Exact q-variations for the True and Simulated Solutions to the Stochastic Heat Equation driven by Additive Gaussian Noise

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Den brusiga diffusionen av värme som utsätts för slumpmässig energi

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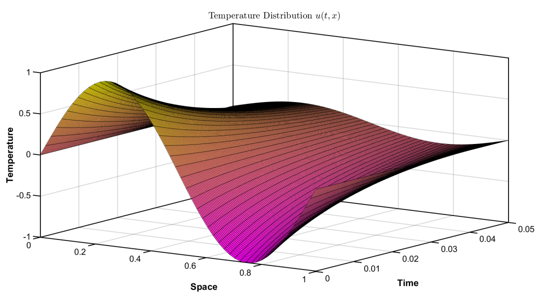
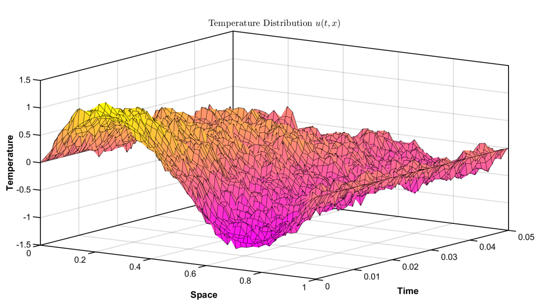
Energi och dess spridning har förutom dess otaliga applikationer inom fysiken och teknologi, en väldigt djup matematisk skönhet som visar sig i ekvationen som beskriver diffusionen. Arbetet har studerat vad som händer när ett system som beskrivs av värmeledningsekvationen påverkas av slumpmässiga fluktueringar, ett så kallat *vitt brus*.



Figur 1. Simulation av värmediffundering i en stav som hettas upp enligt en initial temperaturdistribution och som sedan diffunderar ut över tiden.

Idén bakom vitt brus (förutom en del tekniska krav) är att den totala tillförslade energin är 0, och att energin i två separata regioner i mediet är oberoende av varandra. Fälten som beskriver diffusionen visar sig ha distinkta fraktalbeteenden med kopplingar till andra välkända slumpmässiga processer som kan liknas till slumpvandringar, som exempelvis den Brownska rörelsen och dess generaliseringar.



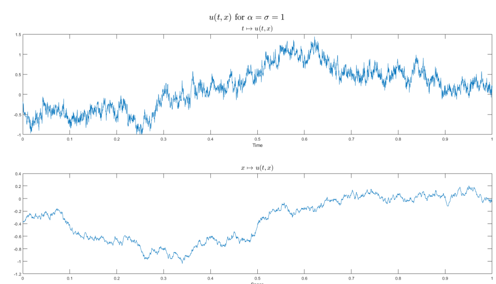
Figur 2. Simulation av värmediffundering i en stav som hettas upp enligt en initial temperaturdistribution och som sedan diffunderar ut över tiden med tillslag av vitt brus.

Dessa resultat gäller även för en del brus som är *färgat*, vilket innebär att energitillförseln i mediet är inte helt oberoende.

Fokuset för den här rapporten ligger i att räkna på *variationen* över tid och rum för diffusionen som påverkas av brus. Variationen är ett slags mått på hur mycket fältet studsar fram och tillbaka, och det vi sammanställer i arbetet är de statistiska egenskaperna för variationen, samt hur datorn kan simulera dessa ytterst ojämna fält.

När det kommer till datorsimuleringarna visar det sig att många av de standardmetoderna som används för att simulera värmeledning och allmän diffusion inte kan fånga den korrekta variationen. Vi demonstrerar att av dessa standardmetoder, finns det endast *en* som kan fånga de korrekta fraktalegenskaperna. Resten av dem antingen slätar ut fältet för mycket, eller gör dem ännu mer irreguljära.

Eftersom fälten däremot är slumpmässiga processer i tid och rum, fås ytterligare en till metod att simulera diffusionens vägar med hjälp av verktyg från sannolikhetsteorin, och denna metod lyckas fånga rätt variation.



Figur 3. De vägvisa fraktalerna av temperaturdiffundering över tid och rum.

Vi belyser dessutom de statistiska egenskaperna av fälten med hjälp av simuleringarna, som bekräftar den härledda teorin.

Acknowledgements

We would like to thank our supervisor Magnus Wiktorsson for all the fruitful discussions and comments we have received along the way. We would also like to thank our closest family and friends for sticking with us throughout this journey.

Abstract

The field of stochastic partial differential equations (SPDEs) has been extensively studied the past decades, with the stochastic heat equation $\frac{\partial}{\partial t}u(t,x) - \alpha\Delta u(t,x) = \sigma\dot{M}(t,x)$ driven by a Gaussian noise that is white in time but perhaps correlated in space being an important example of such SPDEs. The solution u and its exact q-variations is the object of consideration. This thesis rigorously defines the solution u as an isonormal Gaussian process, and we present how it is up to C^1 perturbations a scaled fractional Brownian motion cF^H (fBm) with $c > 0$ and some Hurst parameter H. The $1/H$ variations of u and cF^H will agree, such that with the known variations, estimations of the drift α and diffusion σ can be constructed, by sampling the variation at equidistant points in time and space separately as well as jointly. The solution to the stochastic heat equation is simulated both with the known covariance structure, along with finite difference schemes. We demonstrate how the only one step finite-difference scheme approximation that obtains the correct limiting q-variations is a specific Crank-Nicolson scheme with CFL number $\frac{1}{\pi-2}$. The drift and diffusion estimators' asymptotic normality is illustrated using the simulations.

Contents

Chapter 1

Introduction

1.1 Background

Consider a rod of length $L > 0$ that is heated up according to some initial temperate at starting time $t_0 = 0$, and let $u(t, x)$ denote the temperature distribution of the rod at a point x between 0 and L and time $t > 0$. The change of the temperature at the point t is proportional to the second derivative in space at the point $x \in (0, L)$; in equation form

$$\frac{\partial}{\partial t} u = \alpha \frac{\partial^2}{\partial x^2} u, \tag{1.1}$$

which reflects how the heat diffuses through the rod, where $\alpha > 0$ is a constant describing the rate of diffusion. The solutions to the heat equation are known for their smoothness and regularity. Given sufficiently smooth initial conditions and for example constant periodic Dirichlet boundary conditions $u(t, 0) = u(t, L) = C \in \mathbb{R}$, the solution u spreads out to a steady state due to the diffusive nature of the equation. An illustration of this can be seen in the below Figure 1.1.

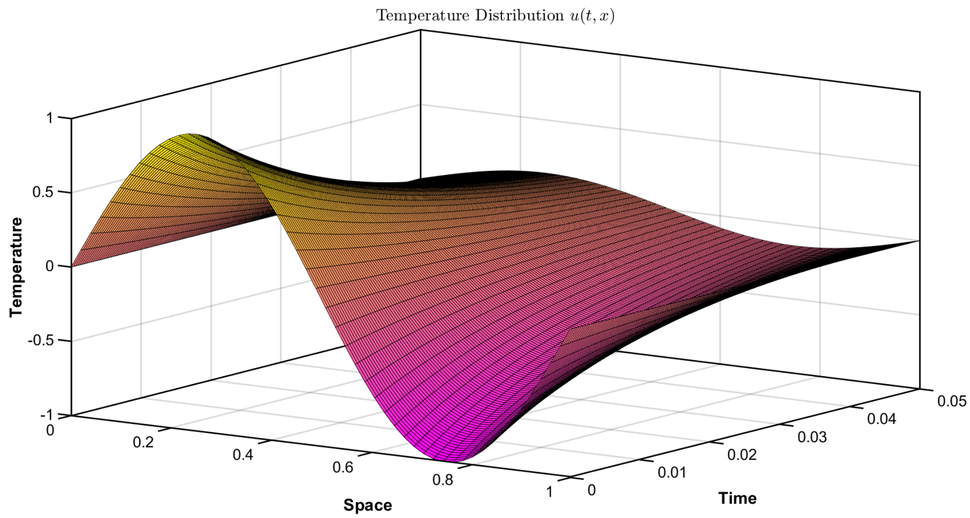


Figure 1.1: Illustration of the solution to the heat equation 1.1 with $\alpha = 1$ on the "rod" with length $L = 1$, constant boundary temperature $u(t, 0) = u(t, 1) = 0$ for all $t > 0$, and initial condition $u(0, x) = \sin(2\pi x)$ for $0 < x < 1$.

The above heat equation (with derivation found in e.g. [Baehr and Stephan, 2011, Chapter 2]) was first introduced by Joseph Fourier in his book "Théorie analytique de la chaleur" (The Analytical Theory of Heat). Addition of a driving term $f(t, x, u(t, x))$ on the right hand side of 1.1 allow us to consider the diffusion in a medium that is provided thermal energy in both time and space according to f.

Applications of the heat equation 1.1 with some driving term are plenty, the obvious one being the description of how heat diffuses in a medium where the solutions u enable us to predict the temperature at a given time and space. Other applications of the heat equation include describing and modelling the following, current flow in electrical conductors, the diffusion of gases, flow in groundwater basins, and option pricing, among many others (see e.g. [Narasimhan, 1999] for a comprehensive exposition of applications and history of the heat equation).

For our thesis, we also aim to consider the heat equation, but with the addition of a *noisy* driving term. We will firstly consider the randomness as a *white noise in time and space*, $\dot{W}(t, x)$, which (heuristically), is Gaussian, adds on average 0 energy to the system, and the energy added in two disjoint regions are independent. An illustration of the solution to the heat equation driven by a space-time white noise can be seen in Figure 1.2 below. Compare it with the smooth solution given in 1.1 above.

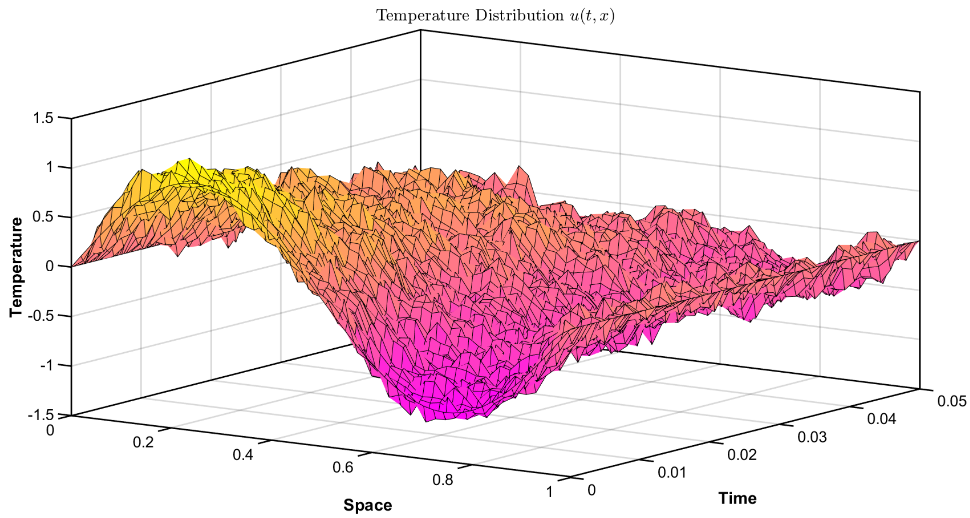


Figure 1.2: Illustration of the solution to the heat equation 1.1 driven by a space-time white noise $\dot{W}(t,x)$ with $\alpha = 1$ on the "rod" with length $L = 1$, constant boundary temperature $u(t,0) = u(t,1) = 0$ for all $t > 0$, and initial condition $u(0,x) = \sin(2\pi x)$ for $0 < x < 1$.

After developing the necessary tools to study this **stochastic** heat equation, we will be able to consider more general noise, which is still independent over time, but has some spatial correlation, which we denote as $\dot{M}(t,x)$. This thesis emphasises the *path-wise roughness* of the solutions to the stochastic heat equation, along with the regularity of its simulations.

1.2 Stochastic processes and their roughness

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. With the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and some arbitrary index set $\mathcal{T}$, a *stochastic process* is given by the real-valued stochastic variables $X(t) := \{X(\omega, t)\ ,\ t \in \mathcal{T}\}$, where $\omega \in \Omega$ is any basic event for all t.

1.2.1 The (Fractional) Brownian motion

The solutions to the stochastic heat equation are heavily related to the *Brownian motion* and the *fractional* Brownian motion (fBm), the latter being a generalisation of the former. The Brownian motion is named after the botanist Robert Brown who observed this process as a jittery motion of pollen grains in water. Brownian motion has since then become a fundamental concept in various scientific fields, including physics, chemistry, biology, and finance. It is used to describe and model the random movement of particles, stocks, and to understand diffusion processes, as well as other phenomena influenced by random fluctuations. Both Louis Bachelier in 1900 and Albert Einstein in 1905 have made rigorous definitions of Brownian motion [Narasimhan, 1999]. For a *standard*[1] Brownian motion we will use the following definition (see [Björk, 2005, Chapter 4, section 1]).

Definition 1.2.1. *A stochastic process W is called a* standard Brownian motion *if the following conditions hold,*

1. $W(0) = 0$.
2. *The process W has independent increments, i.e. if $r < s \leq t < u$ then $W(u) - W(t)$ and $W(s) - W(r)$ are independent random variables.*
3. *For $s < t$, the stochastic variable $W(t) - W(s) \in N(0, t - s)$.*
4. *The process W has continuous sample paths, i.e. for a fix $\omega \in \Omega$, the function $t \mapsto W(\omega, t)$ is continuous.*

It is specifically the last point in the above definition that warrants some attention. The standard Brownian motion has continuous sample paths indeed, but the path $t \mapsto W(\omega, t)$ is almost surely nowhere differentiable as a function of t [Björk, 2005, Theorem 4.2]. See Figure 1.3 below for a plot of a sample path, demonstrating the rugged trajectory.

[1] The emphasis on standard is to highlight the continuous trajectories of the Brownian motion. For general Gaussian processes (and Brownian motions) continuity is not always guaranteed.

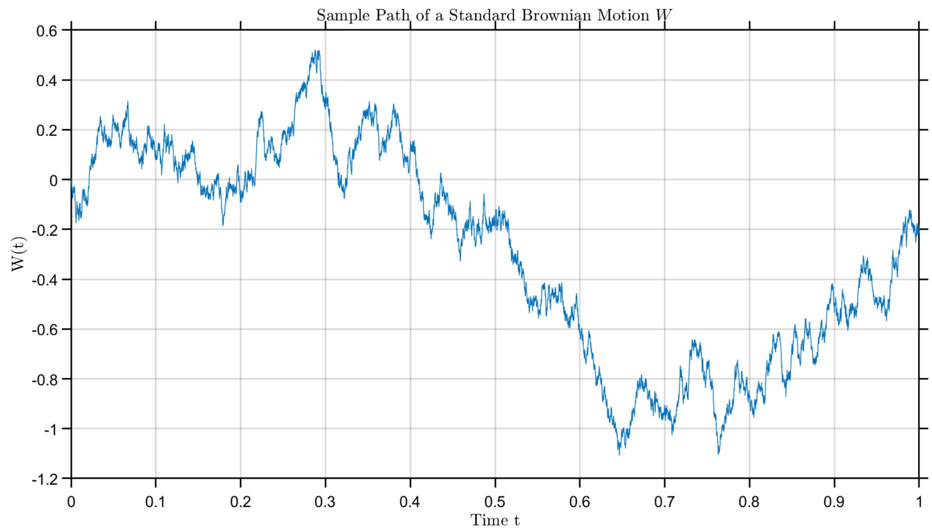


Figure 1.3: A sample path of the standard Brownian motion.

Even though the paths are non-differentiable, they are at least almost $1/2-$Hölder continuous.

Definition 1.2.2 (Local Hölder continuity)**.** *A function* $f : D \to \mathbb{R}$*, where* D *some normed space is locally* γ*-Hölder continuous with Hölder exponent* $\gamma \geq 0$ *if there exists* $C > 0$ *such that*

$$|f(x) - f(y)| \leq C\|x - y\|^{\gamma}$$

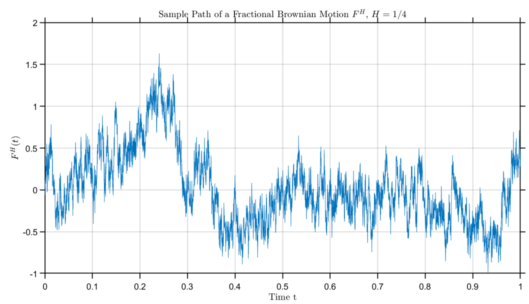
for x*,* $y \in K$ *for all compact sets* $K \subseteq D$*.*

The noise we talked about earlier can be seen in some weak sense to be the derivative of the Brownian motion $\dot{W}(t) = \frac{dW}{dt}(t)$, and for space-time white noise we would have $\dot{W}(t,x) = \frac{\partial W}{\partial t \partial x}(t,x)$, where $W(t,x)$ is the *Brownian sheet*, a higher dimensional generalisation of the Brownian motion.

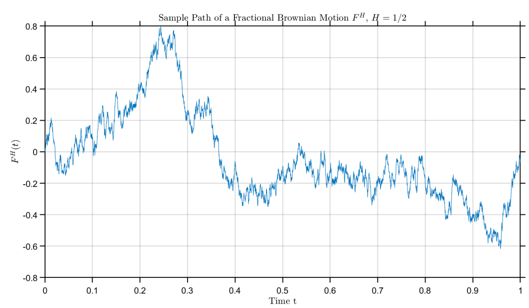
The solutions to the stochastic heat equation will have many of the same properties as the Brownian motion and the fractional Brownian motion (which does away with the independent increments). The fractional Brownian motion was originally defined in [Kolmogorov, 1940] by Kolmogorov in a Hilbert space framework. An example is the usage of fractional Brownian motion for modelling the Nile by Hurst who created the so called Hurst phenomena. The usage of the fractional Brownian- instead of the Brownian motion was because of the fact that they wanted long range dependence in the model [Mason, 2016].

The fBm is denoted F^H, where $H \in (0,1)$ is called the *Hurst* parameter, and with

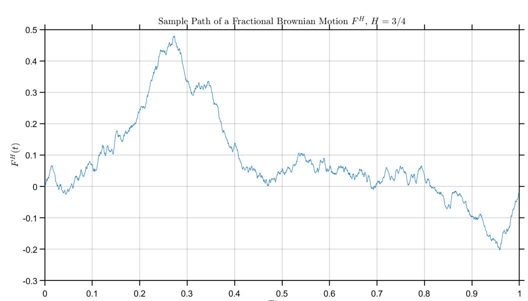
$H = 1/2$ the fBm becomes the Brownian motion. We will give a rigorous definition of an fBm process later in Chapter 3, however we state now that the fBm is also Hölder-continuous, with Hölder-exponent that is almost H. See Figure 1.4 below which shows the irregularity of the fBm for different Hurst parameters H and how its Hölder exponent is connected to the roughness of the paths. The smaller the parameter H, the smaller the Hölder-exponent and the more bumpy the paths become.



(a) Fractional Brownian Motion with Hurst parameter $H = \frac{1}{4}$



(b) Fractional Brownian Motion with Hurst parameter $H = \frac{1}{2}$



(c) Fractional Brownian Motion with Hurst parameter $H = \frac{3}{4}$

Figure 1.4: Fractional Brownian Motion with different Hurst parameters, simulated using the same underlying independent standard Gaussian variables.

1.3 Stochastic Partial Differential Equations (SPDEs)

Often scientists and applied mathematicians use differential equations to model complex interactions in nature, but these equations have a hard time taking into account the inherent randomness that we often perceive when measuring physical phenomenon. One way to mathematically consider this is by the concept of *noise*, which captures the intricacies of randomness present in nature and society. Noise can be observed by the noise in transmitting voice by telephone, mechanical noise from machines, or noise in the temperature from random fluctuations in weather, among others.

By adding noise terms to differential equations, and particularly partial differential equations (PDEs), we obtain what is called **stochastic** partial differential equations or SPDEs for short. We will see that we can solve a subset of SPDEs, with the same methods as for deterministic PDEs. The solutions to SPDEs often have many interesting mathematical and statistical properties, for instance the roughness of their paths, which our thesis emphasises.

1.3.1 What is an SPDE?

In the most general setting, an SPDE of order $k \in \mathbb{N}$ is a *formal*[2] equation that involves the unknown multivariate function (stochastic process) $u : \Omega \times \mathscr{D} \to \mathbb{R}$ and its Jacobian derivatives $\mathcal{D}^i u$, $i = 1, \ldots, k$ such that, for all $x \in \mathscr{D}$ and $\omega \in \Omega$,

$$\begin{aligned} \phi\left(\omega, \mathcal{D}^k u(x), \ldots, \mathcal{D}u(x), u(x), y\right) = \\ \psi\left(\omega, \mathcal{D}^k u(x), \ldots, \mathcal{D}u(x), u(x), x\right) \dot{M}(\omega, x). \end{aligned} \tag{1.2}$$

With initial- and boundary conditions if applicable. Here we have

$$\phi, \psi : \Omega \times \mathbb{R}^{d^k} \times \cdots \times \mathbb{R}^d \times \mathbb{R} \times \mathscr{D} \to \mathbb{R}.$$

The factor $\dot{M}$ is called the *noise* in the system and is often symbolically denoted $\frac{\partial M}{\partial t \partial x}$, furthermore, the initial- and boundary conditions may also be stochastic. The space Ω is the sample space from a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ while $\mathscr{D}$ is usually some subspace of the Euclidean space $\mathbb{R}^d$ for some integer $d \geq 1$.

[2]Because the noise and randomness of the system often does not admit a point-wise (let alone differentiable) solution to 1.2, we call the equation formal. It will become clear later what we mean by a rigorous solution to an SPDE and how it connects to the corresponding PDE without randomness.

1.3.2 Linear SPDE's

To study general equations of the type given in 1.2 require very heavy mathematical machinery that is beyond the scope of this thesis. Instead, the theory we present will be able to handle SPDE's of the unknown process $u : \Omega \times \mathbb{R}_+ \times \mathscr{D} \to \mathbb{R}$ of the form

$$\mathcal{L}u(t,x) = a\Big(t, x, u(t,x)\Big) + \dot{M}(t,x), \;\; t > 0, \; x \in \mathscr{D} \subseteq \mathbb{R}^d, \tag{1.3}$$

with deterministic initial conditions and if necessary also boundary conditions. In the equations and calculations we usually omit the variable $\omega \in \Omega$. The operator $\mathcal{L} = \frac{\partial}{\partial t} - A$ is a linear partial differential operator, where A is a second order differential operator in space with the property of uniform ellipticity, see for example [Polyanin and Nazaikinskii, 2016, Section 16.2.2]. The driving term $\dot{M}$ is a Gaussian noise, where we start with what is called white noise and later expand to a noise which is white in time but "coloured" in space.

The SPDEs of the form 1.3 are often called linear SPDEs driven by *additive* noise. In contrast with SPDEs driven by *multiplicative* noise, which instead would contain a term $b(t,x,u(t,x))\dot{M}(t,x)$ on the right hand side. Equations with multiplicative noise require heavier theory which we will not look at. However we will consider the case of $b \equiv \sigma > 0$ being constant, with σ being the *diffusion* parameter. For the differential operator we will mainly focus on $\mathcal{L} = (\frac{\partial}{\partial t} - \alpha\Delta)$, with $\alpha > 0$ being called the *drift* parameter. We will study the solution on two spaces, $\mathscr{D}$ is either the bounded interval $(0, L)$ for some $L \in \mathbb{R}$ or the entire space $\mathbb{R}^d$ where the integer $d \geq 1$.

From PDE theory (see e.g. [Polyanin and Nazaikinskii, 2016]), if we assumed that equation 1.3 did not contain any stochastic elements and with some further integrability conditions on the functions, we know that there exists a *mild* solution formula for equation 1.3 of the form

$$\begin{aligned} u(t,x) = I_0(t,x) &+ \int_0^t \int_{\mathscr{D}} \Psi(s,y;t,x) a(s,y,u(s,y)) dyds \\ &+ \int_0^t \int_{\mathscr{D}} \Psi(s,y;t,x) M(dyds). \end{aligned} \tag{1.4}$$

I_0 contains the data from the initial- and boundary conditions and $\Psi(s,y;t,x)$ is the fundamental/Green solution to the deterministic PDE version of equation 1.3. When $\mathcal{L} = (\frac{\partial}{\partial t} - \alpha\Delta)$: $\Psi(s,y;t,x) = \Psi(t-s,x,y)$ and furthermore for the fundamental solution on $\mathscr{D} = \mathbb{R}^d$: $\Psi(t-s,x,y) = \Psi(t-s,x-y)$. This is because of the translation invariance in time and space respectively for the Green and fundamental solution to the heat equation.

One big conundrum of this thesis is to give a rigorous meaning to the **stochastic** integral term above,

$$\int_0^t \int_{\mathscr{D}} \Psi(s, y; t, x) M(dyds).$$

We will see that the solutions to an SPDE will be given by the same formula as for the deterministic case, as long as all stochastic integrals in the mild solution are well defined. The definition of the stochastic integral is the big focus of chapter 2.

Remark 1.3.1 (Note on naming convention)**.** *When we are explicitly looking at the solution on $\mathscr{D} = \mathbb{R}^d$ we will denote $\Psi = \Phi$ and call it the* fundamental solution*. If we consider the solution on $\mathscr{D} = (0, L)$ or some other bounded interval we write $\Psi = G$ and call it the* Green function*.*

1.3.3 The (Stochastic) Heat Equation

This section aims to lay down all the needed results for the later analysis of the stochastic heat equation (SHE).

To get an intuition for the noise, first let

$$\frac{\partial M}{\partial t \partial x}(t, x) = \sigma \frac{\partial W}{\partial t \partial x}(t, x),$$

where $\sigma \dot{W} := \sigma \frac{\partial W}{\partial t \partial x}(t, x)$ is a *white noise process* in time and space scaled by the diffusion $\sigma > 0$. The idea is that $\dot{W} \in L^2(\Omega)$ is a zero mean Gaussian stochastic process with finite second moment equal to 1, where the "white" of the noise means that $\dot{W}(t_1, x_1)$ and $\dot{W}(t_2, x_2)$ are independent for $(t_1, x_1) \neq (t_2, x_2)$. However, a process like $\dot{W}(t, x)$ cannot exist. A short motivation is that if it did exist then from these $\dot{W}(t, x)$ we could construct uncountably many random variables in $L^2(\Omega)$ that are mutually orthogonal. But the Hilbert space $L^2(\Omega)$ is separable, and therefore any orthonormal set of vectors has to be at most countable. Which implies that there are no such processes $\dot{W}(t, x)$. This suggests some careful treatment of the noise to make equation 1.5 below rigorous. Nevertheless, we proceed heuristically.

The general heat equation that is the topic for the majority this thesis, on the spatial domain $\mathscr{D} = \mathbb{R}^d$ for some integer $d \geq 1$, or $\mathscr{D} = (0, L)$ where $L \in \mathbb{R}$ is,

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) - \alpha \Delta u(t, x) = \sigma \dot{W}(t, x) & t > 0,\ x \in \mathscr{D} \\ u(0, x) = u_0(x) & x \in \mathscr{D} \\ + \text{ Homogeneous BC if applicable} & x \in \partial \mathscr{D}. \end{cases} \tag{1.5}$$

With $\Delta := \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}$ and $u_0 : \mathscr{D} \to \mathbb{R}$ is some sufficiently integrable deterministic function.

Some important properties that we will show later when we have defined the solution $u(t,x)$ to 1.5 with white noise forcing include that the $u(t,x)$ does not exist point-wise in dimensions $d \geq 2$. And for $d = 1$, u will almost surely not be a differentiable function anywhere. Its sample paths are incredibly rough but they are at least almost 1/2-Hölder continuous in space and almost 1/4-Hölder continuous in time.

One way to force solutions in higher dimensions is to consider a more smooth driving noise $\dot{M}$ in the right hand side, by for example, introducing some homogeneous spatial correlation of the noise.

1.3.3.1 Solutions on $\mathscr{D} = \mathbb{R}^d$

The equation we consider will be

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \alpha\Delta u(t,x) = \sigma\dot{W}(t,x) & t > 0,\ x \in \mathbb{R}^d \\ u(0,x) = u_0(x) & x \in \mathbb{R}^d. \end{cases} \tag{1.6}$$

We give a formal proof of the below proposition in the case of a deterministic and more regular driving term in Appendix C.

Proposition 1.3.2. *The solution to 1.6 is by the superposition principle,*

$$u(t,x) = \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha t}}}{(4\pi\alpha t)^{d/2}} u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha(t-s)}}}{(4\pi\alpha(t-s))^{d/2}} f(s,y)dyds$$

With $f(t,x) = \sigma\dot{W}(t,x) = \sigma\frac{\partial W}{\partial t\partial x}(t,x)$ and using properties of the Riemann-Stiltjes integral we can write the solution as

$$u(t,x) = \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha t}}}{(4\pi\alpha t)^{d/2}} u_0(y)dy + \sigma\int_0^t \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha(t-s)}}}{(4\pi\alpha(t-s))^{d/2}} W(dyds). \tag{1.7}$$

The function $\Phi(s,y;t,x) = \Phi(t-s,x,y) = \Phi(t-s,x-y) := \frac{e^{\frac{-|x-y|^2}{4\alpha(t-s)}}}{(4\pi\alpha(t-s))^{d/2}}$ for $t > s$ and $x, y \in \mathbb{R}^d$ is the fundamental solution to 1.5 above. It solves the differential equation

$$\frac{\partial\Phi}{\partial t}(s,y;t,x) - \alpha\Delta\Phi(s,y;t,x) = 0.$$

With initial condition $\Phi(s,y;t,x) = \delta(x-y)$ for $t > 0$ and $x \in \mathbb{R}^d$ where $s \in [0,t)$ and $y \in \mathbb{R}^d$ are free parameters for above equation. The Dirac-δ initial condition should be seen in a distributional sense, i.e. that

$$\int_{\mathbb{R}^d} u_0(y)\Phi(s,y;t,x)dy \to u_0(x) \text{ as } t \to 0.$$

We emphasise again that the integral $\int_0^t \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha(t-s)}}}{(4\pi\alpha(t-s))^{d/2}} W(dyds)$ is not yet defined. This is the aim of chapter 2.

Remark 1.3.3. *The function*

$$\Phi(t,x) = \frac{e^{\frac{-|x|^2}{4\alpha t}}}{(4\pi\alpha t)^{1/2}}$$

for $t > 0$ and $x \in \mathbb{R}$ can be seen as the probability density function of a zero mean Gaussian variable with variance $2\alpha t$.

A useful property is that for $d = 1$, the function $(s, y) \mapsto \Phi(t-s; x-y)$ lies in $L^2((0,t) \times \mathbb{R})$ for all $t > 0$. To see this note that by a change of variables $t-s$ to s and the translation invariance in y we have

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}} \Phi(t-s, x-y)^2 dyds \\ &= \int_0^t \int_{\mathbb{R}} \left(\frac{e^{\frac{-|x-y|^2}{4\alpha(t-s)}}}{(4\pi\alpha(t-s))^{1/2}} \right)^2 dyds = \int_0^t \int_{\mathbb{R}} \frac{e^{\frac{-|y|^2}{2\alpha s}}}{(4\pi\alpha s)} dyds.\end{aligned}$$

We split the integral over space and time in the following way.

$$= \int_0^t \int_{\mathbb{R}} \frac{e^{\frac{-|y|^2}{2\alpha s}}}{(\sqrt{2\cdot 8}\sqrt{\pi s\alpha}^2)} dyds = \int_0^t \frac{1}{\sqrt{8\pi s\alpha}} \left(\int_{\mathbb{R}} \frac{e^{\frac{-|y|^2}{2\alpha s}}}{(\sqrt{2\pi s\alpha})} dy \right) ds.$$

We know that the fundamental solution $\frac{e^{\frac{-|y|^2}{2\alpha s}}}{(\sqrt{2\pi s\alpha})}$ above can be seen as a probability density of a Gaussian variable, and hence the integral over y is equal to 1. This gives

$$\int_0^t \frac{1}{\sqrt{8\pi s\alpha}} ds = \frac{1}{\sqrt{2\pi\alpha}} \int_0^t \frac{1}{2\sqrt{s}} ds = \sqrt{\frac{t}{2\pi\alpha}} < \infty.$$

The above result actually follows from the semi-group property (cf. [Samuil D. Eidelman, 1998, Property VI.1]) of Φ over $\mathbb{R}^d$.

Proposition 1.3.4. *For all $x, z \in \mathbb{R}^d$, we have*

$$\Phi(s+t, x-z) = \int_{\mathbb{R}^d} \Phi(t, x-y)\Phi(s, y-z)dy.$$

Where we also note the symmetry $\Phi(t, x-z) = \Phi(t, z-x)$.

1.3.3.2 Solutions on $\mathscr{D} = (0, L)$

We prescribe vanishing Dirichlet boundary conditions $u(t,0) = u(t,L) = 0$ along with the initial condition $u(0,x) = u_0(x)$.

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \alpha\Delta u(t,x) = \sigma\dot{W}(t,x) & t > 0,\ x \in (0,L) \\ u(0,x) = u_0(x) & x \in [0,L] \\ u(0,t) = u(L,t) = 0 & t > 0. \end{cases} \tag{1.8}$$

Where $u_0(x)$ is some sufficiently integrable function. We look at the corresponding weak formulation and mild solution (cf. [Walsh, 1986, Chapter 3]). Multiply by a test function $\phi(x)$ and integrate up over the domain $[0,t] \times [0,L]$ gives us weak formulation,

$$\begin{aligned} &\int_0^L u(t,x)\phi(x)dx - \alpha\int_0^t\int_0^L u(s,x)\frac{d^2\phi}{dx^2}(x)dxds \\ = &\int_0^L u_0(x)\phi(x)dx + \sigma\int_0^t\int_0^L \phi(x)W(dxds). \end{aligned}$$

The above weak form implies a so called mild solution of 1.8, for a derivation see for example [Dalang et al., 2009, Chapter 1. Section 6] or [Walsh, 1986, Chapter 3].

$$u(t,x) = \int_0^L G(0,y;t,x)u_0(y)dy + \int_0^t\int_0^L G(s,y;t,x)W(dyds). \tag{1.9}$$

Where $G(t-s,x,y) := G(s,y;t,x) := \frac{2}{L}\sum_{n=1}^{\infty} e^{-\frac{\alpha n^2\pi^2}{L^2}(t-s)}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)$ for $t > s$ and $x, y \in (0,L)$ is called the *Green function* associated to the differential equation

$$\frac{\partial G}{\partial t}(s,y;t,x) - \alpha\Delta G(s,y;t,x) = 0.$$

With initial condition (in distributional sense) $G(s,y;t,x) = \delta(x-y)$ for $t > 0$ and $x \in (0,L)$ and vanishing Dirichlet boundary conditions. The parameters $s \in [0,t)$ and $y \in (0,L)$ are free parameters for the above equation.

Remark 1.3.5 (When $\mathscr{D} = (-L,L)$)**.** *We can find the solutions on $(-L,L)$ by considering the equation on $(0,2L)$ and utilising translation invariance on 1.9. Shift the space one step to the left to get the solution on $(-L,L)$. Effectively replacing all the L's to $2L$ and all the x and y's to $x+L$ and $y+L$ respectively in 1.9.*

For $G(t,x,y) := G(0,y;t,x)$, an important inequality is that

$$G(t,x,y) \le \Phi(t,x-y) \le \frac{1}{\sqrt{4\pi\alpha t}}. \tag{1.10}$$

The Green function also satisfies a semi-group property.

Proposition 1.3.6. *For all $x, z \in (0, L)$, we have*

$$G(t+s, x, z) = \int_0^L G(t, x, y)G(s, y, z)dy. \tag{1.11}$$

Proof. Recall that

$$G(t, x, y) := \frac{2}{L}\sum_{n=1}^{\infty} e^{-\frac{\alpha n^2 \pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right).$$

Then

$$\int_0^L G(t, x, y)G(s, y, z)dy$$
$$= \frac{4}{L^2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} e^{-\frac{\alpha\pi^2}{L^2}(n^2t+k^2s)} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\int_0^L \sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{k\pi y}{L}\right)dy.$$

equation 1.11 follows since,

$$\int_0^L \sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{k\pi y}{L}\right)dy = \begin{cases} L/2, & n = k \\ 0, & n \neq k. \end{cases}$$

□

1.4 The goal of this thesis

One important object of study is the sample path $((t, x) \mapsto u(\omega, t, x))$ regularity of the solution $u(t, x)$ to the stochastic heat equation. One such regularity is the before mentioned Hölder-continuity of u. For the case of the stochastic heat equation with additive white noise, we will show how the solution is almost 1/2-Hölder continuous in space and almost 1/4-Hölder continuous in time. In particular the solution u to the stochastic heat equation has similarly rough paths to the sample paths of the fractional Brownian motion, with the regularity of $F^{\frac{1}{2}}$ corresponding to $x \mapsto u(t, x)$ and $F^{\frac{1}{4}}$ to $t \mapsto u(t, x)$. Below in Figure 1.5 is a simulation of the solution on $\mathbb{R}_+ \times [0, 1]$ with homogeneous Dirichlet conditions- and initial condition.

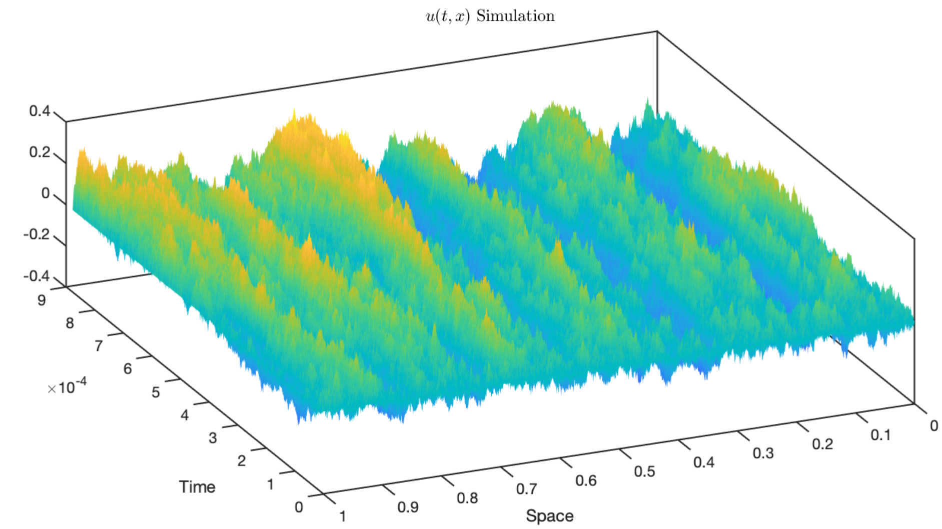


Figure 1.5: Simulation of the random field solution to the stochastic heat equation 1.8.

Fixing one variable we can look at u as a function of one variable, we have the following simulated paths in figures 1.6 and 1.7 (which are from the same simulation as 1.5),

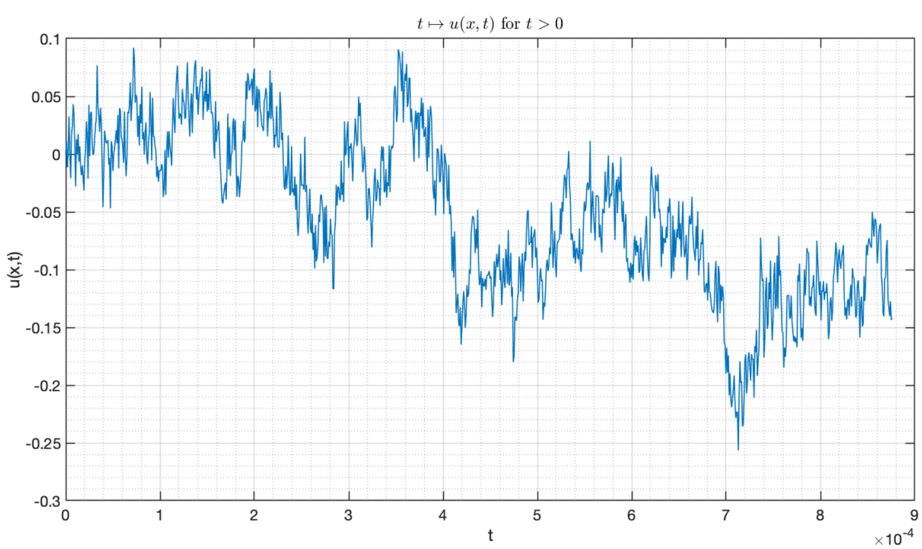


Figure 1.6: Simulation of the random field solution to the stochastic heat equation 1.8 as a function of time.

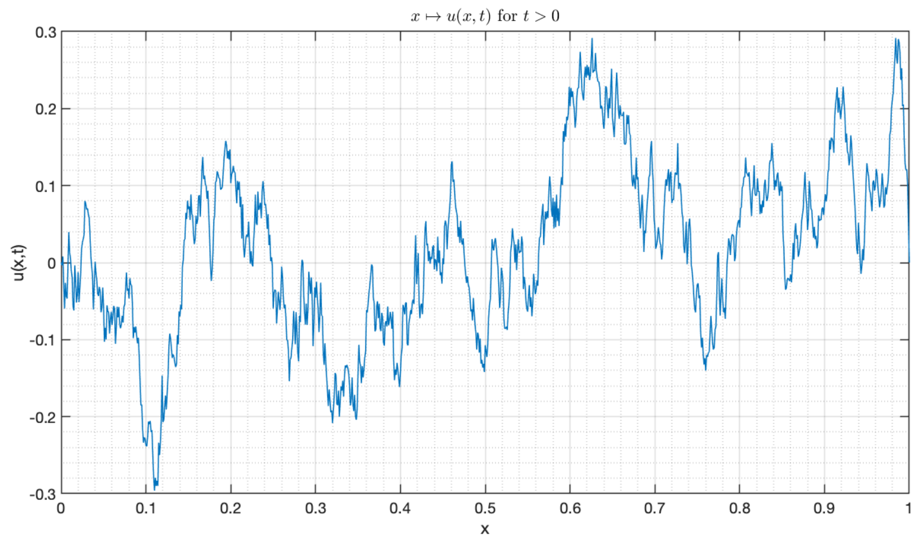


Figure 1.7: Simulation of the random field solution to the stochastic heat equation 1.8 as a function of space.

Of course it doesn't tell us that much to just look at the paths in figures 1.5, 1.6 and 1.7 on their own. For a function f (which can be stochastic) we calculate the *exact* q-variation $V^q_{[a,b]}[f]$ which exists if the limit

$$V^q_{[a,b]}[f] = \lim_{n \to \infty} \sum_{i=1}^{n-1} (f(x_{i+1}) - f(x_i))^q$$

converges in probability for x_i in a uniform partition of the interval $[a, b]$. The function $t \mapsto u(t, x)$ has non-trivial (not equal to zero or ∞) exact 4-variation, which is directly linked to the almost 1/4 Hölder continuity. And $x \mapsto u(t, x)$ has non-trivial 2-variation, which again is thanks to the almost 1/2-Hölder continuity. The variations will depend on the parameters defining the SPDE, especially the drift α and diffusion σ. From these we can construct statistical estimators of the drift and diffusion parameters from observing the field solution by measuring the variations in discrete sampled points.

The main goal of this thesis is to consider the exact q-variation of the true solutions as well as the simulated solutions to the stochastic heat equation.

In Chapter 2 we will give the necessary theory and results needed to study the properties of the solution u. Chapter 3 will show the theory behind the distributions of the solution, how this is connected to the variations, stating and proving the values of the exact q-variations of the solutions, how we can construct estimators of the drift- and diffusion parameters, as well as stating their statistical properties. Chapter 4 will contain the methods to simulate the stochastic processes that make up the solution, like we saw in figures 1.4, 1.5, 1.6, and 1.7. An interesting result here is that many standard ways of simulating solutions to PDEs work for the SPDEs we are considering, the approximations will converge, in the sense that the error goes to zero. But the exact q-variations do *not* converge for general approximation schemes. We finish the thesis with Chapter 5 where we show how to construct the white-coloured noise and its respective stochastic heat equation with solution.

Chapter 2

SPDEs with Gaussian White Noise

This chapter aims to give the central definitions and properties needed to study the stochastic heat equation and its exact q-variations. We will start with some basics on stochastic processes and Gaussian random fields. The presentation is heavily inspired by [Dalang et al., 2009, Chapter 1]. We will be working on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $L^2(\Omega)$ be the set of real-valued random variables with finite second moment, i.e. $\mathbb{E}[X^2] < \infty$. $L^2(\Omega)$ is a Hilbert space with norm $\|X\|_{L^2(\Omega)} := \sqrt{\mathbb{E}[X^2]}$ for $X \in L^2(\Omega)$.

2.1 Gaussian processes

Definition 2.1.1 (Gaussian random vector)**.** *Let $g = (g_1, \ldots, g_d)$ be a random vector. We say that the distribution of g is Gaussian if $\upsilon \cdot g := \sum_{j=1}^{d} \upsilon_j g_j$ is Gaussian random variable for all $\upsilon \in \mathbb{R}^d$.*

Recall that for a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with some arbitrary index set $\mathcal{T}$, a *stochastic process* is given by the real-valued stochastic variables $X(t) := \{X(\omega, t)\ ,\ t \in \mathcal{T}\}$, where $\omega \in \Omega$ is any basic event for all t.

Definition 2.1.2 (Gaussian random field)**.** *The stochastic process $X = \{X(\omega, t)\ ,\ t \in \mathcal{T}\}$ is called a* Gaussian random field *or* Gaussian stochastic process *if for all integers $k \geq 1$ and $t_1, \ldots, t_k \in \mathcal{T}$, the random vector $(X(t_1), X(t_2), \ldots, X(t_k))$ is Gaussian.*

The finite dimensional distributions are the collection of probabilities obtained as follows,

$$p_{t_1,\dots,t_k}(A_1,\dots,A_k) := \mathbb{P}(X(t_1)\in A_1,\dots,X(t_k)\in A_k) \tag{2.1}$$

Definition 2.1.3 (Non-negative Definite Kernel)**.** *Let $\mathcal{T}$ be an arbitrary index set. A symmetric function $K:\mathcal{T}\times\mathcal{T}\to\mathbb{C}$ is called a non-negative definite kernel on the set $\mathcal{T}$ if given $n\in\mathbb{N}$ it holds that*

$$\sum_{k=1}^{n}\sum_{l=1}^{n} c_k K(t_k,t_l)\bar{c}_l \geq 0, \tag{2.2}$$

for each $c_1,\dots c_n\in\mathbb{C}$ and $t_1,\dots,t_n\in\mathcal{T}$.

Remark 2.1.4. *The above definition is equivalent to that every matrix $\mathbf{K}=(K(t_k,t_l))$ created for every $t_1,\dots,t_n\in\mathcal{T}$ and $n\in\mathbb{N}$ is a* non-negative definite matrix*, or equivalently that the eigenvalues to $\mathbf{K}$ are greater than or equal to zero.*

Remark 2.1.5 (Note on naming convention)**.** *If the index set $\mathcal{T}=\mathbb{R}^d$ then the non-negative definite kernel is called a non-negative definite* function*. If we also require that 2.2 is equal to zero if and only if all $c_k=0$ then we call the function/kernel* positive definite*.*

The idea of non-negative definite kernels is especially important for covariance functions. For $s,t\in\mathcal{T}$, the covariance function $C(s,t) := \mathbb{E}(X(s)X(t)) - \mathbb{E}(X(s))\mathbb{E}(X(t))$ is a symmetric ($C(s,t)=C(t,s)$)- and non-negative definite kernel. There is a well known result thanks to Kolmogorov, that guarantees the existence of a Gaussian random field with a given covariance- and mean function.

Lemma 2.1.6. *(1) Let X be a Gaussian random field. The probability measures $p_{t_1,\dots,t_k}$ defined in 2.1 are determined by the mean function $m(t)=\mathbb{E}(X(t))$ and the covariance function $C(s,t)$*
(2) Given functions $m:\mathcal{T}\to\mathbb{R}$ and a symmetric non-negative kernel $C:\mathcal{T}\times\mathcal{T}\to\mathbb{R}$, then there exists a Gaussian random field $X(t)$ with mean function m and covariance function C.

By the above lemma, we have that two Gaussian processes are equal in distribution, if their mean- and covariance functions agree.

Our goal is to represent stochastic integrals with respect to Gaussian random fields. We want to integrate functions h from a separable Hilbert space $\mathscr{H}$ with inner product $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ and induced norm $\|\cdot\|_{\mathscr{H}}$.

Definition 2.1.7. *Let $(\mathscr{H},\langle\cdot,\cdot\rangle_{\mathscr{H}},\|\cdot\|_{\mathscr{H}})$ be a separable inner product space. A stochastic process $I(h)$ indexed by $\mathcal{T}=\mathscr{H}$ on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ is called an* isonormal *Gaussian process if for all $h,g\in\mathscr{H}$, then $I(h)\in N(0,\|h\|^2_{\mathscr{H}})$, and $\mathbb{E}\Big(I(h)I(g)\Big)=\langle h,g\rangle_{\mathscr{H}}$.*

We will see that the isonormal Gaussian processes $I(h)$ indexed by separable Hilbert spaces are a natural interpretation of the stochastic integral of deterministic processes with respect to some Gaussian noise process. Symbolically we want $I(h) = \int_E h dW$ where E is some measure space with measure μ and we have a white noise W that is based on μ.

Proposition 2.1.8. *If $I(h)$ is an isonormal Gaussian process, then the map $I : \mathscr{H} \to L^2(\Omega)$ such that $h \mapsto I(h)$, is a linear isometry.*

Proof. By definition 2.1.7, $\mathbb{V}(I(h)) = \|I(h)\|^2_{L^2(\Omega)} = \|h\|^2_{\mathscr{H}}$. Now we show that it is linear in the sense that

$$I(ah + bg) = aI(h) + bI(g) \text{ a.s}$$

for $a, b \in \mathbb{R}$ and $h, g \in \mathscr{H}$. Both left- and right-hand side have zero mean. It suffices to show that the difference $I(ah + bg) - (aI(h) + bI(g))$ has zero variance.

$$\mathbb{E}\left((I(ah + bg) - aI(h) - bI(g))^2\right) = \|ah + bg\|^2_{\mathscr{H}} + a^2\|h\|^2_{\mathscr{H}} + b^2\|g\|^2_{\mathscr{H}}$$
$$-2a\langle ah + bg, h\rangle_{\mathscr{H}} - 2b\langle ah + bg, g\rangle_{\mathscr{H}} + 2ab\langle h, g\rangle_{\mathscr{H}} = 0$$

□

Proposition 2.1.8 guarantees that an isonormal Gaussian process is indeed a Gaussian random field, since the linear combination $aI(h) + bI(g) \in N(0, \|ag + bh\|^2_{\mathscr{H}})$. This proposition will later also give us the natural property that stochastic integrals are almost surely linear.

2.2 Stochastic calculus with white noise

We first make the definition of stochastic integrals with respect to a **white** noise W based on a measure.

2.2.1 White noise

For our presentation we will use a σ-finite measurable subspace $(E, \mathcal{B}(E), \mu)$ of $\mathbb{R}^d$. Let $\mathcal{B}_b(E)$ be the collection of Borel-measurable subsets of E with finite measure.

Definition 2.2.1 (White Noise)**.** *A* white noise *based on a positive measure μ is a Gaussian random field $W = \{W(A),\ A \in \mathcal{B}_b(E)\}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}\left[W(A)\right] = 0$ and covariance function*

$$C(A, B) = \mathbb{E}\left[W(A)W(B)\right] = \mu(A \cap B).$$

The function C above is indeed a covariance function. It is obviously symmetric $C(A,B) = C(B,A)$. To see that it is non-negative definite, take $x_1, \ldots, x_n \in \mathbb{R}$ and $A_1, \ldots, A_n \in \mathcal{B}_b(E)$. Then

$$\sum_{k,l=1}^{n} x_k x_l C(A_k, A_l) = \sum_{k,l=1}^{n} x_k x_l \mu(A_k \cap A_l)$$
$$= \sum_{k,l=1}^{n} x_k x_l \left(\int_E \mathbb{1}_{A_k}(y) \mathbb{1}_{A_l}(y) \mu(dy) \right) = \int_E \left(\sum_{k=1}^{n} x_k \mathbb{1}_{A_k}(y) \right)^2 \mu(dy) \geq 0.$$

Thus the existence of the Gaussian random field $W(A)$ with index set $\mathcal{T} = \mathcal{B}_b(E)$ follows from Lemma 2.1.6.

Remark 2.2.2. *An important note is that the "white" in "white noise" refers to the covariance structure of the Gaussian random field W. Notice how*

$$C(A,B) = \mu(A \cap B) = \int_E \mathbb{1}_A(y) \mathbb{1}_B(y) \mu(dy) = \langle \mathbb{1}_A, \mathbb{1}_B \rangle_{L^2(E)}.$$

Therefore there exists a natural harmony between the finite second moment space $L^2(\Omega)$ and the Hilbert space $L^2(E)$. This will conclude with the fact that the only functions we will be able to integrate are precicely those which lie in $L^2(E)$.

Proposition 2.2.3. *Let $A, B \in \mathcal{B}_b(E)$ be two disjoint subsets of E. Then $W(A)$ and $W(B)$ are independent and*

$$W(A \cup B) = W(A) + W(B) \qquad a.s.$$

Proof. Since A and B are disjoint, $C(A,B) = \mu(A \cap B) = \mu(\emptyset) = 0$. Therefore they are uncorrelated, and hence independent, since $W(A)$, and $W(B)$ are Gaussian.

To show additivity we check that $W(A \cup B) - (W(A) + W(B))$ has zero variance. By linearity of expectation and the definition of covariance for white noise,

$$\mathbb{E}\left[(W(A \cup B) - W(A) - W(B))^2\right]$$
$$= \mathbb{E}\left[(W(A \cup B)^2\right] + \mathbb{E}\left[W(A)^2\right] + \mathbb{E}\left[W(B)^2\right] - 2\mathbb{E}\left[\left(W(A \cup B) W(A)\right)^2\right]$$
$$-2\mathbb{E}\left[\left(W(A \cup B) W(B)\right)^2\right] + \mathbb{E}\left[W(A) W(B)\right]$$
$$= \mu(A \cup B) + \mu(A) + \mu(B) - 2\mu(A) - 2\mu(B) + 0 = 0.$$

□

Definition 2.2.4 (The Brownian Sheet)**.** *Let $t = (t_1, \dots, t_n)$. The Brownian sheet $\{W(t), t \in \mathbb{R}^d_+\}$ defined by $W_t := W\{(0,t]\} := W\{(0,t_1] \times \ldots \times (0,t_n]\}$. This is a zero mean, Gaussian process with covariance function $E\{W_s W_t\} = \min(s_1,t_2)\cdot \ldots \cdot \min(s_n,t_n)$.*

For example if $d = 2$, the Brownian sheet over rectangles $R = (s,t] \times [x,y]$ (by Proposition 2.2.3) is equal to $W(R) = W_{ty} - W_{tx} - W_{sy} - W_{sx}$.

2.2.2 The stochastic integral

We take a general Lebesgue/Measure theory approach to the construction of a stochastic integral. Starting with integrals of simple functions, and extending with a density argument because of the linear isometry that will be created. Let our integrands come from the Hilbert space $\mathscr{H} = L^2(E,\mu)$. We can now construct an isonormal Gaussian process $I(h)$ on $\mathscr{H}$, given a white noise W based on μ.

Definition 2.2.5. *For simple functions $h = \sum_{k=1}^n a_k \mathbb{1}_{A_k} \in L^2(E,\mu)$ with $a_k \in \mathbb{R}$, and $A_k \in \mathcal{B}_b(E)$ pairwise disjoint. Then*

$$I(h) = I\left(\sum_{k=1}^n a_k \mathbb{1}_{A_k}\right) := \sum_{k=1}^n a_k W(A_k).$$

Proposition 2.2.6. *The process $I(h)$ from the set of simple functions on $L^2(E)$ to random variables in $L^2(\Omega)$ is an isonormal Gaussian process.*

Proof. $I(h)$ is a finite sum of zero-mean normal variables, and hence it is also normal with zero mean. To see that $\mathbb{E}(I(h)I(g)) = \langle I(h), I(g)\rangle_{L^2(\Omega)} = \langle h, g\rangle_{L^2(E)}$, we first observe that I is an isometry,

$$\begin{aligned}
\|I(h)\|^2_{L^2(\Omega)} &= \left\| I\left(\sum_{k=1}^n a_k \mathbb{1}_{A_k}\right)\right\|^2_{L^2(\Omega)} = \mathbb{E}\left(\Big(\sum_{k=1}^n a_k W(A_k)\Big)^2\right) \\
&= \sum_{k=1}^n a_k^2 \mathbb{E}\big(W(A_k)^2\big) = \sum_{k=1}^n a_k^2 \mu(A_k) = \sum_{k=1}^n \int_E a_k^2 \mathbb{1}_{A_k}(y) d\mu(y) \\
&= \int_E \sum_{k=1}^n \Big(a_k^2 \mathbb{1}_{A_k}(y)\Big) d\mu(y) = \int_E \left(\sum_{k=1}^n a_k^2 \mathbb{1}_{A_k}(y)\right)^2 d\mu(y) \\
&= \int_E h^2 d\mu = \|h\|^2_{L^2(E)}.
\end{aligned}$$

Since the map $h \mapsto I(h)$ is an isometry between two inner products spaces, we know that the inner products are preserved in the mapping. Therefore $\mathbb{E}\big(I(h)I(g)\big) = \big\langle I(h), I(g)\big\rangle_{L^2(\Omega)} = \langle h, g\rangle_{L^2(E)}$. □

By the preceeding proposition as well as Proposition 2.1.8 we obtain the following important corollary.

Corollary 2.2.7. *The isonormal Gaussian process $I(h)$ is a linear isometry from the set of simple functions on $L^2(E)$ to $L^2(\Omega)$.*

Remark 2.2.8. *The definition in 2.2.5 is well defined in the sense that if we have another representation of $\tilde{h} = \sum_{l=1}^m b_l \mathbb{1}_{B_l}$ with $b_l \in \mathbb{R}$ and pairwise disjoint $B_l \in \mathcal{B}_b(E)$, such that $h = \tilde{h}$, then $I(h) = I(\tilde{h})$ a.s.*

Proof. (of remark 2.2.8) We show that the difference $I(h) - I(\tilde{h})$ has zero variance.

$$\begin{aligned}
\mathbb{E}\left(\Big(\sum_{k=1}^n a_k W(A_k) - \sum_{l=1}^m b_k W(A_l)\Big)^2\right) &= \mathbb{E}\left(\Big(\sum_{k=1}^n a_k W(A_k)\Big)^2\right) \\
+\mathbb{E}\left(\Big(\sum_{l=1}^m b_l W(B_l)\Big)^2\right) &- 2\mathbb{E}\left(\Big(\sum_{k=1}^n\sum_{l=1}^m a_k b_l W(A_k) W(B_l)\Big)^2\right) \\
= \int_E \left(\sum_{k=1}^n a_k^2 \mathbb{1}_{A_k} + \sum_{l=1}^m b_l^2 \mathbb{1}_{B_l} - 2\sum_{k=1}^n\sum_{l=1}^m a_k b_l \mathbb{1}_{A_k\cap B_l}\right) d\mu \\
= \int_E \left(\sum_{k=1}^n a_k \mathbb{1}_{A_k} - \sum_{l=1}^m b_l \mathbb{1}_{B_l}\right)^2 d\mu = \int_E \left(h - \tilde{h}\right)^2 d\mu = 0.
\end{aligned}$$

□

Since we have a linear isometry from the (dense) set of simple functions on $L^2(E)$ to the complete normed space $L^2(\Omega)$, the map $h \mapsto I(h)$ can be extended uniquely to $L^2(E)$. Take $h \in L^2(E)$ and a sequence of simple functions h_n such that $\|h - h_n\|_{L^2(E)} \to 0$. Then we define

$$\int_E h dW := I(h) := \lim_{n\to\infty} I(h_n).$$

The above definition does not depend on the sequence of simple functions approximating h. To see this, let $h_n \to h$ and $g_n \to h$ be two sequences of simple functions that converge to h. Then by the linear isometry (Corollary 2.2.7), we have

$$\|I(h_n) - I(g_n)\|^2_{L^2(\Omega)} = \|I(h_n - g_n)\|^2_{L^2(\Omega)} = \|h_n - g_n\|^2_{L^2(E)} \to 0.$$

That $I(h)$ for $h \in L^2(E)$ is an isonormal Gaussian process follows directly from the isometry $\|I(h)\|_{L^2(\Omega)} = \|h\|_{L^2(E)}$ for $h \in L^2(E)$ in the same way that we proved it in Proposition 2.2.6.

Remark 2.2.9. *It is vital to note that the stochastic integral $I(h)$ is well defined if and only if $h \in L^2(E)$.*

We will use both $\int_E h dW$, and $\int_E h(x)W(dx)$, to mean the stochastic integral $I(h)$ with respect to space-time white noise W, of the $L^2(E)$ function h, such that $x \mapsto h(x)$. We obtain the following important formula, called *Wiener's isometry*, that shows symbolically how the inner product is preserved,

Theorem 2.2.10 (Wiener's isometry)**.** *For any $h, g \in L^2(E)$,*

$$\mathbb{E}\left(\int_E h(x)W(dx) \int_E g(x)W(dx) \right) = \int_E h(x)g(x)dx.$$

The following corollary can be useful.

Corollary 2.2.11. *Assume $A_1, \ldots, A_n$ are disjoint sets in $\mathcal{B}_b(E)$, then for $f \in L^2(E)$*

$$\mathbb{E}\left(\left[\sum_{i=1}^{n} \int_{A_i} f(x)W(dx) \right]^2 \right) = \sum_{i=1}^{n} \int_{A_i} f(x)^2 dx.$$

Proof. Expand the sum and note that the cross-term multiplications are independent Gaussian variables, then use Wiener's isometry on the square terms. □

Corollary 2.2.7 gives us the almost sure linearity of the stochastic integrals,

Proposition 2.2.12. *The stochastic integral is almost surely linear,*

$$\int_E h + g dW = \int_E h dW + \int_E g dW \quad a.s.$$

2.2.2.1 The case of white noise on $\mathbb{R}_+ \times \mathscr{D}$

In the coming presentation, $E = \mathbb{R}_+ \times \mathscr{D}$, where $\mathscr{D} \subseteq \mathbb{R}^d$ for some integer $d \geq 1$, and for $g = h(s,y)\mathbb{1}_{(0,t)\times\mathscr{D}}(s,y)$ where $h \in L^2((0,t) \times \mathscr{D})$, we will write

$$\int_E g dW = \int_{\mathbb{R}_+ \times \mathscr{D}} g dW =: \int_0^t \int_{\mathscr{D}} h(s,y)W(dyds).$$

The following identity is often useful.

Proposition 2.2.13. *If* $0 < t_1 < t$ *then*

$$\int_0^t \int_{\mathscr{D}} h(s,y) W(dyds) = \int_0^{t_1} \int_{\mathscr{D}} h(s,y) W(dyds) + \int_{t_1}^t \int_{\mathscr{D}} h(s,y) W(dyds) \ \text{a.s}$$

Proof. Let $h(s,y) = \mathbb{1}_{[0,t_1]}(s)h(s,y) + \mathbb{1}_{(t_1,t]}(s)h(s,y)$ and use the almost sure linearity of the stochastic integral. □

2.2.3 Solution to an SPDE driven by white noise

We are now ready to give a definition of a solution to a simplified SPDE of the form presented in the introduction, equation 1.3 with a driving noise $\dot{W}$ represented by the white noise W based on the Lebesgue measure λ. We again consider the space $\mathbb{R}_+ \times \mathscr{D}$ where $\mathscr{D} \subseteq \mathbb{R}^d$. A linear SPDE with additive white driving noise $\dot{W}$ is the equation,

$$\mathcal{L}u(t,x) = \sigma \dot{W}(t,x), \ \ t > 0, \ x \in \mathscr{D}, \tag{2.3}$$

with deterministic initial conditions and if necessary also boundary conditions. The operator $\mathcal{L} = \frac{\partial}{\partial t} - A$ is a linear partial differential operator, where A is a second order differential operator in space with the property of uniform ellipticity, see e.g. [Polyanin and Nazaikinskii, 2016, Section 16.2.2]. The diffusion $\sigma > 0$ is a constant that scales the white noise.

Definition 2.2.14 (Solution to 2.3)**.** *Suppose that there exists a fundamental/Green solution* $\Psi = \Psi(s,y;t,x)$ *to* $\mathcal{L}$ *such that as a function of* s *and* y, $\Psi \in L^2((0,t) \times \mathscr{D})$. *Then the SPDE 2.3 has a solution and it is the random field given by*

$$u(t,x) = I_0(t,x) + \sigma \int_0^t \int_{\mathscr{D}} \Psi(s,y;t,x) W(dyds), \ \ t > 0, x \in \mathscr{D}.$$

I_0 *is the solution to the homogeneous problem* $\mathcal{L}u = 0$ *with the same initial- and boundary conditions as 2.3.*

2.3 The Stochastic Heat Equation

We consider the stochastic heat equation on $\mathscr{D} = \mathbb{R}^d$ for some integer $d \geq 1$, or $\mathscr{D} = (0,L)$ where $L \in \mathbb{R}$, driven by a white noise process W based on the Lebesgue-measure λ on $\mathbb{R}_+ \times \mathscr{D}$. Formally the equation is,

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \Delta u(t,x) = \dot{W}(t,x) & t > 0, \ x \in \mathscr{D} \\ u(0,x) = 0 & x \in \mathscr{D} \\ + \text{ Homogeneous Dirichlet BC if } \mathscr{D} = (0,L) & x \in \partial\mathscr{D}. \end{cases} \tag{2.4}$$

With $\Delta := \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$. We note that for the rest of this chapter we will have the drift term $\alpha = 1$ and diffusion $\sigma = 1$.

Let $\Psi(s, y; t, x) = \Psi(t - s, x, y)$ represent the fundamental and Green solution respectively to 2.4 above. We recall the semi-group property of Ψ stated in chapter 1 propositions 1.3.4 and 1.3.6.

Proposition 2.3.1. *For $x, z \in \mathscr{D}$ and $s, t > 0$*

$$\Psi(s + t, x, z) = \int_{\mathscr{D}} \Psi(t, x, y)\Psi(s, y, z)dy.$$

And we have the symmetry $\Psi(t, x, z) = \Psi(t, z, x)$.

By definition 2.2.14, the solution to 2.4 is given as

$$u(t, x) = \int_0^t \int_{\mathscr{D}} \Psi(t - s, x, y)\, W(dyds). \tag{2.5}$$

The above solution is defined for a given $t > 0$ and $x \in \mathscr{D} \subseteq \mathbb{R}^d$ as long as the function $(s, y) \mapsto \Psi(t - s, x, y)$ lies in $L^2((0, t) \times \mathscr{D})$, which is by definition equivalent to the condition that

$$\|\Psi\|^2_{L^2((0,t)\times\mathscr{D})} < \infty.$$

This holds for $\mathscr{D} = (0, L)$ with $\Psi = G$: indeed by the semi-group property 2.3.1 of the Green function $(s, y) \mapsto G(t - s, x, y)$ and that $G(t, x, y) = G(t, y, x)$, then $\int_0^L G(t - s, x, y)^2 dy = G(2t - 2s, x, x)$. By inequality 1.10 we have that

$$G(2t - 2s, x, x) \leq \frac{1}{\sqrt{4\pi(2t - 2s)}}.$$

And consequently

$$\|G\|^2_{L^2((0,t)\times(0,L))} = \int_0^t G(2t - 2s, x, x)ds$$
$$\leq \int_0^t \frac{1}{\sqrt{4\pi(2t - 2s)}}ds < \infty.$$

On $\mathscr{D} = \mathbb{R}^d$ we have the fundamental solution

$$\Psi(t - s, x, y) = \Phi(t - s, x - y) = \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(4\pi|t - s|)^{d/2}}.$$

By the semi-group property 2.3.1

$$\|\Phi\|^2_{L^2((0,t)\times\mathbb{R}^d)} = \int_0^t \int_{\mathbb{R}^d} \Phi(t-s, x-y)^2 dyds$$
$$= \int_0^t \Phi(2t-2s, 0)ds = \int_0^t \frac{1}{(8\pi|t-s|)^{d/2}} ds = \int_0^t \frac{1}{(8\pi s)^{d/2}} ds.$$

The integral $\int_0^t \frac{1}{(4\pi s)^{d/2}} ds < \infty$ if and only if $d/2 < 1$. Therefore the solution given in 2.5 to equation 2.4 on $\mathscr{D} = \mathbb{R}^d$ exists if and only if $d = 1$. We proceed with the solution for $d = 1$.

It will useful to calculate the integral $\int \frac{e^{\frac{-|x|^2}{ct}}}{t^{1/2}} dt$ for some $c > 0$ when we work with the solution on the entire real line. We have,

$$\int \frac{e^{\frac{-|x|^2}{ct}}}{t^{1/2}} dt = 2\left(\sqrt{t}e^{\frac{-|x|^2}{ct}} - x^2 \int \frac{e^{\frac{-|x|^2}{ct}}}{ct^{3/2}} dt\right).$$

By the variable substitution $s = x/\sqrt{ct}$.

$$\int \frac{e^{\frac{-|x|^2}{ct}}}{ct^{3/2}} dt = -\frac{1}{x\sqrt{c}} \int 2e^{-s^2} ds = -\frac{\sqrt{\pi}}{x\sqrt{c}} \int \frac{2}{\sqrt{\pi}} e^{-s^2} ds$$
$$= -\frac{\sqrt{\pi}}{x\sqrt{c}} \operatorname{erf}(s) = -\frac{\sqrt{\pi}}{x\sqrt{c}} \operatorname{erf}\left(\frac{x}{\sqrt{ct}}\right).$$

Hence

$$\int \frac{e^{\frac{-|x|^2}{ct}}}{t^{1/2}} dt = 2\left(\sqrt{t}e^{\frac{-|x|^2}{ct}} + \frac{\sqrt{\pi}x}{\sqrt{c}} \operatorname{erf}\left(\frac{x}{\sqrt{ct}}\right)\right).$$

We therefore have the following.

Proposition 2.3.2. *For the fundamental solution* $\Phi(t,z) = \frac{e^{-\frac{|z|^2}{4\alpha t}}}{(4\pi\alpha t)^{1/2}}$,

$$\int \Phi(t,z)dt = \frac{1}{\sqrt{\pi\alpha}}\left(\sqrt{t}e^{-\frac{|z|^2}{4\alpha t}} + \frac{z}{2}\sqrt{\frac{\pi}{\alpha}} \operatorname{erf}\left(\frac{z}{2\sqrt{\alpha t}}\right)\right). \tag{2.6}$$

2.3.1 Covariance structure for the solution

By construction, the stochastic integrals $I(h)$ given by the isonormal Gaussian process $h \mapsto I(h) = \int_0^t \int_{\mathscr{D}} h(s,y) W(dyds)$ for $h \in L^2(\mathbb{R}_+ \times \mathscr{D})$ are Gaussian. Therefore the solution $u(t,x)$ at a point $(t,x) \in \mathbb{R}_+ \times \mathscr{D}$, which itself is a stochastic process indexed on $\mathbb{R}_+ \times \mathscr{D}$, is a Gaussian variable, with $\mathbb{E}(u(t,x)) = 0$ and $\mathbb{V}(u(t,x)) = \mathbb{E}(u(t,x)^2) = \|\Psi\|^2_{L^2((0,t)\times\mathscr{D})}$. For two points $(t_1,x_1),(t_2,x_2) \in \mathbb{R}_+ \times \mathscr{D}$ we can explicitly calculate (at least for $\mathscr{D} = \mathbb{R}$) the covariance function $\mathbb{E}\left[u(t_1,x_1)u(t_2,x_2)\right]$. Once again recall that the solution is

$$u(t,x) = \int_0^t \int_{\mathscr{D}} \Psi(t-s,x,y)\, W(dyds).$$

Assuming $t_1 \le t_2$, the covariance can be written as

$$\begin{aligned}
&\mathbb{E}\left[u(t_1,x_1)u(t_2,x_2)\right] \\
=&\mathbb{E}\left[\int_0^{t_1}\int_{\mathscr{D}} \Psi(t_1-s,x_1,y)\, W(dyds) \int_0^{t_2}\int_{\mathscr{D}} \Psi(t_2-s,x_2,y)\, W(dyds)\right] \\
&\mathbb{E}\left[\int_0^{t_2}\int_{\mathscr{D}} \Psi(t_1-s,x_1,y)\mathbb{1}(s\le t_1)\, W(dyds) \int_0^{t_2}\int_{\mathscr{D}} \Psi(t_2-s,x_2,y)\, W(dyds)\right].
\end{aligned}$$

By Wiener's isometry, the above expected value becomes the below Lebesgue integral

$$\begin{aligned}
&\int_0^{t_2}\int_{\mathscr{D}} \Psi(t_1-s,x_1,y)\Psi(t_2-s,x_2,y)\mathbb{1}(s\le t_1)dyds \\
=&\int_0^{t_1}\int_{\mathscr{D}} \Psi(t_1-s,x_1,y)\Psi(t_2-s,x_2,y)dyds.
\end{aligned}$$

If we instead assumed $t_2 < t_1$, we would integrate to t_2. This gives us that the covariance can be simplified to

$$\mathbb{E}\left[u(t_1,x_1)u(t_2,x_2)\right] = \int_0^{t_1\wedge t_2}\int_{\mathscr{D}} \Psi(t_1-s,x_1,y)\Psi(t_2-s,x_2,y)dyds. \tag{2.7}$$

Looking at $\int_{\mathscr{D}} \Psi(t_1-s,x_1,y)\Psi(t_2-s,x_2,y)dy$ we can use the semi-group property 2.3.1 to obtain

$$\begin{aligned}
&\int_0^{t_1\wedge t_2}\int_{\mathscr{D}} \Psi(t_1-s,x_1,y)\Psi(t_2-s,x_2,y)dyds \\
=&\int_0^{t_1\wedge t_2} \Psi(t_1+t_2-2s,x_1,x_2)ds.
\end{aligned}$$

For $\mathscr{D} = \mathbb{R}$, we have $\Psi(t_1+t_2-2s,x_1,x_2) = \Phi(t_1+t_2-2s,x_1-x_2) = \frac{e^{-\frac{|x_1-x_2|^2}{4\alpha u}}}{(4\pi\alpha(t_1+t_2-2s))^{1/2}}$. We make the substitution $\tau = t_1+t_2-2s$ and utilise Proposition 2.3.2:

$$\begin{aligned}
&\int_0^{t_1\wedge t_2} \Phi(t_1+t_2-2s,x_1-x_2)ds = \frac{1}{2}\int_{|t_1-t_2|}^{t_1+t_2} \Phi(\tau,x_1-x_2)d\tau \\
=&\frac{1}{2\sqrt{\pi}}\left[\sqrt{\tau}e^{-\frac{|x_1-x_2|^2}{4\tau}} + \frac{\sqrt{\pi}(x_1-x_2)}{2}\operatorname{erf}\left(\frac{x_1-x_2}{2\sqrt{\tau}}\right)\right]_{|t_1-t_2|}^{t_1+t_2}.
\end{aligned}$$

And there we have

$$= \mathbb{E}\left[u(t_1,x_1)u(t_2,x_2)\right] = \frac{1}{2\sqrt{\pi}}\left(\sqrt{t_1+t_2}e^{-\frac{|x_1-x_2|^2}{4(t_1+t_2)}} - \sqrt{|t_1-t_2|}e^{-\frac{|x_1-x_2|^2}{4|t_1-t_2|}}\right)$$
$$+ \frac{(x_1-x_2)}{4}\left(\operatorname{erf}\left(\frac{x_1-x_2}{2\sqrt{t_1+t_2}}\right) - \operatorname{erf}\left(\frac{x_1-x_2}{2\sqrt{|t_1-t_2|}}\right)\right).$$

2.3.2 Sample path regularity

We gave the definition of local Hölder-continuity in the introduction, we restate it here.

Definition 2.3.3 (Local Hölder continuity)**.** *A function $f : D \to \mathbb{R}$, where D some normed space is locally $\gamma-$Hölder continuous with Hölder exponent $\gamma \geq 0$ if there exists $C > 0$ such that*

$$|f(x) - f(y)| \leq C\|x-y\|^\gamma$$

for x, $y \in K$ for all compact sets $K \subseteq D$.

Definition 2.3.4. *Let $X : \Omega \times \mathcal{T} \to \mathbb{R}$ be a stochastic process on the index set $\mathcal{T}$. A stochastic process $\tilde{X}$ is called a* modification *of X if for all $t \in \mathcal{T}$ it holds that $\mathbb{P}\left(X(t) = \tilde{X}(t)\right) = 1$.*

Two stochastic processes that are modifications of each other have the same finite dimensional law.

The solution to the stochastic heat $u(t,x) = u(\omega,t,x)$ is a stochastic process. Our goal for this section is to show that there exists modifications of the processes $x \mapsto u(t,x)$ and $t \mapsto u(t,x)$ that are Hölder continuous. This was shown for $\mathscr{D} = (0,L)$ in Walsh's lecture notes in [Walsh, 1986, Chapter 3.]. We give a proof of Hölder-continuity in the case of $\mathscr{D} = \mathbb{R}$. We will need Kolmogorov's continuity theorem (statement and proof can be found in e.g [Bell, 2015]).

Theorem 2.3.5. *Let $\{X(t) : t \in \mathcal{T}\}$ be an $\mathbb{R}$-valued stochastic process on the 1-dimensional normed index set $\mathcal{T}$. If there exists constants α, $\beta, C > 0$ such that for all $s,t \in \mathcal{T}$*

$$\mathbb{E}\left[|X(t) - X(s)|^\alpha\right] \leq C|t-s|^{1+\beta}.$$

Then there exists a modification of $X(t)$ which is a.s. continuous and even further a.s locally Hölder γ-continuous for every $\gamma \in (0, \frac{\beta}{\alpha})$.

If a function is γ-Hölder continuous for every $\gamma \in (0, \frac{\beta}{\alpha})$, we will often call it *almost* $\frac{\beta}{\alpha}$-Hölder continuous.

Theorem 2.3.6. *The solution $(t,x) \mapsto u(t,x)$ to the stochastic heat equation 2.4 has a modification which is locally Hölder continuous function with exponent almost $\frac{1}{4}$ in time and almost $\frac{1}{2}$ in space.*

Lemma 2.3.7. *For $Z \sim N(0,\sigma^2)$ we have*

$$\mathbb{E}\left[|Z|^\alpha\right] = \frac{2^{\alpha/2}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right)\mathbb{E}\left[Z^2\right]^{\alpha/2}.$$

Proof. (Of Theorem 2.3.6)

In the proceeding calculations, the value of the constant C may change from line to line. Let $\alpha > 1$. We will look at the Hölder continuity in space and time separately, i.e. look at,

$$\mathbb{E}\left[|u(t,x+h)-u(t,x)|^\alpha\right] \quad \text{and,} \quad \mathbb{E}\left[|u(t+k,x)-u(t,x)|^\alpha\right]. \tag{2.8}$$

We will estimate these two terms separately starting with the first expression, fix $t>0$,

$$\begin{aligned}&\mathbb{E}\left[|u(t,x+h)-u(t,x)|^\alpha\right]\\ &=\frac{2^{\alpha/2}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right)\mathbb{E}\left[\Big(u(t,x+h)-u(t,x)\Big)^2\right]^{\alpha/2}\\ &=\frac{2^{\alpha/2}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right)\left(\int_0^t\int_{\mathbb{R}}(\Phi(t-s,x+h-y)-\Phi(t-s,x-y))^2dyds\right)^{\alpha/2}.\end{aligned}$$

Which is motivated by the fact that $\mathbb{E}\left[|u(t,x+h)-u(t,x)|^\alpha\right]$ is simply is the α:th moment of a zero mean Gaussian random variable, so the constant comes from the calculation of $\mathbb{E}\left[|Z|^\alpha\right]$ where $Z \sim N(0,\sigma^2)$ from Lemma 2.3.7. Looking at the integral $\int_{\mathbb{R}}(\Phi(t-s,x+h-y)-\Phi(t-s,x-y))^2dy$, we have that the integrand is equal to

$$\begin{aligned}&\Big(\Phi(t-s,x+h-y)-\Phi(t-s,x-y)\Big)^2\\ =&\Phi^2(t-s,x+h-y)-2\Phi(t-s,x+h-y)\Phi(t-s,x-y)+\Phi^2(t-s,x-y).\end{aligned}$$

Since we are integrating with respect to y over $\mathbb{R}^d$, we can use the semigroup property in Proposition 2.3.1. Let $C = \frac{2^{\alpha/2}}{\sqrt{\pi}}\Gamma\left(\frac{\alpha+1}{2}\right)$ (which, again, can can change from line to line), we have

$$
\begin{aligned}
&C\left(\int_0^t \int_{\mathbb{R}} (\Phi(t-s,x+h-y) - \Phi(t-s,x-y))^2 dyds\right)^{\alpha/2} \\
&= C\left(\int_0^t \Phi(2t-2s,0) - 2\Phi(2t-2s,h) + \Phi(2t-2s,0)ds\right)^{\alpha/2} \\
&= C\left(\int_0^t 2\Phi(2t-2s,0) - 2\Phi(2t-2s,h)ds\right)^{\alpha/2} \\
&= \{\tau = 2t-2s\} = C\left(\int_0^{2t} \Phi(\tau,0) - \Phi(\tau,h)d\tau\right)^{\alpha/2}.
\end{aligned}
$$

We note that by Proposition 2.3.2,

$$
\begin{aligned}
&\int_0^{2t} \Phi(\tau,0) - \Phi(\tau,h)d\tau \\
&= \left[\frac{1}{\sqrt{\pi}}\left(\sqrt{\tau} - \sqrt{\tau}e^{-\frac{|h|^2}{4\tau}} - \frac{h}{2}\sqrt{\pi}\,\mathrm{erf}\left(\frac{h}{2\sqrt{\tau}}\right)\right)\right]_0^{2t} \\
&= C_1 + C_2 h \leq Ch,
\end{aligned}
$$

because the erf-and exponential functions are bounded. Hence the expected value of an increment in x is bounded in the following way,

$$
\mathbb{E}\left[|u(t,x+h) - u(t,x)|^\alpha\right] \leq Ch^{\alpha/2}.
$$

For the second expression in 2.8 a similar argument is used. We utilise that we know the α-th moment of a zero mean Gaussian variable, fix $x \in \mathbb{R}$

$$
\begin{aligned}
&\mathbb{E}\left[|u(t+k,x) - u(t,x)|^\alpha\right] \\
&= C\mathbb{E}\left[\left(\int_0^{t+k}\int_{\mathbb{R}} \Phi(t+k-s,x-y)\,W(dyds) - \int_0^t\int_{\mathbb{R}} \Phi(t-s,x-y)\,W(dyds)\right)^2\right]^{\alpha/2}.
\end{aligned}
$$

We continue with the expected value of the increment squared, where we use Wiener's isometry, the semi-group property of Φ, and that we know the primitive function of $\Phi(\tau,z)$ with respect to τ. Below we have C, C_1 and C_2 being constants that may change from line to line.

$$\mathbb{E}\left[\left(\int_0^{t+k}\int_{\mathbb{R}}\Phi(t+k-s,x-y)W(dyds)-\int_0^t\int_{\mathbb{R}}\Phi(t-s,x-y)W(dyds)\right)^2\right]$$

$$=C_1\int_0^{t+k}\int_{\mathbb{R}}\Phi^2(t+k-s,x-y)dyds$$

$$+C_2\int_0^{(t+k)\wedge t}\int_{\mathbb{R}}\Phi(t+k-s,x-y)\Phi(t-s,x-y)dyds+C_3\int_0^t\int_{\mathbb{R}}\Phi^2(t-s,x-y)dyds$$

$$=C_1\int_0^{2t+2k}\Phi(\tau,0)d\tau+C_2\int_k^{2t}\Phi(\tau,0)d\tau+C_3\leq C_1+C_2\sqrt{k}\leq C\sqrt{k}.$$

In the second to last inequality we used $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$ for positive numbers a and b. And hence

$$\mathbb{E}\left[|u(t+k,x)-u(t,x)|^{\alpha}\right]\leq Ck^{\alpha/4}.$$

This completes the proof by using Kolmogrov's Theorem 2.3.5 with 1-dimensional index space, $1+\beta$ equal to $\alpha/2$ and $\alpha/4$ in space and time respectively. Therefore the solution has a modification that is Hölder-continuous with Hölder-exponent $\gamma\in(0,1/2-1/\alpha)$ and $\gamma\in(0,1/4-1/\alpha)$ for every $\alpha>0$. Consequently, the Hölder-exponent is almost 1/2 in space and almost 1/4 in time. This is the upper bound of Hölder-continuity. □

For the coming discussions, assume that we are working with the Hölder-continuous modification of u with Hölder-exponent almost 1/2 in space and almost 1/4 in time.

2.3.3 Localisation error

This thesis focuses primarily with the solution on $\mathscr{D}=\mathbb{R}$, but the solutions on $\mathscr{D}=\mathbb{R}$ and $\mathscr{D}=(0,L)$ are closely linked. Because we simulate solution on both bounded $\mathscr{D}$ and $\mathscr{D}=\mathbb{R}$ we have the following nice property that was shown by Candil [Candil, 2022]. As the bounded domain increases to the whole real line, both solutions become arbitrarily close to one another. If we consider $\mathscr{D}=(-L,L)$ for some $L\in\mathbb{R}$, then as $L\to\infty$, the solution to 1.8 actually converges to the solution of the stochastic heat equation on the entire real line, subject to the same initial conditions (where the initial condition can be stochastic as well). First let u_0 satisfy

$$\sup_{x\in\mathbb{R}}\mathbb{E}(|u_0(x)|^p)^{1/p}<\infty. \tag{2.9}$$

For some $p\geq 2$. Below is the localization error, see [Candil, 2022, Theorem 2.4 page 21].

Theorem 2.3.8. *Let $u_L(t,x)$ be the solution to the stochastic heat equation on $\mathbb{R}_+ \times \mathscr{D} = \mathbb{R}_+ \times (-L, L)$, $L > 0$, driven by white noise, drift parameter $\alpha = 1$, and diffusion $\sigma = 1$, and let $u(t,x)$ be the solution to the same differential stochastic heat equation on $\mathbb{R} \times \mathbb{R}_+$ subject to the same initial conditions as u_L. Fix $T > 0$, then*

$$\mathbb{E}(|u(t,x) - u_L(t,x)|^p)^{1/p} \leq \tilde{c}\left(e^{\frac{-(L-x)^2}{8t}} + e^{\frac{-(L+x)^2}{8t}}\right).$$

For all $x \in (-L, L)$ and $t \in (0,T)$. Where $\tilde{c}$ is independent of t, x and L. The p is the same as for the initial condition 2.9.

Chapter 3

Distribution, Exact q-Variation, and Inference of the Solution

Inference on the stochastic heat equation will focus on the estimation of the drift and diffusion of the field solution to the stochastic heat equation,

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \alpha\Delta u(t,x) = \sigma\,\dot{W}(t,x) & t > 0,\ x \in \mathbb{R} \\ u(0,x) = 0 & x \in \mathbb{R}^d. \end{cases} \tag{3.1}$$

Given the field, one can look at the solution for a fixed x or a fixed t which results in a stochastic process whose distribution depends on α and σ. Now let $V : C[a,b] \to \mathbb{R}_+ \cup \{+\infty\}$ be the functional called the exact q-variation. Throughout, Kolmogorov's continuity theorem will be used such that the sample paths are a.s. continuous and hence in the domain of the functional. One of the most important results in this chapter is that the exact q-variation of our solution in time and space will be a known value that depends on the drift and diffusion.

We will have the important property that the functional V is invariant to Lipschitz perturbations, i.e. let the sample paths v be Lipschitz and u be continuous, then $V(u+v) = V(u)$. This is a strong property which will be needed throughout this chapter since we only know the value of the functional V for a specific stochastic process, the fractional Brownian motion (fBm). Hence this chapter begins by representing the solution in time and space with the fBm, which will give us a way to calculate the variation of u, and consequently get some information about the drift and diffusion.

Remark 3.0.1. *Note that the results on the variations of the solution are proved only over the unbounded spatial domain $\mathbb{R}$. However, the results coincide for the solutions on compact intervals $\mathscr{D}$, these proofs are based on the Malliavin calculus and can be found in e.g. [Cialenco and Huang, 2019].*

3.1 Fractional and bi-fractional Brownian motion

This chapter will heavily use the fractional- and bi-fractional Brownian motion. Both are generalisations of the Brownian motion in the sense that they keep some properties such as, being Gaussian, having stationary increments, being self similar and starting from zero. The more generalised, the less properties kept.

Definition 3.1.1. *The* fractional Brownian motion *(fBm) is a continuous time zero mean Gaussian process denoted $F^H(t)$ with covariance*

$$\mathbb{E}\left[F^H(t)F^H(s)\right] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

where $H \in (0,1)$ is the Hurst parameter*.*

A proof that the above covariance function is non-negative definite can be found in e.g [Nourdin, 2012]. Thus the existence of the Gaussian process F^H is guaranteed.

Note that if $H = \frac{1}{2}$ we have normal Brownian motion. Some properties of the fractional Brownian motion,

Proposition 3.1.2. *Fractional Brownian motion is self similar for all $a > 0$,*

$$F^H(at) \stackrel{d}{=} a^H\, F^H(t) \qquad \text{for all } t \in \mathbb{R}_+. \tag{3.2}$$

Proof. Because they are zero mean Gaussian it suffices to prove that,

$$\mathbb{E}\left[(F^H(at))^2\right] = |at|^{2H} = |a|^{2H}|t|^{2H} = \mathbb{E}\left[(a^H\, F^H(t))^2\right].$$

□

Proposition 3.1.3. *Fractional Brownian motion has stationary increments,*

$$F^H(t) - F^H(s) \stackrel{d}{=} F^H(t-s).$$

Proof. Since both the left- and right hand side are zero mean Gaussian processes it suffuses to prove that,

$$\begin{aligned}\mathbb{E}\left[(F^H(t) - F^H(s))^2\right] &= |t|^{2H} + |s|^{2H} - (|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \\ &= |t-s|^{2H} = \mathbb{E}\left[(F^H(t-s))^2\right]\end{aligned}$$

□

The value of H determines the regularity of the sample paths, since for every $T > 0$ and $\epsilon > 0$ there exists a random variable c such that,

$$|F^H(t) - F^H(s)| \leq c|t-s|^{H-\epsilon} \tag{3.3}$$

where $s < t < T$. This will be proven later when needed, but it is easily visualised in Figure 3.1 below where three different values of H are compared.

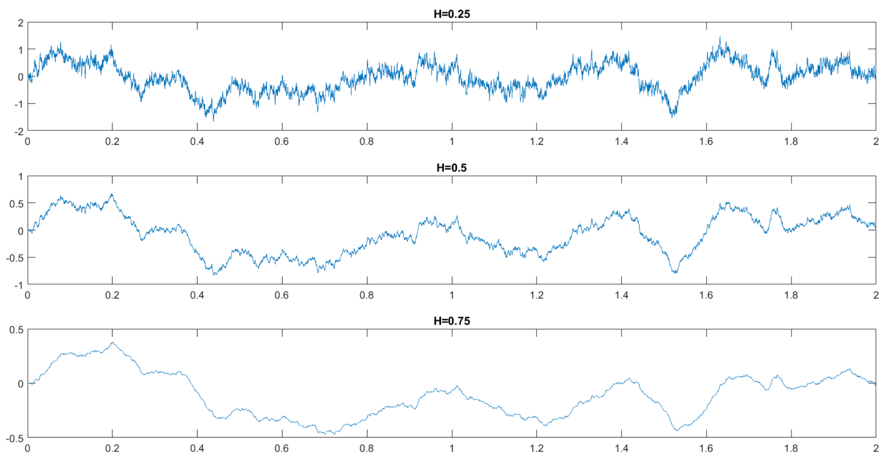


Figure 3.1: Three different values of Hurst parameter for the fractional Brownian motion with the same underlying independent Gaussian variables.

A further generalisation of Brownian motion is the bi-fractional Brownian motion, which was introduced by Houdré and Villa [Houdré and Morales, 2003] to generalise even further, but importantly, it keeps the self-similarity property and lightens the restraint of stationary increments since it was not that useful for modelling purposes.

Definition 3.1.4. *The bi-fractional Brownian motion is a continuous time zero mean Gaussian process denoted $B^{H,K}(t)$ with covariance*

$$\mathbb{E}[B^{H,K}(t)B^{H,K}(s)] = \frac{1}{2^K}\Big((t^{2H} + s^{2H})^K - |t-s|^{2HK}\Big) \tag{3.4}$$

where $H \in (0,1)$ and $K \in (0,1]$.

The covariance function is non-negative definite [Houdré and Morales, 2003, Proposition 2.1].

Note that if $K = 1$ we have fractional Brownian motion. Some properties of the bi-fractional Brownian motion,

Proposition 3.1.5. *The bi-fractional Brownian motion is self similar for all $a > 0$,*

$$B^{H,K}(at) \stackrel{d}{=} a^{HK} B^{H,K}(t) \quad \text{for all } t \in \mathbb{R}_+. \tag{3.5}$$

Proof. Because they are zero mean Gaussian it suffices to prove that,

$$\mathbb{E}\left[(B^{H,K}(at))^2\right] = \frac{2^K}{2^K}(at)^{2HK} = (a)^{2HK}(t)^{2HK} = \mathbb{E}\left[(a^{HK}\, B^{H,K}(t))^2\right].$$

□

Noting that the first term in the covariance for the bi-fractional Brownian motion inhibits it from having stationary increments since the freshman's dream unfortunately is not true in general, the next best thing will have to suffice.

Proposition 3.1.6. *The bi-fractional Brownian motion has approximately stationary increments for small increments.*

Proof. Because they are zero mean Gaussian it suffices to prove that for $n > 0$ and $s = t + \frac{1}{n}$, then

$$\mathbb{E}\left[(B^{H,K}(t) - B^{H,K}(s))^2\right] = |t + \frac{1}{n}|^{2HK} + |t|^{2HK} - 2^{1-K}\Big(|t + \frac{1}{n}|^{2H} + |t|^{2H}\Big)^K + \frac{2^{1-K}}{n^{2HK}}$$

which converges to $\frac{2^{1-K}}{n^{2HK}} = \frac{1}{2^K}|t-s|^{2HK}$ as $n \to \infty$ and since $\frac{1}{2^K}|t-s|^{2HK} = \mathbb{E}\left[(B^{H,K}(t))^2\right]$ this completes the proof. □

3.2 Distribution of $u(t,x)$

Since the goal is to calculate the variation of the stochastic processes given by either fixing t or x, they need to be perturbed such that they have the same distribution as the fractional Brownian motion. To be able to know which process to perturb with, we need to know the distribution of the solution with $\alpha = 1$ and $\sigma = 1$ in time and space.

3.2.1 In time

By fixing x, the solution varying over time will be shown to have the distribution of a known process, the bi-fractional Brownian motion.

Proposition 3.2.1. *The solution $u(t,x)$ to the stochastic heat equation 3.1 with $\alpha = 1$ and $\sigma = 1$ has the same distribution modulo a constant to the bi-fractional Brownian motion with $H = \frac{1}{2}$ and $K = \frac{1}{2}$,*

$$u(t,x) \stackrel{d}{=} (2\pi)^{-\frac{1}{4}}\, B^{\frac{1}{2},\frac{1}{2}}(t) \quad \text{for all } t \in \mathbb{R}_+.$$

Proof. Recall the covariance to the solution of equation 3.1 with a fixed x, $\alpha = 1$ and $\sigma = 1$,

$$\mathbb{E}\left[u(t_1,x)u(t_2,x)\right] = \frac{1}{\sqrt{4\pi}}((t_1+t_2)^{\frac{1}{2}} - |t_1-t_2|^{\frac{1}{2}}),$$

and the covariance for bi-fractional Brownian motion is,

$$\mathbb{E}[B^{H,K}(t)B^{H,K}(s)] = \frac{1}{2^K}(t^{2H} + s^{2H})^K - |t-s|^{2HK}).$$

Hence they match if $K = \frac{1}{2}$, $H = \frac{1}{2}$ and the covariance for bi-fractional is scaled by a factor $\frac{1}{\sqrt{2\pi}}$. Since the covariance of the solution above is the same as the one for the bi-fractional Brownian motion, they have the same distribution by Lemma 2.1.6 (1). □

Remark 3.2.2. *The fact that two Gaussian processes are equal in distribution when their mean and covariance coincide thanks to Lemma 2.1.6 (1) will not be stated throughout the rest of this chapter.*

3.2.2 In space

In space we will stumble upon the fractional Brownian motion instantaneously. The following is inspired by [Foondun et al., 2014, Proposition 3.1],

Theorem 3.2.3. *The solution $u(t,x)$ to the stochastic heat equation with $\alpha = 1$ and $\sigma = 1$ perturbed by the stochastic process*

$$S_t(x) := \int_{(t,\infty)\times\mathbb{R}} \Phi(s,y;0,0) - \Phi(s,y;0,x)W(dyds) \tag{3.6}$$

has the same distribution modulo a constant as the fractional Brownian motion with Hurst parameter $H = \frac{1}{2}$ (i.e. a Brownian motion),

$$u(t,x) - S_t(x) \stackrel{d}{=} \frac{1}{\sqrt{2}}\, F^{\frac{1}{2}}(x) \quad \text{for all } x \in \mathbb{R}. \tag{3.7}$$

The stochastic process $S_t(x)$ is a zero mean Gaussian process in C^1.

Proof. The process $S_t(x)$ is a zero mean Gaussian random field since the kernel is in $L^2((t,\infty)\times\mathbb{R})$. We define the derivative of $S_t(x)$ as,

$$\frac{d}{dx}S_t(x) := \int_t^\infty \int_{\mathbb{R}} \frac{d}{dx}\Phi(0,x;s,y)\,W(dyds). \tag{3.8}$$

The process $\frac{d}{dx}S_t(x)$ has finite second moment (and hence exists in $L^2(\Omega)$),

$$\mathbb{E}\left[\left|\frac{d}{dx}S_t(x)\right|^2\right] = \int_t^\infty \int_{\mathbb{R}} |\frac{d}{dx}\Phi(0,x;s,y)|^2 dyds = \frac{1}{2\pi}\int_t^\infty \int_{\mathbb{R}} |\xi|^2 |\mathscr{F}(\Phi(0,0;s,y))(\xi)|^2 d\xi ds,$$

where the last equality is motivated by the invariance of a change of variable in space and Plancherel's theorem. Continuing,

$$\frac{1}{2\pi}\int_t^\infty \int_{\mathbb{R}} |\xi|^2 e^{-\frac{s|\xi|^2}{2}}(\xi)d\xi ds = \frac{2}{\pi}\int_0^\infty e^{-\frac{t|\xi|^2}{2}}d\xi < \infty.$$

The derivative defined in 3.8 is a.s. the true derivative in an L^2 sense, let

$$\frac{S_t(x+\Delta x)-S_t(x)}{\Delta x} = \int_t^\infty \int_{\mathbb{R}} \frac{\Phi(0,x+\Delta x;s,y)-\Phi(0,x;s,y)}{\Delta x}W(dyds),$$

then,

$$\begin{aligned}
&\lim_{\Delta x\to 0}\mathbb{E}\left[\left|\frac{S_t(x+\Delta x)-S_t(x)}{\Delta x} - \int_t^\infty \int_{\mathbb{R}} \frac{d}{dx}\Phi(0,x;s,y)\,W(dyds)\right|^2\right] \\
&= \lim_{\Delta x\to 0}\mathbb{E}\left[\left|\int_t^\infty \int_{\mathbb{R}} \frac{\Phi(0,x+\Delta x;s,y)-\Phi(0,x;s,y)}{\Delta x} - \frac{d}{dx}\Phi(0,x;s,y)\,W(dyds)\right|^2\right] \\
&= \int_t^\infty \int_{\mathbb{R}} \lim_{\Delta x\to 0}\left|\frac{\Phi(0,x+\Delta x;s,y)-\Phi(0,x;s,y)}{\Delta x} - \frac{d}{dx}\Phi(0,x;s,y)\right|^2 dyds = 0.
\end{aligned}$$

The process $\frac{d}{dx}S_t(x)$ has a continuous modification, with Kolmogorovs continuity theorem,

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{d}{dx}S_t(x_2) - \frac{d}{dx}S_t(x_1)\right|^2\right] &= \frac{1}{2\pi}\int_t^\infty \int_{\mathbb{R}} |\mathscr{F}(\frac{d}{dx}\Phi(0,0;s,y))(\xi)\,(e^{-ix_2\xi}-e^{-ix_1\xi})|^2 d\xi ds \\
&= \frac{1}{2\pi}\int_t^\infty \int_{\mathbb{R}} |\xi|^2 e^{-\frac{s|\xi|^2}{2}}\,|e^{-ix_1\xi}(1-e^{-i\xi\epsilon})|^2 d\xi ds
\end{aligned}$$

where $\epsilon = x_2 - x_1$. Continuing with the fact that $|1-\exp(\theta i)|^2 = 2(1-\cos(\theta))$ and Fubinis theorem,

$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_t^\infty |\xi|^2 e^{-\frac{s|\xi|^2}{2}}(1-\cos(\epsilon\xi))dsd\xi = \frac{1}{\pi}\int_{\mathbb{R}} \frac{|\xi|^2 e^{-\frac{t|\xi|^2}{2}}}{|\xi|^2}(1-\cos(\epsilon\xi))d\xi.$$

The fact that $1-\cos(\theta)\le\theta^2$ and symmetry is used,

$$\le \frac{1}{\pi}\int_{\mathbb{R}} e^{-\frac{t|\xi|^2}{2}}(\epsilon\xi)^2 d\xi = \frac{1}{\pi}\int_0^\infty \xi^2 e^{-2t|\xi|^2} d\xi\, \epsilon^2 = C|x_2-x_1|^2.$$

Now the process $B(x) := u(t,x) - S_t(x)$ can be identified, using the fact that $u(t,x)$ and $S_t(x)$ are integrated over disjoint sets (and hence independent) the following is true,

$$\mathbb{E}\left[|B(x_2)-B(x_1)|^2\right] = \mathbb{E}\left[|u(t,x_2)-u(t,x_1)|^2\right] + \mathbb{E}\left[|S_t(x_2)-S_t(x_1)|^2\right]$$

which by the Wiener isometry is equal to

$$\frac{1}{2\pi}\int_0^t\int_{\mathbb{R}} |\mathscr{F}(\Phi(0,0;s,y))(\xi)\ (e^{-ix_1\xi}-e^{-ix_2\xi})|^2 d\xi ds$$
$$+\frac{1}{2\pi}\int_t^\infty\int_{\mathbb{R}} |\mathscr{F}(\Phi(0,0;s,y))(\xi)\ (e^{-ix_1\xi}-e^{-ix_2\xi})|^2 d\xi ds$$
$$= \frac{1}{\pi}\int_0^\infty\int_{\mathbb{R}} e^{-s|\xi|^2}(1-\cos((x_2-x_1)\xi))d\xi ds = \frac{1}{\pi}\int_0^\infty \frac{(1-\cos((x_2-x_1)\xi))}{\xi^2}d\xi.$$

A change of variables $z=(x_2-x_1)\xi$ gives

$$\frac{|x_2-x_1|}{\pi}\int_0^\infty \frac{(1-\cos(z))}{z^2}dz = \frac{|x_2-x_1|}{2}$$

via Contour Integration and hence the process $B(x)\sqrt{2}$ is a Brownian motion. □

3.3 Distribution of $\sigma u_\alpha(t,x)$

The distributions for the solution with $\alpha=1$ and $\sigma=1$ have been calculated. Now the effects of changing the drift α and the diffusion σ needs to be understood. Since we are working with the heat equation a powerful technique which shifts the drift to the diffusion can be utilised, even in the stochastic case.

The solution to the stochastic heat equation 3.1, is by definition

$$\sigma u_\alpha(t,x) = \int_0^t\int_{\mathbb{R}} \Phi(\alpha s, y; \alpha t, x)\sigma W(dyds). \tag{3.9}$$

Where $\Phi(s,y;t,x)$ is the fundamental solution to the SHE 3.1 with $\alpha=\sigma=1$. We denote solution to the SPDE 3.1 with $\sigma=1$ and general drift as $u_\alpha(t,x)$, and $u_1(t,x)$ is the solution with $\alpha=1$ and $\sigma=1$. Just like in the last part the distributions to $\sigma u_\alpha(t,x)$ must be known such that they can be perturbed into fractional Brownian motions. The shift of drift to diffusion is stated more precisely below. The calculations are inspired by [Mahdi Khalil and Tudor, 2019].

Proposition 3.3.1. *Suppose that the process $\sigma u_\alpha(t,x)$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$ is the solution to 3.1. Define*

$$\sigma v_\alpha(t,x) := \sigma u_\alpha(\frac{t}{\alpha}, x) \quad \text{for all } t \in \mathbb{R}_+, x \in \mathbb{R}$$

then the process $v_\alpha(t,x)$ satisfies the stochastic partial differential equation

$$\frac{\partial}{\partial t} v_\alpha(t,x) - \Delta v_\alpha(t,x) = \alpha^{-\frac{1}{2}} \dot{\tilde{W}}(t,x)$$

where $\dot{\tilde{W}}$ is a space-time white noise.

Proof. Let $t \in \mathbb{R}_+, x \in \mathbb{R}$, then by equation 3.9,

$$\begin{aligned} v_\alpha(t,x) = u_\alpha(\frac{t}{\alpha}, x) = \int_0^{\frac{t}{\alpha}} \int_{\mathbb{R}} \Phi(\alpha s, y; t, x) W(dyds) = \left\{ s = \frac{s'}{\alpha} \right\} \\ \int_0^t \int_{\mathbb{R}} \Phi(s,y;t,x) W(d\frac{s}{\alpha} dy) = \alpha^{-\frac{1}{2}} \int_0^t \int_{\mathbb{R}} \Phi(s,y;t,x) \tilde{W}(dyds) \end{aligned} \tag{3.10}$$

where the second to last equality is motivated by approximating the integrand with a step-function and scaling the rectangles over which we integrate and the last equality is motivated by the self similarly property of Brownian motion. □

3.3.1 In time

Proposition 3.3.2. *For every $x \in \mathbb{R}$ the drift $\alpha > 0$ and diffusion $\sigma > 0$ scales the solution by $\sigma \alpha^{-\frac{1}{4}}$, i.e.*

$$\sigma u_\alpha(t,x) \stackrel{d}{=} \sigma (2\alpha\pi)^{-\frac{1}{4}} \, B^{\frac{1}{2},\frac{1}{2}}(t) \quad \text{for all } t \in \mathbb{R}_+.$$

Proof. Let $x \in \mathbb{R}$, $\sigma > 0$ and $\alpha > 0$, then we have the following equality in distribution

$$\begin{aligned} \mathbb{E}\left[\sigma u_\alpha(t,x) \sigma u_\alpha(s,x)\right] &= \sigma^2 \mathbb{E}\left[v_\alpha(\alpha t, x) v_\alpha(\alpha s, x)\right] \\ &= \sigma^2 \alpha^{-1} \mathbb{E}\left[u_1(\alpha t, x) u_1(\alpha s, x)\right], \end{aligned} \tag{3.11}$$

where the second equality is motivated by same change of variable as in the proof of equation 3.10 and seen below

$$\begin{aligned} v_\alpha(\alpha t, x) = & \int_0^t \int_{\mathbb{R}} \Phi(\alpha s, y; \alpha t, x) \tilde{W}(dyds) \\ &= \int_0^{\alpha t} \int_{\mathbb{R}} \Phi(s, y; \alpha t, x) \tilde{W}(d\frac{s}{\alpha} dy) = \alpha^{-\frac{1}{2}} u_1(\alpha t, x). \end{aligned} \tag{3.12}$$

Continuing on equation 3.11 firstly with the distribution given in Proposition 3.2.1 and secondly the self-similarity property of bi-fractional Brownian motion,

$$\sigma^2\alpha^{-1}\mathbb{E}\left[u_1(\alpha t,x)u_1(\alpha s,x)\right] = \sigma^2\alpha^{-1}(2\pi)^{-\frac{1}{2}}\mathbb{E}\left[B_{\alpha t}^{\frac{1}{2},\frac{1}{2}}B_{\alpha s}^{\frac{1}{2},\frac{1}{2}}\right] = \sigma^2(\alpha 2\pi)^{-\frac{1}{2}}\mathbb{E}\left[B_t^{\frac{1}{2},\frac{1}{2}}B_s^{\frac{1}{2},\frac{1}{2}}\right].$$

□

3.3.2 In space

Proposition 3.3.3. *For every $t \in \mathbb{R}_+$ the drift $\alpha > 0$ and diffusion $\sigma > 0$ scales the perturbed solution by $\sigma\alpha^{-\frac{1}{2}}$, i.e.*

$$\sigma u_\alpha(t,x) - \sigma S_{\alpha t}(x) \stackrel{d}{=} \sigma(2\alpha)^{-\frac{1}{2}}\, F^{\frac{1}{2}}(x) \quad \textit{for all } x \in \mathbb{R}.$$

Proof. Let $t > 0$,

$$\sigma u_\alpha(t,x) = \sigma v_\alpha(\alpha t,x) \stackrel{d}{=} \sigma\alpha^{-\frac{1}{2}}u_1(\alpha t,x) \quad \text{for all } x \in \mathbb{R}$$

where the last equality is motivated by equation 3.12. Continuing the calculations using the distribution given in Lemma 3.2.3,

$$\sigma\alpha^{-\frac{1}{2}}u_1(\alpha t,x) - \sigma S_{\alpha t}(x) \stackrel{d}{=} \sigma(2\alpha)^{-\frac{1}{2}}F^{\frac{1}{2}}(x) \quad \text{for all } x \in \mathbb{R}.$$

□

3.4 Decomposition of the bi-fractional Brownian motion

Since the distribution of the solution in time is the bi-fractional Brownian motion we want to perturb it into the fractional Brownian motion such that the variation can be calculated. The following is inspired by [Lei and Nualart, 2009, Proposition 1].

The theorem is motivated by the decomposition of the covariance function for bi-fractional Brownian motion,

$$\begin{aligned} R^{H,K}(t,s) &= 2^{-K}((t^{2H}+s^{2H})^K - |t-s|^{2HK}) = \\ &\quad 2^{-K}[((t^{2H}+s^{2H})^K - t^{2HK} - s^{2HK}) + (t^{2HK}+s^{2HK}-|t-s|^{2HK})] \end{aligned} \tag{3.13}$$

where the second term is the covariance for the fractional Brownian motion. Now the first term needs to be identified.

Theorem 3.4.1. *The bi-fractional Brownian motion with parameters H and K perturbed by the stochastic process*

$$X^K(t) := \int_0^\infty (1 - e^{-t\alpha})\alpha^{\frac{-1-K}{2}} W(d\alpha)$$

has the same distribution modulo a constant as the fractional Brownian motion with Hurst parameter $H \cdot K$,

$$B^{H,K}(t) + C_1 X^K(t^{2H}) \stackrel{d.}{=} \sqrt{2^{1-K}} F^{HK}(t) \quad \text{for all } t \in \mathbb{R}_+$$

where, $C_1 = \sqrt{\frac{2^{-K}K}{\Gamma(1-K)}}$. The process $X^K(t^{2H})$ is a zero mean Gaussian process in C^1.

Proof. The definition of $X^K(t)$ comes from the fact that the covariance matches the sought after covariance structure in the first term of the decomposition 3.13 with a sign change which can be seen by,

$$\begin{aligned}\mathbb{E}\left[X^K(t)X^K(s)\right] &= \int_0^\infty (1 - e^{-t\alpha})(1 - e^{-s\alpha})\alpha^{-1-K} d\alpha \\ &= \frac{\Gamma(1-K)}{K}(t^K + s^K - (t+s)^K).\end{aligned}$$

Now since these variables are Gaussian the behaviour is governed by the expectation and covariance. Hence the process $F(t) = B^{H,K}(t) + \sqrt{\frac{2^{-HK}HK}{\Gamma(1-HK)}} X^K(t^{2H})$ has the following covariance,

$$\begin{aligned}\mathbb{E}\left[F(t)F(s)\right] =& \mathbb{E}\left[B^{H,K}(t)B^{H,K}(s)\right] + \frac{2^{-K}K}{\Gamma(1-K)}\mathbb{E}\left[X^K(t^{2H})X^K(s^{2H})\right] \\ =& \frac{1}{2^K}((t^{2H} + s^{2H})^K - |t-s|^{2HK}) + \frac{1}{2^K}(t^{2HK} + s^{2HK} - (t^{2H} + s^{2H})^K) \\ =& \frac{1}{2^K}(t^{2HK} + s^{2HK} - |t-s|^{2HK}).\end{aligned}$$

Therefore, the process $F(t)$ is a fractional Brownian motion with Hurst parameter HK, i.e. $F^{HK}(t)$.

To prove that $X^K(t^{2H})$ has a.s. differentiable sample paths, let $Y(t) := \int_0^\infty \theta^{\frac{1-K}{2}} e^{-\theta t} W(d\theta)$ which is well defined since

$$\mathbb{E}\left[Y(t)^2\right] = \int_0^\infty \theta^{2-K-1} e^{-2\theta t} d\theta = \Gamma(2-K)2^{K-2}t^{K-2} < \infty.$$

Applying Fubini and then Cauchy Schwartz inequality with the the moment coefficients from Lemma 2.3.7, we note that $Y(t)$ is locally integrable,

$$\mathbb{E}\left[\int_0^t |Y(s)|ds\right] \leq \sqrt{\frac{2}{\pi}} \int_0^t \sqrt{\mathbb{E}\left[|Y(s)|^2\right]} ds = \sqrt{\frac{2\Gamma(2-K)2^{K-2}}{\pi}} \int_0^t s^{\frac{K-2}{2}} ds < \infty.$$

Applying stochastic Fubini (see e.g. [Protter, 2010, Theorem 64 page 210]),

$$\int_0^t Y(s)ds = \int_0^t \left(\int_0^\infty \theta^{\frac{1-K}{2}} e^{-\theta s} W(d\theta) \right) ds = \int_0^\infty \theta^{\frac{1-K}{2}} \left(\int_0^t e^{-\theta s} ds \right) W(d\theta)$$
$$= \int_0^\infty \theta^{\frac{-1-K}{2}} (1 - e^{-t\theta}) W(d\theta) = X^K(t).$$

Hence, $X^K(t)$ is absolutely continuous and by the fundamental theorem of Lebesgue integral calculus, this is equivalent to the fact that the sample paths of $X^K(t)$ is C^1 with the derivative $Y(t)$ a.s. □

3.5 Variation of perturbed stochastic processes

The functional V mentioned before is called the exact q-variation of a stochastic process over an interval $[a, b]$. Let us define it more precisely.

Definition 3.5.1. *For all $n \geq 1$, let $t_i = A + \frac{i}{n}(B - A)$ for $i = 0, ..., n$. A continuous real valued stochastic process $X(t)$ with $t \in \mathcal{T} \subseteq \mathbb{R}$ admits the exact q-variation*

$$V^q_{[A,B]} \left[X(t) \right] = \lim_{n \to \infty} V^{n,q}_{[A,B]} \left[X(t) \right] = \lim_{n \to \infty} \sum_{i=0}^{n-1} |X(t_{i+1}) - X(t_i)|^q$$

if the sequence $V^{n,q}_{[A,B]} \left[X(t) \right]$ converges in probability as $n \to \infty$.

The importance of the fact that the additive processes $X^{HK}(t)$ and $S_t(x)$ from Theorem 3.3.3 and Theorem 3.4.1 respectively are in C^1 is that they have Lipschitz sample paths and hence do not change the exact q-variation for $q > 1$. The variation in either time or space of the perturbed solution, $u(t, x) + X^{HK}(t)$ or $u(t, x) + S_t(x)$ respectively, are equal to the variations of the solution, i.e.

$$V^q_{[t_1,t_2]}[u(t, x)+X^{HK}(t)] = V^q_{[t_1,t_2]}[u(t, x)], \quad \text{and} \quad V^q_{[x_1,x_2]}[u(t, x)-S_t(x)] = V^q_{[x_1,x_2]}[u(t, x)].$$

This is formalised in the theorem below which is inspired by [Lei and Nualart, 2009, Proposition 4].

Theorem 3.5.2. *Let $q > 1$, then a stochastic process $Y(t)$ with finite exact q-variation perturbed by $A(t)$ which is a zero mean Gaussian process with Lipschitz sample paths has the same exact q-variation as $Y(t)$ itself,*

$$V^q_{[a,b]} \left[Y(t) \right] = V^q_{[a,b]} \left[Y(t) + A(t) \right].$$

Proof. Firstly,

$$\sum_{i=0}^{n-1} |Y(t_{i+1}) + A(t_{i+1}) - Y(t_i) - A(t_i)|^q = \sum_{i=0}^{n-1} |Y(t_{i+1}) - Y(t_i) + A(t_{i+1}) - A(t_i)|^q.$$

The Minkowski inequality can be used to create the following enclosure

$$\begin{aligned}\Big(\sum_{i=0}^{n-1} |Y(t_{i+1}) - Y(t_i)|^q\Big)^{1/q} &- \Big(\sum_{i=0}^{n-1} |A(t_{i+1}) - A(t_i)|^q\Big)^{1/q} \\ &\leq \Big(\sum_{i=0}^{n-1} |Y(t_{i+1}) + A(t_{i+1}) - Y(t_i) - A(t_i)|^q\Big)^{1/q} \\ &\leq \Big(\sum_{i=0}^{n-1} |Y(t_{i+1}) - Y(t_i)|^q\Big)^{1/q} + \Big(\sum_{i=0}^{n-1} |A(t_{i+1}) - A(t_i)|^q\Big)^{1/q}.\end{aligned} \tag{3.14}$$

Continuing by using the fact that $A(t)$ is a.s. Lipschitz continuous where C can change between inequalities,

$$\sum_{i=0}^{n-1} |A(t_{i+1}) - A(t_i)|^q \leq C \sum_{i=0}^{n-1} |t_{i+1} - t_i|^q = C \sum_{i=0}^{n-1} \Big(\frac{1}{n}\Big)^q = Cn^{1-q}.$$

Hence, since the above sum converges to 0 a.s. as $n \to \infty$ since $1 - q < 0$. This completes the proof because the exact q-variation of $Y(t)$ is finite combined with the enclosure in equation 3.14. □

3.6 Variation of the fractional Brownian motion

A specific value of q needs to be chosen for the exact q-variation to have some meaning and not converge to zero or diverge to infinity. This value is one over the Hölder continuity constant γ, i.e. $q = \frac{1}{\gamma}$. The reason is that the variation of a γ-Hölder continuous process $Y(t)$, again let C be changing,

$$\sum_{i=0}^{n-1} |Y(t_{i+1}) - Y(t_i)|^q \leq C \sum_{i=0}^{n-1} (|t_{i+1} - t_i|)^{\gamma q} = C \sum_{i=0}^{n-1} \Big|\frac{B-A}{n}\Big|^{\gamma q} = C\frac{n}{n^{\gamma q}}, \tag{3.15}$$

goes to infinity if $\gamma q < 1$ and to zero if $\gamma q > 1$. Hence to actually get some information about the variation of a γ-Hölder continuous function one needs to use the exact $\frac{1}{\gamma}$-variation.

Kolmogorovs continuity theorem can be applied to the fractional Brownian motion since because of the stationary increments,

$$\mathbb{E}\left[|F^H(t) - F^H(s)|^q\right] \leq |t-s|^{qH} \mathbb{E}\left[|F^H(1)|^q\right].$$

Let $q := \frac{1+R}{H}$ and $\epsilon := qH - 1 = R$ which gives that $F^H(t)$ is γ-Hölder continuity for $0 < \gamma < q/\epsilon = H\frac{R}{1+R}$, and since R was arbitrary, $\gamma \in (0, H)$. Hence the exact $\frac{1}{H}$-variation of the fractional Brownian motion is non trivial and can be seen below (cf. [Rogers, 1997]).

Theorem 3.6.1. *The exact $\frac{1}{H}$-variation of c times a fractional Brownian motion with Hurst parameter H is,*

$$V^{\frac{1}{H}}_{[a,b]}\left[cF^H(t)\right] = c^{\frac{1}{H}}(b-a)\mathbb{E}\left[|Z|^{\frac{1}{H}}\right],$$

where $Z \sim N(0,1)$.

The proof is located in the appendix A since it requires some prerequisites which would disturb the flow of the text. In this appendix the theorem which provides the exact $\frac{1}{H}$-variation of the fractional Brownian motion F^H will be proved. The reasoning behind perturbing into fractional Brownian motion is because it has stationary increments, since if a sequence is constructed from these increments, it is a stationary sequence which enables us to use tools from ergodic theory.

3.7 Exact variation of σu_α

Using everything constructed above the exact q-variation can be calculated.

3.7.1 In time

In time, the solution has the distribution as the bi-fractional Brownian motion with parameters $H = \frac{1}{2}$ and $K = \frac{1}{2}$ given in Proposition 3.3.2. Perturbing this solution with the stochastic process given in Theorem 3.4.1 results in the fractional Brownian motion with Hurst parameter $H = \frac{1}{4}$.

The connection between Hölder continuity and variation implies that the exact 4-variation of the solution should be used. The variation is not changed by the perturbation stated in Theorem 3.5.2. This sum is the fractional Brownian motion and the variation is given by Theorem 3.6.1. In summary,

$$\begin{aligned} V^4_{[s_1,s_2]}\left[\sigma u_\alpha(t,x)\right] &\underbrace{=}_{3.5.2} V^4_{[s_1,s_2]}\left[\sigma u_\alpha(t,x) + \sigma\frac{C_1}{(2\alpha\pi)^{1/4}}X^{\frac{1}{4}}(t)\right] \\ &\underbrace{=}_{3.6.1} V^4_{[s_1,s_2]}\left[\frac{2^{\frac{1}{4}}\sigma}{(2\alpha\pi)^{1/4}}F^{\frac{1}{4}}(t)\right] \underbrace{=}_{3.6.1} \frac{\sigma^4\mathbb{E}\left[|Z|^4\right](s_2-s_1)}{\alpha\pi}. \end{aligned} \tag{3.16}$$

3.7.2 In space

Almost exactly the same argument can be used for the solution in space. The only difference is that the perturbed solution is the fractional Brownian motion with Hurst parameter $H = \frac{1}{2}$ given in Theorem 3.2.3 with coefficients from Theorem 3.3.3. All of which can be expressed in the following equation,

$$\begin{aligned} V^2_{[y_1,y_2]}\left[\sigma u_\alpha(t,x)\right] &\underbrace{=}_{3.5.2} V^2_{[y_1,y_2]}\left[\sigma u_\alpha(t,x) - \sigma S_{\alpha t}(x)\right] \\ \underbrace{=}_{3.6.1} V^2_{[y_1,y_2]}&\left[\frac{\sigma}{(2\alpha)^{1/2}} F^{\frac{1}{2}}(x)\right] \underbrace{=}_{3.6.1} \frac{\sigma^2 \mathbb{E}\left[|Z|^2\right](y_2 - y_1)}{2\alpha} \end{aligned} \tag{3.17}$$

3.8 Estimators

Now that the variation is known and how it depends on the drift and diffusion, estimators can be constructed. We will construct four estimators, two for each path in time and space respectively, $\widehat{\alpha}^{(n)}_{\text{time}}$,$\widehat{\alpha}^{(n)}_{\text{space}}$, $\widehat{\sigma}^{(n)}_{\text{time}}$, and $\widehat{\sigma}^{(n)}_{\text{space}}$.

Recall the theorems 3.5.2 and 3.6.1, for a stochastic process X with Lipschitz sample paths, $V^{\frac{1}{H}}_{[a,b]}\left[cF^H(t) + X(t)\right] = V^{\frac{1}{H}}_{[a,b]}\left[cF^H(t)\right] = c^{\frac{1}{H}}(b-a)\mathbb{E}[|Z|^{\frac{1}{H}}]$ and hence

$$\mathbb{E}[|Z|^{\frac{1}{H}}] = \frac{1}{c^{\frac{1}{H}}(b-a)} V^{\frac{1}{H}}_{[a,b]}\left[cF^H(t) + X(t)\right].$$

Consider therefore the sequence of stochastic variables $V^{n,\frac{1}{H}}_{[a,b]}\left[cF^H(t) + X(t)\right]$ for $n \geq 1$. Replacing $V^{\frac{1}{H}}_{[a,b]}\left[cF^H(t) + X(t)\right]$ by the sequence $V^{n,\frac{1}{H}}_{[a,b]}\left[cF^H(t) + X(t)\right]$ should give an approximation for the value $\mathbb{E}[|Z|^{\frac{1}{H}}]$. The lemma given below is a special case of [Mahdi Khalil and Tudor, 2019, Lemma 1] which shows how well this approximation holds through a central limit type theorem.

Lemma 3.8.1. *Let $cF^H(t)$, with $t \geq 0$, be a fBm with $H \in (0, \frac{1}{2}]$ scaled by the constant $c > 0$. Let X be a zero mean stochastic process with Lipschitz sample paths. Then as $n \to \infty$ we have the following convergence in distribution*

$$\sqrt{n}\left[\frac{V^{n,\frac{1}{H}}_{[a,b]}\left[cF^H(t) + X(t)\right]}{c^{1/H}(b-a)} - \mathbb{E}[|Z|^{\frac{1}{H}}]\right] \to N\left(0, \rho^2_{H,\frac{1}{H}}\right).$$

Where $\rho^2_{H,\frac{1}{H}}$ is an explicit positive constant.

The variance $\rho^2_{H,\frac{1}{H}}$ for the Hurst parameters $H = \frac{1}{4}$ and $H = \frac{1}{2}$ are calculated in appendix B. We present the results here. Let,

$$\kappa_l^2 := \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} r^l(|i-j|), \quad \text{where,} \quad r^l(k) := \left(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}\right)^l.$$

For Hurst parameter $H = \frac{1}{2}$ and $H = \frac{1}{4}$, the variance from Lemma 3.8.1 is,

$$\rho^2_{\frac{1}{2},2} = 2\kappa_2^2, \quad \text{and} \quad \rho^2_{\frac{1}{4},4} = 72\kappa_2^2 + 24\kappa_4^2.$$

3.8.1 Path estimation of drift and diffusion separately

The above lemma will give us a method of determining the asymptotic normality of our estimators of α and σ. By utilising that we know the variations as function of α and σ we can "solve" for one of them if we have the other.

Motivated by equality 3.16, if we calculate the quartic variation over $t \in [0, s]$ for some $s > 0$, the estimator of α can be constructed,

$$\widehat{\alpha}^{(n)}_{\text{time}} := \frac{\sigma^4 s \mathbb{E}[|Z|^4]}{\pi \sum_{i=0}^{n-1} |\sigma u_\alpha(t_{i+1}, x) - \sigma u_\alpha(t_i, x))|^4}.$$

We show the explicit method of computing the asymptotic normality of estimators by utilising Lemma 3.8.1 only for estimator $\widehat{\alpha}^{(n)}_{\text{time}}$ above. The asymptotic normality of the rest of the estimators $\widehat{\alpha}^{(n)}_{\text{space}}$, $\widehat{\sigma}^{(n)}_{\text{time}}$, and $\widehat{\sigma}^{(n)}_{\text{space}}$ will follow by the same procedure and are omitted. We know from equation 3.16 that

$$\mathbb{E}[|Z|^4] = \frac{\pi\alpha}{\sigma^4 s} V^4_{[0,s]}\left[\sigma u_\alpha(t,x)\right] = \frac{V^4_{[0,s]}\left[\sigma u_\alpha(t,x)\right]}{\frac{\sigma^4}{\pi\alpha} s}. \tag{3.18}$$

We can perturb the solution $t \mapsto \sigma u_\alpha(t, x)$ to some fractional Brownian motion,

$$\sigma u_\alpha(t,x) = \sigma u_\alpha(t,x) - X(t) + X(t) \overset{d}{=} cF^H(t) + X(t).$$

Applying Lemma 3.8.1 then gives that the stochastic variable

$$\mathcal{V}_n := \frac{\pi\alpha}{\sigma^4} \frac{V^{n,4}_{[0,s]}\left[\sigma u_\alpha(t,x)\right]}{s},$$

which admits the following asymptotic normality

$$\sqrt{n}\left[\mathcal{V}_n - \mathbb{E}[|Z|^4]\right] \to N\left(0, \rho^2_{\frac{1}{4},4}\right).$$

Multiplying $\mathcal{V}_n$ by $\frac{1}{\mathbb{E}[|Z|^4]\alpha}$ gives that

$$\sqrt{n}\left[\frac{\mathcal{V}_n}{\mathbb{E}[|Z|^4]\alpha} - \frac{1}{\alpha}\right] \to N\left(0, \rho^2_{\frac{1}{4},4}(\alpha\mathbb{E}[|Z|^4])^{-2}\right).$$

The variable $\widehat{\alpha^{-1}}_{(\text{time})} := \frac{\mathcal{V}_n}{\mathbb{E}[|Z|^4]\alpha}$ is an estimator for $1/\alpha$. The well known Delta method (see e.g. [Doob, 1935]) states that if A_n is a sequence of random variables such that $\sqrt{n}(A_n - a_0) \to N(0, \sigma^2)$ and if g is a function such that $g'(a_0)$ exists and is not equal to zero, then $\sqrt{n}(g(A_n) - g(a_0)) \to N(0, \sigma^2 g'(a_0)^2)$. In our case we are looking to take the reciprocal of the variable $\frac{\mathcal{V}_n}{\mathbb{E}[|Z|^4]\alpha}$, by

$$g(\frac{\mathcal{V}_n}{\mathbb{E}[|Z|^4]\alpha}) := \frac{1}{\frac{\mathcal{V}_n}{\mathbb{E}[|Z|^4]\alpha}} = \frac{\sigma^4 s\mathbb{E}[|Z|^4]}{\pi\sum_{i=0}^{n-1}|\sigma u_\alpha(t_{i+1}, x) - \sigma u_\alpha(t_i, x))|^4}.$$

By definition, $\frac{\sigma^4 s\mathbb{E}[|Z|^4]}{\pi\sum_{i=0}^{n-1}|\sigma u_\alpha(t_{i+1}, x) - \sigma u_\alpha(t_i, x))|^4} =: \widehat{\alpha}^{(n)}_{\text{time}}$ and with $g(a) = \frac{1}{a}$, $g'(\frac{1}{\alpha})^2 = \alpha^4$, we have as $n \to \infty$ the following convergence in distribution,

$$\sqrt{n}(\widehat{\alpha}^{(n)}_{\text{time}} - \alpha) \to N\left(0, \rho^2_{\frac{1}{4},4}\left(\alpha^{-1}\mathbb{E}[|Z|^4]\right)^{-2}\right).$$

We collect all results on the estimators below. First for α estimations,

Proposition 3.8.2. *Let*

$$\widehat{\alpha}^{(n)}_{time} := \frac{\sigma^4 s\mathbb{E}[|Z|^4]}{\pi\sum_{i=0}^{n-1}|\sigma u_\alpha(t_{i+1}, x) - \sigma u_\alpha(t_i, x))|^4},$$

be the estimator of the drift α over a path in time $t \in [0, s]$ of the solution $\sigma u_\alpha(t, x)$ of equation 3.1. Then as $n \to \infty$, the following limit converges in distribution

$$\sqrt{n}(\widehat{\alpha}^{(n)}_{time} - \alpha) \to N\left(0, \rho^2_{\frac{1}{4},4}\left(\alpha^{-1}\mathbb{E}[|Z|^4]\right)^{-2}\right).$$

Proposition 3.8.3. *Let*

$$\widehat{\alpha}^{(n)}_{space} := \frac{\sigma^2\mathbb{E}\left[|Z|^2\right](y_2 - y_1)}{2\sum_{i=0}^{n-1}|\sigma u_\alpha(t, x_{i+1}) - \sigma u_\alpha(t, x_i))|^2}.$$

be the estimator of α over a path in space $x \in [y_1, y_2]$ of the solution $\sigma u_\alpha(t, x)$ of equation 3.1. Then as $n \to \infty$, the following limit converges in distribution

$$\sqrt{n}(\widehat{\alpha}^{(n)}_{space} - \alpha) \to N\left(0, \rho^2_{\frac{1}{2},2}\left(\alpha^{-1}\mathbb{E}[|Z|^2]\right)^{-2}\right).$$

The estimations of σ follow by the same procedure.

Proposition 3.8.4. *Let*

$$\widehat{\sigma}^{(n)}_{time} := \left(\frac{\alpha\pi \sum_{i=0}^{n-1} |\sigma u_\alpha(t_{i+1}, x) - \sigma u_\alpha(t_i, x))|^4}{s\mathbb{E}[|Z|^4]} \right)^{1/4}$$

be the estimator of the diffusion σ over a path in time $t \in [0, s]$ of the solution $\sigma u_\alpha(t, x)$ of equation 3.1. Then as $n \to \infty$, the following limit converges in distribution

$$\sqrt{n}(\widehat{\sigma}^{(n)}_{time} - \sigma) \to N\left(0, \frac{1}{16}\rho^2_{\frac{1}{4},4}\left(\sigma^{-1}\mathbb{E}[|Z|^4]\right)^{-2}\right).$$

Proposition 3.8.5. *Let*

$$\widehat{\sigma}^{(n)}_{space} := \left(\frac{2\alpha \sum_{i=0}^{n-1} |\sigma u_\alpha(t, x_{i+1}) - \sigma u_\alpha(t, x_i))|^2}{\mathbb{E}[|Z|^2](y_2 - y_1)} \right)^{1/2}$$

be the estimator of σ over a path in space $x \in [y_1, y_2]$ of the solution $\sigma u_\alpha(t, x)$ of equation 3.1. Then as $n \to \infty$, the following limit converges in distribution

$$\sqrt{n}(\widehat{\sigma}^{(n)}_{space} - \sigma) \to N\left(0, \frac{1}{4}\rho^2_{\frac{1}{2},2}\left(\sigma^{-1}\mathbb{E}[|Z|^2]\right)^{-2}\right).$$

3.8.2 Joint estimation of drift and diffusion

Noting that the two estimators estimate the different ratios $\frac{\sigma^4}{\alpha}$ and $\frac{\sigma^2}{\alpha}$ in time and space respectively gives that both σ and α can be estimated from one stochastic field [Cialenco and Huang, 2019, Section 5].

Let V_s be the exact 2-variation of sample paths of the solution in space given in equation 3.17, then

$$V_s = \frac{\sigma^2\mathbb{E}\left[|Z|^2\right](y_2 - y_1)}{\alpha 2} \qquad \Longleftrightarrow \qquad \frac{\sigma^2}{\alpha} = \frac{2V_s}{\mathbb{E}\left[|Z|^2\right](y_2 - y_1)}.$$

Let V_t be the 4-variation of sample paths to the solution in time given in equation 3.16, then plug in the equation above,

$$V_t = \frac{\sigma^2}{\alpha}\frac{\sigma^2\mathbb{E}\left[|Z|^4\right](s_2 - s_1)}{\pi} = \frac{2V_s\sigma^2\mathbb{E}\left[|Z|^4\right](s_2 - s_1)}{\pi\mathbb{E}\left[|Z|^2\right](y_2 - y_1)}.$$

Again equation 3.17 gives,

$$\frac{\sigma^4}{\alpha} = \frac{\alpha 4V_s^2}{\mathbb{E}\left[|Z|^2\right]^2(y_2 - y_1)^2}.$$

Plugging this ratio into equation 3.16,

$$V_t = \frac{\sigma^4}{\alpha} \frac{\mathbb{E}\left[|Z|^4\right](s_2 - s_1)}{\pi} = \frac{\alpha 4 V_s^2 \mathbb{E}\left[|Z|^4\right](s_2 - s_1)}{\mathbb{E}\left[|Z|^2\right]^2 (y_2 - y_1)^2 \pi}.$$

Algebraic operations gives the two estimators,

$$\hat{\alpha} = \frac{V_t \pi \mathbb{E}\left[|Z|^2\right]^2 (y_2 - y_1)^2}{4 V_s^2 \mathbb{E}\left[|Z|^4\right](s_2 - s_1)}, \quad \text{and} \quad \hat{\sigma}^2 = \frac{V_t \pi \mathbb{E}\left[|Z|^2\right](y_2 - y_1)}{2 V_s \mathbb{E}\left[|Z|^4\right](s_2 - s_1)}. \tag{3.19}$$

According to [Cialenco and Huang, 2019, Section 5] the asymptotic normality of the above estimators remains an open problem. Using finite difference schemes we will simulate a field of our solution in chapter 4 and show that asymptotic normality seems reasonable.

Chapter 4

Simulations of the Stochastic Heat Equation

We will employ two ways to simulate solutions to the stochastic heat equation. Since we have studied the covariance structure in detail, we have been able to find the explicit mean and covariance in time and space separately for the SHE on the entire real line. Therefore, the first method is to utilise that we know how the solution is distributed.

We can use the well known procedure of factorising the covariance matrix $\mathbf{C}$ (with for example a Cholesky decomposition) such that $\mathbf{R}\mathbf{R}^\intercal = \mathbf{C}$. Then $\mathbf{u} = \mathbf{R}\mathbf{z}$, is a way to simulate the solution, where $\mathbf{z} \in N(\mathbf{0}, \mathbf{I})$ are i.i.d normal variables. The draw-back of this method is the computational cost. One would be tempted to simulate a covariance matrix in time and space together (since we have calculated the full covariance for all pairs (t_1, x_1) and (t_2, x_2)). The problem is that if we want to simulate, say, 100 time points and 100 space points, then the covariance matrix $\mathbf{C}$ would be of the huge size $(100 \cdot 100 \times 100 \cdot 100)$. Only simulating the paths $t \mapsto u(t, x)$ and $x \mapsto u(t, x)$ separately is much less demanding. Therefore we can use the paths in time and space separately to calculate the variations and in turn estimate the drift and diffusion.

The second method of simulating consists of utilising the one-step Θ finite difference schemes, which is much more computationally effective if we aim to simulate the entire field of the solution to the stochastic heat equation. These simulations can in turn be employed for the joint estimation of α and σ explained in the previous chapter. The finite-difference schemes are sadly restricted to the SHE on bounded intervals, but the localisation error presented in Theorem 2.3.8 suggests that these approximations will also represent the solution on the entire real line, as long as we make the interval large enough and consider spatial points far away from the boundary. Luckily we also have many shared properties for both the solution on bounded and unbounded domain that we can use. One of these properties is that the exact q-variations agree noted in remark 3.0.1.

We once again state the equations we are considering for the coming simulations. We have the SHE driven by white noise on a compact domain $\mathscr{D} = [a, b]$,

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \alpha\Delta u(t,x) = \sigma\dot{W}(t,x) & t > 0,\ x \in (a,b) \\ u(0,x) = u_0(x) & x \in [a,b] \\ u(t,a) = u(t,b) = 0 & t > 0. \end{cases} \tag{4.1}$$

As well as the corresponding equation for $x \in \mathbb{R}$ (without boundary conditions),

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \alpha\Delta u(t,x) = \sigma\dot{W}(t,x) & t > 0,\ x \in \mathbb{R} \\ u(0,x) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{4.2}$$

4.1 Simulations using the distribution

Recall for $\alpha = \sigma = 1$, the solution $u(t,x)$ to equation 4.2 above has the covariance structure,

$$\begin{aligned} \mathbb{E}\big[u(t_1,x_1)u(t_2,x_2)\big] &= \frac{1}{2\sqrt{\pi}}\left(\sqrt{t_1+t_2}\,e^{-\frac{|x_1-x_2|^2}{4(t_1+t_2)}} - \sqrt{|t_1-t_2|}\,e^{-\frac{|x_1-x_2|^2}{4|t_1-t_2|}}\right) \\ &\quad + \frac{(x_1-x_2)}{4}\left(\operatorname{erf}\Big(\frac{x_1-x_2}{2\sqrt{t_1+t_2}}\Big) - \operatorname{erf}\Big(\frac{x_1-x_2}{2\sqrt{|t_1-t_2|}}\Big)\right). \end{aligned}$$

The solution $\sigma u_\alpha(t,x)$ with general drift and diffusion, by Proposition 3.3.1, has the covariance function

$$\begin{aligned} \mathbb{E}\big[\sigma u_\alpha(t_1,x_1)\sigma u_\alpha(t_2,x_2)\big] &= \frac{\sigma^2}{2\sqrt{\alpha\pi}}\left(\sqrt{t_1+t_2}\,e^{-\frac{|x_1-x_2|^2}{4\alpha(t_1+t_2)}} - \sqrt{|t_1-t_2|}\,e^{-\frac{|x_1-x_2|^2}{4\alpha|t_1-t_2|}}\right) \\ &\quad + \frac{\sigma^2(x_1-x_2)}{4\alpha}\left(\operatorname{erf}\Big(\frac{x_1-x_2}{2\sqrt{\alpha(t_1+t_2)}}\Big) - \operatorname{erf}\Big(\frac{x_1-x_2}{2\sqrt{\alpha|t_1-t_2|}}\Big)\right). \end{aligned}$$

Considering the path in time, $t \mapsto u(t,x)$, for a fixed $x \in \mathbb{R}$, let $x_1 = x_2 = x$, we create the covariance matrix for the paths in time $\mathbf{C}_{ij}^{\text{time}} = \mathbb{E}\big[\sigma u_\alpha(t_i,x)\sigma u_\alpha(t_j,x)\big]$. Similarly by fixing $t > 0$, and $t_1 = t_2 = t$ we get the covariance matrix $\mathbf{C}_{ij}^{\text{space}} = \mathbb{E}\big[\sigma u_\alpha(t,x_i)\sigma u_\alpha(t,x_j)\big]$ for the paths in space, $x \mapsto u(t,x)$.

4.1.1 Paths of the solution

The covariances where calculated for $N = 2000$ uniformly spaced points in the interval $[0, 1]$ for both space and time respectively. An illustration of the simulated paths is found in Figure 4.1 below.

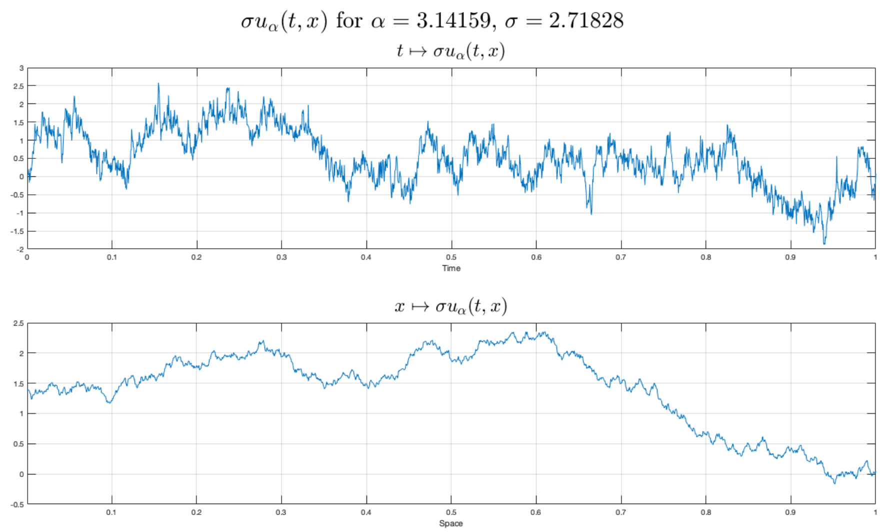


Figure 4.1: Simulated paths $t \mapsto \sigma u_\alpha(t,x)$ and $x \mapsto \sigma u_\alpha(t,x)$ of the solution to equation 4.2 with $\alpha = \pi, \sigma = e$. Simulated with covariance matrix for the paths.

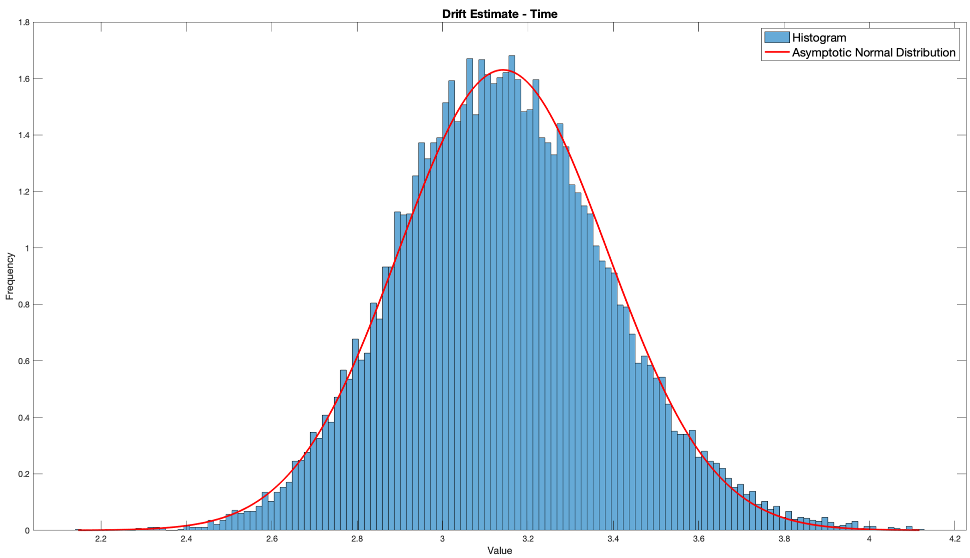
4.1.2 Estimations of drift and diffusion

We can check the normality assumption of employing the estimators $\widehat{\alpha}^{(N)}_{\text{time}}$,$\widehat{\alpha}^{(N)}_{\text{space}}$, $\widehat{\sigma}^{(N)}_{\text{time}}$, and $\widehat{\sigma}^{(N)}_{\text{space}}$, that were defined in the previous chapter, from propositions 3.8.2, 3.8.3, 3.8.4, and 3.8.5. We made $K = 20000$ independent simulated paths in time and space for the solution σu_α with $\sigma = e$ and $\alpha = \pi$ ($N = 2000$ points). The results of these estimations can be found in Table 4.1.

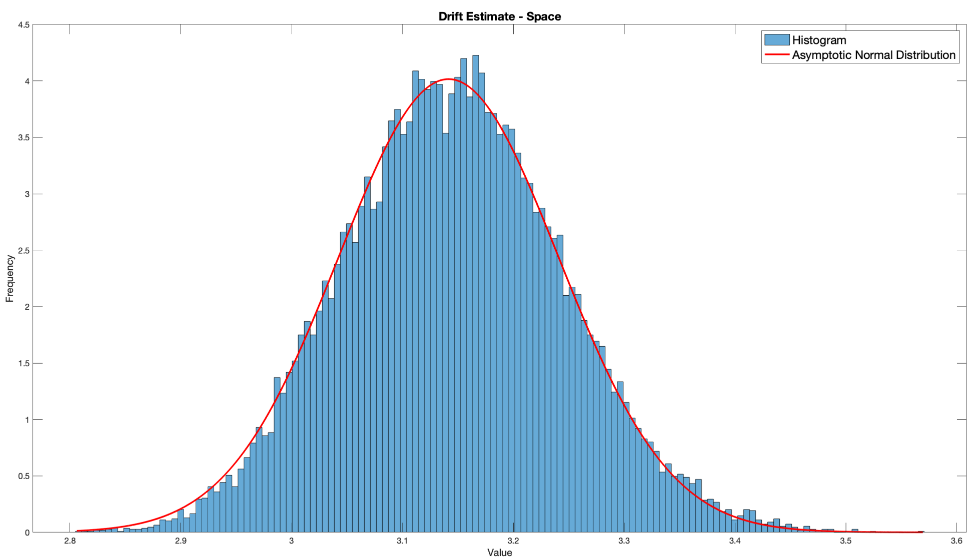
Table 4.1: Estimated mean and variance along with true asymptotic variance of estimators. $N = 2000$ points for simulated path in time $[0, 1]$ and space $[0, 1]$ with $K = 20000$ independent paths respectively. The true variance is calculated from the central limit type convergence given in propositions 3.8.2, 3.8.3, 3.8.4, and 3.8.5, with a scaling factor of $1/N$ to compare it to the estimated variance.

Parameter	Path	Est. Mean	True Mean	Est. Variance	True Variance
Drift Estimate $\hat{\alpha}$	Time	3.1424	3.1416	0.0621	0.0599
Diffusion Estimate $\hat{\sigma}$	Time	2.7208	2.7183	0.0029	0.0028
Drift Estimate $\hat{\alpha}$	Space	3.1462	3.1416	0.0098	0.0099
Diffusion Estimate $\hat{\sigma}$	Space	2.7173	2.7183	0.0018	0.0018

All variances seem to agree. We can even see this in a histogram plot of our simulations found in figures 4.2 and 4.3 below.

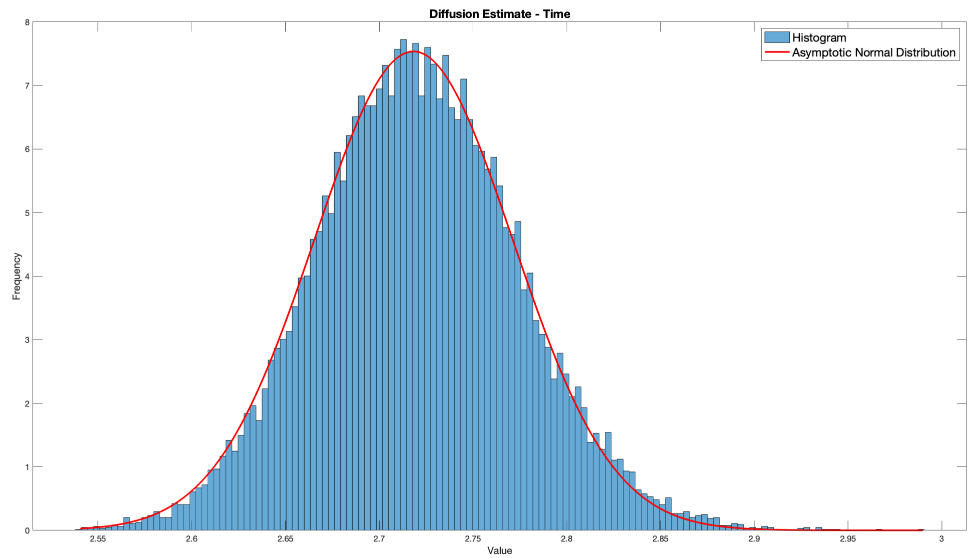


(a) Drift Estimate - Time

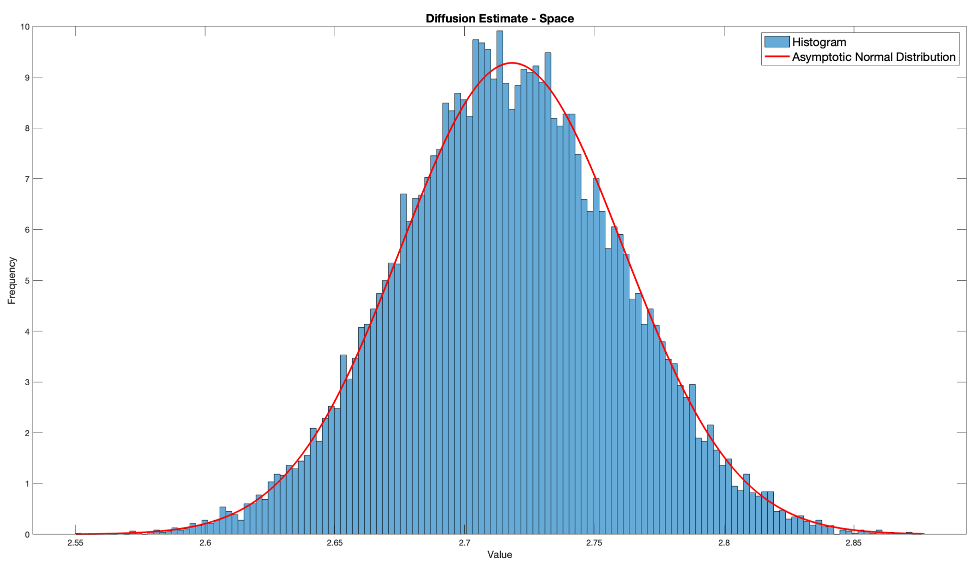


(b) Drift Estimate - Space

Figure 4.2: Asymptotic normality check for drift estimators $\widehat{\alpha}^{(N)}_{\text{time}}$, and $\widehat{\alpha}^{(N)}_{\text{space}}$ of α from the simulated paths $t \mapsto \sigma u_\alpha(t,x)$ and $x \mapsto \sigma u_\alpha(t,x)$ of the solution to equation 4.2 with $\alpha = \pi, \sigma = e$. Simulated in total $K = 20000$ different paths in space and time respectively, with $N = 2000$ points simulated. The normal distributions given by the red lines are created with mean being the true drift and variance from the central limit type convergences given in propositions 3.8.2 and 3.8.3, with a scaling factor of $1/N$ to compare it to the estimated variance.



(a) Diffusion Estimate - Time



(b) Diffusion Estimate - Space

Figure 4.3: Asymptotic normality check for diffusion estimators $\widehat{\sigma}_{\text{time}}^{(N)}$, and $\widehat{\sigma}_{\text{space}}^{(N)}$ of σ from the simulated paths $t \mapsto \sigma u_\alpha(t,x)$ and $x \mapsto \sigma u_\alpha(t,x)$ of the solution to equation 4.2 with $\alpha = \pi, \sigma = e$. Simulated in total $K = 20000$ different paths in space and time respectively, with $N = 2000$ points simulated. The normal distributions given by the red lines are created with mean being the true diffusion and variance from the central limit type convergences given in propositions 3.8.4 and 3.8.5, with a scaling factor of $1/N$ to compare it to the estimated variance.

4.2 One step Θ finite-difference schemes

We will consider the solution on the space $[0, T] \times [a, b]$, with $T > 0$ and $a, b \in \mathbb{R}$. We discretize the time $[0, T]$ and space $[a, b]$ into rectangles $A_j^m = [x_{j-1}, x_j] \times [t_{m-1}, t_m]$, with $j = 1, 2, \ldots, N+1$, and $m = 1, 2, 3, \ldots, M$. The desired number of points of the solution in space and time being $N+2$ and M respectively, and $\Delta t = T/M$, $\Delta x = \frac{b-a}{N+1}$, such that $x_j = a + j\Delta x$ and $t_m = m\Delta t$. We let x_0 and x_{N+1} be the boundaries $[a, b]$ with known values, and $t_0 = 0$ being the initial time where we have an initial condition.

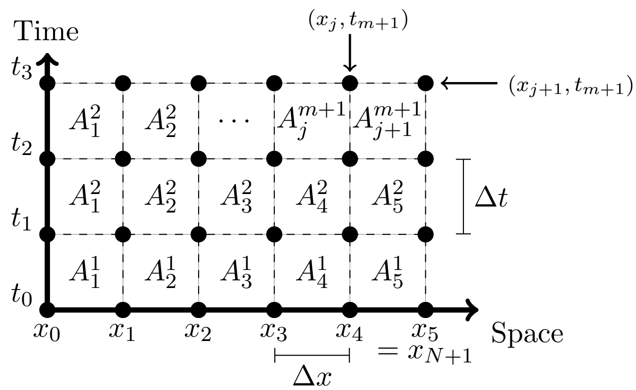


Figure 4.4: Illustration of the grid scheme

We discretize equation 4.1 in the following way,

$$\begin{cases} \frac{u_j^{m+1} - u_j^m}{\Delta t} = \alpha\left(\Theta\frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{\Delta x^2} + (1-\Theta)\frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{\Delta x^2}\right) + \sigma\widehat{\frac{\partial W}{\partial t \partial x}}(t_m, x_j) \\ u_0^m = u_{N+1}^m = 0, \ m = 0, 1, \ldots. \\ u_j^0 = u_0(x_j), \qquad j = 1, 2, \ldots, N. \end{cases} \tag{4.3}$$

Note that the time points $m = 1, 2, \ldots$ continue on forever. In our simulations however we will look at some fixed number of time points M.

We will see how $\frac{W_j^m}{\Delta t \Delta x} := \widehat{\frac{\partial W}{\partial t \partial x}}(t_m, x_j)$, where $W_j^m \in N(0, \Delta t \Delta x)$, is a suitable discretization of the white noise for equation 4.3 above.

Rewriting the equations, with $r_1 = \frac{\Theta\alpha\Delta t}{\Delta x^2}$ and $r_2 = \frac{(1-\Theta)\alpha\Delta t}{\Delta x^2}$ we have for every $j = 1, 2, \ldots, N$ and $m = 0, 1, \ldots, M$.

$$\begin{aligned} &-r_1 u_{j-1}^{m+1} + (1 + 2r_1)u_j^{m+1} - r_1 u_{j+1}^{m+1} \\ &= -r_2 u_{j-1}^m + (1 - 2r_2)u_j^m - r_2 u_{j+1}^m + \frac{W_j^m}{\Delta x}. \end{aligned} \tag{4.4}$$

Let $\mathbf{U}^m = \left(u_1^m, u_2^m, \dots, u_N^m\right)$ and $\mathbf{W}_m = \left(W_1^m, \dots, W_N^m\right)$. We can also write the system 4.4 in matrix form.

$$(\mathbf{I} + r_1\mathbf{A})\mathbf{U}^{m+1} = (\mathbf{I} - r_2\mathbf{A})\mathbf{U}^m + \frac{\mathbf{W}_m}{\Delta x}.$$

With $\mathbf{A}$ being the $N \times N$ tridiagonal matrix,

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \dots & -1 & 2 \end{bmatrix}.$$

4.2.1 Approximating the noise

First our goal is to find a suitable discretisation of the noise process W. The approximations presented here are the ones that are used for the papers in the literature study below.

Employing the following approximation of the mixed derivative $\frac{\partial W(t,x)}{\partial x \partial t}$. The white noise W at point (x_j, t_m) can be approximated as

$$W_j^m := \int_{t_{m-1}}^{t_m} \int_{x_{j-1}}^{x_j} dW(t,x) = W(A_j^m).$$

And we have that

$$W_j^m \in N(0, \Delta t \Delta x),$$

are i.i.d for every $j = 1, 2, \dots, N$ and $m = 1, 2, \dots, M$. Assume for a moment that W is a continuously differentiable function (it is certainly not). Taking inspiration from properties of the Riemann-Stiltjes integral and the mean value theorem, we would find that,

$$\begin{aligned} \int_{t_{m-1}}^{t_m} \int_{x_{j-1}}^{x_j} dW(t,x) &= \int_{t_{m-1}}^{t_m} \int_{x_{j-1}}^{x_j} \frac{\partial W}{\partial x \partial t}(t,x) dx dt \\ &= \Delta x \Delta t \frac{\partial W}{\partial x \partial t}(\xi, \eta). \end{aligned}$$

For some $(\xi, \eta) \in [x_{j-1}, x_j] \times [t_{m-1}, t_m]$. Since the intervals are small, a reasonable approximation for the white noise is therefore

$$\frac{\partial W}{\partial x \partial t}(x_j, t_m) \approx \frac{1}{\Delta x \Delta t} \int_{t_{m-1}}^{t_m} \int_{x_{j-1}}^{x_j} dW(t,x) = \frac{1}{\Delta x \Delta t} W_j^m.$$

4.2.2 Known results on finite-difference schemes for the stochastic heat equation

Finite difference schemes with a one-step Θ method on the solutions to SPDEs is often studied by looking at the *error*: $\mathbb{E}\left[|u_j^m - u(t_m, t_j)|^p\right]$, for some $p \geq 2$. The convergence of the error has been widely studied. It started with Gaines work in [Gaines, 1995], which contained numerical experiments indicating convergence of finite-difference approximations to solutions of the stochastic heat equation driven by white noise. Gyöngy, I. and Nualart, D. in [Gyöngy and Nualart, 1995] showed that the Euler finite-difference schemes for SPDEs with space-time white noise actually converge. Davie and Gaines in [Davie and Gaines, 2001] discovered a universal lower bound for the error of numerical schemes applied to parabolic SPDEs. They found that regardless of whether the scheme is implicit or explicit, the error of the scheme, in terms of space step (Δx) and time step (Δt), will be at least on the order of $O(\Delta x^{1/2} + \Delta t^{1/4})$ (note the exponents and compare it to the Hölder continuity in space and time respectively). This lower bound matches the one proposed by Gyöngi, indicating that even the simple Euler scheme achieves the optimal rate of convergence.

All the mentioned papers demonstrated convergence, but it's important to verify if the numerical approximation accurately reflect the true solution. This was precisely the focus of the paper by Walsh and Chong in [Yuxiang and B, 2012]. They show that the quadratic and quartic variations of the simulations converge to a function of Θ and $\Delta t/(\Delta x)^2$. However, since the true values are unique, it's evident that most schemes do not achieve the correct limit. It is perhaps surprising, but there is only one scheme that produces the correct limiting quadratic and quartic variations. It is the Crank-Nicolson scheme ($\Theta = 1/2$) with CFL number $\alpha \Delta t/\Delta x^2 = \frac{1}{\pi - 2}$.

4.2.3 Variations for Simulations of SHE with white noise

The simulation u_j^m, for $j = 0, 1, 2, \ldots, N+1$ (space) and $m = 0, 1 \ldots, M$ (time) is an approximation of the solution $u(t_m, x_j)$ for t_m and x_j in the grid (see Figure 4.4 of the grid). Like in chapter 3 we can define the corresponding q-variations of the numerical approximation u_j^m.

Definition 4.2.1 (Variations for the numerical approximation.)**.** *Assume u_j^m is the approximation for $j = 1, 2, \ldots, N$ and $m = 1, 2, \ldots$. Let $T > 0$ and $\lfloor T \rfloor$ be the greatest integer less than T. Assume we have approximated time up to time step number $M := \lfloor T \rfloor / \Delta t$. Then*

$$Q_N^{(2)}(T) := \sum_{i=0}^{N} \left| u_{i+1}^M - u_i^M \right|^2 .$$

Let y be a lattice point, i.e. a rational number such that $y = j/(N+1)$ for some

$j = 1, 2, \ldots, N$. *Then*

$$Q_N^{(4)}(y,T) := \sum_{m=0}^{M-1} \left| u_j^{m+1} - u_j^m \right|^4 .$$

The idea is that the q-variations of the simulated solution u_j^m will be the limits (if they exist)

$$Q^{(2)}(T) := \lim_{N\to\infty} Q_N^{(2)}(T).$$

The quartic limit would be defined analogously,

$$Q^{(4)}(y,T) := \lim_{N\to\infty} Q_N^{(4)}(y,T).$$

With the note that there are infinitely many N such that $y = j/(N+1)$ for some j, so the quartic limit is inevitably along such a subsequence. Although, for the quartic variation, we first need to have

$$Q^{(4)}(y,T,\delta) := \lim_{N\to\infty} Q_N^{(4)}(y,T) - Q_N^{(4)}(y,\delta). \tag{4.5}$$

And then

$$Q^{(4)}(y,T) := \lim_{\delta\to 0} Q^{(4)}(y,T,\delta).$$

This is because the numerical approximation fluctuates excessively near $t = 0$ for some schemes, so this corrects it. A reason for this instability could be because of the correlation of two nearby points in time around zero. Recall the covariance structure over two time points t and $t+h$ for a fixed $x_1 = x_2 = y$ is $\frac{1}{2\sqrt{\pi}}\sqrt{2t+h} - \sqrt{|h|}$. As $t \to 0$ and h sufficiently small we have that the derivative of the covariance approaches infinity. This could suggest these fluctuations in the scheme.

The important object of study for this section is the CFL number,

$$c = \frac{\alpha \Delta t}{(\Delta x)^2}.$$

Let the space domain be $[a,b]$ with boundary conditions $u(t,a) = u(t,b) = 0$, and also assume that $u_0 = 0$. The higher order variations of the simulations actually don't depend on the initial condition but it takes some work to show. For a proof of this see [Yuxiang and B, 2012, Section 8].

Remark 4.2.2. *These are some important points for the discussion of convergence of variation for the numerical scheme,*

1. *We will heavily study the CFL number,* $c_N = \frac{\alpha\Delta t}{(\Delta x)^2}$, *where the subscript* N *denotes the dependence of* N *in* $\Delta x = \frac{b-a}{N+1}$. *The time step* Δt *is then* $\frac{(\Delta x)^2 c_N}{\alpha}$ *as a consequence of the choice of sequence* (c_N).

2. *The convergence of the variations are with respect to N.*
3. *We will see that the limiting value (as $N \to \infty$) of the variation for the numerical approximations depends on Θ, T, and $c_\infty := \lim_{N\to\infty} c_N$.*
4. *Spoiler, The only scheme that converges in both variations will be the Crank-Nicolson ($\Theta = 1/2$) with $c_\infty = \frac{1}{\pi-2}$.*
5. ***Although (important)**: According to [Yuxiang and B, 2012] it is an open problem if the conditions we will set on c_N in definition 4.2.3 below actually suffice for convergence of solution. They only show converge of variation. It will converge if $c_\infty(1-\Theta) \leq 1/4$ ([Gyöngy and Nualart, 1995]), so it is not certain if the approximation will converge to the correct solution when $c_\infty = \frac{1}{\pi-2}$. Although it is conjectured that they will.*

Definition 4.2.3. *The sequence (c_N) is said to be a* good *sequence, if*

1. $c_N > 0$;
2. *if $0 \leq \Theta < 1/2$ then there exists an $\epsilon_\Theta > 0$ such that $c_N \leq \frac{1}{2-4\Theta} - \epsilon_\Theta := c_\Theta$;*
3. *if $\Theta = 1/2$, then $c_N \leq \sqrt{N}$;*
4. *if $1/2 < \Theta \leq 1$, then $c_N \leq N$;*

The following two theorems 4.2.4 and 4.2.5 are the main results stating what the limiting variations are for the approximations to the stochastic heat equation. The theorems are only stated here, the proofs can be found in [Yuxiang and B, 2012, Section 5.1].

Theorem 4.2.4 (Convergence of Quadratic Variation)**.** *Let $0 \leq \Theta \leq 1$ and $T > 0$. Assume (c_N) is a good sequence with $c_N \to c_\infty \in [0,\infty]$ (extended). Then the following limit exists in probability*

$$Q^{(2)}(T) := \lim_{N\to\infty} Q_N^{(2)}(T) = \frac{\sigma^2(b-a)}{2\alpha\sqrt{1+2c_\infty(2\Theta-1)}}.$$

Theorem 4.2.5 (Convergence of Quartic Variation)**.** *Let $0 \leq \Theta \leq 1$, $t > 0$ and y a rational number between a and b. Assume (c_N) is a good sequence with $c_N \to c_\infty \in [0,\infty]$ (extended). For $\Theta = 1/2$ suppose also that $c_N/\sqrt{N} \to 0$ and for $\Theta > 1/2$ that $c_N/N^{3/2} \to 0$. Then the following limits exist in probability*

$$\begin{aligned} Q^{(4)}(y,T) &:= \lim_{d\to\infty}\left(\lim_{N\to\infty} Q_N^{(4)}(y,T) - Q_N^{(4)}(y,1/d)\right) \\ &= \frac{3c_\infty\sigma^4 T}{\alpha}\left(\frac{1-2\Theta}{\sqrt{1+2c_\infty(2\Theta-1)}} + \frac{2\Theta}{\sqrt{1+4c_\infty\Theta}}\right)^2. \end{aligned} \tag{4.6}$$

For $\Theta > 1/2$, if $c_\infty = \infty$ then 4.6 is seen as a limit, such that

$$\lim_{c\to\infty} \frac{3cT}{\alpha}\left(\frac{1-2\Theta}{\sqrt{1+2c(2\Theta-1)}} + \frac{2\Theta}{\sqrt{1+4c\Theta}}\right)^2 = \frac{3T}{\alpha}\left(\sqrt{\Theta} - \sqrt{\Theta - 1/2}\right)^2.$$

4.2.4 True variations

We restate the true variations of the solution to the stochastic heat equation 4.2 (solution on the real line), they will agree with the variations for the solution on bounded space of equation 4.1 (see e.g [Cialenco and Huang, 2019]).

$$V^2_{[a,b]}\left[\sigma u_\alpha(t,x)\right] = \frac{\sigma^2(b-a)}{2\alpha}, \quad \text{and} \quad V^4_{[0,T]}\left[\sigma u_\alpha(t,x)\right] = \frac{3T\sigma^4}{\alpha\pi}. \tag{4.7}$$

Now we aim to find for which c_∞ that the variations in theorems 4.2.4 and 4.2.5 match the variations in 4.7.

For the quadratic variation we can see that $Q^{(2)}(T) = \frac{\sigma^2(b-a)}{2\alpha\sqrt{1+2c_\infty(2\Theta-1)}} = \frac{\sigma^2(b-a)}{2\alpha} = V^2_{[a,b]}$ if and only if $c_\infty(2\Theta-1) = 0$. The approximation's quadratic variation $Q^{(2)}$ will therefore agree with the true quadratic variation V^2 if we employ the Crank-Nicolson scheme ($\Theta = 1/2$), or we let $c_\infty = \lim_{n\to\infty} c_N = 0$.

If $c_\infty = 0$, the quartic variation vanishes; $Q^{(4)} = 0$. Assuming that $\Theta = 1/2$, the limiting quartic variations will match if

$$\begin{aligned} Q^4(y,T) &= \frac{3c_\infty\sigma^4 T}{\alpha}\left(\frac{1-2\Theta}{\sqrt{1+2c_\infty(2\Theta-1)}} + \frac{2\Theta}{\sqrt{1+4c_\infty\Theta}}\right)^2 \\ &= \frac{3c_\infty\sigma^4 T}{\alpha(1+2c_\infty)} = V^4_{[0,T]} = \frac{3T\sigma^4}{\alpha\pi} \iff c_\infty = \frac{1}{\pi-2}. \end{aligned}$$

The **only** finite difference scheme that provides both the correct quadratic- and quartic variation that match 4.7 is therefore the Crank-Nicolson scheme ($\Theta = 1/2$) with $c_\infty = \frac{\alpha\Delta t}{\Delta x^2} = \frac{1}{\pi-2}$.

4.2.5 Simulations

First we make an approximation of the stochastic heat equation 4.1 with $[a,b] = [0,1]$, $u_0 \equiv 0$, and $\alpha = \sigma = 1$. using a Crank-Nicolson scheme ($\Theta = 1/2$) as well as

$\Theta = 0.25$ and $\Theta = 1$, we have $M = 5000$ time points and $N = 4999$ space points with $c = \frac{1}{\pi-2} = \frac{\Delta t}{(\Delta x)^2}$, such that $\Delta t = \frac{(\Delta x)^2}{\pi-2}$ and the final stopping time is $T = M\Delta t$.

The computed values of the variations and their theoretical values from theorems 4.2.4 and 4.2.5 can be seen in Table 4.2 below. The true variations for the solution $u(t,x)$ to the stochastic heat equation 4.2 with drift and diffusion $\alpha = \sigma = 1$ and space domain $[0,1]$ are

$$V^2_{[0,1]}\left[u(t,x)\right] = 0.5, \quad \text{and} \quad V^4_{[0,T]}\left[u(t,x)\right] = 0.0008. \tag{4.8}$$

Table 4.2: Comparison of theoretical values $Q^{(2)}(T)$ and $Q^{(4)}(y,T)$ (given in theorems 4.2.4 and 4.2.5) for the higher order variations of simulated solutions in limit $N \to \infty$, and estimated values $Q_N^{(2)}(T)$ and $Q_N^{(4)}(y,T)$ of quadratic and quartic variation for simulations using $M = 5000$ points in time, and $N = 4999$ points in space. $\Delta t/(\Delta x)^2 = \frac{1}{\pi-2}$. With $T = M\Delta t$ and $y = \frac{N\Delta x}{2}$.

Θ	$Q_N^{(2)}(T)$	$Q^{(2)}(T)$ (Theoretical)	$Q_N^{(4)}(y,T)$ (Est.)	$Q^{(4)}(y,T)$ (Theoretical)
0.25	1.4031	1.4197	0.0054	0.0073
0.5	0.4914	0.5000	0.0007	0.0008
1	0.2899	0.3014	0.0003	0.0003

We can actually see how the different Θ-schemes smooth the solution, which will reflect on the variations. The paths that created the values in Table 4.2 above are seen in figures 4.5, 4.6 (for a path in time), 4.7, and 4.8 (for a path in space) below. Note once again that if $c_\infty = \frac{1}{\pi-2}$ and $\Theta = 1$ the condition that $c_\infty(1-\Theta) \leq \frac{1}{4}$ holds and hence the schemes will converge to the correct solution. However, we can not guarantee convergence to the correct solution when $\Theta = 0.25$ and $\Theta = 0.5$ since this is still an open problem. Although, it should be noted that the simulated paths seem to coincide except for the variation in the figures below.

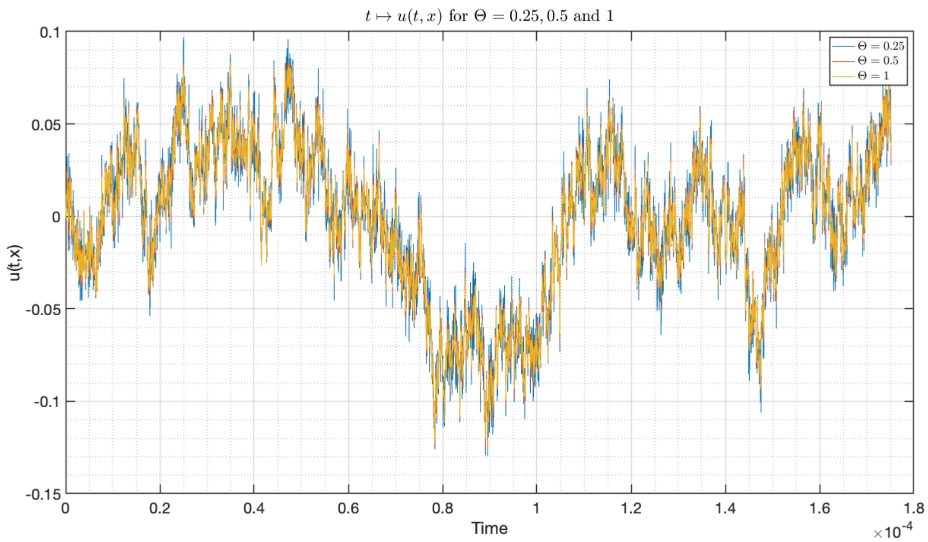


Figure 4.5: Path of solution using $\Theta = 0.25, 0.5$, and $\Theta = 1$ over the time variable at $x = 0.5$.

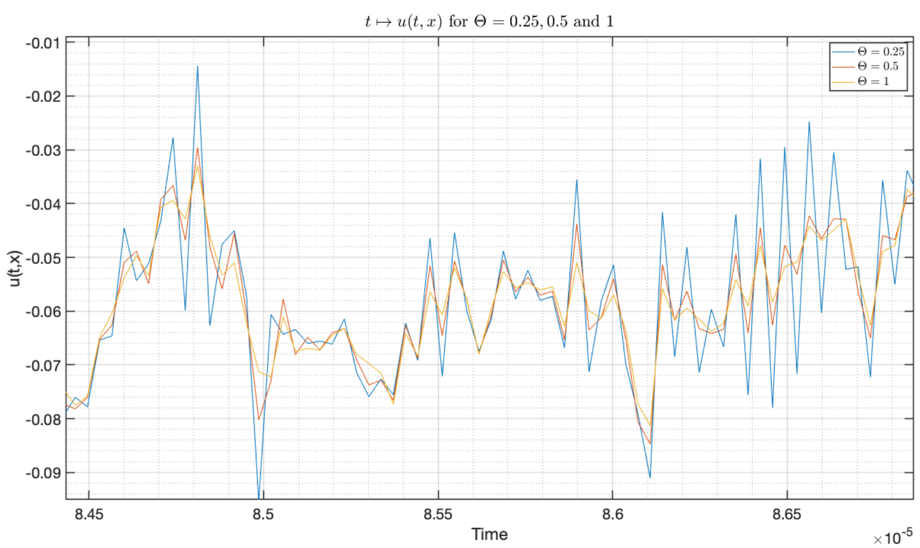


Figure 4.6: Path of solution using $\Theta = 0.25, 0.5$, and $\Theta = 1$ over the time variable at $x = 0.5$. Zoomed in to show the difference in amplitudes.

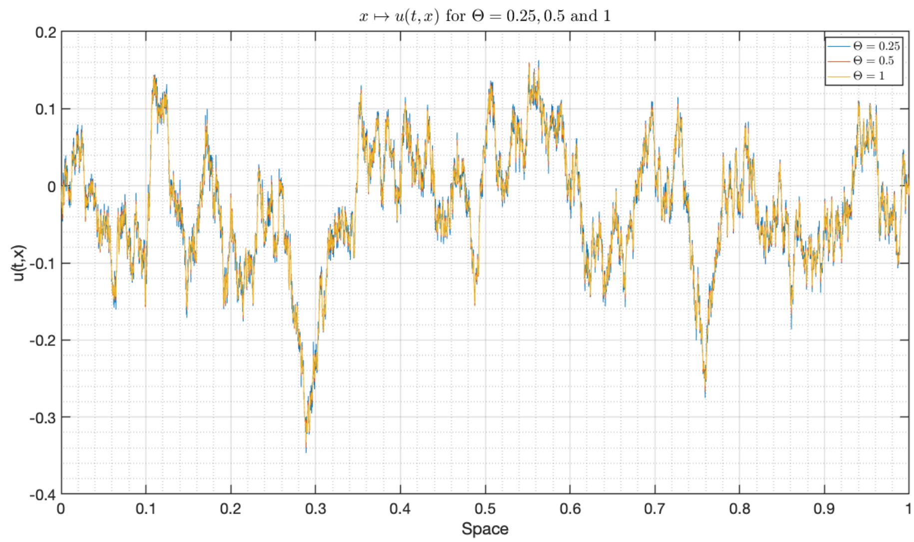


Figure 4.7: Path of solution using $\Theta = 0.25, 0.5$, and $\Theta = 1$ over the space variable at $T = M\Delta t$.

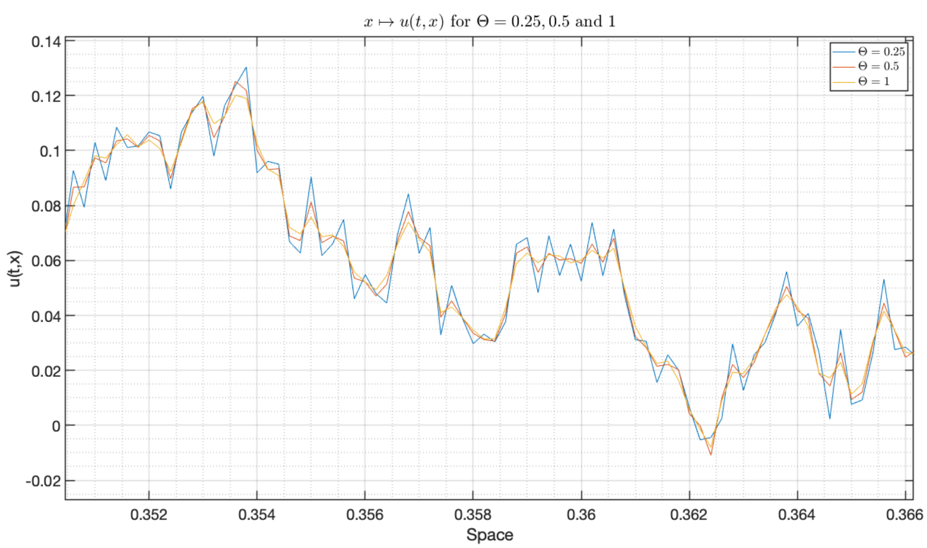


Figure 4.8: Path of solution using $\Theta = 0.25, 0.5$, and $\Theta = 1$ over the space variable at $T = M\Delta t$. Zoomed in to show the difference in amplitudes.

4.2.5.1 Estimations of drift and diffusion

We made another Crank-Nicolson simulation with $c = \frac{1}{\pi - 2}$ with $M = 1000$, $N = 999$, with $\alpha = \pi$ and $\sigma = e$ and simulated these approximations $K = 1000$ times. See Table 4.3 below for the estimations of drift and diffusion. Note that the true asymptotic variances of the joint estimations seems to still be an open problem (see [Cialenco and Huang, 2019]).

Table 4.3: Estimates of drift $\alpha = \pi$ and diffusion $\sigma = e$ over paths in time, space, and jointly respectively, with discretisation $M = 1000, N = 999$ and $K = 1000$ independent simulations. The true variance is calculated from the central limit type convergence given in propositions 3.8.2, 3.8.3, 3.8.4, and 3.8.5, with a scaling factor of $1/N$ to compare it to the estimated variance.

Parameter	Path	Mean	Est. Variance	True Variance
Drift Estimate $\hat{\alpha}$	Time	3.1778	0.1191	0.1199
Diffusion Estimate $\hat{\sigma}$	Time	2.7155	0.0055	0.0056
Drift Estimate $\hat{\alpha}$	Space	3.1744	0.0212	0.0198
Diffusion Estimate $\hat{\sigma}$	Space	2.7064	0.0039	0.0037
Drift Estimate $\hat{\alpha}$	Joint	3.2152	0.2105	-
Diffusion Estimate $\hat{\sigma}$	Joint	2.7281	0.0260	-

For the joint estimation we have the following histogram of the $K = 1000$ simulations in Figure 4.9.

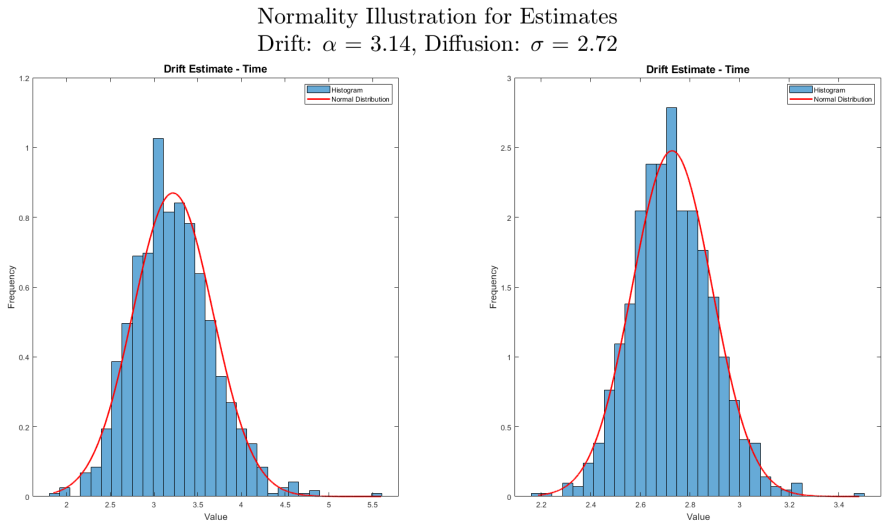


Figure 4.9: Normality check for joint estimators of α and σ from the simulated paths $t \mapsto \sigma u_\alpha(t,x)$ and $x \mapsto \sigma u_\alpha(t,x)$ of the solution to equation 4.2 with $\alpha = \pi, \sigma = e$. Simulated in total $K = 1000$ different paths in space and time respectively, with $N = 999$ points in space and $M = 1000$ points in time simulated.

Chapter 5

A Splash of Colour

We saw in chapter 2 how the stochastic heat equation fails to admit a point-wise solution in $d \geq 2$. This pickle arises from the definition of the stochastic integral, where we recall that $I(h) = \int_{\mathbb{R}\times\mathbb{R}^d} hdW$ is well defined if and only if $h \in L^2(\mathbb{R}\times\mathbb{R}^d)$. In our case this is the function $h(s,y) = \mathbb{1}_{[0,t]}(s)\Psi(t-s,x,y)$ that we need to integrate over that does not lie in this L^2 space. Recall remark 2.2.2 of the white noise definition, where we saw how we acquire an isometry to some Hilbert space depending on the covariance structure of the noise. We finish our thesis with this chapter which aims to show a way to smooth the noise enough to allow solutions in any dimension $d \geq 1$. Our presentation here is heavily inspired by the work of [Dalang, 1999] and [Tudor, 2014].

5.1 White-Coloured noise

Let's restrict ourselves to the measure space $(E,\mu) = \Big(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d), \lambda\Big)$, with λ being the Lebesgue measure and where we use the notation $(t,A) := ([0,t]\times A) \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$. We aim show the construction of a noise process M, and consequently a stochastic integral $\int hdM$ with a bit more regularity, so that we can integrate a larger class of functions $h \in \mathscr{H}$, where $\mathscr{H}$ is this "larger" Hilbert space. The presentation here focuses on $\mathscr{D} = \mathbb{R}^d$, but it is not hard to adapt the discussion for rectangles $\mathscr{D} \subset \mathbb{R}^d$.

For the case of white noise based on λ, using standard results on set algebra and the Lebesgue measure we have that its covariance function $C\Big((t,A),(s,B)\Big)$ can be factorised in a time and space component in the following manner,

$$\begin{aligned} C\Big((t,A),(s,B)\Big) &= \lambda\Big(([0,t]\times A)\cap([0,s]\times B)\Big) = \lambda\Big(([0,t]\cap[0,s])\times(A\cap B)\Big) \\ &= \lambda_{\mathbb{R}}([0,t]\cap[0,s])\cdot\lambda_{\mathbb{R}^d}(A\cap B) = t\wedge s\cdot\lambda_{\mathbb{R}^d}(A\cap B). \end{aligned} \tag{5.1}$$

Where $\lambda_{\mathbb{R}}$ and $\lambda_{\mathbb{R}^d}$ are the Lebesgue measures corresponding to $\mathbb{R}$ and $\mathbb{R}^d$ respectively. Given a smart choice of covariance for the noise, we can extend the stochastic integrals to a larger class of integrands.

One general idea is to establish a spatial parameter to the covariance structure, which is done by changing the spatial factor $\lambda_{\mathbb{R}^d}(A\cap B)$ from calculation 5.1 above. We note that in a generalized sense,

$$\lambda_{\mathbb{R}^d}(A\cap B) = \int_A\int_B \delta(x-y)dxdy.$$

Because we formally have that the derivative of an indicator function is the dirac-δ such that

$$\begin{aligned} \int_A\int_B \delta(x-y)dxdy &= \int_A\left(\int_B \delta(x-y)dx\right)dy \\ = \int_A \mathbb{1}_B(y)dy &= \int_{\mathbb{R}^d}\mathbb{1}_A(y)\mathbb{1}_B(y)dy = \lambda_{\mathbb{R}^d}\left(A\cap B\right). \end{aligned}$$

This motivates the following choice of covariance, which was introduced in Dalang's paper [Dalang, 1999] on the extension of martingale measures in the Walsh sense of SPDEs,

$$C\Big((t,A),(s,B)\Big) = t\wedge s\cdot\int_A\int_B f(x-y)dxdy. \tag{5.2}$$

We can therefore *colour* the spatial component by using an integrable function (or distribution) f as a spatial parameter to the noise.

Let us discuss (a bit informally) the question, for which f is the covariance in 5.2 actually a covariance function? As long as C defined in 5.2 above is non-negative definite and symmetric, we can guarantee by Lemma 2.1.6 that there exists a zero mean Gaussian stochastic process M indexed on the sets $(t,A)\in\mathcal{B}_b(\mathbb{R}_+\times\mathbb{R}^d)$. We note that the factor $t\wedge s$ is itself a covariance function (that of the Brownian motion). Since the product of two covariance functions is once again a covariance function, we only require that the spatial component satisfies, for any $A_1,\dots,A_n\in\mathcal{B}_b(\mathbb{R}^d)$ and $x_1,\dots,x_n\in\mathbb{R}$

$$\sum_{k,l=1}^{n} x_k x_l\left(\int_{A_k}\int_{A_l} f(x-y)dxdy\right)\geq 0. \tag{5.3}$$

Let $g(z) := \sum_{k=1}^n x_k\mathbb{1}_{A_k}(z)\in\mathscr{E}$, where $\mathscr{E}$ is the vector space over $\mathbb{R}$ of simple functions, and denote $\tilde{g}(z) = g(-z)$. Then 5.3 above holds if $\int_{\mathbb{R}^d}(g*\tilde{g})(y)f(y)dy\geq 0$. This can be seen by rewriting equation 5.3 (a bit naively) and utilising convolution formulas,

$$\sum_{k,l=1}^{n} x_k x_l \left(\int_{A_k} \int_{A_l} f(x-y) dx dy \right) = \sum_{k,l=1}^{n} x_k x_l \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{A_k}(x) \mathbb{1}_{A_k}(y) f(x-y) dx dy \right)$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\sum_{k=1}^{n} x_k \mathbb{1}_{A_k}(x) \right) \left(\sum_{l=1}^{n} x_l \mathbb{1}_{A_l}(y) \right) f(x-y) dx dy$$

$$\int_{\mathbb{R}^d} g(x) \left(\int_{\mathbb{R}^d} g(y) f(x-y) dy \right) dx = \int_{\mathbb{R}^d} g(x) \left(\int_{\mathbb{R}^d} g(x-y) f(y) dy \right) dx$$

$$= \int_{\mathbb{R}^d} (g * \tilde{g})(y) f(y) dy.$$

From [Norvidas, 2015, page 19], for any $h \in L^1(\mathbb{R})$, then if f is continuous, $\int_{\mathbb{R}^d}(h * \tilde{h})(y)f(y)dy \geq 0$ is equivalent to that f itself is a non-negative definite function. Since the simple functions $g \in \mathscr{E}$ lie in $L^1(\mathbb{R}^d)$, condition 5.3 is fulfilled for continuous non-negative definite functions.

Bochner's theorem (see e.g. [I.M and N, 1964, Theorem 2, p. 155]) actually characterises all non-negative definite functions as the Fourier transforms of finite non-negative definite measures. We will actually need the concept of tempered measures.

5.1.1 Fourier transforms of tempered measures

Definition 5.1.1. *A non-negative measure ν on $\mathbb{R}^d$ is called a tempered measure if there exists some $k > 0$ such that*

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^k d\nu(\xi) < \infty.$$

We will work Borel measures of the above form since they will guarantee the existence of solutions to our SPDEs with white-coloured noise (more on this later). For any $\phi \in L^1(\mathbb{R}^d)$, we define the Fourier transform

$$\mathcal{F}\phi(\xi) = \widehat{\phi}(\xi) := \int_{\mathbb{R}^d} e^{-i2\pi\xi\cdot y} \phi(y) dy.$$

Let $\mathcal{S}(\mathbb{R}^d)$ be the *Schwartz space*, which consists of the infinitely differentiable functions which are rapidly decreasing as $|x| \to \infty$ together with their derivatives of all orders and let $\mathcal{S}'(\mathbb{R}^d)$ be the corresponding dual containing the distributions on $\mathcal{S}(\mathbb{R}^d)$. (Recall that $\phi(x)$ is rapidly decreasing if $\lim_{|x|\to\infty} |x^k\phi(x)| = 0$ for all k.)

Let f and g be two integrable functions. Then we have the relation

$$\int_{\mathbb{R}^d} \widehat{f}(y)g(y)dy = \int_{\mathbb{R}^d} f(y)\widehat{g}(y)dy.$$

If we have $\phi \in \mathcal{S}$ then every integrable function f induces a distribution $A_f \in \mathcal{S}'$ in the sense that $A_f(\phi) = \langle f, \phi\rangle = \int_{\mathbb{R}^d} f(y)\phi(y)dy$. It is therefore natural to define the Fourier transform on any distribution $A \in \mathcal{S}'(\mathbb{R}^d)$ by $\widehat{A}(\phi) := A(\widehat{\phi})$ for every $\phi \in \mathcal{S}(\mathbb{R}^d)$. A (tempered) measure ν can naturally be seen as a distribution on test functions $\psi \in \mathcal{S}(\mathbb{R}^d)$ as $\psi \mapsto \langle \nu, \psi\rangle := \int_{\mathbb{R}^d} \psi(\xi)d\nu(\xi)$.

Definition 5.1.2 (Fourier transform of a tempered measure)**.** *If $\psi = \widehat{\phi}$ is the Fourier transform of ϕ, then f is called the Fourier transform of the tempered measure ν if*

$$\int_{\mathbb{R}^d} f(x)\phi(x)dx = \int_{\mathbb{R}^d} \widehat{\phi}(\xi)d\nu(\xi) \tag{5.4}$$

holds for all functions $\phi \in \mathcal{S}(\mathbb{R}^d)$.

Remark 5.1.3. *For the proof of one implication in the Bochner theorem we will actually need the equivalent definition of the Fourier transform of a measure in the sense of*

$$f(x) = \hat{\nu} := \int_{\mathbb{R}^d} e^{-i2\pi x\cdot\xi} d\nu(\xi).$$

An important fact that will be used in several calculations is the following.

Proposition 5.1.4. *For any $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$ it holds that*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(x)\phi(y)f(x-y)dxdy = \int_{\mathbb{R}^d} \widehat{\psi}(\xi)\widehat{\phi}^*(\xi)d\nu(\xi)$$

Where z^* denotes the complex conjugate of $z \in \mathbb{C}$.

Proof. Note $\tilde{g}(x) := g(-x)$. We have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(x)\phi(y)f(x-y)dxdy = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \psi(x)\phi(y)f(x-y)dy\right)dx =$$
$$\int_{\mathbb{R}^d} \psi(x)\left(\int_{\mathbb{R}^d} \phi(y)f(x-y)dy\right)dx = \int_{\mathbb{R}^d} \psi(x)\left(\int_{\mathbb{R}^d} \phi(x-y)f(y)dy\right)dx$$
$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(x)\phi(x-y)f(y)dxdy = \int_{\mathbb{R}^d} f(y)\left(\int_{\mathbb{R}^d} \psi(x)\phi(-(y-x))dx\right)dy$$
$$= \int_{\mathbb{R}^d} f(y)(\psi * \tilde{\phi})(y)dy = \int_{\mathbb{R}^d} \widehat{\psi}(\xi)\widehat{\phi}^*(\xi)d\nu(\xi).$$

Where the last equality follows from definition 5.1.2 and the property that the Fourier transform of convolutions is the product of the Fourier transforms. □

Below we state a variant of Bochner's theorem [I.M and N, 1964, Section 2, Theorem 2, page 147].

Theorem 5.1.5 (Bochner's theorem)**.** *A complex valued continuous function f defined on $\mathbb{R}^d$ is non-negative definite if and only if f is the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$ of a tempered measure ν on $\mathbb{R}^d$.*

Proof. For a full proof see [I.M and N, 1964, pages 145-147]. We show the simple implication that the Fourier transform of a tempered measure is non-negative definite. Let f be the Fourier transform of ν and recall that ν is a finite non-negative measure.

$$\begin{aligned}
\sum_{k=1}^{n}\sum_{l=1}^{n} c_k\bar{c}_l f(x_k - x_l) &= \sum_{k=1}^{n}\sum_{l=1}^{n}\left(c_k\bar{c}_l\int_{\mathbb{R}^d} e^{-i2\pi(x_k-x_l)\cdot\xi}d\nu(\xi)\right)\\
=&\sum_{k=1}^{n}\sum_{l=1}^{n}\left(c_k\bar{c}_l\int_{\mathbb{R}^d} e^{-i2\pi x_k\cdot\xi}e^{i2\pi x_l\cdot\xi}d\nu(\xi)\right)\\
=&\int_{\mathbb{R}^d}\left(\sum_{k=1}^{n} c_k e^{-2\pi i x_k\cdot\xi}\sum_{l=1}^{n}\bar{c}_l e^{2\pi i x_l\cdot\xi}\right)d\nu(\xi)\\
=&\int_{\mathbb{R}^d}\left(\sum_{k=1}^{n} c_k e^{-2\pi i x_k\cdot\xi}\sum_{l=1}^{n} c_l e^{-i2\pi x_l\cdot\xi}\right)d\nu(\xi)\\
=&\int_{\mathbb{R}^d}\left|\sum_{k=1}^{n} c_k e^{-i2\pi x_k\cdot\xi}\right|^2 d\nu(\xi)\geq 0.
\end{aligned}$$

□

It follows that the covariance structure defined in 5.2 is actually a covariance function if and only if the continuous spatial parameter function f is the Fourier transform of a tempered measure ν.

5.1.2 Definition of white-coloured noise

Definition 5.1.6 (White-Coloured Noise)**.** *A* spatial coloured noise *that is white in time is a Gaussian random field indexed on the sets with bounded measure of the measure space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$,*

$$M = \{M(t, A),\ t > 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}, \tag{5.5}$$

defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $\mathbb{E}[M(t,A)] = 0$*, and covariance function*

$$C\Big((t,A),(s,B)\Big) = \mathbb{E}\Big[M(t,A)M(s,B)\Big] = t \wedge s \cdot \int_A \int_B f(x-y)dxdy.$$

Where the function f *is the Fourier transform of a tempered measure* ν*.*

A note on the naming convention: A noise that is white in time and white in space will simply be referred to as *white noise*, and if it is white in time but coloured in space it is called *white-coloured noise*. There are several examples of the spatial parameter f, as presented in for example C.A Tudor's paper [Tudor, 2014]. Here is one that we will use as a basis of study (See e.g. [Stein, 1970, Chapter 5, Lemma 1]).

Example 5.1.7. *The Riesz kernel of order* γ*,*

$$f(x) = R_\gamma(x) := 2^{d-\gamma}\pi^{d/2}\frac{\Gamma\Big((d-\gamma)/2\Big)}{\Gamma(\gamma/2)}|x|^{-d+\gamma},\ \ 0 < \gamma < d$$

with $\nu(d\xi) = |\xi|^{-\gamma}d\xi$*.*

Remark 5.1.8. *The reason for the enclosure* $0 < \gamma < d$ *of the value of* γ *is because of the definition of a tempered measure. The order* γ *of the Riesz kernel depends on the space* $\mathscr{D} = \mathbb{R}^d$ *for which it is defined, such that the solution to the stochastic heat equation exists.*

Remark 5.1.9. *As* $\gamma \to 0$*, we will actually approach the white noise case, symbolically,* $\lim_{\gamma\to 0} R_\gamma(x) = \delta(x)$*.*

5.1.3 Stochastic integral with white-coloured noise

Proceeding with the white-coloured noise M we construct a stochastic integral with respect to M. The procedure is the same as for white noise, we define the stochastic process $I(h)$ on deterministic functions $h \in \mathscr{H}$ where the Hilbert space $\mathscr{H}$ is the completion of the set of indicator functions $\mathbb{1}_{(t,A)}$ on $\mathbb{R}_+ \times \mathbb{R}^d$ with inner product

$$\langle \phi, \psi \rangle_{\mathscr{H}} = \int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(t,x)f(x-y)\psi(y,t)dydxdt. \tag{5.6}$$

Call the space of simple functions $\mathscr{E}$. Define the isometry mapping from $\mathscr{E}$ to finite second moment space $L^2(\Omega)$,

$$I : \mathscr{E} \to L^2(\Omega), \qquad g = \sum a_k \mathbb{1}_{(t_k,A_k)} \mapsto \sum a_k M(t_k, A_k) := \int gdM\ . \tag{5.7}$$

Proposition 5.1.10. *The mapping $I : \mathscr{E} \to L^2(\Omega)$ is an isonormal Gaussian process.*

Proof. It suffices to show that I is an isometry. Taking a simple $g = \sum_{k=1}^n a_k \mathbb{1}_{(t_k,A_k)}(t,x) = \sum a_k \mathbb{1}_{(0,t_k]}(t)\mathbb{1}_{A_k}(x)$ we see that

$$\begin{aligned}\|I(g)\|^2_{L^2(\Omega)} &= \left\|\sum a_k M(t_k, A_k)\right\|^2_{L^2(\Omega)} = \mathbb{E}\left[\left(\sum a_k M(t_k, A_k)\right)^2\right] \\ &= \sum_{k=1}^n a_k^2 \mathbb{E} M(t_k, A_k)^2 + 2\sum_{k<l}^n a_k a_l \mathbb{E}\left[M(t_k, A_k)M(t_l, A_l)\right]\end{aligned}$$

Proceeding with the norm of g we find

$$\begin{aligned}\|g\|^2_{\mathscr{E}} &= \int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\Big(\sum_{k=1}^n a_k \mathbb{1}_{(t_k,A_k)}(t,x)\Big)\Big(\sum_{l=1}^n a_l \mathbb{1}_{(t_l,A_l)}(y,t)\Big) dtdxdy \\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\Bigg(\sum_{k=1}^n \Big(a_k^2 \mathbb{1}_{(t_k,A_k)}(t,x)\mathbb{1}_{(t_k,A_k)}(y,t)\Big) \\ &\quad + 2\sum_{k<l}^n \Big(a_k a_l \mathbb{1}_{(t_k,A_k)}(t,x)\mathbb{1}_{(t_l,A_l)}(y,t)\Big)\Bigg) dxdydt\end{aligned}$$

Splitting up the integral over the sum of the two sums we obtain,

$$\begin{aligned}&= \int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)(\sum_{k=1}^n \Big(a_k^2 \mathbb{1}_{(t_k,A_k)}(t,x)\mathbb{1}_{(t_k,A_k)}(y,t)\Big) dxdydt \\ &+ \int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y) 2\sum_{k<l}^n \Big(a_k a_l \mathbb{1}_{(t_k,A_k)}(t,x)\mathbb{1}_{(t_l,A_l)}(y,t)\Big) dxdydt \\ &= \sum_{k=1}^n a_k^2 \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\mathbb{1}_{(t_k,A_k)}(t,x)\mathbb{1}_{(t_k,A_k)}(y,t) dxdydt\right) \\ &+ 2\sum_{k<l}^n a_k a_l \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\mathbb{1}_{(t_k,A_k)}(t,x)\mathbb{1}_{(t_l,A_l)}(y,t) dxdydt\right) \\ &= \sum_{k=1}^n a_k^2 \mathbb{E} M(t_k, A_k)^2 + 2\sum_{k<l}^n a_k a_l \mathbb{E}\left[M(t_k, A_k)M(t_l, A_l)\right],\end{aligned}$$

and hence the norms are equal. □

The rest of the construction follows the white-noise case. We take the completion of $\mathscr{E}$ with respect to the inner-product induced from 5.6 which is the Hilbert space $\mathscr{H}$. Since we have a linear isometry from the set of simple functions, the map $h \mapsto I(h)$

can be extended uniquely to $\mathscr{H}$. Take $h \in \mathscr{H}$ and a sequence of simple functions h_n such that $\|h - h_n\|_{\mathscr{H}} \to 0$. Then we define

$$\int_E hdM := I(h) := \lim_{n\to\infty} I(h_n).$$

Like in the white noise case, the above definition does not depend on the sequence of simple functions approximating h.

5.2 The Stochastic Heat Equation revisited

Consider the stochastic heat equation with white-coloured noise (coloured by the Riesz-kernel in example 5.2.5).

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \frac{1}{2}\Delta u(t,x) = \dot{M}(t,x) & t > 0,\ x \in \mathbb{R}^d \\ u(0,x) = 0 & x \in \mathbb{R}^d. \end{cases} \tag{5.8}$$

As before we define the solution to this equation to be

$$u(t,x) = \int_0^t \int_{\mathbf{R}^d} \frac{e^{-\frac{|x-y|^2}{2(t-s)}}}{(2\pi|t-s|)^{d/2}} M(dsdy), \tag{5.9}$$

as long as the integral above is well defined which is answered by Proposition 5.2.2 below. The following proposition guarantees that the Riesz kernel is actually a non-negative definite function.

Proposition 5.2.1. *The Riesz-kernel f with order γ is a Fourier transform of a tempered measure ν, i.e. $f = \hat{\nu}$, if and only if*

$$d - 2 < \gamma < d.$$

Proof. By definition 5.1.1, a tempered measure has the property that,

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^k d\nu(\xi) < \infty \quad \text{for some } k \geq 1. \tag{5.10}$$

Let $k = 1$ and since $1+|\xi|^2$ acts like a constant around zero and like $|\xi|^2$ at ∞, equation 5.10 is equivalent to

$$\int_{|\xi|\leq 1} \nu(d\xi) < \infty, \quad \text{and} \quad \int_{|\xi|\geq 1} \frac{1}{|\xi|^2}\nu(d\xi) < \infty. \tag{5.11}$$

The fact that $\nu(d\xi) = |\xi|^{-\gamma} d\xi$ gives,

$$\int_{|\xi|\le 1} \frac{1}{|\xi|^{\gamma}} d\xi < \infty, \quad \text{and} \quad \int_{|\xi|\ge 1} \frac{1}{|\xi|^{2+\gamma}} d\xi < \infty.$$

The change of variables to polar coordinates gives the Jacobian

$$J_d = (-1)^{d-1} r^{d-1} \prod_{k=2}^{d-1} \sin^{k-1}\theta_k \le r^{d-1}.$$

given in [Muleshkov and Nguyen, 2016]. Thus the conditions in equation 5.11 can be expressed as,

$$\int_0^1 \frac{1}{r^{\gamma-d+1}} dr < \infty, \quad \text{and} \quad \int_1^\infty \frac{1}{r^{\gamma-d+3}} dr < \infty.$$

From elementary calculus the above integrals are bounded if $\gamma - d + 1 < 1$, hence $\gamma < d$, and $\gamma - d + 3 > 1$, hence $d < \gamma + 2$. Then, $d - 2 < \gamma < d$ is subsequently a necessary and sufficient condition to equation 5.10. □

5.2.1 Existence of solution and covariance

The fundamental solution to the heat equation with drift α, where $x \in \mathbb{R}^d, t > 0$,

$$\Phi(t,x) := \frac{e^{-\frac{|x|^2}{4\alpha t}}}{(4\pi\alpha t)^{d/2}},$$

has the Fourier transform,

$$\mathcal{F}\Phi(t,\cdot)(\xi) = e^{-4\pi^2\alpha|\xi|^2 t}. \tag{5.12}$$

We are ready to show the existence of solution to 5.8.

Proposition 5.2.2. *The equation 5.8 with noise coloured by the Fourier transform of a tempered measure, $f = \hat{\nu}$ admits a unique solution if and only if*

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} d\nu(\xi) < \infty.$$

Proof. We show that the variance (by isometry),

$$\mathbb{E}[u(t,x)^2] = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi(s,y;t,x)\Phi(s,y';t,x) f(y-y') dy dy' ds,$$

is finite. Using Proposition 5.1.4, and a change of variables $\tilde{\xi} = (2\pi\xi)$, the above variance is equal to

$$= (2\pi)^{-d}\int_0^t\int_{\mathbb{R}^d} e^{-\frac{1}{2}(t-s)|\tilde{\xi}|^2}e^{-\frac{1}{2}(t-s)|\tilde{\xi}|^2}d\nu(\tilde{\xi})ds = (2\pi)^{-d}\int_{\mathbb{R}^d}\frac{1}{|\tilde{\xi}|^2}(1-e^{-t|\tilde{\xi}|^2})d\nu(\tilde{\xi}).$$

Since

$$c_{1,t}\frac{1}{1+|\xi|^2} \leq \frac{1}{|\xi|^2}(1-e^{-t|\xi|^2}) \leq c_{2,t}\frac{1}{1+|\xi|^2},$$

the proof is done, since ν is a tempered measure. The thing to note is that $\mathbb{E}[u(t,x)^2] = \|\Phi(s,y;t,x)\|^2_{\mathscr{H}}$ by the isometry which defines the integral of Φ with respect to M. □

Proposition 5.2.3. *Given the Riez kernel of order γ, the solution $u(t,x)$ to 5.8 has the covariance in time for a given $x \in \mathbb{R}^d$,*

$$\mathbb{E}[u(t,x)u(s,x)] = C_0\Big((t+s)^{-\frac{d-\gamma}{2}+1} - (t-s)^{-\frac{d-\gamma}{2}+1}\Big)$$

where

$$C_0 = \Big[(2\pi)^{-d}\int_{\mathbb{R}^d} e^{-\frac{1}{2}|\xi|^2}\frac{1}{-\frac{d-\gamma}{2}+1}\nu(d\xi)\Big]$$

Proof. Let $s \leq t$, starting with the Fourier transform from 5.1.4 and a change of variables to scale away the $(2\pi)^2$ from the Fourier transform of the fundamental solution given in equation 5.12,

$$\begin{aligned}\mathbb{E}[u(t,x)u(s,x)] &= \int_0^{t\wedge s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Phi(\tau,y;t,x)\Phi(\tau,y';s,x)f(y-y')dydy'd\tau \\ &= (2\pi)^{-d}\int_0^s\int_{\mathbb{R}^d} e^{-\frac{1}{2}(t-\tau)|\xi|^2}e^{-\frac{1}{2}(s-\tau)|\xi|^2}d\nu(\xi)d\tau \\ &= (2\pi)^{-d}\int_0^s\int_{\mathbb{R}^d} e^{-\frac{1}{2}(t-\tau)|\xi|^2}e^{-\frac{1}{2}(s-\tau)|\xi|^2}|\xi|^{-\gamma}d\nu(\xi)d\tau.\end{aligned}$$

With the change of variables letting $\tilde{\xi} = \sqrt{t+s-2\tau}\xi$, the functional determinant gives the scaling $(t+s-2\tau)^{-d/2}$.

$$\begin{aligned}\mathbb{E}[u(t,x)u(s,x)] &= (2\pi)^{-d}\int_0^s (t+s-2\tau)^{\frac{-d+\gamma}{2}}\int_{\mathbb{R}^d} e^{-\frac{1}{2}|\xi|^2}d\nu(\xi)d\tau \\ &= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-\frac{1}{2}|\xi|^2}\nu(d\xi)\,\frac{1}{-\frac{d-\gamma}{2}+1}\Big((t+s)^{-\frac{d-\gamma}{2}+1} - |t-s|^{-\frac{d-\gamma}{2}+1}\Big).\end{aligned}$$

Where we let $C_0 := (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-\frac{1}{2}|\xi|^2}\nu(d\xi)\,\frac{1}{-\frac{d-\gamma}{2}+1}$. □

The covariance given in Proposition 5.2.3 is the same as a bi-fractional Brownian motion defined in 3.1.4, with $H = \frac{1}{2}$ and $K = 1 - \frac{d-\gamma}{2}$. This proves the following proposition,

Proposition 5.2.4. *Fix $x \in \mathbb{R}^d$. The solution $u(t,x)$ to the stochastic heat equation 5.8 has the same distribution modulo a constant to the bi-fractional Brownian motion with $H = \frac{1}{2}$ and $K = 1 - \frac{d-\gamma}{2}$,*

$$u(t,x) \overset{d}{=} \sqrt{2^{-1+\frac{d-\gamma}{2}} C_0} B^{\frac{1}{2},1-\frac{d-\gamma}{2}}(t) \quad \textit{for all } t \in \mathbb{R}_+.$$

Proceeding with the above we can utilize the exact same methods as in chapter 3 and 4, we can calculate the exact $\frac{1}{HK} = \frac{1}{\frac{1}{2}-\frac{d-\gamma}{4}}$-variation of our solution. The distribution from Proposition 5.2.4 can of course be calculated for different drift and diffusion in equation 5.8, which in turn motivates the same kind of estimators as before. Simulations of the solution $u(t,x)$ can be created with the same methods as in chapter 4.

5.2.2 White-coloured noise approximation

The finite-difference schemes discussed in the previous chapter can be employed here as well. These types of approximations also converge, see for example [Millet and Morien, 2005] which show convergence of such approximations. The white-coloured noise M at point (t_m, x_j) can be approximated as

$$M_j^m := \int_{t_{m-1}}^{t_m} \int_{x_{j-1}}^{x_j} M(dtdx) = M\left([t_{m-1}, t_m] \times [x_{j-1}, x_j]\right). \tag{5.13}$$

The M_j^m are dependent variables in the spatial direction. The covariance structure is given by

$$\begin{aligned}
&\mathbb{E}(M_j^m M_k^l) = \mathbb{E}\left(M\left([t_{m-1}, t_m] \times [x_{j-1}, x_j]\right) M\left([t_{l-1}, t_l] \times [x_{k-1}, x_k]\right)\right) \\
=&\lambda([t_{m-1}, t_m] \cap [t_{l-1}, t_l]) \int_{x_{j-1}}^{x_j} \int_{x_{k-1}}^{x_k} f(x-y)dxdy \\
=&\begin{cases} \Delta t \int_{x_{k-1}}^{x_k} \int_{x_{l-1}}^{x_l} f(x-y)dxdy & m = l \\ 0 & m \neq l. \end{cases}
\end{aligned} \tag{5.14}$$

And they are normally distributed.

$$M_j^m \in N\left(0, \Delta t \int_{x_{j-1}}^{x_j} \int_{x_{j-1}}^{x_j} f(x-y)dxdy\right). \tag{5.15}$$

Like before the mixed derivative can be approximated as $\frac{\partial}{\partial x \partial t} M(x_j, t_m) \approx \frac{M_j^m}{\Delta t \Delta x}$.

To simulate the Gaussian field we need the covariance matrix which is given by the calculation 5.14 above. Consider Figure 4.4 of the discretization scheme. The points we are simulating the field on are (x_j, t_m) for each $j = 1, \dots, N$ and $m = 1, \dots, M$. We order these M_j^m in a vector as $\mathbf{F} = \Big(M_1^1, \dots, M_N^1, M_1^2, M_2^2 \dots, M_N^2, \dots M_N^M\Big)$.

Let the matrix $\mathbf{K}_{j,k} = \int_{x_{j-1}}^{x_j} \int_{x_{k-1}}^{x_k} f(x-y)dxdy$ be the covariance matrix over the spatial points for a fixed time point t_m. The final covariance matrix over all time and space can be written in the block diagram form with $\mathbf{K}$ on the diagonals and zero matrices everywhere else,

$$\mathbf{Q} = \begin{bmatrix} \mathbf{K} & 0 & \dots & 0 \\ 0 & \mathbf{K} & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & \dots & \mathbf{K} \end{bmatrix}. \tag{5.16}$$

Performing a Cholesky decomposition $\mathbf{Q} = \mathbf{R}\mathbf{R}^\intercal$ we can simulate the noise field as $\mathbf{F} = \mathbf{R}\mathbf{z}$, where $\mathbf{z}$ is a vector of $N \cdot M$ standard i.i.d normal random variables. To speed up computation and save memory we note that the Cholesky decomposition can be reduced to decomposing $\mathbf{K} = \mathbf{L}\mathbf{L}^\intercal$. We then simulate the field $\mathbf{F}_m = \big(M_1^m, \dots, M_N^m\big)$ incrementally over the time $m = 1, 2, \dots, M$ by taking $\mathbf{F}_m = \mathbf{L}\mathbf{z}[(m-1)N + 1 : mN]$ and $\mathbf{F} = (\mathbf{F}_1, \dots \mathbf{F}_M)$.

5.2.2.1 Calculating the covariance integral

We will consider the Riesz-kernel given in example 5.2.5, we state it here again.

Example 5.2.5. *The Riesz kernel of order* γ*,*

$$f(x) = R_\gamma(x) := 2^{d-\gamma}\pi^{d/2}\frac{\Gamma\Big((d-\gamma)/2\Big)}{\Gamma(\gamma/2)}|x|^{-d+\gamma}, \;\; 0 < \gamma < d$$

with $\nu(d\xi) = |\xi|^{-\gamma}d\xi$*.*

We will only look at one spatial dimension, $d = 1$. The goal is to calculate the covariance given in 5.14 and to do so we find the following integral.

$$\int_a^b \int_c^d |x-y|^{\gamma-1}dxdy = \int_a^b \left(\int_c^d |x-y|^{\gamma-1}dy\right) dx.$$

(Note the use of Tonelli's theorem since $g(x) = |x|^{\gamma-1}$ is a positive measurable function for $\gamma \in (0,1)$ and $x \neq 0$.)

We have two cases for our discretization, either $[a,b] \cap [c,d] = \emptyset$, or $[a,b] = [c,d]$. First we assume they are disjoint, and w.l.o.g that $a < b < c < d$. Then $x \in [a,b]$ will always be less than y, so $|x-y| = (y-x)$ which gives

$$\int_c^d |x-y|^{\gamma-1} dy = \int_c^d (y-x)^{\gamma-1} dy$$
$$= \frac{1}{\gamma}\left((d-x)^\gamma - (c-x)^\gamma\right).$$

Thus we obtain for $a < b < c < d$.

$$\int_a^b \left(\int_c^d |x-y|^{\gamma-1} dy\right) dx = \int_a^b \frac{1}{\gamma}\left((d-x)^\gamma - (c-x)^\gamma\right) dx$$
$$= \frac{1}{\gamma(\gamma+1)}\left((c-b)^{\gamma+1} - (d-b)^{\gamma+1} - (c-a)^{\gamma+1} + (d-a)^{\gamma+1}\right).$$

For the general case of $[a,b] \cap [c,d] = \emptyset$ we get

$$\begin{aligned}&\int_a^b \int_c^d |x-y|^{\gamma-1} dx dy \\ =&\frac{1}{\gamma(\gamma+1)}\left(|c-b|^{\gamma+1} - |d-b|^{\gamma+1} - |c-a|^{\gamma+1} + |d-a|^{\gamma+1}\right).\end{aligned} \tag{5.17}$$

Now continuing with $[a,b] = [c,d]$.

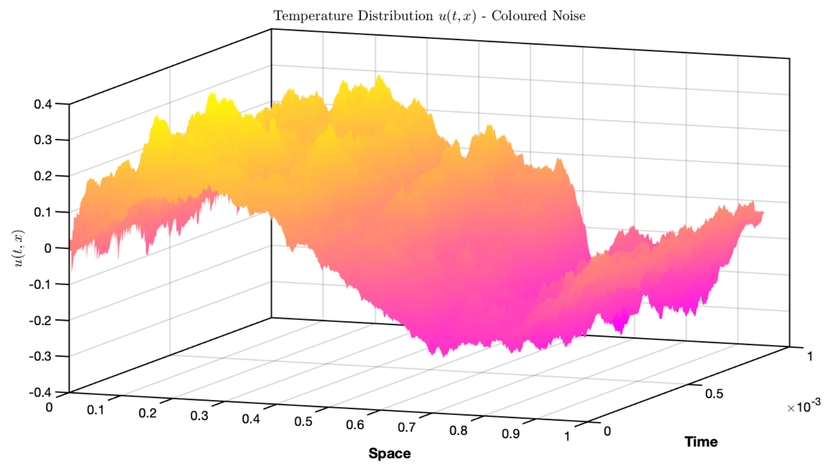
$$\int_a^b \int_a^b |x-y|^{\gamma-1} dx dy = \int_a^b \left(\int_a^b |x-y|^{\gamma-1} dy\right) dx$$
$$= \int_a^b \left(\int_a^x |x-y|^{\gamma-1} dy + \int_x^b |x-y|^{\gamma-1} dy\right) dx$$
$$= \int_a^b \left(\int_a^x (x-y)^{\gamma-1} dy + \int_x^b (y-x)^{\gamma-1} dy\right) dx$$
$$= \int_a^b \frac{1}{\gamma}\left((x-a)^\gamma + (b-x)^\gamma\right) dx$$
$$= \frac{2}{\gamma(\gamma+1)}(b-a)^{\gamma+1}.$$

Which actually corresponds to calculation 5.17 above. For $[a,b] \cap [c,d] = \emptyset$ or $[a,b] = [c,d]$ it holds that,

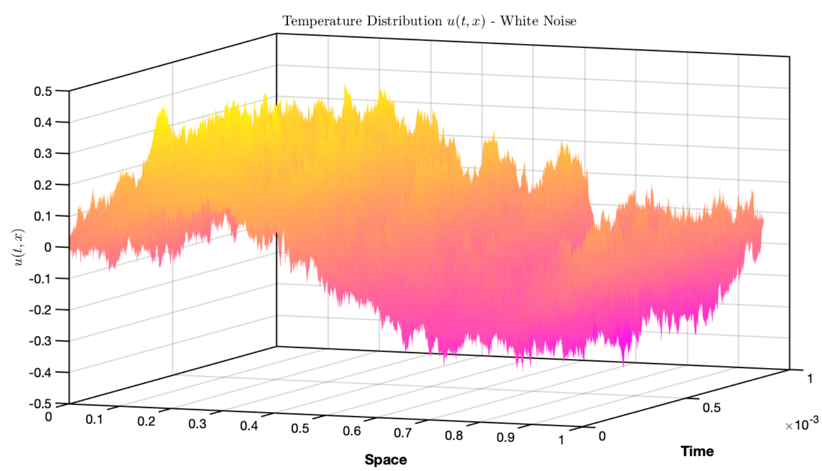
$$\begin{aligned}&\int_a^b \int_c^d |x-y|^{\gamma-1} dx dy \\ =&\frac{1}{\gamma(\gamma+1)}\left(|c-b|^{\gamma+1} - |d-b|^{\gamma+1} - |c-a|^{\gamma+1} + |d-a|^{\gamma+1}\right).\end{aligned} \tag{5.18}$$

5.2.2.2 A simulation

We make a Crank Nicolson finite-difference scheme approximation ($\Theta = 0.5$), with $M = 1000$ points in time, $N = 999$ points in space, and $\alpha\Delta t\ (\Delta x)^2 = \frac{1}{\pi-2}$ to the stochastic heat equation 5.8 with initial condition $u_0 = 0.2\sin(2\pi x)$. The coloured noise is coloured by the Riesz kernel $R_\gamma(x)$ with $\gamma = 3/4$. In Figure 5.1 below are the simulated fields. Figure 5.1a is the coloured noise approximation and 5.1b is the white noise with the same underlying independent Gaussian variables for comparison. We can directly see the smoothing effect of the noise.



(a) Temperature Distribution $u(t,x)$ - Coloured Noise



(b) Temperature Distribution $u(t,x)$ - White Noise

Figure 5.1: Approximation of the solution to the stochastic heat equation 5.8 with both white noise and white-coloured noise, using a Crank-Nicolson scheme.

Appendix A

Proof of Theorem 3.6.1

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A map $\tau : \Omega \to \Omega$ is called *measure preserving* if $\mathbb{P}(\tau^{-1}(A)) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$. For a stochastic process $Y(n)$, we define its' *shifting map* $S_k : \Omega \to \Omega$ such that $Y(S_k(\omega), n) := Y(\omega, n + k)$.

Definition A.0.1. *A map $\tau : \Omega \to \Omega$ is* ergodic *if for every invariant set $A \in \mathcal{F}$ (i.e. $\tau(A) = A$), then $\mathbb{P}(A) = 0$ or $\mathbb{P}(A^c) = 0$.*

A property which implies ergodic is mixing.

Definition A.0.2. *A map $\tau : \Omega \to \Omega$ is* mixing *if*

$$\lim_{n\to\infty} \mathbb{P}(A \cap \tau^n(B)) = \mathbb{P}(A)\mathbb{P}(B),$$

for all $A, B \in \mathcal{F}$.

Mixing implies ergodic since, let A be any invariant set, we have that $\lim_{n\to\infty} \mathbb{P}(\tau^n(A) \cap A) = \mathbb{P}(\tau^n(A))\mathbb{P}(A) = \mathbb{P}(A)\mathbb{P}(A)$ because of mixing and invariance. Hence $\mathbb{P}(A) = \mathbb{P}(A)^2$ which only can be true if $\mathbb{P}(A)$ is equal to 0 or 1.

An equivalent condition for shifting maps S_k from a stationary process $Y(n)$ is the following.

Theorem A.0.3. *The shifting map $S_k : \Omega \to \Omega$ of a stationary process $Y(n)$ is mixing if and only if the autocorrelation function r satisfies,*

$$\lim_{n\to\infty} r(n) = \lim_{n\to\infty} \frac{\mathbb{E}\left[(Y(0) - \mathbb{E}[Y(0)])(Y(n) - \mathbb{E}[Y(0)])\right]}{\mathbb{E}\left[Y(0)^2\right]} = 0.$$

Lastly, the reason for introducing dynamical systems is because of the ergodic theorems that can be utilised. In this proof we will use the following theorem (see e.g. [Pollicott and Yuri, 1998]).

Theorem A.0.4 (Ergodic Theorem of Birkhoff)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. If the map* $\tau : \Omega \to \Omega$ *is ergodic and measure preserving, then for any* $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} X(\tau^i(\omega)) = \int_\Omega X d\mathbb{P}. \qquad a.s.$$

Now the proof of Theorem 3.6.1 can commence,

Proof. The sum can be rewritten as,

$$\sum_{i=0}^{n-1} |cF^H(t_{i+1}) - cF^H(t)|^{1/H} = c^{\frac{1}{H}}(b-a)\frac{1}{n}\sum_{i=0}^{n-1} |F^H(i+1) - F^H(i)|^{1/H}$$

using the self similarity of the fractional Brownian motion. The ergodic Theorem of Birkhoff will be used and hence the shifting map has to measure preserving and ergodic.

Let S_k be the shifting map from $|F^H(1+n) - F^H(n)|^{\frac{1}{H}}$. For any $A \in \mathcal{F}$, let $C \subseteq \mathbb{R}$ such that $\mathbb{P}(A) = \mathbb{P}(|F^H(k+1) - F^H(k)|^{\frac{1}{H}} \in C)$, then

$$\mathbb{P}(A) = \mathbb{P}(|F^H(k+1) - F^H(k)|^{\frac{1}{H}} \in C) = \mathbb{P}(|F^H(1) - F^H(0)|^{\frac{1}{H}} \in C) = \mathbb{P}(S_k^{-1}(A)),$$

where the stationary increments of fractional Brownian motion motivates the second equality. Hence the shifting map is measure preserving.

The shifting map $\tilde{S}_k$ from $F^H(1+k) - F^H(k)$ is mixing since by Theorem A.0.3,

$$\mathbb{E}\left[(F^H(1+k) - F^H(k))F^H(1)\right] = \frac{1}{2}(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}) \to 0 \text{ as } k \to \infty,$$

and hence it is ergodic. Further, the shifting map S_k from $|F^H(1+k) - F^H(k)|^{\frac{1}{H}}$ is ergodic since $x \mapsto |x|^{\frac{1}{H}}$ is a measurable map (the definition of ergodicity rests solely on measures).

Applying the ergodic theorem of Birkhoff with $X := |F^H(1) - F^H(0)|^{\frac{1}{H}} = |F^H(1)|^{\frac{1}{H}}$, $\tau = S_1$, then

$$\begin{aligned}\sum_{i=0}^{n-1} |cF^H(t_{i+1}) - cF^H(t)|^{1/H} =& c^{\frac{1}{H}}(b-a)\frac{1}{n}\sum_{i=0}^{n-1} |F^H(i+1) - F^H(i)|^{1/H} \\ &\to c^{\frac{1}{H}}(b-a)\int_\Omega |F^H(1)|^{\frac{1}{H}} d\mathbb{P} = c^{\frac{1}{H}}(b-a)\mathbb{E}\left[|Z|^{\frac{1}{H}}\right].\end{aligned}$$

With $Z := F^H(1) \in N(0,1)$. □

Appendix B

Asymptotic Variance

The asymptotic variance $\rho^2_{H,\frac{1}{H}}$ from Lemma 3.8.1 actually follows from the below stronger theorem, found in [Cialenco and Huang, 2019][Theorem A.1]. First we define the probabilist Hermite polynomials,

$$\mathrm{He}_j(x) := (-1)^j e^{-\frac{x^2}{2}} \frac{d^j}{dx^j} e^{-\frac{x^2}{2}}. \tag{B.1}$$

The first five polynomials (and the ones we need) are given as

$$\begin{aligned}
\mathrm{He}_0(x) &= 1,\\
\mathrm{He}_1(x) &= x,\\
\mathrm{He}_2(x) &= x^2 - 1,\\
\mathrm{He}_3(x) &= x^3 - 3x,\\
\mathrm{He}_4(x) &= x^4 - 6x^2 + 3.
\end{aligned}$$

A polynomial $H(x;k)$ is said to have *Hermite rank* k if it can be expanded as

$$H(x;k) = \sum_{j=k}^{\infty} c_j \,\mathrm{He}_j(x), \tag{B.2}$$

with $c_k \neq 0$.

Theorem B.0.1. *Let $X(t)$ be a Gaussian process with the following properties,*

1. $X(0) = 0$ *and* $\mathbb{E}[X(t)] = 0$ *for all* t.
2. $X(t+s) - X(t) \in N(0, \sigma^2(s))$, *where* $\sigma(s)$ *is some non-random function of* s.

3. *There exists a constant $\gamma > 0$ such that $X(ct) \stackrel{d}{=} c^{\gamma} X(t)$ for any $c > 0$.*

4. *For any $t \geq 0$, $\Delta t > 0$, the sequence $X_n = X(t + n\Delta t) - X(t + (n-1)\Delta t)$, $n \in \mathbb{N}$ is stationary.*

5. *Let r be the covariance function of the stochastic process $Y_n := \frac{X_n - X_{n-1}}{\sigma(1)}$, $r(n) = \mathbb{E} Y_m Y_{m+n}$ and assume there exists some positive integer k such that $\sum_{n=1}^{\infty} r^k(n) < \infty$.*

Then we have the asymptotic normality

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} H\left(\frac{n^{\gamma}}{\sigma(1)} \left(X_{j/n} - X_{(j-1)/n}\right); k\right) \to N(0, \tilde{\sigma}^2) \tag{B.3}$$

With

$$\tilde{\sigma}^2 = \sum_{l=k}^{\infty} c_l^2 l! \kappa_l^2, \quad \textit{and} \quad \kappa_l^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} r^l(|i-j|).$$

For Lemma 3.8.1 by the above theorem we have

$$r^l(k) := \left(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}\right)^l.$$

And the variances are given as

$$\rho^2_{\frac{1}{2},2} = 2\kappa_2^2, \quad \text{and} \quad \rho^2_{\frac{1}{4},4} = 72\kappa_2^2 + 24\kappa_4^2.$$

For $\rho^2_{\frac{1}{4},4}$ the constants 72 and 36 comes from the fact that the $x^4 - \mathbb{E}\left[|Z|^4\right] = x^4 - 3 = 6\,\mathrm{He}_2 + \mathrm{He}_4$ with coefficients $c_2 = 6$ and $c_4 = 1$. All this is because the terms in the sum defining the exact 4-variation subtracted by $\mathbb{E}\left[|Z|^4\right]$ can be written as a polynomial with Hermite rank 2,

$$\rho^2_{\frac{1}{4},4} = \sum_{l=2}^{\infty} c_l^2\, l!\, \kappa_l^2 = 6^2 \cdot 2!\, \kappa_2^2 + 4!\, \kappa_4^2 = 72\kappa_2^2 + 24\kappa_4^2.$$

For $\rho^2_{\frac{1}{2},2}$ the quadratic variation minus the centring constant $x^2 - \mathbb{E}\left[|Z|^2\right] = x^2 - 1$ is equal to He_2. Hence

$$\rho^2_{\frac{1}{2},2} = \sum_{l=2}^{\infty} c_l^2\, l!\, \kappa_l^2 = 2!\, \kappa_2^2 = 2\kappa_2^2.$$

Appendix C

Derivation of Solution to Heat Equation

Set $\sigma \dot{W} = f(t,x)$. Assume that all coming manipulations are well defined for u, u_0 and f (like the Fourier transforms, inverse transform, all integrals, etc.).

We have the equation,

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) - \alpha \Delta u(t,x) = f(t,x) & t > 0,\ x \in \mathbb{R}^d \\ u(0,x) = u_0(x) & x \in \mathbb{R}^d. \end{cases} \tag{C.1}$$

To solve 1.6 we solve first the homogeneous problem

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) - \alpha \Delta u(t,x) = 0 & t > 0,\ x \in \mathbb{R}^d \\ u(0,x) = u_0(x) & x \in \mathbb{R}^d. \end{cases} \tag{C.2}$$

Recall the Fourier transform

$$\mathscr{F}\Big(u(t,x)\Big)(t,\xi) = \widehat{u}(t,\xi) := \int_{\mathbb{R}^d} u(t,x) e^{-2\pi \xi \cdot x} dx.$$

Where $\xi \cdot x$ is the inner product on $\mathbb{R}^d$. Using that $\mathscr{F}\Big(\Delta u\Big) = -|\xi|^2 \widehat{u}$ and the convolution identity $\mathscr{F}(f * g) = \hat{f} \cdot \hat{g}$. We will do a Fourier transform on C.2 to get the corresponding ODE

$$\begin{cases} \frac{d}{dt} \widehat{u}(\xi,t) + |\xi|^2 \alpha \widehat{u}(\xi,t) = 0 & t > 0,\ \xi \in \mathbb{R}^d \\ \widehat{u}(\xi,0) = \widehat{u_0}(\xi) & \xi \in \mathbb{R}^d. \end{cases}$$

We continue with an inverse transform, obtaining the homogeneous solution

$$u_h(t,x) = \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha t}}}{(4\pi\alpha t)^{d/2}} u_0(y) dy,\ \ x \in \mathbb{R}^d\ , t > 0.$$

Proceed to the in-homogeneous problem with zero initial value to complete the superposition,

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \alpha\Delta u(t,x) = f(t,x) & t > 0,\ x \in \mathbb{R}^d \\ u(0,x) = 0 & x \in \mathbb{R}^d. \end{cases} \tag{C.3}$$

To solve the equation above we will invoke **Duhamel's principle** (see e.g. [Evans, 2010, Page 49]. Which gives a method to solve the equation above with homogeneous initial conditions, but with a non-zero driving term. We illustrate it like this: Consider once again a homogeneous equation, for $0 < s < t$, of the form,

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) - \alpha\Delta u(t,x) = 0 & t > s,\ x \in \mathbb{R}^d \\ u(s,x) = f(s,x) & x \in \mathbb{R}^d. \end{cases} \tag{C.4}$$

By a translation $t' = t - s$ we obtain a PDE of the form in C.2, which admits the solution,

$$u_s(t,x) = \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha(t-s)}}}{(4\pi\alpha(t-s))^{d/2}} f(s,y)dy,\ \ x \in \mathbb{R}^d\ , t > s.$$

Duhamel's principle gives us that the solution to C.3 is simply to integrate u_s with respect to s for $0 < s < t$, which gives the particular solution

$$u_p(t,x) = \int_0^t \int_{\mathbb{R}^d} \frac{e^{\frac{-|x-y|^2}{4\alpha(t-s)}}}{(4\pi\alpha(t-s))^{d/2}} f(s,y)dyds,\ \ x \in \mathbb{R}^d\ , t > 0.$$

The solution is given by the superposition principle, $u(t,x) = u_p(t,x) + u_h(t,x)$.

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