Current-based metrology in a quantum thermal machine

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Abstract

Quantum metrology is a rapidly growing field of quantum information science that aims to exploit the counter-intuitive properties of quantum systems to estimate physical quantities with better-than-classical precision. The field primarily deals with the problem of estimating system parameters, which may or may not directly be associated to an observable. A prominent result of quantum metrology is the *quantum Cramér-Rao bound* (QCRB), which establishes a fundamental lower bound on the attainable uncertainty in a parameter estimation problem. Nevertheless, this bound is in most cases unreachable by experiments. Hence, the study of estimation strategies that are simultaneously precise and experimentally feasible is essential for the development of quantum sensing devices.

In this work, the problem of parameter estimation is explored in a two-qubit autonomous thermal machine interacting weakly with two thermal reservoirs. The qubits are coupled to each other and their energy gaps are separated by a small detuning. The estimation precision is calculated from the particle currents that naturally arise between the system and the reservoirs. In addition, violations of classical thermodynamic inequalties, called *thermodynamic uncertainty relations* (TUR's), are investigated and connected to a possible quantum advantage in parameter estimation. Finally, the current-based estimation precision is compared to the quantum Cramér-Rao bound.

A classical thermodynamic bound on the current precision is derived for the inter-qubit coupling and the detuning. It is found that TUR violations are possible in different regions of parameter space, and that they can lead to an increase in current precision. Furthermore, it is observed that, for vanishing detuning, the current precision for the inter-qubit coupling can approximate the quantum Cramér-Rao bound to a high degree.

List of abbreviations

TUR - Thermodynamic uncertainty relation

NESS - Non-equilibrium steady state

FCS - Full counting statistics

FI - Fisher information

QFI - Quantum Fisher information

QCRB - Quantum Cramèr-Rao bound

CPTP - Completely positive and trace preserving

PVM - Projection-valued measure

POVM - Positive operator-valued measure

SLD - Symmetric logarithmic derivative

Contents

1 Introduction

Metrology is the science of measurement, encompassing a broad range of disciplines aimed at ensuring accuracy, precision, and reliability in quantitative measurements. Its aim is to provide effective methods to accurately estimate physical quantities, both theoretically and experimentally. The characterisation of the limits to such methods is also a fundamental question in metrology.

Quantum mechanics, besides bringing about a paradigm shift in the understanding of the universe, has allowed for the development of quantum information processing, which aims to exploit the properties of quantum theory to achieve a computational advantage with respect to classical computers. The emerging field of *quantum metrology* has an analogous goal, and has seen great progress both theoretically and experimentally thus far. As an example, for an ensemble of N classical particles used to measure a certain physical quantity, the lower bound on the uncertainty on the estimation scales with $1/\sqrt{N}$. Nevertheless, it has been shown that with the aid of quantum entanglement the scaling of the uncertainty with the number of particles can be reduced to as low as $1/N$ [1, 2]. Furthermore, the quantum advantage in metrology has been observed experimentally in interferometry [3], magnetometry [4], and even in the detection of gravitational waves [5, 6]. A key result in quantum metrology is the *quantum Cramér-rao bound* (QCRB) [7], which sets a universal lower bound on the uncertainty of quantum parameter estimation, although the estimation protocol that allows to attain the bound is mostly experimentally inaccessible.

In its early days, non-relativistic quantum theory only provided a description of the dynamics of quantum states in isolated and closed systems. Nonetheless, the majority of physical scenarios feature open systems, which are capable of exchanging information with an external environment. Hence, the vast field of *open quantum systems* extends the study of quantum mechanics to open systems. Pivotal quantities used to characterise the dynamics of open quantum systems are the *particle currents* between the system and the environment, as they carry fundamental information about the time evolution of the system [8].

One of the most prominent theoretical setups studied by open quantum systems is *quantum thermal machines*, which are physical devices capable of performing certain tasks by interacting with the environment. They can function as heat engines [9, 10], refrigerators [11], and as generators of quantum entanglement [12, 13]. Hence, they have numerous applications in quantum technology and quantum information processing.

In this thesis, the problem of parameter estimation is explored in an autonomous quantum thermal machine consisting of a pair of two-level systems (qubits) coupled to each other and to fermionic thermal reservoirs. In particular, the estimation precision for the system parameters is calculated by using the mean particle current and its fluctuations. Furthermore, the violations of classical thermodynamic inequalities, referred to as *thermodynamic uncertainty relations* (TUR's), are explored and related to a non-classical advantage in parameter estimation. Lastly, the current-based precision is compared to the quantum Cramér-Rao bound. The entire analysis is done in the *non-equilibrium steady state* (NESS), in which the expec-

tation values of the system's observables do not change with time. The NESS is reached by the machine in the long time limit.

The findings of this work are intended to be included in a future publication.

2 Theory

In this section and throughout the rest of this work, the Planck units $\hbar = c = k_B = 1$ are used.

2.1 Density operator

In quantum mechanics, a *pure state* is a quantum state that can be represented by a ket vector. The concept of state can be extended to *mixed states*, which are a statistical ensemble of pure states. Mixed states are best mathematically described by the *density operator* formalism. Given a Hilbert space $\mathcal{H}$, a density operator can be generally defined as

$$\hat{\rho} = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \tag{2.1}$$

where the $\left|\psi_i\right\rangle \in \mathcal{H}$ are the pure states that form the ensemble and the $p_i \geq 0$ are their respective statistical weights, which are such that

$$\sum_i^N p_i = 1. \tag{2.2}$$

Hence, a density operator has to be *normalised*, which implies that

$$\mathrm{Tr}(\hat{\rho}) = 1. \tag{2.3}$$

Furthermore, by the definition in 2.1, for any vector $\left|\phi\right\rangle$ in $\mathcal{H}$,

$$\left\langle\phi\right| \hat{\rho} \left|\phi\right\rangle = \sum_i^N p_i \left|\left\langle\psi_i|\phi\right\rangle\right|^2, \tag{2.4}$$

and since $\left|\left\langle\psi_i|\phi\right\rangle\right| \geq 0$ for all $\left|\psi_i\right\rangle$ and $\left|\phi\right\rangle$, a density operator is positive semi-definite. Moreover, for any two vectors $\left|\phi\right\rangle$ and $\left|\varphi\right\rangle$,

$$\left\langle\phi\right| \hat{\rho} \left|\varphi\right\rangle = \sum_i^N p_i \left\langle\phi|\psi_i\right\rangle \left\langle\psi_i|\varphi\right\rangle = \sum_i^N p_i \left\langle\varphi|\psi_i\right\rangle^* \left\langle\psi_i|\phi\right\rangle^* = \left\langle\varphi\right| \hat{\rho} \left|\phi\right\rangle, \tag{2.5}$$

hence $\hat{\rho}$ is Hermitian. To summarise, a linear operator $\mathcal{O}$ on a Hilbert space $\mathcal{H}$ can be identified as a density operator if and only if

- it is normalised ($\mathrm{Tr}(\mathcal{O}) = 1$);
- it is positive semi-definite ($\left\langle\phi\right| \mathcal{O} \left|\phi\right\rangle \geq 0 \quad \forall \left|\phi\right\rangle \in \mathcal{H}$);

- it is Hermitian ($\mathcal{O}^\dagger = \mathcal{O}$).

If $\mathcal{H}$ is finite-dimensional, $\hat{\rho}$ can be written in matrix form, and thus is referred to as *density matrix*. Hence, by choosing an orthonormal basis for H, the diagonal elements of a density matrix can be identified as the populations of the basis vectors and the off-diagonal ones bear the name *coherences* and can be generally thought as the overlap between the ensemble's states in the chosen basis. In this formalism, the expectation value of an operator $\hat{A}$ in a general mixed state described by a density operator $\hat{\rho}$ is

$$\langle \hat{A} \rangle_{\hat{\rho}} = \mathrm{Tr}\Big(\hat{\rho}\hat{A}\Big). \tag{2.6}$$

From this point onwards, the terms '*state*' and '*density matrix*' will be used interchangeably.

2.1.1 Partial trace

In certain physical situations, a system described by quantum states in a Hilbert space $\mathcal{H}$ can be partitioned in multiple parts $\{A_k\}$, with corresponding Hilbert spaces $\{\mathcal{H}_k\}$. $\mathcal{H}$ can thus be rewritten as the tensor product $\mathcal{H} = \bigotimes_j \mathcal{H}_j$.

A density operator $\hat{\rho}$ on $\mathcal{H}$ cannot in general be factorised as a tensor product of density matrices on the individual subspaces $\mathcal{H}_k$. Nevertheless, it would still be beneficial to obtain a density matrix that belongs to the subspace of interest and that retains some information of $\hat{\rho}$. Under this premise, for two partitions A and B with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the *reduced density matrix* $\hat{\rho}_A$, without loss of generality, is defined as

$$\hat{\rho}_A = \mathrm{Tr}_B\, \hat{\rho}, \tag{2.7}$$

where $\mathrm{Tr}_B(\cdot)$ is the *partial trace* over subsystem B. By choosing bases $\{|a_k\rangle\}$ and $\{|b_k\rangle\}$ for $\mathcal{H}_A$ and $\mathcal{H}_B$, $\hat{\rho}$ can be written as

$$\hat{\rho} = \sum_{\alpha\beta\gamma\delta} c_{\alpha\beta\gamma\delta} \left|a_\alpha\right\rangle \left\langle a_\beta\right| \otimes \left|b_\gamma\right\rangle \left\langle b_\delta\right|, \tag{2.8}$$

$c_{\alpha\beta\gamma\delta}$ are complex coefficients. Hence, $\hat{\rho}_A = \mathrm{Tr}_B(\hat{\rho})$ can then be computed as

$$\hat{\rho}_A = \sum_{\alpha\beta} \left[\sum_{\gamma} c_{\alpha\beta\gamma\gamma} \right] \left|a_\alpha\right\rangle \left\langle a_\beta\right|. \tag{2.9}$$

The partial trace is a *completely positive and trace-preserving* map (CPTP) [14], which guarantees that $\hat{\rho}_A$ is a density operator and represents a quantum state.

2.2 Time evolution of closed and isolated systems

In non-relativistic quantum mechanics, the time evolution of an isolated system described by a pure state $|\psi\rangle$ is governed by Schrödinger's equation, which reads

$$i\frac{d}{dt}\left|\psi(t)\right\rangle = \hat{H}\left|\psi(t)\right\rangle, \tag{2.10}$$

where $\hat{H}$ is the Hamiltonian of the system. If the state $|\psi(t_0)\rangle$ is known for some instant t_0, the solution to Eq. 2.10 is given by

$$|\psi(t)\rangle = \hat{U}(t, t_0)\,|\psi(t_0)\rangle\,, \tag{2.11}$$

where $\hat{U}(t, t_0)$ is the *time evolution operator* for the system. Since the system is isolated, probability is conserved and always equal to unity, and hence $\hat{U}(t, t_0)$ is unitary. If $\hat{H}$ is constant,

$$U(t, t_0) = \exp\Big(-i\hat{H}(t - t_0)\Big), \tag{2.12}$$

otherwise, in the case of a closed system with a time-dependent Hamiltonian, the time evolution operator can be expressed as

$$\hat{U}(t, t_0) = \hat{\tau}\exp\left(-i\int_{t_0}^{t}\hat{H}(s)ds\right), \tag{2.13}$$

where $\hat{\tau}$ is the *time ordering* operator, which assures that a product of operators is ordered such that their time arguments decrease from left to right.

By substituting Eq. 2.11 into Eq. 2.1 for all the pure states in the mixture, the time evolution of a mixed state may now also be described as

$$\hat{\rho}(t) = \sum_i \hat{U}(t, t_0)[p_i\,|\psi_i(t_0)\rangle\,\langle\psi_i(t_0)|]\hat{U}^\dagger(t, t_0) = \hat{U}(t, t_0)\hat{\rho}(t_0)\hat{U}^\dagger(t, t_0), \tag{2.14}$$

where $\hat{\rho}(t_0) = \sum_i p_i\,|\psi_i(t_0)\rangle\,\langle\psi_i(t_0)|$. Given a general and possibly time-dependent Hamiltonian, differentiating with respect to time both sides of Eq. 2.14 yields

$$\dot{\hat{\rho}}(t) = -i\hat{H}(t)\hat{\rho}(t) + i\hat{\rho}(t)\hat{H}(t) = -i[H(t), \hat{\rho}(t)], \tag{2.15}$$

which is referred to as the *Liouville-Von Neumann equation*. Eq. 2.15 can be rewritten as

$$\dot{\hat{\rho}} = \mathcal{L}(t)\hat{\rho}(t), \tag{2.16}$$

where $\mathcal{L}(t)$ is called *Liouvillian superoperator*, or *Liouvillian* for short.

2.3 Time evolution of open systems

In a given isolated system, it might be desirable to describe only the dynamics of a subsystem S, which is coupled to the complementary subsystem E, which usually takes the name of *environment*. Such physical scenarios are known as *open quantum systems*. The Hamiltonian for the combined system may be written as

$$\hat{H}_{SE} = \hat{H}_S \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}_E + \hat{H}_I(t), \tag{2.17}$$

where H_S and H_E are time-independent Hamiltonians on the respective Hilbert spaces of S and E and encode the internal dynamics of system and environment, $H_I(t)$ is a possibly time-dependent term that describes the interaction between S and E and $\mathbb{1}$ is the identity

matrix.
By employing the partial trace, given the density operator $\hat{\rho}_{SE}$ of the combined system $S+E$, it may be possible to obtained the reduced density matrix for S:

$$\hat{\rho}_S = \mathrm{Tr}_B\, \hat{\rho}_{SE}. \tag{2.18}$$

and hence, according to Eq. 2.16 the time evolution of $\hat{\rho}_S$ is governed by

$$\dot{\hat{\rho}}_S = \mathrm{Tr}_B[\mathcal{L}(t)\hat{\rho}_{SE}(t)], \tag{2.19}$$

In order to describe the non-unitary evolution of $\hat{\rho}_S$, for a general partition of system and environment, Eq. 2.19 requires complete knowledge of the total density matrix $\hat{\rho}_{SE}$. This is in several cases highly impractical, as the environment possibly consists of an infinite number of degrees of freedom. Hence, it might be beneficial to have a description of the time evolution of $\hat{\rho}_S$ which does not depend on all the possible interactions between all the degrees of freedom of S and E and which treats the exchange of information between system and environment as dissipative noise.

Nevertheless, this is possible under certain assumptions: firstly, it is assumed that at an initial time $t = t_0$, $\hat{\rho}_{SE} = \hat{\rho}_S(t_0)\otimes\hat{\rho}_E$, and that the environment is much larger than the system and that the coupling between them is small enough so that the environment's reduced state $\hat{\rho}_E$ stays constant throughout the evolution. This approximation is commonly known as the *Born approximation* or *weak coupling limit*; the second assumption is that the time taken by environment excitations to decay is much shorter than average time scale of the evolution of the system τ_S, under which the expectation values of the system's observables change appreciatively [15]. Hence, the environment retains no memory of the system's internal changes and thus the evolution of S does not depend of the full history of $\hat{\rho}_S$. This property is called *Markovianity*. As it was first shown by Gorini, Kossakowski and Sundarshan in [16] and independently by Lindblad in [17], the general form for a Liouvillian for an open quantum system obeying the aforementioned assumptions is

$$\mathcal{L}^{(\mathrm{GKSL})}\hat{\rho}_S = -i[\hat{H}_S, \hat{\rho}_S] + \sum_i \gamma_i \left(\hat{L}_i \hat{\rho}_S \hat{L}_i^\dagger + \frac{1}{2}\{\hat{L}_i^\dagger \hat{L}_i, \hat{\rho}_S\} \right), \tag{2.20}$$

where $\{A, B\} = AB + BA$ is the anticommutator, the L_i are the *Jump operators*, which belong to the space of operators on the system's Hilbert space and describe the different ways through which the system can exchange information with the environment, and the $\gamma_i \geq 0$ are the associated *damping rates*. $\mathcal{L}^{(\mathrm{GKSL})}$ is a CPTP map, and thus Eq. 2.20 takes the name of *Markovian master equation*.

2.4 Thermodynamic uncertainty relations

A system in thermal equilibrium with its environment may be described by the *canonical ensemble*, in which the system's states, labelled by their energy, are distributed according to the *Boltzmann distribution* [18]. In such cases, the response of the system to external perturbations is encoded in the fluctuations of its quantities through the *fluctuation-dissipation theorem*. This is no longer possible in non-equilibrium scenarios, in which characterising

fluctuations is a highly non-trivial task. This and several other problems related to non-equilibrium setups are the subject of interest of *stochastic thermodynamics*, a branch of statistical mechanics that models out-of-equilibrium phenomena as stochastic processes, in which the system's state transitions are governed by probability distributions.

A certain class of stochastic processes, called *jump processes*, describe a discrete random variable X over a set S that changes randomly in time through instantaneous random jumps. If the time intervals between jumps follow an exponential distribution, the process is Markovian. If $(d/dt)\langle X\rangle = 0$, the process is said to be in the *steady state*. The transitions between $X_i \to X_j$ with $X_i, X_j \in S$ are described by rates k_{ij}. For a given element X_α in S, if at the initial time t_0 the process is in the steady state, the average amount $N_\alpha(t)$ of jumps through X_α is given by

$$\langle N(t)\rangle = (t - t_0)\sum_{i,j}(k_{\alpha i} - k_{j\alpha}), \tag{2.21}$$

where by convention the number of exits from the state X_α is positive while the number of entrances is negative. From Eq. 2.21, *mean current* may be defined as

$$\langle J_\alpha\rangle \equiv \frac{d}{dt}\langle N_\alpha(t)\rangle = \sum_{i,j}(k_{\alpha i} - k_{j\alpha}). \tag{2.22}$$

Recent findings in the study of fluctuations in classical microscopic systems [19, 20, 21] have shown that for classical jump-like processes in non-equilibrium thermodynamics,

$$\frac{\mathrm{Var}(J_\alpha)}{\langle J_\alpha\rangle^2} \geq \frac{2}{\sigma}, \tag{2.23}$$

where k_B is Boltzmann's constant and σ is the *entropy production rate*. Inequalities of this kind are known as *thermodynamic uncertainty relations*, as they set a trade-off between the fluctuations of the current, quantified by the variance of J_α, its average strength $\langle J_\alpha\rangle$, and the dissipation, encoded in σ.

2.5 Quantum parameter estimation

In quantum mechanics, the measurement of a physical quantity A corresponding to an observable $\hat{A}$ with eigenvalues $\{\lambda_k\}$ and eigenvectors $\{|\lambda_k\rangle\}$ mathematically corresponds to the set of operators $\{P_k = |\lambda_k\rangle\langle\lambda_k|\}$, such that

$$\sum_k P_k = \mathbb{1}, \tag{2.24}$$

and

$$\mathrm{Tr}(P_i P_j) = \delta_{ij}. \tag{2.25}$$

All measurements induced by observables are *projection-valued measures* (PVM's), as they can be interpreted as the act of projecting the state onto the observable's eigenvectors. Nonetheless, if the ortho-normality condition in Eq. 2.25 is relaxed, quantum mechanical

measurements can be generalised to *positive operator-valued measures* (POVM's), which need only statisfy the completeness condition in Eq. 2.24. In general, POVM's are not induced by observables and thus do not always correspond to a physical measurement, although it is possible to relate any POVM with finite number of operators F_i to a PVM in a larger Hilbert space through Naimark's *dilation theorem* [22]. For all measurements, the probability of obtaining measurement outcome $m = \lambda_i$ by performing a measurement on a state $\hat{\rho}$ is given by the *Born rule*

$$p(m_i) = \mathrm{Tr}\left(\hat{\rho}\hat{F}_i\right). \tag{2.26}$$

As mentioned in the Introduction, given a state $\hat{\rho}(\theta)$, the fundamental problem in quantum metrology is estimating the value of θ, and quantifying the precision and accuracy of such estimation. An estimation problem is formulated with the following protocol:

1. A state $\hat{\rho}$ is prepared.
2. $\hat{\rho}$ is mapped to $\hat{\rho}(\theta)$ by a CPTP map, so that $\hat{\rho}(\theta)$ retains all the properties of a density operator.
3. A POVM measurement $\Pi : \{F_i\}$ is performed and an outcome m_i is obtained.
4. A function $E(m)$, called *estimator*, is used to estimate θ from the measurement outcome
5. The procedure is repeated N times

An ideal estimator would be such that $E(m) = \theta$ for all measurement outcomes, although in most scenarios there is no such estimator. A easier condition to achieve is requiring that theta is equal to the average value of E over all measurements. In the asymptotic limit ($N \to \infty$), the average value of E coincides with the mean value $\langle E \rangle$ and the condition reduces to $\langle E \rangle = \theta$. All such estimators are called *unbiased*.

In order to quantify the amount of information about θ contained in the measurement of Π, the classical *Fisher information* (FI) associated to Π is defined as

$$\mathcal{I}[\Pi, \theta] = \sum_i \frac{1}{p(m_i|\theta)} \left(\frac{\partial p(m_i|\theta)}{\partial \theta} \right)^2, \tag{2.27}$$

where $p(m_i|\theta) = \mathrm{Tr}(F_i\hat{\rho}(\theta))$.

For any POVM and any unbiased estimator $E(m)$, the variance $\mathrm{Var}(E)$, which provides information about the uncertainty on θ, is bounded by [7]

$$\mathrm{Var}(E) \geq \frac{1}{N\mathcal{F}(\hat{\rho}(\theta))}, \tag{2.28}$$

where $\mathcal{F}(\hat{\rho}(\theta))$ is the *Quantum Fisher Information* (QFI), which is obtained by maximising the FI over all possible POVM's

$$\mathcal{F}(\hat{\rho}(\theta)) = \max_{\Pi} \{\mathcal{I}[\Pi, \theta]\} \tag{2.29}$$

The inequality in Eq. 2.28 is known as the *quantum Cramér-Rao bound* (QCRB), and it is a pivotal result in quantum parameter estimation theory, as it establishes a universal lower bound on the estimation uncertainty of a system parameter. The quantum Cramér-Rao bound is in fact independent of the measurement or the estimation strategies that one may use.

Even unbiased estimators may be difficult to find in certain estimation problems. Hence, for biased estimators $\mathcal{E}(m)$, such that $\langle \mathcal{E} \rangle = \theta + b(\theta)$, where $b(\theta)$ is the *bias*, the Cramér-Rao bound reads [8]

$$\mathrm{Var}(\mathcal{E}) \geq \frac{(1 + \partial_\theta b(\theta))^2}{N\mathcal{F}(\hat{\rho}(\theta))}, \tag{2.30}$$

and given that $\partial_\theta b(\theta) = \partial_\theta(\langle \mathcal{E} \rangle - \theta) = \partial_\theta \langle \mathcal{E} \rangle - 1$, rearranging Eq. 2.30 yields

$$\frac{\mathrm{Var}(\mathcal{E})}{(\partial_\theta \langle \mathcal{E} \rangle)^2} \geq \frac{1}{\mathcal{F}(\hat{\rho}(\theta))}. \tag{2.31}$$

The quantum Cramér-Rao bound is always saturated by a certain optimal POVM, which, in the case of the discrete measurement protocol described above, is a projective measurement in the eigenbasis of the *symmetric logarithmic derivative* (SLD) $\hat{L}_{SLD}$, implicitly defined as [23]

$$\frac{\hat{L}_{SLD}\hat{\rho}(\theta) + \hat{\rho}(\theta)\hat{L}_{SLD}}{2} = \frac{\partial}{\partial\theta}\hat{\rho}(\theta) \tag{2.32}$$

Nonetheless, the optimal measurement is often impractical or impossible to realise experimentally.

3 Methods

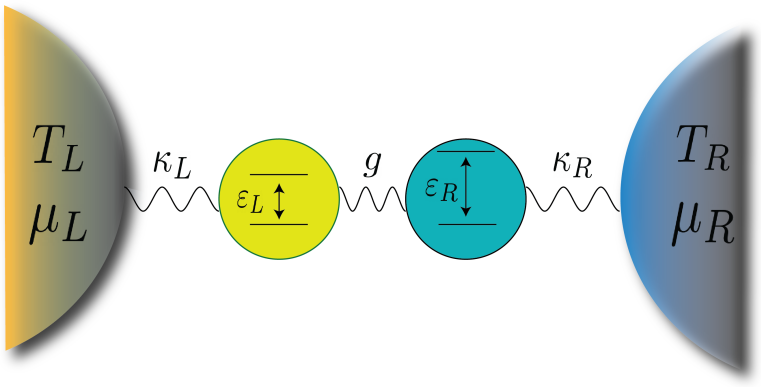

Figure 1: Schematic representation of the two qubit thermal machine. The excited states of each qubit are coupled to the ground states of the other with coupling strength g. Each qubit has its own energy gap ε_α ($\varepsilon_R \geq \varepsilon_L$) and is coupled with strength κ_α to thermal bath α at temperature T_α and chemical potential μ_α.

3.1 Local master equation

The system taken into consideration is a thermal machine formed by a pair of qubits (two-level systems) coupled to each other and to two fermionic reservoirs in thermal equilibrium. The Hamiltonian for the system reads

$$\hat{H}_S = \varepsilon_L \hat{\sigma}_L^+ \hat{\sigma}_L^- + \varepsilon_R \hat{\sigma}_R^+ \hat{\sigma}_R^- + g(\hat{\sigma}_L^+ \hat{\sigma}_R^- + \hat{\sigma}_R^+ \hat{\sigma}_L^-) \tag{3.1}$$

where ϵ_α, with $\alpha = \{L, R\}$ are the energies of the excited states of the two qubits, g is the inter-qubit coupling, and $\hat{\sigma}_\alpha^\pm$ are creation and annihilation operators on the qubits. In the computational bases $\{|0\rangle_L, |1\rangle_L\}$ and $|0\rangle_R, |1\rangle_R\}$,

$$\begin{aligned} \hat{\sigma}_L^- &= |0\rangle_L \langle 1|_L \otimes \mathbb{1} & \hat{\sigma}_L^+ &= |1\rangle_L \langle 0|_L \otimes \mathbb{1} \\ \hat{\sigma}_R^- &= \mathbb{1} \otimes |0\rangle_R \langle 1|_R & \hat{\sigma}_R^+ &= \mathbb{1} \otimes |1\rangle_R \langle 0|_R . \end{aligned} \tag{3.2}$$

Furthermore, it is assumed that the two qubit energies are separated by a small *detuning* δ such that $\varepsilon_R - \varepsilon_L = \delta$. Hence, it is convenient to rewrite $\varepsilon_L = \varepsilon$ and $\varepsilon_R = \varepsilon + \delta$.
The two reservoirs, also referred to as baths, have Hamiltonians

$$\hat{H}_\alpha^{\text{bath}} = \sum_j \omega_{j,\alpha} \hat{a}_{j,\alpha}^\dagger \hat{a}_{j,\alpha}, \tag{3.3}$$

with

$$\{\hat{a}_{i,\alpha}^\dagger, \hat{a}_{j,\alpha}\} = \delta_{ij} \tag{3.4}$$

and are in *thermal states* [24]

$$\hat{\rho}_\alpha = \frac{e^{(-(\hat{H}_\alpha^{\text{bath}} - \mu_\alpha \hat{N}_\alpha)/T_\alpha)}}{\text{Tr}\left(e^{(-(\hat{H}_\alpha^{\text{bath}} - \mu_\alpha \hat{N}_\alpha)/T_\alpha)}\right)}, \tag{3.5}$$

where μ_α are the chemical potentials, T_α are the temperatures, and $\hat{N}_\alpha$ are the number operators

$$\hat{N}_\alpha = \hat{a}_{j,\alpha}^\dagger \hat{a}_{j,\alpha}. \tag{3.6}$$

The system-bath interactions are described by

$$\hat{H}_\alpha^{(I)} = \sum_j t_{\alpha,j} (\hat{a}_{j,\alpha}^\dagger \hat{\sigma}_{j,\alpha}^- + \hat{\sigma}_\alpha^+ \hat{a}_{j,\alpha}). \tag{3.7}$$

If $\kappa_{\alpha,j} \ll \max\{\epsilon_\alpha - \mu_\alpha, T_\alpha\}$, the system is Markovian and thus the evolution of the system density matrix $\hat{\rho}$ can be described by a master equation in GKSL form (Eq. 2.20). By further assuming that $\sqrt{g^2 + \delta^2} \ll \max\{T_\alpha, |\varepsilon_\alpha - \mu_\alpha|\}$, the master equation for the system density matrix $\hat{\rho}$ can be expressed as [25, 26]

$$\dot{\hat{\rho}} = -i[\hat{H}_S, \hat{\rho}] + \sum_\alpha \left(\kappa_\alpha (1 - N_\alpha(\bar{\varepsilon})) \mathcal{D}[\sigma_\alpha^-] \hat{\rho} + \kappa_\alpha N_\alpha(\bar{\varepsilon}) \mathcal{D}[\sigma_\alpha^+] \hat{\rho} \right) \tag{3.8}$$

where $N_\alpha(\bar{\varepsilon})$ are Fermi-Dirac distributions

$$N_\alpha(\epsilon) = \frac{1}{1 + e^{(\epsilon-\mu_\alpha)/T_\alpha}} \tag{3.9}$$

evaluated at $\bar{\varepsilon} = (\varepsilon_L + \varepsilon_R)/2 = \varepsilon + \delta/2$, κ_α are the bath damping rates, and $\mathcal{D}[\hat{\sigma}_\alpha^\pm]\hat{\rho}$ are the *dissipators*, which read

$$\mathcal{D}[\sigma_\alpha^\pm]\hat{\rho} = \hat{\sigma}_\alpha^\pm \hat{\rho} \hat{\sigma}_\alpha^\mp + \frac{1}{2}\{\hat{\sigma}_\alpha^\mp \hat{\sigma}_\alpha^\pm, \hat{\rho}\}. \tag{3.10}$$

Eq. 3.8 is a *local master equation*, as the interactions described by the dissipators, emission and absorption to and from the baths, are localised on each one of the qubits. Hence, the possible transitions, in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, are $|00\rangle \longleftrightarrow |01\rangle$, $|00\rangle \longleftrightarrow |10\rangle$, $|10\rangle \longleftrightarrow |11\rangle$ and $|01\rangle \longleftrightarrow |11\rangle$. Furthermore, the evaluation of the Fermi-Dirac distributions at $\bar{\varepsilon} = (\varepsilon_L + \varepsilon_R)/2$ ensures consistency with the second law of thermodynamics [26]. It is important to stress that the local master equation is only valid for small inter-qubit coupling, as this allows the baths to distinguish the two qubits. For larger values of g, this is no longer possible, and the system is better described by a *global master equation* [25], in which the baths couple to the eigenstates of the two-qubit system.

3.2 Solving the master equation

Given a density operator $\hat{\rho}$ on a N-dimensional Hilbert space $\mathcal{H}$, a Liouvillian superoperator $\mathcal{L}$ acts on the N^2-dimensional space of operators $B(\mathcal{H})$. A master equation is a matrix equation, and solving for $\hat{\rho}$ directly is highly impractical in most cases. Nevertheless, the Markovian master equation in Eq. 2.20 can be cast into a vectorised form expressed as

$$\dot{\mathbf{p}} = \tilde{\mathcal{L}}\mathbf{p}, \tag{3.11}$$

where $\mathbf{p}$ is the N-dimensional column vector whose coordinates are the elements $\hat{\rho}_{ij}$ of the density matrix $\hat{\rho}$ and $\tilde{\mathcal{L}}$ is a $N^2 \times N^2$ matrix such that [27]

$$\tilde{\mathcal{L}} = -i(\hat{H} \otimes \mathbb{1} - \mathbb{1} \otimes \hat{H}^T) + \sum_i \gamma_i \left(\hat{L}_i \otimes \hat{L}_i^* - \frac{1}{2}(\hat{L}_i^\dagger \hat{L}_i \otimes \mathbb{1} + \mathbb{1} \otimes \hat{L}_i^T \hat{L}_i^*) \right), \tag{3.12}$$

where $\hat{H}$ and $\{L_i\}$ are the Hamiltonian and the jump operators in their matrix representation, and $(\cdot)^T$, $(\cdot)^*$, and $\otimes$ respectively denote transposition, complex conjugation and the Kronecker product. The vector $\mathbf{p}$ belongs to a N^2-dimensional vector space called *Liouville space* [28].

The steady state $\hat{\rho}_{SS}$ is such that

$$\mathcal{L}\hat{\rho}_{SS} = 0, \tag{3.13}$$

and is thus the eigenoperator associated to the zero eigenvalue $\lambda_0 = 0$ of $\mathcal{L}$. For a time independent Liouvillian on a finite dimensional operator space, λ_0 is guaranteed to be an eigenvalue, and there always exists a steady state $\hat{\rho}_{SS}$, which can be reached in long-time limit [29]. In vectorised form, the steady state vector $\mathbf{p}_{SS}$ is obtain by solving $\tilde{\mathcal{L}}\mathbf{p}_{SS} = 0$, which is a system of N^2 equations.

3.3 Full counting statistics

The transfer of fermions between the two-qubit system and the baths can be modelled as a stochastic process. The system density matrix $\hat{\rho}$ does not directly provide information about the number of jumps n from and into either bath $\alpha = \{L, R\}$. Hence, in order to track n throughout the system's evolution, following the approach in [14], it is convenient to introduce the *conditional density matrix* $\hat{\rho}^{(n_\alpha)}(t)$, which provides information about the number of particles exchanged between the system and a particular bath α at any given time t. $\hat{\rho}^{(n_\alpha)}(t)$ is such that the probability that n particles have been exchanged between the system and α until time t is

$$P^{(n_\alpha)}(t) = \mathrm{Tr}\big[\hat{\rho}^{(n_\alpha)}(t)\big]. \tag{3.14}$$

In order to study the time evolution of $\hat{\rho}^{(n_\alpha)}(t)$ Eq. 2.20 can be rewritten as

$$\dot{\hat{\rho}}^{(n_\alpha)}(t) = \mathcal{L}\hat{\rho}^{(n_\alpha)} = \mathcal{L}^0\hat{\rho}^{(n_\alpha)}(t) + \mathcal{L}^-\hat{\rho}^{(n_\alpha+1)}(t) + \mathcal{L}^+\hat{\rho}^{(n_\alpha-1)}(t), \tag{3.15}$$

where $\mathcal{L}^+\hat{\rho}^{(n_\alpha-1)}$ and $\mathcal{L}^-\hat{\rho}^{(n_\alpha+1)}$ account for the jumps in and out of bath α, whilst $\mathcal{L}^0\hat{\rho}^{(n_\alpha)}$ accounts for the terms that do not change the number of particles in the system. It is important to remark that the jumps are assumed to be instantaneous [30]. Eq. 3.17 represents a system of a possibly infinite number of equations. Shifting the description to Fourier space can reduce the system to one single equation, and thus the *counting density matrix* is defined as

$$\hat{\rho}(\chi, t) = \sum_n \hat{\rho}^{(n_\alpha)}(t)e^{i\chi n}, \tag{3.16}$$

where χ is referred to as the *counting field*. Hence, Eq. 3.15 can be rewritten as

$$\dot{\hat{\rho}}(\chi, t) = \mathcal{L}^0(t)\hat{\rho}(\chi, t) + e^{-i\chi}\mathcal{L}^-_\alpha\hat{\rho}(\chi, t) + e^{i\chi}\mathcal{L}^+_\alpha\hat{\rho}(\chi, t) = \mathcal{L}(\chi)\hat{\rho}(\chi, t), \tag{3.17}$$

where $\mathcal{L}(0)$ corresponds to the original Liouvillian. Since $\mathcal{L}(\chi)$ is time-independent, for an initial density operator $\hat{\rho}_0$,

$$\hat{\rho}(\chi, t) = \exp[\mathcal{L}(\chi)t]\hat{\rho}_0, \tag{3.18}$$

where it is assumed that the are no jumps at the initial time t_0 and thus that $\hat{\rho}(\chi, t_0) = \hat{\rho}_0$.

3.3.1 Mean current

In general, for a given discrete random variable $\mathbf{X}$ with probability mass function $P(x)$, a statistical moment of order k is defined as

$$\langle X^k \rangle = \sum_i x_i^k P(x_i). \tag{3.19}$$

In the case of the number of particles exchanged between the system and R, the probability mass function at time t is given by Eq. 3.14. By rewriting $\hat{\rho}(\chi, t)$ as in Eq. 3.16, the following quantity can be defined:

$$\mathcal{M}(\chi,t) = \mathrm{Tr}[\hat{\rho}(\chi,t)] = \mathrm{Tr}\left[\sum_{n_\alpha} \hat{\rho}^{(n_\alpha)}(t) e^{i\chi n_\alpha}\right] = \sum_{n_\alpha} P_{n_\alpha}(t) e^{i\chi n_\alpha}. \tag{3.20}$$

Furthermore, the following holds:

$$(-i)^k \frac{\partial^k}{\partial \chi^k} \mathcal{M}(\chi,t)\Big|_{\chi=0} = (-i)^k \sum_{n_\alpha} (i n_\alpha)^k P_{n_\alpha}(t) e^{i\chi n_\alpha}\Big|_{\chi=0} = \sum_n n^k P_{n_\alpha}(t) = \langle n_\alpha^k(t)\rangle. \tag{3.21}$$

Hence, $\mathcal{M}(\chi,t)$ is referred to as the *moment generating function* [31]. The first order moment corresponds to the mean of the random variable. Thus, the *mean jump current* through bath α reads

$$\langle I_\alpha(t)\rangle = \frac{d}{dt}\langle n_\alpha(t)\rangle. \tag{3.22}$$

If the initial density operator is chosen to be the steady state $\hat{\rho}_{SS}$, $\langle \hat{N}\rangle$ does not change, where $\hat{N}$ is the number operator for the system, and thus the mean currents for the two baths are equal in absolute value. The α subscript is henceforth dropped. The *stationary current* $\langle I\rangle$ for either of the baths can be then calculated as

$$\begin{aligned}\langle I\rangle = -i\frac{\partial}{\partial\chi}\frac{d}{dt}\mathcal{M}(\chi,t)\Big|_{\chi=0} &= -i\frac{\partial}{\partial\chi}\frac{d}{dt}\,\mathrm{Tr}[\hat{\rho}(\chi)]\Big|_{\chi=0} = -i\frac{\partial}{\partial\chi}\frac{d}{dt}\,\mathrm{Tr}\{\exp[\mathcal{L}(\chi)t]\hat{\rho}_{SS}\}\Big|_{\chi=0} \\ &= -i\frac{\partial}{\partial\chi}\,\mathrm{Tr}\{\mathcal{L}(\chi)\exp[\mathcal{L}(\chi)t]\hat{\rho}_{SS}\}\Big|_{\chi=0} \\ &\quad .\end{aligned} \tag{3.23}$$

Replacing $\exp[\mathcal{L}(\chi)t]$ with its Taylor expansion

$$\exp[\mathcal{L}(\chi)t] = \sum_{j=0}^{\infty} \frac{[\mathcal{L}(\chi)t]^j}{j!},$$

and using that $\mathcal{L}(0)\hat{\rho}_{SS} = 0$ and $\exp[\mathcal{L}(0)t]\hat{\rho}_{SS} = \hat{\rho}_{SS}$ yield

$$\langle I\rangle = -i\,\mathrm{Tr}\left\{\sum_{j=0}^{\infty}\mathcal{L}'(\chi)\frac{[\mathcal{L}(\chi)t]^j}{j!}\hat{\rho}_{SS} + \sum_{j=0}^{\infty}\mathcal{L}'(\chi)[\mathcal{L}(\chi)t]^{j+1}\hat{\rho}_{SS}\right\}\Big|_{\chi=0} = -i\,\mathrm{Tr}[\mathcal{L}'(0)\hat{\rho}_{SS}] \tag{3.24}$$

3.3.2 Current fluctuations

Another important quantity that characterizes the total particle number or charge is its variance

$$\mathrm{Var}[n(t)] = \langle n^2(t)\rangle - \langle n(t)\rangle^2, \tag{3.25}$$

which provides information on the statistical fluctuations of the measured number of particle jumps.
From this, it is useful to define the *noise*

$$D(t) = \frac{d}{dt}\mathrm{Var}[n(t)], \tag{3.26}$$

as, in the steady state, for a time interval τ, [8]

$$\frac{\mathrm{Var}[n(t)]}{\langle n(t)\rangle^2} = \frac{D(t)}{\tau\langle I\rangle^2}, \tag{3.27}$$

which is akin to the quantity bounded by the TUR bound in 2.23. Furthermore, in the long time limit, $D(t)$ becomes independent of time, and thus it can be rewritten as D. In order to derive an expression for D from $\mathcal{L}(\chi, t)$, it is convenient to define the *cumulant* of order k of a random variable $\mathbf{X}$, expressed through

$$\langle\!\langle X^k \rangle\!\rangle = (-i)^k \frac{\partial^k}{\partial \chi^k}\mathcal{C}(\chi)\bigg|_{\chi=0}, \tag{3.28}$$

where $\mathcal{C}(\chi)$ reads

$$\mathcal{C}(\chi) = \ln[\mathcal{M}(\chi)], \tag{3.29}$$

and takes the name of *cumulant generating function*. The first and second order cumulants correspond respectively to the mean and the variance of the distribution. In the case for $n(t)$,

$$\mathcal{C}(\chi, t) = \ln\{\mathrm{Tr}[\hat{\rho}(\chi, t)]\}. \tag{3.30}$$

In the long-time limit, when the system reaches the steady state, the dominant contribution to $\mathcal{C}(\chi, t)$ is provided by the eigenvalue $\lambda_0(\chi)$ of $\mathcal{L}(\chi)$ with the largest real part. Hence,

$$\mathcal{C}(\chi, t) \approx \lambda_0(\chi)t, \tag{3.31}$$

where $\lambda_0(\chi)$ is such that

$$\lambda_0(0) = 0, \tag{3.32}$$

In general, it is not straight-forward to determine $\lambda_0(\chi)$ analytically, and thus it may be more convenient to follow the approach shown in [32], where the following quantity is defined:

$$P(\chi, x) = \det\Big(\tilde{\mathcal{L}}(\chi) - x\mathbb{1}\Big), \tag{3.33}$$

where $\tilde{\mathcal{L}}(\chi)$ is the matrix representation of the counting Liouvillian, defined in the same way as in Eq. 3.11. $P(\chi, x)$ the characteristic polynomial of $\tilde{\mathcal{L}}(\chi)$, and as such it can be expanded as

$$P(\chi, x) = \sum_{j}^{N} a_j(\chi) x^j, \tag{3.34}$$

where N is the dimension of L and the coefficients $a_j(\chi) = (1/j!)\partial_x^j P(x,\chi)|_{x=0}$ can be replaced by their Taylor expansions

$$a_j(\chi) = \sum_{k}^{\infty} a_j^{(k)} \frac{\chi^k}{k!}, \tag{3.35}$$

where

$$a_j^{(k)} = \partial_\chi^k a_j(\chi)|_{\chi=0}. \tag{3.36}$$

Eq. 3.34 can then be rewritten as

$$P(\chi, x) = \sum_{j=0}^{N} \sum_{k=0}^{\infty} a_j^{(k)} \frac{\chi^k}{k!} x^j. \tag{3.37}$$

The jump fluctuations can then be expressed as (Appendix B.1)

$$\langle\langle n(t) \rangle\rangle = \mathrm{Var}[n(t)] = i \frac{\langle n(t) \rangle}{a_1^{(0)}} \left(\frac{a_0^{(2)} a_1^{(0)}}{a_0^{(1)}} + \frac{2 a_0^{(1)} a_2^{(0)}}{a_1^{(0)}} - 2 a_1^{(1)} \right) \tag{3.38}$$

Finally, since $D = (d/dt)\langle\langle n(t) \rangle\rangle$

$$D = -i \frac{\langle I \rangle}{a_1^{(0)}} \left(\frac{a_0^{(2)} a_1^{(0)}}{a_0^{(1)}} + \frac{2 a_0^{(1)} a_2^{(0)}}{a_1^{(0)}} - 2 a_1^{(1)} \right). \tag{3.39}$$

Hence, the TUR bound in Eq. 2.23 can be rewritten as

$$\frac{\langle I \rangle^2}{D \langle \sigma \rangle} \leq \frac{1}{2}. \tag{3.40}$$

3.3.3 Estimation precision

The expressions for the mean current $\langle I \rangle$ and the noise D may now be used to calculate the information about a system parameter θ encoded in the jump current. To this end, for a given measurement time τ, in the steady state, the *current precision* can be expressed as the error propagation formula [1]

$$(\Delta\theta)^{-2} = \frac{(\partial_\theta \langle I \rangle)^2}{D} \tau. \tag{3.41}$$

Under this definition, the current precision is expressed in a similar form as the reciprocal of the quantity bounded by the Cramér-Rao bound in Eq. 2.31.

3.4 Entropy production and TUR violations

The mean entropy production rate for the two qubit system is [33]

$$\langle\sigma\rangle = \partial_t S(\hat{\rho}) - \left(\frac{J_L}{T_L} + \frac{J_R}{T_R}\right), \tag{3.42}$$

where $S(\hat{\rho})$ is the *von Neumann entropy*

$$S(\hat{\rho}) = \mathrm{Tr}(\hat{\rho}\ln(\hat{\rho})), \tag{3.43}$$

and J_α are the heat currents between the system and the two baths. In the steady state $\hat{\rho}_{SS}$, the von Neumann entropy is constant, and thus $\langle\sigma\rangle$ reduces to

$$\langle\sigma\rangle = \frac{J_L}{T_L} + \frac{J_R}{T_R}, \tag{3.44}$$

where [26]

$$J_\alpha = \pm\langle I\rangle(\varepsilon_\alpha - \mu_\alpha). \tag{3.45}$$

In order to study such violations, the amount of TUR *violation* V_{TUR} can be defined as [34]

$$V_{TUR} = \max\{0, \frac{\langle I\rangle^2}{D\langle\sigma\rangle} - \frac{1}{2}\}. \tag{3.46}$$

3.5 Continuous measurement QFI

In order to measure the jump current, one needs to perform some kind of measurement that continuously monitors the baths and system. This aspect is captured by the current precision in Eq. 3.41, which, in the steady state, scales with the measurement time.

Alternatively to the discrete measurement protocol highlighted in Sec. 2.5, the quantum Fisher information may also be defined and computed for continuous measurements. If the encoding CPTP map for a parameter θ is generated by a Markovian Liouvillian written in GKSL form, the continuous measurement QFI in the steady state for a large measurement time τ reads [8, 35]

$$\mathcal{F}(\hat{\rho}_{SS}(\theta)) = 4\tau\left(\sum_j \langle(\partial_\theta \hat{L}_j^\dagger)(\partial_\theta \hat{L}_j)\rangle - \mathbf{1}^T\tilde{\mathcal{V}}_L\tilde{\mathcal{L}}_D\tilde{\mathcal{V}}_R\mathbf{p}_{SS} - \mathbf{1}^T\tilde{\mathcal{V}}_R\tilde{\mathcal{L}}_D\tilde{\mathcal{V}}_L\mathbf{p}_{SS}\right), \tag{3.47}$$

where $\tilde{\mathcal{V}}_{L/R}$ are the matrix representations of the superoperators

$$\mathcal{V}_L\hat{\rho} = -i\partial_\theta\left(\hat{H} - \frac{i}{2}\sum_j \hat{L}_j^\dagger\hat{L}_j\right)\hat{\rho} + \sum_j(\partial_\theta\hat{L}_j)\hat{\rho}\hat{L}_j^\dagger \tag{3.48}$$

$$\mathcal{V}_R\hat{\rho} = i\hat{\rho}\partial_\theta\left(\hat{H}^\dagger - \frac{i}{2}\sum_j \hat{L}_j^\dagger\hat{L}_j\right) + \sum_j \hat{L}_j\hat{\rho}(\partial_\theta\hat{L}_j^\dagger), \tag{3.49}$$

$\mathbf{1} = \sum_j |j\rangle^* \otimes |j\rangle$ is the *vectorised identity* in the basis $\{|j\rangle\}$, and $\tilde{\mathcal{L}}_D$ is the *Drazin inverse* of the matrix representation of the Liouvillian. As explained in Sec. 3.2, a time independent Liouvillian has at least one steady state and hence at least one density matrix belonging to the zero eigenvalue. Thus, for an Liouville space of dimension N^2, $\text{rank}(\tilde{\mathcal{L}}) \leq N^2 - 1$. The Drazin inverse is hence defined as the unique matrix $\tilde{\mathcal{L}}_D$ such that [36]

$$\tilde{\mathcal{L}}^{k+1}\tilde{\mathcal{L}}_D = \tilde{\mathcal{L}}^k, \quad \tilde{\mathcal{L}}_D\tilde{\mathcal{L}}\tilde{\mathcal{L}}_D = \tilde{\mathcal{L}}_D, \quad \tilde{\mathcal{L}}_D\tilde{\mathcal{L}} = \tilde{\mathcal{L}}\tilde{\mathcal{L}}_D, \tag{3.50}$$

where k is the smallest integer such that $\text{rank}(\tilde{\mathcal{L}}^{k+1}) = \text{rank}(\tilde{\mathcal{L}}^k)$. By rewriting $\tilde{\mathcal{L}}$ as $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_0 + \tilde{\mathcal{J}}$, where $\tilde{\mathcal{L}}_0$ and $\tilde{\mathcal{J}}$ are respectively the *no-jump* term

$$\tilde{\mathcal{L}}_0 = -i(\hat{H} \otimes \mathbb{1} - \mathbb{1} \otimes \hat{H}^T) - \frac{1}{2}\sum_i \gamma_i \left(\hat{L}_i^\dagger \hat{L}_i \otimes \mathbb{1} + \mathbb{1} \otimes \hat{L}_i^T \hat{L}_i^* \right), \tag{3.51}$$

and the *jump* term

$$\tilde{\mathcal{J}} = \sum_i \gamma_i \hat{L}_i \otimes \hat{L}_i^*, \tag{3.52}$$

the Drazin inverse can be computed as [8]

$$\tilde{\mathcal{L}}_D = (\mathbb{1} + \mathcal{Q}\tilde{\mathcal{L}}_0^{-1}\tilde{\mathcal{J}})^{-1}\mathcal{Q}\tilde{\mathcal{L}}_0^{-1}\mathcal{Q}, \tag{3.53}$$

where $\mathcal{Q} = \mathbb{1} - \mathbf{p}_{SS}\mathbf{1}^T$.

In the case of the local master equation in Eq. 3.8, the operators $\sigma_\alpha^\pm$ do not depend on the system parameters, and thus Eq. 3.47 reduces to

$$\mathcal{F}(\hat{\rho}_{SS}(\theta)) = -4\tau(\mathbf{1}^T\tilde{\mathcal{V}}_L\tilde{\mathcal{L}}_D\tilde{\mathcal{V}}_R\mathbf{p}_{SS} + \mathbf{1}^T\tilde{\mathcal{V}}_R\tilde{\mathcal{L}}_D\tilde{\mathcal{V}}_L\mathbf{p}_{SS}), \tag{3.54}$$

with $\tilde{\mathcal{V}}_L = -i(\partial_\theta\hat{H})\hat{\rho}$ and $\tilde{\mathcal{V}}_R = i\hat{\rho}(\partial_\theta\hat{H}^\dagger)$.

4 Results and discussion

In this section, analytical expressions for the steady state density matrix $\hat{\rho}_{SS}$, the mean current $\langle I \rangle$, and the noise D are shown. Furthermore, the precision of parameter estimation in the system and its non-classicality are studied through three figures of merit: the current precision $(\Delta\theta)^{-2}$, violations of the TUR in Eq. 3.40, and the QFI for continuous measurements on the system. In particular, the current precision and the continuous measurement QFI are analysed for the inter-qubit coupling g and the detuning δ. For all numerical results, the condition $\kappa_L = \kappa_R = \kappa$ is applied.

4.1 Steady state density matrix, mean current and fluctuations

In the steady state, the Hamiltonian is such that, in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the only non-vanishing coherences are $\hat{\rho}_{12}$ and $\hat{\rho}_{21}$ [37]. Hence, the steady state density matrix $\hat{\rho}_{SS}$ can be written as

$$\hat{\rho}_{\rm SS} = \begin{pmatrix} p_0 & 0 & 0 & 0 \\ 0 & p_1 & c & 0 \\ 0 & c^* & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix}.$$

By following the procedure in 3.2, the non-vanishing elements of $\hat{\rho}_{SS}$ read

$$p_0 = \frac{4g^2(\gamma_R^- + \gamma_L^-)^2 + \gamma_R^-\gamma_L^-(4\delta^2 + \Gamma^2)}{\chi} \tag{4.1}$$

$$p_1 = \frac{4g^2(\gamma_R^- + \gamma_L^-)(\gamma_R^+ + \gamma_L^+) + \gamma_R^+\gamma_L^-(4\delta^2 + \Gamma^2)}{\chi} \tag{4.2}$$

$$p_2 = \frac{4g^2(\gamma_R^- + \gamma_L^-)(\gamma_R^+ + \gamma_L^+) + \gamma_R^-\gamma_L^+(4\delta^2 + \Gamma^2)}{\chi} \tag{4.3}$$

$$p_3 = \frac{4g^2(\gamma_R^+ + \gamma_L^+)^2 + \gamma_R^+\gamma_L^+(4\delta^2 + \Gamma^2)}{\chi} \tag{4.4}$$

$$c = \frac{2g(\gamma_R^-\gamma_L^+ - \gamma_R^+\gamma_L^-)(2\delta - i\Gamma)}{\chi}, \tag{4.5}$$

where

$$\gamma_\alpha^+ := \kappa_\alpha N_\alpha(\bar{\varepsilon}), \gamma_\alpha^- := \kappa_\alpha(1 - N_\alpha(\bar{\varepsilon}))$$

$$\Gamma = \kappa_L + \kappa_R$$

$$\chi = \Gamma^2(4g^2 + \kappa_L\kappa_R) + 4\delta^2\kappa_L\kappa_R$$

The mean current and the noise, calculated with the full counting statistics scheme in Sec. 3.3, read (Appendix C)

$$\langle I \rangle = \frac{4g^2\kappa_L\kappa_R(\kappa_L + \kappa_R)(N_L - N_R)}{(\kappa_L + \kappa_R)^2(4g^2 + \kappa_L\kappa_R) + 4\delta^2\kappa_L\kappa_R} \tag{4.6}$$

$$D = \frac{N_L + N_R - 2N_LN_R}{N_L - N_R}\langle I \rangle - \frac{2}{\kappa_L + \kappa_R}\langle I \rangle^2\left(1 + \frac{(\kappa_L + \kappa_R)^2 + 4\delta^2(\kappa_L - \kappa_R)^2/(\kappa_L + \kappa_R)^2}{4g^2 + \kappa_L\kappa_R + 4\delta^2\kappa_L\kappa_R/(\kappa_L + \kappa_R)^2}\right) \tag{4.7}$$

4.2 Current-based estimation precision

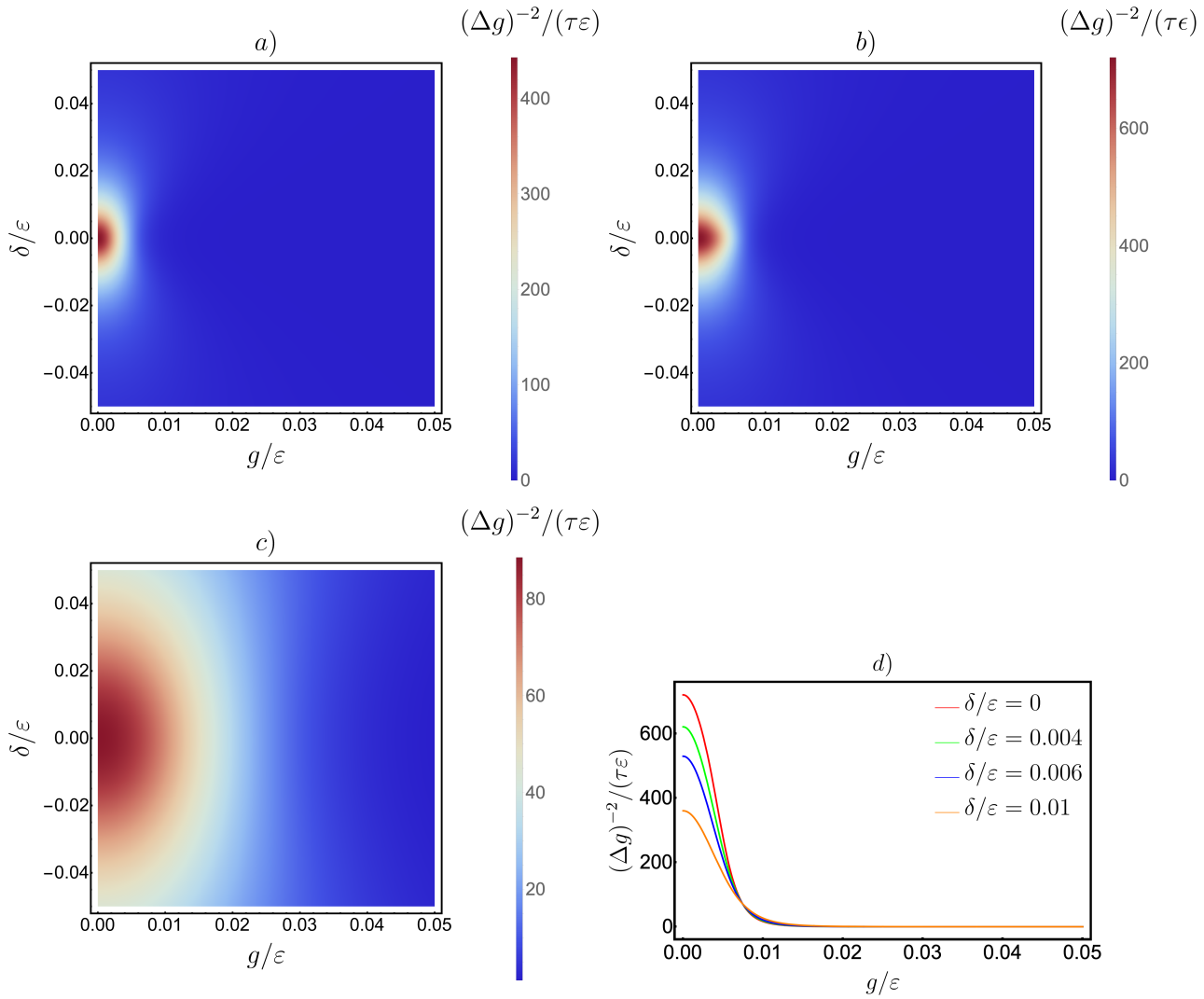

Figure 2: Current precision for the inter-qubit coupling g as a function of g and δ [a), b), c)]. Cross section of b) for different values of δ [d)]. Common parameters to all plots: $T_L/\varepsilon = T_R/\varepsilon = T/\varepsilon = 1$. Parameters for individual plots: $\kappa_L/\varepsilon = \kappa_R/\varepsilon = \kappa/\varepsilon = 0.01, \mu_L/\varepsilon = 2, \mu_R/\varepsilon = -1$ [a)], $\kappa_L/\varepsilon = \kappa_R/\varepsilon = \kappa/\varepsilon = 0.01, \mu_L/\varepsilon = 4, \mu_R/\varepsilon = -2$ [b), d)],$\kappa_L/\varepsilon = \kappa_R/\varepsilon = \kappa/\varepsilon = 0.05, \mu_L/\varepsilon = 2, \mu_R/\varepsilon = -1$ [c)]

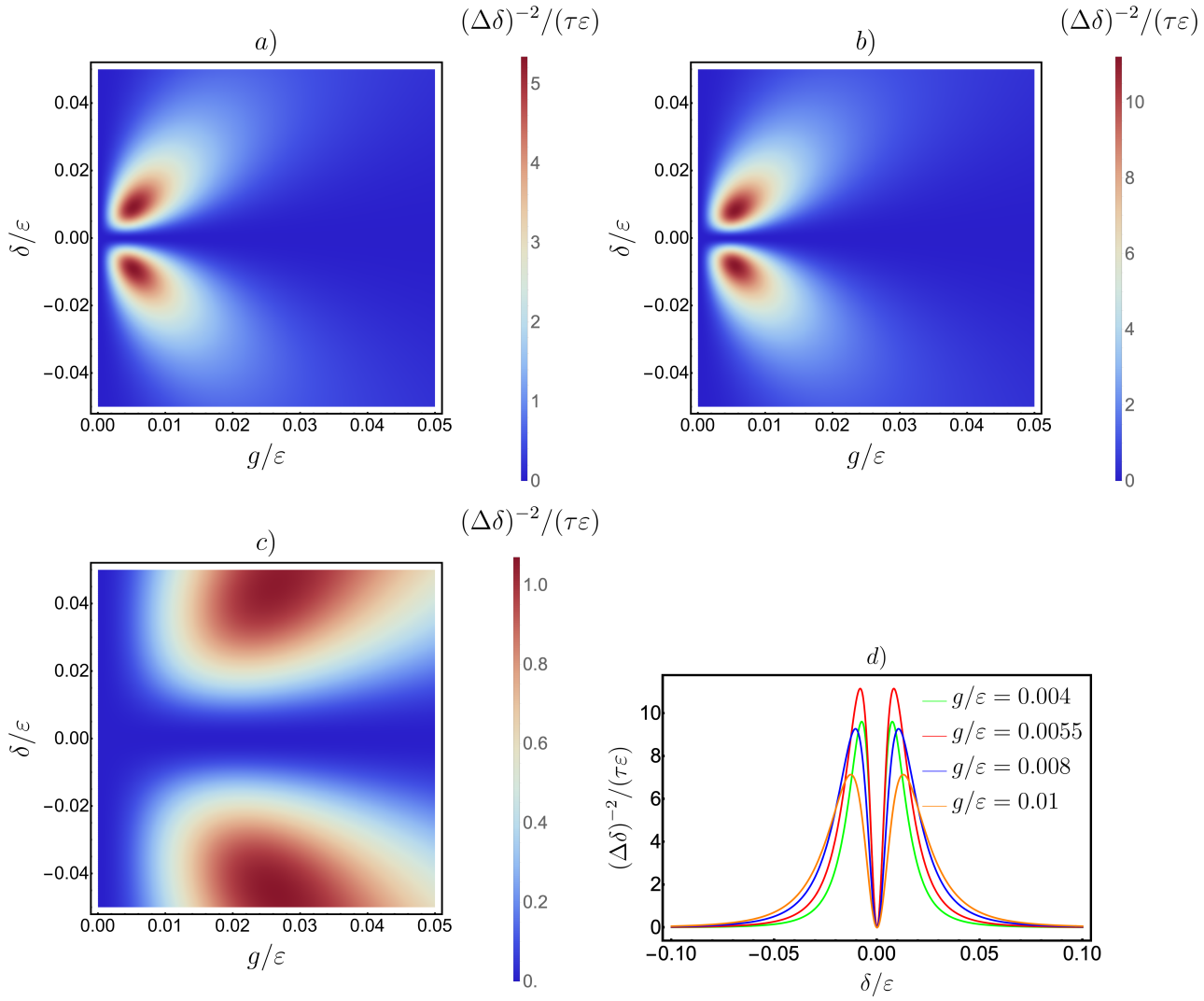

Figure 3: Current precision for the detuning δ as a function of g and δ [a), b), c)]. Cross section of b) for different values of δ [d)]. Common parameters to all plots: $T_L/\varepsilon = T_R/\varepsilon = T/\varepsilon = 1$. Parameters for individual plots: $\kappa_L/\varepsilon = \kappa_R/\varepsilon = \kappa/\varepsilon = 0.01, \mu_L/\varepsilon = 2, \mu_R/\varepsilon = -1$ [a)], $\kappa_L/\varepsilon = \kappa_R/\varepsilon = \kappa/\varepsilon = 0.01, \mu_L/\varepsilon = 4, \mu_R/\varepsilon = -2$ [b), d)],$\kappa_L/\varepsilon = \kappa_R/\varepsilon = \kappa/\varepsilon = 0.05, \mu_L/\varepsilon = 2, \mu_R/\varepsilon = -1$ [c)]

As it can be noticed in Fig. 2, the current precision for the inter-qubit coupling is maximal for $g, \delta \ll \kappa$, and rapidly decays with increasing g until $g \simeq \kappa$. In fact, as highlighted in [34], from a transport physics perspective, g is the rate of quantum tunnelling between the two qubits, and as g and the bath damping rates $\kappa_L = \kappa_R = \kappa$ become comparable in magnitude, the inter-qubit tunnelling and the jump process between the system and the baths become equally likely. As g increases beyond κ, the influx of fermions from the baths bottlenecks the inter-qubit tunnelling, and thus the mean current $\langle I \rangle$ no longer increases with g. Hence, $\partial_g \langle I \rangle$ vanishes with increasing g beyond κ, and so does the current precision for g.

On the other hand, as it is shown in Fig. 3, the current precision for the detuning δ

vanishes around $\delta = 0$, but rapidly increases until $|\delta|$ becomes comparable with κ, where it reaches its maximum. Interestingly, the maxima occur at non-vanishing value of g. For the parameter values in Figs. 3b) and 3d), the optimal value of the inter-qubit coupling is $g/\varepsilon = 0.0055$.

4.3 TUR violations and effects on current precision

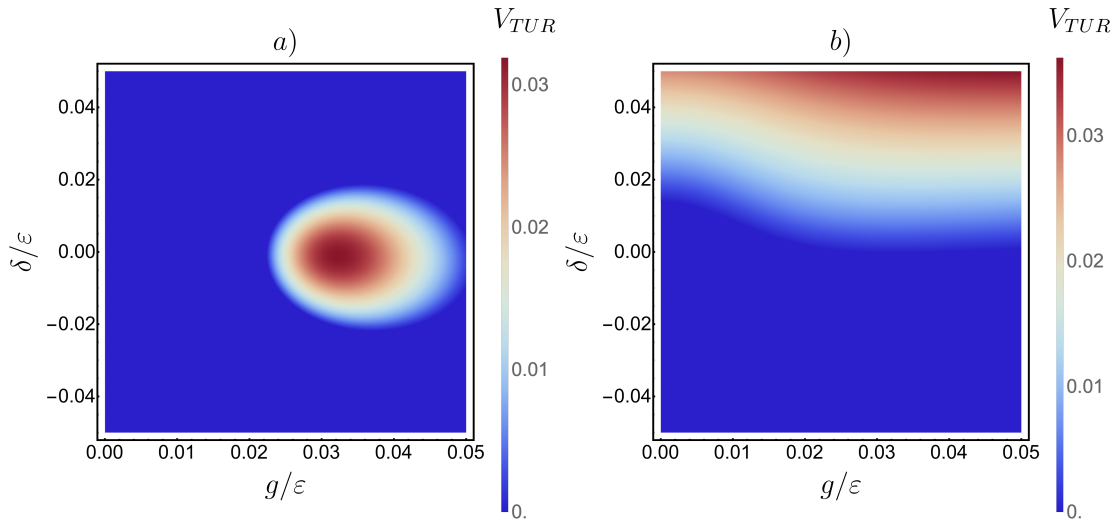

Figure 4: V_{TUR} as a function of g and δ in the case of an applied bias [a)] and a temperature gradient [b)]. Common parameters: $\kappa_L/\varepsilon = \kappa_R/\varepsilon = \kappa/\varepsilon = 0.05$. Parameters for individual plots: $T_L/\varepsilon = T_R/\varepsilon = T/\varepsilon = 1$, $\mu_L/\varepsilon = 2, \mu_R/\varepsilon = -1$ [a)], $\mu_L/\varepsilon =, \mu_R/\varepsilon = 0, T_L/\varepsilon = 1, T_R/\varepsilon = 2$ [b)].

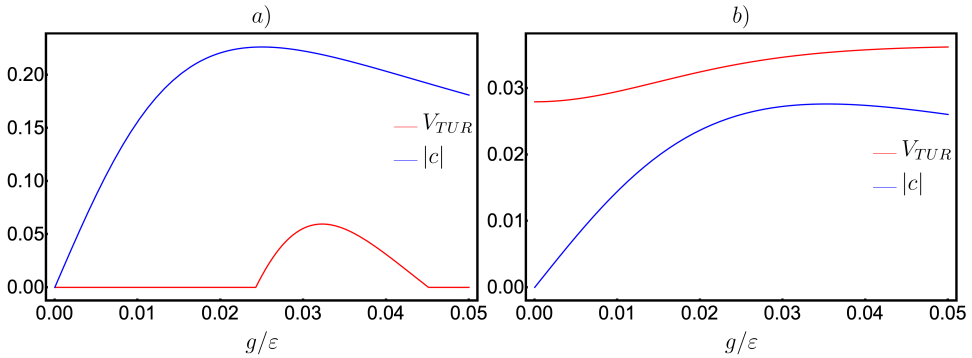

Figure 5: Comparisons between V_{TUR} and the norm of coherence $|c|$ as a function of g and δ in the case of an applied bias [a)] and a temperature gradient [b)]. Common parameters: $\kappa_L/\varepsilon = \kappa_R/\varepsilon = \kappa/\varepsilon = 0.05$. Parameters for individual plots: $T_L/\varepsilon = T_R/\varepsilon = T/\varepsilon = 1$, $\mu_L/\varepsilon = 2, \mu_R/\varepsilon = -1$ [a)], $\mu_L/\varepsilon =, \mu_R/\varepsilon = 0, T_L/\varepsilon = 1, T_R/\varepsilon = 2$ [b)].

As shown in Sec. 4.1, $\hat{\rho}_{SS}$ presents coherences between states $|01\rangle$ and $|10\rangle$ in the computational basis, and thus the jump process between the qubits and the baths cannot be

reduced to a classical continuous-time Markov chain. Violations of TUR were found, and are presented in Fig. 4. It is found that TUR violations can occur both in the presence of a difference in chemical potential between the baths, achieved by applying an external voltage gradient (*bias*), and with a temperature gradient. In the former case, the violation peaks at $g = 0.0321$, whilst in the latter the violation is maximal for $\delta = 0.05$. Interestingly, as shown in Fig. 5, the increase of norm of the coherence $|c|$ with varying g seems to correspond to the occurrence of TUR violations in some degree. Hence, coherences in the steady state play a significant role in TUR violations.

It is now shown analytically how the TUR bound induces an upper thermodynamic bound on the current precision for g and δ, and the subsequent precision advantage allowed by TUR violations. The partial derivative of the mean current with respect to δ reads

$$\partial_\delta \langle I \rangle = -\frac{32 g^2 \kappa_L^2 \kappa_R^2 (\kappa_L + \kappa_R)(N_L - N_R)\delta}{[(\kappa_L + \kappa_R)^2 (4g^2 + \kappa_L \kappa_R) + 4\delta^2 \kappa_L \kappa_R]^2}. \tag{4.8}$$

By comparison with the mean current in Eq. 4.6,

$$\partial_\delta \langle I \rangle = \frac{8\kappa_L \kappa_R \delta}{(\kappa_L + \kappa_R)^2 (4g^2 + \kappa_L \kappa_R) + 4\delta^2 \kappa_L \kappa_R} \langle I \rangle, \tag{4.9}$$

and thus the precision $(\Delta\delta)^{-2}/\tau$ may now be expressed as

$$(\Delta\delta)^{-2}/\tau = \left(\frac{8\kappa_L \kappa_R \delta}{(\kappa_L + \kappa_R)^2 (4g^2 + \kappa_L \kappa_R) + 4\delta^2 \kappa_L \kappa_R} \right)^2 \frac{\langle I \rangle^2}{D}. \tag{4.10}$$

The current precision for δ, now expressed in terms of $\langle I \rangle^2 / D$, can now be directly connected to the TUR bound in Eq. 3.40, and $(\Delta\delta)^{-2}/\tau$ is thus bounded by

$$\begin{aligned} (\Delta\delta)^{-2}/\tau &= \left(\frac{8\kappa_L \kappa_R \delta}{(\kappa_L + \kappa_R)^2 (4g^2 + \kappa_L \kappa_R) + 4\delta^2 \kappa_L \kappa_R} \right)^2 \frac{\langle I \rangle^2}{D} \frac{\langle \sigma \rangle}{\langle \sigma \rangle} \\ &\leq (\Delta\delta)^{-2}_{\mathrm{TUR}} \equiv \frac{1}{2} \left(\frac{8\kappa_L \kappa_R \delta}{(\kappa_L + \kappa_R)^2 (4g^2 + \kappa_L \kappa_R) + 4\delta^2 \kappa_L \kappa_R} \right)^2 \langle \sigma \rangle \equiv f(\delta) \langle \sigma \rangle, \end{aligned} \tag{4.11}$$

which represents the largest current precision for δ allowed by classical stochastic thermodynamics. Hence, achieving a higher precision implies a TUR violation and the presence of non-classical thermodynamic behaviour.

A similar procedure can be repeated for $(\Delta g)^{-2}/\tau$, which is bounded by $(\Delta g)^{-2}/\tau \leq h(g)\langle \sigma \rangle$, where

$$h(g) = \frac{1}{2} \left(\frac{2\kappa_L \kappa_R (4\delta^2 + (\kappa_L + \kappa_R)^2)}{g[(\kappa_L + \kappa_R)^2 (4g^2 + \kappa_L \kappa_R) + 4\delta^2 \kappa_L \kappa_R]} \right)^2. \tag{4.12}$$

Fig. 6 shows how the current precision outperforms the thermodynamic precision bound in the presence of TUR violations.

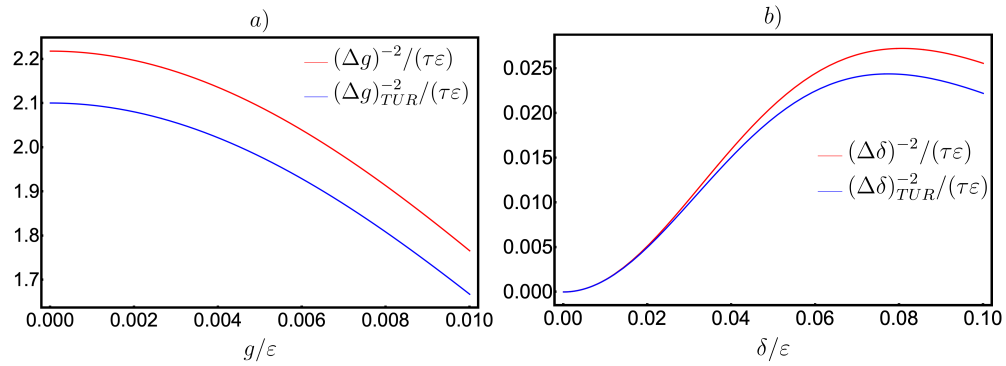

Figure 6: Classical thermodynamic precision bounds compared with current precision for g [a)] and δ [b)]. Parameters: $\kappa_L/\varepsilon = \kappa_R/\varepsilon = \kappa/\varepsilon = 0.05, T_L/\varepsilon = T_R/\varepsilon = T/\varepsilon = 1, \mu_L/\varepsilon = 0, \mu_R/\varepsilon = 0, T_L/\varepsilon = 1, T_R/\varepsilon = 2$.

4.4 Continuous measurement QFI

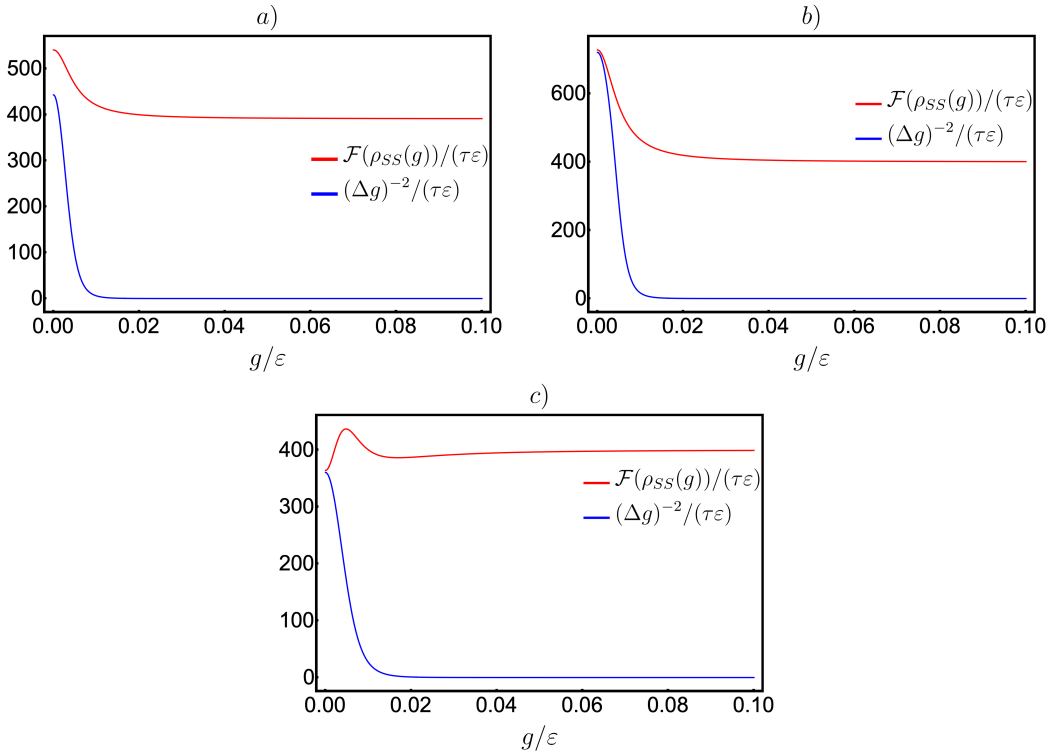

Figure 7: QFI for continuous measurements compared to the current precision in the steady state for the inter-qubit coupling g. Common parameters to all plots: $T_L/\varepsilon = T_R/\varepsilon = T/\varepsilon = 1, \kappa_L/\varepsilon = \kappa_R/\varepsilon = \kappa/\varepsilon = 0.01$. Parameters for individual plots: $\mu_L/\varepsilon = 2, \mu_R/\varepsilon = -1, \delta/\varepsilon = 0$ [a)], $\mu_L/\varepsilon = 4, \mu_R/\varepsilon = -2, \delta/\varepsilon = 0$ [b)], $\mu_L/\varepsilon = 4, \mu_R/\varepsilon = -2, \delta/\varepsilon = 0.01$ [c)].

As expected from the Cramér-Rao bound, the QFI bounds the current precision from above. As shown in Fig. 7, for intermediate bias ($\mu_L = 2, \mu_R = -1$) and vanishing detuning, the

current precision presents a similar behaviour as the steady state continuous measurement QFI around $g = 0$. Nevertheless, for $g > \kappa$, the QFI converges to a finite non-vanishing value.

Interestingly, for higher bias, the current precision nearly coincides with the QFI around $g = 0$, which implies that for $g \ll \kappa$ and vanishing detuning the jump current carries nearly as much information about g as the optimal measurement that saturates the Cramér-Rao bound. Hence, this suggests that in the aforementioned conditions the estimation of the inter-qubit coupling can be performed very effectively by measuring the jump current. As detuning is switched on, the QFI for g presents a different behaviour with increasing g. For the parameters in Fig. 7c), the QFI presents a peak for intermediate coupling ($g \simeq \kappa/2$), in correspondence with the drop in current precision. In this case, the jump current does no longer seem to provide a near-optimal estimation performance for the inter-qubit coupling.

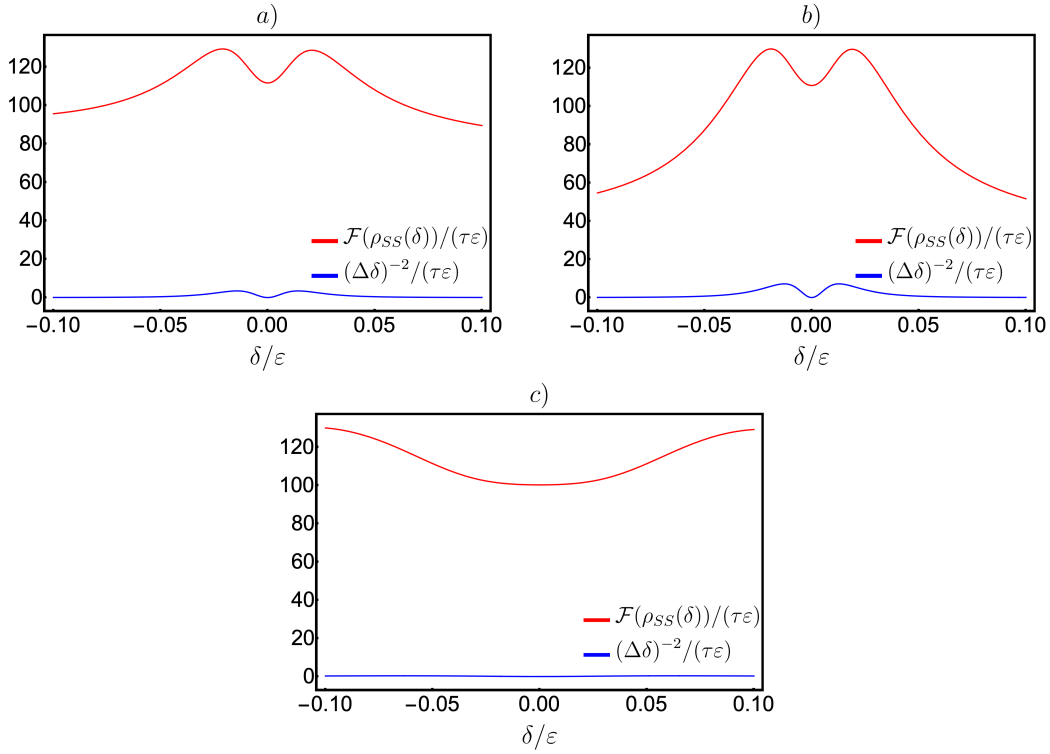

Figure 8: QFI for continuous measurements compared to the current precision in the steady state for the detuning δ. Common parameters to all plots: $T_L/\varepsilon = T_R/\varepsilon = T/\varepsilon = 1$, $k_L/\varepsilon = k_R/\varepsilon = k/\varepsilon = 0.01$. Parameters for individual plots: $\mu_L/\varepsilon = 2, \mu_R/\varepsilon = -1, g/\varepsilon = 0.01$ [a)], $\mu_L/\varepsilon = 4, \mu_R/\varepsilon = -2, g/\varepsilon = 0.01$ [b)], $\mu_L/\varepsilon = 4, \mu_R/\varepsilon = -2, g/\varepsilon = 0.05$ [c)].

In the case of the estimation precision for the detuning δ, the current precision seems to perform rather poorly in comparison to the maximal precision allowed by the Cramér-Rao bound. As shown in Fig. 8, the current precision is systematically lower than the QFI by at least one order of magnitude. Nevertheless, for the choices of parameters in Figs. 8a) and 8b), the current precision and the QFI show a similar behaviour with varying δ. Unlike the case for the inter-qubit coupling, a change in bias or in inter-qubit coupling does not have a significant effect on the maximum attainable value of the QFI for δ.

5 Conclusions and outlook

In this thesis, the precision of parameter estimation in the non-equilibrium steady state of a quantum thermal machine of two qubits couple to thermal baths was studied under the Born-Markov approximation. In particular, this work explored the possibility of using the steady state particle current between the system and the baths to estimate system parameters, and the potential advantage deriving from the quantum mechanical description of the system.

It was shown that, with a local master equation approach (Eq. 3.8), the thermodynamic uncertainty relation in Eq. 2.23 can be violated by the system, and how such violations are linked to the presence of coherence in the steady state density matrix. Furthermore, classical thermodynamic bounds on the on the estimation precisions of the inter-qubit coupling and the detuning were derived, and thus it was shown that TUR violations allow for higher current precision. Moreover, the current precision was compared to the quantum Fisher information, and it was found that for vanishing detuning it is possible for the current precision to nearly saturate the quantum Cramér-Rao bound for the inter-qubit coupling. Hence, in this model, it is possible to measure the coupling with near-maximal precision by measuring the steady state particle current.

There are several possible research directions that may be explored. The non-classical estimation advantage in the system has been linked to coherence, which is an inherent feature of quantum mechanics. Nonetheless, quantum entanglement is undoubtedly the phenomenon that sets quantum mechanics apart from any other classical theory, and thus studying the relation between entanglement and the non-classical gain in estimation precision could be worthwhile. Furthermore, in this work, violations of thermodynamic uncertainty relations were explored in order to achieve a better-than-classical estimation performance. Nevertheless, there exists other types of bounds in classical Markovian jump processes referred to as *Kinetic uncertainty relations* (KUR's), which set a trade-off between the mean current, its fluctuations and the mean number of jump events [38]. Hence, it might be interesting to study the non-classical advantage in parameter estimation that may arise from KUR violations. Moreover, the work could be extended to the transient regime, in which the two-qubit thermal machine could be explored through the lens of autonomous quantum clocks [39]. In addition, the Markovian master equation approach may be relinquished in favour of other methods which do not rely on the Born-Markov approximation, in order to generalise the research to out-of-equilibrium environments.

A Steady state density matrix

A.1 Calculation of $\hat{\rho}_{SS}$

The matrix representation of the system Hamiltonian, and of the creation and annihilation operators in the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis read

$$\hat{H} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon+\delta & g & 0 \\ 0 & g & \varepsilon & 0 \\ 0 & 0 & 0 & 2\varepsilon+\delta \end{pmatrix}, \tag{A.1}$$

$$\hat{\sigma}_L^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \hat{\sigma}_L^- = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.2}$$

$$\hat{\sigma}_R^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_R^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.3}$$

Given that the matrix representation of $\hat{\rho}_{SS}$ simplifies to Eq. 4.1, the master equation can be reduced to a linear system of 6 equations. The Liouvillian matrix $\tilde{\mathcal{L}}$, computed as prescribed in Eq. 3.12 in the $\{p_0, p_1, p_2, p_3, c, c^\star\}$ basis, reads

$$\tilde{\mathcal{L}} = \begin{pmatrix} -\gamma_L^+ - \gamma_R^+ & \gamma_R^- & \gamma_L^- & 0 & 0 & 0 \\ \gamma_R^+ & -\gamma_R^- - \gamma_L^+ & 0 & \gamma_L^- & ig & -ig \\ \gamma_L^+ & 0 & -\gamma_L^- - \gamma_R^+ & \gamma_R^- & -ig & ig \\ 0 & \gamma_R^+ & \gamma_L^+ & -\gamma_L^- - \gamma_R^- & 0 & 0 \\ 0 & ig & -ig & 0 & -\frac{\Gamma}{2} - i\delta & 0 \\ 0 & -ig & ig & 0 & 0 & -\frac{\Gamma}{2} + i\delta \end{pmatrix}. \tag{A.4}$$

Furthermore, given the the normalization constraint $p_{LR} = 1 - p_2 - p_1 - p_0$, there can be an additional reduction of one equation. Thus, the vectorised master equation may be rewritten as

$$\dot{\mathbf{p}} = M\mathbf{p} + \mathbf{v}, \tag{A.5}$$

where $\mathbf{v}$ is a constant vector and M is a matrix such that $\tilde{\mathcal{L}}\mathbf{p} = M\mathbf{p}+\mathbf{v}$. In the $\{p_0, p_1, p_2, c, c^*\}$ basis, the matrix M and the vector $\mathbf{v}$ (Eq. A.5) read

$$M = \begin{pmatrix} -\gamma_L^+ - \gamma_R^+ & \gamma_R^- & \gamma_L^- & 0 & 0 \\ \gamma_R^+ - \gamma_L^- & -\gamma_R^- - \gamma_L^- - \gamma_L^+ & -\gamma_L^- & ig & -ig \\ \gamma_L^+ - \gamma_R^- & -\gamma_R^- & -\gamma_L^- - \gamma_R^- - \gamma_R^+ & -ig & ig \\ 0 & ig & -ig & -\frac{\Gamma}{2} - i\delta & 0 \\ 0 & -ig & ig & 0 & -\frac{\Gamma}{2} + i\delta \end{pmatrix} \tag{A.6}$$

$$\mathbf{v} = \begin{pmatrix} 0 \\ \gamma_L^- \\ \gamma_R^- \\ 0 \\ 0 \end{pmatrix}. \tag{A.7}$$

Hence, given that $M\mathbf{p}_{SS} + \mathbf{v} = 0$, the steady state vector $\mathbf{p}_{SS}$ is solved for as

$$\mathbf{p}_{SS} = -M^{-1}\mathbf{v} \tag{A.8}$$

B Mean current and fluctuations

B.1 Derivation of Eq. 3.38

By definition [32],

$$P(\chi, \lambda_0(\chi)) = 0, \tag{B.1}$$

and thus

$$\frac{d^l}{d\chi^l} P(\chi, \lambda_0(\chi))\bigg|_{\chi=0} = 0 \qquad \forall l \geq 1. \tag{B.2}$$

For $l = 1$,

$$\begin{aligned} \frac{d}{d\chi} P(\chi, \lambda_0(\chi))\bigg|_{\chi=0} &= \sum_{j=0}^{N}\sum_{k=1}^{\infty}\left(a_j^{(k)}\frac{\chi^{k-1}}{k!}\lambda_0^j(\chi)\right) \\ &+ \sum_{j=1}^{N}\sum_{k=0}^{\infty}\left(a_j^{(k)} j\frac{\chi^{k}}{k!}\lambda_0'(\chi)\lambda_0^{j-1}(\chi)\right)\bigg|_{\chi=0} = 0 \end{aligned} \tag{B.3}$$

By the condition in Eq. 3.32, there only are non-vanishing terms for $\{j,k\} = \{1,0\}$ and $\{j,k\} = \{0,1\}$:

$$\frac{d}{d\chi} P(\chi, \lambda(\chi))\bigg|_{\chi=0} = a_0^{(1)} + a_1^{(0)}\lambda_0'(0) = 0. \tag{B.4}$$

The same procedure can be repeated for $l = 2$ to obtain

$$\frac{d^2}{d\chi^2} P(\chi, \lambda(\chi))\bigg|_{\chi=0} = a_0^{(2)} + a_1^{(0)}\lambda_0''(0) + 2a_1^{(1)}\lambda_0'(0) + 2a_2^{(0)}(\lambda_0'(0))^2 = 0. \tag{B.5}$$

Eqs. 3.28 and 3.31 imply that, in the steady state,

$$\langle n(t)\rangle = -i\lambda_0'(0)t \qquad \text{and} \qquad \langle\langle n(t)\rangle\rangle = -\lambda_0''(0)t. \tag{B.6}$$

Solving for $\lambda_0''(0)$ in terms of $\lambda_0'(0) = \langle n(t)\rangle$ in eqs. B.4 and B.5 thus yields

$$\langle n(t)\rangle = i\frac{a_0^{(1)}}{a_1^{(0)}}t \implies \langle I\rangle = i\frac{a_0^{(1)}}{a_1^{(0)}} \tag{B.7}$$

$$\langle\langle n(t)\rangle\rangle = \mathrm{Var}[n(t)] = -i\frac{\langle n(t)\rangle}{a_1^{(0)}}\left(\frac{a_0^{(2)}a_1^{(0)}}{a_0^{(1)}} + \frac{2a_0^{(1)}a_2^{(0)}}{a_1^{(0)}} - 2a_1^{(1)}\right)$$

$$\implies D = -i\frac{\langle I\rangle}{a_1^{(0)}}\left(\frac{a_0^{(2)}a_1^{(0)}}{a_0^{(1)}} + \frac{2a_0^{(1)}a_2^{(0)}}{a_1^{(0)}} - 2a_1^{(1)}\right) \quad \text{(B.8)}$$

C Calculation of mean current and fluctuations

If the counting is performed on the left qubit, the matrix representation $\tilde{\mathcal{L}}(\chi)$ of the counting Liouvillian in the $\{p_0, p_1, p_2, p_3, c, c^*\}$ basis reads

$$\tilde{\mathcal{L}}(\chi) = \begin{pmatrix} -\gamma_L^+ - \gamma_R^+ & \gamma_R^- & \gamma_L^- e^{-i\chi} & 0 & 0 & 0 \\ \gamma_R^+ & -\gamma_R^- - \gamma_L^+ & 0 & \gamma_L^- e^{-i\chi} & ig & -ig \\ \gamma_L^+ e^{i\chi} & 0 & -\gamma_L^- - \gamma_R^+ & \gamma_R^- & -ig & ig \\ 0 & \gamma_R^+ & \gamma_L^+ e^{i\chi} & -\gamma_L^- - \gamma_R^- & 0 & 0 \\ 0 & ig & -ig & 0 & -\frac{\Gamma}{2} - i\delta & 0 \\ 0 & -ig & ig & 0 & 0 & -\frac{\Gamma}{2} + i\delta \end{pmatrix}. \quad \text{(C.1)}$$

The coefficients $a_j^{(k)}$ defined in Eq. 3.36 required to calculate the mean current and its fluctuations read

$$a_0^{(1)} = -ig^2\kappa_L\kappa_R(\kappa_L + \kappa_R)^2(N_L - N_R) \quad \text{(C.2)}$$

$$a_1^{(0)} = \frac{1}{4}(\kappa_L + \kappa_R)^3\left(4g^2 + \kappa_L\kappa_R\right) + \kappa_L\kappa_R(\kappa_L + \kappa_R)\delta^2 \quad \text{(C.3)}$$

$$a_0^{(2)} = g^2\kappa_L\kappa_R(\kappa_L + \kappa_L)^2(N_L + N_R + 2N_LN_R) \quad \text{(C.4)}$$

$$a_2^{(0)} = \frac{1}{4}(\kappa_L + \kappa_R)^2(20g^2 + \kappa_L^2 + 7\kappa_L\kappa_R + \kappa_R^2) + (\kappa_L^2 + 3\kappa_L\kappa_R + \kappa_R^2)\delta^2 \quad \text{(C.5)}$$

$$a_1^{(1)} = 4ig^2\kappa_L\kappa_R(\kappa_L + \kappa_R)(N_L - N_R) \quad \text{(C.6)}$$

Hence, by inserting the coefficients into Eqs. B.7 and B.8,

$$\langle I\rangle = \frac{4g^2\kappa_L\kappa_R(\kappa_L + \kappa_R)(N_L - N_R)}{(\kappa_L + \kappa_R)^2\left(4g^2 + \kappa_L\kappa_R\right) + 4\delta^2\kappa_L\kappa_R} \quad \text{(C.7)}$$

$$\begin{aligned} D = &\Bigg(- \frac{8g^2\kappa_L\kappa_R(N_L - N_R)\left((\kappa_L + \kappa_R)^2\left(20g^2 + \kappa_L^2 + 7\kappa_L\kappa_R + \kappa_R^2\right) + 4\delta^2\left(\kappa_L^2 + 3\kappa_L\kappa_R + \kappa_R^2\right)\right)}{\left((\kappa_L + \kappa_R)^2\left(4g^2 + \kappa_L\kappa_R\right) + 4\delta^2\kappa_L\kappa_R\right)^2} \\ &+ \frac{32g^2\kappa_L\kappa_R(\kappa_L + \kappa_R)(N_L - N_R)}{(\kappa_L + \kappa_R)^2\left(4g^2 + \kappa_L\kappa_R\right) + 4\delta^2\kappa_L\kappa_R} + \frac{N_L + N_R - 2N_LN_R}{N_L - N_R}\Bigg)\langle I\rangle i, \end{aligned} \quad \text{(C.8)}$$

where D can be further simplified into

$$D = \frac{N_L + N_R - 2N_L N_R}{N_L - N_R}\langle I\rangle - \frac{2}{\kappa_L + \kappa_R}\langle I\rangle^2\left(1 + \frac{(\kappa_L+\kappa_R)^2 + 4\delta^2(\kappa_L-\kappa_R)^2/(\kappa_L+\kappa_R)^2}{4g^2 + \kappa_L\kappa_R + 4\delta^2\kappa_L\kappa_R/(\kappa_L+\kappa_R)^2}\right). \tag{C.9}$$

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