility. The model showing good performance for the term structure at the beginning of 2019 does not necessarily mean it will perform well for other periods. Especially those with high and less predictable volatility like the 2008 financial crisis or the stock market crash at the beginning of the COVID-19 pandemic.

There are also other specifications of the FHJM model that could be used. First of all, there are many possible specifications of the basis functions. As discussed by both Lyashenko and Goncharov [3] and Sepp and Rakhmonov [20], the Nelson Siegel basis could, for example, be a suitable choice due to its vast use and testing in $\mathbb{P}$ modelling of the yield curve. As pointed out by Lyashenko and Goncharov however, the choice of basis should have a limited effect on performance assuming it is complex enough to capture the full family of possible yield curves.

A more relevant area of exploration would thus be the volatility specification. As previously discussed, the FHJM framework allows for time-varying and even stochastic volatility processes. While these specifications potentially increase flexibility and performance, they typically come with the drawback of added model complexity, both analytically and computationally. Even the comparatively simple specification used in this thesis has implementation challenges and a relatively high level of complexity compared to many short-rate models used in practice. Considering this, the loss of analytical tractability that comes with more advanced volatility structures might be reason enough for many industry practitioners to stick with constant volatility.

One possible middle-of-the-road solution, increasing flexibility while retaining analytical tractability is adding a piece wise constant scaling factor to the volatility process. The model specification of $X(t)$ would then be

$$dX(t) = DX(t) + \alpha(t)\mathbf{\Sigma}dW^{\mathbb{Q}}(t) \tag{10.1}$$

where $\alpha(t)$ is a piece wise constant one-dimensional process. This specification would also result in $\Omega(t)$ being time-varying as

$$\tilde{B}(\tau)\Omega(t) = B(\tau)\alpha^2(t)\mathbf{\Sigma}\mathbf{\Sigma}^* \left(\int_0^\tau B(u)du\right)^* . \tag{10.2}$$

To avoid specifications of the volatility with either scaled $\alpha(t)$ or scaled $\mathbf{\Sigma}$ values being equivalent, one of the values in $\mathbf{\Sigma}$ could be set to one, essentially taking on the value of $\alpha(t)$. The underlying assumption of this version of the model is that the volatility relationship between tenors remains constant over time but that the overall volatility level changes. This specification arguably enhances the model's interpretability. After calibration to swaptions, the evolution of $\alpha(t)$ can be interpreted as the market's expectation of how volatility will change over time. With that said, simply comparing Black volatility for different tenors is a significantly simpler way to achieve the same result.

10.2 Prepayment Model

As currently constituted, the prepayment model is most likely the bottleneck in the pricing performance. While the model clearly has some predicting power, as discussed in section 8.6 the model has a clear bias overestimating low and underestimating high prepayment rates. There is also a significant variance in the model with points with similar actual prepayment rates receiving a significant spread of predictions. There are many possible reasons for this. As previously discussed, it is hard to determine if the β-variables are optimal. While the consistency in optimisation convergence regardless of initiation, carries some indication of a global optimum it is very possible that a finer grid search or another optimisation method all together could provide a better result.

There are many other predictive variables that could be explored. These include loan-specific variables available to banks and financial institutions that were not accessible for this thesis. Examples include bond-specific tax rates, service fees, and more detailed borrower pools, among others. Additionally, macroeconomic variables could be considered. It is reasonable to assume that borrowers not only take the current refinancing variables into account when deciding to prepay but also the financial outlook. If they, for example, believe the rates will drop further in the coming quarter it could make sense to wait with refinancing as doing it twice would result in additional service fees. The general economic outlook could also have an effect. If, for example, the cost of living or unemployment is high individuals might be more prone to look for potential savings. Some macroeconomic variables that could be added to the model are; unemployment rate, central bank rates and predicted rates, a variety of housing market indicators, consumer confidence and spending, and inflation.

As discussed in section 8.7 it is also possible to use a different prepayment model. The fact that the categorical boosting model achieved impressive performance compared to the probit model with very limited features and hyperparameter tuning implies that machine learning models might provide a significant performance improvement. This potential also grows when considering the extended variable space discussed above. The prepayment rates connection to, for example, macroeconomic factors might not be as direct as the one to the refinancing gain. As such, a non-linear machine learning model might be better than the probit model at adapting to the more complex dependence structure. There are, however, some downsides to using these models. Most importantly they come with a reduced level of explainability and an increased risk of over fitting. Simpler models with lower bias, like tree-based ensemble methods, should therefore be primarily considered.

It is also possible to take a different approach to the prepayment modelling all together. Instead of creating a prepayment model that is then used to price the bond, the bond prices can either be modelled directly or the market's expected prepayment rates can be inferred from the prices of the bonds. This approach can be likened to calibrating the volatility process of a yield curve model to swaption data, adapting to market expectations rather than historically observed values. The main arguments for this approach stem from the uncertainty in the prepayment model and the necessity of adding an option-adjusted spread to the pricing model discussed above. Calibrating directly to market prices would ensure that the model at least accurately prices the bonds at the point of calibration. With that said the approach comes with some significant downside. Compared to the previously discussed models it reduces the intuitiveness and explainability. The model constructed in chapter 8 tries to predict actual prepayment rates using logical variables. The connection between market prices and prepayments is much more subdued. Calibrating to market prices means calibrating not only to pure data but also to other market actor's pricing and prepayment models. If all market participants where to use this approach the pricing models would become self-fulfilling prophesies without much connection to actual cash flow expectations. While this argument holds for calibration to swaptions as well, they are significantly simpler contracts meaning irregularities are easier to spot.

11 Conclusion

This thesis has explored the complexities of modelling yield curves and pricing Danish callable mortgage bonds, focusing on bridging the divide between real-world and pricing modelling. By implementing and calibrating a factor Heath-Jarrow-Morton (HJM) model, we aimed to create a framework that accounts for both no-arbitrage pricing and real-world elements. The implementation in Python, along with the calibration to forward rates and swaption premiums enabled us to simulate yield curves and subsequently calculate refinance gains. The implemented FHJM model seems to calibrate well to market volatility and rates.

Through the use of a probit model, the output from the FHJM model were used to estimate prepayment rates. When plotting the resulting rates from the best model variation, against actual historical prepayment rates, it can be seen that the model indeed has some predicting power. The best model (model B), had an R^2 value of around 0.45, meaning approximately 45% of the variability in the prepayment rates can be explained by the independent variables included in model B. As mentioned before however, there also exist systematic biases and significant areas of improvement in the model. It is not certain whether the calibrated β vector was indeed a global optimum, or whether other optimisation methods and more rigorous testing would have proven otherwise. Furthermore, some simplifying assumptions about variables like tax and administration margin were made, which could have negatively impacted the results. To further improve the predicting power of the model, more detailed data about each specific bond and debtor group, as well as macroeconomic factors, could be explored.

While the factor HJM model seems to calibrate well to the market, it is currently unlikely that it will be used for things like pricing callable mortgage bonds. A model is only as good as its worst part, and having a very complex interest rate model is therefore unlikely to make a significant impact on the results when the prepayment model is kept rather simple. Furthermore, practitioners might not feel the need to bridge the divide between financial modelling in real world and pricing applications, since it is more of a theoretical issue and is often overseen. In theory, however, this should be a step in the right direction of keeping risk neutral pricing models more grounded in the real world.

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Appendix A

UML-Diagram for FHJM Model Implementation

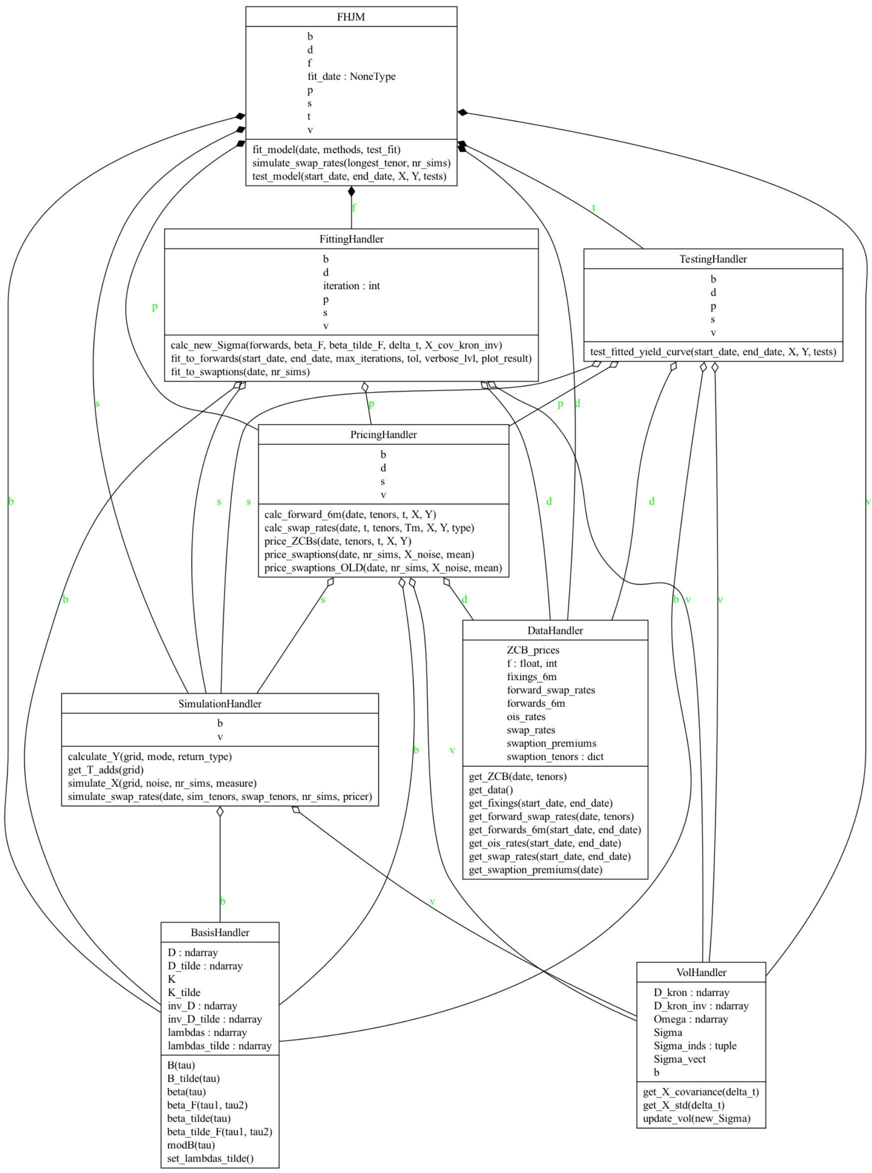

Figure A.1: UML-diagram of the implementation of the FHJM-model

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