May 2024

Isospin breaking at two loops in three-flavour Chiral Perturbation Theory

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Master thesis (60 hp) supervised by Johan Bijnens

Abstract

In this thesis, we first give an introduction to the framework of Chiral Perturbation Theory. Following this, we proceed to conduct a detailed two-loop order calculation within Chiral Perturbation Theory, aimed at analyzing pseudoscalar meson masses and decay constants while incorporating isospin breaking effects. Our calculations account for both quark mass and charge isospin breaking effects in determining the masses, while focusing solely on quark mass correction for the decay constants. Throughout this work, we provide explicit expressions for both the masses and decay constants.

Populärvetenskaplig beskrivning

In the vast landscape of quantum field theories, such as the electroweak theory and quantum chromodynamics (QCD), lies a fundamental question: how do we effectively describe phenomena at lower energy scales when the true nature of high-energy physics remains elusive? This puzzle is elegantly addressed by Effective Field Theories (EFT), which sift through the complexities of high-energy interactions to focus on the essential physics at play in the low-energy realm.

Think of EFT as a skilled cartographer simplifying a map: just as cities are represented by dots, EFT distills the behavior of particles and forces to their most relevant aspects. This approach allows physicists to zoom in on specific features, separating the essential from the extraneous much like examining a map in detail.

Take, for instance but also what we investigate in this thesis, the realm of QCD, where the coupling constant decreases as energy scale increases. This characteristic enables the use of perturbation theory at high energies. However, in the low-energy domain, where the coupling constant becomes much bigger than one, perturbation theory fails to suffice. The non-perturbative nature of QCD in the low-energy region is actually not difficult to understand, here, the degrees of freedom shift from quarks and gluons to hadrons, similar to focusing on cities rather than the intricate details of their architecture.

The low-energy behavior of QCD can be understood through the lens of Chiral Perturbation Theory (ChPT), an effective field theory specifically tailored to describe the interactions of hadrons. ChPT is constructed such that its shares the same approximate chiral symmetry as QCD, just as the spatial arrangement of cities is preserved in the distribution of points on a map.

As technological advancements propel experiments probing low-energy QCD to unprecedented precision, the need for equally precise theoretical calculations intensifies. Despite past research yielding invaluable insights, there are still some uncertainties that persist, particularly the isospin breaking effects, which stem from differences in masses and charges between u and d quarks. In this work, we've incorporated these effects and corrected the calculations for the masses and decay rates of particles, thereby establishing formulas to predict these observable properties more accurately.

Contents

Acronyms

$\overline{MS}$ Modified Minimal Subtraction.

1PI One-Particle-Irreducible.

ChPT Chiral Perturbation Theory.

EFT Effective Field Theory.

EM Electromagnetic.

LECs Low-Energy Constants.

LO Leading Order.

LSZ Lehmann–Symanzik–Zimmermann.

NGB Nambu-Goldstone Boson.

NLO Next-to-Leading Order.

NNLO Next-to-Next-to-Leading Order.

pNGB pseudo-Nambu-Goldstone Boson.

QCD Quantum Chromodynamics.

SSB Spontaneous Symmetry Breaking.

1 Introduction

Being a small branch of the whole physics world, particle physics holds the honor of being the most fundamental part by studying the fundamental constituents of matter and the forces acting upon them at the smallest scales imaginable.

In this thesis, we mainly focus on an essential component of particle physics, mesons. Mesons are composite particles, consisting of a quark and an anti-quark, or in their superposition, bound together by the strong nuclear force (as illustrated in Fig. 1).

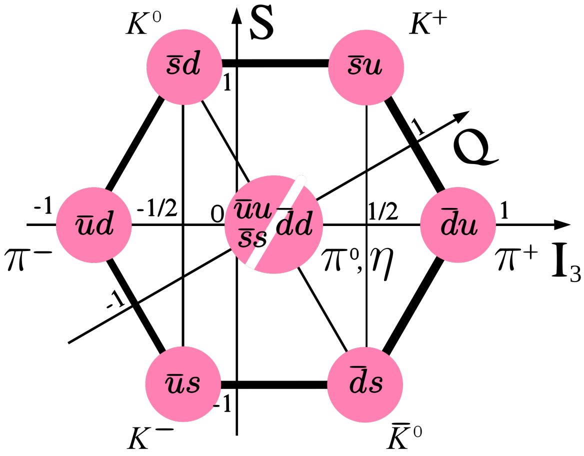


Figure 1: The meson octet, arranged by isospin I_3, strangeness S, and electric charge Q.

The theory that describes this strong nuclear force is Quantum Chromodynamics (QCD), whose coupling constant g has a special property, asymptotic freedom. The renormalization group equation for g demonstrates that g increases as the scale decreases. This property validates the use of perturbation theory in the high-energy regime, but also leads to the fact that the theory is completely non-perturbative in the low energy regime.

The non-perturbative nature of QCD in the low-energy region is not difficult to understand, because the physical degrees of freedom in this part are not quarks and gluons but hadrons. Therefore, a low-energy effective field theory named Chiral Perturbation Theory (ChPT) has been developed to describe hadrons.

Recently, as the precision of low-energy QCD experiments continues to advance, corresponding higher order calculations are needed to enhance the accuracy of our theoretical prediction for the measurements. In the previous research, observables (e.g., mesons' mass and decay constants) have already been calculated up to two-loop order in ChPT [1, 2, 3].

Nevertheless, the significance of isospin breaking effects remained a notable uncertainty. Early studies incorporated quark mass difference corrections up to two-loop order [4] and Electromagnetic (EM) corrections up to one-loop order [5, 6]. This thesis extends these studies by evaluating both quark mass and EM isospin breaking effects in meson masses and quark mass corrections in decay constants to Next-to-Next-to-Leading Order (NNLO).

The organization of this thesis is as follows: Sec. 2 revisits the prerequisite concepts of ChPT, including symmetry breaking and the Goldstone theorem. Sec. 3 introduces the underlying principles of effective theory and the fundamental theory of ChPT, QCD. The ChPT Lagrangian is constructed then in the fourth section. In Sec. 5, we discuss the theoretical expressions for phenomena such as mass and decay constants in the isospin symmetric limit, and Sec. 6 details the corrections due to isospin breaking effects. Sec. 7 provides a summary of the results, and Sec. 8 concludes the thesis, while the Appendices contain the explicit results and a list of functions employed in the analysis.

2 Symmetry Breaking and the Goldstone Theorem

In this section, we will briefly review Spontaneous Symmetry Breaking (SSB), Goldstone's theorem and explicit symmetry breaking, Our derivations roughly follow Refs. [7, 8, 9, 10].

2.1 Spontaneous Symmetry Breaking in the Linear Sigma Model

Consider a complex scalar field ϕ with Lagrangian

$$\mathscr{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) + m^2 \phi\phi^\dagger - \frac{\lambda}{4}\phi^2\phi^{\dagger 2}. \tag{2.1}$$

This Lagrangian is invariant under the global $U(1)$ transformation $\phi(x) \to \mathrm{e}^{i\alpha}\phi(x)$ for constant α. For $m^2 > 0$ the theory is unstable around $\phi = 0$. We can obtain the vacuum by simply minimizing the potential

$$V_0(\phi) = -m^2|\phi|^2 + \frac{\lambda}{4}|\phi|^4, \tag{2.2}$$

this is satisfied when

$$|\phi|^2 = \frac{2m^2}{\lambda}. \tag{2.3}$$

So now there are an infinite number of equivalent vacua $|\Omega_\theta\rangle$ with vacuum expectation value:

$$\langle\Omega_\theta|\,\phi\,|\Omega_\theta\rangle = \sqrt{\frac{2m^2}{\lambda}}\mathrm{e}^{i\theta} \tag{2.4}$$

for any constant θ. Since all the vacua are equivalent (by symmetry), we can pick any convenient parameterization. One common choice is the $|\Omega\rangle$ that makes $\langle\Omega|\,\phi\,|\Omega\rangle$ real:

$$\langle\Omega|\,\phi\,|\Omega\rangle = v = \sqrt{\frac{2m^2}{\lambda}}. \tag{2.5}$$

Instead of writing $\phi(x) = v + \widetilde{\phi}(x)$, with $\widetilde{\phi}(x)$ a complex field as in many textbooks, it is often more convenient to expand around v by parameterizing $\phi(x)$ in terms of two fields $\sigma(x)$ and $\pi(x)$ as

$$\phi(x) = \left(v + \frac{1}{\sqrt{2}}\sigma(x)\right) \mathrm{e}^{i\frac{\pi(x)}{F_\pi}}, \tag{2.6}$$

with F_π a constant and has the same dimension as π, just as we can write a vector in Cartesian or polar coordinates. Then the $V(x)$ only depend on $\sigma(x)$, and not on $\pi(x)$. Plug our new fields into the Lagrangian,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \left(v + \frac{1}{\sqrt{2}}\sigma(x)\right)^2 \frac{(\partial_\mu\pi)^2}{F_\pi^2} - \left(-\frac{m^4}{\lambda} + m^2\sigma^2 + \frac{1}{2}\sqrt{\lambda}m\sigma^3 + \frac{1}{16}\lambda\sigma^4\right). \tag{2.7}$$

By choosing $F_\pi = \sqrt{2}v$ we can make the π kinetic term canonically normalized. This theory is called a linear sigma model.

Upon inspection of the terms quadratic in the fields, one finds after spontaneous symmetry breaking one massless boson and one massive boson:

$$\begin{aligned} m_\sigma^2 &= 2m^2, \\ m_\pi^2 &= 0, \end{aligned} \tag{2.8}$$

the massless boson is often referred to as the Nambu-Goldstone Boson (NGB) and corresponds to the broken symmetry ($U(1)$ in this case).

2.2 Goldstone's Theorem

We will now present a comprehensive overview and proof of the Goldstone theorem, as referenced in Zee [8], which was exemplified in the preceding subsection. The theorem states that the spontaneous breaking of a continuous symmetry within a theory yields a massless boson for each generator of the symmetry that has been broken. The proof is as follows.

From Noether's theorem, we know that every continuous symmetry is associated with a conserved charge Q, which commutes with the Hamiltonian

$$[H, Q] = 0. \tag{2.9}$$

Denote the ground state by $|\Omega\rangle$, by adding an appropriate constant to the Hamiltonian $H \to H + c$, we can always have

$$H|\Omega\rangle = 0. \tag{2.10}$$

When the symmetry is still valid, the ground state is also invariant under the symmetry transformation, $\mathrm{e}^{i\theta Q}|\Omega\rangle = |\Omega\rangle$, or in other words, $Q|\Omega\rangle = 0$.

However, after the symmetry is spontaneously broken, which means that we have chosen a particular ground state, the vacuum is not invariant under the symmetry transformation, $e^{i\theta Q} |\Omega\rangle \neq |\Omega\rangle$, in other words, $Q |\Omega\rangle \neq 0$.

Consider the state $Q |\Omega\rangle$ in spontaneously broken symmetry case, the eigenenergy of this state is

$$HQ |\Omega\rangle = -QH |\Omega\rangle + [H, Q] |\Omega\rangle = 0, \tag{2.11}$$

the second equality follows from Eq. 2.9 and Eq. 2.10. Thus, we have found another state $Q |\Omega\rangle$ with the same energy as $|\Omega\rangle$.

In quantum field theory, the conserved charge Q is associated with a local current, that is

$$Q = \int \mathrm{d}^d x J^0(\vec{x}, t) \tag{2.12}$$

where d denotes the dimension of space and the conservation of Q indicates that the integral can be evaluated at any time. Consider the state

$$|s\rangle = \int \mathrm{d}^d x \mathrm{e}^{-i\vec{k}\cdot\vec{x}} J^0(\vec{x}, t) |\Omega\rangle \tag{2.13}$$

which has spatial momentum $\vec{k}$ since

$$P^i |s\rangle = \int \mathrm{d}^D x \mathrm{e}^{-i\vec{k}\cdot\vec{x}} (-J^0(\vec{x}, t) P^i + [P^i, J^0(\vec{x}, t)]) |\Omega\rangle = k^i |\Omega\rangle \tag{2.14}$$

where in the second equality, we use the fact that $P^i |\Omega\rangle = 0$, $[P^\mu, \phi(x)] = -i\partial^\mu \phi(x)$ and then integrating by parts. As the momentum $\vec{k}$ tends to zero, $|s\rangle$ goes over to $Q |\Omega\rangle$, which has zero energy. This implies that as the momentum of the state $|s\rangle$ approaches zero, its energy goes to zero. Referring to the relativistic dispersion relation

$$E^2 = \vec{p}^2 + m^2, \tag{2.15}$$

this indicates that $|s\rangle$ describes a massless particle. Thus, a spontaneously broken symmetry generator Q yields a massless scalar $|s\rangle$, which is also called the Goldstone state.

Since quantum field theory is a Lorentz invariant theory, we can always perform a Lorentz boost $J^0 \to J^\mu$. Therefore, the matrix element of the current between the vacuum and the Goldstone state $|s\rangle$ is nonzero, $\langle\Omega| J^\mu(x) |s\rangle \neq 0$. Assuming the four-momentum of the state $|s\rangle$ is p_s^μ, Lorentz invariance require the matrix element to take the form [7]

$$\langle\Omega| J^\mu(x) |s\rangle = i\sqrt{2} F p_s^\mu \mathrm{e}^{ip_s\cdot x}, \tag{2.16}$$

here $\sqrt{2}$ is introduced due to convention and F is a constant coefficient with the dimensions of energy which reflects the "strength" of symmetry breaking, which is often referred to as "decay" constant since it is proportional to the amplitude between vacuum and a one-NGB state.

As demonstrated, each NGB corresponds to a conserved charge that does not leave the vacuum invariant. That is, for each broken charge Q^a, we can construct a Goldstone state with the ground state $Q^a |\Omega\rangle$. In the specific example discussed in Sec. 2.1, the $U(1)$ symmetry is associated with a single charge $Q = \alpha$ that alters the vacuum, resulting in the emergence of only one NGB. In general, if the Lagrangian is left invariant by a symmetry group G with $n(G)$ generators, but the ground state only remains invariant under a subgroup H of G with $n(H)$ generators, then there are $n(G) - n(H)$ ($\dim(G/H)$) NGBs, in other words, the NGBs live in the quotient G/H .

To illustrate this statement, consider a general symmetry breaking $G \to H$, we denote the vacuum which is invariant under H by $\vec{V}$. By Goldstone's theorem, this will give rise to $n(G) - n(H)$ NGBs, which we label $\vec{\phi} = (\phi_1, \dots \phi_i)$. Since we are only interested in these modes, i.e., in the low energy region, the fluctuations of the massive fields can be neglected[1]. In this limit, recalling that before a specific choice of vacuum was made, the entire manifold was symmetric under G, and since we have ignore the massive components in $\vec{V}$, the Goldstone field $\vec{\phi}$ can be parameterized in terms of a matrix representation of G, $R(g)$, acting on $\vec{V}$:

$$\phi_i = R_{ij}(g)V_j = R_{ij}(g)R_{jk}(h)V_k = R_{ij}(gh)V_j, \quad g \in G, h \in H, \tag{2.17}$$

where in the second equality I used the fact that $\vec{V}$ is invariant under H. This shows that $\vec{\phi}$ is a function of the elements of the coset space $gH \equiv \{gh|h \in H\}$ or we can say that ϕ_i live in the quotient G/H. This association is unique due to the property of cosets, namely that non-identical cosets are disjoint.

2.3 Explicit Symmetry Breaking

We will now consider the effect of adding small symmetry-breaking terms to the action which explicitly breaks the symmetry. As we shall see, the spontaneous breakdown of an approximate symmetry does not result in the emergence of massless NGBs but rather gives rise to low-mass spinless particles, commonly referred to as pseudo-Nambu-Goldstone Boson (pNGB).

Consider adding a small perturbation to our Lagrangian of Eq. 2.1 and modify the potential, here we parameterize the field by $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ for better illustration:

$$V(\phi) = V_0(\phi) + V_1(\phi) = -\frac{m^2}{2}\phi_i\phi_i + \frac{\lambda}{16}(\phi_i\phi_i)^2 + a\phi_2, \quad i = 1, 2 \tag{2.18}$$

where $V_1(\phi) = a\phi_2$ is a small correction and breaks the $U(1)$ symmetry of the Lagrangian. The conditions for the new minimum of the potential read

$$\phi_1 = 0, \quad \frac{\lambda}{4}\phi_2^3 - m^2\phi_2 + a = 0, \tag{2.19}$$

[1] I will illustrate it in detail in Sec. 3.1

where we have chosen a particular ground state. Since we introduce a small correction, we can solve this cubic equation for ϕ_2 using perturbative ansatz

$$\phi_2 = \phi_2^{(0)} + a\phi_2^{(1)} + \mathcal{O}(a^2). \tag{2.20}$$

The solution reads

$$\phi_2^{(0)} = \pm\sqrt{\frac{4m^2}{\lambda}}, \quad \phi_2^{(1)} = -\frac{1}{2m^2}, \tag{2.21}$$

the condition for a minimum excludes $\phi_2^{(0)} = -\sqrt{\frac{4m^2}{\lambda}}$. Expanding the potential with $\phi(x) = \sqrt{\frac{4m^2}{\lambda}} - \frac{a}{2m^2} + \widetilde{\phi}(x)$ we obtain, for the masses

$$\begin{aligned} m_{\phi_1}^2 &= a\sqrt{\frac{\lambda}{4m^2}}, \\ m_{\phi_2}^2 &= 2m^2 + 3a\sqrt{\frac{\lambda}{4m^2}}. \end{aligned} \tag{2.22}$$

An important characteristic observed is that the original NGBs, denoted as ϕ_1, now acquire mass, with the squared mass being proportional to the symmetry breaking parameter a. Such particles are identified as pNGBs. As will be demonstrated in Sec. 4, the eight mesons depicted in Fig. 1 are exactly examples of such pNGBs.

3 Effective Theory and Chiral Symmetry of QCD

3.1 Effective Field Theory

Before delving into ChPT, it is crucial to outline the principal characteristics of the Effective Field Theory (EFT) approach. EFT is fundamentally designed to provide a low-energy approximation of a physical system, capturing essential features while excluding the complexities of high-energy processes. This methodology enables precise predictions within a specific energy regime, while avoiding unnecessary complications from the irrelevant degrees of freedom.

The construction of an EFT is tailored to the physical problem under investigation. Often, in scenarios where the EFT serves as a weakly coupled, low-energy approximation of a more comprehensive ultraviolet theory, it suffices to consider only the light fields. However, the identification of relevant degrees of freedom can sometimes pose challenges, as illustrated by ChPT in Section 4.

A pivotal aspect of utilizing EFT is ensuring that observables retain a consistent correlation with those from the fundamental theory. This coherence is achieved by developing, as outlined by S. Weinberg, the most general possible Lagrangian that is consistent with the symmetries of the underlying theory [11].

However, given that we are employing the most general Lagrangian, it introduces potentially an infinite number of terms, complicating the computation of physical observables. To avoid having an infinite number of contributions to physical observable, EFT imposes two key constraints: first, it demands only finite precision for the results; second, it confines itself to a specified energy domain. The EFT is then used to calculate physical quantities in terms of an expansion in p/Λ, where p stands for energy, momenta or masses that are smaller than the scale Λ related to the underlying theory. Therefore, for a specified accuracy (such as to a certain order), we only need to compute a finite number of terms tailored to the precision required.

Take, for example, our Lagrangian (Eq. 2.7) in Sec. 2.1, if we consider a process that occurs at energies significantly below the σ mass m_σ, i.e., $q \ll m_\sigma$, then σ clearly cannot appear in the initial or final state but only in the internal lines (propagator) which are experimentally un-observable; the propagator of σ, $\Delta = i/(q^2 - m_\sigma^2)$ will be replaced by the lowest-order expansion in the small quantity q/m_σ, namely

$$\frac{i}{q^2 - m_\sigma^2} \to -\frac{i}{m_\sigma^2} + \mathcal{O}(m_\sigma^{-4}). \tag{3.1}$$

In this way, we end up with a theory that only involves the π field, given that we can treat the σ propagator as a constant, and the original effect of the σ will be replaced by additional modes of π self-interactions whose couplings involve factors of $1/m_\sigma^2$. The effective Lagrangian describing only π in the LO is,

$$\mathscr{L}_{eff} = \frac{1}{2}\partial_\mu \pi \partial^\mu \pi + \frac{1}{4v^2 m_\sigma^2}(\partial_\mu \pi \partial^\mu \pi)^2 + \mathcal{O}(m_\sigma^{-4}) \tag{3.2}$$

To verify this, consider the process $\pi(p_1) + \pi(p_2) \to \pi(p_3) + \pi(p_4)$. In the original theory, the scattering amplitude is determined by three Feynman diagrams (Fig. 2a, 2b and 2c) in the tree level and can be directly read out from the Lagrangian (Eq. 2.7):

$$\mathscr{M}_{fund} = -\frac{2}{v^2}\left(\frac{i(p_1 \cdot p_2)(p_3 \cdot p_4)}{s - m_\sigma^2} + \frac{i(p_1 \cdot p_3)(p_2 \cdot p_4)}{t - m_\sigma^2} + \frac{i(p_1 \cdot p_4)(p_2 \cdot p_3)}{u - m_\sigma^2}\right), \tag{3.3}$$

where $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$ are Mandelstam variables. Since we have restrict ourselves to very low energies, such that $\{s, |t|, |u|\} \ll m_\sigma^2$, the amplitude can then be approximated as

$$\mathscr{M}_{fund} = \frac{2i}{v^2 m_\sigma^2}[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)] + \mathcal{O}(m_\sigma^{-4}). \tag{3.4}$$

In the effective theory, we only need to consider one diagram (Fig. 2d), thus the corresponding amplitude is

$$\begin{aligned}\mathscr{M}_{eff} &= 2 \times 4 \times \frac{i}{4v^2 m_\sigma^2}[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)] \\ &= \frac{2i}{v^2 m_\sigma^2}[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)],\end{aligned} \tag{3.5}$$

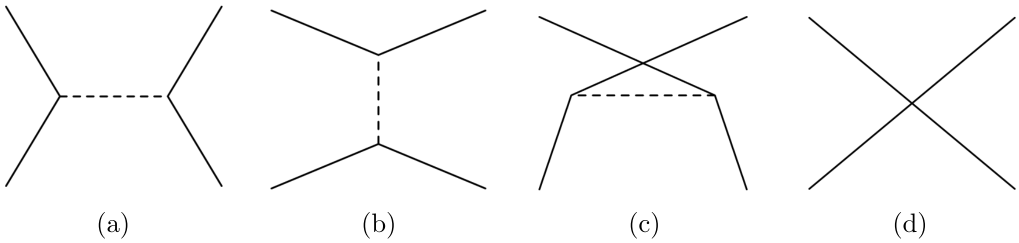


Figure 2: (**a**)(**b**)(**c**). Diagrams contributing to $\pi(p_1) + \pi(p_2) \to \pi(p_3) + \pi(p_4)$ in the fundamental theory, plain line and dash line denote π field and σ field, respectively. (**d**). Diagram contributing to $\pi(p_1) + \pi(p_2) \to \pi(p_3) + \pi(p_4)$ in the effective theory, plain line denotes π field.

which exactly reproduces the leading-order contribution in the fundamental theory.

If we want to distingush our NGB π in this theory from the massive field σ, we can take the limit that $m_\sigma \to \infty$ and $\lambda \to \infty$, but keeping $F_\pi = \frac{2m}{\sqrt{\lambda}}$ fixed to decouple the σ field. Then the effective Lagrangian (Eq. 3.2) reduces to

$$\mathscr{L}_{eff} = \frac{1}{2}\partial_\mu \pi \partial^\mu \pi, \tag{3.6}$$

which is a theory of a free π. This Lagrangian is an example of a nonlinear sigma model, which is the linear sigma model in which the σ field has been decoupled [9].

3.2 Power-Counting Scheme

To practically apply Weinberg's theorem on constructing EFT, one must first establish a scheme for organizing the effective Lagrangian. This should be followed by a systematic approach to assess the significance of diagrams generated by the interaction terms of this Lagrangian when calculating a physical matrix element.

In general, an effective Lagrangian can be organized as a string of terms[2] with an increasing power of momentum p,

$$\mathscr{L}_{eff} = \mathscr{L}^{(1)} + \mathscr{L}^{(2)} + \mathscr{L}^{(3)} + \cdots, \tag{3.7}$$

where the superscript refer to the order of the terms. To illustrate this, consider the effective Lagrangian (Eq. 3.2). Here, the superscript (1) denotes the terms like $\partial_\mu \pi \partial^\mu \pi$, that have the Leading Order (LO), $\mathcal{O}(p^2)$ of momentum. Correspondingly, the superscript (2), represents the term, such as $(\partial_\mu \pi \partial^\mu \pi)^2$, that have the Next-to-Leading Order (NLO) dimension, $\mathcal{O}(p^4)$.

[2]The term we are talking about here does not include the coefficients in the Lagrangian

Now that we have derived the form of the most general Lagrangian, we need a methodology to identify which Feynman diagrams contribute to a specific order in a given process. To that end, we analyze each diagram under a simultaneous re-scaling of all external momenta, $q_i \to tq_i$, and the masses, $m_i \to tm_i$. As we will show below, this results in an overall re-scaling of the amplitude $\mathscr{M}$ of a given diagram,

$$\mathscr{M}(tp_i, tm_i) = t^D \mathscr{M}(p_i, m_i), \tag{3.8}$$

in which the dimension D is determined by

$$D = nN_L + \sum_i N_i d_i - \sum_f 2I_f, \tag{3.9}$$

where n is the number of space-time dimensions, N_L the number of independent loops, N_i the number of vertices of interaction type i with d_i derivatives or masses and I_f the number of internal lines of field f (Here we only consider the scalar field).

To verify this power-counting formula, we can first note that the propagator $\Delta_f(k, m)$ of a field of type f in the form

$$\Delta_f(k, m) \sim \frac{i}{k^2 - m^2}. \tag{3.10}$$

Internal lines are described by a propagator in n dimensions which under re-scaling behaves as

$$\begin{aligned} \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{i}{k^2 - m^2} &\to \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{i}{t^2((k/t))^2 - m^2)} \\ &\overset{k=tk'}{=} t^{n-2} \int \frac{\mathrm{d}^n k'}{(2\pi)^n} \frac{i}{k'^2 - m^2}. \end{aligned} \tag{3.11}$$

Also, the derivatives and masses in each interaction of type i introduce d_i momentum factors into the integrand, can be re-scaled as

$$\delta^n(q) q^{d_i} \to \delta^n(tq)(tq)^{d_i} = t^{d_i - n} \delta^n(q) q^{d_i}. \tag{3.12}$$

Adding the contributions Eq. 3.11 and Eq. 3.12, and an additional factor n from the overall momentum-conserving delta function. We can read the dimension

$$D = n + \sum_i N_i(d_i - n) + \sum_f I_f(n-2). \tag{3.13}$$

Combining the relationship between loops, internal lines and vertices

$$N_L = \sum_f I_f - \left(\sum_i N_i - 1 \right), \tag{3.14}$$

results in Eq. 3.9.

$$D = nN_L + \sum_i N_i d_i - \sum_f 2I_f. \tag{3.15}$$

Using this power-counting formula alongside a specified accuracy requirement for a physical quantity, we can subsequently identify all the Feynman diagrams that require calculation.

3.3 The QCD Lagrangian

As the EFT Lagrangian is constrained by the symmetries of the underlying theory, the initial step involves identifying the symmetries inherent in the fundamental theory, QCD. Consider the QCD Lagrangian,

$$\mathscr{L} = \sum_{f=1}^{6} \bar{q}_f (i\not{D} - m_f) q_f - \frac{1}{2}\mathrm{Tr} G_{\mu\nu} G^{\mu\nu}, \tag{3.16}$$

where $D_\mu = \partial_\mu + igA_\mu$ is the covariant derivative, q_f are the quark fields with f being the flavor index (we have hidden the Dirac-spinor and color indices), $A_\mu = A_\mu^a T_a$ are the eight gluon fields with T_a being $SU(3)$ generators, and $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ being the gluon field strength.

The quark mass spectrum exhibits an intriguing feature: the six quark flavors can be divided into the three light quarks u, d and s and the three heavy flavors c, b and t,

$$m_u, m_d, m_s \ll \Lambda \sim 1\mathrm{GeV} \leq m_c, m_b, m_t, \tag{3.17}$$

where the scale Λ is associated with the masses of the lightest hadrons containing light quarks, e.g., $m_\rho = 770\mathrm{MeV}$. For low-energy QCD, we need only focus on three light quarks, thereby reducing the dimension of the flavor space from six dimensions to three. The Lagrangian of interest is as follows,

$$\mathscr{L} = \sum_{i=1}^{3} \bar{q}_i (i\not{D} - m_i) q_i - \frac{1}{2}\mathrm{Tr} G_{\mu\nu} G^{\mu\nu}. \tag{3.18}$$

Note that since the Dirac spinor is the direct sum of Weyl spinors, we can use the projection operator in the spinor space to perform chiral decomposition of the quark field,

$$q = q_L + q_R, \quad \text{where} \quad q_L = \frac{1}{2}(1-\gamma^5)q, \quad q_R = \frac{1}{2}(1+\gamma^5)q, \tag{3.19}$$

the Lagrangian can then be rewritten to

$$\mathscr{L} = \sum_{i=1}^{3} (\bar{q}_{Li} i\not{D} q_{Li} + \bar{q}_{Ri} i\not{D} q_{Ri} - \bar{q}_{Li} m_i q_{Ri} - \bar{q}_{Ri} m_i q_{Li}) - \frac{1}{2}\mathrm{Tr} G_{\mu\nu} G^{\mu\nu}. \tag{3.20}$$

We observe that the kinetic term decouples the left-hand and right-hand fields, while the mass terms couple them together. Considering the small mass of light quarks, we can make the approximation that the quark mass is zero, leading to the Lagrangian in the chiral limit,

$$\mathscr{L}^0 = \sum_{i=1}^{3} (\bar{q}_{Li} i\not{D} q_{Li} + \bar{q}_{Ri} i\not{D} q_{Ri}) - \frac{1}{2}\mathrm{Tr} G_{\mu\nu} G^{\mu\nu}. \tag{3.21}$$

This Lagrangian by itself evidently respects a $U(3)_L \times U(3)_R$ symmetry, it is invariant under transformation

$$q_L \to \mathcal{L} q_L \quad \text{and} \quad q_R \to \mathcal{R} q_R, \tag{3.22}$$

where $\mathcal{L} \in U(3)_L$ and $\mathcal{R} \in U(3)_R$ are unitary transformations. This can be seen by,

$$\begin{aligned} \bar{q}_L i \not{D} q_L &\to \bar{q}_L \mathcal{L}^\dagger i \not{D} \mathcal{L} q_L = \bar{q}_L i \not{D} q_L \quad \text{and} \\ \bar{q}_R i \not{D} q_R &\to \bar{q}_R \mathcal{R}^\dagger i \not{D} \mathcal{R} q_R = \bar{q}_R i \not{D} q_R. \end{aligned} \tag{3.23}$$

The symmetry group $U(3)_L \times U(3)_R$ can be further decomposed into $SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$. Within this decomposition, the $U(1)_A$ transformation, where $q \to \mathrm{e}^{i\alpha\gamma_5} q$, is not preserved due to quantum corrections, and is often referred to as anomalies [12]. While it is a symmetry of the Lagrangian, it does not manifest in observable phenomena, leaving the residual symmetry group as $SU(3)_L \times SU(3)_R \times U(1)_V$.

The $U(1)_V$ symmetry corresponds to the baryon number conservation, under which both left- and right-handed quarks of all flavors pick up a common phase. The remaining symmetry, $SU(3)_L \times SU(3)_R$, is known as "chiral symmetry," under which the left- and right-handed fields transform according to their respective $SU(3)_L$ and $SU(3)_R$ matrices.

4 Chiral Perturbation Theory

4.1 Breaking of Chiral Symmetry

Since the QCD Lagrangian in the chiral limit $\mathscr{L}^0$ possesses a chiral symmetry $SU(3)_L \times SU(3)_R$, from symmetry consideration involving the Hamiltonian H^0 only, one would expect that all of the hadrons should follow the same symmetry group. Therefore, we can define vector and axial-vector charge operator

$$Q_{Va} \equiv Q_{Ra} + Q_{La} \xrightarrow{P} Q_{Va} \tag{4.1}$$

and

$$Q_{Aa} \equiv Q_{Ra} - Q_{La} \xrightarrow{P} -Q_{Aa} \tag{4.2}$$

through the linear combinations of charge operator Q_{Ra} and Q_{La} which commute with H^0 since they correspond to the chiral symmetry, and thus Q_{Va} and Q_{Aa} also commute with H^0

$$[H^0, Q_{Va}] = [H^0, Q_{Aa}] = 0. \tag{4.3}$$

From the expression of these two operators, one can tell that they have opposite parity, and therefore for states of positive parity one would always expect the existence of degenerate states of negative parity which can be seen as follows.

Define $|i,+\rangle$ being an eigenstate of H^0 with positive parity denotes by $'+'$ and eigenvalue E_i,

$$\begin{aligned} H^0 |i,+\rangle &= E_i |i,+\rangle \\ P |i,+\rangle &= + |i,+\rangle , \end{aligned} \tag{4.4}$$

such as, e.g., a member of the lowest-lying baryon octet (in the chiral limit). Now we can construct a new state $|\psi_{ai}\rangle = Q_{Aa} |i,+\rangle$, from Eq. 4.2, Eq. 4.3 and Eq. 4.4, we have

$$\begin{aligned} H^0 |\psi_{ai}\rangle &= H^0 Q_{Aa} |i,+\rangle = Q_{Aa} H^0 |i,+\rangle = E_i Q_{Aa} |i,+\rangle = E_i |\psi_{ai}\rangle , \\ P |\psi_{ai}\rangle &= P Q_{Aa} |i,+\rangle = P Q_{Aa} P^{-1} P |i,+\rangle = -Q_{Aa}(+|i,+\rangle) = -|\psi_{ai}\rangle . \end{aligned} \tag{4.5}$$

If we insert a set of complete basis

$$|\psi_{ai}\rangle = Q_{Aa} |i,+\rangle = \sum_j |j,-\rangle \langle j,-| Q_{Aa} |i,+\rangle = -\sum_j t_{a,ij} |j,-\rangle , \tag{4.6}$$

where $-t_{a,ij}$ represent the matrix elements of $\langle j,-| Q_{Aa} |i,+\rangle$, we can then expand the state $|\psi_{ai}\rangle$ in terms of the members of a multiplet with negative parity.

However, the low-energy spectrum of baryons does not contain a degenerate baryon octet of negative parity, which means that the above arguments are incomplete. Indeed, we have tacitly assumed that the ground state of QCD is annihilated by the generators Q_{Aa}. Let $a_i^\dagger$ and $b_j^\dagger$ denote operators creating states $|i,+\rangle$ and $|j,-\rangle$, with opposite parity respectively, i.e., $|i,+\rangle = a_i^\dagger |\Omega\rangle$ and $|j,-\rangle = b_j^\dagger |\Omega\rangle$. We assume that under $SU(3)_L \times SU(3)_R$ the creation operators are related by[3]

$$[Q_{Aa}, a_i^\dagger] = -t_{a,ij} b_j^\dagger . \tag{4.7}$$

The usual chain of arguments then works as

$$|\psi_{ai}\rangle = Q_{Aa} |i,+\rangle = Q_{Aa} a_i^\dagger |\Omega\rangle = \Big([Q_{Aa}, a_i^\dagger] + a_i^\dagger Q_{Aa} \Big) |\Omega\rangle = -t_{a,ij} b_j^\dagger |\Omega\rangle = -t_{a,ij} |j,-\rangle \tag{4.8}$$

where in the forth equality, we naively use $Q_{Aa} |\Omega\rangle = 0$. However, if the ground state is not annihilated by Q_{Aa}, $Q_{Aa} |\Omega\rangle \neq 0$, the reasoning of Eq. 4.8 does no longer apply, then the state $|\psi_{ai}\rangle$ can not be expanded to negative parity states, which explains why we do not observe them in the experiments.

If we recall our arguments in Sec. 2.2, we can identify that this is exactly the spontaneous symmetry breaking, the ground state is now only annihilated by Q_{Va} but not Q_{Aa}, which means that $SU(3)_V$ instead of $SU(3)_L \times SU(3)_R$ is approximately realized as a symmetry of the ground states (hadrons). In addition, the hadron spectrum also suggests that the octet of the pseudoscalar mesons is special in the sense that the masses of its members

[3]Here, we analogize the commutator of charge and field operators, $[Q_a(t), \Phi_i(\vec{y},t)] = -t_{a,ij}\Phi_j(\vec{y},t)$, which can be calculated through Noether's theorem for infinitesimal transformations which are linear in the fields, $\Phi_i(x) \to \Phi_i'(x) = \Phi_i(x) - i\epsilon_a(x) t_{a,ij} \Phi_j(x)$.

are small in comparison with the other mesons, which indicates that they are candidates for the NGBs of the spontaneous chiral symmetry breaking, and the small mass of the pseudoscalar mesons can be explained by the explicit chiral symmetry breaking caused by small quark masses.

This chiral symmetry breaking happened 14 billion years ago, when the temperature of the universe cooled below $T_C \sim \Lambda_{QCD}$ and confined hadrons instead of quark-gluon plasma appeared. Although it has not been proven from QCD itself, the ground state of QCD apparently has a non-zero expectation value for the quark bilinears (quark condensate):

$$3V^3 = \langle\Omega|\, \bar{q}q \,|\Omega\rangle = \langle\bar{q}q\rangle = 3\,\langle\bar{u}u\rangle = 3\,\langle\bar{d}d\rangle = 3\,\langle\bar{s}s\rangle\,, \tag{4.9}$$

where the approximate $SU(3)_V$ symmetry of the ground state in chiral limit suggests that $\langle\bar{u}u\rangle$, $\langle\bar{d}d\rangle$ and $\langle\bar{s}s\rangle$ have the same expectation value. If we rewrite the quark condensate in terms of left- and right-handed fields

$$\langle\bar{q}q\rangle = \langle\bar{q}_L q_R\rangle + \langle\bar{q}_R q_L\rangle\,, \tag{4.10}$$

we observe that the quark condensate is not invariant under chiral symmetry but remains invariant under $SU(3)_V$, which rotates left- and right-handed fields the same way. Like $\langle\Omega|\, \phi \,|\Omega\rangle = v$ in Sec. 2.1, this specific ground state breaks the $SU(3)_L \times SU(3)_R$ symmetry to $SU(3)_V$.

4.2 Parameterization of the Field and Construction of Effective Lagrangian

In Sec. 2.2, we have shown that the NGBs ϕ_i live in the quotient G/H which means that $\vec{\phi}$ is a representation of the coset gH.[4] Now, the symmetry group relevant to the application in QCD are

$$\begin{aligned} &G = SU(3)_L \times SU(3)_R = \{(\mathcal{L},\mathcal{R})|\mathcal{L} \in SU(3)_L, \mathcal{R} \in SU(3)_R\} \\ &H = \{(\mathcal{V},\mathcal{V})|\mathcal{V} \in SU(3)\} \cong SU(3)_V, \end{aligned} \tag{4.11}$$

the manifold G/H is the same as the manifold of the group $SU(3)$ locally and gives rise to $\dim(G/H) = 8$ NGBs. This follows easily by observing that a typical coset is

$$gH = (\mathcal{L},\mathcal{R})H = (\mathcal{L},\mathcal{R})(\mathcal{R}^\dagger,\mathcal{R}^\dagger)H = (\mathcal{L}\mathcal{R}^\dagger, \mathbf{1})H \quad g \in G, \tag{4.12}$$

where we use the rearrangement theorem from group theory in the second equality. Hence the coset can be characterized by the elements $\mathcal{L}\mathcal{R}^\dagger = U \sim SU(3)$, since the second

[4] It is common to refer to particles as representations of certain groups. For instance, Weinberg identifies the state of a single particle as irreducible representations of Poincaré group in Ref. [13]. But actually, it's important to note that this terminology does not imply that the particle itself is a specific representation of the group, but rather that the particle field serves as the basis for a particular representation.

argument is a unit matrix. We can obtain the transformation behavior of U under G by multiplying the coset gH from the left with $g \in G$:

$$\widetilde{g}gH = (\widetilde{\mathcal{L}}, \widetilde{\mathcal{R}})(\mathcal{L}\mathcal{R}^\dagger, \mathbf{1})H = (\widetilde{\mathcal{L}}\mathcal{L}\mathcal{R}^\dagger, \widetilde{\mathcal{R}})(\widetilde{\mathcal{R}}^\dagger, \widetilde{\mathcal{R}}^\dagger)H = (\widetilde{\mathcal{L}}(\mathcal{L}\mathcal{R}^\dagger)\widetilde{\mathcal{R}}^\dagger, \mathbf{1})H, \tag{4.13}$$

thus

$$U = \mathcal{L}\mathcal{R}^\dagger \to U' = \widetilde{\mathcal{L}}(\mathcal{L}\mathcal{R}^\dagger)\widetilde{\mathcal{R}}^\dagger = \widetilde{\mathcal{L}}U\widetilde{\mathcal{R}}^\dagger. \tag{4.14}$$

The matrix representation U of $SU(3)$ can be parameterized in terms of a set of coordinates $\phi_i(x)$ in the coset space G/H, as:

$$\begin{aligned} U =& \exp\left(i\frac{\sum_{i=1}^{8}\phi_i \cdot \lambda_i}{F_0}\right) = \exp\left(i\frac{\phi}{F_0}\right) \\ =& \exp\left[\frac{i}{F_0}\begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}\right] \end{aligned} \tag{4.15}$$

where λ_a are Gell-Mann matrices and the constant F_0 is introduced to make the argument of the exponential function dimensionless as it has to be. Since a bosonic field has the dimension of energy, F_0 also has the dimension of energy. We express the final result of U in terms of physical fields, e.g.,

$$\pi^+ = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \tag{4.16}$$

Now, with nothing but symmetry to guide us, we are able to construct the most general Lagrangian involving U that is invariant under $SU(3)_L \times SU(3)_R$, that is

$$\begin{aligned} \mathscr{L} =& \mathscr{L}_{p^2} + \mathscr{L}_{p^4} + \mathscr{L}_{p^6} + \cdots \\ =& \frac{F_0^2}{4}\mathrm{Tr}[(\partial_\mu U)^\dagger(\partial^\mu U)] \\ &+ \{L_1 \mathrm{Tr}[(\partial_\mu U)^\dagger(\partial^\mu U)]^2 + L_2 \mathrm{Tr}[(\partial_\mu U)^\dagger(\partial_\nu U)]\mathrm{Tr}[(\partial^\mu U)^\dagger(\partial^\nu U) + \ldots\} + \ldots, \end{aligned} \tag{4.17}$$

where the terms displayed between curly brackets are of $\mathcal{O}(p^4)$ and '. . . ' in the end consists of terms with higher order momentum. The L_i are referred to as Low-Energy Constants (LECs) whose numerical values can not be determined by chiral symmetry but should, in principle, be determined by the parameters in QCD or from experiments. The purpose of the multiplicative constant $F_0^2/4$ is to generate the standard form of the kinetic term $(\partial\phi_a)^2/2$.

4.3 The ChPT Lagrangian

So far, we have developed the majority of ChPT, and now only the final "capping work" remains. The effective Lagrangian constructed in the last subsection is based on the ideal

$SU(3)_L \times SU(3)_R$ symmetry. However, in reality, this symmetry is broken explicitly by the quark masses. We have shown in Sec. 2.3, through a simple example, that explicit symmetry breaking lead to pNGBs with finite masses.

Now, we proceed to systematically incorporate the slight breaking of chiral symmetry into the framework, following the approach outlined by Gasser & Leutwyler [14, 15]. This is achieved by utilizing the external field method within the assumption that QCD is chirally symmetric. We first need to introduce the Lagrangian of QCD with external fields.

$$\mathscr{L} = \mathscr{L}^0 + \bar{q}\left[\gamma^\mu\left(v_\mu + \gamma_5 a_\mu\right) - (s - i\gamma_5 p)\right]q \tag{4.18}$$

where v_μ, a_μ, s, p are external c-number fields that couple to vector and axial-vector currents, as well as scalar and pseudoscalar quark densities, respectively. In practice, these external fields can be replaced depending on the phenomenon under investigation. For instance, the electroweak interaction can be introduced through v_μ and a_μ; through the scalar sector s, the mass terms of quark can be introduced by Higgs interaction.

To apply these external fields in the construction of ChPT Lagrangian, we need to encapsulate these fields in blocks whose transformation properties are consistent with chiral symmetry. To that end, we first rewrite the QCD Lagrangian by left- and right-handed fields

$$\mathscr{L} = \mathscr{L}^0 + \bar{q}_L\gamma^\mu l_\mu q_L + \bar{q}_R\gamma^\mu r_\mu q_R - \frac{1}{2B}[\bar{q}_R\chi q_L - \bar{q}_L\chi^\dagger q_R], \tag{4.19}$$

where constant B is introduced for future use, and

$$l_\mu \equiv v_\mu - a_\mu, \qquad r_\mu \equiv v_\mu + a_\mu, \qquad \chi \equiv 2B(s + ip), \tag{4.20}$$

Thus, under transformation $(\mathcal{L}, \mathcal{R}) \in SU(3)_L \times SU(3)_R$, they must change as

$$l_\mu \to \mathcal{L} l_\mu \mathcal{L}^\dagger + i\mathcal{L}\partial_\mu\mathcal{L}^\dagger, \quad r_\mu \to \mathcal{R} l_\mu \mathcal{R}^\dagger + i\mathcal{R}\partial_\mu\mathcal{R}^\dagger, \quad \chi \to \mathcal{L}\chi\mathcal{R}^\dagger, \tag{4.21}$$

so that the Lagrangian is invariant under a chiral transformation. By analogy to gauge theory, for any object $\mathcal{A}$ transforming as $\mathcal{L}\mathcal{A}\mathcal{R}^\dagger$, e.g., U and χ, we can introduce the covariant derivative as

$$D_\mu\mathcal{A} \equiv \partial_\mu\mathcal{A} - ir_\mu\mathcal{A} + i\mathcal{A}l_\mu. \tag{4.22}$$

In addition to this, we can also borrow forms and terminology in gauge theory to define the field strength tensors

$$\begin{aligned} F_L^{\mu\nu} &\equiv \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu] \quad &\to \quad &\mathcal{L}F_L^{\mu\nu}\mathcal{L}^\dagger \\ F_R^{\mu\nu} &\equiv \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu] \quad &\to \quad &\mathcal{R}F_R^{\mu\nu}\mathcal{R}^\dagger. \end{aligned} \tag{4.23}$$

whose transformation properties are much simpler than l_μ and r_μ, allowing for easier tracking when we constructing the Lagrangian.

Now, the ChPT Lagrangian can be written in terms of $U, \chi, F_L^{\mu\nu}, F_R^{\mu\nu}$ and covariant derivatives D_μ. Following the underlying chiral symmetry $SU(3)_L \times SU(3)_R$, we can obtain the ChPT Lagrangian:

$$
\begin{aligned}
\mathscr{L}_\chi =&\mathscr{L}_{p^2} + \mathscr{L}_{p^4} + \mathscr{L}_{p^6} + \cdots \\
=&\frac{F_0^2}{4} \langle (\partial_\mu U)^\dagger (\partial^\mu U) \rangle + \frac{F_0^2}{4} \langle \chi U^\dagger + U \chi^\dagger \rangle \\
&+ \Big\{ L_1 \langle (D_\mu U)^\dagger (D^\mu U) \rangle^2 + L_2 \langle (D_\mu U)^\dagger (D_\nu U) \rangle \langle (D^\mu U)^\dagger (D^\nu U) \rangle \\
&+ L_3 \langle (D_\mu U)^\dagger (D^\mu U)(D_\nu U)^\dagger (D^\nu U) \rangle + L_4 \langle (D_\mu U)^\dagger (D^\mu U) \rangle \langle \chi U^\dagger + U \chi^\dagger \rangle \\
&+ L_5 \langle (D_\mu U)^\dagger (D^\mu U)(\chi U^\dagger + U \chi^\dagger) \rangle + L_6 \langle \chi U^\dagger + U \chi^\dagger \rangle^2 \\
&+ L_7 \langle \chi U^\dagger - U \chi^\dagger \rangle^2 + L_8 \langle \chi U^\dagger \chi U^\dagger + U \chi^\dagger U \chi^\dagger \rangle \\
&- i L_9 \langle F_R^{\mu\nu} (D_\mu U)(D_\nu U)^\dagger + F_L^{\mu\nu} (D_\mu U)^\dagger (D_\nu U) \rangle + L_{10} \langle U F^L_{\mu\nu} U^\dagger F_R^{\mu\nu} \rangle \\
&+ H_1 \langle F^R_{\mu\nu} F_R^{\mu\nu} + F^L_{\mu\nu} F_L^{\mu\nu} \rangle + H_2 \langle \chi \chi^\dagger \rangle \Big\} \\
&+ \ldots .
\end{aligned}
\tag{4.24}
$$

Again, the terms displayed between curly brackets are of $\mathcal{O}(p^4)$ (From Ref. [14, 15]) and '. . . ' in the end consists of terms with higher order momentum. The derivation and explicit expression of $\mathcal{O}(p^6)$ terms can be found in Ref. [16][5]. And from now on, we will use $\langle \cdot \rangle$ to denote trace $\mathrm{Tr}(\cdot)$.

5 Phenomenology in Isospin Symmetric Limit

Now that we have established ChPT on theoretical grounds, it is time to depict its connections to real-world phenomena.

5.1 The Leading-Order Masses

Comparing Eq. 4.19 and Eq. 3.20, we observe that the quark masses are introduced by replacing s with $M \equiv \mathrm{diag}(m_u, m_d, m_s)$. We can employ a similar approach in our ChPT Lagrangian. By doing so, however, we explicitly break the chiral symmetry, as we substitute a constant matrix for a field. The NGBs then naturally acquire a small mass and become pNGBs.

Before tackling the issue of masses, we must address some "historical legacy issues," namely the undetermined constant B. One approach to determine it is by utilizing the QCD vacuum energy density $\langle H_{QCD} \rangle$. In the original QCD theory, this value is

$$
\langle H_{QCD} \rangle = \langle \bar{q} M q \rangle = m_u \langle \bar{u} u \rangle + m_d \langle \bar{d} d \rangle + m_s \langle \bar{s} s \rangle . \tag{5.1}
$$

[5] In this article, different building blocks are used to construct the ChPT Lagrangian, but in general they are equivalent.

For the effective ChPT, consider the ground state ($U_{min} = 1$), the vacuum expectation value of energy density is

$$\langle H_\chi \rangle = -\frac{F_0^2}{4} \langle (\chi + \chi^\dagger) \rangle = -F_0^2 B(m_u + m_d + m_s). \tag{5.2}$$

Comparing their derivative with respect to (any of) the light-quark masses m_q, and combining Eq. 4.9, we have

$$\begin{aligned} \left.\frac{\partial \langle H_{QCD} \rangle}{\partial m_q}\right|_{m_u=m_d=m_s=0} &= \frac{1}{3} \langle \bar{q}q \rangle = V^3, \\ \left.\frac{\partial \langle H_\chi \rangle}{\partial m_q}\right|_{m_u=m_d=m_s=0} &= -F_0^2 B. \end{aligned} \tag{5.3}$$

The constant B is thus related to the scalar singlet quark condensate and decay constant F by

$$B = -\frac{V^3}{F_0^2} \tag{5.4}$$

Now if we expanding the exponential $U = 1 + i\phi/F_0 + \cdots$, and identifying the quadratic terms in the LO ChPT Lagrangian,

$$\begin{aligned} \mathscr{L}_\chi \supset &-\frac{B}{2} \langle \phi^2 M \rangle \\ = &-\frac{B}{2} \Big[2(m_u + m_d)\pi^+\pi^- + 2(m_u + m_s)K^+K^- + 2(m_d + m_s)K^0\bar{K}^0 \\ &+(m_u + m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u - m_d)\pi^0\eta + \frac{1}{3}(m_u + m_d + 4m_s)\eta^2 \Big], \end{aligned} \tag{5.5}$$

note that there exists a term that mixes the π^0 and η. For simplicity, we will consider the isospin symmetric limit from this point onward in this section, i.e., $m_u = m_d$. Therefore, we can directly read off that[6]

$$\begin{aligned} m^2_{\pi,p^2} &= m^2_{\pi^0,p^2} = m^2_{\pi^\pm,p^2} = 2B\hat{m}, \\ m^2_{K,p^2} &= m^2_{K^0,p^2} = m^2_{\bar{K}^0,p^2} = m^2_{K^\pm,p^2} = B(\hat{m} + m_s), \\ m^2_{\eta,p^2} &= \frac{2}{3}B(\hat{m} + 2m_s), \end{aligned} \tag{5.6}$$

where $\hat{m} = (m_u + m_d)/2 = m_u = m_d$ and the p^2 in subscript indicates the LO.

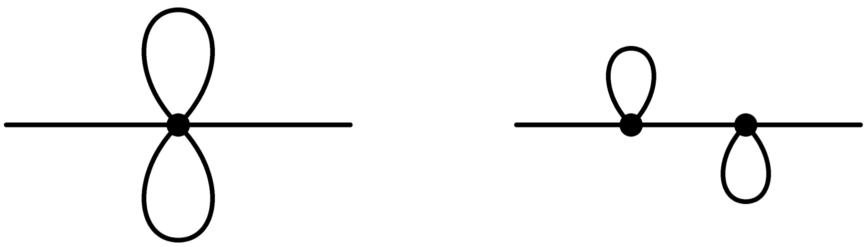
[6]Note that $\pi^\pm$, $K^\pm$ and K^0 are physical fields. We need first express them in ϕ_i, and then calculate their masses.

5.2 Exact Propagator and Pole Mass

Now that we have obtained the LO masses, we can define the lowest order propagator as

$$P_\phi(p^2) = \frac{1}{p^2 - m^2_{\phi,p^2} + i0^+}, \quad \phi = \pi, K, \eta. \tag{5.7}$$

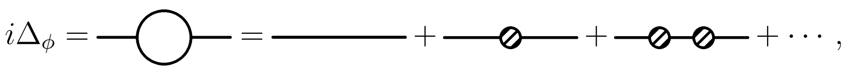
In general, the exact propagator Δ_ϕ acquires corrections from an infinite number of diagrams, and can be expressed as the sum of a single geometric series constructed by connecting graphs that cannot be split by cutting a single propagator. Such graphs are called One-Particle-Irreducible (1PI) diagrams. For example, Fig. 3a is a 1PI, but Fig. 3b is not.



(a) (b)

Figure 3: (**a**). Example of one-particle-irreducible graphs. (**b**). Example of non-1PI, the graph falls apart into two separate pieces when cutting an internal line.

Therefore, the exact propagator is



(5.8)

where the filled circles correspond to the sum of 1PI and the lines to lowest order propagator. Defining Π_ϕ as the sum of all 1PI, which is often referred to as self-energy, we find

$$\begin{aligned} i\Delta_\phi =& iP_\phi + iP_\phi(i\Pi_\phi iP_\phi) + iP_\phi(i\Pi_\phi iP_\phi)^2 + \cdots = iP_\phi(1 + \Pi_\phi P_\phi)^{-1} \\ =& \frac{i}{p^2 - m^2_{\phi,p^2} + \Pi_\phi + i0^+} \end{aligned} \tag{5.9}$$

On the other hand, the exact propagator can also be expressed in the Källén–Lehmann form [17, 18]:

$$i\Delta_\phi(p^2) = \frac{i}{p^2 - m^2_{\phi,phy} + i0^+} + \int_{4m^2_{\phi,phy}}^{\infty} \mathrm{d}s\rho(s)\frac{i}{p^2 - s + i0^+}, \tag{5.10}$$

where $m^2_{\phi,phy}$ is the physical mass of the field and $\rho(s) = \sum_n |\langle p, n| \phi(0) |\Omega\rangle|^2 \delta(s - M^2)$ is the spectral density, in which M is the mass of the multi-particle state $|p, n\rangle$, specified by a three-momentum $\vec{p}$ and other parameters (such as relative momenta among the different particles) that we will collectively denote as n. We particularly observe that that $i\Delta_\phi$ has an isolated pole at $p^2 = m^2_{\phi,phy}$ with residue one.

Since these two forms of propagators are equivalent, we expect that Eq. 5.9 also has a pole at $p^2 = m^2_{\phi,phy}$ with residue one,

$$m^2_{\phi,phy} - m^2_{\phi,p^2} + \Pi(m^2_{\phi,phy}) = 0. \tag{5.11}$$

Considering our use of the lowest order propagator, to clearly illustrate the concept of residue one, we can expand Π_ϕ around $p^2 = \bar{\lambda}^2$,

$$\Pi_\phi(p^2) = \Pi_\phi(\bar{\lambda}^2) + (p^2 - \bar{\lambda}^2)\Pi'_\phi(\bar{\lambda}^2) + \widetilde{\Pi}_\phi(p^2, \bar{\lambda}^2), \tag{5.12}$$

where the remainder $\widetilde{\Pi}_\phi(p^2, \bar{\lambda}^2)$ depends on the choice of $\bar{\lambda}^2$ and by definition satisfies $\widetilde{\Pi}_\phi(\bar{\lambda}^2, \bar{\lambda}^2) = \widetilde{\Pi}'_\phi(\bar{\lambda}^2, \bar{\lambda}^2) = 0$. Taking $\bar{\lambda}^2 = m^2_{\phi,phy}$ and applying the condition of Eq. 5.11, we then obtain for the propagator

$$\begin{aligned} i\Delta_\phi(p^2) =& \frac{i}{(p^2 - m^2_{\phi,phy})[1 + \Pi'_\phi(m^2_{\phi,phy})] + \widetilde{\Pi}_\phi(p^2) + i0^+} \\ =& \frac{iZ_\phi}{p^2 - m^2_{\phi,phy} + Z_\phi\widetilde{\Pi}_\phi(p^2) + i0^+}, \end{aligned} \tag{5.13}$$

where we have introduced the wave function renormalization constant via the residue of the exact propagator

$$\begin{aligned} Z_\phi =& \mathrm{Res}[\Delta_\phi(p^2), m^2_{\phi,phy}] = \lim_{p^2 \to m^2_{\phi,phy}} (p^2 - m^2_{\phi,phy})\Delta_\phi(p^2) \\ =& \frac{1}{1 + \Pi'_\phi(m^2_{\phi,phy})}. \end{aligned} \tag{5.14}$$

Introducing renormalized field as $\phi_R = (Z_\phi)^{-1/2}\phi$, the renormalized propagator is given by

$$\begin{aligned} i\Delta^R_\phi(p^2) =& \int \mathrm{d}^4x \mathrm{e}^{ip\cdot x} \langle T[\phi_R(0)\phi_R(0)]\rangle \\ =& \frac{i}{p^2 - m^2_{\phi,phy} + Z_\phi\widetilde{\Pi}_\phi(p^2) + i0^+} \end{aligned} \tag{5.15}$$

which corresponds to the propagator in Lehmann–Symanzik–Zimmermann (LSZ) reduction formulae [10].

In practical applications, it is more convenient to use the non-renormalized propagator since ChPT is a non-renormalizable effective theory. Expand the physical mass and the self-energy in chiral orders

$$\begin{aligned} m^2_{\phi,phy} &= m^2_{\phi,p^2} + m^2_{\phi,p^4} + m^2_{\phi,p^6} + \mathcal{O}(p^8), \\ \Pi_\phi(p^2) &= \Pi^{(4)}_\phi(p^2) + \Pi^{(6)}_\phi(p^2) + \mathcal{O}(p^8), \end{aligned} \tag{5.16}$$

applying this expression in the Eq. 5.11, then using Taylor expansion around m^2_{ϕ,p^2}

$$f(m^2_{\phi,phy}, p^2) = f(m^2_{\phi,p^2}, p^2) + (m^2_{\phi,phy} - m^2_{\phi,p^2})\frac{\partial}{\partial p^2} f(p^2)\bigg|_{p^2=m^2_{\phi,p^2}} \tag{5.17}$$

to express the arguments in lowest quantities, we obtain (by comparing chiral order)

$$\begin{aligned} m^2_{\phi,p^4} &= -\Pi^{(4)}_{\phi}(m^2_{\phi,p^2}), \\ m^2_{\phi,p^6} &= -\Pi^{(6)}_{\phi}(m^2_{\phi,p^2}) - m^2_{\phi,p^4}\frac{\partial}{\partial p^2}\Pi^{(4)}_{\phi}(p^2)\bigg|_{p^2=m^2_{\phi,p^2}}, \end{aligned} \tag{5.18}$$

where the second term in m^2_{ϕ,p^6} comes from the expansion of $\Pi^{(4)}_{\phi}(p^2)$ around m^2_{ϕ,p^2}.

5.3 Higher Order Masses

Having established the expression for higher order masses, we can focus our efforts on calculating the self-energy $\Pi^{(2n)}_{\phi}$. Following the power counting scheme, the Feynman diagrams underlying our loop and counterterm analyses are the ones depicted in Fig. 4. According to the argument presented in the last section, we only need to consider the pseudoscalar for both incoming and outgoing cases (Fig. 4c) when evaluating masses.

The one-loop analysis was originally carried out by Gasser and Leutwyler [14] in $SU(2)$ basis (where we only consider up and down quark) and $SU(3)$ in [15]. Here we shall briefly summarize the theoretical analysis within the $SU(3)$ basis for one-loop level.

At the one loop level, there are two amplitudes that contribute to the self-energy, the unitarity diagram from the chiral Lagrangian $\mathscr{L}_{p^4}$ and the tadpole diagram from $\mathscr{L}_{p^2}$ (Fig. 4e and 4f). To calculate these diagrams, we first need to extract the relevant vertices from the Lagrangian, which can be read off as

$$\mathscr{L} = \mathscr{L}^{4\phi}_{p^2} + \mathscr{L}^{2\phi}_{p^4}, \tag{5.19}$$

where $\mathscr{L}^{4\phi}_{p^2}$ and $\mathscr{L}^{2\phi}_{p^4}$ are given by

$$\begin{aligned} \mathscr{L}^{4\phi}_{p^2} =& \frac{1}{48F_0^2}\left[\langle\phi^4\chi\rangle - 2\langle\phi^2\partial_\mu\phi\partial^\mu\phi\rangle + 2\langle\phi\partial_\mu\phi\phi\partial^\mu\phi\rangle\right], \\ \mathscr{L}^{2\phi}_{p^4} =& \frac{1}{F_0^2}\left\{2L_4\langle\chi\rangle\langle\partial_\mu\phi\partial^\mu\phi\rangle + 2L_5\langle\chi\partial_\mu\phi\partial^\mu\phi\rangle - 4L_6\langle\chi\rangle\langle\phi^2\chi\rangle\right. \\ & \left. -4L_7\langle\phi\chi\rangle^2 - 2L_8[\langle\phi^2\chi^2\rangle + \langle\phi\chi\phi\chi\rangle]\right\} \end{aligned} \tag{5.20}$$

Consider the special cases where the incoming and outcoming particle are both π^0 fields, applying the Feynman rules for scalar particles allows us to obtain the self-energy at the one-loop level:

$$
\begin{aligned}
\Pi^{(4)}_{\pi^0\pi^0}(p^2) =& - i\mathscr{M}_{\pi^0\pi^0} \\
=& - \frac{m^2_{\pi,p^2}}{6F_0^2}\left[A(m^2_{\pi,p^2}) + A(m^2_{\eta,p^2}) + 2A(m^2_{K,p^2})\right] + \frac{p^2}{3F_0^2}[2A(m^2_{\pi,p^2}) + A(m^2_{K,p^2})] \\
& - \frac{32Bm^2_{\pi,p^2}}{F_0^2}[(2\hat{m}+m_s)L_6 + \hat{m}L_8] + \frac{16Bp^2}{F_0^2}[(2\hat{m}+m_s)L_4 + \hat{m}L_5]
\end{aligned}
\tag{5.21}
$$

where $A(m^2)$ represents the relevant Feynman integral, and is defined in Appendix. C along with all other Feynman integrals we will encounter at one- and two-loop level.

We observe that $A(m^2)$ is a divergent integral, and through the dimensional regularization, its infinite part are expressed in Appendix. C.2. To that end, we have to use renormalization scheme to subtract these divergence. In this work, we have employed the version of Modified Minimal Subtraction ($\overline{MS}$) that is customary in ChPT [2, 14, 15, 19, 20].

Following $\overline{MS}$, let L_i denote any of the $\mathcal{O}(p^4)$ low-energy constants, we can redefine it as

$$
L_i \equiv (\mu c)^{-2\epsilon}\left(\frac{-1}{32\pi^2\epsilon}\Gamma_i + L_i^r(\mu)\right) = (\mu)^{-2\epsilon}\left(\frac{-1}{32\pi^2}\Gamma_i\lambda + L_i^r(\mu) + \mathcal{O}(\epsilon)\right), \tag{5.22}
$$

where $\ln c = -1/2(\ln 4\pi - \gamma + 1)$, μ is the renormalization scale and we have suppressed the explicit μ-dependence of the L_i^r in the expression hereafter. The coefficients Γ_i are given in Ref. [15]. Analogously, the coefficients at $\mathcal{O}(p^6)$ are given by

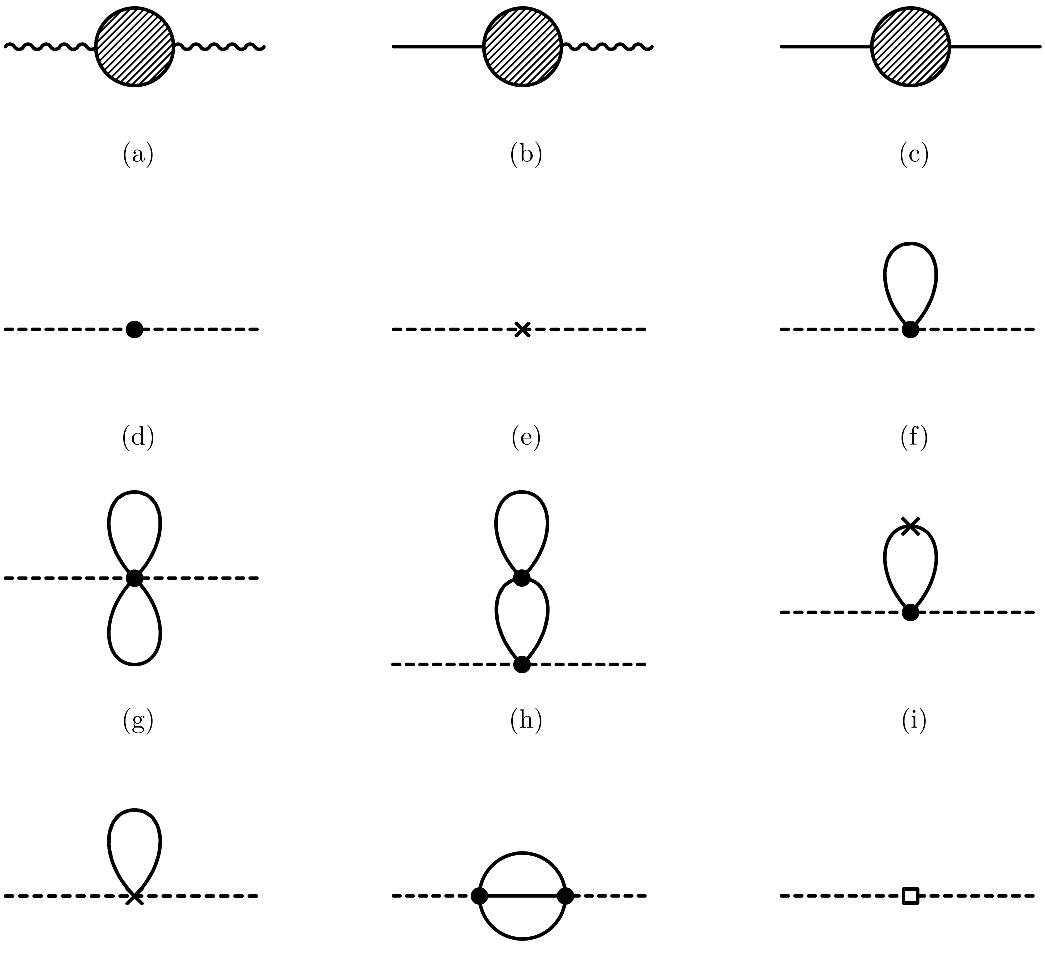
$$
C_i \equiv (\mu c)^{-4\epsilon}\left(\frac{\gamma_{2i}}{\epsilon^2} + \frac{\gamma_{1i}}{\epsilon^1} + C_i^r(\mu)\right) = (\mu)^{-4\epsilon}\left(\gamma_{2i}\lambda_2 + \gamma_{1i}\lambda_1 + C_i^r(\mu) + \mathcal{O}(\epsilon)\right), \tag{5.23}
$$

the coefficients γ_{2i} and γ_{1i} are given in Ref. [16]. The divergent quantities λ, λ_1 and λ_2 are consistent with that in Appendix. C.

Since the coefficients Γ_i are properly chosen so that they can cancel all the infinities in the loop integral, if we apply Eq. 5.22 and Eq. C.9 into Eq. 5.21, we can get the result for $m^2_{\pi^0,p^4}$ as

$$
\begin{aligned}
m^2_{\pi^0,p^4} =& - \Pi^{(4)}_{\pi^0\pi^0}(m^2_{\pi,p^2}) \\
=& m^2_{\pi,p^2}\left\{\frac{m^2_{\pi,p^2}}{32\pi^2F_0^2}\ln m^2_{\pi,p^2} - \frac{m^2_{\eta,p^2}}{96\pi^2F_0^2}\ln m^2_{\eta,p^2}\right. \\
& \left. + \frac{16B}{F_0^2}[(2\hat{m}+m_s)(2L_6^r - L_4^r) + \hat{m}(2L_8^r - L_5^r)]\right\}
\end{aligned}
\tag{5.24}
$$

in which we can find that the expression for the mass is finite now (Note that we have suppressed the explicit μ-dependence of the expression). The realization in $\mathcal{O}(p^6)$ is similar but involves much more terms and more complicated integrals.



(a) (b) (c) (d) (e) (f) (g) (h) (i) (j) (k) (l)

Figure 4: The set of diagrams contributing to the 1PI. (**a**). axial-vector (in) and axial-vector (out), (**b**). pseudoscalar (in) and axial-vector (out), (**c**). pseudoscalar (in) and pseudoscalar (out), (**d**) – (**l**). the respective diagrams when the dashed lines are replaced with the external legs of (**a**), (**b**) or (**c**). The plain line is a pNGB propagator, a dot, a cross and a square represent a vertex of $\mathcal{O}(p^2)$, $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$, respectively.

5.4 Decay Constant

The decay constant of NGBs is defined via Eq. 2.16. In our ChPT case, the current that corresponds to the broken symmetry is $J^{\mu}_{Aa} = J^{\mu}_{Ra} - J^{\mu}_{La}$. To express this in the leading order, we consider the infinitesimal transformations

$$\begin{aligned}\mathcal{L} &= 1 - i\varepsilon_{La}(x)\frac{\lambda_a}{2},\\ \mathcal{R} &= 1 - i\varepsilon_{Ra}(x)\frac{\lambda_a}{2}.\end{aligned} \tag{5.25}$$

First, we set $\varepsilon_{Ra} = 0$ and utilize Noether's theorem to construct J^μ_{La}

$$J^\mu_{La} = \frac{\partial \delta \mathscr{L}_{p^2}}{\partial(\partial_\mu \varepsilon_{La})} = i\frac{F_0^2}{4}\left\langle \lambda_a \partial^\mu U^\dagger U\right\rangle, \tag{5.26}$$

then analogously, by setting $\varepsilon_{La} = 0$, we obtain

$$J^\mu_{Ra} = \frac{\partial \delta \mathscr{L}_{p^2}}{\partial(\partial_\mu \varepsilon_{Ra})} = -i\frac{F_0^2}{4}\left\langle \lambda_a U \partial^\mu U^\dagger\right\rangle, \tag{5.27}$$

combining Eq. 5.26 and Eq. 5.27, the axial-vector reads

$$J^\mu_{Aa} = J^\mu_{Ra} - J^\mu_{La} = -i\frac{F_0^2}{4}\left\langle \lambda_a \{U, \partial^\mu U^\dagger\}\right\rangle = -\sqrt{2}F_0 \partial^\mu \phi_a + \cdots, \tag{5.28}$$

where in the third equality, we expand U and retain the leading term. Now evaluate the matrix element between vacuum and a one-NGB state we find (in isospin symmetric limit):

$$\begin{aligned}\langle \Omega | J^\mu_{Aa}(x) |\phi_b(p)\rangle &= \langle \Omega | -\sqrt{2}F_0\partial^\mu\phi_a |\phi_b(p)\rangle = -\sqrt{2}F_0\partial^\mu \mathrm{e}^{-ip\cdot x}\delta_{ab}\\ &= i\sqrt{2}F_0 p^\mu \mathrm{e}^{-ip\cdot x}\delta_{ab},\end{aligned} \tag{5.29}$$

which indicates that F_0 is the lowest order value of the real decay constant F_ϕ.

To calculate this observable to higher orders, we follow the usual LSZ reduction formalism. The matrix-element in momentum space for any n incoming and n' outgoing states is

$$\begin{aligned}\mathcal{A}_{i_1\cdots i_n, f_1 c\ldots f_{n'}} &= \langle f|i\rangle\\ &= (-i)^{n+n'}\prod_{i=1}^{n+n'}\lim_{p_i^2\to m_i^2}(p_i^2 - m_i^2)G^R_{i_1\cdots i_n, f_1\cdots f_{n'}}(p_1,\cdots,p_n,p_{n+1},\cdots,p_{n'}).\end{aligned} \tag{5.30}$$

The function G^R is the full $(n+n')$-point Green function generated by the $n+n'$ fields $\phi_{i_1}(p_1), \cdots \phi_{f_1}(p_{n+1}), \cdots$.The LSZ formula is valid provided that the field obeys (Note that the requirements here are defined in position space)

$$\langle \Omega | \phi^R(x) |\Omega\rangle = 0 \quad \text{and} \quad \langle p | \phi^R(x) |\Omega\rangle = \mathrm{e}^{-ipx}. \tag{5.31}$$

However, these normalization conditions conflict with our original field in ChPT Lagrangian Eq. 4.24, which we shall use to calculate the Green function. To that end, we utilize the renormalized field we defined in Sec. 5.2 and rewrite our formula accordingly as

$$\mathcal{A}_{i_1\cdots i_n, f_1\cdots f_{n'}} = \frac{(-i)^{n+n'}}{\sqrt{Z_{i_1}\cdots Z_{f_1}}}\prod_{i=1}^{n+n'}\lim_{p_i^2\to m_i^2}(p_i^2 - m_i^2)G_{i_1\cdots i_n, f_1\cdots f_{n'}}(p_1,\cdots,p_{n+1},\cdots), \tag{5.32}$$

the Green function G is now generated by bare fields $\phi_i(p_i)$ in the ChPT Lagrangian.

For the decay constant, we need only consider the case where one of the external legs correspond to the axial current and another is pNGB field. However, due to one external leg being an axial-vector, the true propagator does not conform to a 'geometric progression'. Consequently, we model this propagator by combining a true pseudoscalar-pseudoscalar propagator (Δ_ϕ) with the axial-vector–pseudoscalar one-particle-irreducible component ($\Pi^{\mu}_{\phi,A}$), as illustrated in Fig. 5. Therefore, we get

$$\begin{aligned}\mathcal{A}^{\mu}_{a,A} =& \langle\Omega|\, J^{\mu}_{Ac}(0)\,|\phi_a(p)\rangle = i\sqrt{2}F_\phi p^\mu \delta_{ac} \\ =& \frac{-i}{\sqrt{Z_a}} \lim_{p_a^2\to m_a^2} (p_a^2 - m_a^2)\Delta^{ab}_{\phi}(p_a^2)\Pi^{b\mu}_{\phi,A}(p_a^2)\delta_{ac},\end{aligned} \tag{5.33}$$

with an implicit summation over b. The diagrams that contribute to $\Pi^{\mu}_{\phi,A}$ are those illustrated in Fig. 4b.

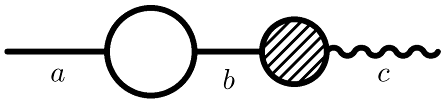


Figure 5: Exact axial-vector–pseudoscalar propagator, which contribute to the decay constant. True pseudoscalar-pseudoscalar propagator Δ_ϕ is denoted by the empty circle on the left-hand-side, and the left-hand-side is the axial-vector–pseudoscalar 1PI. a, b and c denotes the type of pseudoscalar or axial-vector.

In the isospin symmetric limit, the exact propagator Δ_ϕ is diagonal because there are no mixing fields. Applying the definition of wave-function-renormalization constant (Eq. 5.14), and define $\mathscr{A}_\phi(p^2)$ through $-\Pi^{\mu}_{\phi,A}(p^2) = \sqrt{2}\mathscr{A}_\phi(p^2)p^\mu$, the expression for the real decay constant is

$$F_\phi = \sqrt{Z_\phi(p^2)}\mathscr{A}_\phi(p^2) \tag{5.34}$$

Similar to the approach utilized in evaluating mass, we expand F_ϕ, Π'_ϕ and $\mathscr{A}_\phi$ in chiral orders.

$$\begin{aligned}F_\phi &= F_{\phi,p^2} + F_{\phi,p^4} + F_{\phi,p^6} + \mathcal{O}(p^8), \\ \mathcal{Z}_\phi(p^2) &\equiv \frac{\partial}{\partial p^2}\Pi_\phi = \mathcal{Z}^{(2)}_\phi(p^2) + \mathcal{Z}^{(4)}_\phi(p^2) + \mathcal{Z}^{(6)}_\phi(p^2) + \mathcal{O}(p^8), \\ \mathscr{A}_\phi(p^2) &= \mathscr{A}^{(2)}_\phi + \mathscr{A}^{(4)}_\phi(p^2) + \mathscr{A}^{(6)}_\phi(p^2) + \mathcal{O}(p^8),\end{aligned} \tag{5.35}$$

combining with the Taylor expansion of $\sqrt{Z_\phi}$ (since $\mathscr{A}_\phi$ starts from the second chiral order, we only need to expand to $\mathcal{O}(p^4)$ here)

$$\begin{aligned}\sqrt{Z_\phi}(p^2) &= \left(1+\Pi'_\phi(p^2)\right)^{-1/2} \\ &= 1-\frac{1}{2}\Pi'_\phi(p^2)+\frac{3}{8}\left(\Pi'_\phi(p^2)\right)^2+\cdots \\ &= 1-\frac{1}{2}\left(\mathcal{Z}_\phi^{(2)}(p^2)+\mathcal{Z}_\phi^{(4)}(p^2)\right)+\frac{3}{8}\left(\mathcal{Z}_\phi^{(2)}(p^2)\right)^2+\mathcal{O}(p^6),\end{aligned} \tag{5.36}$$

applying this expression in the Eq. 5.34 and performing the Taylor expansion to express the arguments in lowest order, we obtain,

$$\begin{aligned}F_{\phi,p^2} &= \mathscr{A}_\phi^{(2)} = F_0 \\ F_{\phi,p^4} &= -\frac{1}{2}\mathcal{Z}_\phi^{(2)}\mathscr{A}_\phi^{(2)}+\mathscr{A}_\phi^{(4)}, \\ F_{\phi,p^6} &= -\frac{1}{2}\mathcal{Z}_\phi^{(4)}\mathscr{A}_\phi^{(2)}+\frac{3}{8}\left(\mathcal{Z}_\phi^{(2)}\right)^2\mathscr{A}_\phi^{(2)}-\frac{1}{2}\mathcal{Z}_\phi^{(2)}\mathscr{A}_\phi^{(4)}+\mathscr{A}_\phi^{(6)},\end{aligned} \tag{5.37}$$

where we have utilized the fact that $\partial\Pi_\phi^{'(4)}/\partial p^2 = \partial\mathscr{A}_\phi^{(4)}/\partial p^2 = 0$, and all right-hand sides are evaluated at $p^2 = m^2_{\phi,p^2}$. Similarly to the procedure for calculating masses, following two-loop integral calculations, dimensional regularization, and renormalization, we will arrive at a finite value for the decay constant F_ϕ.

6 QCD Isospin Breaking Effects

In this section, we will extend the analysis of masses and decay constants to include isospin breaking effects. Previously, in Sec. 5, we operated under the assumption that up and down quarks are identical. However, in practice, these quarks can be differentiated by their distinct masses and charges. Consequently, isospin breaking arises from two primary sources: the difference in quark masses, $m_u - m_d$, and EM interactions [4].

6.1 Quark Mass Correction

Given the distinct treatment of up and down quarks, an immediate issue arises: the mass term in Eq. 5.5 induces a mixing contribution between the π^0 and η meson fields, expressed as $\mathscr{L}_{mix} = (-2B/\sqrt{3})(m_u - m_d)\pi^0\eta$. A significant consequence of this mixing is that the exact pseudoscalar-pseudoscalar propagator, Δ_ϕ, is no longer diagonal, since the incoming state does not necessarily coincide with the outgoing state. The pole of the propagator can be found now as places where the determinant of inverse of Δ vanishes

$$\det\Delta^{-1} = \det\left(P^{-1}+\Pi\right) = 0, \tag{6.1}$$

note that we now use matrices Δ, P and Π with elements Δ_{ij}, P_{ij} and Π_{ij} to denote the exact propagator, lowest order propagator and self-energy, respectively.

However, before proceeding with further calculations, the kinetic term must be brought to the canonical form. Working at lowest order, we can diagonalize Δ through a field rotation (since there is only mixing between π^0 and η, only this block is non-diagonal and needs to be considered.)

$$\begin{pmatrix}\pi^0\\ \eta\end{pmatrix} \xrightarrow{\mathcal{R}} \begin{pmatrix}\pi_3^0\\ \eta_8\end{pmatrix} = \begin{pmatrix}\cos\epsilon & \sin\epsilon\\ -\sin\epsilon & \cos\epsilon\end{pmatrix}\begin{pmatrix}\pi^0\\ \eta\end{pmatrix}, \tag{6.2}$$

where the mixing angle satisfies

$$\tan 2\epsilon = -\sqrt{3}\frac{\widetilde{m}}{m_s - \hat{m}}, \tag{6.3}$$

in which $\widetilde{m} = 1/2(m_u - m_d)$ is the quark mass difference. After carrying out this $\pi^0\eta$ rotation, the LO mass can be directly calculated as

$$\begin{aligned}
m^2_{\pi^+,p^2} =& m^2_{\pi^-,p^2} = 2B\hat{m},\\
m^2_{\pi^0_3,p^2} =& 2B\hat{m} + \frac{4B}{3}(m_s - \hat{m})\sin^2\epsilon + \frac{4B}{\sqrt{3}}\widetilde{m}\sin\epsilon\cos\epsilon,\\
m^2_{K^+,p^2} =& m^2_{K^-,p^2} = (m_s + m_u)B,\\
m^2_{K^0,p^2} =& m^2_{\bar{K}^0,p^2} = (m_s + m_d)B,\\
m^2_{\eta_8,p^2} =& \frac{2}{3}B(\hat{m} + 2m_s) - \frac{4B}{3}(m_s - \hat{m})\sin^2\epsilon - \frac{4B}{\sqrt{3}}\widetilde{m}\sin\epsilon\cos\epsilon,
\end{aligned} \tag{6.4}$$

Consequentially, in this new basis $(\pi^\pm, \pi^0_3, K^\pm, K^0, \bar{K}^0, \eta_8)$, the lowest order propagator P is also diagonalized

$$P_{ij} = \frac{\delta_{ij}}{p^2 - m^2_{i,p^2}}, \tag{6.5}$$

yet the self-energy Π only starts, by definition, at NLO, so it does not need to be diagonal. We have only used the fact here that a lowest order mixing angle ϵ is well-defined. There is, of course, no reason for this to hold true at higher orders and indeed it is not the case already at $\mathcal{O}(p^4)$.

Performing the same expansion of the real masses $m^2_{\phi,phys}$ and self-energy Π_{ij} as in Eq. 5.16 (where the subscripts i and j represent the incoming and outgoing particles, respectively), and combining the pole equation Eq. 6.1, we have

$$\begin{aligned}
0 =& \det\left(P^{-1} + \Pi\right) = \det\begin{pmatrix}P^{-1}_{33} + \Pi_{33} & \Pi_{38}\\ \Pi_{83} & P^{-1}_{88} + \Pi_{88}\end{pmatrix}\\
=& (P^{-1}_{33} + \Pi_{33})(P^{-1}_{88} + \Pi_{88}) - \Pi^2_{38}\\
=& (m^2_{\phi,phys} - m^2_{3,p^2} + \Pi^{(4)}_{33} + \Pi^{(6)}_{33})(m^2_{\phi,phys} - m^2_{8,p^2} + \Pi^{(4)}_{88} + \Pi^{(6)}_{88})\\
&- (\Pi^{(4)}_{38} + \Pi^{(6)}_{38})^2 + \mathcal{O}(p^8)
\end{aligned} \tag{6.6}$$

in which we use 3 and 8 to represent π_3^0 and η_8 fields defined in Eq. 6.2, respectively. Additionally, we use the fact that the matrix Π is symmetric. Solving this equation by expanding around $m^2_{\phi,phy}$ and fitting the chiral order, we obtain for π_3^0 cases

$$
\begin{aligned}
m^2_{3,p^4} =& -\Pi^{(4)}_{33}(m^2_{3,p^2}),\\
m^2_{3,p^6} =& -\Pi^{(6)}_{33}(m^2_{3,p^2}) - m^2_{3,p^4}\frac{\partial}{\partial p^2}\Pi^{(4)}_{33}(p^2)\Big|_{p^2=m^2_{3,p^2}} + \frac{1}{\Delta m^2_{38}}(\Pi^{(4)}_{38}(m^2_{3,p^2}))^2,
\end{aligned}
\tag{6.7}
$$

where $\Delta m^2_{38} = m^2_{3,p^2} - m^2_{8,p^2}$. The mass for η_8 can be obtained by using the same formula, but interchanging $3 \leftrightarrow 8$, and the remaining masses can be obtained by using the formula in the isospin symmetric limit (Eq. 5.18).

Similarly, our approach to calculating the decay constant also needs to be improved, as now the self-energy Π is no longer a diagonal matrix. However, from the derivation of the LO decay constant Eq. 5.28, Eq. 5.29 and the fact that the self-energy starts from $\mathcal{O}(p^4)$, we can still safely normalize our decay constant F_ϕ to F_0 at the lowest order.

In order to obtain the higher order decay constant, we first need to adjust our quantities in Eq. 5.33 to accommodate isospin-breaking cases. For simplification, we can define $\mathcal{P} \equiv P^{-1} + \Pi$, then the propagator Δ is the inverse of $\mathcal{P}$ (analogous to Eq. 5.9)

$$
i\Delta(p^2) = i\mathcal{P}^{-1}(p^2) = \frac{i}{\det\mathcal{P}(p^2)}\begin{pmatrix}\mathcal{P}_{88}(p^2) & -\mathcal{P}_{38}(p^2)\\ -\mathcal{P}_{38}(p^2) & \mathcal{P}_{33}(p^2)\end{pmatrix}. \tag{6.8}
$$

With this expression of the propagator and the definition of wave-function-renormalization constant in Eq. 5.14, we obtain, considering π_0 first

$$
\begin{aligned}
Z_3 =& \mathrm{Res}[\Delta_{33}(p^2), m^2_{3,phy}] = \lim_{p^2\to m^2_{3,phy}}(p^2 - m^2_{3,phy})\Delta_{33}(p^2)\\
=& \frac{1}{\frac{\partial}{\partial p^2}(\det\mathcal{P}(p^2))|_{p^2=m^2_{3,phy}}}\mathcal{P}_{88}(m^2_{3,phy}),
\end{aligned}
\tag{6.9}
$$

and analogously, for η_8 case

$$
Z_8 = \frac{1}{\frac{\partial}{\partial p^2}(\det\mathcal{P}(p^2))|_{p^2=m^2_{8,phy}}}\mathcal{P}_{33}(m^2_{8,phy}). \tag{6.10}
$$

Applying this expression in Eq. 5.33, the decay constant in the isospin-breaking case is then given by (considering the π_3^0 case)

$$
\begin{aligned}
F_3 =& \frac{1}{\sqrt{Z_3}}\lim_{p^2\to m^2_{3,phy}}(p^2 - m^2_{3,phy})[\Delta_{33}(p^2)\mathscr{A}_3(p^2) + \Delta_{38}(p^2)\mathscr{A}_8(p^2)]\\
=& \frac{1}{\sqrt{Z_3}}\left[\mathcal{P}_{88}(m^2_{3,phy})\mathscr{A}_3(m^2_{3,phy}) - \mathcal{P}_{38}(m^2_{3,phy})\mathscr{A}_8(m^2_{3,phy})\right]\lim_{p^2\to m^2_{3,phy}}\frac{p^2 - m^2_{3,phy}}{\det\mathcal{P}(p^2)}\\
=& \frac{1}{\sqrt{\mathcal{P}_{88}\frac{\partial}{\partial p^2}(\det\mathcal{P})}}(\mathcal{P}_{88}\mathscr{A}_3 - \mathcal{P}_{38}\mathscr{A}_8),
\end{aligned}
\tag{6.11}
$$

where all functions are taken at $p^2 = m^2_{3,phy}$.

As usual, we perform a chiral expansion to the quantities

$$
\begin{aligned}
F_\phi &= F_{\phi,p^2} + F_{\phi,p^4} + F_{\phi,p^6} + \mathcal{O}(p^8), \\
\mathscr{A}_\phi(p^2) &= \mathscr{A}_\phi^{(2)} + \mathscr{A}_\phi^{(4)}(p^2) + \mathscr{A}_\phi^{(6)}(p^2) + \mathcal{O}(p^8) \\
\Pi_{ij}(p^2) &= \Pi_{ij}^{(4)}(p^2) + \Pi_{ij}^{(6)}(p^2) + \mathcal{O}(p^8) \\
\mathcal{Z}_{ij}(p^2) &= \frac{\partial}{\partial p^2}\Pi_{ij} \equiv \mathcal{Z}_{ij}^{(2)}(p^2) + \mathcal{Z}_{ij}^{(4)}(p^2) + \mathcal{Z}_{ij}^{(6)}(p^2) + \mathcal{O}(p^8),
\end{aligned}
\tag{6.12}
$$

after expanding around m^2_{3,p^2}, one explicitly obtains for the π^0_3 case

$$
\begin{aligned}
F_{3,p^2} =& \mathscr{A}_3^{(2)} \\
F_{3,p^4} =& \mathscr{A}_3^{(4)} - \frac{1}{2}\mathcal{Z}_{33}^{(2)}\mathscr{A}_3^{(2)} - \frac{\Pi_{38}^{(4)}}{\Delta m_{38}^2}\mathscr{A}_8^{(2)} \\
F_{3,p^6} =& \mathscr{A}_\phi^{(6)} - \frac{1}{2}\mathcal{Z}_{33}^{(4)}\mathscr{A}_3^{(2)} - \frac{1}{2}\mathcal{Z}_{33}^{(2)}\mathscr{A}_3^{(4)} + \frac{3}{8}\left(\mathcal{Z}_{33}^{(2)}\right)^2\mathscr{A}_3^{(2)} \\
&+ \frac{\mathcal{Z}_{38}^{(2)}\Pi_{38}^{(4)}}{\Delta m_{38}^2}\mathscr{A}_3^{(2)} - \frac{1}{2}\left(\frac{\Pi_{38}^{(4)}}{\Delta m_{38}^2}\right)^2\mathscr{A}_3^{(2)} - \frac{\Pi_{38}^{(4)}}{\Delta m_{38}^2}\mathscr{A}_8^{(4)} \\
&- \frac{\Pi_{38}^{(6)}}{\Delta m_{38}^2}\mathscr{A}_8^{(2)} + \frac{\Pi_{38}^{(4)}\Pi_{88}^{(4)}}{(\Delta m_{38}^2)^2}\mathscr{A}_8^{(2)} + \frac{1}{2}\mathcal{Z}_{33}^{(2)}\frac{\Pi_{38}^{(4)}}{\Delta m_{38}^2}\mathscr{A}_8^{(2)},
\end{aligned}
\tag{6.13}
$$

where all functions are evaluated at $p^2 = m^2_{3,p^2}$. Again, the case for η_8 can be obtained by using the same formula with interchanging $3 \leftrightarrow 8$, and the remaining decay constants can be obtained by using the formula in the isospin symmetric limit (Eq. 5.37).

6.2 EM Correction

Another notable distinction between up and down quarks is their differing electric charges. Consequently, when considering the coupling of quarks to an external EM four-vector potential A_μ, the corresponding field is given by

$$
r_\mu = l_\mu = -A_\mu Q, \tag{6.14}
$$

where $Q = \mathrm{diag}(2e/3, -e/3, -e/3)$ is the quark-charge matrix and e being the elementary charge. The external field part of QCD Lagrangian is therefore (focusing on EM only)

$$
\begin{aligned}
\mathscr{L}_{ext} =& \bar{q}_L\gamma^\mu l_\mu q_L + \bar{q}_R\gamma^\mu r_\mu q_R \\
=& A_\mu(\bar{q}_L\gamma^\mu Q q_L + \bar{q}_R\gamma^\mu Q q_R).
\end{aligned}
\tag{6.15}
$$

However, since Q is not proportional to the unit matrix, this term is not invariant under chiral transformation. To that end, we employ a similar trick as introducing the quark mass matrix into the ChPT Lagrangian.

To ensure the chiral symmetry of the QCD Lagrangian, we introduce local spurious Q_R and Q_L instead of the charge matrix Q[21], which transform under $SU(3)_L \times SU(3)_R$ as

$$\begin{aligned} Q_L &\rightarrow \mathcal{L} Q_L \mathcal{L}^\dagger, \\ Q_R &\rightarrow \mathcal{R} Q_R \mathcal{R}^\dagger. \end{aligned} \tag{6.16}$$

The Lagrangian is modified,

$$\mathscr{L}_{ext} = A_\mu(\bar{q}_L \gamma^\mu Q_L q_L + \bar{q}_R \gamma^\mu Q_R q_R), \tag{6.17}$$

and is now invariant under chiral transformation.

We have now established the coupling of photons to quarks. For low-energy photons, this coupling is illustrated in Fig. 6a, where photons depart from the quark. The corresponding vertices in ChPT are depicted in Fig. 6b, with the photon coupling expressed in the covariant derivative, treating the photon as an external field. Conversely, for high-energy virtual photons, although the vertex configuration remains the same as that for low-energy photons, these photons are not observable as they are immediately absorbed by another quark, as shown in Fig. 7a. For such photons, it is necessary to construct a new type of effective Lagrangian, represented by the shadow vertex in Fig. 7b.

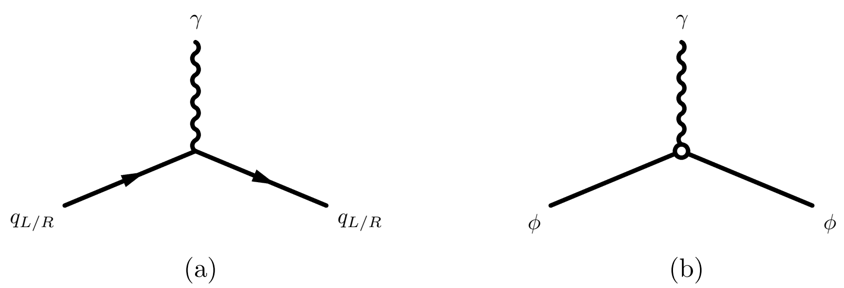


Figure 6: Low-energy photon coupling. **(a)**. The coupling of soft photon with quarks from QCD. **(b)**. The coupling of soft photon with corresponding pNGB from ChPT, the empty circle represents the vertices from Eq. 4.24.

Hence, considering that the low-energy effective Lagrangian must encompass all permissible terms consistent with the symmetry, the inclusion of these new blocks $Q_{L,R}$ entails additional terms in the lowest-order ChPT Lagrangian. Now incorporating EM interactions (considering also the relevant kinetic term),

$$\begin{aligned} \mathscr{L}_\chi =& \mathscr{L}_{e^2} + \mathscr{L}_{p^2} \\ =& -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{\lambda}{2} (\partial_\mu A^\mu)^2 + C \left\langle Q_R U Q_L U^\dagger \right\rangle + \mathscr{L}_{p^2} \\ =& -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2 + C \left\langle Q U Q U^\dagger \right\rangle + \mathscr{L}_{p^2}, \end{aligned} \tag{6.18}$$

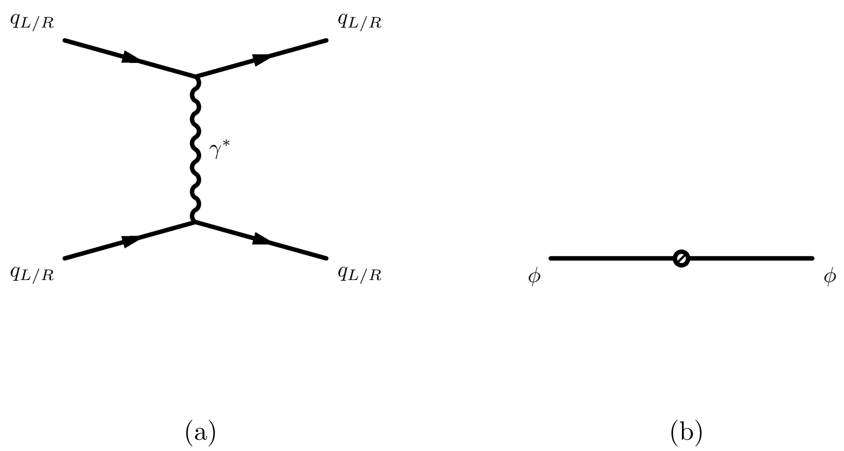


Figure 7: High-energy photon coupling. **(a)**. The coupling of high-energy virtual photon with quarks from QCD. **(b)**. The coupling of virtual photon with corresponding pNGB from ChPT, the shaded circle represents the new vertex in LO.

where in the second equality we choose Feynman gauge ($\lambda = 1$). $F_{\mu\nu}$ is the field strength tensor of the photon field A_μ, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, λ is the gauge fixing parameter and C is a low-energy constant which is independent of the quark masses and not fixed by chiral symmetry alone. At the end of the expression, we identify $Q_{L.R}$ with the charge matrix Q just as we replace s with M in the case of mass.

Now at lowest-order, the quadratic terms of this extra Lagrangian is

$$\mathscr{L}_{e^2} \supset -\frac{2e^2 C}{F_0^2}(\pi^+\pi^- + K^+K^-), \tag{6.19}$$

thus it contributes an equal amount to the square of the masses of charged fields $\pi^\pm$ and $K^\pm$,

$$m^2_{\pi^\pm, e^2} = m^2_{K^\pm, e^2} = 2e^2 \frac{C}{F_0^2}, \tag{6.20}$$

but does not contribute to the mass of neutral fields π^0, K^0, $\bar{K}^0$ or η, nor does it affect $\pi^0 - \eta$ mixing.

The calculations conducted so far in this subsection have been confined to the lowest order. Importantly, there is no foundational reason to assume that EM corrections affect only charged fields, especially at higher orders. Indeed, this assumption does not hold even at the one-loop level.

In this thesis, we limit our analysis to the one-photon contribution to the mass, corresponding to the e^2 order, Therefore, we can ignore any terms at $\mathcal{O}(e^4)$, which also helps us get rid of the e^4 order divergence. However, prior to calculating the additional one-photon loop, it is necessary to incorporate contributions from chiral invariant local terms of order

e^2. The e^2p^2 order Lagrangian was derived by Urech [6], which consists of 14 terms, each with an LEC K_i, has the form:

$$\begin{aligned}\mathscr{L}_{e^2p^2} =&K_1F_0^2 \langle (D_\mu U)^\dagger (D^\mu U)\rangle \langle Q^2\rangle + K_2F_0^2 \langle (D_\mu U)^\dagger (D^\mu U)\rangle \langle QUQU^\dagger\rangle \\ &+K_3F_0^2[\langle (D_\mu U)^\dagger QU\rangle \langle (D^\mu U)^\dagger QU\rangle + \langle (D_\mu U)QU^\dagger\rangle \langle (D^\mu U)QU^\dagger\rangle] \\ &+K_4F_0^2 \langle (D_\mu U)^\dagger QU\rangle \langle (D^\mu U)QU^\dagger\rangle \\ &+K_5F_0^2 \langle ((D_\mu U)^\dagger (D^\mu U) + (D_\mu U)(D^\mu U^\dagger))Q^2\rangle \\ &+K_6F_0^2 \langle (D_\mu U)^\dagger (D^\mu U)QU^\dagger QU + (D_\mu U)(D^\mu U^\dagger)QUQU^\dagger\rangle \\ &+K_7F_0^2 \langle \chi U^\dagger + U\chi^\dagger\rangle \langle Q^2\rangle + K_8F_0^2 \langle \chi U^\dagger + U\chi^\dagger\rangle \langle QUQU^\dagger\rangle \\ &+K_9F_0^2 \langle (\chi U^\dagger + U\chi^\dagger + \chi^\dagger U + U^\dagger\chi)Q^2\rangle \\ &+K_{10}F_0^2 \langle (\chi U^\dagger + U\chi^\dagger)QUQU^\dagger + (\chi^\dagger U + U^\dagger\chi)QU^\dagger QU\rangle \\ &+K_{11}F_0^2 \langle (\chi U^\dagger - U\chi^\dagger)QUQU^\dagger + (\chi^\dagger U - U^\dagger\chi)QU^\dagger QU\rangle \\ &+K_{12}F_0^2 \langle (D^\mu U)^\dagger [c_\mu^R Q, Q]U + (D^\mu U)[c_\mu^L Q, Q]U^\dagger\rangle \\ &+K_{13}F_0^2 \langle c^{R\mu}QUc_\mu^L QU^\dagger\rangle + K_{14}F_0^2 \langle c^{R\mu}Qc_\mu^R Q + c^{L\mu}Qc_\mu^L Q\rangle ,\end{aligned} \tag{6.21}$$

in which $c_\mu^R Q = -i[r_\mu, Q]$ and $c_\mu^L Q = -i[l_\mu, Q]$. In the same vein as the definition of L_i, the coupling constant K_i are characterized as

$$K_i = (\mu)^{-2\epsilon}\left(\frac{-1}{32\pi^2}\Sigma_i\lambda + K_i^r(\mu) + \mathcal{O}(\epsilon)\right), \tag{6.22}$$

the coefficients Σ_i are given in Ref. [6].

The methodology for calculating masses with EM contributions closely mirrors the original approach. At the one-loop level, this process requires integrating two additional diagrams, as illustrated in Figures 8a and 8b. Additionally, to capture the EM correction terms, we must utilize a modified Lagrangian with extra EM correction terms instead of the old one to calculate the original one-loop again to obtain additional terms.

In fact, we can also only consider the new contributions and treat them as the EM correction to the original mass. Similarly, we need first to identify the vertices that are involved in the one-photon loop calculation from the original Lagrangian (Eq. 4.24), which are

$$\begin{aligned}\mathscr{L}_{p^2}^{2\phi A_\mu} &= -\frac{1}{2}\langle \phi\partial^\mu\phi QA_\mu\rangle + \frac{1}{2}\langle \phi QA_\mu\partial^\mu\phi\rangle , \\ \mathscr{L}_{p^2}^{2\phi 2A_\mu} &= \frac{1}{2}\langle \phi^2Q^2A^\mu A_\mu\rangle - \frac{1}{2}\langle \phi QA^\mu\phi QA_\mu\rangle .\end{aligned} \tag{6.23}$$

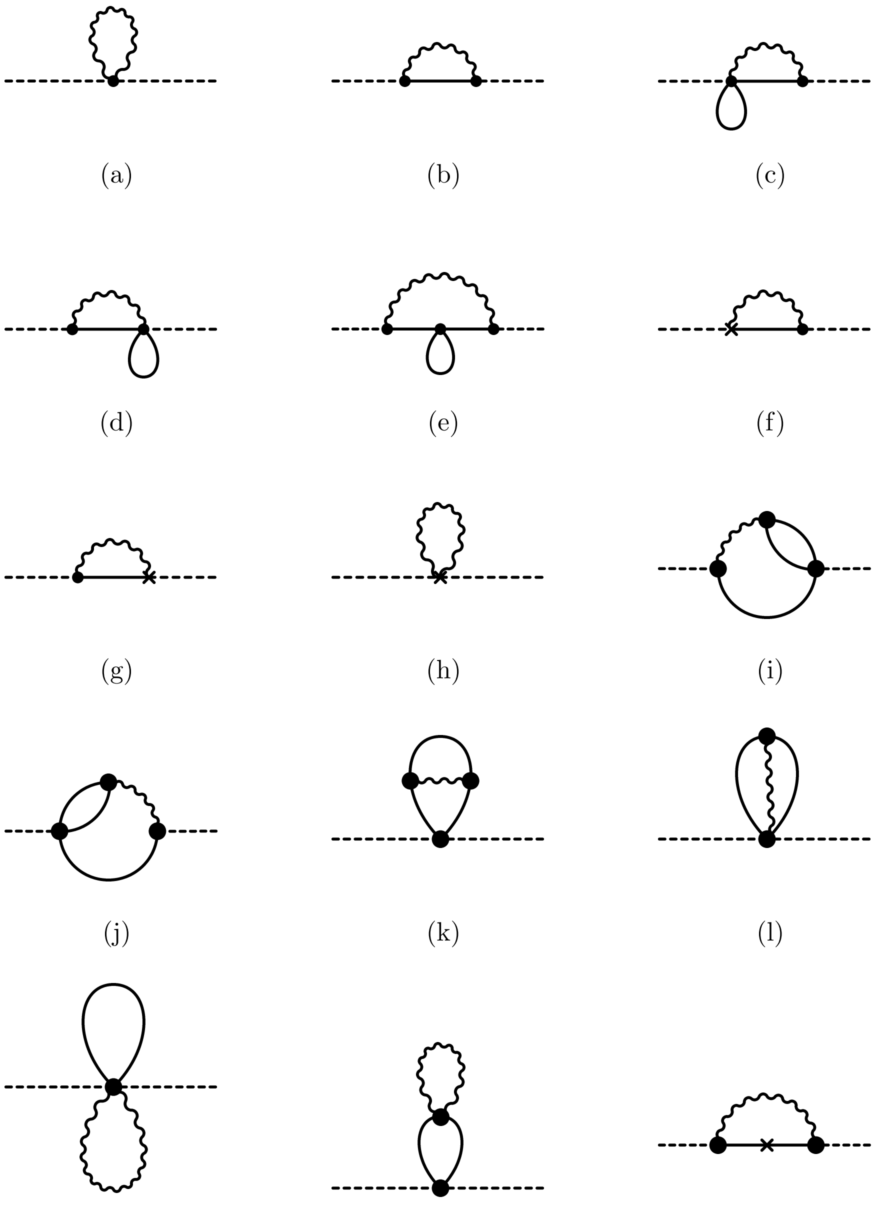

(a) (b) (c) (d) (e) (f) (g) (h) (i) (j) (k) (l) (m) (n) (o)

Figure 8: The set of additional diagrams contributing to the 1PI when we consider the contribution of one-photon. (**a**) $-$ (**o**). The respective diagrams for the calculation of mass are those the dashed lines are replaced with the external legs of Fig. 4c . The plain line is a pNGB propagator, the wiggly line is a photon propagator, a dot and a cross represent a vertex of $\mathcal{O}(p^2)$ and $\mathcal{O}(p^4)$, respectively.

In addition with the vertices from the extra Lagrangian (Eq. 6.18 and Eq. 6.21)

$$
\begin{aligned}
\mathscr{L}_{e^2}^{4\phi} =& \frac{C}{12F_0^4}\left(\langle Q^2\phi^4\rangle - 4\langle Q\phi Q\phi^3\rangle + \langle Q\phi^2 Q\phi^2\rangle\right), \\
\mathscr{L}_{e^2p^2}^{2\phi} =& (K_1+K_2)\langle Q^2\rangle\langle\partial^\mu\phi\partial_\mu\phi\rangle + (-2K_3+K_4)\langle Q\partial^\mu\phi\rangle\langle Q\partial_\mu\phi\rangle \\
&+ 2(K_5+K_6)\langle Q^2\partial^\mu\phi\partial_\mu\phi\rangle - K_7\langle Q^2\rangle\langle\phi^2\chi\rangle \\
&- K_8\left(\langle Q^2\rangle\langle\phi^2\chi\rangle + 2\langle\chi\rangle\langle Q^2\phi^2\rangle - 2\langle\chi\rangle\langle Q\phi Q\phi\rangle\right) \\
&- K_9\left(\langle Q^2\phi^2\chi\rangle + \langle Q^2\chi\phi^2\rangle\right) - 2K_{10}\langle Q\phi^2 Q\chi\rangle \\
&+ (K_{10}+K_{11})\left(-\langle Q^2\phi^2\chi\rangle - 2\langle Q^2\phi\chi\phi\rangle - \langle Q^2\chi\phi^2\rangle\right. \\
&\left.+2\langle Q\phi Q\phi\chi\rangle + 2\langle Q\phi Q\chi\phi\rangle\right).
\end{aligned}
\tag{6.24}
$$

Consider the cases where the incoming and outgoing particle are both $\pi^\pm$ fields, after applying the Feynman rules for scalar particle and photon to the diagrams that are involved (Fig. 4d-4f and Fig. 8a-8b), the self energy can be obtained as

$$
\begin{aligned}
\Pi^{(4)}_{\pi^\pm\pi^\pm,e^2p^2}(p^2) =& -i\mathscr{M}_{\pi^\pm\pi^\pm,e^2p^2} \\
=& -\frac{e^2C}{3F_0^4}[16A(m^2_{\pi^\pm,p^2}) + 2A(m^2_{\pi^0,p^2}) + 8A(m^2_{K^\pm,p^2}) + A(m^2_{K^0,p^2}) \\
&+ 2A(m^2_{\eta,p^2})\sin^2\epsilon] - \frac{e^2}{2}A(m_{A_\mu}) + e^2[m_{A_\mu}B(m^2_{\pi^\pm,p^2}, m_{A_\mu}, p^2) \\
&- 2m^2_{\pi^\pm,p^2}B(m^2_{\pi^\pm,p^2}, m_{A_\mu}, p^2) - 2A(m_{A_\mu}) + A(m^2_{\pi^\pm,p^2}) \\
&- 2p^2B(m^2_{\pi^\pm,p^2}, m_{A_\mu}, p^2)] + \frac{4}{9}e^2p^2(6K_1+6K_2+5K_5+5K_6) \\
&- 4e^2\left[\frac{2}{3}\hat{m}K_7 + \left(m_s + \frac{8}{3}\hat{m}\right)K_8 + \left(\frac{1}{3}\widetilde{m} + \frac{5}{9}\hat{m}\right)K_9\right. \\
&\left.+\left(\frac{1}{3}\widetilde{m} + \frac{23}{9}\hat{m}\right)K_{10} + 2\hat{m}K_{11}\right].
\end{aligned}
\tag{6.25}
$$

Combining the regularization and renormalization procedure, we can get the result for EM correction at one-loop order

$$
\begin{aligned}
m^2_{\pi^\pm,e^2p^2} =& -\Pi^{(4)}_{\pi^\pm\pi^\pm,e^2p^2}(m^2_{\pi^\pm,p^2}) \\
=& -\frac{e^2}{8\pi^2}\frac{C}{F_0^4}\left[2m^2_{\pi,p^2}\ln m^2_{\pi^\pm,p^2} + m^2_{K^\pm,p^2}\ln m^2_{K^\pm,p^2}\right] \\
&- \frac{e^2}{16\pi^2}m^2_{\pi^\pm,p^2}\left(3\ln m^2_{\pi^\pm,p^2} - 4\right) + \frac{4e^2}{3}\widetilde{m}\left(-3K_8^r + K_9^r + K_{10}^r\right) \\
&- \frac{4e^2}{9}m^2_{\pi^\pm,p^2}(6K_1^r + 6K_2^r + 5K_5^r + 5K_6^r - 6K_7^r \\
&- 15K_8^r - 5K_9^r - 23K_{10}^r - 18K_{11}^r) + 8e^2m^2_{K^\pm,p^2}K_8^r \\
&- \frac{16e^2C}{F_0^4}\left[\left(m^2_{\pi^\pm,p^2} + 2m^2_{K^\pm,p^2} - \widetilde{m}\right)L_4^r + m^2_{\pi^\pm,p^2}L_5^r\right],
\end{aligned}
\tag{6.26}
$$

in which the mass terms with superscript or subscript (i.e. $m^2_{\pi^\pm,p^2}$ or $m^2_{\eta_8,p^2}$) are the bare mass with quark mass correction in Eq. 6.4. While setting $\widetilde{m}=0$, we recover the results of Ref. [6].

For the $\mathcal{O}(e^2p^4)$ order, the basic principle is the same, we need first obtain all the vertices we need and performing the calculation of all the two-loop diagrams. However, since the most general $\mathscr{L}_{e^2p^4}$ has not been constructed yet, we have to introduce a local counterterm based on power counting to make the finial amplitude finite.

For $\mathcal{O}(e^2p^4)$ order, all the possible counter terms are

$$\begin{aligned} -\Pi^{(6)}_{e^2p^4,\phi}\Big|_{CT} =& R_{1,\phi}m^4_{\pi^\pm,p^2}e^2 + R_{2,\phi}m^4_{K^\pm,p^2}e^2 + R_{3,\phi}\widetilde{m}^2e^2 \\ &+ R_{4,\phi}m^2_{\pi^\pm,p^2}m^2_{K^\pm,p^2}e^2 + R_{5,\phi}m^2_{\pi^\pm,p^2}\widetilde{m}e^2 + R_{6,\phi}m^2_{K^\pm,p^2}\widetilde{m}e^2 \end{aligned} \tag{6.27}$$

in which the coefficients are defined as

$$\begin{aligned} R_{i,\phi} \equiv& (\mu c)^{-4\varepsilon}\left(\frac{\beta_{2i,\phi}}{\varepsilon^2} + \frac{\beta_{1i,\phi}}{\varepsilon^1} + R^r_{i,\phi}(\mu)\right) \\ =& (\mu)^{-4\varepsilon}\left(\beta_{2i,\phi}\lambda^2 + \beta_{1i,\phi}\lambda + R'^r_{i,\phi}(\mu) + \mathcal{O}(\varepsilon)\right), \end{aligned} \tag{6.28}$$

to cancel the divergent terms in loop integrals. Note that we utilize λ and λ^2, rather than $\lambda_{1,2}$, as used in Eq. 5.22 and 5.23. This adjustment is achieved by shifting the finite term $R^r_{i,\phi}(\mu)$ to $R'^r_{i,\phi}(\mu)$. The detailed expressions for $\beta_{2i,\phi}$ and $\beta_{1i,\phi}$ are listed in Appendix. A.

7 Analytical Results

Given the extensive length of the explicit formulae for the two-loop order results, this section will only present a sample of the one-loop contributions. The set of one-loop results is detailed in Appendix B. Meanwhile, the two-loop results are compiled in the supplementary material.

7.1 Masses

In general, the physical masses can be split to two-loop order as follows:

$$m^2_{\phi,phy} = m^2_{\phi,p^2} + m^2_{\phi,e^2} + m^2_{\phi,p^4} + m^2_{\phi,e^2p^2} + m^2_{\phi,p^6} + m^2_{\phi,e^2p^4} + \mathcal{O}(e^4,p^8,e^2p^6), \tag{7.1}$$

in which the quark masses correction are considered, and m^2_{ϕ,p^2} and m^2_{ϕ,e^2} have already been given in Eq. 6.4 and Eq. 6.20, respectively. In this thesis, quark mass corrections at the $\mathcal{O}(e^2p^4)$ order are not included, as their inclusion would result in excessively lengthy expression.

For simplicity, we only give the formulae for charged pion here, since the $m^2_{\pi^\pm,e^2p^2}$ has been given in Eq. 6.26, the remaining one-loop correction from $\mathcal{O}(p^4)$ order is

$$\begin{aligned} m^2_{\pi^\pm,p^4} =& \frac{m^2_{\pi^\pm,p^2}}{F_0^2}\left\{\left[\frac{m^2_{\pi^\pm,p^2}}{32\pi^2} - \frac{\sin^2\epsilon}{72\pi^2}(m^2_{K^\pm,p^2} + 2m^2_{\pi^\pm,p^2} - \frac{\widetilde{m}}{2}) + \frac{\widetilde{m}\sin 2\epsilon}{96\sqrt{3}\pi^2}\right]\ln m^2_{\pi_3^0,p^2}\right. \\ & \left[-\frac{\cos^2\epsilon}{144\pi^2}(2m^2_{K^\pm,p^2} - \widetilde{m}) + \frac{m^2_{\pi^\pm,p^2}}{288\pi^2}(1+8\sin^2\epsilon) - \frac{\widetilde{m}\sin 2\epsilon}{96\sqrt{3}\pi^2}\right]\ln m^2_{\eta_8,p^2} \\ & \left. +8[(2m^2_{K^\pm,p^2} + m^2_{\pi^\pm,p^2} - \widetilde{m})(2L_6^r - L_4^r) + m^2_{\pi^\pm,p^2}(2L_8^r - L_5^r)]\right\} \end{aligned} \tag{7.2}$$

Again, the mass terms with superscript or subscript (i.e. $m^2_{\pi^\pm,p^2}$) are the bare mass with quark mass correction. Setting $\widetilde{m} = 0$, we find that this result agrees with that presented in Eq. 5.24.

7.2 Decay Constants

Since the electromagnetic correction to the decay constant involves more kinds of external legs than we showed in Fig. 4a [22], we only calculated the decay constants with quark mass correction here.

Analogously, the decay constants have the form

$$F_\phi = F_{\phi,p^2} + F_{\phi,p^4} + F_{\phi,p^6} + \mathcal{O}(p^8), \tag{7.3}$$

where the LO F_{ϕ,p^2} is just the LEC F_0 in the ChPT Lagrangian. In this subsection, we will use $F_{\pi^\pm}$ as an illustration, the one-loop contribution is

$$\begin{aligned} F_{\pi^\pm,p^4} =& + 4(2m^2_{K^\pm,p^2} - \widetilde{m})\frac{L_4^r}{F_0} + 8m^2_{\pi^\pm,p^2}\frac{L_4^r}{F_0} + 4m^2_{\pi^\pm,p^2}\frac{L_5^r}{F_0} \\ & - \frac{1}{32F_0\pi^2}m^2_{\pi^\pm,p^2}\ln m^2_{\pi^\pm,p^2} \\ & - \frac{1}{192F_0\pi^2}\left(\sqrt{3}\widetilde{m}\sin 2\epsilon + 6m^2_{\pi^\pm,p^2} - 6m^2_{\pi^\pm,p^2}\sin^2\epsilon\right)\ln m^2_{\pi_3^0,p^2} \\ & - \frac{1}{64F_0\pi^2}m^2_{K^\pm,p^2}\ln m^2_{K^\pm,p^2} \\ & + \frac{1}{64F_0\pi^2}(\widetilde{m} - m^2_{K^\pm,p^2})\ln m^2_{K^0,p^2} \\ & + \frac{1}{192F_0\pi^2}(\sqrt{3}\widetilde{m}\sin 2\epsilon - 6m^2_{\pi^\pm,p^2}\sin^2\epsilon)\ln m^2_{\eta_8,p^2}. \end{aligned} \tag{7.4}$$

Several properties emerge from our results. Firstly, we observe that the expressions for masses and decay constants are finite. This finiteness is achieved as the divergent parts of the LECs in the Lagrangian and the local counterterms, which we introduced manually, effectively cancel the infinite terms arising from the loop integrals.

Furthermore, the analysis reveals generally two types of terms in the results. First, there are terms that are analytic in the bare meson masses, exemplified by expressions like m^2_{ϕ,p^2} multiplied by the renormalized LECs, i.e., L^r_i. However, there are also non-analytic terms of the type, such as $\ln m^2_{\phi,p^2}$ (the so-called chiral logarithms) or $\mathrm{Li}_2(x)$ which are contained within $\bar{H}(m_1, m_2, m_3, p^2)$ (which arise from the two-loop contributions and are not shown here), which do not introduce new parameters. Such a behavior is an illustration of the mechanism found by Li and Pagels [23], who noticed that a perturbation theory around a symmetry which is realized in the Nambu-Goldstone mode results in both analytic as well as non-analytic expressions in the perturbation theory [10].

8 Conclusion

In this thesis, we have performed detailed calculations of the masses and decay constants of eight mesons within ChPT up to NNLO, with a specific focus on incorporating isospin breaking effects due to quark mass differences and EM interactions. Our analysis of the meson masses included both quark mass and EM corrections. By comparing our results with existing studies, notably the works referenced [3, 4, 6], we confirmed the consistency of our findings with previous research, thereby strengthening the validity of our computational approaches and theoretical models.

For the EM corrections at the two-loop level, despite the absence of a fully developed $\mathcal{O}(e^2p^4)$ order ChPT Lagrangian, our confidence in the results remains strong and stems from several factors. Firstly, we successfully canceled all logarithmic and non-local divergences in the infinite parts of the additional amplitude. This ensures that our local counterterm effectively cancels all divergences from the loop integral. Moreover, in the analysis of the loop integrals in diagrams shown in Fig. 8i and 8j, we identified and confirmed the cancellation of λ^3 terms that arise due to the recurrence relation when simplifying the tensor integral to a scalar integral. This aspect of our research highlights the need for ongoing development in the theoretical structure of ChPT, especially regarding higher-order corrections.

The calculation of decay constants proved more challenging, primarily due to the increased complexity of two-loop diagrams involving virtual photons. Given these complexities, we restricted our corrections to those involving quark masses only and used the results from Ref. [24] to systematically remove all divergent terms from the amplitudes, affirming the accuracy of our corrections.

Looking forward, while in this thesis the counterterms for the two-loop calculations are relatively straightforward, but owing to the current gaps in the $\mathcal{O}(e^2p^4)$ order ChPT Lagrangian, there is a clear and pressing need for further theoretical advancements. The development of this new Lagrangian will be crucial for enhancing our understanding of mesonic properties at higher orders. Additionally, EM corrections to the decay constants also require additional study. Presently, we have only considered $\mathcal{O}(e^2)$ order EM corrections. As experimental techniques advance and the precision of measurements improves, addressing higher-order EM corrections, which involve more complex multi-photon diagrams, will become increasingly important. This not only poses a significant theoretical challenge but also a substantial opportunity to deepen our understanding of QCD in the non-perturbative regime.

Overall, the work presented in this thesis contributes to the broader effort of improving the precision and scope of theoretical predictions in particle physics, offering insights that could be pivotal for future experimental and theoretical explorations in the field.

Acknowledgments

In the end of this thesis, I would like to express my heartfelt gratitude to all those who have contributed, directly or indirectly, to the completion of this thesis. Foremost, I extend my deepest appreciation to my supervisor, Johan Bijnens, whose guidance and support have been invaluable throughout this project. His kindness, wisdom, and hard work have inspired me and significantly contributed to my growth as a scientist.

I am also grateful to my colleague August, with whom I shared an office and engaged in numerous discussions about research and life. His companionship and insights have enriched my research experience. Additionally, I extend my thanks to all my friends who have accompanied me during my time in Lund, making my life vibrant and enjoyable.

Finally, I would like to thank my family for their unwavering support. Their encouragement and understanding have been a constant source of strength for me. Without them, I would not have been able to pursue my studies and enjoy the academic journey. Their love and support mean the world to me.

A $\mathcal{O}(e^2p^4)$ Counterterms

In this Appendix, we list the coefficients for our local counterterm as detailed in Eq. 6.28. However, as noted in Section 7, the quark mass corrections at the $\mathcal{O}(e^2p^4)$ order are not included in this thesis. Therefore, we have that

$$R_{3,\phi} = R_{5,\phi} = R_{6,\phi} = 0. \tag{A.1}$$

For the charged pion, we have

$$\begin{aligned}
\beta_{21,\pi^\pm} =& \frac{1}{F_0^2(16\pi)^2}\left(\frac{35}{3} - \frac{41}{18}\frac{C}{F_0^4}\right), \\
\beta_{11,\pi^\pm} =& -\frac{185}{4608}\frac{C}{F_0^6\pi^4} - \frac{205}{3072}\frac{e^2}{F_0^2\pi^4} \\
&- \left(15L_8^r + 19L_6^r - \frac{33}{2}L_5^r - \frac{43}{2}L_4^r - \frac{53}{27}L_3^r - \frac{17}{9}L_2^r - \frac{10}{3}L_1^r\right)\frac{C}{F_0^6\pi^2} \\
&- \left(\frac{1}{2}L_9^r + \frac{3}{2}L_8^r + \frac{3}{2}L_6^r + \frac{3}{4}L_5^r + \frac{3}{4}L_4^r\right)\frac{1}{F_0^2\pi^2} \\
&- \left(\frac{38}{9}K_{11}^r + \frac{1411}{324}K_{10}^r + \frac{43}{324}K_9^r + \frac{185}{54}K_8^r + \frac{4}{27}K_7^r - \frac{25}{18}K_6^r - \frac{7}{54}K_5^r\right. \\
&\left. - \frac{53}{216}K_4^r - \frac{55}{108}K_3^r - \frac{77}{54}K_2^r - \frac{4}{27}K_1^r\right)\frac{1}{F_0^2\pi^2}.
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\beta_{22,\pi^\pm} =& \frac{1}{F_0^2(16\pi)^2}\left(\frac{3}{2} + \frac{2}{9}\frac{C}{F_0^4}\right), \\
\beta_{12,\pi^\pm} =& -\frac{5}{768}\frac{1}{F_0^2\pi^4} - \left(4L_8^r + 8L_6^r - 4L_5^r - 8L_4^r\right)\frac{C}{F_0^6\pi^2} \\
&- \left(K_{11}^r + K_{10}^r + \frac{22}{9}K_8^r - \frac{35}{54}K_6^r - \frac{13}{9}K_2^r\right)\frac{1}{F_0^2\pi^2}.
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\beta_{24,\pi^\pm} =& \frac{1}{F_0^2(16\pi)^2}\left(\frac{29}{6} - \frac{47}{6}\frac{C}{F_0^4}\right), \\
\beta_{14,\pi^\pm} =& -\frac{131}{9216}\frac{C}{F_0^6\pi^4} - \frac{23}{3072}\frac{1}{F_0^2\pi^4} - \left(3L_6^r + \frac{3}{2}L_4^r\right)\frac{1}{F_0^2\pi^2} \\
&- \left(2L_8^r + 30L_6^r - \frac{7}{2}L_5^r - 41L_4^r - \frac{233}{54}L_3^r - \frac{46}{9}L_2^r - \frac{40}{3}L_1^r\right)\frac{C}{F_0^6\pi^2} \\
&- \left(\frac{49}{36}K_{11}^r + \frac{125}{81}K_{10}^r + \frac{59}{324}K_9^r + \frac{163}{54}K_8^r + \frac{13}{54}K_7^r - \frac{85}{216}K_6^r - \frac{47}{216}K_5^r\right. \\
&\left. - \frac{1}{54}K_4^r + \frac{1}{27}K_3^r - \frac{14}{27}K_2^r - \frac{13}{54}K_1^r\right)\frac{1}{F_0^2\pi^2}.
\end{aligned} \tag{A.4}$$

For the neutral pion we obtain

$$
\begin{aligned}
\beta_{21,\pi^0} =& \frac{1}{F_0^2(16\pi)^2}\left(-3-\frac{157}{18}\frac{C}{F_0^4}\right), \\
\beta_{11,\pi^0} =& -\frac{47}{3072}\frac{C}{F_0^6\pi^4}+\frac{1}{384}\frac{1}{F_0^2\pi^4} \\
& -\left(5L_8^r+7L_6^r-\frac{7}{2}L_5^r-\frac{13}{2}L_4^r+4L_3^r+2L_2^r+8L_1^r\right)\frac{C}{F_0^6\pi^2} \\
& -\left(\frac{43}{324}K_{10}^r+\frac{43}{324}K_9^r+\frac{43}{108}K_8^r+\frac{4}{27}K_7^r-\frac{17}{27}K_6^r-\frac{7}{54}K_5^r-\frac{67}{108}K_4^r\right. \\
& \left.+\frac{67}{54}K_3^r-\frac{35}{54}K_2^r-\frac{4}{27}K_1^r\right)\frac{1}{F_0^2\pi^2}.
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
\beta_{22,\pi^0} =& 0, \\
\beta_{12,\pi^0} =& -\frac{3}{2}\frac{1}{F_0^2(16\pi)^2}.
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
\beta_{24,\pi^0} =& -\frac{31}{18}\frac{C}{F_0^6(16\pi)^2}, \\
\beta_{14,\pi^0} =& -\frac{17}{1024}\frac{C}{F_0^6\pi^4}-\left(2L_8^r+10L_6^r-\frac{3}{2}L_5^r-7L_4^r+\frac{5}{2}L_3^r+2L_2^r+8L_1^r\right)\frac{C}{F_0^6\pi^2} \\
& -\left(\frac{1}{4}K_{11}^r+\frac{35}{81}K_{10}^r+\frac{59}{324}K_9^r+\frac{13}{54}K_8^r+\frac{13}{54}K_7^r-\frac{101}{216}K_6^r-\frac{47}{216}K_5^r\right. \\
& \left.-\frac{29}{108}K_4^r+\frac{29}{54}K_3^r-\frac{20}{27}K_2^r-\frac{13}{54}K_1^r\right)\frac{1}{F_0^2\pi^2}.
\end{aligned}
\tag{A.7}
$$

For charged kaon, we obtain

$$
\begin{aligned}
\beta_{21,K^\pm} =& \frac{3}{2}\frac{1}{F_0^2(16\pi)^2}, \\
\beta_{11,K^\pm} =& -\frac{1}{384}\frac{1}{F_0^2\pi^4}-\left(4L_7^r+3L_6^r-\frac{11}{3}L_5^r-3L_4^r\right)\frac{C}{F_0^6\pi^2} \\
& -\left(\frac{11}{12}K_{11}^r+\frac{209}{216}K_{10}^r+\frac{11}{216}K_9^r+\frac{83}{72}K_8^r-\frac{11}{24}K_6^r-\frac{13}{216}K_5^r\right. \\
& \left.-\frac{5}{72}K_4^r+\frac{5}{36}K_3^r-\frac{7}{9}K_2^r\right)\frac{1}{F_0^2\pi^2}.
\end{aligned}
\tag{A.8}
$$

$$
\begin{aligned}
\beta_{22,K^\pm} =& \frac{1}{F_0^2(16\pi)^2}\left(\frac{40}{3}-\frac{119}{12}\frac{C}{F_0^4}\right),\\
\beta_{12,K^\pm} =& -\frac{823}{18432}\frac{C}{F_0^6\pi^4}-\frac{875}{9216}\frac{1}{F_0^2\pi^4}-\left(\frac{1}{2}L_9^r+\frac{3}{2}L_8^r+3L_6^r+\frac{3}{4}L_5^r+\frac{3}{2}L_4^r\right)\frac{1}{F_0^2\pi^2}\\
&-\left(18L_8^r+4L_7^r+32L_6^r-\frac{115}{6}L_5^r-39L_4^r-\frac{145}{27}L_3^r-\frac{49}{9}L_2^r-14L_1^r\right)\frac{C}{F_0^6\pi^2}\\
&-\left(\frac{89}{18}K_{11}^r+\frac{1631}{324}K_{10}^r+\frac{29}{324}K_9^r+\frac{295}{54}K_8^r+\frac{29}{108}K_7^r-\frac{16}{9}K_6^r-\frac{17}{108}K_5^r\right.\\
&\left.-\frac{29}{108}K_4^r-\frac{25}{54}K_3^r-\frac{239}{108}K_2^r-\frac{29}{108}K_1^r\right)\frac{1}{F_0^2\pi^2}.
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
\beta_{24,K^\pm} =& \frac{1}{F_0^2(16\pi)^2}\left(\frac{19}{6}+\frac{1}{36}\frac{C}{F_0^4}\right),\\
\beta_{14,K^\pm} =& -\frac{179}{18432}\frac{C}{F_0^6\pi^4}-\frac{121}{9216}\frac{1}{F_0^2\pi^4}-\left(\frac{3}{2}L_6^r+\frac{3}{4}L_4^r\right)\frac{1}{F_0^2\pi^2}\\
&-\left(-2L_8^r-8L_7^r+22L_6^r-\frac{7}{6}L_5^r-\frac{57}{2}L_4^r-\frac{49}{54}L_3^r-\frac{14}{9}L_2^r-\frac{8}{3}L_1^r\right)\frac{C}{F_0^6\pi^2}\\
&-\left(\frac{13}{18}K_{11}^r+\frac{581}{648}K_{10}^r+\frac{113}{648}K_9^r+\frac{491}{216}K_8^r+\frac{13}{108}K_7^r-\frac{7}{36}K_6^r-\frac{7}{54}K_5^r\right.\\
&\left.+\frac{2}{27}K_4^r-\frac{4}{27}K_3^r-\frac{43}{108}K_2^r-\frac{13}{108}K_1^r\right)\frac{1}{F_0^2\pi^2}.
\end{aligned}
\tag{A.10}
$$

For the neutral kaon we obtain

$$
\begin{aligned}
\beta_{21,K^0} =& 0,\\
\beta_{11,K^0} =& \frac{1}{3}\frac{1}{F_0^2(16\pi)^2}.
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
\beta_{22,K^0} =& \frac{8}{9}\frac{C}{F_0^6(16\pi)^2},\\
\beta_{12,K^0} =& -\frac{23}{1536}\frac{C}{F_0^6\pi^4}-\frac{1}{384}\frac{1}{F_0^2\pi^4}\\
&-\left(4L_8^r+8L_6^r-\frac{5}{2}L_5^r-8L_4^r+\frac{5}{2}L_3^r+2L_2^r+8L_1^r\right)\frac{C}{F_0^6\pi^2}\\
&-\left(\frac{1}{4}K_{11}^r+\frac{55}{162}K_{10}^r+\frac{29}{324}K_9^r+\frac{83}{108}K_8^r+\frac{29}{108}K_7^r-\frac{67}{216}K_6^r-\frac{13}{216}K_5^r\right.\\
&\left.-\frac{1}{54}K_4^r+\frac{1}{27}K_3^r-\frac{83}{108}K_2^r-\frac{29}{108}K_1^r\right)\frac{1}{F_0^2\pi^2}.
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
\beta_{24,K^0} =& \frac{4}{3}\frac{C}{F_0^6(16\pi)^2}, \\
\beta_{14,K^0} =& -\frac{5}{512}\frac{C}{F_0^6\pi^4} - \left(2L_8^r + 6L_6^r - \frac{3}{2}L_5^r - 7L_4^r + \frac{5}{2}L_3^r + 2L_2^r + 8L_1^r\right)\frac{C}{F_0^6\pi^2} \\
& - \left(\frac{1}{4}K_{11}^r + \frac{25}{81}K_{10}^r + \frac{19}{324}K_9^r + \frac{67}{108}K_8^r + \frac{13}{108}K_7^r - \frac{59}{216}K_6^r - \frac{5}{216}K_5^r\right. \\
& \left. -\frac{1}{27}K_4^r + \frac{2}{27}K_3^r - \frac{67}{108}K_2^r - \frac{13}{108}K_1^r\right)\frac{1}{F_0^2\pi^2}.
\end{aligned}
\tag{A.13}
$$

For the eta, we have

$$
\begin{aligned}
\beta_{21,\eta} =& \frac{1}{F_0^2(16\pi)^2}\left(1 + \frac{71}{54}\frac{C}{F_0^4}\right), \\
\beta_{11,\eta} =& -\frac{47}{9216}\frac{C}{F_0^6\pi^4} + \frac{1}{384}\frac{1}{F_0^2\pi^4} \\
& - \left(\frac{17}{3}L_8^r + \frac{40}{3}L_7^r - \frac{5}{3}L_6^r - \frac{25}{18}L_5^r + \frac{11}{6}L_4^r - \frac{2}{3}L_3^r - \frac{2}{3}L_2^r - \frac{8}{3}L_1^r\right)\frac{C}{F_0^6\pi^2} \\
& - \left(\frac{1}{3}K_{11}^r + \frac{461}{972}K_{10}^r + \frac{137}{972}K_9^r + \frac{41}{324}K_8^r + \frac{17}{81}K_7^r - \frac{17}{162}K_6^r - \frac{13}{81}K_5^r\right. \\
& \left. -\frac{17}{324}K_4^r + \frac{17}{162}K_3^r - \frac{7}{162}K_2^r - \frac{17}{81}K_1^r\right)\frac{1}{F_0^2\pi^2}.
\end{aligned}
\tag{A.14}
$$

$$
\begin{aligned}
\beta_{22,\eta} =& \frac{1}{F_0^2(16\pi)^2}\left(-2 - \frac{130}{27}\frac{C}{F_0^4}\right), \\
\beta_{12,\eta} =& -\frac{77}{2304}\frac{C}{F_0^6\pi^4} - \frac{1}{384}\frac{1}{F_0^2\pi^4} \\
& - \left(\frac{32}{3}L_8^r + \frac{32}{3}L_7^r + \frac{32}{3}L_6^r - \frac{46}{9}L_5^r - \frac{28}{3}L_4^r + \frac{14}{3}L_3^r + \frac{8}{3}L_2^r + \frac{32}{3}L_1^r\right)\frac{C}{F_0^6\pi^2} \\
& - \left(\frac{1}{3}K_{11}^r + \frac{131}{243}K_{10}^r + \frac{50}{243}K_9^r + \frac{77}{81}K_8^r + \frac{50}{81}K_7^r - \frac{125}{162}K_6^r - \frac{35}{162}K_5^r\right. \\
& \left. -\frac{35}{81}K_4^r + \frac{70}{81}K_3^r - \frac{104}{81}K_2^r - \frac{50}{81}K_1^r\right)\frac{1}{F_0^2\pi^2}.
\end{aligned}
\tag{A.15}
$$

$$
\begin{aligned}
\beta_{24,\eta} =& \frac{3}{2}\frac{C}{F_0^6(16\pi)^2}, \\
\beta_{14,\eta} =& -\left(-10L_8^r - 24L_7^r + 6L_6^r + \frac{13}{6}L_5^r - 7L_4^r + \frac{3}{2}L_3^r + 2L_2^r + 8L_1^r\right)\frac{C}{F_0^6\pi^2} \\
&- \left(-\frac{1}{4}K_{11}^r - \frac{191}{486}K_{10}^r - \frac{139}{972}K_9^r + \frac{5}{81}K_8^r - \frac{71}{162}K_7^r + \frac{79}{648}K_6^r + \frac{133}{648}K_5^r\right. \\
&\left.+\frac{49}{324}K_4^r - \frac{49}{162}K_3^r - \frac{5}{81}K_2^r + \frac{71}{162}K_1^r\right)\frac{1}{F_0^2\pi^2} + \frac{35}{3072}\frac{C}{F_0^6\pi^4}.
\end{aligned} \tag{A.16}
$$

B Explicit results for the masses and decay constants

In this appendix, we will present the result of $\mathcal{O}(p^4, e^2p^2)$, the two-loop order are collected in the supplement file since they are to lengthy. All the LO mass terms presented in the results represent bare masses, which include corrections for quark masses as detailed in Eq. 6.4. The results for the neutral pion π_3^0 and eta η, due to their length, are also included in the supplementary material together with the two-loop results.

B.1 Masses

The expression for mass are given in Eq. 7.1 combine with the result for $\pi^\pm$ in Sec. 7, and we will show all the left results here.

For charged kaon, we obtain

$$
\begin{aligned}
m^2_{K^\pm,p^4} =& + \frac{8m^2_{K^\pm,p^2}}{F_0^2}\left[(2m^2_{K^\pm,p^2} + m^2_{K^\pm,p^2} - \widetilde{m})(2L_6^r - L_4^r) + m^2_{K^\pm,p^2}(2L_8^r - L_5^r)\right] \\
&+ \frac{\ln m^2_{\pi_3^0,p^2}}{F_0^2\pi^2}\left(\frac{5}{144\sqrt{3}}\sin 2\epsilon m^4_{K^\pm,p^2} - \frac{1}{144\sqrt{3}}\sin 2\epsilon m^2_{\pi^\pm,p^2}m^2_{K^\pm,p^2}\right. \\
&+\frac{1}{288\sqrt{3}}\sin 2\epsilon m^4_{\pi^\pm,p^2} + \frac{1}{36}\sin^2\epsilon m^4_{K^\pm,p^2} - \frac{1}{144}\sin^2\epsilon m^2_{\pi^\pm,p^2}m^2_{K^\pm,p^2} \\
&-\frac{1}{72\sqrt{3}}\widetilde{m}\sin 2\epsilon m^2_{K^\pm,p^2} + \frac{5}{576\sqrt{3}}\widetilde{m}\sin 2\epsilon m^2_{\pi^\pm,p^2} + \frac{5}{288}\widetilde{m}m^2_{K^\pm,p^2} \\
&-\frac{1}{576}\widetilde{m}m^2_{\pi^\pm,p^2} - \frac{1}{36}\widetilde{m}\sin^2\epsilon m^2_{K^\pm,p^2} + \frac{1}{288}\widetilde{m}\sin^2\epsilon m^2_{\pi^\pm,p^2} \\
&\left.+\frac{1}{288\sqrt{3}}\widetilde{m}^2\sin 2\epsilon - \frac{1}{288}\widetilde{m}^2 + \frac{1}{144}\widetilde{m}^2\sin^2\epsilon\right) \\
&+ \frac{\ln m^2_{\eta_8,p^2}}{F_0^2\pi^2}\left(-\frac{5}{144\sqrt{3}}\sin 2\epsilon m^4_{K^\pm,p^2} + \frac{1}{144\sqrt{3}}\sin 2\epsilon m^2_{\pi^\pm,p^2}m^2_{K^\pm,p^2}\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{288\sqrt{3}}\sin 2\epsilon m_{\pi^\pm,p^2}^4+\frac{1}{36}m_{K^\pm,p^2}^4-\frac{1}{144}m_{\pi^\pm,p^2}^2 m_{K^\pm,p^2}^2\\
&-\frac{1}{36}\sin^2\epsilon m_{K^\pm,p^2}^4+\frac{1}{144}\sin^2\epsilon m_{\pi^\pm,p^2}^2 m_{K^\pm,p^2}^2\\
&+\frac{1}{72\sqrt{3}}\widetilde{m}\sin 2\epsilon m_{K^\pm,p^2}^2-\frac{5}{576}\widetilde{m}\sqrt{3}^{-1}\sin 2\epsilon m_{\pi^\pm,p^2}^2\\
&-\frac{1}{96}\widetilde{m}m_{K^\pm,p^2}^2+\frac{1}{576}\widetilde{m}m_{\pi^\pm,p^2}^2+\frac{1}{36}\widetilde{m}\sin^2\epsilon m_{K^\pm,p^2}^2\\
&-\frac{1}{288}\widetilde{m}\sin^2\epsilon m_{\pi^\pm,p^2}^2-\frac{1}{288\sqrt{3}}\widetilde{m}^2\sin 2\epsilon+\frac{1}{288}\widetilde{m}^2-\frac{1}{144}\widetilde{m}^2\sin^2\epsilon\Bigg). \qquad \text{(B.1)}
\end{aligned}
$$

$$
\begin{aligned}
m_{K^\pm,e^2p^2}^2 =&-\frac{e^2}{8\pi^2}\frac{C}{F_0^4}\left[m_{\pi,p^2}^2\ln m_{\pi^\pm,p^2}^2+2m_{K^\pm,p^2}^2\ln m_{K^\pm,p^2}^2\right]\\
&-\frac{e^2}{16\pi^2}m_{K^\pm,p^2}^2\left(3\ln m_{K^\pm,p^2}^2-4\right)+\frac{4e^2}{3}\widetilde{m}\left(-3K_8^r+K_9^r+K_{10}^r\right)\\
&-\frac{4e^2}{9}m_{K^\pm,p^2}^2(6K_1^r+6K_2^r+5K_5^r+5K_6^r-6K_7^r\\
&-24K_8^r-2K_9^r-20K_{10}^r-18K_{11}^r)\\
&+\frac{4}{3}e^2 m_{K^\pm,p^2}^2(3K_8^r+K_9^r+K_{10}^r)\\
&-\frac{16e^2C}{F_0^4}\left[\left(m_{\pi^\pm,p^2}^2+2m_{K^\pm,p^2}^2-\widetilde{m}\right)L_4^r+m_{\pi^\pm,p^2}^2L_5^r\right].
\end{aligned} \qquad \text{(B.2)}
$$

For neutral kaon, we have

$$
\begin{aligned}
m_{K^0,p^4}^2 =&+\frac{8m_{K^\pm,p^2}^2}{F_0^2}\left[(2m_{K^\pm,p^2}^2+m_{K^\pm,p^2}^2-3\widetilde{m})(2L_6^r-L_4^r)\right.\\
&\left.+(m_{K^\pm,p^2}^2-2\widetilde{m})(2L_8^r-L_5^r)\right]-\frac{8m_{\pi^\pm,p^2}^2}{F_0^2}(2L_6^r-L_4^r)\\
&+8\widetilde{m}^2(-L_4^r-L_5^r+2L_6^r+2L_8^r)\\
&+\frac{\ln m_{\pi_3^0,p^2}^2}{F_0^2\pi^2}\Bigg(-\frac{5}{144\sqrt{3}}\sin 2\epsilon m_{K^\pm,p^2}^4+\frac{1}{144\sqrt{3}}\sin 2\epsilon m_{\pi^\pm,p^2}^2 m_{K^\pm,p^2}^2\\
&-\frac{1}{288\sqrt{3}}\sin 2\epsilon m_{\pi^\pm,p^2}^4+\frac{1}{36}\sin^2\epsilon m_{K^\pm,p^2}^4-\frac{1}{144}\sin^2\epsilon m_{\pi^\pm,p^2}^2 m_{K^\pm,p^2}^2\\
&+\frac{1}{18\sqrt{3}}\widetilde{m}\sin 2\epsilon m_{K^\pm,p^2}^2+\frac{1}{576\sqrt{3}}\widetilde{m}\sin 2\epsilon m_{\pi^\pm,p^2}^2-\frac{5}{288}\widetilde{m}m_{K^\pm,p^2}^2\\
&+\frac{1}{576}\widetilde{m}m_{\pi^\pm,p^2}^2-\frac{1}{36}\widetilde{m}\sin^2\epsilon m_{K^\pm,p^2}^2+\frac{1}{288}\widetilde{m}\sin^2\epsilon m_{\pi^\pm,p^2}^2\\
&-\frac{7}{288\sqrt{3}}\widetilde{m}^2\sin 2\epsilon+\frac{1}{72}\widetilde{m}^2+\frac{1}{144}\widetilde{m}^2\sin^2\epsilon\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\ln m^2_{\eta_8,p^2}}{F_0^2\pi^2}\Bigg(+\frac{5}{144\sqrt{3}}\sin 2\epsilon m^4_{K^\pm,p^2}-\frac{1}{144\sqrt{3}}\sin 2\epsilon m^2_{\pi^\pm,p^2}m^2_{K^\pm,p^2}\\
&+\frac{1}{288\sqrt{3}}\sin 2\epsilon m^4_{\pi^\pm,p^2}+\frac{1}{36}m^4_{K^\pm,p^2}-\frac{1}{144}m^2_{\pi^\pm,p^2}m^2_{K^\pm,p^2}\\
&-\frac{1}{36}\sin^2\epsilon m^4_{K^\pm,p^2}+\frac{1}{144}\sin^2\epsilon m^2_{\pi^\pm,p^2}m^2_{K^\pm,p^2}\\
&-\frac{1}{18\sqrt{3}}\widetilde{m}\sin 2\epsilon m^2_{K^\pm,p^2}-\frac{1}{576}\widetilde{m}\sqrt{3}^{-1}\sin 2\epsilon m^2_{\pi^\pm,p^2}\\
&-\frac{13}{288}\widetilde{m}m^2_{K^\pm,p^2}+\frac{1}{192}\widetilde{m}m^2_{\pi^\pm,p^2}+\frac{1}{36}\widetilde{m}\sin^2\epsilon m^2_{K^\pm,p^2}\\
&-\frac{1}{288}\widetilde{m}\sin^2\epsilon m^2_{\pi^\pm,p^2}+\frac{7}{288\sqrt{3}}\widetilde{m}^2\sin 2\epsilon+\frac{1}{48}\widetilde{m}^2-\frac{1}{144}\widetilde{m}^2\sin^2\epsilon\Bigg).
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
m^2_{K^0,e^2p^2}=&+\frac{8e^2}{9}\widetilde{m}\left(3K_1^r+3K_2^r+K_5^r+K_6^r-3K_7^r-3K_8^r-K_9^r-K_{10}^r\right)\\
&-\frac{8e^2}{9}m^2_{K^0,p^2}\left(3K_1^r+3K_2^r+K_5^r+K_6^r-3K_7^r-3K_8^r-K_9^r-K_{10}^r\right)
\end{aligned}
\tag{B.4}
$$

B.2 Decay Constants

The expression for decay constants are given in Eq.7.3 with the result for $\pi^\pm$ in Sec. 7. For charged kaon, we have

$$
\begin{aligned}
F_{K^\pm,p^4}=&+\frac{4}{F_0}[(2m^2_{K^\pm,p^2}+m^2_{\pi^\pm,p^2}-\widetilde{m})L_4^r+m^2_{K^\pm,p^2}L_5^r]\\
&-\frac{1}{64F_0\pi^2}m^2_{\pi^\pm,p^2}\ln m^2_{\pi^\pm,p^2}\\
&-\frac{1}{192F_0\pi^2}\Big[\sqrt{3}(\widetilde{m}+m^2_{\pi^\pm,p^2}+2m^2_{K^\pm,p^2})\sin 2\epsilon+\frac{3}{2}(m^2_{\pi^\pm,p^2}+\widetilde{m})\\
&+3(2m^2_{K^\pm,p^2}-m^2_{\pi^\pm,p^2}+\widetilde{m})\sin^2\epsilon\Big]\ln m^2_{\pi^0_3,p^2}\\
&-\frac{1}{32F_0\pi^2}m^2_{K^\pm,p^2}\ln m^2_{K^\pm,p^2}\\
&+\frac{1}{64F_0\pi^2}(\widetilde{m}-m^2_{K^\pm,p^2})\ln m^2_{K^0,p^2}\\
&+\frac{1}{192F_0\pi^2}\Big[\sqrt{3}(\widetilde{m}+m^2_{\pi^\pm,p^2}+2m^2_{K^\pm,p^2})\sin 2\epsilon+\frac{3}{2}(m^2_{\pi^\pm,p^2}+\widetilde{m}\\
&-4m^2_{K^\pm,p^2})+3(2m^2_{K^\pm,p^2}-m^2_{\pi^\pm,p^2}+\widetilde{m})\sin^2\epsilon\Big]\ln m^2_{\eta_8,p^2}.
\end{aligned}
\tag{B.5}
$$

For neutral pion, we obtain

$$
\begin{aligned}
F_{K^0,p^4} =& + \frac{4}{F_0}[(2m^2_{K^\pm,p^2} + m^2_{\pi^\pm,p^2} - \widetilde{m})L^r_4 + (m^2_{K^\pm,p^2} - \widetilde{m})L^r_5] \\
& - \frac{1}{64F_0\pi^2} m^2_{\pi^\pm,p^2} \ln m^2_{\pi^\pm,p^2} \\
& + \frac{1}{384F_0\pi^2}\Big[\sqrt{3}(6\widetilde{m} + m^2_{\pi^\pm,p^2} + 2m^2_{K^\pm,p^2})\sin 2\epsilon + 3(-m^2_{\pi^\pm,p^2} + \widetilde{m}) \\
& +6(-2m^2_{K^\pm,p^2} + m^2_{\pi^\pm,p^2} + \widetilde{m})\sin^2\epsilon\Big] \ln m^2_{\pi^0_3,p^2} \\
& - \frac{1}{64F_0\pi^2} m^2_{K^\pm,p^2} \ln m^2_{K^\pm,p^2} \\
& + \frac{1}{32F_0\pi^2}(\widetilde{m} - m^2_{K^\pm,p^2}) \ln m^2_{K^0,p^2} \\
& + \frac{1}{384F_0\pi^2}\Big[\sqrt{3}(\widetilde{m} + m^2_{\pi^\pm,p^2} + 2m^2_{K^\pm,p^2})\sin 2\epsilon + 3(m^2_{\pi^\pm,p^2} + 3\widetilde{m} \\
& -4m^2_{K^\pm,p^2}) + 6(2m^2_{K^\pm,p^2} - m^2_{\pi^\pm,p^2} - \widetilde{m})\sin^2\epsilon\Big] \ln m^2_{\eta_8,p^2}.
\end{aligned}
\tag{B.6}
$$

C Loop integrals

In this Appendix, we collect all the Feynman integral we encountered with their definitions and properties. We use dimensional regularization here throughout in d dimensions with $d = 4 - 2\varepsilon$. We will start by discussing the general form of loop integrals that need to be addressed in this paper and introduce the procedure to simplify them into tensor integrals. Then, with the help of Kira program [25], the tensor integrals can be written as a combination of scalar integrals, which will be listed in the end.

C.1 General Integral and Simplification

In general, the integrals encountered in this thesis can be summarized in the following form. A general one-loop integral is

$$
I^{(d)}(v_1, v_2, p^2) = \frac{1}{i}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{a_1^{v_1}a_2^{v_2}} N(k^2, kp), \tag{C.1}
$$

where $N(k^2, kp)$ is a polynomial and

$$
a_1 = k^2 - m_1^2, \quad a_2 = (k-p)^2 - m_2^2. \tag{C.2}
$$

A general two-loop integral is

$$
I^{(d)}(v_1, v_2, v_3, v_4, v_5, p^2) = \frac{1}{i^2}\int \frac{\mathrm{d}^d k_1}{(2\pi)^d}\frac{\mathrm{d}^d k_2}{(2\pi)^d}\frac{1}{c_1^{v_1}c_2^{v_2}c_3^{v_3}c_4^{v_4}c_5^{v_5}} N(k_1^2, k_2^2, k_1p, k_2p, k_1k_2), \tag{C.3}
$$

where $N(k_1^2, k_2^2, k_1p, k_2p, k_1k_2)$ is also a polynomial and

$$\begin{aligned} c_1 &= k_1^2 - m_1^2, \quad c_3 = (k_1 - p)^2 - m_3^2, \quad c_5 = (k_1 - k_2)^2 - m_5^2, \\ c_2 &= k_2^2 - m_2^2, \quad c_4 = (k_2 - p)^2 - m_4^2. \end{aligned} \tag{C.4}$$

Individual one-loop and two-loop integrals in this thesis can be specified by the power of denominators v_i and by the function N. Tarasove [26] has provided a solution to the problem of reducing two-loop integral to a scalar integral ($N = 1$), the evaluation of the integrals is performed in four steps in his paper. However, with the help of Kira program, one only needs to simplify the integrand as much as possible to a tensor integral, the rest can be automatically completed by Kira to obtain a scalar integral.

The simplification procedure is just a matter of tedious algebra. For one-loop integral, we can perform

$$\begin{aligned} \frac{(kp)^\alpha}{a_2^{v_2}} &= \frac{(kp)^\alpha}{a_2^{v_2}} \left(\frac{k^2 - a_2 + p^2 - m_2^2}{2kp} \right)^{\min(\alpha, v_2)}, \\ \frac{(k^2)^\alpha}{a_1^{v_1}} &= \frac{(k^2)^\alpha}{a_1^{v_1}} \left(\frac{a_1 + m_1^2}{k^2} \right)^{\min(\alpha, v_1)}. \end{aligned} \tag{C.5}$$

After this progress, the one-loop integral will become tensor or scalar integral. For scalar integral, when $v_1 = 1, v_2 = 0$ or $v_1 = v_2 = 1$, these correspond to the one-loop integral A and B in Appendix. C.2. As for other scalar cases, one can obtain them by derivation w.r.t m_i^2 of A or B until the denominator matches the certain order. This derivation method is also valid when we are dealing with two-loop scalar integral.

For two-loop integral, we can still start the simplification with the repeated use of the substitutions:

$$\begin{aligned} \frac{(k_1k_2)^\alpha}{c_5^{v_5}} &= \frac{(k_1k_2)^\alpha}{c_5^{v_5}} \left(\frac{k_1^2 - c_5 + k_2^2 - m_5^2}{2k_1k_2} \right)^{\min(\alpha, v_5)}, \\ \frac{(k_2p)^\alpha}{c_4^{v_4}} &= \frac{(k_2p)^\alpha}{c_4^{v_4}} \left(\frac{k_2^2 - c_4 + p^2 - m_4^2}{2k_2p} \right)^{\min(\alpha, v_4)}, \\ \frac{(k_1p)^\alpha}{c_3^{v_3}} &= \frac{(k_1p)^\alpha}{c_3^{v_3}} \left(\frac{k_1^2 - c_3 + p^2 - m_3^2}{2k_1p} \right)^{\min(\alpha, v_3)}, \\ \frac{(k_2^2)^\alpha}{c_2^{v_2}} &= \frac{(k_2^2)^\alpha}{c_2^{v_2}} \left(\frac{c_2 + m_2^2}{k_2^2} \right)^{\min(\alpha, v_2)}, \\ \frac{(k_1^2)^\alpha}{c_1^{v_1}} &= \frac{(k_1^2)^\alpha}{c_1^{v_1}} \left(\frac{c_1 + m_1^2}{k_1^2} \right)^{\min(\alpha, v_1)}. \end{aligned} \tag{C.6}$$

After performing these substitutions, the two-loop integral will become either tensor or scalar integral, as in the one-loop cases. In this thesis's calculation, we will only encounter the two-loop scalar integral with $v_2 = v_3 = v_5 = 1, v_1 = v_4 = 0$ (integral H in Appendix. C.3), or its derivative w.r.t masses.

These simplification procedures are not universal, and sometimes we will end up with integrals with irreducible numerators. Such integrals are often referred to as tensor integrals, which have the form:

$$\begin{aligned}
&\frac{1}{i}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{a_1^{v_1}a_2^{v_2}}k_{\mu_1}\cdots k_{\mu_n},\\
&\frac{1}{i^2}\int\frac{\mathrm{d}^dk_1}{(2\pi)^d}\frac{\mathrm{d}^dk_2}{(2\pi)^d}\frac{1}{c_1^{v_1}c_2^{v_2}c_3^{v_3}c_4^{v_4}c_5^{v_5}}k_{1_{\mu_1}}\cdots k_{1_{\mu_n}}k_{2_{\nu_1}}\cdots k_{2_{\nu_n}}.
\end{aligned}\tag{C.7}$$

Here, in our thesis, all the internal momentum contract with the external momentum p.

Using recurrence relation, integration-by-parts and Lorentz-invariance identities, program Kira can perform the reduction of the tensor integrals to scalar integrals and further reduce the results to their simplest form. Here, we have used this package to assist us in processing the integrals. The `integralfamilies.yaml` file for Kira in our cases is defined as follows:

```
integralfamilies:
  - name: "twoloop1"
    loop_momenta: [r,s]
    top_level_sectors: [31]
    propagators:
      - [ "r",       0]
      - [ "r-p1",    m1]
      - [ "s",       m1]
      - [ "s-p1",    m1]
      - [ "r+s",     m1]
  - name: "twoloop2"
    loop_momenta: [r,s]
    top_level_sectors: [31]
    propagators:
      - [ "r",       0]
      - [ "r-p1",    m1]
      - [ "s",       m2]
      - [ "s-p1",    m1]
      - [ "r+s",     m2]
```

As these two types of integrals represent all the two-loop integrals we encountered. And the kinematic relation information are given in the file `kinematics.yaml`

```
kinematics:
  incoming_momenta: [p1]
  outgoing_momenta: [q1]
  momentum_conservation: [p1,q1]
  kinematic_invariants:
```

```
    - [m1, 2]
    - [m2, 2]
  scalarproduct_rules:
    - [[p1,p1], m1]
    - [[q1,q1], m1]
  symbol_to_replace_by_one: m1
```

Then, with the assistance of Kira, Feynman integrals are simplified to scalar integrals.

C.2 One-Loop Scalar Integral

The simplest one for one-loop scalar integral is what we defined A integral:

$$A(m^2) = \frac{1}{i}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{q^2-m^2}. \tag{C.8}$$

The explicit expression for A is well known,

$$A(m^2) = \frac{m^2}{16\pi^2}[\lambda - \ln m^2 + \mathcal{O}(\varepsilon)], \tag{C.9}$$

where $\lambda = 1/\varepsilon + C_0$ and $C_0 = \ln(4\pi) + 1 + \Gamma'(1)$ is a constant. The divergence of A is isolated in λ and the rest is the finite part. Note that we have suppressed the explicit μ-dependence of the logarithm term $\ln(m^2/\mu^2)$.

Next, we define the scalar integral with two different propagators

$$B(m_1^2, m_2^2, p^2) = \frac{1}{i}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{(q^2-m_1^2)((q-p)^2-m_2^2)}, \tag{C.10}$$

The evaluation procedure is standard and we obtain

$$B(m_1^2, m_2^2, p^2) =\frac{1}{16\pi^2}(\lambda - 1) + \bar{B}(m_1^2, m_2^2, p^2) + \mathcal{O}(\varepsilon) \tag{C.11}$$

where

$$\bar{B}(m_1^2, m_2^2, p^2) = -\frac{1}{16\pi^2}\int_0^1 \mathrm{d}x \ln\left(m_1^2 x + m_2^2(1-x) - x(1-x)p^2\right). \tag{C.12}$$

C.3 Two-Loop Scalar Integral

In this thesis, we need only deal with one kind of two-loop scalar integral (and its derivative w.r.t mass), which is defined as,

$$H(m_1^2, m_2^2, m_3^2, p^2) = \frac{1}{i^2}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\mathrm{d}^d r}{(2\pi)^d}\frac{1}{(q^2-m_1^2)(r^2-m_2^2)((q+r-p)^2-m_3^2)}, \tag{C.13}$$

After the extraction of the divergent parts, we have (from Ref. [3])

$$\begin{aligned} H(m_1^2, m_2^2, m_3^2, p^2) =& \frac{1}{(16\pi^2)^2} \left\{ \frac{1}{2}\lambda_2(m_1^2 + m_2^2 + m_3^2) + \frac{1}{2}\lambda_1 \left[m_1^2(1 - 2\ln m_1^2) \right.\right. \\ & \left. + m_2^2(1 - 2\ln m_2^2) + m_3^2(1 - 2\ln m_3^2) - \frac{p^2}{2} \right] \\ & \left. + \bar{H}(m_1^2, m_2^2, m_3^2, p^2) \right\} + \mathcal{O}(\varepsilon), \end{aligned} \tag{C.14}$$

in which $\lambda_2 = \lambda^2 + C_0^2$, $\lambda_1 = \lambda + C_0$ and the finite part $\bar{H}$ is

$$\begin{aligned} \bar{H}(m_1^2, m_2^2, m_3^2, p^2) = & \left(\frac{m_1^2 m_2^2 m_3^2}{\lambda_m^2} - \frac{1}{2} \right) \Psi(m_1^2, m_2^2, m_3^2) + \frac{1}{8} + \frac{m_1^2 \delta_m}{2\lambda_m} \ln m_1^2 \\ & + \frac{m_2^2}{2\lambda_m}(m_2^2 - m_1^2 - m_3^2) \ln m_2^2 + \frac{m_3^2}{2\lambda_m}(m_3^2 - m_1^2 - m_2^2) \ln m_3^2 \\ & + m_1^2 \left[\frac{1}{12}\pi^2 + \frac{3}{2} - \ln m_1^2 + \frac{1}{2}\left(- \ln m_2^2 \ln m_3^2 + \ln m_1^2 \ln m_4^2 \right) \right] \\ & + m_2^2 \left[\frac{1}{12}\pi^2 + \frac{3}{2} - \ln m_2^2 + \frac{1}{2}\left(- \ln m_1^2 \ln m_3^2 + \ln m_2^2 \ln m_4^2 \right) \right] \\ & + m_3^2 \left[\frac{1}{12}\pi^2 + \frac{3}{2} - \ln m_3^2 + \frac{1}{2}\left(- \ln m_1^2 \ln m_2^2 + \ln m_3^2 \ln m_4^2 \right) \right] \\ & + J(m_1^2, m_2^2, m_3^2, p^2). \end{aligned} \tag{C.15}$$

Here $m_4^2 = m_1^2 m_2^2 m_3^2$, $\delta_m = m_1^2 - m_2^2 - m_3^2$ and $\lambda_m = \lambda(m_1^2, m_2^2, m_3^2)$, where

$$\lambda(x, y, z) = (x - y - z)^2 - 4yz \tag{C.16}$$

is the Källén function.

The expression of the function Ψ is dependent on the relation between its three variables. In the case when $\lambda_m \geq 0$, there are three possible situations. First, we consider $m_1 + m_2 \leq m_3$, the expression of Ψ is

$$\begin{aligned} & \Psi(m_1^2, m_2^2, m_3^2) = \\ & \quad - \sqrt{\lambda_m} \left\{ 2\ln x_1 \ln x_2 - \ln \frac{m_1^2}{m_3^2} \ln \frac{m_2^2}{m_3^2} + \frac{\pi^2}{3} - 2\mathrm{Li}_2(x_1) - 2\mathrm{Li}_2(x_2) \right\}, \end{aligned} \tag{C.17}$$

with

$$x_1 = \frac{m_3^2 + m_1^2 - m_2^2 - \sqrt{\lambda_m}}{2m_3^2}, \quad x_2 = \frac{m_3^2 + m_2^2 - m_1^2 - \sqrt{\lambda_m}}{2m_3^2}, \tag{C.18}$$

and the dilogarithm $\mathrm{Li}_2(x)$ defined by

$$\mathrm{Li}_2(x) = - \int_0^x \frac{\mathrm{d}t}{t} \ln(1 - xt). \tag{C.19}$$

For the other two cases $m_1 + m_3 \leq m_2$ and $m_2 + m_3 \leq m_1$, one can obtain them by relabelling masses in Eq. C.17.

When $\lambda_m \leq 0$,

$$\Psi(m_1^2, m_2^2, m_3^2) = 2\sqrt{-\lambda_m}\left\{\mathrm{Cl}_2(2\arccos z_1) + \mathrm{Cl}_2(2\arccos z_2) + \mathrm{Cl}_2(2\arccos z_3)\right\}, \quad \text{(C.20)}$$

with

$$z_1 = \frac{-m_1^2 + m_2^2 + m_3^2}{2m_2m_3}, \quad , z_2 = \frac{-m_2^2 + m_3^2 + m_1^2}{2m_1m_3}, \quad z_3 = \frac{-m_3^2 + m_1^2 + m_2^2}{2m_1m_2}. \quad \text{(C.21)}$$

The $\mathrm{Cl}_2(x)$ in the expression is Clausen's function, and is defined by

$$\mathrm{Cl}_2(x) = -\int_0^x \mathrm{d}t \ln\left|2\sin\frac{t}{2}\right| = -i(\mathrm{Li}_2(\mathrm{e}^{ix}) - \mathrm{Li}_2(1)) - \frac{\pi x}{2} + i\frac{x^2}{4}. \quad \text{(C.22)}$$

In practice, the derivative of Ψ with respect to the mass m_i^2 is also important. Here we show the case with m_1^2, the other two can be obtained by switch the variables.

$$\begin{aligned}&\frac{\partial}{\partial m_1^2}\Psi(m_1^2, m_2^2, m_3^2) = \\ &\quad \frac{m_1^2 - m_2^2 - m_3^2}{\lambda_m}\Psi(m_1^2, m_2^2, m_3^2) - \ln\frac{m_1^4}{m_2^2m_3^2} + \frac{m_2^2 - m_3^2}{m_1^2}\ln\frac{m_2^2}{m_3^2},\end{aligned} \quad \text{(C.23)}$$

which will allow one to easily evaluate all the necessary derivatives.

The expression of the function J also depend on the relation of its four variables, below the threshold $p^2 < (m_1 + m_2 + m_3)^2$, we have

$$J(m_1^2, m_2^2, m_3^2, p^2) = \int_{(m_2+m_3)^2}^{\infty} \mathrm{d}\sigma \sqrt{\lambda\left(1, \frac{m_2^2}{\sigma}, \frac{m_3^2}{\sigma}\right)} \times \int_0^1 \mathrm{d}x\, \mathscr{J}_2(x, \sigma, p^2), \quad \text{(C.24)}$$

where

$$\mathscr{J}_2(x, \sigma, p^2) = \ln\frac{m_1^2(1-x) + \sigma x - x(1-x)p^2}{m_1^2(1-x) + \sigma x} + \frac{p^2 x(1-x)}{m_1^2(1-x) + \sigma x}. \quad \text{(C.25)}$$

Above the threshold $p^2 \geq (m_1 + m_2 + m_3)^2$, the function J develops imaginary parts and they can then be evaluated from its dispersive representation

$$J(m_1^2, m_2^2, m_3^2, p^2) = \frac{p^4}{\pi}\int_{(m_1+m_2+m_3)^2}^{\infty} \frac{\mathrm{d}z}{z^2}\frac{\mathrm{Im}J(m_1^2, m_2^2, m_3^2, z)}{z - p^2}. \quad \text{(C.26)}$$

The imaginary parts are given by (in d=4)

$$\mathrm{Im}J(m_1^2, m_2^2, m_3^2, p^2) = -2\pi\int_{m_1}^{E_1^{\max}} \mathrm{d}E_1 \frac{\sqrt{\lambda(p^2, m_1^2, m_{23}^2)\lambda(m_{23}^2, m_2^2, m_3^2)}}{m_{23}^2|p|}, \quad \text{(C.27)}$$

with

$$
\begin{aligned}
m_{23}^2 &= p^2 + m_1^2 - 2|p|E_1, \\
E_1^{\max} &= \frac{1}{2|p|}\left[p^2 + m_1^2 - (m_2+m_3)^2\right].
\end{aligned}
\tag{C.28}
$$

In this thesis, due to the structure of the ChPT Lagrangian, identical NGBs fields must appear in pairs at each vertex. Consequently, all $H(m_1^2, m_2^2, m_3^2, p^2)$ functions encountered by us have m_2 and m_3 that are necessarily the same. Therefore, even though we only consider the on-shell case where p^2 equals the real mass of incoming particles, we can still use the results from the off-shell case, where the external momentum $p^2 = (m_1 - m_2 + m_3)^2$.

At this pseudo-threshold value, $p^2 = (m_1 - m_2 + m_3)^2$, the explicit expression of the finite part $\bar{H}(m_1^2, m_2^2, m_3^2, p^2)$, as given by Caffo *et al.*[27], is

$$
\begin{aligned}
&\bar{H}(m_1^2, m_2^2, m_3^2, (m_1+m_2-m_3)^2) = \\
&\frac{2}{(m_3-m_1-m_2)^2}\left[m_1^3(m_1+2m_2)\mathcal{I}(m_1,m_2,m_3) + m_2^3(2m_1+m_2)\mathcal{I}(m_2,m_1,m_3)\right] \\
&+\frac{4}{(m_3-m_1-m_2)}\Big[(m_1+m_2)\left(m_1^2\mathcal{I}(m_1,m_2,m_3) + m_2^2\mathcal{I}(m_2,m_1,m_3)\right) \\
&+\frac{1}{4}(m_1^2+m_2^2+m_1m_2)\left(m_1\left(\ln m_3^2 - \ln m_1^2\right) + m_2\left(\ln m_3^2 - \ln m_2^2\right)\right)\Big] \\
&-2(m_3^2-m_1^2-m_2^2)\left(\mathcal{I}(m_1,m_2,m_3) + \mathcal{I}(m_2,m_1,m_3)\right) \\
&+\frac{1}{2}\Big[m_1^2\ln^2(m_1^2) + m_2^2\ln^2(m_2^2) + m_3^2\ln^2(m_3^2) + (m_1^2+m_2^2-m_3^2)\ln(m_1^2)\ln(m_2^2) \\
&+(m_1^2-m_2^2+m_3^2)\ln(m_1^2)\ln(m_3^2) + (-m_1^2+m_2^2+m_3^2)\ln(m_2^2)\ln(m_3^2) \\
&-m_1(3m_1+2m_3)\ln(m_1^2) - m_2(3m_2+2m_3)\ln(m_2^2) \\
&+(2m_1^2+2m_1m_2+2m_2^2-m_3^2)\ln(m_3^2)\Big] \\
&+\left(\frac{\pi^2}{12}-\frac{5}{8}\right)(m_1^2+m_2^2+m_3^2) + \frac{13}{4}(-m_1m_2+m_1m_3+m_2m_3) \\
&+\frac{1}{2}(m_1+m_2-m_3)^2,
\end{aligned}
\tag{C.29}
$$

in which

$$
\mathcal{I}(m_1,m_2,m_3) = \mathrm{Li}_2\left(1-\frac{m_3}{m_2}\right) - \mathrm{Li}_2\left(-\frac{m_1}{m_2}\right) + \log\left(\frac{m_3}{m_1+m_2}\right)\log\left(\frac{m_1}{m_2}\right). \tag{C.30}
$$

The diagrams in Figure 8i and 8j introduce additional terms proportional to λ. Consequently, terms of the order $\mathcal{O}(\varepsilon)$ in the H functions are necessary. Although these results can be found in Ref. [28, 29], in the results of this thesis, we use $\bar{H}(\varepsilon, m_1, m_2, m_3, p^2)$ to denote such terms.

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