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Removing QED Feynman diagrams with chirality flow

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Abstract

This thesis explores the simplifications the chirality-flow formalism brings the calculations of scattering amplitudes in massless QED. We explain Feynman diagrams and chirality-flow diagrams to emphasize how the calculations of scattering amplitudes differ. It is studied how applying the chirality-flow formalism simplifies the calculations. The work focuses on calculations with and without external photons, separately. The probability of a diagram contributing to the amplitude is calculated for different cases - including varying the number of photons and fermions. The results from the thesis offers a deeper understanding of interactions in massless QED.

Populärvetenskaplig beskrivning

Inom partikelfysik, går de teoriska och experimentella aspekterna hand i hand. För att förstå hur partiklar växelverkar med varandra och kunna förstå experimentella resultat, behövs en teoretisk modell av interaktionerna. Feynmandiagram erbjuder exakt detta, ett verktyg för att simpelt visualisera partikelporcesser. Med Feynmandiagram är det möjligt att förstå vilka processer som är möjliga och räkna ut sannolikheten att de sker, med en såkallad spridningsamplitud. Till varje process existerar det ett eller flera Feynmandiagram som alla inkluderas till beräkningen av spridningsamplituden. Beräkningarna med Feynmandiagram kan bli väldigt komplexa för processer med många partiklar. I ett försök att förenkla beräkningarna, används kiralitetsflödesformalismen.

Kiralitetsflödesformalismen handlar, förenklat, om att studera hur en partikels höger- och vänster-hänthet ändras när den växelverar med andra partiklar. Teorin baseras på spinn-helicitet formalismen och ger nya regler för hur olika partiklar representeras grafiskt. Detta genererar nya slags diagram som, likt Feynmandiagram, alla inkluderas i spridningsamplituden. Däremot, minskar antalet bidragande diagram när dessa regler tillämpas, vilket i sin tur förenklar beräkningarna av spridningsamplituden.

Genom att effektivisera beräkningar av spridningsamplituder är det möjligt att studera mer komplexa processer som annars skulle ta för lång tid eller inte vara möjliga. Effektiviseringen gör det möjligt för forskare att utvidga de områden de studerar och bidra till en djupare förståelse av partikelfysik. Att förstå partikelfysik är en fundamental del av att förstå universums byggstenar. Människan har ständigt letat efter svar kring varför och hur universum ser ut som det gör idag. Inom partikelfysik finns det många frågor som inte är besvarade ännu och alla effektiviseringar är ett steg närmare dessa svar.

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1 Introduction

In this thesis, the simplifications that the chirality-flow formalism [2] brings to theoretical particle physics are explored. Traditionally, Feynman diagrams are used to visualize interactions and calculate scattering amplitudes [3]. A scattering amplitude gives a measure of how probable an interaction is to happen. It gives the probability of a certain outcome with respect to the incoming particles. When considering processes including many particles, the calculations given by a Feynman diagram become very complex.

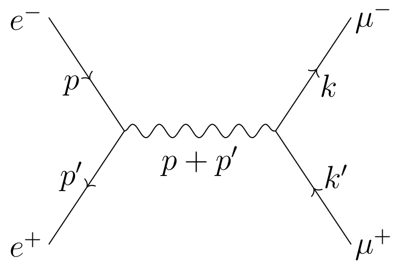
Algorithms used to calculate scattering amplitudes based on Feynman diagrams have previously been optimized [2]. The optimization consists of minimising the number of contributing diagrams and re-using parts of calculations to streamline the calculation time.

The purpose of this thesis is to study and quantify if and how the calculations can be simplified. This is done by using the chirality-flow formalism. Firstly, the rules of building diagrams depicting the flow of chirality are explained. This includes fermions, anti-fermions, external photons and propagators. This thesis is limited to massless fermions in QED, which makes the correlation between helicity and chirality very simple. By considering the left- and right-handness of massless fermions, this thesis studies how the number of contributing diagrams can be reduced. This is done by introducing the basic concepts of the chirality-flow formalism, based on the spinor-helicity formalism. Furthermore, the difference between including and excluding external photons is noticed and the work is divided, accordingly. Simple examples are considered to give an understanding of how the chirality-flow formalism is applied and simplifies calculations.

This thesis begins with explaining Feynman diagrams in QED, where a simple process is used as an example in Section 2. Section 3 explains the basics of the spinor-helicity formalism, including the notation needed to understand the chirality-flow formalism. Concluding the theory, the notation for vertices and propagators is explained. Additionally, the chirality-flow formalism is applied to a simple process to calculate the scattering amplitude, which is presented in Section 3.4. The work is done considering one fermion-line and is divided into two parts: Section 4.1 which only includes external fermions and Section 4.2 which includes external photons. In both sections, the number of contributing diagrams is calculated. The result when considering external photons is then generalized in Section 4.3, as it is summed over all possible helicities. Finally, Section 4.4 consideres more than one fermion-line. The conclusion is then presented, whoch states that there is a simplification due to applying the chirality-flow formalism. The calculations become further simplified for many photons and many fermion-lines.

2 Feynman diagrams in QED

Quantum electrodynamics include interactions between charged fermions via the exchange of a virtual photon. In general, fermions in QED are identified as particles with spin $\frac{1}{2}$. An interaction between fermions can result in a number of final states. All possible interactions are commonly described by Feynman diagrams, following the set of rules given in Table 1. Given a Feynman diagram and properties, such as electric charge of included particles, the scattering amplitude can be calculated. The scattering amplitude is a measure of probability of the final state relative to the incoming state. Considering the process $e^+e^- \longrightarrow \mu^+\mu^-$



,

the amplitude is given by

$$M = \frac{e^2}{q^2}(\overline{v}(p')\gamma^\mu u(p)) \times (\overline{u}(k)\gamma_\mu v(k')), \tag{1}$$

where γ^μ is given by

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \tag{2}$$

in the chiral representation. The new object σ is introduced, which represents the Pauli matrices defined as

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{3}$$

$$\sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4}$$

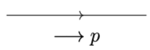
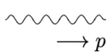
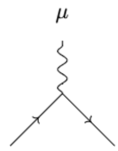
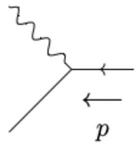
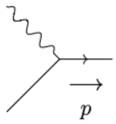
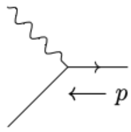
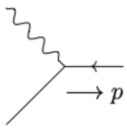
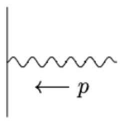
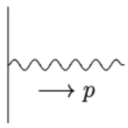
Dirac propagator	 $= \frac{i\not{p}}{p^2+i\epsilon}$
Photon propagator	 $= \frac{-ig_{\mu\nu}}{p^2+i\epsilon}$
QED vertex	 $= iQe\gamma^{\mu}$
Initial, external fermion	 $= u(p)$
Final, external fermion	 $= \bar{u}(p)$
Initial, external anti-fermion	 $= \bar{v}(p)$
Final, external anti-fermion	 $= v(p)$
Initial, external photon	 $= \epsilon_{\mu}(p)$
Final, external photon	 $= \epsilon^{*}_{\mu}(p)$

Table 1: Table containing the Feynman rules for QED. Here, 'final' refers to the particle being a result of process and 'initial' refers to the particle entering the process. Further, $u(p)$ and $v(p)$ are spinors, Q is the electric charge of the particle and p is the momentum of the particle. Finally, ϵ is the polarization vector and $\not{p} = \gamma^{\mu}p_{\mu}$.

Additionaly, $\bar{\sigma}$ is defined as

$$\bar{\sigma}^0 = \sigma^0 \qquad , \qquad \bar{\sigma}^i = -\sigma^i. \tag{5}$$

As seen in Eq. (2), γ^μ and γ_μ are 4x4-matrices. The spinors u and v are 4-vectors and the multiplications have to be done for every μ. The result from Eq. (1) is summed over all values of μ, resulting in a complex calculation.

The Feynman rules for QED are summarized in TAble 1.As seen in Table 1, each fermion-line has an arrow indicating if it is a fermion or an anti-fermion. The arrow goes parallell to the momentum of a fermion but in the opposite direction for an anti-fermion. Since quantum numbers, such as lepton number and charge, need to be conserved, the direction of the arrows becomes crucial. The number of incoming arrows has to equal the number of outgoing arrows [1]. Finally, momentum is conserved in each interaction vertex.

The process $e^+e^- \longrightarrow \mu^+\mu^-$ is very simple and has only one contributing diagram. However, processes normally have multiple contributing diagrams when considering all possible variations of interactions. The total amplitude is obtained by summing all contributing diagrams. At last, the sum is squared to calculate the probability amplitude of the process [1].

3 The chirality-flow formalism

3.1 Helicity and Chirality

The helicity of a particle is defined as the projection of its spin onto its momentum and defines if the particle is right-handed or left-handed:

$$\begin{aligned} \overline{\sigma} \cdot \frac{\overline{p}}{|p|} &= 1 \qquad \Longrightarrow \qquad \text{right-handed,} \\ \overline{\sigma} \cdot \frac{\overline{p}}{|p|} &= -1 \qquad \Longrightarrow \qquad \text{left-handed.} \end{aligned} \tag{6}$$

Equation 6 gives the spin of the particle projected onto the momentum, $\bar{p}$. In general, an object is said to be chiral if it is distinct from its mirror image. The chirality of a particle is divided into left- and right-chiral, but it is a different property from the helicity. A fermion can be divided into a left-chiral part and a right-chiral part using projection operators given by the fifth gamma-matrix;

$$P_R = \frac{1}{2}(1+\gamma^5), \qquad P_L = \frac{1}{2}(1-\gamma^5), \tag{7}$$

where

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Considering the massless case, the fermions will travel at the speed of light allowing the chirality to be given by the helicity [3]. This means that the direction of the momentum will be the same for all observers.

3.2 Basics of spinor-helicity formalism

All fermions can be described using Dirac spinors [1], which solve the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \tag{8}$$

Here, ψ is the plane-wave solution containing the Dirac spinors, m is the mass of the particle and ∂_μ is the derivative-operator with respect to the μ^{th} n n n component. Since only massless fermions are considered in this paper, Eq. (8) simplifies to

$$i\partial^\mu \sigma_\mu \psi = 0 \qquad \text{and} \qquad i\partial^\mu \bar{\sigma}_\mu \psi = 0, \tag{9}$$

which are known as the Weyl equations. In the chiral representation, the Dirac spinors are given by

$$u(p) = \begin{bmatrix} u_L \\ u_R \end{bmatrix} = \begin{bmatrix} \tilde{\lambda}^{\dot{\alpha}}_p \\ \lambda_{\beta,p} \end{bmatrix}, \qquad v(p) = \begin{bmatrix} v_L \\ v_R \end{bmatrix} = \begin{bmatrix} \tilde{\lambda}^{\dot{\alpha}}_p \\ \lambda_{\beta,p} \end{bmatrix}, \tag{10}$$

where $\tilde{\lambda}^{\dot{\alpha}}$ transforms under the left-chiral representation, $\left(\frac{1}{2}, 0\right)$, and λ_β transforms under right-chiral representation $\left(0, \frac{1}{2}\right)$.

In the massless case, Eq. (9) gives a different solution for each chirality [1]. The Dirac spinors are simplified further in the massless limit. The solutions in Eq. (10) then become

$$\begin{aligned} u^+(p) = v^-(p) &= \begin{bmatrix} 0 \\ |p\rangle \end{bmatrix} \quad , \quad u^-(p) = v^+(p) = \begin{bmatrix} |p] \\ 0 \end{bmatrix}, \\ \bar{u}^+(p) = \bar{v}^-(p) &= \begin{bmatrix} [p| & 0 \end{bmatrix} \quad , \quad \bar{u}^-(p) = \bar{v}^+(p) = \begin{bmatrix} 0 & \langle p| \end{bmatrix}, \end{aligned} \tag{11}$$

where the bra-ket notation has been introduced to easily differentiate between fermions of different chirality [2]. Here, $+$ and $-$ denote right-handedness and left-handedness, respectively. The following holds:

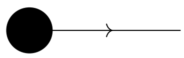
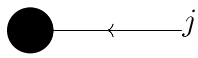
$$\lambda^\alpha_p \longleftrightarrow \langle p| \qquad \tilde{\lambda}_{p,\alpha} \longleftrightarrow |p\rangle$$

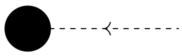
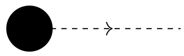
$$\lambda_{p,\dot{\alpha}} \longleftrightarrow [p| \qquad \tilde{\lambda}^{\dot{\alpha}}_p \longleftrightarrow |p]$$

Here it is seen that the dotted indices are used to denote left-chiral fermions and undotted indices are used to denote right-chiral fermions. The Weyl spinors are related by the Hermitian conjugate,

$$|p\rangle^\dagger = [p|, \quad |p]^\dagger = \langle p|. \tag{12}$$

A new graphic representation for the spinors is now introduced [2]:

$\langle p_i| = \langle i| =$  i $\qquad$ $\langle p_j| = \langle j| =$  $j,$

$[p_i| = [i| =$  i $\qquad$ $[p_j| = [j| =$  $j.$

The dashed lines in a diagram now represent either a fermion or an anti-fermion with left chirality. The solid line in a diagram represent either a fermion or an anti-fermion with right chirality.

It is now possible to define what we call the 'flow' of chirality.

$$i \longrightarrow j = \epsilon^{\alpha\beta}\lambda_{i,\beta}\lambda_{j,\alpha} = \langle ij\rangle.$$

$$i \dashrightarrow j = \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_i^{\dot{\beta}}\tilde{\lambda}_j^{\dot{\alpha}} = [ij].$$

In this step, we introduced the Levi-Civita tensor to lower and raise indices,

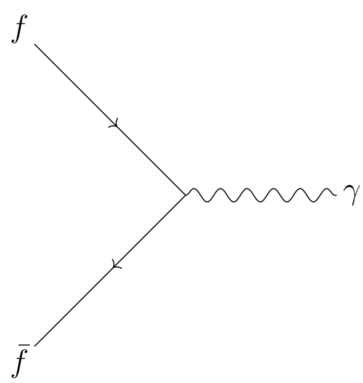
$$\epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{13}$$

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{14}$$

Thus, the amplitude for a given process is given by combinations of the contractions $[ij]$ and $\langle ij\rangle$, as the contractions are the basic Lorentz invariant quantities [2]. As seen in Eq. (13) and Eq. (14), the Levi-Civita tensor is anti-symmetric. Consequently, $[pp] = \langle pp\rangle = 0$.

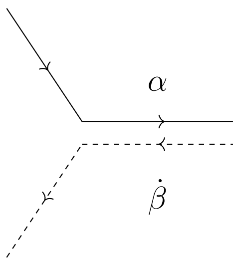
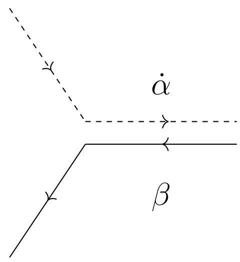
3.3 Vertices and Propagators

Consider the following fermion-photon vertex given by the Feynman rules

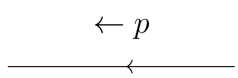


$$= ieQ_f\gamma^\mu = ieQ_f \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}.$$

In the chiral representation of the γ-matrices, it is divided into two parts, one with σ^μ and one with $\bar{\sigma}^\mu$ [2].

$ieQ_f\sqrt{2}$  and $ieQ_f\sqrt{2}$ 

Since the included fermions can have different chiralities, only the diagram with the correct chirality contributes to the amplitude. Intuitively, fermion propagators are also divided into two parts in the chiral representation. Consider the fermion propagator

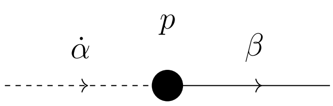


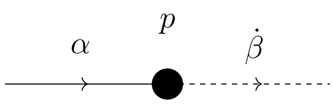
$$= \frac{ip_\mu\gamma^\mu}{p^2} = \frac{i}{p^2}\begin{bmatrix} 0 & \not{p} \\ \bar{\not{p}} & 0 \end{bmatrix}$$

Here, a new notation is introduced to simplify the expression,

$$\begin{aligned} \bar{\not{p}} &= p_\mu\bar{\sigma}^\mu, \\ \not{p} &= p_\mu\sigma^\mu. \end{aligned} \tag{15}$$

The first part of the fermion propagator in chiral-representation contains the factor $\frac{i}{p^2}\not{p}$, while the second part contains $\frac{i}{p^2}\bar{\not{p}}$. Introducing the momentum-dot notation [2], the fermion propagators are given by

$p_\mu\sigma^\mu \quad \longrightarrow \quad \frac{i}{p^2}$  ,

$p_\mu\bar{\sigma}^\mu \quad \longrightarrow \quad \frac{i}{p^2}$  .

The value of the momentum-dot is the sum of all contributing momenta.

Moreover, the factor given by a propagating photon with momentum p is defined as

$$-i\frac{g_{\mu\nu}}{p^2}.$$

The chirality-flow formalism of the propagating photon is, like a fermion, divided into two parts [2]. However, only the part with allowed chirality contributes to the amplitude. The allowed chiralities depend on the chiralities of the fermions in the process.

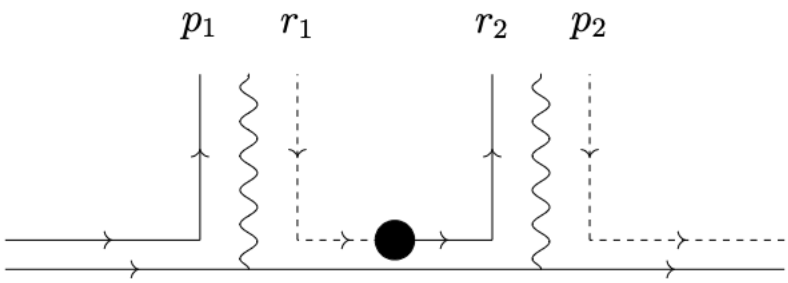
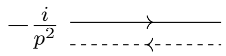
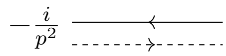


Figure 1: Figure of one fermion line with two external photons. The figure depicts how the chirality of a fermion changes because of photons and where the momentum-dot is located.



and



Only the diagram with allowed arrow direction will contribute to the amplitude. An interaction can also contain external photons, that interact through a vertex. Figure 1 shows how an external photon changes the chirality along the fermion line through a momentum-dot. The external photon is denoted by its momemtum p and its reference vector r [2]. The reference vector can be chosen arbitrarily given it follows the requirements

$$r^2 = 0 \qquad , \qquad p \cdot r \neq 0. \tag{16}$$

The reference vector is directly related to the polarization vector. The polarization vector is given by

$$\epsilon_L^\mu(p,r) = \frac{|r\rangle[p|}{\langle rp\rangle} \qquad \text{or} \qquad \frac{|p]\langle r|}{\langle rp\rangle}, \tag{17}$$

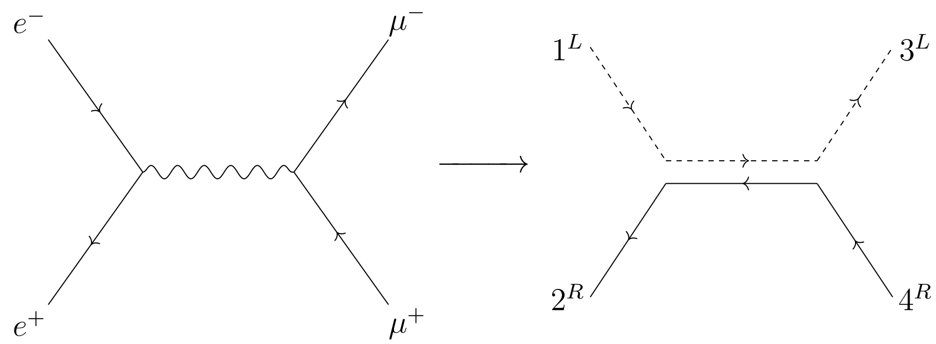
$$\epsilon_R^\mu(p,r) = \frac{|r]\langle p|}{[pr]} \qquad \text{or} \qquad \frac{|p\rangle[r|}{[pr]}, \tag{18}$$

where ϵ_L^μ denotes an incoming photon with negative helicity or an outgoing photon with positive helicity. Similarly, ϵ_R^μ denotes an incoming photon with positive helicity or an outgoing photon with negative helicity.

3.4 Applying the chirality-flow formalism

It is now possible to calculate the amplitude of an interaction using the chirality-flow formalism. When studying any interaction of massless fermions, each fermion can have either left or right chirality, as explained in Section 2. Hence, the calculations are done by assuming the chirality of each fermion. As explained in Sections 3.3 and 3.2, the chirality of an outgoing fermion-line is dependent on the chirality of the connected incoming fermion-line. As shown in Figure 1, an incoming right-chiral fermion changes to a left-chiral fermion, and vice versa.

Re-visiting the example calculated in Section 2, the chirality-flow diagram can now be defined.

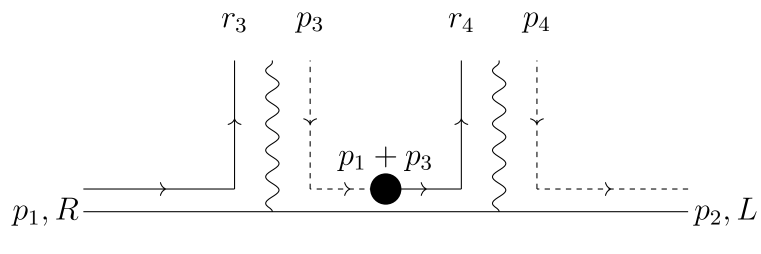


In this case, the fermions are assumed to be left-chiral and the anti-fermions are assumed to be right-chiral. The calculations become

$$(\sqrt{2}ei)^2 \times \frac{-i}{s_{12}} \times [13]\langle 24\rangle,$$

where $s_{12} = (p_1 + p_2)^2 = (p_3 + p_4)^2$ comes from momentum conservation. Dissecting the calculations, each vertex contributes with a factor $(\sqrt{2}ei)^2$ where the electric charge of the included particle is e. The factor $\frac{-i}{s_{12}}$ comes from the propagating photon, [13] comes from following the chirality-flow of the dashed line and $\langle 24\rangle$ comes from following the chirality-flow of the un-dashed line [2]. This is an extreme simplification from using the standard Feynman rules.

Now, consider an interaction involving external photons. Re-using the example from Section 3.4 with defined chiralities,



The calculation of the scattering amplitude becomes

$$(\sqrt{2}ei)^2 \times \frac{1}{(p_1 + p_3)^2} \times \frac{1}{\langle 3r_3\rangle[r_4 4]} \times \langle 1r_3\rangle \times \Big(\langle 31\rangle[1r_4] + \langle 33\rangle[3r_4]\Big) \times [42]. \tag{19}$$

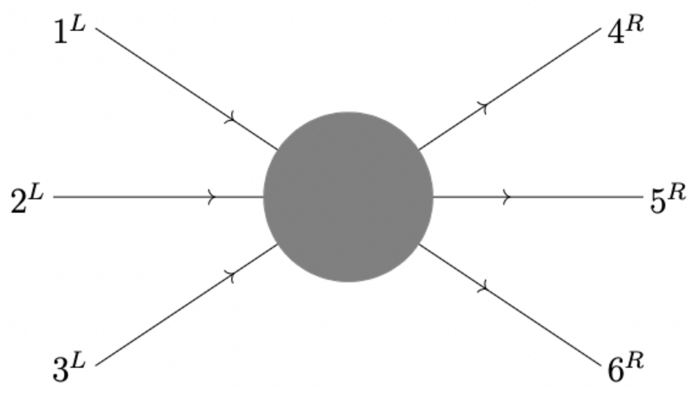


Figure 2: Depiction of process only including external fermions, showing the direction of the chirality-flow arrows.

Letting $r_3 = p_1$ would make the Eq. (19) equal zero and the diagram would not contribute to the total scattering amplitude. This is a consequence of the Levi-Civita tensor being anti-symmetric.

4 Reducing the number of diagrams using chirality-flow

Considering the previously presented theory, the number of contributing diagrams can be reduced. The interesting question is, how many diagrams can be removed?

This result is found by calculating the probability of all photons having the correct chirality for the diagram to contribute to the amplitude. All diagrams are not contributing since each fermion and photon can be either left- or right chiral.

As seen in Eq. (19), the calculations are more involved when considering a process with external photons. Only including external fermions results in simpler calculations and the number of contributing diagrams is reduced because the chirality-flow arrows have to match. When including external photons, the reference vector can be chosen arbitrarily, which means that it can be chosen such that diagrams equal zero and do not contribute to the amplitude.

4.1 Reducing the number of diagrams including external fermions

Here, the amount of simplification using the chirality-flow formalism compared to standard Feynman rules is calculated, considering processes with only external fermions. A visual presentation of the process is given in Figure 2. Considering the chiralities of the interacting fermions limits the number of possible diagrams.

In Section 3.2, it was explained how the fermions are contracted and how the chirality of a fermion changes in an interaction. Another way of formulating this is by stating that the number of incoming arrows has to equal the number of outgoing arrows. An incoming fermion will change chirality once when interacting with a photon. In the case of multiple photons, there will always exist a momentum-dot that also changes the chirality once. This applies for any interaction between the particles. This means

$$n_{L,\text{in}} = n_{R,\text{out}}, \qquad n_{R,\text{in}} = n_{L_{\text{out}}}. \tag{20}$$

Without applying the chirality-flow formalism, the number of possible diagrams is given by the possible outcomes, for a given set of incoming particles. This means the number of contributing diagrams is given by the number of ways the incoming arrows can be arranged in. Without considering the chirality of fermions, this becomes the factorial of the number of incoming fermions,

$$(n_{f,in})!, \tag{21}$$

Since each incoming arrow can connect to any outgoing arrow, the number of possible diagrams is instead given by

$$(n_{L,in})! \times (n_{R,in})!. \tag{22}$$

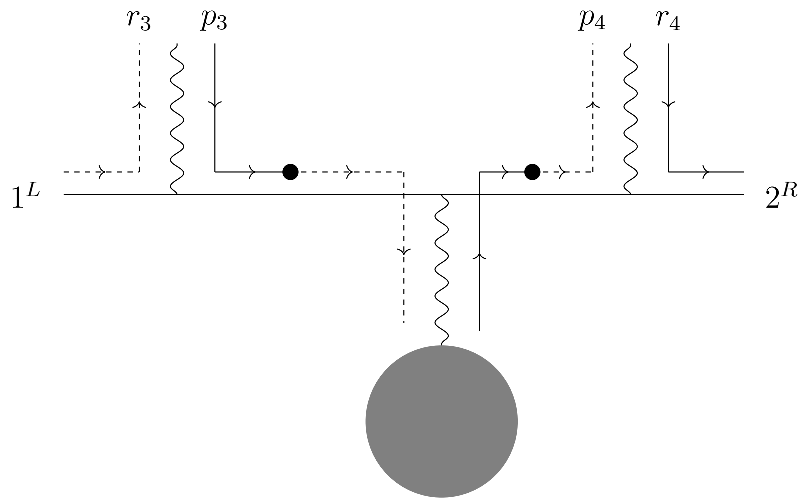
The result could be given in terms of $n_{R,\text{out}}$ and $n_{L,\text{out}}$, but the result is symmetric under $n_L \leftrightarrow n_R$. This means, the only case where the number of contributing diagrams is not decreased is when all incoming particles have the same chirality. Additionally, there is no simplification if all fermions have different flavor since it would limit the number of ways the fermions can be arranged in. The effect of hacing different flavors is seen in the example used in Section 3.4, $e^-e^+ \longrightarrow \mu^-\mu^+$. Since this process has only one diagram, there will be no simplification.

4.2 Reducing the number of diagrams using chirality of photons

Here, the simplification that the chiralty-flow formalism brings to processes including external photons, is calculated. Unless stated otherwise, the calculations are done for one fermion-line. Let n_L be the number of left-chiral photons and n_R the number of right-chiral photons. n_γ is defined as $n_\gamma = n_L + n_R$. To confirm that the results are correct, they are obtained using two different methods.

Method 1

The first method entails identifying the probability of a diagram being non-vanishing and contributing to the amplitude. It is assumed that external photons and one propagating photon are included in the interaction. Although the calculations could be done for many propagating photons, only one is included for simplicity. The interaction has the following form,



The gray circle is used to show that the propagating photon could be connected to anything. Although only one fermion-line is considered, it could be part of a bigger process. The gray circle could be ignored and the result would be the same, but it is used to make it clear that one photon is a propagating photon. Since the fermions have fixed chiralities, the chiralities of the left and right photons, on the fermion-line, are already determined for the diagram to be non-vanishing. Following the flow of chirality, the first requirement for the diagram to be non-vanishing is for the left photon to be left-handed. This comes from setting $r_3 = p_1$. The reference vector is chosen to equal the momentum it is connected to, such that the contraction becomes 0. This choice of reference vector is specific for this interaction and the optimal choice of reference vector would change if any involved momenta change. The probability of this happening is calculated by identifying which photons could be the left photon to give a non-vanishing diagram. A non-vanishing diagram is called a surviving diagram and a diagram with a left-handed left photon is said to survive the left side.

Now, it is being studied in how many ways the diagram can survive the left side. The left photon can either be a left-chiral photon or the propagating photon. This results in $n_L + 1$ possibilities for the left photon. Without requiring the diagram to survive the left side, the left photon could be any of the external photons, n_γ or the propagator. This gives a total of $n_\gamma + 1$ possibilities. Hence, the probability of the diagram surviving the left side is given by

$$P_{\text{survive,left}} = \frac{n_L + 1}{n_\gamma + 1}. \tag{23}$$

Assuming the diagram survives the left side, the diagram needs to survive the right side as well. Thus, the right photon needs to be right-handed. Which photon the right photon could be depends on the left photon, since the propagating photon could be either of them in the case of a surviving diagram. This can be divided into two cases:

First case: The left photon is assumed to be one of the external photons, n_L. Assuming the diagram survives the left side, the total number of possibilities for the left photon is $n_L + 1$. The probability of the left photon being an external photon becomes

$$P_{\text{left,case 1}} = \frac{n_L}{n_L + 1}. \tag{24}$$

The right photon could be one of n_R or the propagating photon. Requiring the diagram to survive, the total number of possibilities for the right photon is $n_R + 1$. Without requiring the diagram to survive the right side, it could be any of the remaining photons, n_γ. The probability of the diagram surviving the right side then becomes

$$P_{\text{right,case 1}} = \frac{n_R + 1}{n_\gamma}. \tag{25}$$

The probability of the diagram surviving the right side after surviving the left side, in the first case, becomes

$$P_{\text{case 1}} = P_{\text{left,case 1}} \times P_{\text{right,case1}} = \frac{n_L}{n_L + 1} \times \frac{n_R + 1}{n_\gamma}. \tag{26}$$

Second case: The left photon is assumed to be the propagating photon. As in the first case, there are $n_L + 1$ possibilities for the left photon. The probability of the left photon being the propagating photon is

$$P_{\text{left, case2}} = \frac{1}{n_L + 1}. \tag{27}$$

The right photon can now be one of the n_R photons, for the diagram to survive the right side. Without requiring the diagram to survive, the right photon could be any of n_γ photons. The probability of the diagram surviving the right side is

$$P_{\text{right, case 2}} = \frac{n_R}{n_\gamma}. \tag{28}$$

The probability of the diagram to survive the right side after surviving the left side, in the second case, becomes

$$P_{\text{case 2}} = P_{\text{left, case 2}} \times P_{\text{right, case 2}} = \frac{1}{n_L + 1} \times \frac{n_R}{n_\gamma}. \tag{29}$$

The probability of the diagram surviving the right side in any of the cases is given by the sum

$$P_{\text{survive,right}} = P_{\text{case 1}} + P_{\text{case 2}} = \frac{n_L(n_R + 1)}{n_\gamma(n_L + 1)} + \frac{n_R}{n_\gamma(n_L + 1)}. \tag{30}$$

For a diagram to survive in a process both sides need to survive simultaneously. Requiring this, the total probability of a diagram being non-vanishing is given by

$$P_{\text{survive}} = P_{\text{survive,left}} \times P_{\text{survive,right}} = \frac{n_L + 1}{n_\gamma + 1} \times \frac{n_L n_R + n_L + n_R}{n_\gamma(n_L + 1)} = \frac{n_L n_R + n_\gamma}{n_\gamma(n_\gamma + 1)}. \tag{31}$$

Throughout the calculations, we note that the chiralities of the photons are assumed. However, the result in Eq. (31) is symmetric under the change $n_L \leftrightarrow n_R$, as it must be.

Method 2

The second method is to calculate the probability of a diagram being non-vanishing by considering the configuration of photons with assumed chiralities. That is, calculating

$$P_{\text{survive}} = \frac{\#\text{ of non-vanishing diagrams}}{\text{total }\#\text{ of diagrams}}. \tag{32}$$

On one fermion-line, the number of possible diagrams is given by the number of ways the photons can be arranged. Assuming there is only one propagating photon apart from the external photons, the total number of diagrams is $(n_\gamma + 1)!$ in all cases.

To obtain the number of diagrams that are non-vanishing the chiralities of the fermions have to be considered. For a diagram to survive both sides, the left photon needs to be left-handed and the right photon needs to the right-handed. The remaining photons can be arranged in any order. As in Method 1, the propagating photon can be either the left- or right photon in the case of a surviving diagram. This means that the right photon is still dependent on how the left side survives and the calculations are divided into two cases, accordingly.

First case: The left photon is assumed to be an external photon, n_L, meaning there are $n_R + 1$ possibilities for the right photon. After fixing the left and right photons, there are $n_\gamma - 1$ left that can be arranged in any order. Hence, the total number of surviving diagrams, in this case, is given by

$$\#\text{ of surviving diagrams} = n_L(n_R + 1)(n_\gamma - 1). \tag{33}$$

The probability of a diagram surviving, in this case, becomes

$$P_{\text{survive},1} = \frac{n_L(n_R + 1)(n_\gamma - 1)!}{(n_\gamma + 1)!}. \tag{34}$$

Second case: The left photon is assumed to be the propagating photon, leaving n_R possibilities for the right photon. As in case 1, there are $n_\gamma - 1$ photons remaining that can be arranged in any order. The total number of diagrams, in this case, is given by

$$\#\text{ of surviving diagrams} = 1 \cdot n_R(n_\gamma - 1)!. \tag{35}$$

The probability of a diagram surviving, in this case, becomes

$$P_{\text{survive},2} = \frac{1 \cdot n_R(n_\gamma - 1)!}{(n_\gamma + 1)!}. \tag{36}$$

The total probability of a diagram surviving in either of the cases is given by the sum

$$\begin{aligned} P_{\text{survive}} = P_{\text{survive},1} + P_{\text{survive},2} &= \frac{n_L(n_R + 1)(n_\gamma - 1)! + n_R(n_\gamma - 1)!}{(n_\gamma + 1)!} = \\ &= \frac{n_L(n_R + 1) + n_R}{(n_\gamma + 1)n_\gamma} = \frac{n_\gamma + n_L n_R}{n_\gamma(n_\gamma + 1)}. \end{aligned} \tag{37}$$

This is the same result as in Method 1. The results are intuitively confirmed by first letting $n_L = n_R$, giving

$$n_L = n_R = \frac{n_\gamma}{2} \implies \frac{n_\gamma + n_L n_R}{n_\gamma(n_\gamma + 1)} = \frac{n_\gamma + \frac{n_\gamma^2}{4}}{n_\gamma^2 + n_\gamma}. \tag{38}$$

Taking the limit when letting the number of photons increase to infinity results in

$$\lim_{n_\gamma \to \infty} \frac{n_\gamma + \frac{n_\gamma^2}{4}}{n_\gamma^2 + n_\gamma} \longrightarrow \frac{1}{4}. \tag{39}$$

When considering a diagram that includes a very large number of photons, the probability of the diagram surviving is given by the probability of the left and right side surviving simultaneously. Since each photon can be either left-handed or right-handed the probability of a diagram surviving should approach $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, in agreement with the above.

4.3 Summing over helicities

The results in Eq. (31) and Eq. (37) give the probability of a diagram surviving assuming chiralities of all photons. To generalize the result and obtain a probability without knowing the configuration of chiralities, the result is summed over all possible chirality-configurations. Summing over all possible chiralities is equivalent to letting the number of left-handed photons go from zero to n_γ. Which photons are left-handed does not matter.

For a given number of left-handed photons, the number of ways they can be distributed, throughout the total number of photons, is given by the binomial coefficient

$$\binom{n_L}{n_\gamma} = \frac{n_\gamma!}{n_L!(n_\gamma - n_L)!}. \tag{40}$$

Here, the number of left-handed photons is still assumed to be known. The probability of having a specific amount of left-handed photons is given by

$$\frac{n_\gamma!}{n_L!(n_\gamma - n_L)!} \times \frac{1}{2^{n_\gamma}}, \tag{41}$$

where 2^{n_γ} is the total number of chirality-configurations considering that each photon can be either left- or right-handed. The probability of the diagram surviving, for any value of n_L, becomes

$$\sum_{n_L=0}^{n_\gamma}\left[\frac{n_L!}{n_L!(n_\gamma - n_L)!}\times\frac{1}{2^{n_\gamma}}\times\frac{n_\gamma + n_L n_R}{n_\gamma(n_\gamma+1)}\right], \tag{42}$$

where n_R can be expressed as $n_\gamma - n_L$. Summing over all possible values of n_L make the calculations only depend on n_γ. As the result for a specific chirality-configuration, this result is symmetric under $n_L \leftrightarrow n_R$. Letting $n_\gamma \to \infty$ and $n_L = n_R$, Eq. (42) approaches the value $\frac{1}{4}$. To see the dependence on the number of photons, the probability of a diagram surviving is plotted as a function of n_γ in Figure 3. It should be mentioned that the chiralities of the fermiosn are fixed and could differ. If the incoming fermion were to be right-chiral the outgoing fermion needs to be left-chiral, meaning the chirality of each photon needs to change. Thus, the calculations would be the same if the chirality of the fermions change.

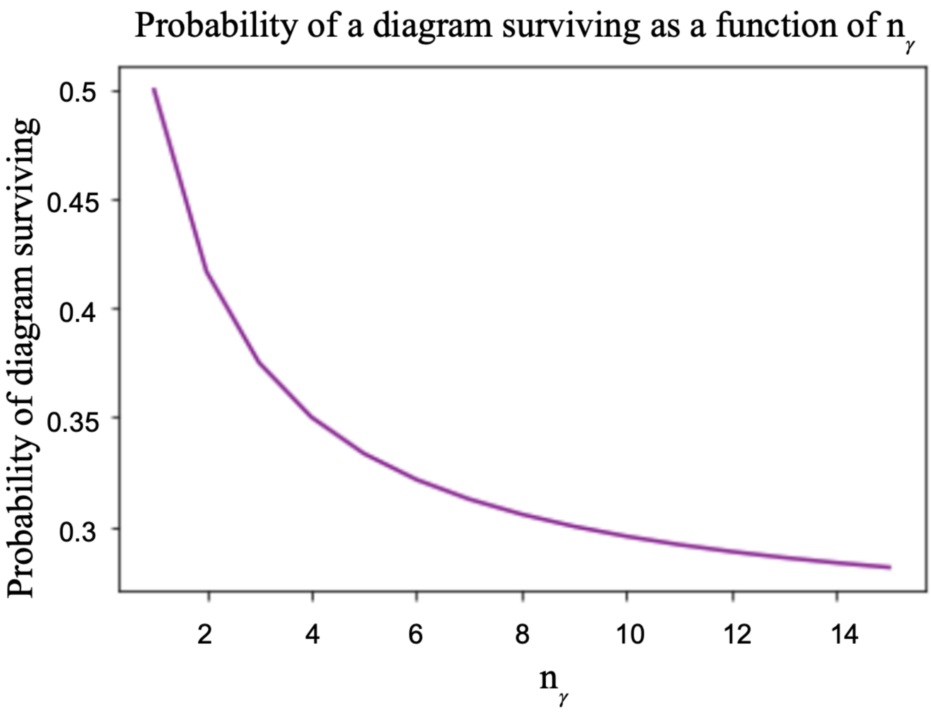


Figure 3: Plot of the probability of a diagram with one fermion-line surviving as a function of the number of photons on the fermion-line. The plot shows the probability when the number of external photons goes from 0 to 15. The propagating photon is not included in the external photons, meaning there will always be at minimum one photon on the fermion-line.

4.4 Considering more than one fermion-line

The results in Sections 4.2 and 4.3 are true for processes with only one fermion-line. Continuing, the results can be generalized further by considering more than one fermion-line. Such a process is shown in Figure 4.

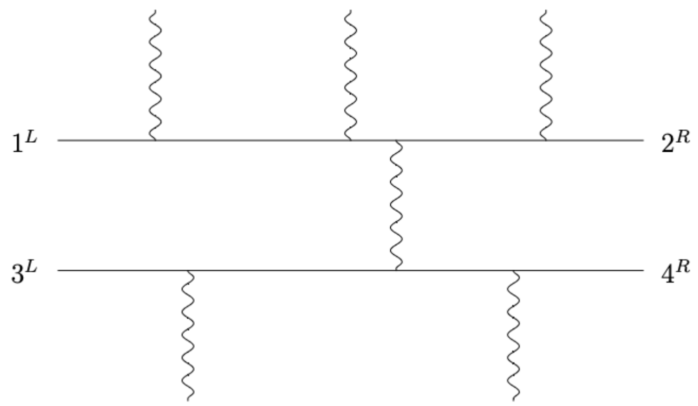


Figure 4: Depiction of how a process with two fermion-lines could look like. The number of photons on each line is allowed to vary.

As seen in Figure 4, there is still only one propagating photon between the lines. This means the results in Sections 4.2 and 4.3 are true for each line. However, the photons could be on any of the fermion-lines, which needs to be included in the total probability of the diagram surviving. It is assumed that only the total number of photons is known, no photons are fixed to a specific line. Let n_1 be the photons on the first line and n_2 be the photons on the second line. The total number of photons is $n_\gamma = n_1 + n_2$.

Firstly, the probability of surviving the first and second fermion-line is calculated. Within n_1 and n_2 there are a number of left-handed and a number of right-handed photons. The probability to survive one of the lines is given in Eq. (37). The probability of the first line surviving, for any number of left-chiral photons, is

$$P_{\text{line1}} = \sum_{n_L=0}^{n_1} \left[\frac{n_L!}{n_L!(n_1 - n_L)!} \times \frac{1}{2^{n_1}} \times \frac{n_1 + n_L n_R}{n_1(n_1+1)} \right], \tag{43}$$

and the probability of surviving the second line is

$$P_{\text{line2}} = \sum_{n_L=0}^{n_2} \left[\frac{n_L!}{n_L!(n_2 - n_L)!} \times \frac{1}{2^{n_2}} \times \frac{n_2 + n_L n_R}{n_2(n_2+1)} \right], \tag{44}$$

In Equation 43 and Equation 44, n_L and n_R are different for each fermion-line. Next, the possible ways the photons can be distributed between the fermion-lines is calculated. Since

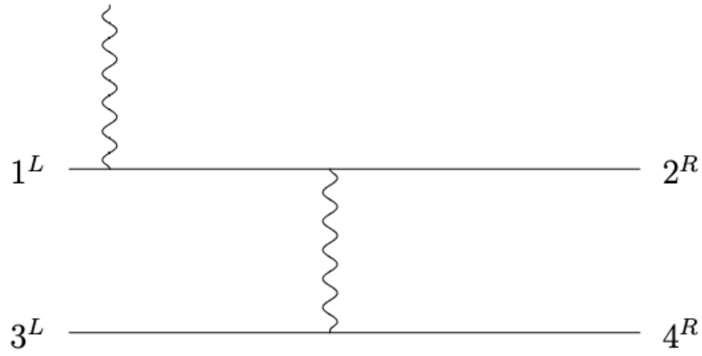


Figure 5: Depiction of a configuration with $n_1 = 1$ photons on the first fermion-line.

$n_2 = n_\gamma - n_1$, it is enough to calculate how many values n_1 can take. Without caring about the order of the photons, the numbers of ways n_1 photons can be picked out from n_γ is

$$\binom{n_1}{n_\gamma} = \frac{n_\gamma!}{n_1!(n_\gamma - n_1)!}. \tag{45}$$

Since each photon can be on one of two fermion-lines, the total number of configurations is 2^{n_γ}. The probability of having a specific configuration becomes

$$\frac{n_\gamma!}{n_1!(n_\gamma - n_1)!} \times \frac{1}{2^{n_\gamma}}. \tag{46}$$

For a diagram to survive with a certain configuration of photons, the probability becomes

$$P_{\text{line1}} \times P_{\text{line2}} \times \frac{n_\gamma!}{n_1!(n_\gamma - n_1)!} \times \frac{1}{2^{n_\gamma}}. \tag{47}$$

Finally, the result is summed over all possible values of n_1. The result becomes

$$\sum_{n_1=0}^{n_\gamma} \left(P_{\text{line1}} \times P_{\text{line2}} \times \frac{n_\gamma!}{n_1!(n_\gamma - n_1)!} \times \frac{1}{2^{n_\gamma}} \right). \tag{48}$$

Figure 5 shows a process when $n_\gamma = 1$, where each photons could have any chirality. The result of Eq. (48) for $n_\gamma = 1$ is $\frac{1}{2}$, which is reasonable and expected. The limit when letting n_γ grow to infinity is $\left(\frac{1}{4}\right)^2$, as expected. Again, to see the dependence on the number of photons, the probability of a diagram surviving is plotted as a function of the total number of photons on the fermion-lines in Figure 6. If the chiralities would change on both fermion-lines, the calculcations would be the same, following the same reasoning as for one fermion-line.

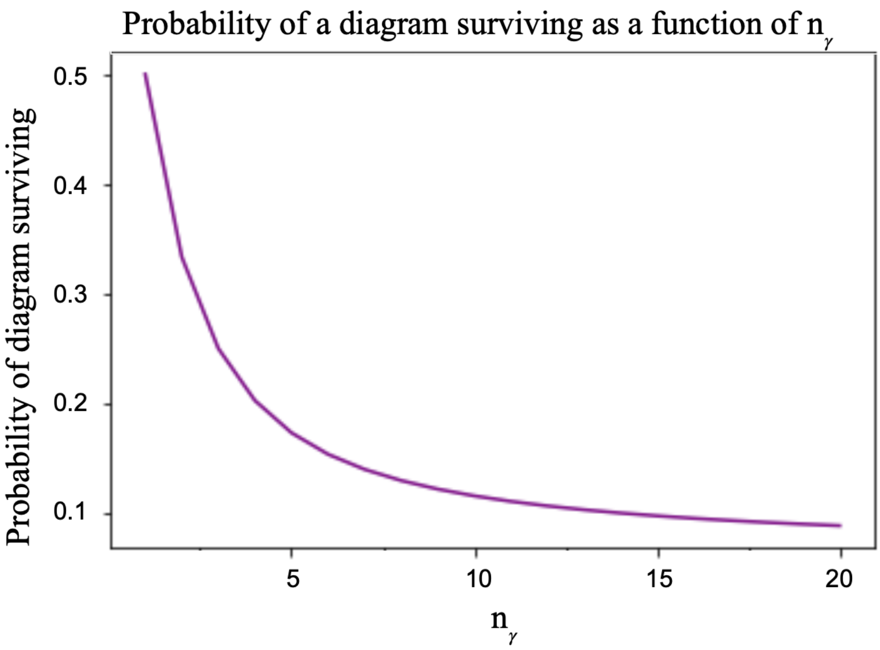


Figure 6: Plot of the probability of a diagram with two fermion-lines surviving as a function of the total number of photons on the fermion-lines. The plot shows the probability when the number of external photons goes from 0 to 20.

As explained earlier, including two fermions-lines means performing the same calculations twice, for different number of photons. Including three fermion-lines would become more complicated since there are two propagators attached to one of the fermion-lines. Such a configuration is shown in Figure 7. The factor $(n + 1)$, which is seen in Eq. (43) and Eq. (44), would become $(n + 2)$ for the third fermion-line. Hence, there is an additional configuration the third fermion-line could have for the diagram to survive. Considering a larger number of fermion-lines, the same reasoning would be applied for each line.

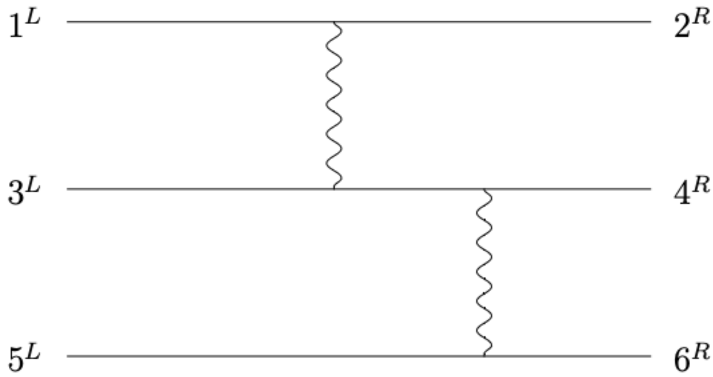


Figure 7: Depiction of a configuration with three fermion-lines, showing one of the lines must have two propagators attached to it.

5 Conclusion

In conclusion, this thesis has given a way to understand and calculate the simplifications the chirality-flow formalism brings to massless QED. This has been done by calculating the number of contributing diagrams when using the chirality-flow formalism compared to the standard Feynman rules. The work began with noticing the calculation of the scattering amplitude look different when including and excluding external photons. This meant the calculations can be simplified in different ways. For fermions, the number of contributing diagrams is reduced due to the constraints on their chirality. When including photons, the number of contributing diagrams reduces due to a limited number of chiralities resulting a contributing diagram. The results are first presented by assuming a configuration of photons and chiralities of fermions and later summed over all possible configurations.

The results presented in this thesis prove that using the chirality-flow formalism brings significant simplifications. However, the work only considers simplification through reducing the number of contributing diagrams and does not explore how the algorithms used to calculate the scattering amplitudes can be simplified for the surviving diagrams. The applications of the chirality-fow formalism in QCD has not been considered in this thesis. Including QCD would result in very complex diagrams and involved calculations because the diagrams can become much more involved when including gluons.

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