Asymptotic behaviour of the Hausdorff dimension of Julia set for quadratic polynomials around -2

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Den fraktal dimensionen av den invarianta mängden för ett dynamiskt system

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Euklidisk geometri var den första geometrin som man stötte på i grundskolan, allt från trianglar till cirklar. Samtidigt fick man lära sig att inget av dessa objekt finns i verkligheten. Det var inte förrän Benoît Mandelbrot (1924-2010) insåg att många naturfenomen är så oregelbundna och komplexa att de inte kan beskrivas med euklidisk geometri. I detta sammanhang verkar Mandelbrots citat vara mest lämpligt att nämna.

> Why is geometry often described as "cold" and "dry"? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

Mandelbrot var motiverad av flera matematiker och deras arbete för att introducera ett nytt ramverk från vilket man kan studera denna mera komplexa geometri. Exempelvis Andrej Kolmogorovs (1903-1987) teori om turbulens samt hans definition av "capacity" av ett geometrisk objekt, samt den polska matematikern Felix Hausdorffs (1868-1942) definition av fraktal dimension (Hausdorff dimension). En fraktal är, på ett ungefär, ett geometriskt objekt vars struktur och mönster upprepas om och om igen då man förstorar den.

Mandelbrot nämner en varierande mängd av olika fraktaler i hans kända bok "The Fractal Geometry of Nature". I detta arbete studeras hur den fraktala dimensionen av en viktig invariant mängd (s.k. Julia mängden) för ett dynamiskt system påverkas då man stör systemet genom att ändra på parametern. I detta arbete undersöker vi mer precist hur dynamiken av $f_c(z) = z^2 + c$ ändras när $c = -2 - \epsilon$ där ϵ är något litet reellt eller imaginärt tal. I ett ostört läge som vi startar med, dvs. $c = -2$ så är Julia mängden ett intervall alltså en fraktal dimension 1 men när systemet störs, dvs. när ϵ inte är noll längre, får vi något likt följande figur, med en fraktal dimension mindre än 1.

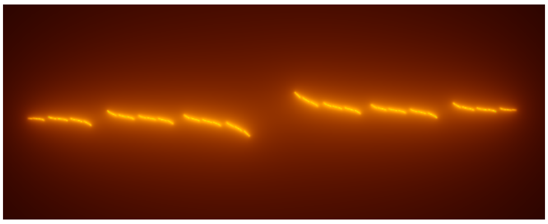


Figure 1: Julia mängden för $f(z) = z^2 - 2 + 0.1i$

Målet var att undersöka hur fort den fraktala dimensionen av Julia mängden konvergerade till 1 då störningen ϵ går mot noll i vänstra halvplanet. Vi bevisar ett nytt resultat om hur snabbt denna konvergens sker, då denna störning är rent imaginär. Arbetet var bl.a. inspirerat av Ludwik Jaksztas artikel [1].

[1] On the directional derivative of the Hausdorff dimension of quadratic polynomial Julia sets at -2

Contents

Abstract

Let $\dim_H(\mathcal{J}_\epsilon)$ and $\dim_H(\mathcal{J}_{i\epsilon})$ denote the Hausdorff dimension of the Julia set of the polynomials $f_\epsilon(z) = z^2 - 2 - \epsilon$ and $f_{i\epsilon}(z) = z^2 - 2 + i\epsilon$ receptively for small $\epsilon > 0$. This thesis contains two main Theorems, both dealing with the upper bound for the asymptotic behaviour of $\dim_H(\mathcal{J}_\epsilon)$ and $\dim_H(\mathcal{J}_{i\epsilon})$ when $\epsilon \to 0$. The novelty of this thesis lays in the imaginary perturbation case. Before proving the main Theorems, we introduce the general framework and techniques to calculate Hausdorff measure.

1 Introduction

Let f be a complex polynomial in one variable of degree at least 2. We define the filled-in Julia set $\mathcal{K}_f$ as the set of point that stay bounded under iterations of f

$$\mathcal{K}_f := \{z \in \mathbb{C} : f^n(z) \not\to \infty\}.$$

The Julia set $\mathcal{J}_f$ is the boundary of the filled-in Julia set $\mathcal{J}_f := \partial\mathcal{K}_f$. The Julia set exhibits elegant fractal structures that have captured the interest of mathematicians for centuries. The approach of fractals has gone through increasingly rigorous mathematical treatment. One example is to study the fractal dimension (Hausdorff dimension), the framework of this thesis, which was first introduced by Felix Hausdorff (1886–1942). Fractals were reintroduced to the mathematical scene by Benoit Mandelbrot (1924-2010) who used Hausdorff dimension as a new tool to study more complex geometry. His motivation was partly because nature is too complex to be modelled by Euclidean geometry. In his book The Fractal Geometry of Nature [Mandelbrot (1983)] Mandelbrot says:

> Why is geometry often described as "cold" and "dry"? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

There have recently been multiple statistical applications using fractals to further extract structures from experimental data or as a tool from computer visuals. In their book, Novak & Dewey list a small portion of papers from vastly different areas of science that have utilised techniques from fractals geometry [Novak & Dewey (1997)]. From the perspective of practical application, it is very important to study and develop these tools.

In this thesis, we specifically study how the Hausdorff dimension of the Julia set generated by the quadratic maps $(z \mapsto z^2 + c)$ varies while perturbing the parameter $c \in \mathbb{C}$. We begin by introducing the tools required for this analysis such as Hausdorff dimensions and their properties, Iterated function systems (IFS). Then I define and prove some equivalent definitions for the Julia set. Lastly in Section 5 I work to prove Grötzsch inequality. The scope of this thesis began as an attempt to complete the details in [Dobbs et al. (2022)] proof sketch for the negative real perturbation around $c = -2$, which is done in Theorem 6.8. The novelty of this thesis is presented in the proof and statement of Theorem 6.10.

First we introduce the framework on which the thesis rests. Section 2 contains a short introduction to Hausdroff measure and Hausdorff dimension. This section is without any proofs and only serves as a collection of results to refer to during the reading of the thesis. Section 3 contains some results relating to Iterated function systems (IFS), which can be used to estimate the Hausdorff dimensions. Section 4 contains, with proofs, the basic terminology for complex dynamics, for which the goal is to prove the four equivalent definitions of the Julia set. Section 5 is on quasiconformal mappings and moduli of curve families which will serve as the vital instruments to prove Theorem 6.10. The last section will combine all previous sections and estimate Hausdorff dimension for Julia sets depending on the choice of the parameter.

I wish to acknowledge and express my deepest gratitude for my advisor Magnus Aspenberg for his time and engaging conversation about the subject as well as surrounding matters. I also want to thank Victor Ufnarovski and Jörg Schmeling for engaging my interest in this subject already in the second year of my bachelors during the complex analysis seminar classes. At the same time, I want to extend this gratitude to my family and friends for their understanding and compassion during the troubling and uncertain times while I was writing this thesis.

2 Hausdorff measure and dimension

In this section, the Hausdorff dimension and the corresponding maps which preserve dimension are introduced along with some of its elementary properties. First, an extension of the usual Lebesgue measure, called the Hausdorff measure, is presented. No proofs will be presented. For a full treatment consult Falconer (2013) and Evans & Gariepy (2015).

2.1 Hausdorff measure

Definition 2.1. ***(Diameter of a set in metric spaces)***
Let (X, d) be a metric space and $E \subset X$, we define the diameter of E as

$$|E| = \sup\{d(x, y) : x, y \in E\}.$$

Definition 2.2. ***(δ-cover of a set)***
A δ-cover of a set E is a countable (or finite) collection of sets $\{U_i\}_{i=1}^{\infty}$ with $0 < |U_i| \leq \delta$ for each i such that

$$E \subset \bigcup_{i=1}^{\infty} U_i.$$

Definition 2.3. ***(Hausdorff measure)*** *Let (X, d) be some metric space, $s \geq 0$ and $E \subset X$ the Hausdorff measure of E is,*

$$\mathcal{H}^s_\delta(A) = \inf\{\sum_{i=1}^{\infty} |E_i|^s : \{E_i\}_i^{\infty} \text{ is a } \delta\text{-covering of } A\}$$

where infimum is taken over all possible δ-covers. When $\delta \to 0$ we get the s-dimensional Hausdorff measure

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

In words, Hausdorff measure is the smallest s-dimensional "volume" of δ-covers for E. While s-dimensional Hausdorff measure is passing the Hausdorff measure to infinitesimally fine δ-covers. Note that, as the δ-covers become finer, the set of possible covers decreases, hence making the s-dimensional Hausdorff measure increases. Hausdorff measure has many properties, some of which will be stated below. First, we define Hölder continuous maps, which will serve as a starting point for the maps that preserve Hausdorff dimension.

Definition 2.4. ***(Hölder continuous)*** *Let $E \subset \mathbb{R}^n$ a map $f : E \to \mathbb{R}^m$ is Hölder continuous with exponent $0 < \alpha$ if the following is satisfied*

$$|f(x) - f(y)| \leq C|x - y|^\alpha,$$

for each $x, y \in E$ and for some universal constant C.

Theorem 2.5. ***[Properties of the Hausdorff measure]***
Let $E \subset \mathbb{R}^n$ and $s \geq 0$.

i) For all $0 \leq s < \infty$, $\mathcal{H}^s$ is a Borel regular outer measure in $\mathbb{R}^n$.

ii) $\mathcal{H}^0$ is the counting measure.

iii) $\mathcal{H}^1 = \mathcal{L}^1$ *i.e. the Hausdorff measure coincides with the Lebesgue measure in* $\mathbb{R}^1$.

iv) *If* $\lambda > 0$ *and* $\lambda E = \{\lambda x : x \in E\}$. *Then*

$$\mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E).$$

v) *Let* $f : E \to \mathbb{R}^m$ *be a mapping with Hölder condition of exponent* $\alpha > 0$. *For each* s

$$\mathcal{H}^{s/\alpha}(f(E)) \leq c^{s/\alpha}\mathcal{H}^s(E).$$

2.2 Hausdorff dimension

Here, we motivate and define the Hausdorff dimension with the maps which preserves the dimension.

Consider a set $F \subset \mathbb{R}^n$ and let $\delta < 1$, the Hausdorff measure is non-increasing with s, and by extension, is also true for the s-dimensional Hausdorff measure. Assume now that if $t > s$ and $\{U_i\}$ is a δ-cover of F, then

$$\sum_{i=1}^{\infty} |U_i|^t = \sum_{i=1}^{\infty} |U_i|^{t-s}|U_i|^s \leq \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s.$$

Taking the infimum of both sides, we get $\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F)$. Since the Hausdorff measure is non-decreasing in δ, with $\delta \to 0$, we get

$$\mathcal{H}^s(F) < \infty \implies \mathcal{H}^t(F) = 0.$$

Plotting the s-Hausdorff measure against s, we get

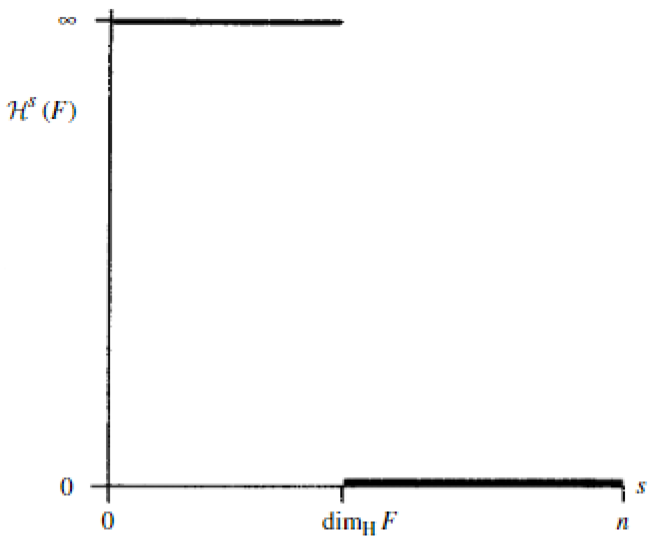


Figure 1: The s-Hausdorff measure with respect to s.

The s for which we get discontinuity is defined to be the Hausdorff dimension of the set F.

Definition 2.6. ***(Hausdorff dimension & s-sets)***
Let $E \subset \mathbb{R}^n$. The Hausdorff dimension of F is

$$\dim_H E = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(E) = \infty\}.$$

Note that

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{if } 0 \leq s < \dim_H(F), \\ 0 & \text{if } s > \dim_H(F). \end{cases}$$

The Hausdorff measure at $\dim_H(F)$ can attain any value $0 \leq \mathcal{H}^{dim_H(F)}(F) \leq \infty$. Sets for which $0 < \mathcal{H}^{\dim_H(F)}(f) < \infty$ are called s-sets.

Intuitively, the definition tells us the optimal s-dimension intervals that is needed to cover the set F. For instance, if we let $F = [0,1]^2$ be the unit square in $\mathbb{R}^2$, then a δ-cover with $s = 1$ has infinite measure. But with $s = 3$, the Hausdorff measure of F is zero. The Hausdorff dimension gives the "adequate" space for which the set belongs to. We will now state some properties of the Hausdorff dimension.

Proposition 2.7. ***(Properties of Hausdorff dimension)***

(i) ***Monotonicity:*** *If $E \subset F$ then $dim_H E \leq dim_H F$.*

(ii) ***Countably stability:*** *If $\{F_i\}$ is a countable sequence of sets then*

$$dim_H \cup_{i=1}^{\infty} F_i = \sup_{1 \leq i < \infty} \{dim_H F_i\}.$$

(iii) ***Countable sets:*** *If F is countable then $dim_H(F) = 0$.*

(iv) ***Open sets:*** *If $F \subset \mathbb{R}^n$ is open and non-empty, then $dim_H F = n$.*

Next, we introduce the morphisms that preserve the Hausdorff dimension.

Proposition 2.8. ***(Hausdorff dimension under Hölder condition maps)***
Let $F \subset \mathbb{R}^n$ and a map $f : F \to \mathbb{R}^n$ which satisfies α-Hölder condition, then $dim_H f(F) \leq \frac{1}{\alpha} dim_H F$.

Definition 2.9. ***(bi-Lipschitz)***
A mapping $f : F \to \mathbb{R}^n$ is said to be a ***bi-Lipschitz*** *transformation if there exists c_1 and c_2 such that $0 < c_1 \leq c_2 < \infty$ and,*

$$c_1|x-y| \leq |f(x) - f(y)| \leq c_2|x-y| \quad (\forall x, y \in F).$$

Combining this with Proposition 2.8 gives.

Proposition 2.10. ***(Hausdorff dimension preserving maps)***

(i) If $f : F \to \mathbb{R}^n$ is a Lipschitz transformation then $dim_H f(F) \leq dim_H F$.

(ii) If f is bi-Lipschitz transformation, then $dim_H f(H) = dim_H F$.

We end this section by stating a proposition that connects the value of the Hausdorff dimension to the geometry of the set.

Proposition 2.11. ***(Geometric property of sub 1 Hausdorff dimensional sets)***
A set $F \subset \mathbb{R}^n$ with $dim_H F < 1$ is totally disconnected.

3 Iterated function systems (IFS)

In this section, we introduce a core technique for estimating the Hausdorff dimension. To begin, consider a system of functions $\{S_i\}_{i=1}^m$ with $m \geq 2$. Iterating $\{S_i\}_{i=1}^m$ in all possible combinations on the domain yields what is called an Iterative function system (IFS). Let F be the sub-set of the domain which stays invariant under iterations. This is called the invariant set or the attractor set. The attractor set F usually exhibit fractal-like behaviour. Further in some certain settings, the Hausdorff dimension can be calculated precisely or be given an upper and lower bounds depending on the nature of these maps. The material in this chapter follows Falconer (2013) and presented here for the convenient of the reader.

Definition 3.1. ***(Contraction and similarities)***
Let $D \subseteq \mathbb{R}^n$ be closed. A map $S : D \to D$ is called a contraction map if there exists some $0 < c < 1$ such that

$$|S(x) - S(y)| \leq c|x - y| \quad \forall x, y \in D.$$

If $|S(x) - S(y)| = c|x - y|$ instead, then S is called a contracting similarity.

If an IFS $\{S_i\}_{i=1}^m$ is comprised of contractions, then the attractor set F fulfils the following

$$F = \bigcup_{i=1}^{m} S_i(F).$$

The following Theorem tells us that the attractor set exist, is unique, and can be found by intersecting iterations of some non-empty compact set. It can be viewed as an extension of the famous Banach's fixed point Theorem.

Theorem 3.2. ***(The fundamental Theorem for IFS)***
Consider the IFS given by the contractions $\{S_1, \ldots, S_m\}$ on $D \subset \mathbb{R}^n$ such that

$$|S_i(x) - S_i(y)| \leq c_i|x - y|$$

with $c_i < 1$ for each i. Then, there is a unique attractor set F, i.e. a non-empty compact set, such that

$$F = \bigcup_{i=1}^{m} S_i(F).$$

Moreover, if we define a transformation S on the class $\tilde{\mathcal{S}}$ of non-empty compact sets by

$$S(E) = \bigcup_{i=1}^{m} S_i(E)$$

for $E \in \tilde{\mathcal{S}}$, and write S^k to be the kth iteration of the set E, then

$$F = \bigcap_{k=0}^{\infty} S^k(E)$$

for every set $E \in \tilde{\mathcal{S}}$ such that $S_i(E) \subset E$.

The proof uses the Banach's contraction mapping Theorem on the complete metric space $(\tilde{\mathcal{S}}, d_H)$ where $d_H : \tilde{\mathcal{S}} \times \tilde{\mathcal{S}} \to \mathbb{R}$ is the Hausdorff metric- that is,

$$d_H(A, B) = \inf\{\delta : A \subset B_\delta \text{ and } B \subset A_\delta\}$$

where $A_\delta = \{x \in D : |x - a| \leq \delta \ for\ some\ a\}$. A full proof can be found in [Falconer (2013),Theorem 9.1]. Note that if we assume that the union $\bigcup_{i=1}^m S_i(F)$ is disjoint we will get

$$\mathcal{H}^s(F) = \mathcal{H}^s(\bigcup_{i=1}^m S_i(F)) = \sum_{i=1}^m \mathcal{H}^s(S_i(F)) = \sum_{i=1}^m c_i^s \mathcal{H}^s(F) \implies \sum_{i=1}^m c_i^s = 1.$$

Where in the last implication, we assumed that F has positive finite s-dimensional Hausdorff measure. If one manages to find the s such that $\sum_{i=1}^m c_i^s = 1$ then one has found the Hausdorff dimension. But, since the union is not disjoint we will require a weaker criteria called the open set condition. We say S_i satisfies the open set condition if there exists some non-empty bounded open set V such that

$$\bigcup_{i=1}^m S_i(V) \subset V$$

with the union disjoint. With this condition, we are ready to state the Theorem calculating the Hausdorff dimension.

Theorem 3.3. ***(Hausdorff dimension of IFS with similarities)***
Suppose that the open set condition holds for the similarities S_i on $\mathbb{R}^n$ with ratios $0 < c_i < 1$ for $1 \leq i \leq m$. If F is the attractor set of the IFS $\{S_1, \dots, S_m\}$,

$$F = \bigcup_{i=1}^m S_i(F) \tag{3.1}$$

then $dim_H F = s$, where s is given by

$$\sum_{i=1}^m c_i^s = 1. \tag{3.2}$$

Moreover, the s-Hausdorff measure will be positive and finite.

Example 3.4. ***(Hausdorff dimension for the Sierpiński triangle)***
For the Sierpiński triangle, the three similarities scale down a triangle into three equal-sized triangles (figure 2), hence $c_i = \frac{1}{2}$. Previous Theorem give us $\sum_{i=1}^3 (\frac{1}{2})^s = 1$, then $\dim_H F = s = \frac{\log(3)}{\log(2)} \approx 1.585\dots$.

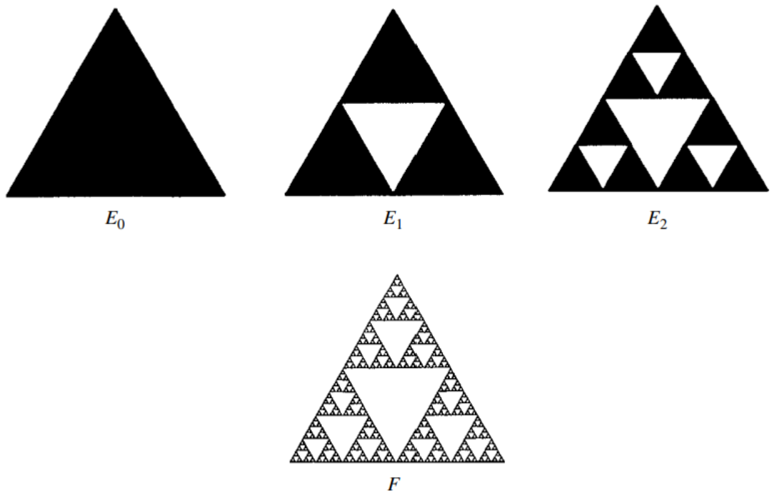


Figure 2: Construction of the Sierpinski triangle ($\dim_H F = \frac{\log 3}{\log 2} = 1.585\ldots$)

Theorem 3.5. ***(Hausdorff dimension of contracting IFS)***
Let $\{S_i\}_{i=1}^m$ be an IFS consisting of contractions on a closed subset D of $\mathbb{R}^n$ i.e.

$$|S_i(x) - S_i(y)| \leq c_i|x - y| \quad (x, y \in D).$$

Now, let F be the attractor for the IFS. Let s be such that $\sum_{i=1}^m c_i^s = 1$, then,

$$\dim_H F \leq s.$$

Next, we obtain a lower bound for the case when $F = \bigcup_{i=1}^m S_i(F)$ is a disjoint union.

Theorem 3.6. ***(Lower bound for the Hausdorff dimension)***
Consider the IFS $\{S_i\}_{i=1}^m$ on a closed subset D of $\mathbb{R}^n$, such that

$$b_i|x - y| \leq |S_i(x) - S_i(y)| \quad (x, y \in D).$$

with $0 < b_i < 1$ for each i. Assume that the union $F = \bigcup_{i=1}^m S_i(F)$ is disjoint. Let s be such that

$$\sum_{i=1}^m b_i^s = 1.$$

Then F is totally disconnected and $s \leq \dim_H F$.

4 Concept in complex dynamics

This section contains a basic introduction with proofs for complex dynamics. It will be the framework based on which the rest of the essay will be presented. We begin by defining the Julia set for a complex polynomial. The goal is to prove four equivalent definitions for the Julia set.

4.1 Julia sets and their properties

The Julia set can be defined in four equivalent ways: (1) as the boundary of the filled-in Julia set, (2) as the set of points $z \in \mathbb{C}$ for which the sequence $\{f^k\}_k^\infty$ fails to be normal, (3) as the closure of the repelling periodic points, and (4) if we have a polynomial with a critical point (points with derivative is equal to zero) and an attractive periodic point, then as the boundary of the corresponding basin of attraction. These definitions provide different perspectives on the same object, with some being more favourable than others depending on the context.

First, we introduce some notation and terminology from dynamics. Let $f \in \mathbb{C}[z]$ be a complex polynomial. We will write $f^k = f \circ \cdots \circ f$ as the k-th composition of the function f while the k-th derivative will be written inside a parenthesis i.e. $f^{(k)} = \frac{d^k f}{dz^k}$ for $k \geq 3$. Recall that we call w a periodic point of period p for f if $f^p(w) = w$ where p is the smallest non-zero integer with this property. If $p = 1$, then we call w a fixed point, now with $(f^p)'(w) = \lambda$.

Definition 4.1. ***(Attractive/repelling fix points)***

1. *Attractive periodic point:* $0 < |\lambda| < 1$.
2. *Repelling periodic point:* $|\lambda| > 1$.

An attractive periodic point "pulls" nearby points to it under iteration of f while repelling periodic points "pushes" nearby points. If $\lambda = 0$, we call it superattracting. Part of studying the dynamics for some complex function is to classify what initial points stay bounded under iteration, this set of point is called the **filled-in Julia set**- named after the famous mathematician Gaston Maurice Julia.

Definition 4.2. ***(Filled-in Julia set)***
For a complex polynomial f, we define the filled-in Julia set to be the points in the complex plane that do not diverge.

$$\mathcal{K}(f) = \{z \in \mathbb{C} : f^k(z) \nrightarrow \infty\}.$$

Note that this set contains points that neither diverge nor converge. For instance, we can have periodic points or non-returning points. The next definition is more analytical, which will play a role in describing uniformity of a family of functions in an open set or around a specific point.

Definition 4.3. ***(Normal family of analytic functions)***
Let U be an open set in $\mathbb{C}$, and

$$\mathcal{F} = \{g : U \to \mathbb{C} : g \ \text{analytic on} \ U\},$$

a family of analytic functions on U. We call the family $\mathcal{F}$ normal if, for every sequence g_k in $\mathcal{F}$ and every compact subset $\tilde{U}$ of U, there exists some subsequence $\{g_{n_k}\}$ which converges uniformly on $\tilde{U}$, *either to some bounded analytic function or ∞. The family $\mathcal{F}$ is **normal at a point** $z \in U$ if there exists an open set $V \subset U$ containing z such that the family is normal on V.*

Definition 4.4. ***(Basin of attraction)***
If w is an attractive fixed point of $f \in \mathbb{C}[z]$, the basin of attraction is

$$A(w) = \{z \in \mathbb{C} : f^k(z) \to w \text{ as } k \to \infty\}.$$

Definition 4.5. ***(Julia set)*** *Let $f \in \mathbb{C}[z]$. The Julia set is defined to be the boundary of the filled-in Julia set.*

$$\mathcal{J}(f) = \partial\mathcal{K}(f).$$

The complement of the Julia set is the Fatou set, named after the famous French mathematician Pierre Fatou (1878-1929).

Definition 4.6. ***(Fatou set)***
The Fatou set is the complement of the Julia set.

$$\mathcal{F}(f) = \mathcal{J}^c.$$

We are now equipped to state the main Theorem of this section.

Theorem 4.7. ***(Equivalence of Julia set)***
Let $f \in \mathbb{C}[z]$, the following are equivalent,

i) The boundary of the filled-in Julia set.

$$\mathcal{J}(f) = \partial\mathcal{K}(f)$$

ii) The set of points on the complex plane for which $\{f^k\}_k^\infty$ fails to be normal on.

$$\mathcal{J}(f) = \{z \in \mathbb{C} : \{f^k\}_{k=0}^\infty \textit{ is not normal at } z\}.$$

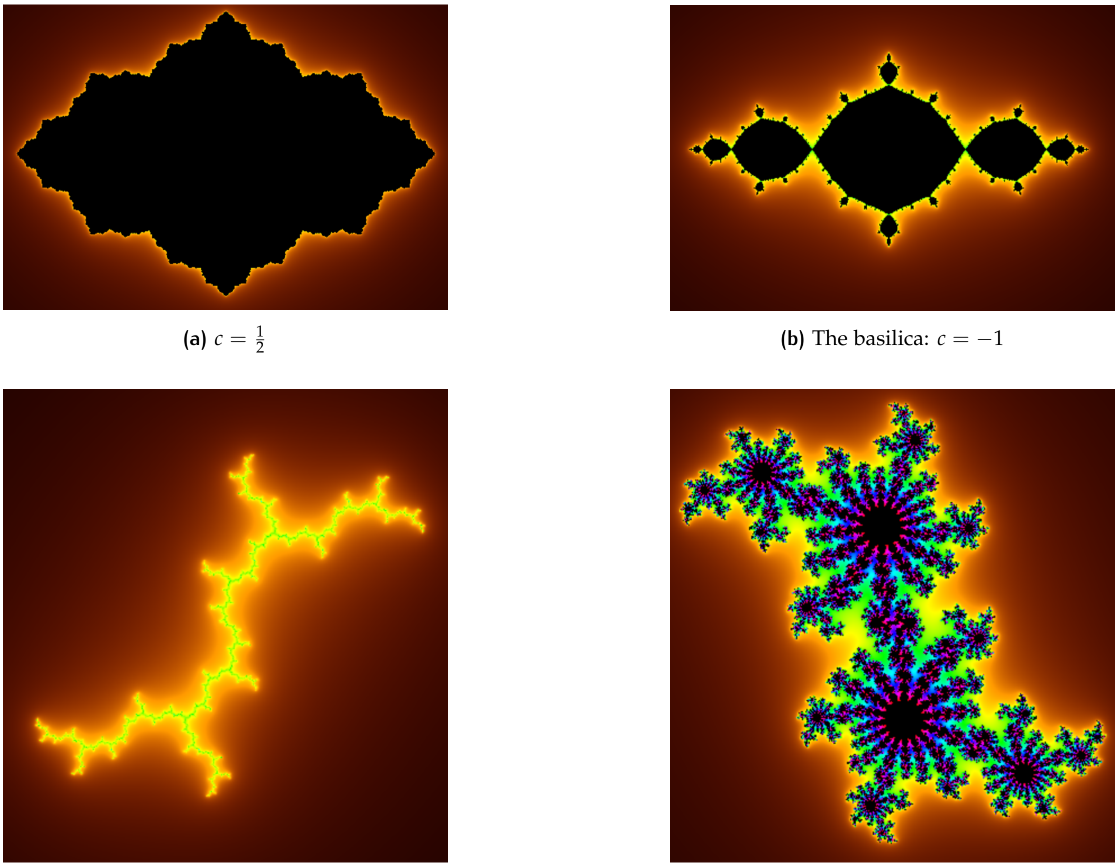
iii) The closure of the repelling periodic points.

iv) The boundary of the basin for any attractive fixed point w of f.

Note that the four definitions serve different roles when studying the Julia set. The first and last serves as geometric interpretation, and most importantly, as an instrument to calculate the Julia set. The second enables us to use the theory of complex functions, and in particular, Montel's theorem, while the third connects us to dynamics . To prove this Theorem we need to establish some properties of the Julia set.

This Theorem also tells us that we can view the Fatou set as the complement to the boundary of the Filled Julia set or as the collection of point for which the family of composition of $\{f^k\}_k^\infty$ are normal on. Here is a simple example to demonstrate the equivalences.

Example 4.8. ***(Julia set for z^2)***
Let $f(z) = z^2$. Then, it follows that $f^k(z) = z^{2^k}$. Hence, for $|z| < 1$ we get that $f^k(z) \to 0$ while $f^k(z) \to \infty$ when $|z| > 1$. The filled-in Julia set is then the disc. The Julia set is hence $\mathcal{J}(f) = \partial\mathcal{K}(f) = \mathbb{S}^1$. In the same way, any open neighbourhood around a point on the circle will not be normal under $\{f^k\}_k^\infty$. The boundary of the respective basins for the two attractive fixed points, origin and ∞, is, in both cases, the unit circle. Lastly, the repelling periodic points are all the point on the circle.



(a) $c = \frac{1}{2}$

(b) The basilica: $c = -1$

(c) The dendrite: $c = i$

(d) $c = 0.2 - 0.56i$

Figure 3: Examples of Julia sets for the quadratic map $f(z) = z^2 - c$ for different c

To be able to prove the equivalences in Theorem 4.7, a number of general statements will be needed. The first one gives us an estimate of the disc that contains the filled-in Julia set. We move on to prove some topological properties of $\mathcal{J}(f)$ and $\mathcal{K}(f)$, then two statements of the invariance of the Julia set.

Lemma 4.9. *Let $f \in \mathbb{C}[z]$ with degree n, with $f(z) = \sum_{i=0}^{n} a_i z^i$. There exists some $r > 0$ such that*

$$|z| \geq r \implies |f(z)| \geq 2|z|.$$

In particular, if $|f^m(z)| \geq r$ for some m, then $f^m(z) \to \infty$.

Proof. Choose r such that if $|z| \geq r$ then $2|z| \leq \frac{1}{2}|a_n||z|^n$ and $\sum_{i=0}^{n-1} |a_i z^i| \leq \frac{1}{2}|a_n||z|^n$, simple calculations shows that the following r suffices:

$$r \geq \max \left\{ \left(\frac{2|z|}{|a_n|} \right)^{1/n}, \left(\frac{2}{|a_n|} \sum_{i=0}^{n-1} |a_i z^i| \right)^{1/n} \right\}.$$

Then, for $|z| \geq r$ we have

$$|f(z)| \geq |a_n||z|^n - \sum_{i=0}^{n-1} |a_i z^i| \geq |a_n||z|^n - \sum_{i=0}^{n-1} |a_i z^i| \geq \frac{1}{2}|a_n||z|^n \geq 2|z|.$$

For the second statement, note that if $f^m(z)| \geq r$ for some m, then, applying the previous result inductively, we get $|f^{m+k}(z) \geq 2^m |f^k(z)| \geq r$ thus $f^k(z) \to \infty$. □

Proposition 4.10. ***[Compactness of the (filled-in) Julia set]***
Let $f \in \mathbb{C}[z]$ and $\deg(f) \geq 2$. Then, both the filled Julia set and the Julia set are non-empty compact sets with $\mathcal{J}(f) \subset \mathcal{K}(f)$. Furthermore, $\mathcal{J}(f)$ has an empty interior.

Proof. We prove compactness via Heine–Borel Theorem, which is sufficient to prove closed and boundedness. Using r from Lemma 4.9, we can see that $\mathcal{K}(f)$ is contained in a disc of radius r, hence bounded. Then, $\mathcal{J}(f) = \partial\mathcal{K}(f)$ will be bounded by the same disc.
To show that $\mathcal{K}(f)$ is closed, we show that the complement is open. Take $z \notin \mathcal{K}(f)$ then $f^k(z) \to \infty$ so $|f^m(z)| > r$ for some integer m. Now, since f is continuous, there exists an open ball B around z such that $f^k(w) \to \infty$ for any $w \in B$ i.e. $w \notin \mathcal{K}(f)$. Hence $\mathcal{K}(f)^c$ is open and $\mathcal{K}(f)$ is closed. Since $\mathcal{J}(f)$ is the boundary, it is also closed.

The complex numbers are algebraically closed, hence $f(z) = z$ has at least one solution z_0, thus giving us $f^k(z_0) = z_0$ for all k. This means $z_0 \in \mathcal{K}(f)$, and hence, is non-empty. Let $z_1 \in \mathbb{C} \setminus \mathcal{K}(f)$. Taking the line $\lambda z_0 + (1-\lambda)z_1$ contacting z_0 and z_1, the line will intersect the boundary for some λ. Taking the

$$\lambda = \inf\{\lambda : \lambda z_0 + (1-\lambda)z_1, \lambda \in [0,1]\}$$

will suffice for such a λ. Thus, $\mathcal{J}(f) = \partial\mathcal{K}(f)$ is non-empty.
To prove that the Julia set has an empty interior, assume first that U is a non-empty open subset of $\mathcal{J}(f) \subset \mathcal{K}(f)$ □

Note that the restriction on the polynomial's degree ensures a solution to $f(z) = z$. This can be relaxed to simply requiring $f(z) = z$ to have a solution.

Proposition 4.11. ***[invariance of the Julia set]***
The Julia set is invariant under forward and backward iterations of f. The Julia set is also invariant if the underlying function is a composition of f.

- $\mathcal{J}(f) = f(\mathcal{J}(f)) = f^{-1}(\mathcal{J}(f))$
- $\mathcal{J}(f^p) = \mathcal{J}(f)$ *for any positive integer p.*

Proof. If $z \in \mathcal{J}(f)$, then $f^k(z) \not\to \infty$, and since $\mathcal{J}(f)$ is closed, we may find $w_n \to z$ with $f^k(w_n) \to \infty$ for all n. Now, since f is continuous, one can choose $f(w_n)$ arbitrarily close to $f(z)$. Since $f^k(f(w_n)) \to \infty$ and $f^k(f(z)) \not\to \infty$, that means $f(z) \in \mathcal{J}(f)$, hence $f(\mathcal{J}(f)) \subset \mathcal{J}(f)$. Taking the inverse image of the inclusion we get $\mathcal{J}(f) \subset f^{-1}(f(\mathcal{J}(f))) \subset f^{-1}(\mathcal{J}(f))$.

In the same way, let z and w_n be as above with z_0 such that $f(z_0) = z$. Since f is a complex polynomial, one may find $v_n \to z_0$ such that $f(v_n) = w_n$ for all n. Hence, by the previous arguments, we get that, $f^k(z_0) = f^{k-1}(z) \not\to \infty$ and $f^k(v_n) = f^{k-1}(w_n) \to \infty$ as $k \to \infty$, which is that $z_0 \in \mathcal{J}(f)$, and for that matter, $f^{-1} \subset \mathcal{J}(f)$, which implies as above, $J \subset f(f^{-1}(\mathcal{J}(f))) \subset f(\mathcal{J}(f))$, finishing the proof of the first statement.

The proof of the latter statement follows from the fact that $f^k(z) \to \infty$ iff $(f^p)^k(z) = f^{kp}(z) \to \infty$. Thus, both filled Julia sets are identical, i.e. $\mathcal{K}(f) = \mathcal{K}(f^p)$,and thus, the same Julia set. □

To prove the first equivalence we need Montel's theorem, which describes the behaviour of abnormal family of analytical functions by considering images for the family of functions.

Theorem 4.12. ***[Montel's theorem]***
Let $\{g_k\}_{k=1}^{\infty}$ be a family of analytic functions on an open domain U. If the family is not normal on U, then for each $w \in \mathbb{C}$ with possible one exception, there exist some $z \in U$ and $\hat{k}$ such that $g_{\hat{k}}(z) = w$.

We are now ready to prove the first equivalence of the main Theorem.

Proof. **[Theorem 4.7 (i) $\iff$ (ii)]**

$(i \implies ii)$
Let $z \in \mathcal{J}(f)$. Then for each open neighbourhood V contains some point w that diverges under iteration of the complex polynomial, $f^k(w) \to \infty$, while $f^k(z)$ stays bounded. Thus, no subsequence of $\{f^k\}_k^{\infty}$ converges uniformly on all compact subsets of V. Thus, the family of compositions are not normal on z.

$(ii \implies i)$
We lead with a contra positive proof. Assume that $z \notin \mathcal{J}(f)$, then either $z \in int(\mathcal{K}(f))$ or $z \in \mathbb{C} \setminus \mathcal{K}(f)$. For the first case, take any open neighbourhood V around z such that $z \in V \subset int(\mathcal{K}(f))$.Since every point in the filled Julia set stay bounded under iteration of f. In particular we have, $f^k(w) \in \mathcal{K}$ for all $w \in V$ and k[1]. By Montel's Theorem 4.12 we have that $\{f^k\}_k^{\infty}$ is normal.
In the other case $z \in \mathbb{C} \setminus \mathcal{K}(f)$, taking a open neighbourhood which contains z, we get that there exist some k such that $|f^k(w)| > r$ for some r via Lemma 4.9 for all $w \in V$. Thus, by the same Lemma, we get that $f^k(w) \to \infty$ uniformly on V. Thus, $\{f^k\}_k^{\infty}$ is normal. □

Remark 4.13. *Using normality gives a natural extension for the definitions of Julia sets to more general settings such as rational functions or meromorphic functions. Furthermore, the function $f(z) = ((z-2)/z)^2$ gives an example of a non-bounded Julia set $\mathcal{J}(f) = \mathbb{C}$.*

With Montel's Theorem 4.12 we get a remarkable result. That is, for any point $z \in \mathcal{J}(f)$ and any open neighbourhood V around z, we can cover the whole complex plane with iterations of f with possibly one exception.

Proposition 4.14. ***[$\mathcal{J}$ is mixing]***
If f is a complex polynomial and $w \in \mathcal{J}(f)$, then for each open set V containing w we have that $W = \bigcup_{k=j}^{\infty} f^k(V)$ is the whole of $\mathbb{C}$ expect possibly one point, this point will not be in the Julia set for f, and moreover, is independent of both w and V.

Proof. Since the family $\{f^k\}_k^{\infty}$ is not normal at w. Further $\{f^k\}_{k=j}^{\infty}$ is not normal at w. By Montel's Theorem, we get that for every open set V containing w, there exist some $\hat{w} \in V$ and some p such that $f^p(\hat{w}) = z$ for any $z \in \mathbb{C}$ with possible one exception. This is the same as. W covering the complex plane with possibly one exception. Now, assume that $v \notin W$ and $f(W) \subset W$, then if $f(z) = v$, then $z \notin W$. Hence, $v = z$ (since $\mathbb{C} \setminus W$ contains at most one point). Since f is an n dimensional polynomial, $f(z) - v = 0$ implies $f(z) - v = c(z-v)^n$ where c

[1] One sees this since, otherwise, the sequence will diverge if it lands outside the filled Julia set and contradict the initial point that was inside the filled Julia set.

is some complex constant. Taking z to be sufficiently close to v, we get that $f^k(z) - v \to 0$ uniformly.

$$f^k(z) - v = f(f^{k-1}(z)) - v = c(f^{k-1}(z) - v)^n + v - v = c(f^{k-1}(z) - v)^n = \cdots = c^k(z - v)^{kn}.$$

If we take z to be in, for instance, $\{z : |z - v| < (2c)^{-1/n}\}$, then we get that the convergence is uniform thus $v \notin \mathcal{J}(f)$. □

A further classification of Julia sets is, if W omits one point, then the Julia set is the a circle with centre v and radius $c^{-1/(n-1)}$. Now we state a Theorem that stands as a basis for the reason why one can create computer pictures of the Julia sets. The idea is that one point in the Julia set can "generate" the whole of the Julia set. Making computation easier since one only need to find one point in the Julia set.

Proposition 4.15. ***[Computation of the Julia set]***

- *For all $z \in \mathbb{C}$ with at most one exception: if V is an open set intersecting $\mathcal{J}(f)$, then $f^{-k}(z)$ intersects V for infinitely many values of k.*
- *If $z \in \mathcal{J}(f)$ then $\mathcal{J}(f) = \overline{\bigcup_{k=1}^{\infty} f^{-k}(z)}$*

Proof. The first statement follows from the previous Proposition 4.15. Assume $z \in W$ is not the exceptional point, then $z \in f^k(V)$ for all k, hence $f^{-k}(z)$ intersects V for infinitely many k. For the second statement, let $z \in \mathcal{J}(f)$, and since the Julia set is backwards invariant under f, then $f^{-k}(Z) \in \mathcal{J}(f)$ for all k. Hence, $\bigcup_{k=1}^{\infty} f^{-k}(z)$ is also contained in the Julia set. Because the Julia set is a closed set, then the closure is also contained in the Julia set. On the other hand, let V be an open set containing $z \in \mathcal{J}(f)$, then by the first part of this proposition, $f^{-k}(z)$ intersects V for some value k and z is not the exception point so $\mathcal{J}(f) = \overline{\bigcup_{k=1}^{\infty} f^{-k}(z)}$ □

To further understand the topological properties of the Julia set, we will prove that it is a perfect set.

Definition 4.16. ***[Perfect set]***
The set A in a topological space X is called perfect if it is both closed and contains no isolated points.

Lemma 4.17. *Every non-empty perfect set is uncountable*

Proposition 4.18. ***[The Julia set is a perfect set]***
The Julia set $\mathcal{J}(f)$ has no isolated point, hence it's a perfect set.

Proof. Let $v \in \mathcal{J}(f)$ with an open neighbourhood V of v. The aim is to show there exist other points of $\mathcal{J}(f)$ in V. We consider three cases,

- If v is not a fixed point. Since the Julia set is invariant under f, the second statement in proposition 4.15 gives that $f^{-k}(v) \subset \mathcal{J}(f)$ for some $k \geq 1$. This point will different from v.
- v is a fixed point, $f(v) = v$. If $f(z) = v$ has no solution other than v, then $v \notin \mathcal{J}(f)$. Let $w \neq v$ and $f(w) = v$, again by proposition 4.15, V contains a point of $f^{-k}(w) = f^{-k-1}(v)$ for some $k \geq 1$. Any of such point is in the Julia set because of backwards invariance and if further distinct from v.

- If we have a periodic point of order $p > 1$, $f^p(v) = v$. Since $\mathcal{J}(f^p) = \mathcal{J}(f)$, we can employ the same argument as in the last point, since v is a fixed point for the function $g = f^p$.

□

We are now ready to prove the remaining equivalences in the main Theorem of this section.

Theorem 4.19. ***(Uniform limit of holomorphic functions is holomorphic)***
Let U be an open subset of the complex plane, and $f_n : U \to \mathbb{C}$ be a sequence of holomorphic functions converging uniformly on every compact subsets of U to f. Then, f is holomorphic moreover $f_n^{(k)} \to f^{(k)}$ uniformly on compact subsets of U.

Proof. **[Theorem 4.7 (ii) $\iff$ (iii)]**

Let w be a repelling periodic point of f of period p. Then, it is a repelling fixed point of $g = f^p$, since the Julia set of g and f is the same we can study g. Now, assume that $w \notin \mathcal{J}(f)$ that is $\{g^k\}$ is normal at w, that is there exists some open set V containing w such that for every subsequence $\{g^{k_i}\}$ and every compact subset of V, $\{g^{k_i}\}$ converges to some analytic function g_0. Since $g^k(w) = w$ for all k, then $g_0 \neq \infty$. Via Theorem 4.19 the derivatives also converge $(g^{k_i})'(z) \to g_0'(z)$ if $z \in V$. However, since w is a repelling fixed point for g, $|g(z)'| > 1$. Using the chain rule we get that $|(g^{k_i})'(w)| = |(g'(w))^{k_i}| \to \infty$, contradicting faintness of $g_0'(w)$. Thus, $\{g^k\}$ cannot be normal at w, which implies that $w \in \mathcal{J}(g) = \mathcal{J}(f^p) = \mathcal{J}(f)$. Since the Julia set is a closed set, it also contains the closure of the repelling periodic points.

For the opposite inclusion, let $E = \{w \in \mathcal{J}(f) : \exists v \neq w \text{ with } f(v) = w \text{ and } f'(v) \neq 0\}$, that is the set of points w in the Julia set which are the images of some non-critical point $v \neq w$. Let $w \in E$, which implies $f'(w) \neq 0$ and there exist some open neighbourhood V of w with some local analytic inverse $f^{-1} : V \to \mathbb{C} \setminus V$ such that $f^{-1}(w) = v \neq w$ by choosing the values of $f^{-1}(z)$ in a continuous manner. Now, define a new family of functions

$$h_k(z) = \frac{f^k(z) - z}{f^{-1}(z) - z}.$$

Let U be any open subset of V which contains w. Since $\{f^k\}$ is not normal on U, then $\{h_k\}$ is also not normal on U. Now, using the key Theorem by Montel, $h_k(z)$ must take either the value 0 or 1 for some k_0 and $z_0 \in U$.

- In the first case, we get, $h_{k_0}(z_0) = 0$, which implies that $f^{k_0}(z_0) = z_0$ for $z_0 \in U$.
- In the latter case, we get $h_{k_0}(z_0) = 1$, which implies that $f^{k_0}(z_0) = f^{-1}(z_0)$ so $f^{k_0+1}(z_0) = z_0$.

Thus, any open subset U of V contains a periodic point of f. Furthermore, since f is a polynomial, E contains all of $\mathcal{J}(f)$ expect for a finite number of points. This combined with the fact that the Julia set is a perfect set, that is, it contains no isolated points- hence, we can chose a sequence of periodic points p_n that accumulate to w. Mitsuhiro Shishukur proved [Shishikura (1987)] that for a rational function the number of non-repelling periodic points is finite. More precisely, for a rational map f with degree d, the non-repelling orbits was proved to be sharply bounded above with $2d - 2$. Using this, we can remove the finite set of non-repelling periodic point from p_n leaving us with a sequence of repelling periodic points accumulating to w. Hence, any $w \in E$ is in the closure of the repelling periodic points. But, since f is a polynomial, then E will contain all of $\mathcal{J}(f)$ except of finite number of points. Thus $\mathcal{J}(f) \subset \bar{E} \subset \overline{\{repelling\ periodic\ points\}}$, and we are done. □

One can prove similar equivalence for more general families of functions. Fatou proved that, when f is a rational function with degree $d \geq 2$ then the equivalence holds [Fatou (1919)][2].

Proof. **[Theorem 4.7 (iii) $\iff$ (iv)]**

First, assume $z \in \mathcal{J}(f)$. Since the Julia set is invariant under iterations of f, we get that $f^k(z) \in \mathcal{J}(f)$ for all k. Now, let U be some open set containing z. Then, $f^k(U)$ contains points of $A(w)$ that is, there exist arbitrary close points to z that iterate to w. Thus, $z \in \overline{A(w)}$, since $z \notin A(w)$, hence $z \in \partial A$.

Suppose $z \in \partial A(w)$, but $z \notin \mathcal{J}(f)$. Thus, there exist some connected open set around z on which $\{f^k\}$ has a subsequence that converges uniformly to some analytical function or ∞. Note that on the open and connected set $V \cap A(w)$, the sequence converges to w. Note that analytical functions are constant on connected set if they are constant on any open subset. But this means that all the points in V gets mapped to $A(w)$ under iterates of f, which contradicts $z \in \partial A(w)$. □

[2] For a proof in English, consult [Schwick (1997)] where W.SCHWICK proved the case when f is an entire function. Consult also chapter five in Blanchard [Blanchard (1984)] for similar results on the Riemann sphere.

4.2 The Mandelbrot set for quadratic polynomials

If we consider the family of functions f_c depending on a complex constant c, then a natural question is "how the Julia set varies with the parameter?". In this thesis, we will study the family of complex quadratic functions. We assert that it's enough to study polynomials of the form

$$f_c(z) = z^2 + c.$$

This is the case since for any other polynomial g of degree 2, we can find an affine map $h = \alpha z + \beta$ which conjugate with f_c,

$$g = h^{-1} \circ f_c \circ h \implies g^k = h^{-1} \circ f_c^k \circ h.$$

This in turns, means that the Julia set $\mathcal{J}(g)$ is the same as $\mathcal{J}(f)$ up to some affine translation. We only need to pick α and β correctly according to the following explicit formula.

$$h^{-1}(f_c(h(z))) = \frac{\alpha^2 z^2 + 2\alpha\beta z + \beta^2 + c - \beta}{\alpha}.$$

Example 4.20. *Let $g : z \mapsto z^2 + 2z$. If we wishes to find the Julia set, we observe that g conjugates with $f_0 : z \mapsto z^2$ via, $h : z \mapsto z + 1$.*

$$g(z) = z^2 + 2z = (z+1)^2 - 1 = (h(z))^2 - 1 = (h^{-1} \circ f_0 \circ h)(z).$$

From example 4.8, we know that the Julia set of f_0 is the circle centred at zero. Hence, the Julia set for g becomes

$$\mathcal{J}(g) = h^{-1}(\mathcal{J}(f)) = \{z - 1 : |z| = 1\}.$$

Definition 4.21. ***(Mandelbrot set)***
Let f_c be a family of complex function depending on the complex parameter c.

$$\mathcal{M} = \{c \in \mathbb{C} : \mathcal{J}(f_c) \text{ is connected}\}.$$

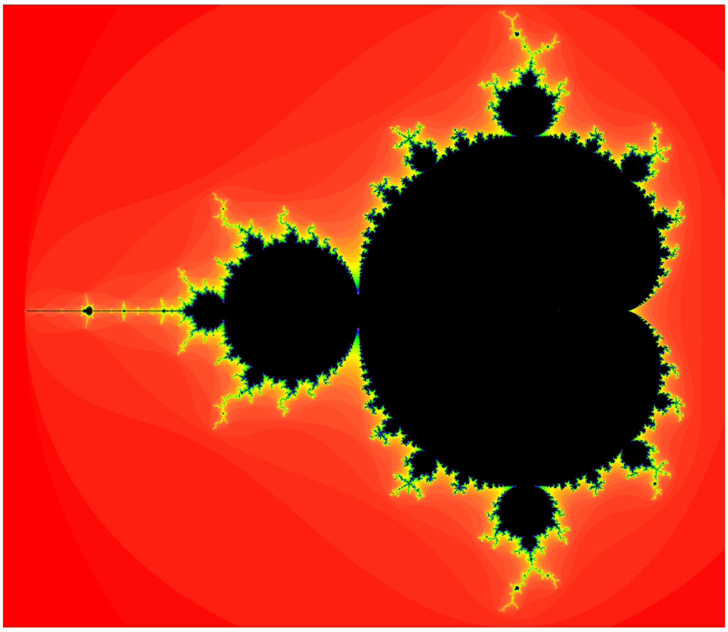


Figure 4: The Mandelbrot set

This definition is difficult to work with, especially when checking if some c is in the Mandelbrot set. We hence give an equivalent characterisation of the Mandelbrot set in terms of iterates of the origin.

Theorem 4.22.

$$\mathcal{M} = \{c \in \mathbb{C} : \{f_c^k(0)\}_{k\geq 1} \textit{ is bounded}\}.$$

Since the origin is a critical point for f_c for each c (i.e. $f_c'(0) = 0$), it plays an integral role in determining the structure of the Mandelbrot set. It's exactly at this point for which f_c fails to be a local bijection. This will be a crucial part of the proof. To prove this Theorem, we need the following Lemma concerning the inverse image under f_c of loops in relation to the placement of c.

Lemma 4.23. *Let C be a loop that is a smooth, closed, non-self-intersecting curve in the complex plane.*

i) *If c is inside of C then $f_c^{-1}(C)$ is a loop, with the inverse image of the interior of C as the interior of $f_c^{-1}(C)$.*

ii) *If $c \in C$, then $f^{-1}(C)$ is of a figure eight self intersecting at 0. Again, the inverse image of the interior of C is the interior of the two loops.*

iii) *If c is outside C, then $f^{-1}(C)$ comprises two disjoint loops, with the inverse image of the interior of C the interior of the two loops.*

Proof. **(Theorem 4.22)**

First, note that $\{f_c^k(0)\}$ is bounded if and only if $f_c^k(0) \not\to \infty$. We will begin by proving that if $\{f_c^k(0)\}$ is bounded, then the corresponding Julia set is connected. Begin by choosing C

to be a great circle that contains all of $\{f_c^k(0)\}$ such that $f_c^{-1}(C)$ is interior to C and with all points outside C iterating to infinity. Since $c = f_c(0)$ is inside C, the same follows for $f_c^k(c)$, Lemma 4.23 (i) gives that $f^{-1}(C)$ will be mapped onto a loop contained inside of C. Because the exterior of C is mapped to the exterior of $f_c^{-1}(C)$, hence c is inside of $f_c^{-1}(C)$. Applying the same argument yields a nested sequence $\{f_c^{-k}(C)\}$ of loops, refer to Figure 5 (a). Now, let K denote the closed set that constitutes all the points that are on or inside of $f_c^{-k}(C)$ for all k. Then

$$A(\infty) = \{z : f_c^k(z) \to \infty\} = \mathbb{C} \setminus K.$$

This is because, if $z \notin K$, then for some iterate k $f_c^k(z) \notin C$, hence diverges. The boundary of K is the Julia set. Since K is attained by intersecting a decreasing sequence of loops, then K is simply connected, and hence, the boundary is connected.

The opposite implication follows in the same manner. Pick a circle C in the same way as the previous proof with the added conduction that for some p, then $f_c^{p-1}(c) = f_c^p(0) \in C$. That is the forward orbit of the critical point hits the circle. The decreasing sequence of loops $\{f_c^{-k}(C)\}$ (figure 5 (b)). The previous argument breaks down at step $1-p$. This is since $c \in f_c^{1-p}(C))$, applying the inverse image again will, by Lemma 4.23 (b), result in an "eight" shaped curve with a self-intersection at the origin, for brevity let $E = f_c^{-p}(C)$. The Lemma also tells us, each half of E gets mapped by f_c onto $f_c^{1-p}(C)$. The Julia set must be contained in the interior of E. Since the origin diverges under iterations, hence $0 \notin \mathcal{J}(f)$. Either the Julia set is contained completely in one of the halves, or it's disconnected. Since the Julia set is invariant under f_c^{-1},parts of it must be contained in each of the loops of E, proving that the Julia set is disconnected.

□

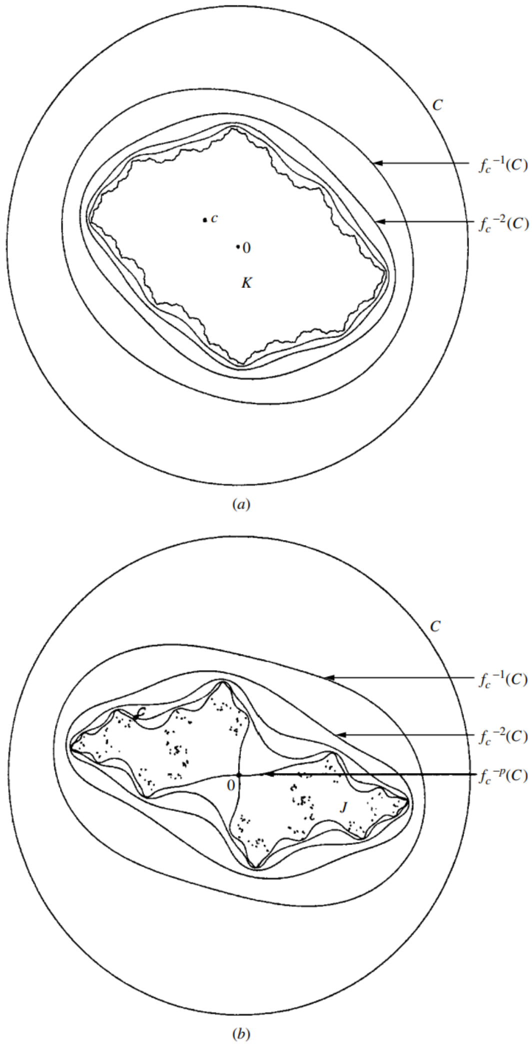


Figure 5: The inverse image of a great circle C, (a) $c = -0.3 + 0.3i$; (b) $c = -0.9 + 0.5i$.

The next Theorem states some of the general properties for the Mandelbrot set.

Theorem 4.24. *The Mandelbrot set $\mathcal{M}$ is a closed set that is contained within the closed disk of radius 2. Moreover, $\mathcal{M} \cap \mathbb{R} = [-2, 1/4]$.*

Proof. To prove that the Mandelbrot set is contained in the closed disk of radius 2 assume $|\hat{c}| = 2 + \epsilon$ for some $\epsilon > 0$. The aim is to show via induction that

$$|f_{\hat{c}}^k(0)| \geq 2 + (2^k - 1)\epsilon \quad k \geq 1.$$

The base case follows directly

$$|f_{\hat{c}}(0)| = |c| = 2 + \epsilon \geq 2 + (2^1 - 1)\epsilon.$$

Assume now that

$$|f_{\hat{c}}^k(0)| \geq 2 + (2^k - 1)\epsilon.$$

Using the triangle inequality gives

$$|f_{\hat{c}}^{k+1}(0)| = |(f_{\hat{c}}^k(0))^2 + \hat{c}| \geq |f_{\hat{c}}^k(0)|^2 - |\hat{c}| \geq |(2 + (2^k - 1)\epsilon)^2| - |\hat{c}| =$$
$$= 4 + 4(2^k - 1)\epsilon + \underbrace{(2^k - 1)^2\epsilon^2}_{>0} \geq 2(2 + (2^{k+1} - 2)\epsilon) \geq 2 + (2^{k+1} - 1)\epsilon.$$

Hence, $|f_{\hat{c}}^k(0)| \to \infty$, since for any $M > 2$ choosing $N \geq \log_2(\frac{M-2}{\epsilon}) - 1$ will give $|f_c^k(0)| \geq M$, for $k \geq N$.

The Mandelbrot set is closed. Since $f_c(0)$ is a continuous function in c, the Mandelbrot is written as follows

$$\mathcal{M} = \bigcap_{k=1}^{\infty} \{c \in \mathbb{C} : |f_c^k(0)| \leq 2\}.$$

The right side is an intersection of closed sets (continuous functions on $\mathbb{C}$ are closed under compositions), hence is closed.

$\mathcal{M} \cap \mathbb{R} = [-2, 1/4]$: When c is real then $f_c(x) - x$ has no real fixed roots. Hence, for any $c > 1/4$ then $f_c^k(0)$ diverges, since otherwise, the finite convergence point will be a fixed point and hence contradicting the non-existence of real roots for $f_c(x)$. When $c \leq 1/4$ let $a = (1 + \sqrt{1 - 4c})/2$ be the larger real root. If $c \geq -2$

$$|c| = |f_c(0)| \leq a.$$

Then $|f_c^n(0)| \leq a$ implies

$$|f_c^{n+1}(0)| = |f_c^n(0) + c| \leq a^2 + c = \frac{1 + 1 - 4c + 2\sqrt{1 - 4c} + 4c}{4} = \frac{1 + \sqrt{1 - 4c}}{2} = a.$$

Hence the sequence is uniformly bounded by a, and $\mathcal{M} \cap \mathbb{R} = [-2, 1/4]$. □

Mitsuhiro Shishikura proved the following remarkable Theorem,

Theorem 4.25. *[Shishikura (1998)]*

$$\dim_H(\partial\mathcal{M}) = 2$$

Moreover, for any open set U intersecting $\partial\mathcal{M}$, has $\dim_H(\partial \cap U) = 2$.

But it is still unknown what the Hausdorff measure of the boundary is.

5 Quasiconformal mappings & moduli of curve families

Quasiconformal maps arose historically as a consequence of Grötzsch problem, which is stated as follows. Let R and R' be two rectangles, with vertices (A, B, C, D) and (A_1, B_1, C_1, D_1) with side length (a, b) and (a_1, b_1) respectively. Consider $f \in C^1$ such that $f : R \to R_1$ which maps all the vertices from R onto R_1.

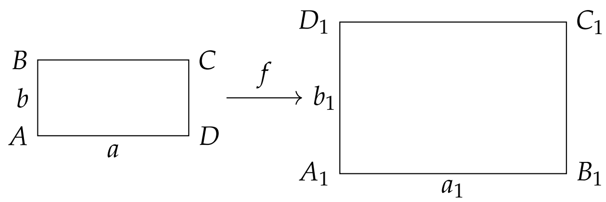


Figure 6: Grötzsch problem

The problem is how "close" f can be made into a conformal map, that is, f is a homeomorphism and is holomorphic. Herbert Grötzsch proved in 1928 that f cannot be conformal unless $a/b = a_1/b_1$. In fact, there only exists such map a when $a/b = a_1/b_1$. We will now define what it means to be close to a conformal map, which in turn, will define the notion of a Quasiconformal maps.

Let Ω be a connected plane domain with a smooth boundary in the complex plane. Also let $f : \Omega \to f(\Omega)$ be a C^1-diffeomorphism. We expand $f = u(x, y) + iv(x, y)$. The differential defines a linear map

$$du = u_x dx + v_y dy,$$

$$dv = v_x dx + v_y dy.$$

Or in a compact form

$$df = f_z dz + f_{\bar{z}} d\bar{z}.$$

With f_z and $f_{\bar{z}}$ are the complex derivatives. Since f is a diffeomorphism, we can approximate its action via the linear maps. The linear map df sends circles onto ellipses with major axis of length α and minor axis of length β.

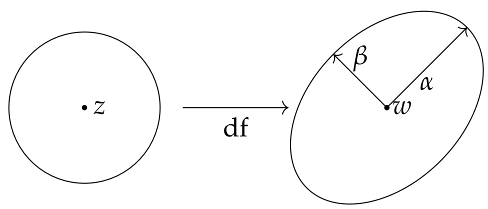


Figure 7: Distortion on infinitesimal circles

The distortion at a point z_0 is defined to be

$$D_f := \frac{\alpha}{\beta} = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \geq 1.$$

Now, by Cauchy–Riemann equations, if $f \in C^1$ and f is conformal, it then implies that $f_{\bar{z}} \equiv 0$, which gives $D_f = 1$. Geometrically, this means conformal functions map infinitesimal circles onto infinitesimal circles.

Definition 5.1. *Let $f : \Omega \to \mathbb{C}$ be a diffeomorphism. We say that f is a quasiconformal map if the distortion is bounded in Ω. Further, if the distortion is bounded by some $1 \leq K < \infty$, then f is called K-quasiconformal. Let K_f be the infimum of all such K.*

The set of all C^1-diffeomorphism that are compactly contained in Ω are quasiconformal. Equipped with this, the larger K_f is the further away a map is from being conformal. It's also true that a map is conformal if, and only if, it is 1-quasiconformal. Hence, the precise formulation of Grötzsch problem is to find a homeomorphism f which maps R into R_1 preserving vertices for which $K_f \leq K_g$ for any other admissible homeomorphism g, and if this map exist, to find what is K_f. For a full treatment and solution of the Grötzsch problem, refer to any of the following books Fletcher & Markovic (2007) or Ahlfors (1966). We will instead move on to define module of family of curves and extremal length.

We start with fixing notation and listing some basic definitions.

Definition 5.2. ***[Curve]*** *Let $I \subset \mathbb{R}$ be an interval. A continuous mapping $\gamma : I \to \mathbb{R}^n$, is called a* ***curve****. The curve is called open or closed if I is open or closed.*

Definition 5.3. ***[Length of curve & (locally) rectifiable curves]*** *Let $\gamma : [a, b] \to \mathbb{R}^n$ be a curve and let $\{t_k\}_{k=0}^n$ be an ordered partition of $[a, b]$. The* ***length of curve*** *γ is defined by*

$$l(\gamma) = \sup \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})|.$$

With supremum taken over all possible ordered partitions. Note that $0 \leq l(\gamma) \leq \infty$. A curve γ is called ***rectifiable*** *if the length if $l(\gamma) < \infty$ and* ***locally rectifiable*** *if γ, restricted to each closed sub-interval of I, is rectifiable.*

Let Γ be a family of locally rectifiable curves in the plane. We shall now introduce a geometric quantity called the extremal length $\lambda(\Gamma)$, which is the average of the minimum length. This quantity will later prove to be an invariant under conformal mappings and bounded under quasiconformal mappings.

Definition 5.4. *A function ρ, defined on the whole plane, is an* ***admissible metric*** *if*

- $\rho \geq 0$
- *ρ is measurable*
- $A(\rho) = \int\int_{\mathbb{C}} \rho^2 dxdy \neq 0$ *and* $\neq \infty$

The length of a curve $\gamma \in \Gamma$ with respect to ρ is

$$L_\gamma(\rho) = \int_\gamma \rho |dz|$$

Let $L(\rho)$ define the minimum over Γ that is

$$L(\rho) := \inf_{\gamma \in \Gamma} L_\gamma(\rho)$$

*The **extremal length** is defined by*

$$\lambda(\Gamma) := \sup_{\rho} \frac{L(\rho)^2}{A(\rho)},$$

with supremum taken over all admissible metrics.

Note that $L_\gamma(1)$ coincides with the length of the curve given in Definition 5.3. We are now ready to define the modulus of a curve family.

Definition 5.5. *Let Γ be a family of locally rectifiable curves in $\mathbb{R}^n$, let*

$$F(\Gamma) := \{\rho : \mathbb{R}^n \to R : \rho \text{ is non-negative, Borel measurable, and } \int_\gamma \rho ds \geq 1 \;\; \forall \gamma \in \Gamma\}.$$

A function in $F(\Gamma)$ is called an admissible function, denote by $d\mathcal{L}^m$ the m-dimensional Lebesgue measure. Now for $p \geq 1$ we set

$$M_p(\Gamma) := \inf_{\rho \in F(\Gamma)} \int\int \rho^p d\mathcal{L}^2(z)$$

*The number $0 \leq M_p(\Gamma) \leq \infty$ is called the **p-module of the family** Γ. In this thesis, we will only consider the case when $p = n = 2$.*

Following this definition, we remark that, for a given Γ as above, we get $\lambda(\Gamma) = \frac{1}{M(\Gamma)}$. It is more natural to work with the module, for it has the following measure-theoretic property:

Theorem 5.6. *Let $p \geq 1$ then M_p is an outer measure in the space of all curves in $\mathbb{R}^n$. That is*

i) $M_p(\emptyset) = 0$

ii) *If* $\Gamma_1 \subset \Gamma_2$ *then* $M_p(\Gamma_1) \leq M_p(\Gamma_2)$

iii) $M_p\left(\bigcup_{i=i}^{\infty} \Gamma_i\right) \leq \sum_{i=1}^{\infty} M_p(\Gamma_i)$

Proof. Since the zero function belongs to $F(\emptyset)$ then $M_p(\emptyset) = 0$. If $\Gamma_1 \subset \Gamma_2$, then $F(\Gamma_2) \subset F(\Gamma_1)$, hence $M_p(\Gamma_1) \leq M_p(\Gamma_2)$. For the last, we may assume that $M_p(\Gamma_i) < \infty$ for each i. For every $\epsilon > 0$ pick $\rho_i \in F(\Gamma_i)$ such that

$$\int \rho_i^p d\mathcal{L}^m(z) < M_p(\Gamma_i) + \frac{\epsilon}{2^i}.$$

Then the function $\rho = (\sum \rho_i^p)^{1/p}$ belongs to $F(\cup_{i=1}^{\infty}\Gamma_i)$, since for all i, we have $\rho \geq \rho_i$. Thus

$$M_p(\Gamma) \leq \int \rho^p d\mathcal{L}^m(z) = \sum_{i=1}^{\infty} \int \rho_i^p d\mathcal{L}^m(z) < \epsilon + \sum_{i=1}^{\infty} M_p(\Gamma_i).$$

Let $\epsilon \to 0$ gives the desired inequality. □

Theorem 5.7. ***(Composition of of quasiconformal maps)*** *[Fletcher & Markovic (2007)]*
If f is a K_1-quasiconformal map and g a K_2-quasiconformal map, then both $f \circ g$ and $g \circ f$ are K_1K_2 quasiconformal

Lemma 5.8. ***[Grötzsch inequality]*** *Let Γ_1, Γ_2 be two disjoint families of curves. Then*

$$M_2(\Gamma_1 \cup \Gamma_2) \geq M_2(\Gamma_1) + M_2(\Gamma_2).$$

Proof. Note this is the same as saying $\lambda(\Gamma_1 \cup \Gamma_2)^{-1} \geq \lambda(\Gamma_1)^{-1} + \lambda(\Gamma_2)^{-1}$. Without loss of generality we can assume $\lambda(\Gamma_1 \cup \Gamma_2) \neq 0$. Consider an admissible ρ with $L(\rho) > 0$ and set $\rho_i = \rho$ on E_i for $i = 1, 2$ and $\rho_i = 0$ outside of E_i. With E_1 and E_2 are complementary measurable sets such that $\Gamma_1 \subset E_1$ and $\Gamma_2 \subset E_2$. Then, we have $L_{\Gamma_i}(\rho_1) \geq L(\rho)$, and $A(\rho) = A(\rho_1) + A(\rho_2)$ hence

$$\frac{A(\rho)}{L^2(\rho)} \geq \frac{A(\rho_1)}{L^2_{\Gamma_1}(\rho_1)} + \frac{A(\rho_2)}{L^2_{\Gamma_2}(\rho_2)}.$$

□

Now, we are ready to prove the important property which is invariance under conformal mappings.

Theorem 5.9. *For a map $f : \Omega \to \mathbb{C}$, $\Omega \in \mathbb{C}$, the following is true.*

i) If f is conformal, then $M_2(\Gamma) = M_2(f(\Gamma))$.

ii) If f is quasiconformal, then $\frac{1}{K} M_2(\Gamma) \leq M_2(f(\Gamma)) \leq K M_2(\Gamma)$.

Proof. We will only prove the second statement, since the first follows with $K = 1$. We will prove the similar statement for extremal lengths. Let $\xi = f(z)$ and ρ be an admissible metric, define

$$\rho_1(\xi) := \left(\frac{\rho}{|f_z| - |f_{\bar{z}}|} \circ f^{-1} \right)(\xi)$$

and zero outside $f(\Omega)$. Then

$$\int\int_{f(\Omega)} \rho_1^2 |d\xi|^2 = \int\int_\Omega \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \rho^2 dxdy \leq \int\int_\Omega \underbrace{\frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}}_{=D_f} dxdy \int\int_\Omega \rho^2 dxdy \leq K_f A(\rho) < \infty.$$

In the first equality, we transform to the domain Ω and multiply by the Jacobin $J_{f^{-1}}$. The last inequality follows that f is quasiconformal and ρ is admissible, hence ρ_1 is admissible. If $\gamma_1 = f(\gamma)$ for some $\gamma \in \Gamma$ and $\gamma_1 \in f(\Gamma)$, then

$$\int_{\gamma_1} \rho_1 |d\xi| \geq \int_\gamma \rho |dz|$$

which implies

$$L_{\gamma_1}(\rho_1) \geq L_\gamma(\rho) \implies L(\rho_1) \geq L(\rho).$$

Finally

$$\lambda(\Gamma_1) = \sup_{\rho_1} \frac{L^2(\rho_1)}{A(\rho_1)} \geq \sup_\rho \frac{L^2(\rho)}{K_f A(\rho)} = \frac{1}{K_f} \lambda(\Gamma) \implies K_f M_2(f(\Gamma)) \leq M_2(\Gamma).$$

The other inequality follows from same consideration but with the inverse f^{-1}, since the inverse is also quasiconformal. □

Example 5.10. ***[The module of an annulus]***
An annulus consists of the region between two concentric circles. Let $A = \{z = re^{i\theta} : r_1 < |z| = r < r_2, 0 < \theta < 2\pi\}$ be the annulus, denoted by Γ the family of locally rectifiable curves that joins the

boundaries of the annulus A which do not leave $\overline{A}$ (see Figure 8). Let $\rho = \frac{1}{r}$, we want to show that this is indeed an admissible metric for Γ, then, via Cauchy–Schwarz inequality, bound the 2-module. Let $\gamma \in \Gamma$, we obtain

$$\int_\gamma \rho ds = \frac{1}{\log(\frac{r_2}{r_2})}\int_\gamma \frac{dt}{r} = \frac{1}{\log(\frac{r_2}{r_2})}\int_{r_1}^{r_2} \frac{|\dot\gamma(t)|}{|\gamma(t)|}dt \geq \frac{1}{\log(\frac{r_2}{r_2})}\left|\int_{r_1}^{r_2} \frac{\dot\gamma(t)}{\gamma(t)}dt\right| = \frac{1}{\log(\frac{r_2}{r_2})}\left|log(\frac{r_2}{r_1})\right| = 1$$

The inequality is the triangle inequality for integrals. Hence ρ is admissible for Γ. Let $\Gamma_0 \subset \Gamma$ be the radial curves. Let $\gamma_0 \in \Gamma_0$ with following parametrization $\gamma_0 = (t\cos\theta, t\sin\theta)$ for $t \in [r_1, r_2]$. Then

$$\left|\frac{d\gamma_0(t)}{dt}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

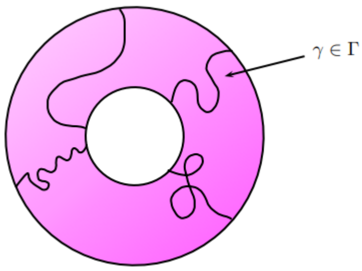


Figure 8: Annulus with arcs joining outer and inner circle with radius r_2 respectively r_1.

Hence we get the following

$$\int_{\gamma_0} \rho ds = \int_{r_1}^{r_2} \frac{1}{r\log(\frac{r_2}{r_1})}\left|\frac{d\gamma_0(t)}{dt}\right|dt = \frac{1}{\log(\frac{r_2}{r_1})}\int_{r_1}^{r_2}\frac{1}{r}dt = 1.$$

for any $\gamma_0 \in \Gamma_0$. Moreover, for and $\gamma \in F(\Gamma)$ and any $\Gamma \ni \gamma : [r_1, r_2] \to A$

$$1 \leq \int_\gamma \rho dr = \int_{r_1}^{r_2} \rho(\gamma(r))\left|\frac{d\gamma(r)}{dr}\right|dr = \int_{r_1}^{r_2} \rho(re^{i\theta})r^{1/2}r^{-1/2}dr.$$

Now, we integrate over $\theta \in [0, 2\pi]$, giving

$$2\pi \leq \int_0^{2\pi}\int_{r_1}^{r_2} \rho(re^{i\theta})r^{1/2}r^{-1/2}drd\theta = \int_0^{2\pi}\int_{r_1}^{r_2} \rho(re^{i\theta})r^{1/2}r^{-1/2}drd\theta.$$

Applying the Cauchy-Shwarz inequality, we see

$$\begin{aligned} 2\pi &\leq \left(\int_0^{2\pi}\int_{r_1}^{r_2} \rho^2(re^{i\theta})rdrd\theta\right)^{1/2} \cdot \left(\int_0^{2\pi}\int_{r_1}^{r_2} r^{-1}drd\theta\right)^{1/2} \\ &= \left(\int_0^{2\pi}\int_{r_1}^{r_2} \rho^2(re^{i\theta})rdrd\theta\right)^{1/2} \cdot \left(2\pi\log(\frac{r_2}{r_1})\right)^{1/2}. \end{aligned}$$

Squaring gives

$$\frac{2\pi}{\log(\frac{r_2}{r_1})} \leq \int \int_A \rho^2 d\mathcal{L}^2(z).$$

Since this holds for every $\rho \in F(\Gamma)$, it also holds when taking the infimum over $F(\Gamma)$, that is,

$$\frac{2\pi}{\log(\frac{r_2}{r_1})} \leq M_2(\Gamma).$$

Since we proved that $\rho_0 = \frac{1}{r log(\frac{r_2}{r_1})}$ is admissible for Γ, we also have

$$M_2(\Gamma) = \inf_{\rho \in F(\Gamma)} \int \int \rho^2 d\mathcal{L}^2(z) \leq \int \int \rho^2 d\mathcal{L}^2(z) = \frac{2\pi}{\log(\frac{r_2}{r_1})}.$$

Altogether $M_2(\Gamma) = \frac{2\pi}{\log(\frac{r_2}{r_1})}$.

In the next section, we will estimate the moduli of doubly connected regions to bound the distortion of the quadratic map using the Koebe distortion theorem. The moduli, being invariant under conformal mappings (as established by the previous theorem), have been calculated for an annulus. Thus, it is relevant to determine if there exists a conformal map from a doubly connected region to two concentric disks. We recall that Riemann's mapping Theorem gives such a map from one open simply connected region onto the unit disk. A more general statement is needed. The following is a simplification of the Theorem presented in Lars Ahlfor's book: Complex analysis.

Theorem 5.11. ***[Ahlfors (2021), Chapter 6 Theorem 10]***
Let Ω be a n-multiply connected region (that is the complement of Ω contains exactly n connected components), $n > 1$. There exists a conformal mapping of Ω onto some annulus.

This theorem, combined with the previous theory, ensures that we can calculate the moduli of a doubly connected region by first conformally mapping it onto two concentric circles. The moduli will then be determined by the radii of these concentric circles.

We will now state two distortion Theorems that will be used in the next section:

Theorem 5.12. ***(Koebe distortion Theorem)***
Let $f : \mathbb{C} \to \mathbb{C}$ be some univalent function on some open disk $\mathbb{B}(a, r)$. Furthermore let $\mathbb{B}(a, \hat{r}) \subset \mathbb{B}(a, r)$. Then, there exist some constant that only depends on $\hat{r}$ such that,

$$\left| \frac{Df(z)}{Df(w)} \right| \leq C_{\hat{r}}, \qquad z, w \in \mathbb{B}(a, \hat{r}).$$

Theorem 5.13. *[Carleson & Gamelin (1993), theorem 1.6]**(Distortion Theorem)***
If $f \in \mathcal{S}$, then

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |f'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3},$$

where $\mathcal{S}$ is the collection of univalent functions in the open disk $\Delta = \{|z| < 1\}$ such that $f(0) = 0$ and $f'(0) = 1$.

6 Julia set for quadratic polynomials

In this section, we combine the previous sections to calculate the Hausdorff dimension for different Julia sets. The importance of the Mandelbrot set on the geometry of Julia sets will also become more apparent. The attractive periodic point plays a crucial role in the form of the Julia set. The following Theorem gives us a way to classify the parameters c.

Theorem 6.1. *If $\mathbb{C} \ni w \neq \infty$ is an attractive periodic point of a polynomial f_c then $A(w)$ contains some critical point.*

Proof. **Claim 1:** The quadratic polynomial f_c has at most one attractive fixed-point w with $c \in \mathcal{A}(w)$.
Assume, for a contradiction, that $c \notin \mathcal{A}(w)$. Now, let $w \in U \subset \mathcal{A}(w)$ be an open set. Then, $f_c^k(c) \notin U$ for all k. Pick for each k the branch of the inverse f_c^{-k} to be the continuous analytic function with $f_c^{-k}(w) = w$. Now, since $f_c^{-k}(U) \subset \mathcal{A}(w)$ for all k, and the attractive basin of w is a bounded subset of $\mathbb{C}$, then, Montel theorem implies that $\{f_c^{-k}\}_{k=0}^{\infty}$ is normal on U. However, this is a contradiction, since w is a repelling fixed point for f_c^{-1}. Note that, by the chain rule, we get $(f_c^{-k})'(w) = ((f_c^{-1})'(w))^k \to \infty$. Hence, any subsequence cannot converge uniformly to some analytical function. Hence, $c \in \mathcal{A}(w)$. Further, w is unique since it cannot be in two different basins of attraction.

Claim 2: Let w be an attracting fixed point for some general polynomial $f \in \mathbb{C}[z]$, then $\mathcal{A}(w)$ contains some critical point $\hat{z}$.
This proof follows the same structure as the above. Assume, for the sake of contradiction, that $\mathcal{A}(w)$ lacks all the critical points. Pick some open disc $w \in U \subset \mathcal{A}(w)$. Hence, $f^k(\hat{z}) \notin U$ for all k and all critical point $\hat{z}$. Hence, we can choose for each k a branch of the inverse f^{-k} on U that is a continuous analytic function with $f^{-k}(w) = w$ for each k. Since U is invariant within the basin of attraction under iterates of the inverse map, i.e. $f^{-k}(U) \subset \mathcal{A}(w)$ for each k and $\mathcal{A}(w)$ is bounded. We have again, by Montel's theorem, that $\{f^{-k}\}_{k=0}^{\infty}$ is a normal family on U, which is a contradiction, since w is a repelling fixed point of f^{-1}. Thus $\hat{z} \in \mathcal{A}(w)$ for some critical point $\hat{z}$.

Now, the Theorem follows from applying claim 2 on the function $g = f^p$ where p is the order of w. □

The only critical point of f_c is 0. Then, f_c can only have one attracting periodic orbit. Note that if $c \notin \mathcal{M}$, then we have no attracting periodic orbits. This follows from both Theorem 4.22 and Theorem 6.1. Hence, one can categorise the points in the Mandelbrot set according to the order of the periodic orbit p. Any points with different p are considered to be in separate regions of the Mandelbrot set.

To begin with calculating the Hausdorff dimension, let $\{S_1, S_2\}$ be the IFS consisting of the two different branches of f_c^{-1}. We will suppose that c is large enough so that the attractor for the IFS $\{S_1, S_2\}$ is totally disconnected and satisfy the conditions for Theorem 3.5 and Theorem 3.6.

Theorem 6.2. ***(Hausdorff dimension for large c)***
Suppose $|c| > \frac{1}{4}(5 + 2\sqrt{6}) \approx 2.475\ldots$ Then $\mathcal{J}(f_c)$ is totally disconnected, and is the attractor set of

the contractions given by the two inverse branches $f_c^{-1}(z) = \pm(z-c)^{1/2}$. The Hausdorff dimension is given by

$$\frac{2\log 2}{\log 4(|c| + |2c|^{1/2})} \le \dim_H(F) \le \frac{2\log 2}{\log 4(|c| - |2c|^{1/2})}.$$

Proof. Let $C = \{z \in \mathbb{C} : |z| = |c|\}$, and $D = \{z \in \mathbb{C} : |z| < |c|\}$. Since $c \in C$ then by Lemma 4.23, the preimage of C will be an eight shaped curve that self intersects at zero (see Figure 9). Notice that $f_c^{-1}(C) \subset D$, since if $|z| > |C| > 2$, then $|f_c(z)| \ge |z^2| - |c| \ge |c^2| - |c| > |c|$. Each interior for both loops of $f_c^{-1}(C)$ is mapped under f_c bijectivity onto D. Now let $S_1, S_2 : D \to D$ be the two branches of f_c^{-1}, then $S_1(D)$ and $S_2(D)$ are the interior of the two loops. Now, define $V := \{z : |z| < r_V = |2c|^{1/2}\}$ hence $f_c^{-1}(C) \subset V$.

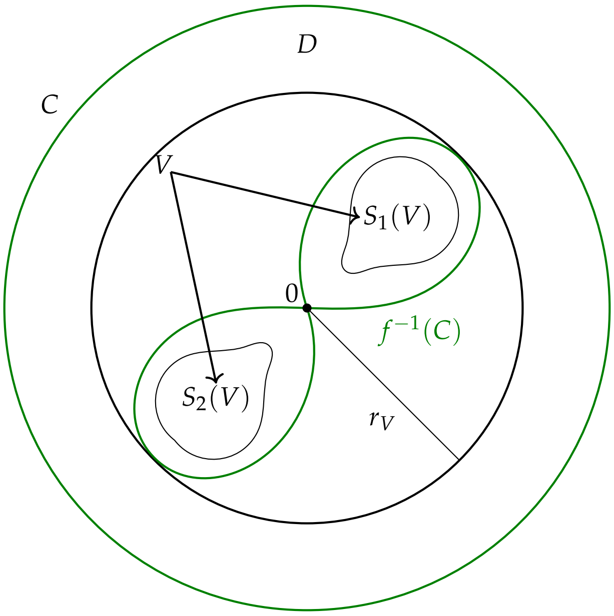


Figure 9: Iteration of the circle C under f_c^{-1}

Now the $S_i(\overline{V})$ are disjoint, we have for both S_i and for the principal valued square root branch

$$\begin{aligned}|S_i(z_1) - S_i(z_2)| &= |(z_1 - c)^{1/2} - (z_2 - c)^{1/2}| \\ &= \frac{|(z_1 - c)^{1/2} + (z_2 - c)^{1/2}|}{|(z_1 - c)^{1/2} + (z_2 - c)^{1/2}|}|(z_1 - c)^{1/2} - (z_2 - c)^{1/2}| \\ &= \frac{|z_1 - z_2|}{|(z_1 - c)^{1/2} + (z_2 - c)^{1/2}|}.\end{aligned}$$

The scaling factor of the maps S_i are

$$\frac{|S_i(z_1) - S_i(z_2)|}{|z_1 - z_2|} = \frac{1}{|(z_1 - c)^{1/2} + (z_2 - c)^{1/2}|}.$$

Maximising this over the condition that $z_1, z_2 \in \overline{V}$ can be found by minimising the denominator. While the minimum is found when z_1 and z_2 are as close to c as possible, (see Figure 10). Hence, the maximum will be $1/2\sqrt{|c| - |2c|^{1/2}}$. In the same manner, the maximum for the denominator is the furthest away z_i can be placed from c, i.e the minimum will be $1/2\sqrt{|c| + |2c|^{1/2}}$.

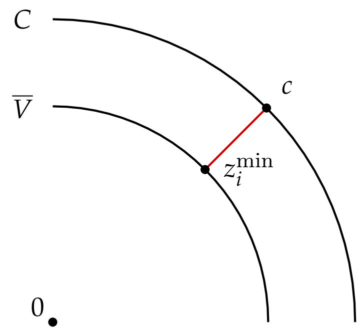


Figure 10: Maximising the scaling factor for S_i, the red line has magnitude $2\sqrt{|c| - |2c|^{1/2}}$.

$$\frac{1}{2\sqrt{|c| + |2c|^{1/2}}} \leq \frac{|S_i(z_1) - S_i(z_2)|}{|z_1 - z_2|} \leq \frac{1}{2\sqrt{|c| - |2c|^{1/2}}}.$$

The IFS is a contraction if the upper bound less than 1,

$$|c| > \frac{1}{4}(5 + 2\sqrt{6}) \implies \frac{1}{4(|c| - |2c|^{1/2})} < 1 \implies \frac{1}{2\sqrt{|c| - |2c|^{1/2}}} < 1$$

Hence by Theorem 3.2, there exist some non-empty, compact, totally disconnected attractor set F, which is invariant under S_i,

$$F = S_1(F) \cup S_2(F).$$

Furthermore, this attractor set is the Julia set, since $\overline{V}$ contains some repelling periodic point $z \in \overline{V}$ of f_c. By the equivalence Theorem 4.7 of Julia sets we have, $\mathcal{J}(f_c) = \overline{\bigcup_{k=1}^{\infty} f_c^{-k}(z)} \subset \overline{V}$. Since the Julia set is non-empty compact invariant subset of $\overline{V}$, then $\mathcal{J}(f_c) = F$, since F was unique. Now, a simple calculation to estimate the Hausdorff dimension, relying on Theorem 3.5 and Theorem 3.6, is

$$\sum_{i=1}^{m} c_i^s = 1 \implies 2\left(\frac{1}{2\sqrt{|c| - |2c|^{1/2}}}\right)^s = 1 \implies s = \frac{2\log 2}{\log 4(|c| + |2c|^{1/2})}.$$

The lower bound can be found in the similar way and we get in summary,

$$\frac{2\log 2}{\log 4(|c| + |2c|^{1/2})} \leq \dim_H(F) \leq \frac{2\log 2}{\log 4(|c| - |2c|^{1/2})}.$$

Which was to be proven. □

The following corollary follows easily from the previous Theorem 6.2.

Corollary 6.3.

$$|c| \to \infty \implies \dim_H \mathcal{J}(f_c) \to 0$$

For small c, we have the following Theorem proved by David Ruelle.

Theorem 6.4 (Ruelle (1982))**.** *The Hausdorff dimension of the Julia set for f_c where c is in the main cardioid (the big bulb containing $1/4$) of the Mandelbrot set is*

$$\dim_H(\mathcal{J}(f_c)) = 1 + \frac{|c|^2}{4\log 2} + O(|c|^3).$$

In fact Ruelle proves a more general statement. Consider maps $f_{p,q} : z \to z^q - p$, then

$$\dim_H(\mathcal{J}(f_{c,q})) = 1 + \frac{|p|^2}{4\log(q)} + O(p^3)$$

A parameter point of specific interest is $c = -2$, since it's a Misiurewicz's parameter that is a the critical point is strictly preperiodic (that is preperiodic but not periodic), in this case $0 \mapsto -2 \mapsto 2$, with 2 a fixed point. Further by the following proposition the corresponding Julia set is the interval $[-2, 2]$.

Proposition 6.5. *When $c = -2$ the Julia set is the compact interval $\mathcal{J}_{-2} = [-2, 2]$.*

Proof. In Section 4.2 we showed that two conformally conjugate functions will provide the same Julia set. Consider the conformal map, $h : \{\xi \in \mathbb{C} : |\xi| > 1\} \to \mathbb{C} \setminus [-2, 2]$ that maps according to $h(\xi) = \xi + 1/\xi$. Then

$$f_{-2}(h(\xi)) = (\xi + \frac{1}{\xi})^2 - 2 = \xi^2 + \frac{1}{\xi^2} = h(\xi^2) \implies h^{-1} \circ f_{-2} \circ h = \xi^2.$$

This tells us that all the points outside of $[-2, 2]$ are in the attracting basin of ∞ for the map f_{-2}, since they behave like z^2. Furthermore, the interval $[-2, 2]$ is invariant under f_{-2}. By the equivalence of Julia sets, the proposition is proven. □

Proposition 6.6. *[Dobbs et al. (2022), Proposition 1.1]**(Discontinuity of $\dim_H(\mathcal{J}_c)$ at -2)** The dimension function $c \in \mathcal{M} \cap \mathbb{R} \mapsto \dim_H(\mathcal{J}_c)$ is discontinuous at -2 and moreover*

$$\limsup_{c \to -2} \dim_H(\mathcal{J}_c) = \sup_{c \in \mathcal{M} \cap \mathbb{R}} \dim_H(\mathcal{J}_c) > 1 = \liminf_{c \to -2^+} \dim_H(\mathcal{J}_c)$$

The remaining of this thesis will concentrate on estimating Hausdorff dimension for Julia sets when it perturbed around -2. Specifically we will consider negative real perturbations and imaginary perturbations.

Lemma 6.7. [Rudin (1987)] *If $(v_1, \ldots, v_N)$ are complex numbers, and if*

$$P_N = \prod_{k=1}^{N}(1 + v_k) \qquad P_N^* = \prod_{k=1}^{N}(1 + |v_k|)$$

then

$$P_N^* \leq e^{\sum_{k=1}^{N} |v_k|}$$

and

$$|P_N - 1| \leq P_N^* - 1.$$

Proof. By the power series for e^x we get the inequality $1 + x \leq e^x$, hence

$$1 + |u_k| \leq e^{u_k} \quad \forall k \implies \prod_{n=1}^{N}(1 + |u_n|) \leq \prod_{n=1}^{N} e^{|u_n|} = e^{\sum_{n=1}^{N} |u_n|}.$$

This gives the first statement. For the second statement, we proceed by induction: for $N = 1$ is trivial and for the general case

$$P_{k+1} - 1 = P_k(1 + u_{k+1}) - 1 = (p_k - 1)(1 + u_{k+1}) + u_{k+1},$$

hence if the inequality holds for N, then also

$$|p_{k+1} - 1| \leq (p_k^* - 1)(1 + |u_{k+1}|) + |u_{k+1}| = p_{k+1}^* - 1.$$

□

The following Theorem is proved in [Fan et al. (2005)] by studying the geometry of Cantor systems, and ultimately, they bound the $\mathcal{GAP}_\epsilon$ geometry. For a given Cantor system $\mathcal{G} = \{\mathcal{I}, \mathcal{G}\}$ the $\mathcal{GAP}$ geometry is defined to be the set of ratios of the diameter of the gaps over diameters of all the bridges.

$$\mathcal{GAP} := \bigcup_{n=0}^{\infty} \left\{ \frac{|G|}{|J|} : I = L \cup G \cup R \in \mathcal{I}_n \text{ with } J = L \text{ or } R \in \mathcal{I}_{n+1} \text{ and } G \in \mathcal{G}_{n+1} \right\}$$

Using techniques from real analysis [Jiang (1999)]. The proof in this paper relies on the outline presented in [Dobbs et al. (2022)]. To the best of my knowledge, the initial conjecture originated from the doctoral dissertation of Y. Jiang.

Theorem 6.8. ***(Real valued perturbation)***
Let $f_\epsilon = z^2 - 2 - \epsilon$, with sufficiently small $\epsilon > 0$, such that $c \in (-3, -2)$. There exist some universal C' such that

$$\dim_H(\mathcal{J}_\epsilon) \leq 1 - C'\sqrt{\epsilon}.$$

Proof. Let p_ϵ be the repelling fixed point of f_ϵ for which the orientation of $\mathbb{R}$ is preserved. Solving $f_\epsilon(x) = x$ conditioned on $|Df_\epsilon(x)| > 1$ preserving the orientation gives the following

$$f_\epsilon(x) = x \implies x^2 - 2 - \epsilon = x \implies x = \frac{1}{2}(1 \pm \sqrt{9 + 4\epsilon})$$

Checking the magnitude of the derivative for respective fixed points gives that only the positive root is repelling and orientation-preserving for all ϵ. Hence, $p_\epsilon = \frac{1}{2}(1 + \sqrt{9 + 4\epsilon})$, denoted by $\pm y$ the preimage of $-p_\epsilon$, that is $f_\epsilon^{-1}(-p_\epsilon) = \{-y, y\}$. To get an estimate for $\sqrt{y}$, we calculate the distance $|c - p_\epsilon|$. Taylor expands the pre-image gives $|y| \sim \sqrt{\epsilon}$,

$$|c - p_\epsilon| = |-2 - \epsilon - \frac{1 + \sqrt{9 + 4\epsilon}}{2}| = |-2 - \epsilon - \frac{1 + 3 + \frac{2\epsilon}{3} + O(\epsilon^2)}{2}| = |\frac{1}{3}\epsilon + O(\epsilon^2)|$$

Taking the pre-image of this will yield the desired estimation - that is, $|y| \sim \sqrt{\epsilon}$, denoted by $I = [-p_\epsilon, p_\epsilon]$. Since we have an quadratic polynomial which is expanding and if we assume $|z| > |p_\epsilon|$, then the future orbit of z will be strictly bounded away from p_ϵ, $|z^2 - 2 - \epsilon| \geq |p_\epsilon^2| - |-2 - \epsilon| = p_\epsilon$ and diverge - hence not in the Julia set. In this case, the Julia set is a cantor set. To see this, consider the one iteration of the origin $f_\epsilon(0) = -2-\epsilon < -p_\epsilon$ combined with the fact that f_ϵ is a continuous map. Thus the open set $(-y, y)$ will also be mapped outside of I. Now, all of the pre-images of the interval $(-y, y)$ will not be in the Julia set. Hence the Julia set is

$$\mathcal{J}_\epsilon = \bigcap_{n\geq 0} f_\epsilon^{-n}(I).$$

(See figure 11 of a virtual representation for I mapped under f_ϵ). Informally, one can view this process as repeating folding I in the middle and placing 0 at c and cutting out the excess- the remaining will be the Cantor set. Further iterations of f_ϵ will be a process of folding.

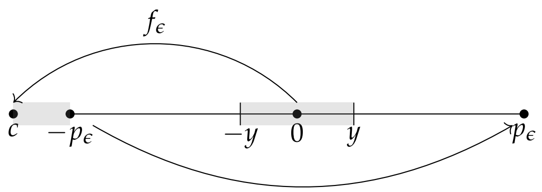


Figure 11: Iteration of f_ϵ^{-1} on the interval I

Define $\mathcal{I}_{n,\epsilon}$ to consist of all intervals in (i.e. connected components of) $f_\epsilon^{-n}(I)$. And let $A \in \mathcal{I}_{n,\epsilon}$, Since $A \in \mathcal{I}_{n,\epsilon}$, there exists at most two iterates $k(0 \leq k \leq n-1)$ for which $f_\epsilon^k(\partial A) \cap \{\pm y\} \neq \emptyset$, one of which equals $n-1$. Denote by k_0 the other one. We can thus decompose f_ϵ^n in into

$$f_\epsilon^n = f_\epsilon \circ g_1 \circ f_\epsilon \circ g_2,$$

where $g_2 = f_\epsilon^{k_0}$ and $g_1 = f_\epsilon^m$ such that $k_0 + m + 2 = n$. If k_0 does not exist, then $f_\epsilon^n = f_\epsilon \circ g_1$, with $g_1 = f_\epsilon^{n-1}$. This is the case if, and only if, $\partial A \cap \{\pm y\} \neq \emptyset$. Now, we need to find upper bounds on the distortion for the maps.

The proof that g_2 has bounded distortion is a consequence from Koebe distortion Theorem 5.12. Since first k_0 iterations of A are separated from $\{-y, y\}$, one can find some $B \in \mathcal{I}_{k_0,\epsilon}$ such that $A \subset B$ and B is mapped diffeomorphically onto I, and $g_2(A)$ is far from ∂I. Hence, by Koebe, the distortion is bounded (The notation $DF = \frac{dF}{dz}$ is just the normal derivative),

$$\left|\frac{Dg_2(z)}{Dg_2(w)}\right| \leq C_{\hat{r}} \qquad \text{for all } \; z, w \in A.$$

To show that g_1 has bounded distortion we use Lemma 6.7

$$|\prod_{k=1}^{N}(1+v_k) - 1| = |P_N - 1| \leq P_N^* - 1 \leq e^{\sum_{k=1}^N |v_k|} - 1$$

that is

$$|\prod_{k=1}^{N}(1+v_k) - 1| \leq e^{\sum_{k=1}^N |v_k|} - 1.$$

With the substitution $v_k = u_k - 1$, we get

$$|\prod_{k=1}^{N}(u_k) - 1| \leq e^{\sum_{k=1}^N |u_k - 1|} - 1.$$

Now, we calculate the distortion for g_1. Let $A \in \mathcal{I}_{n-k_0-1,\epsilon}$ that contains p_ϵ and let $z, w \in A$. We need the following quantity to be small.

$$\left|\frac{Df_\epsilon^m(z)}{Df_\epsilon^m(w)} - 1\right| = \left|\prod_{k=0}^{m-1} \frac{Df_\epsilon(z_k)}{Df_\epsilon(w_k)} - 1\right| \qquad z, w \in A$$

With $z_k = f^k(z)$, $w_k = f^k(w)$, that is the derivative along the orbits of z and w respectively. Let $u_k = \frac{Df_\epsilon(z_k)}{Df_\epsilon(w_k)}$, using Lemma 6.7.

$$\left|\prod_{k=0}^{m-1} \frac{Df_\epsilon(z_k)}{Df_\epsilon(w_k)} - 1\right| \le e^{\sum_{k=1}^{m-1}|u_k - 1|} - 1.$$

Hence $e^{\sum_{k=1}^{m-1}|u_k-1|}$ needs to be bounded that is equivalent to bound $\sum_{k=1}^{m-1}|u_k - 1|$.

$$\sum_{k=1}^{m-1}|u_k - 1| = \sum_{k=1}^{m-1}\left|\frac{Df_\epsilon(z_k)}{Df_\epsilon(w_k)} - 1\right| = \sum_{k=1}^{m-1}\left|\frac{Df_\epsilon(z_k) - Df_\epsilon(w_k)}{Df_\epsilon(w_k)}\right| = \sum_{k=1}^{m-1}\left|\frac{z_k - w_k}{w_k}\right|.$$

Since p_ϵ is a fixed point, $z_k = f_\epsilon^k(z_0) = p_\epsilon$. Furthermore $w_k = f_\epsilon^k(w_0)$ which is bounded by 1. Combined with that, w_k is expanding further away from p_ϵ i.e. $|p_\epsilon - w_k| \le \lambda^{-1}|p_\epsilon - w_{k+1}|$ for some $\lambda \in (0,1)$. We have the following sequence of inequalities

$$\sum_{k=1}^{m-1}\left|\frac{z_k - w_k}{w_k}\right| \le \sum_{k=1}^{m-1}|z_k - w_k| \le \sum_{k=1}^{m-1}|z_k - w_{m-1}|\lambda^{-k} \le \frac{|p_\epsilon - w_{n-1}|}{1-\lambda} \le \frac{1}{1-\lambda} < \infty.$$

The previous proves that g_1 and g_2 have bounded distortions. All that is left is to estimate the length of $f^{-n}(I)$ via the distortions provided by each iteration. Consider the sets

$$E_j = \{x \in f_\epsilon^j(A) : f_\epsilon^{n-j}(x) \in [-y, y]\} \qquad for \qquad j \le n.$$

Now, both $E_n = [-y, y]$ and E_{n-1} will have measure $\sim \sqrt{\epsilon}$. By bounded distortion of g_1, we have that $|E_{k_0+1}|/|f_\epsilon^{k_0+1}(A)| \sim \sqrt{\epsilon}$. Pulling back once more gives $|E_{k_0}|/|f_\epsilon^{k_0}(A)| \sim \sqrt{\epsilon}$. Finally by bounded distortion of g_2, $|E_0|/|A| \sim \sqrt{\epsilon}$. One can view this quantity as the probability that a certain point inside of A diverges, since E_0 contains all the points inside of A that in n-th iteration which will be mapped into $[-y, y]$. Hence, $|f_\epsilon^{-n}(I)| \le (1 - C\sqrt{\epsilon})^n$ where C is some uniform constant. We have that $f_\epsilon^{-n}(I)$ contains 2^n connected components like A. By Hölder's inequality $p = \frac{1}{1-\alpha}$ and $q = \frac{1}{\alpha}$,

$$\sum |A|^\alpha \le 2^{(1-\alpha)n}(1 - C\sqrt{\epsilon})^{\alpha n} \le 1.$$

For when $\alpha \ge 1 - C'\sqrt{\epsilon}$. Hence the Hausdorff measure of $f_\epsilon^{-n}(I)$ is bounded above for $\alpha \ge 1 - C'\sqrt{\epsilon}$. This means the Hausdorff dimension is less than α, proving $\dim_H(\mathcal{J}_c) \le \alpha$. Another justification for the totally disconnectedness of $\mathcal{J}_c$ is that the Hausdorff dimension is less than 1 and Proposition 2.11. □

A. Fan, Y. Jiang, and J. Wu gave also in [Fan et al. (2005)] a lower bound.

Theorem 6.9. *[Fan et al. (2005)] There exists constants $K > 0$ and $\epsilon_1 > 0$ with $c = -2 - \epsilon_1$, such that*

$$1 - K^{-1}\sqrt{|\delta|} \le \dim_H(\mathcal{J}_c) \le 1 - K\sqrt{|\delta|}.$$

The case with perturbation from inside the Mandelbrot set to -2 is proved in [Dobbs et al. (2022)], that is letting $\epsilon < 0$ in Thorem 6.8. Now we consider the purely imaginary valued perturbation. Below is a figure demonstrating how the Julia set for such parameter looks like.

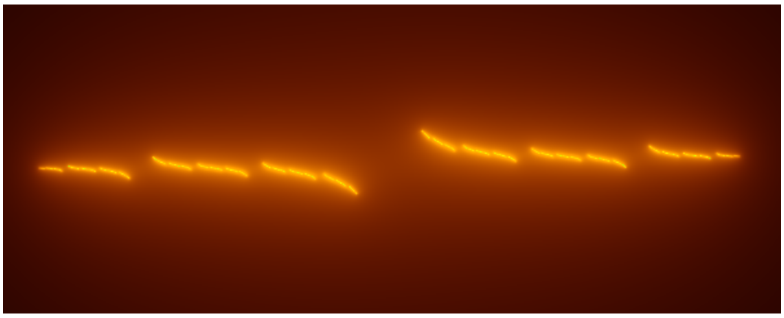


Figure 12: Julia set for $c = -2 + 0.1i$

Theorem 6.10. ***(Imaginary valued perturbation)*** *Let $f_\epsilon = z^2 - 2 + i\epsilon$, with sufficiently small $|\epsilon| > 0$. There exist some universal C' such that*

$$\dim_H(\mathcal{J}_\epsilon) = \dim_H(\mathcal{J}(f_\epsilon)) \leq 1 - C'\sqrt{\epsilon}.$$

Proof. The proof of this Theorem will follow the same structure as in the case with real valued perturbation, Theorem 6.8. Let p_ϵ be the repelling fixed point for f_ϵ with largest modulus, and denote $\{y, -y\} = f_\epsilon^{-1}(-p_\epsilon)$. Following the same calculations as in Theorem 6.8 gives $|y| \sim \sqrt{\epsilon}$. For $c = -2 + i\epsilon \notin \mathcal{M}$, the Julia set is totally disconnected, and the origin diverges, $f_\epsilon^k(0) \to \infty$. Since f_ϵ is continuous, the open ball $B_1 = \mathbb{B}(0, |y|)$ will be mapped by the function f_ϵ onto an open ball $B_2 = \mathbb{B}(c, r')$ centred at c. Now, let $\mathbb{D} = \mathbb{B}(0, |c|)$. The Julia set can be written as

$$\mathcal{J}_\epsilon = \bigcap_{n \geq 0} f_\epsilon^{-n}(\mathbb{D}).$$

The underlying iterated function system will look like the figure below, which follows from Lemma 4.23

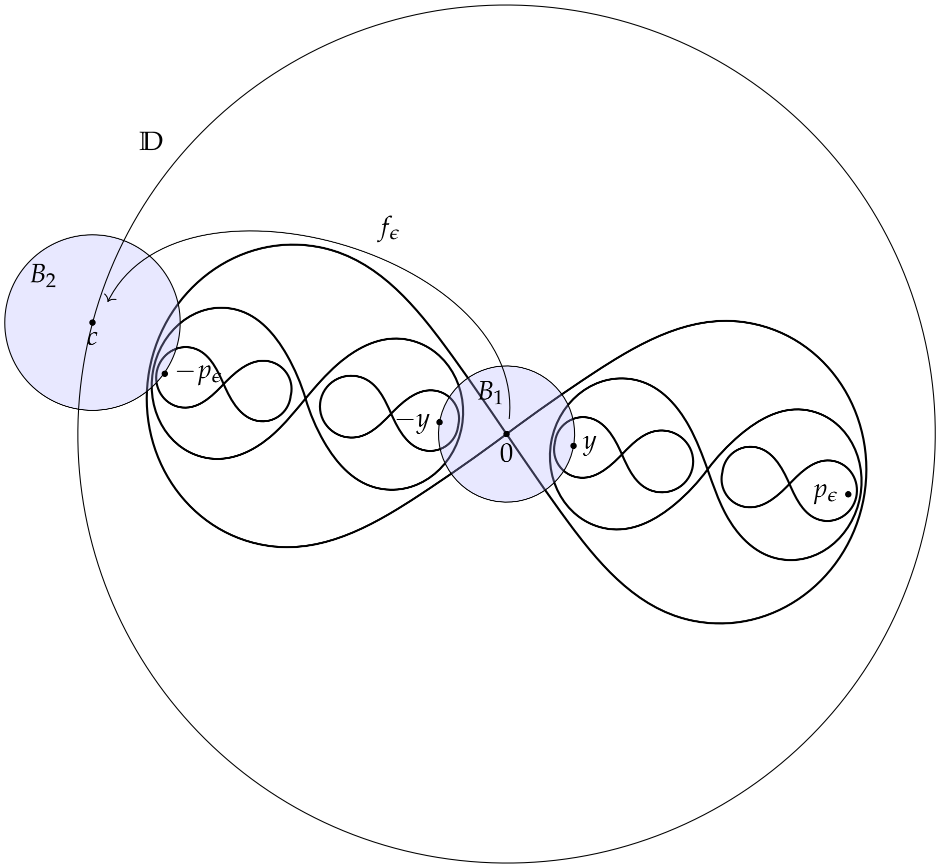


Figure 13: Iterations of f_ϵ^{-1} on $\mathbb{D}$.

Note that the figure will continue in the same manner in each loop. We will get "infinity" curves all the way down. Let $\mathcal{I}_{n,\epsilon}$ consist of all the connected components of $f_\epsilon^{-n}(\mathbb{D})$. Now let $A \in \mathcal{I}_{n,\epsilon}$, and define A_L and A_R to be the "left" respectively "right" connected components, (see Figure 13). A will be an "infinity" sign at the nth level of iteration. In the same way as before, we decompose f_ϵ^n in four parts.

$$f_\epsilon^n = f_\epsilon \circ g_1 \circ f_\epsilon \circ g_2.$$

(or $f^n = f \circ g_1$ if k_0 does not exist) Where $g_2 = f_\epsilon^{k_0}$ with k_0 as the first instance for which we return to $\{-y, y\}$, that is $f_\epsilon^{k_0}(A) \cap \{-y, y\} \neq \emptyset$. Additionally, $g_1 = f_\epsilon^m$ such that $k_0 + m + 2 = n$. Note that after g_2, then $p_\epsilon \in f_\epsilon^k \circ f_\epsilon \circ g_2(A)$ for all $k \leq n - k_0 - 1$. Since first k_0 iterations of A are separated from $\{-y, y\}$ one can find some $B \in \mathcal{I}_{k_0}$ such that $A \subset B$ and B is mapped diffeomorphically onto $\mathbb{D}$. By Theorem 5.11 there exists a conformal mapping ϕ that maps A and B onto two concentric disks with radius r and 1. We study the map $\tilde{g}_2 = g_2 \circ \phi$, since $\tilde{g}_2 \in \mathcal{S}$ from Theorem 5.13, and $g_2 = \tilde{g}_2 \circ \phi^{-1}$. By Theorem 5.7 we get that

$$|D_{g_2}| = |D_{\tilde{g}_2 \circ \phi^{-1}}| \leq D_{\tilde{g}_2} \cdot D_{\phi^{-1}}.$$

It now follows that the distortion is bounded since ϕ is conformal (the inverse also has bounded distortion) and applying the distortion Theorem 5.13 on $\tilde{g}_2$

$$D_{\tilde{g}_2} = \left|\frac{\tilde{g}_2'(z)}{\tilde{g}_2'(w)}\right| \underbrace{\leq}_{5.13} \left|\frac{\frac{1+|z|}{(1-|z|)^3}}{\frac{1-|w|}{(1+|w|)^3}}\right| = \left|\frac{(1+|z|)(1+|w|)^3}{(1-|z|)^3(1-|w|)}\right| \leq \frac{2^4}{(1-r)^4} = K(r).$$

We decompose $g_2(A)$ into the right and left connected components $g_2(A)_R$ and $g_2(A)_L$. We bound the module from below for both loops separately. Let Γ_R be the set of locally rectifiable curves joining $\partial g_2(A)_R$ and $\partial\mathbb{D}$. Note that $g_2(A)_R$ is contained in the disk with radius $\sqrt{2}$, since otherwise $p_\epsilon \in g_2(A)$ which contradicts minimality of k_0. Hence denote by Γ_1 the curves joining $\partial g_2(A)_R$ and $\mathbb{B}(0,\sqrt{2})$ and Γ_2 joining $\mathbb{B}(0,\sqrt{2})$ with $\mathbb{D}$. Bounding Modulus of Γ gives,

$$M_2(\Gamma_R) \underbrace{\geq}_{\text{Monotonicity}} M_2(\Gamma_1 \cup \Gamma_2) \underbrace{\geq}_{\text{Grötzch ineq } 5.8} M_2(\Gamma_1) + M_2(\Gamma_2) \geq M_2(\Gamma_2) \underbrace{=}_{Ex:5.10} \frac{2\pi}{\log(\frac{|c|}{\sqrt{2}})} > 0.$$

The same inequality holds for Γ_L. The combined family of curves will also be bounded from below. Since the modulus is bounded from below it follows that $r < 1$ and $K(r) < \infty$. Now we bound the distortion of g_1 in the same manner as in previous Theorem. First define $\gamma_n \in \mathcal{I}_{n,\epsilon}$ such that $p_\epsilon \in \gamma_n$ that is the furthest loop to the right, $n = 0, 1, 2 \ldots n$. This family of sets are chained under iterations of f_ϵ, that is $f_\epsilon(\gamma_n) = \gamma_{n-1}$ diffeomorphically. In our case we have $p_\epsilon \in f \circ g_2(A)$ hence $f \circ g_2(A) = \gamma_{n-k_0-1}$ which means that further iteration will send γ_{n-k_0-1} to γ_{n-k_0-2} until we reach γ_1,

$$\gamma_{n-k_0-1} \xrightarrow{f_\epsilon} \gamma_{n-k_0-2} \xrightarrow{f_\epsilon} \ldots \xrightarrow{f_\epsilon} \gamma_2 \xrightarrow{f_\epsilon} \gamma_1.$$

Figure 14: Mapping γ_{n-k_0-1} under $g_2 = f_\epsilon^m$.

Following the same calculations as before we get

$$\left|\frac{Df_\epsilon^m(z)}{Df_\epsilon^m(w)} - 1\right| \leq e^{\sum_{k=1}^{m-1}|u_k - 1|} - 1.$$

Again $u_k = \frac{Df_\epsilon(z_k)}{Df_\epsilon(w_k)}$,

$$\sum_{k=1}^{m-1} |u_k - 1| = \sum_{k=1}^{m-1} \left|\frac{z_k - w_k}{w_k}\right| \leq \sum_{k=1}^{m-1} |z_k - y|\lambda^{-k} \leq \frac{|p_\epsilon - y|}{1-\lambda} \leq \frac{1}{1-\lambda} < \infty.$$

The remaining calculation is the same as in Theorem 6.8, giving the desired asymptotic result. □

I believe that this proof can be used to prove the same rate of convergences for more general curves approaching -2, as long as the sequence of complex numbers is outside the closed ball of radius 2. The conjectured Theorem will look similarly as Theorem 6.11, with the stated rate of convergences.

We complete this section with stating three further results concerning the behaviour of $\dim(\mathcal{J}(f))$ for more general approach to -2. First there is Rivera-Letelier's article in which it he proves that for suitably "good" approach to a parameter c_0 the Hausdorff dimension converges.

Theorem 6.11. *[Rivera-Letelier (2001)]*
Let $c_0 \in \partial\mathcal{M}_d := \partial\{c \in \mathbb{C} | \mathcal{J}_c^d = \mathcal{J}(z^d + c)$ is connected$\}$ be such that $f_{c_0}(z) = z^d + c_0$ is semi hyperbolic (the critical point 0 is not recurrent and $0 \in \mathcal{J}_c^d$). Then there is some $C > 0$ such that if a sequence $c_n \to c_0$ is such that

$$d(c_n, \mathcal{M}_d) \geq C|c_n - c_0|^{1+1/d} \implies \dim_H(\mathcal{J}_{c_n}) \to \dim_H(\mathcal{J}_{c_0})$$

Further there are new results concerning the directional derivative for the Hausdorff dimension, proved in [Jaksztas (2023)], the main is the following.

Theorem 6.12. *[Jaksztas (2023)]*
For $\alpha \in (0, \pi]$, we write

$$\Omega_{-2}(\alpha) := \frac{1}{\sqrt{6}\log 2}\left(\cos\alpha - \frac{1}{2}\sqrt{\sin\alpha}\int_\alpha^\pi \sqrt{\sin x}dx\right).$$

If $\alpha \in (\pi, 2\pi)$ we define $\Omega_{-2} := \Omega_{-2}(2\pi - \alpha)$. Let $f_\epsilon(z) = z^2 - 2 + \epsilon$, $\epsilon \in \mathbb{C}$. For every $\alpha \in (0, 2\pi)$ we have

$$\lim_{|\epsilon| \to 0} \sqrt{|\epsilon|} \cdot \dim_v'(\mathcal{J}(f_\epsilon)) = \Omega_{-2}(\alpha)$$

with $\alpha = \arg\delta$ and $v = e^{i\alpha}$.

One can with this formula plug in different α and integrate to get formulas for the asymptotic behaviour for example look at the Corollary 1.2 in [Jaksztas (2023)] for the case of small $\epsilon \in \mathbb{R}$. Lastly we state a famous Theorem which calculates the Hausdorff dimension for a dense subset of $\partial\mathcal{M}$.

Definition 6.13. ***(Nowhere dense)*** *A set is said to be nowhere dense if the interior of the set closure of X is the empty set.*

For example the Cantor set is nowhere dense.

Definition 6.14. ***(Residual set)*** *In a complete metric space, a countable union of nowhere dense sets is said to be **meager**; the complement of such a set is a **residual set**.*

Theorem 6.15. *[Shishikura (1998)]*
There exist a residual (hence dense) subset $\mathcal{R}$ of $\partial\mathcal{M}$, such that if $c \in \mathcal{R}$ then

$$\dim_H \mathcal{J}_c = 2.$$

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