# Hall Algebras

Sebastian Westerlund

Bachelor's thesis
2024:K12

Lund University

Faculty of Science
Centre for Mathematical Sciences
Mathematics

CENTRUM SCIENTIARUM MATHEMATICARUM

# Hall Algebras

by Sebastian Westerlund

24th June 2024

Lund University
Faculty of Science
Department of Mathematics

Bachelor's Thesis in Mathematics
Thesis advisor: Gustavo Jasso

**Abstract**

This thesis uses the strategy of Dyckerhoff-Kapranov to show the associativity and unitality of Hall Algebras for proto-abelian categories. After a more classical introduction to Hall algebras, we show that the Waldhausen S-construction of a proto-abelian category is a 2-Segal simplicial groupoid, and use this to construct a monoid object in the category of spans in groupoids. We then use this monoid object to recover the classical Hall Algebra. The thesis also includes an interlude on the category $\mathbf{Vect}_{\mathbb{F}_1}$ of vector spaces over the field with one element. In particular, we compute the Hall algebra of this category.

# Popular science description

Matematiker undersöker ofta talsystem som är annorlunda från de vanliga talen vi är vana vid. Ett exempel på ett sådant talsystem är timmarna på en klocka, där $13 = 1$ och $9 + 6 = 3$. Ett mindre självklart exempel är sekvenser av drag på en Rubik's kub. Man kan addera ihop två dragsekvenser genom att göra dem efter varandra, och det går att göra algebraiska manipulationer med dessa drag precis som med vanliga tal. Det matematiska området som kallas abstrakt algebra handlar om att inte anta något om sitt talsystem utom vilka algebraiska regler som gäller. På så sätt kan man bevisa påståenden som inte bara gäller för tal, utan även för klockor och för Rubik's kub. Det finns dock flera olika sätt att göra detta på: Har vi bara addition, eller multiplikation också? Ska vi kräva att $a + b = b + a$? (Detta är inte alltid sant för en Rubik's kub). Beroende på vilka val som görs så får vi olika sorters så kallade *algebraiska strukturer*.

Om man väljer en sorts algebraisk struktur och tittar på alla möjliga talsystem av den sorten så bildar de vad som kallas för en *kategori*. Kategorier är väldigt generella objekt som dyker upp i många matematiska områden, inte bara algebra. Kategorier studeras i det matematiska ämnet *kategoriteori*.

Matematikern Philip Hall undersökte 1959 kategorin av en speciell typ av algebraisk struktur, och upptäckte han att man kunde göra ett talsystem av kategorin själv. Han listade alltså ut ett sätt att addera och multiplicera ihop talsystem på ett sätt som uppfyller de vanliga algebraiska räknereglerna. Hall var inte den första att göra denna upptäckt, men detta talsystem fick namn efter honom och kallades för *Hall-Algebran*. En naturlig fråga är vad som var så speciellt med just den kategorin som Hall använde. Fungerar konstruktionen även i andra fall? Mycket riktigt lyckades andra matematiker göra liknande konstruktioner av andra kategorier. På 2010-talet har matematikerna Tobias Dyckerhoff och Mikhail Kapranov systematiserat förståelsen av dessa olika sorters Hall-Algebror genom att använda kategoriteoretiska verktyg. Den här uppsatsen följer en text skriven av Dyckerhoff och använder deras strategi för att bevisa att Halls konstruktion fungerar för en relativt stor generell klass av kategorier. Den innehåller även en explicit beräkning av en Hall-algebra annan än den Hall själv undersökte.

## Acknowledgements

I would like to give many thanks to my incredible advisor Gustavo Jasso, who has given me invaluable support, help, and advice with regards to writing this thesis and other aspects of my studies. He has had an enormously positive impact on me as a student, and I am very grateful for the opportunity to work with him.

I would also like to thank my friends Bror Persson and Arthur Faxén for discussing the subject matter with me, it has been great to work on my thesis in parallel with you. Lastly, I want to thank Filip Bar for giving his time to be my examiner.

## Disclaimer

The author does not claim to be the first to prove or discover any result in this thesis, and has made his best effort to cite all used sources appropriately. When a result is unlabelled, this should be taken as the result either being judged well-known or simple enough that a citation was not needed, or that the statement is formulated by the author, but that he expects it to be considered well-known among the subject matter experts.

# Contents

# Introduction

Hall Algebras are named after Philip Hall [Hal59], who discovered that the combinatorics of flags of finite abelian $p$-groups gave rise to a unital and associative algebra. This fact was actually already discussed more than 50 years earlier by Steinitz [Ste01], but the construction was eventually named after Hall.

Explicitly, fix a prime number $p$ and let $\mathbf{FinAb}(p)$ be the category of finite abelian $p$-groups. Consider the abelian group

$$\bigoplus_{[G]\in\pi_0(\mathbf{FinAb}(p))} \mathbb{Z}[G]$$

of formal $\mathbb{Z}$-linear combinations of isomorphism classes of finite abelian $p$-groups. Hall noticed that this can be turned into a ring if we define the multiplication

$$[A][B] = \sum_{C\in\pi_0(\mathbf{FinAb}(p))} g^C_{A,B}[C].$$

and extend it bilinearly. Here $g^C_{A,B}$ denotes the number of subgroups $M \subseteq C$ such that $M \cong B$ and $C/M \cong A$. This ring is called the *Hall algebra of partitions*, or the *Hall algebra of finite abelian p-groups*.

Once the associativity and unitality of this algebra has been established, it is natural to wonder what about the category $\mathbf{FinAb}(p)$ allowed this construction to work. Indeed, Ringel [Rin90] managed to generalize the construction to abelian categories satisfying certain finiteness conditions, called *finitary* abelian categories. Ringel's contributions to the area are significant, to the point that Hall Algebras for categories other than $\mathbf{FinAb}(p)$ are sometimes called *Ringel-Hall algebras*. This thesis is based on the work of Dyckerhoff and Kapranov, who argue that *"Hall algebras, as previously studied, are only the shadow of a much richer structure governed by a system of higher coherences captured in the datum of a 2-Segal space."* ([DK12], abstract). In particular, it closely follows the exposition of [Dyc18], whose proof strategy we follow to show that associativity and unitality for the Hall algebra of so-called *finitary proto-abelian* categories, which is a condition strictly weaker than that of finitary abelian categories.

The structure of the thesis is as follows:

- Chapter 1 is mostly concerned with the classical view of Hall algebras, before introducing the machinery required to view the situation from Dyckerhoff-Kapranov's more abstract perspective
  - In Section 1.1, we prove that the classical Hall algebra of partitions, introduced above, is associative and unital.
  - In Section 1.2, we define the notion of a *proto-abelian* category. We prove a sufficient condition for a category to be proto-abelian, which is oftentimes easier to prove in practice. We also define the Hall Algebra of a general proto-abelian category.
  - In Section 1.3, we investigate the category $\mathbf{Vect}_{\mathbb{F}_1}$ - a specific example of a proto-abelian category which is not abelian - and compute its Hall algebra.
  - In Section 1.4, we introduce the notion of a *wing*, which is a type of diagram which will become very important in the second chapter
- Chapter 2 starts with the introduction of the more abstract notions requires to understand Dyckerhoff-Kapranov's approach to Hall Algebras.
  - In Section 2.1, we introduce the notion of a simplicial object, which is a fundamental building object of $\infty$-category theory. We show that the collection of wing form a simplicial groupoid $\mathcal{S}_\bullet$, which also appears in algebraic $K$-theory under the name of the *Waldhausen S-construction*.
  - In Section 2.2, we introduce the notions of *2-Pullback* and *isofibration*, which become important later in the chapter.
  - In Section 2.3, we introduce the *2-Segal* conditions and show that the simplicial groupoid $\mathcal{S}_\bullet$ satisfies them.
  - In Section 2.4, we introduce the so called *Abstract Hall algebra*, as an algebra object in the monoidal category of spans in groupoids. We show it is associative and unital.
  - In Section 2.5, we construct a functor which, upon application to the Abstract Hall Algebra, yields the Hall Algebra of a proto-abelian category.

# Notation and Conventions

- Rings are taken to be untial and ring homomorphisms preserve units.
- The natural numbers $\mathbb{N}$ include 0.
- We do not discuss set-theoretical issues of category theory.
- for a category $\mathcal{C}$, we denote by $\pi_0(\mathcal{C})$ the set of isomorphism classes of objects in $\mathcal{C}$. Of course there is the issue that this might not always be a set. As indicated in the previous point, we will not justify the usage of this notation.
- Throughout, hooked arrows (as in "$\hookrightarrow$") denote monomorphisms, and double-headed arrows (as in "$\twoheadrightarrow$") denote epimorphisms.
- This thesis assumes the reader has is already familiar with category theory, and in particular the notion of an abelian category. For a resource on general category theory we refer to [Lei16], and for abelian categories we refer to [Osb00].

# Chapter 1

# Hall Algebras and Proto-Abelian Categories

## 1.1 Hall's Algebra of Partitions

As described in the introduction, we fix a prime number $p$ and let $\mathbf{FinAb}(p)$ be the category of finite abelian $p$-groups. The *Hall algebra of partitions* is the ring with underlying abelian group

$$\bigoplus_{[G]\in\pi_0(\mathbf{FinAb}(p))} \mathbb{Z}[G]. \tag{1.1.1}$$

Endowed with the multiplication

$$[A][B] := \sum_{C\in\pi_0(\mathbf{FinAb}(p))} g^C_{A,B}[C]. \tag{1.1.2}$$

extended bilinearly. Again, $g^C_{A,B}$ denotes the number of subgroups $M \subseteq C$ such that $M \cong B$ and $C/M \cong A$.

*Remark* 1.1.3. One might wonder why we use the word *algebra* for a ring, but the terminology is justified: For a commutative ring $R$ there is the notion of an $R$-algebra, which is a ring equipped with a compatible $R$-module structure (in the same way an algebra in the usual sense is a ring equipped with a compatible vector space structure). Just like how $\mathbb{Z}$-modules are exactly abelian groups, $\mathbb{Z}$-algebras are exactly rings.

If one replaces the ring $\mathbb{Z}$ in (1.1.1) with the field $\mathbb{Q}$, one obtains an algebra in the usual sense. Since the structure constants $g^C_{A,B}$ are all integers, the associativity and unitality of the Hall algebra do not depend on the choice of $\mathbb{Z}$ or $\mathbb{Q}$, so the difference between the two cases is not discussed in depth.

**Theorem 1.1.4.** *The Hall algebra of partitions is a well-defined ring. In particular, the sum in the multiplication 1.1.2 only has finitely many nonzero terms, each constant $g^C_{A,B}$ is finite, and the multiplication is associative and unital, with the trivial group as unit.*

*Proof.* We first show that the sum 1.1.2 only has finitely many nonzero terms. Fix $A, B \in \mathbf{FinAb}(p)$, and note that if $M \cong B$ is a subgroup of $C$ with $C/M \cong A$, then $|C| = |M||A| = |B||A|$. There are only finitely many finite abelian $p$-groups of each order, so the constant $g^C_{A,B}$ is nonzero for all but finitely many $C$.

Each group $C \in \mathbf{FinAb}(p)$ only has finitely many subgroups, so the constants $g^C_{A,B}$ are all finite.

We now show that the trivial group $\{e\}$ is the unit of the multiplication. Let $A \in \mathbf{FinAb}(p)$ be arbitrary. Then $g^C_{A,\{e\}}$ is only only nonzero for $C \cong A$, since we need $C \cong C/\{e\} \cong A$. The trivial group appears as a subgroup of each group exactly once. So we have

$$[A][\{e\}] = [A],$$

as desired. Similarly, $g^C_{\{e\},B}$ is only nonzero when $B \cong C$, and each group has itself as a subgroup exactly once. So we have

$$[\{e\}][B] = [B].$$

Now we turn to associativity, which is the main difficulty of the proof. We compute:

$$\begin{aligned}([A][B])[C] &= \left(\sum_{[G]} g^G_{A,B}[G]\right)[C] = \sum_{[G]} g^G_{A,B}[G][C] = \\ &= \sum_{[G]} g^G_{A,B} \sum_{[H]} g^H_{G,C}[H] = \sum_{[G]}\sum_{[H]} g^G_{A,B} g^H_{G,C}[H].\end{aligned}$$

We know that $g^G_{A,B}$ is only non-zero for finitely many $G$, and for each of these the summand is only non-zero for finitely many $H$, so the entire sum is finite and we may exchange the order of summation:

$$([A][B])[C] = \sum_{[H]}\sum_{[G]} g^G_{A,B} g^H_{G,C}[H]. \tag{1.1.5}$$

We abandon this choice of bracketing for now and instead compute

$$[A]([B][C]) = [A]\left(\sum_{[G]} g^G_{B,C}[G]\right) = \sum_{[G]} g^G_{B,C}[A][G] =$$
$$= \sum_{[G]} g^G_{B,C} \sum_{[H]} g^H_{A,G}[H] = \sum_{[G]}\sum_{[H]} g^G_{B,C} g^H_{A,G}[H]$$

By a similar argument as before, we may exchange the order of summation:

$$[A]([B][C]) = \sum_{[H]}\sum_{[G]} g^G_{B,C} g^H_{A,G}[H] \tag{1.1.6}$$

in light of the equalities 1.1.5 and 1.1.6, associativity of the multiplication comes down to the equality

$$\sum_{[G]} g^G_{B,C} g^H_{A,G} = \sum_{[G]} g^G_{A,B} g^H_{G,C} \tag{1.1.7}$$

for all H. We will establish this by claiming that both of these describe the number of sequences $X \subseteq Y \subseteq H$ with $H/Y \cong A$, $Y/X \cong B$, and $X \cong C$. We denote this constant $g^H_{A,B,C}$.

For the left-hand side this is clear. For the right-hand side, we have that $G \cong H/C$, and want to investigate subgroups $B \subseteq G$. We use that these subgroups correspond bijectively to intermediate subgroups of $C$ and $H$, in such a way that for each $B \subseteq G$ bijectively corresponds to a $Y$ with $C \subseteq Y \subseteq H$ and $Y/C \cong B \subseteq G \cong H/C$.

With this notation, $G/B \cong \frac{H/C}{Y/C} \cong H/Y$, so the condition $G/B \cong A$ is equivalent to $H/Y \cong A$.

In summary, fixing a $C$ with $H/C \cong G$, we have that $g^G_{A,B}$ is equal to the number of intermediate subgroups $C \subseteq Y \subseteq H$ with $H/Y \cong A$ and $Y/C \cong B$. Since $g^G_{A,B}$ is a constant, this does not depend on the choice of $C \subseteq H$. We conclude that the right hand side of 1.1.7 also equals $g^H_{A,B,C}$. □

Now when we have established this associativity, the following question becomes natural:

*Question.* What properties of the category **FinAb**$(p)$ made the proof work?

And in fact answering this question is a large part of the motivation for this thesis. The answer we will be giving (again, following Dyckerhoff in [Dyc18]) is the following:

## 1.2 Proto-Abelian Categories

**Definition 1.2.1.** [Dyc18, Def. 2.2] A category $\mathcal{C}$ is called *proto-abelian* if it satisfies the following:

i) $\mathcal{C}$ is a pointed category (i.e, it has a zero object which is both initial and final).

ii) Every diagram on the form

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, two heads] & B \\ C & \end{tikzcd}$$

can be completed to a pushout diagram on the form

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, two heads] & B \arrow[d, two heads] \\ C \arrow[r, hook] & D \end{tikzcd}$$

iii) Every diagram on the form

$$\begin{tikzcd} & B \arrow[d, two heads] \\ C \arrow[r, hook] & D \end{tikzcd}$$

Can be completed to a pullback diagram on the form

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, two heads] & B \arrow[d, two heads] \\ C \arrow[r, hook] & D \end{tikzcd}$$

iv) Commutative squares as in ii) and iii) are pushout if and only if they are pullback. (Being both pushout and pullback is sometimes called being *bicartesian*)

*Remark* 1.2.2. A bicartesian square on the form

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, two heads] & B \arrow[d, two heads] \\ 0 \arrow[r, hook] & D \end{tikzcd}$$

amounts to a short exact sequence $A \hookrightarrow B \twoheadrightarrow D$. The pullback property of the square is exactly the universal property of the kernel, and the pushout property is exactly the universal property of the cokernel. A short exact sequence $A \hookrightarrow B \twoheadrightarrow D$ is also called an *extension of $D$ by $A$*.

The above definition is convenient for proving theorems *about* proto-abelian categories, but it can be inconvenient to show that a category satisfies it. In particular, one needs to construct pushouts and pullbacks and show that they behave well with regard to monomorphisms and epimorphisms. To aid us, we introduce the following sufficient condition for a category to be proto-abelian, which is often easier to prove.

### 1.2.1 A Sufficient Condition for Proto-abelian Categories

**Theorem 1.2.3.** *Let $\mathcal{C}$ be a category. Suppose the following statements hold:*

*i) $\mathcal{C}$ is a pointed category.*

*ii) Every morphism in $\mathcal{C}$ has a kernel.*

*iii) Every morphism in $\mathcal{C}$ has a cokernel.*

*iv) Every monomorphism in $\mathcal{C}$ is the kernel of its cokernel.*

*v) Every epimorhism in $\mathcal{C}$ is the cokernel of its kernel.*

*vi) For each morphism $f$, there exists a monomorphism $i$ and an epimorphism $p$ so that $f = i \circ p$.*

*Then $\mathcal{C}$ is a proto-abelian category.*

*Remark* 1.2.4. These conditions are all true for abelian categories. Notably absent is the existence of finite products and coproduct, which usually taken as an axiom for abelian categories. The conditions of Theorem 1.2.3 are in fact strictly weaker than the axioms of abelian categories. In Section 1.3, we will study a category which satisfies these conditions but isn't abelian. Furthermore, Theorem 1.2.3, once proven, immediately implies that abelian categories are proto-abelian.

We need several lemmas before we can prove Theorem 1.2.3. The following three lemmas and proofs are from chapter 7 of [Osb00], adapted only for paraphrasing and changing the initial assumptions.

**Lemma 1.2.5.** [Osb00, Lemma 7.35] *Let $\mathcal{C}$ be a pointed category. Consider the diagram*

$$\begin{tikzcd} K \arrow[r, hook, "j"] & A \arrow[r, hook, "f"] & B \arrow[r, "\varphi"] & C \end{tikzcd}$$

*where $f$ and $j$ are a monomorphisms. Suppose that $f \circ j$ is a kernel of $\varphi$. Then $j$ is a kernel of $\varphi \circ f$.*

*Proof.* We check the universal property of the kernel on $j$. Let $X \in \mathcal{C}$ and $g\colon X \to A$ be such that $(\varphi \circ f) \circ g = 0$.

$$\begin{tikzcd} K \arrow[r, hook, "j"] & A \arrow[r, hook, "f"] & B \arrow[r, "\varphi"] & C \\ & X \arrow[ul, dotted, "\exists!\psi"] \arrow[u, "g"] \arrow[urr, "0"'] & & \end{tikzcd}$$

Then, since $\varphi \circ (f \circ g) = 0$, by the universal property of $f \circ j$ there is a unique map $\psi\colon X \to K$ such that $f \circ g = f \circ j \circ \psi$. Since $f$ is monic, this implies $g = j \circ \psi$. □

**Lemma 1.2.6.** [Osb00, Prop. 7.36] *Let $\mathcal{C}$ be a category satisfying the assumptions of Theorem 1.2.3. Let $\pi\colon A \to B$ be epic, and let $\varphi \in \operatorname{Hom}(B, C)$. Suppose $\pi$ and $\varphi \circ \pi$ have the same kernel. Then $\varphi$ is mono.*

*Proof.* Suppose first that $\varphi$ is epic. Then $\varphi \circ \pi$ is epic. Let $j\colon K \to A$ denote a kernel for both $\pi$ and $\varphi \circ \pi$.

$$\begin{tikzcd} & & B \arrow[dd, dotted, "\exists\psi"] \\ K \arrow[r, hook, "j"] & A \arrow[ur, two heads, "\pi"] \arrow[dr, two heads, "\varphi\pi"'] & \\ & & C \end{tikzcd}$$

Since epimorphisms are cokernels of their kernels, both $\pi$ and $\varphi \circ \pi$ are cokernels of $j$. By uniqueness of cokernels, there exists an isomorphism $\psi\colon B \to C$ making the above diagram commute. Since $\pi$ is an epimorphism, the equality $\psi \circ \pi = \varphi \circ \pi$ implies $\varphi = \psi$, so $\varphi$ is an isomorphism. In particular, it is also mono.

For general $\varphi$ we split it up into $\varphi = f \circ p$ where $f$ is mono and $p$ is epi.

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, two heads, "p"'] & & C \\ & \operatorname{im}\varphi \arrow[ur, hook, "f"'] & \end{tikzcd}$$

(One of the assumptions on $\mathcal{C}$ is that this is possible). By the same argument as when $\varphi$ was epic, we have that $p$ is an isomorphism. Thus $f \circ p = \varphi$ is a monomorphism. $\square$

**Lemma 1.2.7.** *Let $\mathcal{C}$ be a category satifsying the assumptions of Theorem 1.2.3. Suppose the rows in the commutative diagram*

$$\begin{tikzcd}
A \arrow[r, hook, "j"] \arrow[d, equal] & B \arrow[r, two heads, "p"] \arrow[d, hook, "\psi"] & C \arrow[d, "\eta"] \\
A \arrow[r, hook, "j'"] & B' \arrow[r, two heads, "p'"] & C'
\end{tikzcd}$$

*are exact in the sense of Remark 1.2.2. Suppose also that $\psi$ is monic. Then $\eta$ is monic.*

*Proof. ([Osb00] Proof of Prop. 7.37).* We want to use Lemma 1.2.6 to show that $\eta$ is monic. To do this, we need that $j$ is the kernel of both $p$ and $\eta \circ p$. The first of these is true since the top row is exact. The second follows from the fact that $\psi \circ j = j'$ is a kernel for $p'$, so by Lemma 1.2.5 $j$ is a kernel for $p' \circ \varphi = \eta \circ p$. $\square$

Now we are ready to prove Theorem 1.2.3.

*Proof of Theorem 1.2.3.* The first condition for a category to be proto-abelian is that it is pointed, which is true by assumption.

We will first show the remaining conditions for the special case where the bottom-left object of the sqares involved is the zero object.

Given any diagram on the form

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[d, two heads] & B \\
0 &
\end{tikzcd}$$

We can compete it to a pushout sqaure

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[d, two heads] & B \arrow[d, two heads] \\
0 \arrow[r, hook] & \operatorname{coker} i
\end{tikzcd}$$

The universal property of the pushout corresponds in this case precisely to that of the cokernel. Dually, given any diagram on the form

$$\begin{tikzcd} & B \arrow[d, two heads, "q"] \\ 0 \arrow[r, hook] & D \end{tikzcd}$$

We can complete it to a pullback square by taking the kernel.

$$\begin{tikzcd} \ker q \arrow[r, hook] \arrow[d, two heads] & B \arrow[d, two heads, "q"] \\ 0 \arrow[r, hook] & D \end{tikzcd}$$

Now consider general squares on the form

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, two heads] & B \arrow[d, two heads, "q"] \\ 0 \arrow[r, hook] & D \end{tikzcd}$$

By assumption on $\mathcal{C}$, if $i$ is the kernel of $q$ then $q$ is the cokernel of its kernel $i$. Similarly, if $q$ is the cokernel of $i$, then $i$ is the kernel of $q$. In other words, squares on this form are pushout if and only if they are pullback.

Now we consider the general case. Consider a diagram on the form

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, two heads, "p"'] & B \\ C & \end{tikzcd}$$

and construct the diagram

$$\begin{tikzcd} \ker p \arrow[r, hook, "j"] \arrow[d, two heads] & A \arrow[r, hook, "i"] \arrow[d, two heads, "p"'] & B \arrow[d, two heads] \\ 0 \arrow[r, hook] \arrow[rr, bend right, hook] & C \arrow[r, dotted, "f"] & \operatorname{coker} i \circ j \end{tikzcd}$$

by first taking the kernel of $p$, calling the inclusion of the kernel $j$, and then taking the cokernel of $i \circ j$. By the previous discussion, the left square and the outer rectangle are both bicartesian. From the pushout property of the left square we obtain a map $f : C \to \operatorname{coker} i \circ j$, which commutes with the diagram. We can now use the pasting lemma A.0.1 for pushouts to conclude that the right square is pushout. It remains to show that $f$ is mono. To do this, we draw the diagram in a different way:

$$\begin{tikzcd}
\ker p \arrow[r, hook, "j"'] \arrow[d, equal] & A \arrow[r, two heads, "p"'] \arrow[d, hook, "i"] & C \arrow[d, "f"'] \\
\ker p \arrow[r, hook, "i\circ j"'] & B \arrow[r, two heads] & \operatorname{coker} i\circ j
\end{tikzcd}$$

Both rows are exact by the previous discussion, so by Lemma 1.2.7 $f$ is a monomorphism.

We thus managed to extend our span $C \twoheadleftarrow A \hookrightarrow B$ to a pushout square

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, two heads] & B \arrow[d, two heads] \\
C \arrow[r, hook, "j"] & D
\end{tikzcd}$$

We claim that such a square must also be pullback. Indeed, consider the cokernel of $j$ as in the following diagram

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, two heads] & B \arrow[d, two heads] \\
C \arrow[r, hook, "j"] \arrow[d, two heads] & D \arrow[d, two heads] \\
0 \arrow[r, hook] & \operatorname{coker} j
\end{tikzcd}$$

The bottom square was shown to be bicartesian in the first half of the proof. Since the top square is pushout, we conclude by Lemma A.0.1 that the outer rectangle is pushout. This outer rectangle has the object 0 in its lower left-hand corner, and we have established that such commutative squares are pushout if and only if they are pullback. Hence the outer rectangle is also pullback. Since the bottom square and the outer rectangle are both pullback, the pasting lemma A.0.2 gives us that the upper square is pullback, as desired.

We have now shown that the pushouts required in the definition of proto-abelian categories exist, and that such pushout squares are also pullback. The duals to these statements remain to be shown: That the corresponding pullbacks exist, and that such pullback squares are also pushout. To do this, we note that the assumptions on $\mathcal{C}$ are self-dual, such that if $\mathcal{C}$ satisfies the assumptions of the theorem $\mathcal{C}^{\mathrm{op}}$ does as well. Applying what we have proven to $\mathcal{C}^{\mathrm{op}}$ and taking the opposite, we obtain the desired dual statements, and our proof is complete. □

**Corollary 1.2.8.** *Abelian categories are proto-abelian.*

*Proof.* Abelian categories satisfy all the conditions of theorem 1.2.3 □

### 1.2.2 The Hall Algebra of a Proto-Abelian Category

It turns out that the construction of the Hall algebra of partitions extends almost verbatim to proto-abelian categories, but first we need to introduce some notions which allow us to properly count in arbitrary categories.

**Definition 1.2.9.** In a proto-abelian category $\mathcal{C}$, we introduce an equivalence relation on extensions. Two extensions $A \hookrightarrow B \twoheadrightarrow D$ and $A \hookrightarrow B' \twoheadrightarrow D$ of $D$ by $A$ considered equivalent (sometimes called Yoneda-equivalent) if there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{id}"'] & B \arrow[r] \arrow[d, "\cong"'] & D \arrow[d, "\mathrm{id}"] \\ A \arrow[r] & B' \arrow[r] & D \end{tikzcd}$$

where the map $B \to B'$ is an isomorphism. The set of equivalence classes under this equivalence relation is denoted $\mathrm{Ext}_{\mathcal{C}}(D, A)$

**Definition 1.2.10.** A proto-abelian category $\mathcal{C}$ is called *finitary* if, for each $A, D \in \mathcal{C}$, the sets $\mathrm{Hom}_{\mathcal{C}}(A, D)$ and $\mathrm{Ext}_{\mathcal{C}}(D, A)$ are finite.

*Example* 1.2.11. Examples of finitary proto-abelian categories are the categories $\mathbf{FinAb}(p)$ of finite abelian $p$-groups, and $\mathbf{Vect}_{\mathbb{F}_q}$ of vector spaces over a finite field $\mathbb{F}_q$. These are proto-abelian because they are abelian, and that they are finitary can easily be checked.

**Definition 1.2.12.** Let $\mathcal{C}$ be a category and $A \in \mathcal{C}$. A *subobject* of $A$ is an equivalence class of pairs $(B, i)$ where $i\colon B \hookrightarrow A$ is a monomorphism. Two pairs $(B, i)$, $(B', j)$ are considered equivalent if there is an isomorphism $\varphi\colon B \to B'$ such that $j \circ \varphi = i$, i.e such that the diagram.

$$\begin{tikzcd} B \arrow[r, hook, "i"] \arrow[d, "\varphi"'] & A \\ B' \arrow[ur, hook, "j"'] & \end{tikzcd}$$

commutes.

By abuse of notation, we often just say that $B$ is a subobject of $A$ and leave the inclusion morphism implicit until it is needed. This is also denoted $B \subseteq A$.

Now we are finally able to state the definition of the Hall algebra for a proto-abelian category.

**Theorem 1.2.13.** *Let $\mathcal{C}$ be a finitary proto-abelian category. The* Hall algebra *of $\mathcal{C}$ is the algebra*

$$\mathrm{Hall}_{\mathbb{Z}}(\mathcal{C}) = \bigoplus_{[A]\in\pi_0(\mathcal{C})} \mathbb{Z}[A] \tag{1.2.14}$$

*with multiplication defined by bilinearly extending*

$$[A][B] := \sum_{[C]\in\pi_0(\mathcal{C})} g^C_{A,B}[C],$$

*where $g^C_{A,B}$ denotes the number of subobjects $B \subseteq C$ with $C/B \cong A$. This is a well-defined (unital, associative) algebra. In particular, for each $A, B$, the constants $g^C_{A,B}$ are finite and only finitely many of them are nonzero.*

*Proof.* The main aim of this thesis is to prove this claim in a structured way, utilizing the perspective of Dyckerhoff-Kapranov. □

*Remark* 1.2.15. When we prove this theorem, we will actually use the version of the Hall algebra with $\mathbb{Z}$ in 1.2.14 replaced by $\mathbb{Q}$ - which is denoted by $\mathrm{Hall}_{\mathbb{Q}}(\mathcal{C})$. As described in Remark 1.1.3, the statements for $\mathbb{Z}$ and $\mathbb{Q}$ are equivalent.

## 1.3 Vector Spaces Over the Field With One Element

We now provide an example (taken from [Dyc18]) of a proto-abelian category which is not abelian. One advantage with this category is that the Hall algebra is fairly easy to compute explicitly.

**Definition 1.3.1.** A *finite-dimensional* $\mathbb{F}_1$ *- vector space*, or a *finite-dimensional vector space over the field with one element* is a finite pointed set

$$V = \{*, v_1, v_2, \ldots, v_n\}.$$

The element $*$ is distinguished and sometimes called the *zero element* of $V$. We call $n$ the *dimension* of $V$. A *morphism* or *linear transformation* between $\mathbb{F}_1$-vector spaces $V$ and $W$ is a set-theoretic function $V \longrightarrow W$ which maps $* \in V$ to $* \in W$, and with the property that

$$f\big|_{V\setminus f^{-1}(*)} : V \hookrightarrow W.$$

is an injective function. In other words: $* \mapsto *$, and multiple elements in $V$ may map to $*$, but otherwise the function is required to be injective.

*Remark* 1.3.2. The composition of $\mathbb{F}_1$-linear maps is $\mathbb{F}_1$-linear, and so is the identity map from a $\mathbb{F}_1$-vector space to itself. Therefore the category of finite-dimensional $\mathbb{F}_1$-vector spaces is a well-defined category. We denote this category by $\mathbf{Vect}_{\mathbb{F}_1}$. A $\mathbb{F}_1$-linear map $f$ also has an image $\mathrm{im}(f)$ defined as its set-theoretic image, and a kernel $\ker(f)$ defined as the preimage $f^{-1}(*)$. Despite this, $\mathbf{Vect}_{\mathbb{F}_1}$ is not an abelian category. For example, binary products do not exist, as shown in Proposition 1.3.10.

*Remark* 1.3.3. One should not be tricked into thinking that $\mathbb{F}_1$-vector spaces are vector spaces in the usual sense. Indeed, they lack both abelian group structure and scalar multiplication. Similarly, fields by definition require at least two elements, so there is no such thing as a "field with one element" $\mathbb{F}_1$. The name is justified due to qualitative similarities to ordinary vector spaces, such as the following:

**Proposition 1.3.4** (Rank-Nullity for $\mathbb{F}_1$-vector spaces)**.** *Let $V$ and $W$ be $\mathbb{F}_1$-vector spaces and $f\colon V \longrightarrow W$ a $\mathbb{F}_1$-linear map between them. Then*

$$\dim(\ker(f)) + \dim(\mathrm{im}(f)) = \dim(V).$$

*Proof.* By definition, $\dim(\ker(f))$ is the number of nonzero elements of $V$ whose image is $*$. Since $f$ is $\mathbb{F}_1$-linear, $\dim(\mathrm{im}(f))$ is the number of nonzero elements of $V$ whose image is not $*$. These two numbers add upp to $\dim(V)$. □

*Remark* 1.3.5. This proof is morally the same as for normal vector spaces, the main difference is not having to make a clever choice of basis at the start of the proof.

We introduce some more definitions:

**Definition 1.3.6.** For a $\mathbb{F}_1$-vector space $V$, a subset $W \subseteq V$ is a *subspace* of $V$ if it is also a $\mathbb{F}_1$-vector space (in other words, if $* \in W$).

**Definition 1.3.7.** For $\mathbb{F}_1$-vector spaces $W \subseteq V$, we define the quotient space $V/W$ as the space $\{*\} \cup (V \setminus W)$. It comes equipped with the canonical projection map $\pi\colon V \longrightarrow V/W$ which maps the elements of $W$ to $*$ and leaves the elements of $V \setminus W$ unchanged.

**Definition 1.3.8.** Given a morphism of $\mathbb{F}_1$-vector spaces $f\colon V \longrightarrow W$, we define the cokernel $\mathrm{coker}\, f$ as the quotient $W/\mathrm{im}(f)$. The cokernel map is the canonical projection map $W \longrightarrow \mathrm{coker}\, f$.

It is easy to show that these definitions of kernel and cokernel indeed satisfy the corresponding universal properties.

Of course, the reason we are considering this category is the following theorem:

**Theorem 1.3.9.** *The category* $\mathbf{Vect}_{\mathbb{F}_1}$ *is proto-abelian.*

*Proof.* We use Theorem 1.2.3. The zero object is the zero vector space $\{*\}$. We have already established that cokernels and kernels exist. It is clear that monomorphisms are kernels of their cokernels, and that epimorphisms are cokernels of their kernels. For the final condition, that being the epi-mono factorisation, we can rewrite a morphism $V \xrightarrow{f} W$ as $V \twoheadrightarrow \operatorname{im}(f) \hookrightarrow W$. We have established that $\mathbf{Vect}_{\mathbb{F}_1}$ satisfies the conditions of Theorem 1.2.3, and therefore it is proto-abelian. □

It should be noted that $\mathbf{Vect}_{\mathbb{F}_1}$ is not abelian, and therefore the conditions of Theorem 1.2.3 do not imply that the category is abelian. This can be seen in the following quick result.

**Proposition 1.3.10.** *The category* $\mathbf{Vect}_{\mathbb{F}_1}$ *does not have finite products, and is therefore not an abelian category.*

*Proof.* Suppose for the sake of contradiction that $\mathbf{Vect}_{\mathbb{F}_1}$ has finite products. Let $A = \{*, 1\}$. The universal property of the product gives us a bijection

$$\Psi\colon \operatorname{Hom}(A, A) \times \operatorname{Hom}(A, A) \xrightarrow{\sim} \operatorname{Hom}(A, A \times A)$$

and two canonical $\mathbb{F}_1$-linear projection maps $\pi_1, \pi_2 : A \times A \to A$.

A map out of $A = \{*, 1\}$ is uniquely determined by where it sends the element 1. Thus the two different maps $\Psi((\mathrm{id}_A, \mathrm{id}_A)), \Psi((\mathrm{id}_A, 0))\colon A \to A \times A$ send $1 \in A$ to two different values in $A \times A$. Denote these values by $x$ and $y$, respectively. Then $\pi_1(x) = \pi_1(y) = 1$ by definition of $\Psi$, but this is a contradiction since $\pi$ is $\mathbb{F}_1$-linear, so two nonzero elements cannot have the same image. (Note that $x, y \neq *$ since their image under $\pi_1$ is nonzero). □

**Proposition 1.3.11.** *The category* $\mathbf{Vect}_{\mathbb{F}_1}$ *is finitary.*

*Proof.* Let $V$ and $W$ be $\mathbb{F}_1$-vector spaces. The set $\operatorname{Hom}(V, W)$ is finite since its cardinality is bounded above by the number $|W|^{|V|}$ of set-theoretic functions $V \to W$. It remains to show that the set $\operatorname{Ext}(W, V)$ is finite. All possible middle terms of a extension of $W$ by $V$ have dimension $n := \dim(V) + \dim(W)$ by Rank-Nullity (Proposition 1.3.4), and each equivalence class in $\operatorname{Ext}(W, V)$ contains an extension where the middle term is exactly the $\mathbb{F}_1$-vector space $\{*, 1, \ldots n\}$. Then it is clear that $\operatorname{Ext}(W, V)$ is finite, since there are only finitely many injections $V \hookrightarrow \{*, \ldots, n\}$. □

**Corollary 1.3.12.** $\mathbf{Vect}_{\mathbb{F}_1}$ *has a well-defined Associative Hall algebra.*

*Proof.* $\mathbf{Vect}_{F_1}$ is finitary and proto-abelian, so by Theorem 1.2.13 the Hall Algebra is well-defined. Note that we haven't proven Theorem 1.2.13 yet, so this result should be taken on faith for now. By the end of the thesis Theorem 1.2.13 will be proven, and this corollary will not be used in the sequel except implicitly when calculating the Hall Algebra of $\mathbf{Vect}_{\mathbb{F}_1}$. □

### 1.3.1 Computing an Example Hall Algebra

The following calculation is taken from [Dyc18].

It turns out that it is not too difficult to explicitly calculate the Hall algebra of $\mathbf{Vect}_{\mathbb{F}_1}$. Each isomorphism class of $\mathbb{F}_1$-vector spaces can be identified with its dimension, so we let $[n]$ for a nonnegative integer $n$ denote the isomorphism class of $n$-dimensional $\mathbb{F}_1$-vector spaces. The underlying abelian group structure of $\mathrm{Hall}(\mathbf{Vect}_{\mathbb{F}_1})$ is therefore

$$\mathrm{Hall}(\mathbf{Vect}_{\mathbb{F}_1}) = \bigoplus_{[n]\in\mathbb{N}} \mathbb{Z}[n]$$

with multiplication given by

$$[n][m] = \sum_{k\in\mathbb{N}} g^k_{n,m}\mathbb{Z}[k]$$

where $g^k_{n,m}$ is the number of $m$-dimensional subspaces of each $k$-dimensional vector space whose quotient space is $n$-dimensional. Clearly we have

$$g^k_{n,m} = \begin{cases} \binom{n+m}{m} & k = n+m \\ 0 & \text{otherwise.} \end{cases}$$

So that the multiplication is given by

$$[n][m] = \binom{n+m}{m}[n+m] = \frac{(n+m)!}{n!m!}[n+m]$$

This suggests an association

$$\begin{aligned} \varphi\colon\ \mathrm{Hall}_{\mathbb{Z}}(\mathbf{Vect}_{\mathbb{F}_1}) &\xrightarrow{\sim} \mathbb{Z}\left[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \dots\right] \\ [n] &\mapsto \frac{x^n}{n!} \qquad \text{for } n \geq 0 \end{aligned}$$

which indeed is an isomorphism. Clearly $\varphi$ is an isomorphism of the underlying groups, since both rings are direct sums indexed over $\mathbb{N}$, and $\varphi$ maps the basis elements bijectively. To show it is a ring isomorphism, we note that

$$
\begin{aligned}
\varphi([n][m]) = \varphi\left(\frac{(n+m)!}{n!m!}[n+m]\right) = \frac{(n+m)!}{n!m!}\,\frac{x^{n+m}}{(n+m)!} = \\
= \frac{x^{n+m}}{n!m!} = \varphi([n])\varphi([m]).
\end{aligned}
$$

## 1.4 Wings

**Definition 1.4.1.** For a pointed category $\mathcal{C}$, A *flag of length $n$* in $\mathcal{C}$ is a diagram on the form

$$0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \ldots \hookrightarrow A_n$$

*Remark* 1.4.2. Let $(X, \leq)$ be a partially ordered set (also referred to as a *poset*). Recall that we can view this as a category by letting its objects be the elements of $X$, and for each pair of objects $a, b$ in $X$ a unique morphism $a \to b$ if and only if $a \leq b$.

Now consider the ordered set $[n] := \{0, 1, \ldots, n\}$ with the usual ordering. From this perspective, a flag of length $n$ is a functor $[n] \to \mathcal{C}$ which maps $0 \in [n]$ to $0 \in \mathcal{C}$, and each morphism in $[n]$ to a monomorphism in $\mathcal{C}$.

In many cases, such as that of proto-abelian categories, we can obtain more data from a flag by taking quotients, taking quotients of those quotients, and so on. All of this data can be conveniently displayed in what we call a *wing*

**Definition 1.4.3.** For a pointed category $\mathcal{C}$, a *wing of size $n$* in $\mathcal{C}$ is a diagram on the form

$$
\begin{tikzcd}
0 \arrow[r, hook] & A_{0,1} \arrow[r, hook] \arrow[d, two heads] & A_{0,2} \arrow[r, hook] \arrow[d, two heads] & \ldots \arrow[r, hook] & A_{0,n-1} \arrow[r, hook] \arrow[d, two heads] & A_{0,n} \arrow[d, two heads] \\
& 0 \arrow[r, hook] & A_{1,2} \arrow[r, hook] \arrow[d, two heads] & \ldots \arrow[r, hook] & A_{1,n-1} \arrow[r, hook] \arrow[d, two heads] & A_{1,n} \arrow[d, two heads] \\
& & 0 & \ddots & \vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] \\
& & & 0 \arrow[r, hook] & A_{n-1,n-1} \arrow[r, hook] \arrow[d, two heads] & A_{n-1,n} \arrow[d, two heads] \\
& & & & 0 \arrow[r, hook] & A_{n,n} \arrow[d, two heads] \\
& & & & & 0
\end{tikzcd}
$$

In which each $1 \times 1$- square is bicartesian. By convention, we sometimes denote the zeroes on the diagonal as $A_{i,i} \cong 0$ for $0 \leq i \leq n$.

*Remark* 1.4.4. Using the word *wing* to refer to these diagrams is nonstandard in the sense that it is not used in [Dyc18], which this thesis is largely modelled after. It is the author's understanding that such diagrams are sometimes called *wings* in representation theory, hence the terminology.

*Remark* 1.4.5. As in Remark 1.4.2, a wing can be viewed as a functor from a poset. We define

$$\langle n \rangle := \{(i,j) \in [n]^2 : i \leq j\}$$

with $(i,j) \leq (m,n)$ if and only if $i \leq m$ and $j \leq n$.

Wiewed as categories, $\langle n \rangle$ is the full subcategory of the product cateogory $[n]^2$ generated by the objects $(i,j)$ with $i \leq j$. In terms of this notation, a wing is a functor $A \colon \langle n \rangle \to \mathcal{C}$ where $A(k,k) = 0$ for all $k$, where the horizontal morphisms $A(k,i) \hookrightarrow A(k,j)$ are monomorphisms, the vertical morphisms $A(i,k) \twoheadrightarrow A(j,k)$ are epimorphisms, and each $1 \times 1$-square is bicartesian.

With this in mind, we find a natural notion of isomorphism between wings, namely natural isomorphism of functors.

A wing can be understood as a being generated by the flag which constitutes its top row. Our current goal is to justify this point of view, and to generally understand the basics of wings.

**Corollary 1.4.6.** *In a wing, every commutative rectangle is bicartesian.*

*Proof.* Each $1 \times 1$ rectangle of the wing is bicartesian by the definition of a wing. By pasting together $k$ such squares, according to lemmas A.0.1 and A.0.2, we have that every $k \times 1$ - rectangle is bicartesian. By pasting $l$ of these together, we have that each $k \times l$ - rectangle is bicartesian, as desired. □

Now we can describe each object $A_{i,j}$ in a wing. Note that every such object is part of a bicartesian square

$$\begin{tikzcd} A_{0,i} \arrow[r, hook] \arrow[d, two heads] & A_{0,j} \arrow[d, two heads] \\ A_{i,i} = 0 \arrow[r, hook] & A_{i,j} \end{tikzcd}$$

As in remark 1.2.2, this amounts to a short exact sequence

$$A_{0,i} \hookrightarrow A_{0,j} \twoheadrightarrow A_{i,j}$$

. We can therefore identify $A_{i,j}$ with the quotient $A_{0,j}/A_{0,i}$. In this sense, we can view the wing as the data of all quotients of objects in the flag

$$0 \hookrightarrow A_{0,1} \hookrightarrow A_{0,2} \hookrightarrow \dots \hookrightarrow A_{0,n}$$

with the canonical maps between them. Interestingly, the wing also contains quotients of quotients. For concreteness, consider a wing of size 3:

$$\begin{tikzcd}
0 \arrow[r, hook] & A_{0,1} \arrow[r, hook] \arrow[d, two heads] & A_{0,2} \arrow[r, hook] \arrow[d, two heads] & A_{0,3} \arrow[d, two heads] \\
& 0 \arrow[r, hook] & A_{1,2} \arrow[r, hook] \arrow[d, two heads] & A_{1,3} \arrow[d, two heads] \\
& & 0 \arrow[r, hook] & A_{2,3} \arrow[d, two heads] \\
& & & 0
\end{tikzcd}$$

In the above point of view, we can identify $A_{2,3}$ both as the quotient $A_{1,3}/A_{1,2}$ and the quotient $A_{0,3}/A_{0,2}$. We can also identify $A_{1,3}$ and $A_{1,2}$ as quotients of objects in the top row. Filling in this information, we obtain the following diagram:

$$\begin{tikzcd}
0 \arrow[r, hook] & A_{0,1} \arrow[r, hook] \arrow[d, two heads] & A_{0,2} \arrow[r, hook] \arrow[d, two heads] & A_{0,3} \arrow[d, two heads] \\
& 0 \arrow[r, hook] & A_{0,2}/A_{0,1} \arrow[r, hook] \arrow[d, two heads] & A_{0,3}/A_{0,1} \arrow[d, two heads] \\
& & 0 \arrow[r, hook] & A_{0,3}/A_{0,2} \cong \frac{A_{0,3}/A_{0,1}}{A_{0,2}/A_{0,1}} \arrow[d, two heads] \\
& & & 0
\end{tikzcd}$$

The two different ways of viewing $A_{2,3}$ as a quotient amount to an isomorphism

$$A_{0,3}/A_{0,2} \cong \frac{A_{0,3}/A_{0,1}}{A_{0,2}/A_{0,1}}.$$

This is the statement of the third isomorphism theorem! That such a theorem arises as a simple shift in perspective should hopefully convince the reader that wings are a powerful way of organising the data of all possible quotients generated by a flag.

We now justify the existence and uniqueness of wings in a proto-abelian category.

**Theorem 1.4.7.** *Let $\mathcal{C}$ be a proto-abelian category and fix a flag $\mathbf{A} = \{0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_n\}$. Then there exists a wing in $\mathcal{C}$ with this flag as its top row, and this wing is unique up to a natural isomorphism of wings which induces the identity on* $\mathbf{A}$.

*Proof.* We use the convention that $A_{i,i} = 0$ for all $i$. For existence, we construct the wing inductively, row by row, left to right. We start by setting the top row to the flag $\mathbf{A}$, letting $A_{0,k} := A_k$. After this we define $A_{i,j}$, for $0 < i < n$ and $j > i$, as the pushout of the diagram

$$\begin{tikzcd}
A_{i-1,j-1} \arrow[r, hook] \arrow[d, two heads] & A_{i-1,j} \\
A_{i,j-1} &
\end{tikzcd}$$

where the these objects have already been constructed at this stage in the induction. The axioms of proto-abelian categories guarantee that such pushouts always exist, and that the morphisms that need to be mono or epi are such.

We turn to proving uniqueness. Suppose $\{A_{i,j}\}_{0\le i\le j\le n}$ and $\{B_{i,j}\}_{0\le i\le j\le n}$ are both flags with $\mathbf{A}$ as their top rows, so that $A_{0,k} = B_{0,k} = A_k$. We need to establish that there is a natural isomorphism between these wings, which induces the identity on $\mathbf{A}$. Construct the maps $\{\varepsilon_{i,j} \colon A_{i,j} \to B_{i,j}\}$ by setting $\varepsilon_{0,k} = id_{A_k}$ otherwise $\varepsilon_{i,j}$ as the isomorphism given by the universal property of the pushout

$$\begin{tikzcd}
A_{0,i} \arrow[r, hook] \arrow[d, two heads] & A_{0,j} \arrow[d, two heads] \arrow[ddr, bend left, two heads] & \\
0 \arrow[r, hook] \arrow[drr, bend right, hook] & A_{i,j} \arrow[dr, dashed, "\varepsilon_{i,j}"] & \\
& & B_{i,j}
\end{tikzcd}$$

These are isomorphisms since ince $A_{i,j}$ and $B_{i,j}$ are both pushouts of 0 and $A_{0,j}$ along $A_{0,i}$. We now need to show these maps are natural, i.e showing that every diagram on the form

$$\begin{tikzcd}
A_{i,j} \arrow[r, "\varepsilon_{i,j}"] \arrow[d, "f"] & B_{i,j} \arrow[d, "g"] \\
A_{m,n} \arrow[r, "\varepsilon_{m,n}"'] & B_{m,n}
\end{tikzcd}$$

commutes. Note that a map $A_{i,j} \to A_{m,n}$ only exists if $i \leq m$ and $j \leq n$, and if it exists the map is unique, i.e. $f$ and $g$ are uniquely determined by their domain and codomain. Consider the diagram

$$\begin{tikzcd}
A_{0,j} \arrow[rr, hook, "i"'] \arrow[dr, two heads, "p_B"] \arrow[dd, two heads, "p_A"'] & & A_{0,n} \arrow[dr, two heads, "q_B"] \arrow[dd, two heads, "q_A"] & \\
& B_{i,j} \arrow[rr, "g"] & & B_{m,n} \\
A_{i,j} \arrow[ur, "\varepsilon_{i,j}"] \arrow[rr, "f"] & & A_{m,n} \arrow[ur, "\varepsilon_{m,n}"'] &
\end{tikzcd}$$

The triangles in this diagram are commutative by the definition of $\varepsilon$. We also have that $q_A \circ i = f \circ p_A$ and $g \circ p_B = q_b \circ i$ from the commutativity of our wings. Now we compute:

$$g \circ \varepsilon_{i,j} \circ p_A = g \circ p_B = q_B \circ i = \varepsilon_{m,n} \circ q_A \circ i = \varepsilon_{m,n} \circ f \circ p_A.$$

Since $p_A$ is an epimorphism we conclude $g \circ \varepsilon_{i,j} = \varepsilon_{m,n} \circ f$ as desired. □

It will eventually become useful to consider the dual version of this theorem, so we introduce the dual notions.

**Definition 1.4.8.** Let $\mathcal{C}$ be a pointed category. A *coflag* in $\mathcal{C}$ is a a diagram on the form

$$A_0 \twoheadrightarrow A_1 \twoheadrightarrow A_2 \twoheadrightarrow \ldots \twoheadrightarrow A_{n-1} \twoheadrightarrow 0$$

Clearly, each column of a wing is a coflag. While it is most natural to think of a wing as being generated by the flag of its top row, it is actually equivalent to think of it as being generated by the coflag of its rightmost column, as the following theorem shows.

**Theorem 1.4.9.** *Let $\mathcal{C}$ be a proto-abelian category and fix a coflag* $\mathbf{A} = \{A_0 \twoheadrightarrow \ldots \twoheadrightarrow A_{n-1} \twoheadrightarrow 0\}$. *Then there exists a wing in $\mathcal{C}$ with this coflag as its rightmost column, and this wing is unique up to a natural isomorphism of wings which induces the identity on* $\mathbf{A}$.

*Proof.* Dual of Theorem 1.4.7. □

# Chapter 2

# A More Abstract Perspective

## 2.1 Simplicial Objects

This part of the thesis aims to view the Hall Algebras in chapter 1 in a more systematic way. First, we introduce the notion of a simplicial set, which is a fundamental building block of $\infty$-category theory. The exposition is based on that in [Rie11].

**Definition 2.1.1** (Simplex Category)**.** Let $\mathbf{\Delta}$ be the category whose objects are the sets $[n] := \{0, 1, \ldots, n\}$ for $n \in \mathbb{N}$, and whose morphisms are order-preserving maps between these sets. $\mathbf{\Delta}$ is called the *Simplex category*.

*Remark* 2.1.2. If we use the usual construction of considering ordered sets as categories, we obtain another (isomorphic) way of defining the Simplex category. Let $[n]$ be the (unique up to unique isomorphism) category which represents a totally ordered set with $n$ elements. Then $\mathbf{\Delta}$ is the full subcategory of **Cat** generated by $[n]$ for $n \in \mathbb{N}$. In other words, $\mathbf{\Delta}$ is the category where the objects are the categories $[n]$ and the morphisms are functors between them.

There are some morphisms in the simplex category which merit special attention. First of these are the maps $d^i : [n-1] \to [n]$ which are injective everywhere and surjective everywhere except at $i \in [n]$. These are called coface maps.

The second distinguished set of maps are the maps $s^i : [n+1] \to [n]$ which are surjective everywhere and injective except at $i \in [n]$. These are called codegeneracy maps.

It is not hard to verify that every map in the simplex category can be written as a composition of coface and codegeneracy maps. We omit a proof here, referring to [Lan10, Section VII.5]. We will not use this result other than for intuition.

**Definition 2.1.3** (Simplicial Sets)**.** A *Simplicial set X* is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. The image under $X$ of the coface maps are called face maps, and the image under $X$ of the codegeneracy maps are called degeneracy maps.

*Remark* 2.1.4. To understand a simplicial set, it is in some sense often enough to understand what it does to the coface and codegeneracy maps, since these generate all of $\mathbf{\Delta}$.

*Remark* 2.1.5. If the codomain of the functor is something other than **Set**, we adjust the terminology accordingly. An important example for our purposes will be that functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Grpd}$ are called *simplicial groupoids*. The collective name is *simplicial objects*

*Remark* 2.1.6. For a simplicial set $X$ we often, but not always, denote the image $X[n]$ by $X_n$ or $X_{\{0,\ldots,n\}}$. Since keeping track of the indices of (co-)face and (co-)degeneracy maps can be tricky, the following abuse of notation is often useful: We allow ourselves to write in the subscript of $X$ other finite totally ordered sets than $[n]$. Within each isomorphism class, exactly which ordered set we choose does not matter on the level of objects, but the choice communicates which morphisms we are referring to. For example, in terms of objects $X_{\{0,1,2\}} = X_{\{0,2,3\}}$, but when we write

$$X_{\{0,1,2,3\}} \xrightarrow{f} X_{\{0,1,2\}}$$

we mean that $f : X_3 \to X_2$ is the image under $X$ of the inclusion $\{0,1,2\} \hookrightarrow \{0,1,2,3\}$. On the other hand, when we write

$$X_{\{0,1,2,3\}} \xrightarrow{g} X_{\{0,2,3\}}$$

we mean that $g : X_3 \to X_2$ is the image under $X$ of the injection $\{0,1,2\} \to \{0,1,2,3\}$ which corresponds to the inclusion $\{0,2,3\} \hookrightarrow \{0,1,2,3\}$

In order to understand wings, we first consider a simpler example which generalizes to our desired point of view

*Example* 2.1.7 (Nerve of a category). Given a (small) category $\mathcal{C}$ we define a simplicial set $N : \mathbf{\Delta} \to \mathbf{Set}$ by:

- $N_0 = \mathrm{ob}(\mathcal{C})$.
- $N_1 =$ the set $\{\bullet \to \bullet\}$ of morphisms in $\mathcal{C}$.
- $N_2 =$ the set $\{\bullet \to \bullet \to \bullet\}$ of composable strings of length 2 in $\mathcal{C}$.
- $N_3 =$ the set $\{\bullet \to \bullet \to \bullet \to \bullet\}$ of composable strings of length 3 in $\mathcal{C}$.

- $N_n$ = the set of composable strings of length $n$ in $\mathcal{C}$.

The $i^{th}$ face map $N_{n+1} \to N_n$ maps each string

$$A_0 \to A_1 \to \cdots \to A_n \to A_{n+1}.$$

to the corresponding string generated by deleting the $i^{th}$ object and composing the relevant morphisms:

$$A_0 \to \cdots \to A_{i-1} \to A_{i+1} \to \cdots \to A_{n+1}.$$

The $i^{th}$ degeneracy map $N_{n-1} \to N_n$ maps each string

$$A_0 \to A_1 \to \cdots \to A_{n-2} \to A_{n-1}$$

to the string generated by duplicating the $i^{th}$ object and adding an identity morphism:

$$A_0 \to \cdots \to A_i \xrightarrow{\mathrm{id}} A_i \to \cdots \to A_{n-1}$$

Since $\mathbf{\Delta}$ is generated by the coface and codegeneracy maps, the rest of the morphisms of $N$ are a composition of these operations. It is, howver, not completely obvious that this is indeed a functor. It becomes easier if we adopt the perspective of remark 2.1.2. In this case we view strings

$$A_0 \to \cdots \to A_n$$

as functors $[n] \to \mathcal{C}$. Thus our functor $N$ sends $[n]$ to $\mathrm{Hom}([n], \mathcal{C})$, and the face and degeneracy maps described above arise as precomposition by the coface and codegeneracy functors. All in all, we have the equality of functors

$$N = \mathrm{Hom}(-, \mathcal{C}) \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$$

and in particular $N$ is a well-defined functor.

The nerve of a category satisfies the following important condition:

**Definition 2.1.8.** A simplicial set is said to be *Segal*, or to satisfy the *Segal condition*, if for each $0 \leq k \leq n$, the commutative diagram

$$\begin{tikzcd} X_{\{0,\ldots,n\}} \arrow[r] \arrow[d] & X_{\{0,\ldots,k\}} \arrow[d] \\ X_{\{k,\ldots,n\}} \arrow[r] & X_{\{k\}} \end{tikzcd} \tag{2.1.9}$$

is a pullback diagram. An equivalent condition for this is that there is a canonical isomorphism

$$X_{\{0,\ldots,n\}} \xrightarrow{\sim} X_{\{0,\ldots,k\}} \times_{X_{\{k\}}} X_{\{k,\ldots,n\}} \tag{2.1.10}$$

where the right-hand side is the usual pullback in the category of sets. The isomorphism should be canonical in the sense that it commutes with the square 2.1.9 in the natural way.

**Proposition 2.1.11.** *For any category $\mathcal{C}$, the nerve $N(\mathcal{C})$ is Segal.*

*Proof.* The left-hand side of the isomorphism 2.1.10 is the set of strings of length $n$, while the right-hand side is the set of pairs of strings $l, m$ of lengths $k$ and $n-k$, where the target object of the first is the source of the second. The desired bijection is given by composing the two strings. □

*Remark* 2.1.12. The following insight, paraphrased from the introduction of [DK19], provides a striking analogy between properties of the Segal condition and what we will later do with the more involved 2-Segal condition. For a Segal simplicial set $X$, by repeatedly applying isomorphisms like 2.1.10, we obtain for each $n$ an isomorphism

$$f_n\colon X_{\{0,\dots,n\}} \xrightarrow{\sim} X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} \times_{X_{\{2\}}} \cdots \times_{X_{\{n-1\}}} X_{\{n-1,n\}}$$

If we take the objects in $X_0$ to be objects of a category, and the objects in $X_1$ to be morphisms between them, then the span

$$X_{\{0,1\}} \times X_{\{1,2\}} \leftarrow X_{\{0,1,2\}} \rightarrow X_{\{0,2\}}$$

can be seen as defining a composition law on the morphisms. The fact that composition of morphisms is associative can be seen as a consequence of the fact that $f_3$ is bijective.

In the next section we will meet the simplicial groupoid $\mathcal{S}_\bullet$, and we will eventually show that it satisfies the the so-called *2-Segal conditions*. Then, the span

$$\mathcal{S}_{\{0,1\}} \times \mathcal{S}_{\{1,2\}} \leftarrow \mathcal{S}_{\{0,1,2\}} \rightarrow \mathcal{S}_{\{0,2\}}$$

will be what defines the multiplication of the Hall Algebra, and the 2-Segal conditions for $n = 3$ will be what shows that the multiplication is associative.

### 2.1.1 Wings as a Simplicial Groupoid

**Definition 2.1.13.** Let $\mathcal{C}$ be a proto-abelian category. Then we denote by $\mathcal{S}_n$ the maximal groupoid in the category of wings of size $n$ in $\mathcal{C}$ as described by Remark 1.4.5. Recall that morphisms between wings in $\mathcal{S}_n$ are natural isomorphisms of functors.

*Remark* 2.1.14. $\mathcal{S}_0$ consists of the zero groupoid. $\mathcal{S}_1$ is (isomorphic to) the maximal groupoid of objects in $\mathcal{C}$. $\mathcal{S}_2$ is the groupoid of short exact sequences in $\mathcal{C}$ where an isomorphism of two sequences $A \hookrightarrow B \twoheadrightarrow C$ and $A' \hookrightarrow B' \twoheadrightarrow C'$ is a triple of isomorphisms $u, v, w$ such that the diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "u"] & B \arrow[r, two heads] \arrow[d, "v"] & C \arrow[d, "w"] \\ A' \arrow[r, hook] & B' \arrow[r, two heads] & C' \end{tikzcd}$$

commutes.

The crucial insight is that the groupoids $\mathcal{S}_\bullet$ together form a simplicial groupoid.

**Theorem 2.1.15.** *There is a simplicial groupoid $\mathcal{S}_\bullet$ where $\mathcal{S}_n$ is the groupoid of wings of size n.*

*Proof.* Let $\nabla$ denote the full subcategory of **Cat** generated by the ordered sets $\langle n \rangle$ as defined in Remark 1.4.5. Let $\mathrm{Hom}^*(\langle n \rangle, \mathcal{C})$ denote the groupoid of wings as defined in Remark 1.4.5. Consider the composition of functors

$$\mathbf{\Delta} \xrightarrow{\quad F \quad} \nabla \xrightarrow{\quad \mathrm{Hom}^*(-,\mathcal{C}) \quad} \mathbf{Grpd}$$

$$\begin{tikzcd} {[n]} \arrow[r] \arrow[d, "f"] & \langle n \rangle \arrow[r] \arrow[d, "Ff\colon (i,j)\mapsto(f(i),f(j))"] & \mathrm{Hom}^*(\langle n \rangle, \mathcal{C}) \\ {[m]} \arrow[r] & \langle m \rangle \arrow[r] & \mathrm{Hom}^*(\langle m \rangle, \mathcal{C}) \arrow[u, "-\circ(Ff)"'] \end{tikzcd}$$

If $f$ is monotone $[n] \to [m]$, then clearly $(i, j) \mapsto (f(i), f(j))$ is monotone $\langle n \rangle \to \langle m \rangle$. Functoriality follows from uniqueness of maps in a poset-category. Thus $F$ is a functor.

To show that $\mathrm{Hom}^*(-, \mathcal{C})$ is a (contravariant) functor, we appeal to the fact that $\mathrm{Hom}(-, \mathcal{C})\colon \mathbf{Cat} \to \mathbf{Grpd}$ is one. It remains to show that for each wing $\mathbf{A} \in \mathrm{Hom}^*(\langle m \rangle, \mathcal{C})$ and each $f \in \mathrm{Hom}_{\mathbf{\Delta}}([n], [m])$, the composition $\mathbf{A} \circ (Ff)$ is a wing (that is, an element of $\mathrm{Hom}^*(\langle n \rangle, \mathcal{C})$). For this we need to show that:

- For each $k$, $(\mathbf{A} \circ (Ff))_{k,k} = 0$. This is true because

$$(\mathbf{A} \circ (Ff))_{k,k} = \mathbf{A}_{f(k),f(k)} = 0$$

.

- For each $k \leq l \leq l'$, the morphism
$$(\mathbf{A} \circ (Ff))_{k,l} \to (\mathbf{A} \circ (Ff))_{k,l'}$$
is mono. But this morphism is the same as the morphism.
$$\mathbf{A}_{f(k),f(l)} \to \mathbf{A}_{f(k),f(l')}$$
which is mono since $\mathbf{A}$ is a wing.
- For each $k \leq k' \leq l$, the morphism
$$(\mathbf{A} \circ (Ff))_{k,l} \to (\mathbf{A} \circ (Ff))_{k',l}$$
is epi. The argument is the same as in the mono case.
- Each $1 \times 1$ - square of $\mathbf{A} \circ (Ff)$ is bicartesian. This is true because the $1 \times 1$-squares of $\mathbf{A} \circ (Ff)$ are $n \times m$-rectangles in $\mathbf{A}$, where $n$ and $m$ are natural numbers (possibly 0). If $n$ and $m$ are not zero the result follows from Corollary 1.4.6. If at least one of them is, the diagram is on the form
$$\begin{tikzcd} a \arrow[r] \arrow[d, "id"] & b \arrow[d, "id"] \\ a \arrow[r] & b \end{tikzcd}$$
but such diagrams are always bicartesian.

Now that we have shown that $F$ and $\mathrm{Hom}^*(-, \mathcal{C})$ are functors, their composition is one. This is the simplicial groupoid $\mathcal{S}_\bullet$. □

One can think of $\mathcal{S}_\bullet$ as a sort of two-dimensional version of the nerve $N$, which is based on two-dimensional posets rather than one-dimensional.

*Remark* 2.1.16. To better understand the structure of $\mathcal{S}_\bullet$, we examine the face and degeneracy maps. The $i$th face map "deletes" the $i$th row and column of the wing, composing the relevant maps. In general, if $X$ is a subset of $\{0, 1, \ldots n\}$, the map
$$\mathcal{S}_{\{0,1,\ldots n\}} \to \mathcal{S}_X$$
is a forgetful functor which for each wing forgets those objects whose coordinates are not both in $X$, and then composes the relevant morphisms.

The $i$th degeneracy map $\sigma_i \colon \mathcal{S}_n \to \mathcal{S}_{n+1}$ is given by replacing the $i$th row with two rows connected by identities, and replacing the $i$th column with two columns connected by identities. For example, the first degeneracy map $\sigma_1 \colon \mathcal{S}_2 \to \mathcal{S}_3$ would map

$$
\begin{tikzcd}
0 \arrow[r, hook] & A_{0,1} \arrow[r, hook] \arrow[d, two heads] & A_{0,2} \arrow[d, two heads] \\
& 0 \arrow[r, hook] & A_{1,2} \arrow[d, two heads] \\
& & 0
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
0 \arrow[r, hook] & A_{0,1} \arrow[r, hook, "\mathrm{id}"] \arrow[d, two heads] & A_{0,1} \arrow[r, hook] \arrow[d, two heads] & A_{0,2} \arrow[d, two heads] \\
& 0 \arrow[r, hook, "\mathrm{id}"] & 0 \arrow[r, hook] \arrow[d, "\mathrm{id}"] & A_{1,2} \arrow[d, "\mathrm{id}"] \\
& & 0 \arrow[r, hook] & A_{1,2} \arrow[d, two heads] \\
& & & 0
\end{tikzcd}
$$

## 2.2 2-Pullbacks and Isofibrations

We are now starting to work within the category **Grpd** of groupoids, and will soon want to take pullbacks.

**Definition 2.2.1.** Given a diagram $\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$ of groupoids, the *pullback* $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is the groupoid with objects given by pairs $(a, b)$ where $a \in \mathcal{A}, b \in \mathcal{B}$ such that $F(a) = G(b)$. A morphism $(a, b) \to (a', b')$ is a pair of morphisms $f \colon a \to a', g \colon b \to b'$ such that $Ff = Gg$

This definition involves an equality of objects $F(a) = G(b)$. This is not desirable. In general we do not want to care if two objects are equal, only if they are isomorphic. For this, we need the more subtle notion of a *2-pullback*.

**Definition 2.2.2.** [Dyc18, p. 20] Given a diagram $\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$ of groupoids, we introduce the *2-pullback* $\mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B}$ to be the groupoid with objects given by triplets $(a, b, \varphi)$, where $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $\varphi \colon F(a) \xrightarrow{\sim} G(b)$. A morphism $(a, b, \varphi) \to (a', b', \varphi')$ is a pair of morphisms $f \colon a \to a', g \colon b \to b'$, such that the diagram

$$
\begin{tikzcd}
F(a) \arrow[r, "\varphi"] \arrow[d, "Ff"] & G(b) \arrow[d, "Gg"] \\
F(a') \arrow[r, "\varphi'"] & G(b')
\end{tikzcd}
$$

commutes.

**Proposition 2.2.3.** *Consider a diagram on the form.*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F'"] \arrow[d, "G'"'] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] \arrow[ur, phantom, "\overset{\eta}{\nearrow}"] & \mathcal{C} \end{tikzcd}$$

*where $\eta\colon FG' \to GF'$ is a natural isomorphism. Then the map*

$$\Phi\colon \mathcal{X} \longrightarrow \mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}$$

$$\begin{tikzcd} x \arrow[r] \arrow[d, "\varphi"] & (G'(x), F'(x), \eta_x) \arrow[d, "(G'\varphi, F'\varphi)"] \\ y \arrow[r] & (G'(y), F'(y), \eta_y) \end{tikzcd}$$

*is a functor.*

*Proof.* That this morphism preserves identities and is functorial follows directly from $F$ and $G$ being functors. It only remains to show that $(G'(x), F'(x), \eta_x)$ is an object of $\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}$, and that $(G'\varphi, F'\varphi)$ is a morphism in $\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}$. For the first of these, we recall that since $\eta$ is a natural isomorphism, $\eta_x$ is an isomorphism $FG'(x) \xrightarrow{\sim} GF'(x)$. For the second, we need the commutativity of the diagram

$$\begin{tikzcd} F(G'(x)) \arrow[r, "\eta_x"] \arrow[d, "FG'\varphi"] & G(F'(x)) \arrow[d, "GF'\varphi"] \\ F(G'(y)) \arrow[r, "\eta_y"] & G(F'(y)) \end{tikzcd}$$

which is precisely the condition for $\eta$ to be a natural transformation. □

**Definition 2.2.4.** In the setting of Proposition 2.2.3, we call the diagram a *2-pullback diagram* if the morphism $\Phi\colon \mathcal{X} \to \mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}$ is an equivalence of categories.

*Remark* 2.2.5. 2-pullbacks may or may not be pullbacks in the usual sense. Conversely, pullbacks are not necessarily 2-pullbacks.

**Definition 2.2.6.** [Dyc18, p. 20] A functor $F\colon \mathcal{A} \to \mathcal{B}$ is called an *isofibration* if for each $a \in \mathcal{A}$ and each isomorphism $\varphi\colon F(a) \xrightarrow{\sim} b$ in $\mathcal{B}$, there is a object $a'$ and an isomorphism $\psi\colon a \xrightarrow{\sim} a'$ such that $F(a') = b$ and $F\psi = \varphi$.

*Remark* 2.2.7. One can think of the property of being an isofibration as being orthogonal in some sense to that of essential surjectivity. If a functor in **Grpd** is essentially surjective and an isofibration, then it is surjective in the sense that every object and morphism has a preimage.

*Remark* 2.2.8. As a rule of thumb, forgetful functors are isofibrations. This is of course an informal statement since forgetful functors are an informal concept. For example, consider the forgetful functor $F\colon \mathbf{Grp} \to \mathbf{Set}$. Then for a group $G$ if we have a bijection of sets $F(G) \xrightarrow{\sim} X$, we can use this bijection to induce a group structure on $X$ isomorphic to that of $G$. This shows that $F$ is an isofibration.

The main reasons for why we are interested in isofibrations are the following propositions, which allow us to use regular pullbacks instead of the more complicated 2-pullbacks:

**Proposition 2.2.9.** [Dyc18, p. 20] *Let*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F'"] \arrow[d, "G'"'] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

*be a pullback diagram of groupoids. If $F$ is an isofibration, then the diagram is 2-pullback in the sense of Definition 2.2.4.*

*Proof.* By uniqueness of pullbacks, we assume without loss of generality that $\mathcal{X} = \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$. Since $FG' = GF'$ there is the identity natural isomorphism between them. As in Proposition 2.2.3, we obtain a functor

$$\Phi\colon \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \longrightarrow \mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B}$$

$$\begin{tikzcd} (a,b) \arrow[r] \arrow[d, "(f,g)"] & (a,b,\mathrm{id}_{F(a)}) \arrow[d, "(f,g)"] \\ (x,y) \arrow[r] & (x,y,\mathrm{id}_{F(x)}) \end{tikzcd}$$

We claim this functor is an equivalence. Clearly it is full and faithful, since maps $(a,b) \to (x,y)$ and maps $(a,b,\mathrm{id}) \to (x,y,\mathrm{id})$ are the same thing, and $\Phi$ acts as the identity on morphisms. It remains to show that $\Phi$ is essentially surjective. Let $(a,b,\varphi) \in \mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B}$ be arbitrary. Since $F$ is an isofibration there is a $a'$ in $\mathcal{A}$ and an isomorphism $\psi\colon a \xrightarrow{\sim} a'$ such that $F(a') = G(b)$ and $F\psi = \varphi$. Then $(a',b,\mathrm{id})$ is an object in $\mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B}$, which is in the image of $\Psi$. We have an isomorphism $(\psi, \mathrm{id}_b)\colon (a,b,\varphi) \xrightarrow{\sim} (a',b,\mathrm{id})$ since the diagram

$$\begin{tikzcd} F(a) \arrow[r, "\varphi"] \arrow[d, "F\psi"] & G(b) \arrow[d, "\mathrm{id}"] \\ F(a') \arrow[r, "\mathrm{id}"] & G(b) \end{tikzcd}$$

commutes. The isomorphism has inverse $(\psi^{-1}, \mathrm{id}_b)$. This shows that $\Psi$ is essentially surjective, finishing the proof. □

**Proposition 2.2.10.** *Let*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F'"] \arrow[d, "G'"'] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

*be a commutative square of groupoids, where $F$ is an isofibration. Suppose further that the functor*

$$\Psi\colon \mathcal{X} \longrightarrow \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$$

$$\begin{tikzcd} x \arrow[r] \arrow[d, "\varphi"] & (G'(x), F'(x)) \arrow[d, "(G'\varphi, F'\varphi)"] \\ y \arrow[r] & (G'(y), F'(y)) \end{tikzcd}$$

*is an equivalence. Then the square is a $2-$pullback square.*

*Proof.* There is a composition of functors

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Psi"] & \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \arrow[r, "\Phi"] & \mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B} \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r] \arrow[d, "f"] & (G'(x), F'(x)) \arrow[r] \arrow[d, "(G'f, F'f)"] & (G'(x), F'(x), \mathrm{id}_{FG'(x)}) \arrow[d, "(G'f, F'f)"'] \\ y \arrow[r] & (G'(y), F'(y)) \arrow[r] & (G'(y), F'(y), \mathrm{id}_{FG'(y)}) \end{tikzcd}$$

where $\Phi$ is from the proof of Proposition 2.2.9. We have shown that $\Phi$ is an equivalence, and $\Psi$ is an equivalence by assumption. Since the composition of equivalences is an equivalence, our initial commutative square is a 2-pullback square by Definition 2.2.4 □

To use the prevoius lemmas we need to know maps that are isofibrations. Luckily many of the maps in $\mathcal{S}_\bullet$ are, as can be seen in the following useful proposition:

**Proposition 2.2.11.** *Let $\iota\colon [m] \hookrightarrow [n]$ be an injective map in $\Delta$. Then the morphism $F\colon \mathcal{S}_n \to \mathcal{S}_m$ given by $\iota$ is an isofibration.*

*Remark* 2.2.12. We should not be surprised that these maps are isofibrations, since one can see them as forgetful functors as in Remark 2.1.16.

*Proof.* Recall that for each $A\colon \langle n\rangle \to \mathcal{C}$ in $\mathcal{S}_n$, by definition we have $F(A)_{i,j} = A_{\iota(i),\iota(j)}$. Let $B\colon \langle m\rangle \to \mathcal{C}$ be an object in $\mathcal{S}_m$, and $\varphi\colon F(A) \xrightarrow{\sim} B$ be an isomorphism in $\mathcal{S}_m$. Our aim is to show that there is a wing $\overline{B} \in \mathcal{S}_n$ and an isomorphism $\Phi\colon A \to \overline{B}$ such that $F(\overline{B}) = B$ and $F(\Phi) = \varphi$. Let $X = \operatorname{im}\iota$. Since $\iota$ is injective, there is a function $r\colon X \to [m]$ such that $r \circ \iota = \mathrm{id}_{[m]}$.

Consider the components $\varphi_{i,j}\colon F(A)_{i,j} = A_{\iota(i),\iota(j)} \to B_{i,j}$. For simplicity, we will for the rest of this proof use the following abuse of notation: We re-index $\varphi$ and $B$ in such a way that when we write $\varphi_{i,j}$ we mean $\varphi_{r(i),r(j)}$, and when we write $B_{i,j}$ we mean $B_{r(i),r(j)}$. This way, we have morphisms $\varphi_{i,j}\colon A_{i,j} \to B_{i,j}$. If $i$ or $j$ are not elements of $X$, then the terms $\varphi_{i,j}$ and $B_{i,j}$ are left undefined. Essentially what this abuse of notation does is view $B$ as a subwing of a wing of size $n$, taking up the coordinates determined by $X$.

Now we define $\overline{B}$:

$$\overline{B}_{i,j} = \begin{cases} A_{i,j} & \text{if } i \notin X \text{ or } j \notin X \\ B_{i,j} & \text{otherwise.} \end{cases}$$

We now need to describe the morphisms in $\overline{B}$. Notice that the data $(A, B, \varphi\colon F(A) \xrightarrow{\sim} B)$ gives us several different types of morphisms. For each $k, k', l, l' \in [n]$ where $(k \le k'), (l \le l')$, we have

- A unique morphism $A_{k,l} \to A_{k',l'}$ from $A$
- If $k, k', l, l' \in X$, a unique morphism $B_{k,l} \to B_{k',l'}$ from $B$
- If $k, l \in X$, two unique inverse isomorphisms

  $$A_{k,l} \xrightarrow{\varphi_{k,l}} B_{k,l} \quad \text{and} \quad B_{k,l} \xrightarrow{\varphi_{k,l}^{-1}} A_{k,l}$$

  from $\varphi$.

That $\varphi$ is a natural transformation tells us that composing morphisms of the different types is done uniquely, in the sense that each choice of domain and codomain uniquely determines the morphism between them, if one exists. Explicitly, if $k', l' \in X$, we have a unique morphism

$$A_{k,l} \to B_{k',l'}$$

and if $k, l \in X$ we have a unique morphism

$$B_{k,l} \to A_{k',l'}.$$

We also have unique morphisms between objects in $A$, even if we allow for arbitrary compositions of the three types of morphisms given to us by $(A, B, \varphi)$. The same is true for unique morphisms between objects in $B$. Letting all these morphisms be the morphisms of $\overline{B}$, we obtain that $\overline{B}$ is a commutative diagram. We also obtain that if we define $\Phi$ as

$$\Phi_{i,j} = \begin{cases} \mathrm{id}_{A_{i,j}} & \text{if } i \notin X \text{ or } j \notin X \\ \varphi_{i,j} & \text{otherwise,} \end{cases}$$

then $\Phi$ is a natural isomorphism. This is because each component of $\Phi$ is an isomorphism, and for each diagram on the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & Y \end{tikzcd}$$

which $\Phi$ needs to make commute, there is only one morphism $X \to Y$ which can be obtained from our data, so both paths in the diagram must represent this morphism. Also note that when $i, j \in X$, then $\Phi_{i,j} = \phi_{i,j}$, so $F(\Phi) = \phi$ as desired. It remains to show that $\overline{B}$ satisfies the rest of the conditions for a wing. We need to show that:

- For any $k$, the object $\overline{B}_{k,k}$ is 0. This is true since $\overline{B}_{k,k}$ is either $A_{k,k}$ or $B_{k,k}$, and both of these are 0.
- For any $k \leq l \leq l'$, the morphism

  $$\overline{B}_{k,l} \to \overline{B}_{k,l'}$$

  is mono. This is true because it can be written as a composite of (some of) the following:

  1. Morphisms $A_{k,i} \hookrightarrow A_{k,j}$ for some $i \leq j$, which are all mono.

2. Morphisms $B_{k,i} \hookrightarrow B_{k,j}$ for some $i \leq j$ (if defined). These are also all mono.
3. Isomorphisms $A_{k,i} \xrightarrow{\sim} B_{k,i}$ for some $i$ (if defined).

Regardless of how the composition looks, it will be mono.

- For any $k \leq k' \leq l$, the morphism

$$\overline{B}_{k,l} \to \overline{B}_{k',l}$$

is epi. The reasoning is the same as in the mono case.

- For each $k \leq k'$ and $l \leq l'$, the diagram

$$\begin{tikzcd} \overline{B}_{k,l} \arrow[r, hook] \arrow[d, two heads] & \overline{B}_{k,l'} \arrow[d, two heads] \\ \overline{B}_{k',l} \arrow[r, hook] & \overline{B}_{k',l'} \end{tikzcd}$$

is bicartesian. This follows because $\Phi$ induces a natural isomorphism between this square and the corresponding square in $A$, and natural isomorphisms of pushout/pullback squares are pushout/pullback. □

*Remark* 2.2.13. This proof strategy actually obtains something stronger: Namely that for any $i, j$ not both in $X$, the object $\overline{B}_{i,j} = A_{i,j}$, and the isomorphism $\Phi \colon A \to \overline{B}$ induces the identity on these objects. This will come in use in the proof of 2.3.5.

## 2.3 The 2-Segal condition

We will now introduce the 2-segal condition, which was introduced by Dyckerhoff and Kapranov in [DK12]. The exposition is based on that in [Dyc18].

Let $\mathcal{X}$ be a simplicial groupoid. Consider a $(n+1)$-gon with vertices labelled by $0, 1, \ldots, n$. Let $i < j$ be a diagonal of $P$. This subdivides the polygon into two polygons, with labels $\{0, 1 \ldots i, j, j+1, \ldots n\}$ and $\{i, i+1, \ldots j\}$, respectively. This subdivision corresponds to the following commutative square:

$$\begin{tikzcd} X_{\{0,\ldots,n\}} \arrow[r] \arrow[d] & X_{\{0,\ldots,i,j,\ldots n\}} \arrow[d] \\ X_{\{i,\ldots,j\}} \arrow[r] & X_{\{i,j\}} \end{tikzcd} \tag{2.3.1}$$

Also, for each $0 \leq i < n$ we have a commutative square

$$\begin{tikzcd} X_{\{0,\dots,n-1\}} \arrow[r] \arrow[d, "\sigma_i"] & X_{\{i\}} \arrow[d] \\ X_{\{0,1,\dots,n\}} \arrow[r] & X_{\{i,i+1\}} \end{tikzcd} \tag{2.3.2}$$

where $\sigma_i$ denotes the $i$th degeneracy map.

*Remark* 2.3.3. Notice that diagonals in a $(n+1)$-gon with vertices labelled by $0, 1, \dots n$ correspond naturally to the objects of a wing that are not the zeroes on the diagonal. Both these sets are indexed by unordered pairs of distinct integers from 0 to $n$ (or, equivalently, ordered pairs where the smallest value is first).

**Definition 2.3.4.** [Dyc18, Def. 3.5]
A simplicial groupoid $\mathcal{X}$ is called *2-Segal* if:

a) For each polygon $P$ with vertices labelled by $0, 1, \dots, n$, and each diagonal $i < j$ of $P$, the corresponding square on the form of diagram 2.3 is 2-pullback.

b) For each $0 \leq i \leq n$, the corresponding square on the from of diagram 2.3.2 is 2-pullback.

The main result about these conditions is that $\mathcal{S}_\bullet$ satisfies them. But first, we should take a look at what that might look like. Consider the polygonal subdivision of the pentagon $\{0, 1, 2, 3, 4\}$ by the diagonal $\{1, 3\}$. The 2-Segal conditions tell us to consider the commutative diagram

$$\begin{tikzcd} \mathcal{S}_{\{0,1,2,3,4\}} \arrow[r] \arrow[d] & \mathcal{S}_{\{0,1,3,4\}} \arrow[d] \\ \mathcal{S}_{\{1,2,3\}} \arrow[r] & \mathcal{S}_{\{1,3\}} \end{tikzcd}$$

The way this commutative diagram acts on the level of objects can be seen in Fig. 2.1. The blue, dashed outline denotes the parts of the diagram which belong to the $\mathcal{S}_{\{0,1,3,4\}}$-component, and the red, solid outline denotes the part of the diagram which belongs to the $\mathcal{S}_{\{1,2,3\}}$-component. The statement that $\mathcal{S}_\bullet$ is 2-Segal can be interpreted as the statement that the these two outlined components are enough to reconstruct the original wing in $\mathcal{S}_4$ in a sufficiently unique way that this yields an equivalence of categories.

**Theorem 2.3.5.** [Dyc18, Thm. 3.7] *Let $\mathcal{C}$ be a proto-abelian category. Then the simplicial groupoid $\mathcal{S}_\bullet$ of groupoids of wings in $\mathcal{C}$ is 2-Segal.*

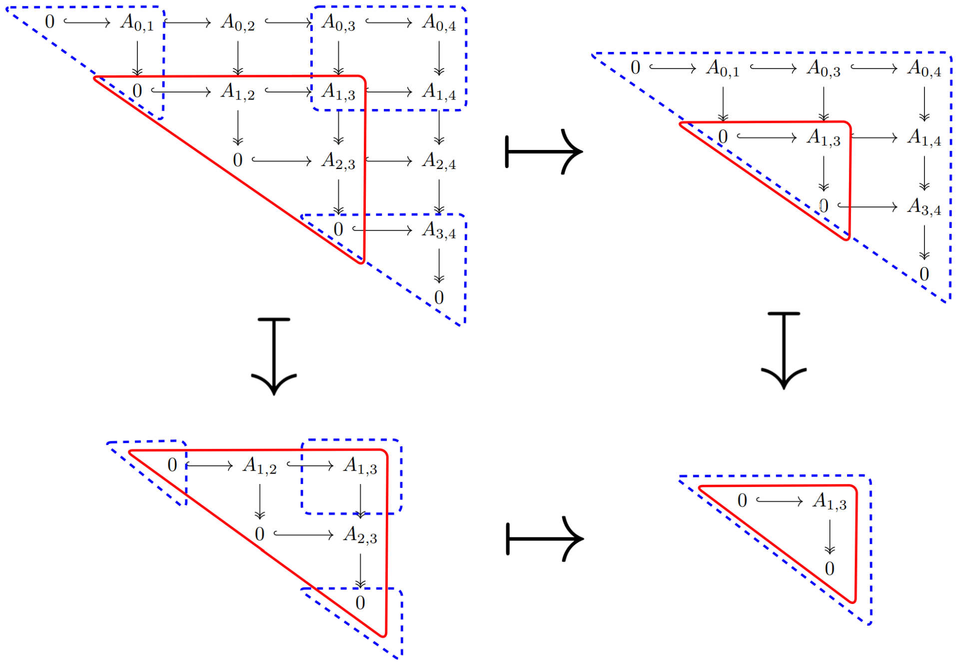



Figure 2.1: According to the 2-segal condition, this square is 2-pullback.

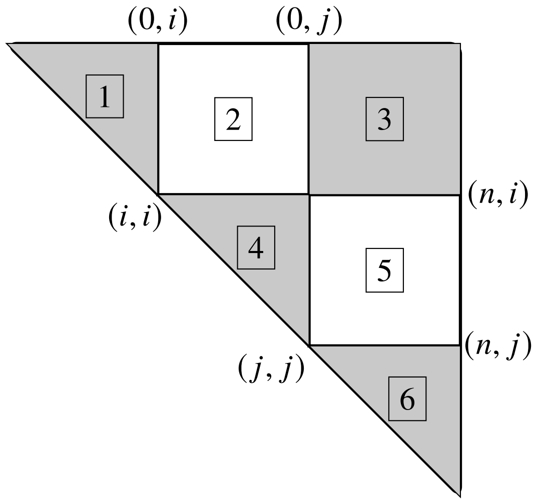
This following proof will be more of a sketch than other proofs in this thesis. This is because a fully rigorous proof would be very long and, in the author's opinion, the amount of casework and notation necessary would serve to obscure the comparative simplicity of the ideas involved. More rigorous proofs using similar, oftentimes identical ideas can be found in Proposition 2.2.11 and Theorem 1.4.7. The author hopes that the reader will be convinced that this argument can be made rigorous at the cost of length and readability. The proof is based on the sketch found in [Dyc18].

*Proof of 2-Segal condition* a). Let $i < j$ be a diagonal in a polygon labelled by $\{0, 1, \ldots, n\}$. The maps in diagram 2.3 are isofibrations, so in light of Proposition 2.2.9 we work with 1-pullbacks. We want to check that the canonical functor

$$\Psi\colon \mathcal{S}_{\{0,\ldots,n\}} \to \mathcal{S}_{\{0,\ldots i,j,\ldots,n\}} \times_{\mathcal{S}_{\{i,j\}}} \mathcal{S}_{\{i,i+1,\ldots,j\}} \qquad (2.3.6)$$

is an equivalence. The objects of the right-hand side consist of pairs ow wings $(B, C)$ which agree on a certain subwing. Throughout this proof, we will use an abuse of notation similar to that in Proposition 2.2.11. We re-index $B$ and $C$ so as to represent subwings of a wing in $\mathcal{S}_n$ in the natural way. One might say that $B \in \mathcal{S}_{\{0,1,\dots,i,j,\dots,n\}}$, and that '$C \in \mathcal{S}_{\{i,\dots,j\}}$. In other words, the indices of an object in $B$ must be from the set $\{0,\dots,i,j,\dots,n\}$, and the indices of an object in $C$ must be from the set $\{i, i+1,\dots,j\}$. This allows us to write equations like $A_{i,j} = B_{i,j}$, which would be comparatively tricky if we did not do this re-indexing.

Our goal is to construct a pseudoinverse $\Phi$ to $\Psi$, so it will do us good to understand what $\Psi$ does to a wing. It is forgetful functor which, to each wing $C$ remembers the objects whose indices are either both in $\{0, ...i, j,\dots,n\}$ or both in $\{i, i+1, ..., j\}$. This means that $\Psi$ forgets exactly those objects whose indices correspond to diagonals of the polygon $\{0,1,\dots,n\}$ that cross the diagonal $\{i,j\}$. One of the main ideas here will be that none of the diagonals containing $i$ or $j$ cross the diagonal $i,j$. Hence every object in $A$ where either of the coordinates is $i$ or $j$ is remembered by $\Psi$. Consider the following illustration:




Fix wings $A, B, C$ such that $\Psi(A) = (B, C)$. The illustration shows the different parts of $A$ as $\Psi$ is applied to it. The solid lines denote the rows and columns $i$ and $j$. The wing is subdivided into areas (1) through (6), and are affected by $\Psi$ as follows:

- Areas (1), (3) and (6) are remembered by $B$.
- Area (4) is remembered by $C$
- Areas (2) and (5) are forgotten by $\Psi$.

The convention is that shaded areas are remembered, and non-shaded areas are forgotten. There is another important convention: The shaded areas contain their boundaries to other areas, **but the non-shaded areas do not**. For example, for some $0 \leq k < i$, the object $A_{k,i}$ is in the shaded area (1), but not in the area (2). Also notice that $A_{i,i}$, $A_{i,j}$, and $A_{j,j}$ each belong to two different gray areas. This reflects how these objects are remembered by both $B$ and $C$. In fact, an arbitrary object $(B', C')$ in $\mathcal{S}_{\{0,\ldots i,j,\ldots,n\}} \times_{\mathcal{S}_{\{i,j\}}} \mathcal{S}_{\{i,i+1,\ldots,j\}}$ can be viewed as comprising the shaded areas of the illustration, and the fiber product over $\mathcal{S}_{\{i,j\}}$ represents that the shaded areas $B'$ and $C'$ agree on their intersection.

Another thing to notice is that the shaded areas are never empty, but the non-shaded ones might be (for example, if $i = 0$, $j = n$).

Our task is now to show that one can reconstruct an entire wing of size $n$ from the shaded areas in the illustration, and that this can be done in an essentially unique way. Let $(B, C)$ be an object of $\mathcal{S}_{\{0,\ldots i,j,\ldots,n\}} \times_{\mathcal{S}_{\{i,j\}}} \mathcal{S}_{\{i,i+1,\ldots,j\}}$, and keep in mind that this represents specifying the shaded areas of the illustration. We use Theorem 1.4.7 to obtain a wing $Q$ in $\mathcal{S}_{\{i,i+1,\ldots,n\}}$ with row $i$ as its top row. We use Proposition 2.2.11 to choose such a $Q$ which agrees with $C$ on area (4). In fact, in light of Remark 2.2.13 we may choose to generate a wing $Q \in \mathcal{S}_{\{i,i+1,\ldots,n\}}$ such that $Q$ agrees with the datum $(B, C)$ on areas (4), (6) and row $i$ (to see this, use Proposition 2.2.11 twice on areas (4) and (6) in turn, keeping in mind Remark 2.2.13). Such a wing is unique up to unique isomorphism which induces the identity on areas (4),(6) and row $i$. Now do the same with column $j$, using Theorem 1.4.9 and Remark 2.2.13 to generate a wing $P$ in $\mathcal{S}_{\{1,\ldots,j\}}$ which agrees with the datum $(B, C)$ on areas (1), (4), and column $j$. This is also unique up to a unique isomorphism inducing the identity on the appropriate areas.

Now we claim that if we paste together $P, Q$, and the area (3) (which is included in the wing $B$), we obtain a wing in $\mathcal{S}_n$. This diagram will be commutative because pasting together commutative diagrams yields commutative diagrams. That the correct morphisms are mono and epi, and that each $1 \times 1$-square is bicartesian follows immediately from $P, Q$ being wings and the area (3) being part of a wing. We let $\Phi((B, C))$ be an (arbitrarily chosen) such pasted together diagram.

We need to describe how $\Psi$ acts on morphisms. We claim that each natural isomorphism which is already defined on the shaded areas of the illustration can be extended to one that is defined also in the non-shaded ones. We do this analogously to the proof of Theorem 1.4.7, using that each object in area (5) is a pushout of two objects in row $i$ along a 0 object in the main diagonal. Just like in said proof, the universal properties of pushouts and pullbacks tell us that each component of our extension exists, is unique, and that the resulting set of maps is a natural isomorphism. Now, given a map $f\colon (B, C) \to (B', C')$, we define $\Psi(f)$ to be the unique morphism we obtain by extending this way. That this operation is unital and functorial follows from the uniqueness of this construction

Now we need to show that $\Phi$ and $\Psi$ are mutually pseudoinverse. It is clear that

$$\Psi \circ \Phi = \mathbb{1}_{\mathcal{S}_{\{0,\ldots i,j,\ldots,n\}} \times_{\mathcal{S}_{\{i,j\}}} \mathcal{S}_{\{i,i+1,\ldots,j\}}}$$

So we need to construct a natural isomorphism

$$\eta\colon \mathbb{1}_{\mathcal{S}_{\{1,\ldots,n\}}} \xrightarrow{\sim} \Phi \circ \Psi.$$

For each $A \in \mathcal{S}_{\{1,\ldots,n\}}$, $(\Phi \circ \Psi)(A)$ is isomorphic to $A$ up to a unique isomorphism inducing the identity on the shaded areas. This isomorphism can be constructed by extending the morphism $\mathrm{id}\colon \Psi(A) \to \Psi(A)$ to a (unique) morphism $A \to \Phi(\Psi(A))$. We take this morphism to be the component of $\eta$ at $A$. Let $f\colon A \to A'$ be a morphism in $\mathcal{S}_{\{1,\ldots,n\}}$. Then we need to show that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"] & A' \arrow[d, "\eta_{A'}"'] \\
\Phi(\Psi(A)) \arrow[r] & \Phi(\Psi(A'))
\end{tikzcd}$$

We view this diagram as illustrating two different morphisms $\Phi(\Psi(A)) \to \Phi(\Psi(A'))$, which we need to show are equal. But if we restrict both these morphisms to the grey areas of the illustration, we obtain in both cases the map

$$\Psi(f)\colon \Psi(A) \to \Psi(A')$$

As discussed above, this map can only be extended to a map $\Phi(\Psi(A)) \to \Phi(\Psi(A'))$ in one way, so the diagram in question commutes, and $\eta$ is a natural isomorphism, as desired. $\square$

*Proof of 2-segal condition* b). We need to show that for each pair of integers $0 \le i \le n$ the diagram

$$\begin{tikzcd}
\mathcal{S}_{\{0,\ldots,n-1\}} \arrow[r] \arrow[d, "\sigma_i"] & \mathcal{S}_{\{i\}} \arrow[d] \\
\mathcal{S}_{\{0,1,\ldots,n\}} \arrow[r] & \mathcal{S}_{\{i,i+1\}}
\end{tikzcd}$$

is 2-pullback. Once again, in light of Proposition 2.2.10 we can work with normal pullbacks instead. Since $\mathcal{S}_{\{i\}} = \mathcal{S}_0$ is the trivial groupoid, the category $\mathcal{S}_{\{1,2,\ldots,n\}} \times_{\mathcal{S}_{\{i,i+1\}}} \mathcal{S}_{\{i\}}$ is isomorphic to the category given by the wings $A \in \mathcal{S}_n$ with $A_{i,i+1} = 0$. Denote this category by $\mathcal{S}_n^0$. We claim that the condition $A_{i,i+1} = 0$ implies that $A$ is isomorphic to a wing in the image of the degeneracy map $\sigma_i$, so that the map $\sigma_i \colon \mathcal{S}_{n-1} \to \mathcal{S}_n^0$ is essentially surjective. Indeed, for each $k \leq i$, we have that the commutative square

$$\begin{tikzcd} A_{k,i} \arrow[r, hook, "\iota_k"] \arrow[d, two heads] & A_{k,i+1} \arrow[d, two heads] \\ 0 \arrow[r, hook] & 0 \end{tikzcd}$$

is bicartesian, and in particular pullback. This means that $\iota_k \colon A_{k,i} \to A_{k,i+1}$ is an isomorphism. Let $A'$ be the wing obtained by replacing each object $A_{k,i}$ with $A_{k,i+1}$, and composing the relevant morphisms. It is clear that $A'$ is indeed a wing. Then we claim that the map $A \to A'$ given by $\iota_k$ on the objects $A_{k,i}$ and identity otherwise is a natural isomorphism. This follows from the commutativity of the diagram

$$\begin{tikzcd} A_\bullet^{(1)} \arrow[r] \arrow[d, "\mathrm{id}"'] & A_{k,i} \arrow[r] \arrow[d, "\iota_k"] & A_\bullet^{(2)} \arrow[d, "\mathrm{id}"'] \\ A_\bullet^{(1)} \arrow[r] & A_{k,i+1} \arrow[r] & A_\bullet^{(2)} \end{tikzcd}$$

where $A_\bullet^{(1)}$ and $A_\bullet^{(2)}$ denote arbitrary objects in $A$, and the unlabelled morphisms are uniquely determined since $A$ is a commutative diagram.

We apply the same argument to row $i$ (as opposed to column $i$), and obtain that $A'$ is isomorphic to a diagram $A''$, where for each $l \geq i$, we have $A_{i,l} = A_{i+1,l}$, and that the morphism between them is the identity. Then $A''$ is in the image of $\sigma_i$, so that $\sigma_i$ is essentially surjective. We show that it is full and faithful. For $X, Y \in \mathcal{S}_{n-1}$, the function

$$\sigma_i \colon \operatorname{Hom}_{\mathcal{S}_{n-1}}(X, Y) \to \operatorname{Hom}_{\mathcal{S}_n^0}(\sigma_i(X), \sigma_i(Y))$$

acts by duplicating the components at row and column $i$. For simplicity, write $\sigma_i(X) = A, \sigma_i(Y) = B$. Let $f$ be a morphism $A \to B$. Then for each $k \leq i$, we have that the diagram

$$\begin{tikzcd} A_{k,i} \arrow[r, "\mathrm{id}"] \arrow[d, "f_{k,i}"'] & A_{k,i+1} \arrow[d, "f_{k,i+1}"] \\ B_{k,i} \arrow[r, "\mathrm{id}"] & B_{k,i+1} \end{tikzcd}$$

commutes. But this means that $f_{k,i} = f_{k,i+1}$. Similarly, for each $l \geq i$, we have $f_{i,l} = f_{i+1,l}$. But then it is clear that $\sigma_i$ is surjective onto $\operatorname{Hom}_{\mathcal{S}_n^0}(\sigma_i(X), \sigma_i(Y))$, and it is clearly injective. Hence the functor

$$\sigma_i : \mathcal{S}_{n-1} \to \mathcal{S}_n^0 \cong \mathcal{S}_{\{1,2,\ldots,n\}} \times_{\mathcal{S}_{\{i,i+1\}}} \mathcal{S}_{\{i\}}$$

is an equivalence. We have shown that $\mathcal{S}_\bullet$ satisfies the second part of the 2-Segal condition. □

## 2.4 The Abstract Hall Algebra

### 2.4.1 Spans of Groupoids

The category Span(**Grpd**) of *spans in groupoids* is defined as follows: The objects in Span(**Grpd**) are groupoids. For groupoids $\mathcal{A}, \mathcal{B}$ we define

$$\mathrm{Hom}(\mathcal{A}, \mathcal{B}) = \{\mathcal{A} \leftarrow \mathcal{X} \rightarrow \mathcal{B}\}/\sim$$

In other words, morphisms are equivalence classes of diagrams on the form $\mathcal{A} \leftarrow \mathcal{X} \rightarrow \mathcal{B}$ modulo an equivalence relation. The relation in question is this: two spans $\mathcal{A} \leftarrow \mathcal{X} \rightarrow \mathcal{B}$ and $\mathcal{A} \leftarrow \mathcal{X}' \rightarrow \mathcal{B}$ are considered equivalent if there exists a diagram

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "P_1"'] \arrow[dr, "P_2"] \arrow[dd, "F"] & \\ \mathcal{A} \arrow[r, phantom, "\Uparrow"] & \phantom{x} \arrow[r, phantom, "\Uparrow"] & \mathcal{B} \\ & \mathcal{X}' \arrow[ul, "Q_1"] \arrow[ur, "Q_2"'] & \end{tikzcd} \tag{2.4.1}$$

where $F$ is an equivalence and the double arrows denote natural isomorphisms of functors.

**Proposition 2.4.2.** *The above relation is indeed an equivalence relation.*

*Proof.* Reflexivity is clear, both $F$ and the two natural transformations are identities. Now consider symmetry. Since $F$ is an equivalence, it has an inverse $F^{-1}$ such that $FF^{-1} \simeq \mathbb{1}_{\mathcal{X}'}$. We claim that replacing $F$ in diagram 4 by $F^{-1}$ works. Indeed,

$$Q_1 F \simeq P_1 \implies Q_1 F F^{-1} \simeq P_1 F^{-1} \implies Q_1 \simeq P_1 F^{-1}$$

and by symmetry $Q_2 \simeq P_2 F^{-1}$, as desired. For transitivity, consider the following diagram

$$\begin{tikzcd}
 & \mathcal{X} \arrow[dl, "P_1"'] \arrow[d, "F"] \arrow[dr, "P_2"] & \\
\mathcal{A} \arrow[ur, phantom, "\Leftarrow" near end] & \mathcal{X}' \arrow[l, "Q_1"'] \arrow[r, "Q_2"] \arrow[d, "G"] & \mathcal{B} \arrow[ul, phantom, "\Rightarrow" near end] \\
 & \mathcal{X}'' \arrow[ul, "R_1"] \arrow[ur, "R_2"'] \arrow[ul, phantom, "\Leftarrow" near end] \arrow[ur, phantom, "\Rightarrow" near end] &
\end{tikzcd}$$

where $R_1GF \simeq Q_1F \simeq P_1$ and symmetrically $R_2GF \simeq P_2$ □

We still need to describe composition in this category. Given morphisms $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ we represent them as spans $\mathcal{A} \leftarrow \mathcal{X} \to \mathcal{B}$ and $\mathcal{B} \leftarrow \mathcal{Y} \to \mathcal{C}$ and form the 2-pullback:

$$\begin{tikzcd}
 & & \mathcal{Z} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{X} \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\Longrightarrow"] & & \mathcal{Y} \arrow[dl] \arrow[dr] & \\
\mathcal{A} & & \mathcal{B} & & \mathcal{C}
\end{tikzcd}$$

where the upper square is a 2-pullback square. The composite morphism $g \circ f : \mathcal{A} \to \mathcal{C}$ is the span $\mathcal{A} \leftarrow \mathcal{Z} \to \mathcal{C}$.

**Proposition 2.4.3.** *This is a well-defined operation. Explicitly, the choices of equivalence class representatives $\mathcal{A} \leftarrow \mathcal{X} \to \mathcal{B}$ and $\mathcal{B} \leftarrow \mathcal{Y} \to \mathcal{C}$ do not matter, and neither does the choice of 2-pullback.*

*Proof.* We first consider the statement that the choice of 2-pullback doesn't matter. We note that in fact the following stronger statement is true: That for every diagram on the form

$$\begin{tikzcd}
 & \mathcal{X}' \arrow[d, "F"'] & \\
 & \mathcal{X} \arrow[dl, "P"] \arrow[dr, "Q"'] & \\
\mathcal{A} & & \mathcal{B}
\end{tikzcd}$$

where F is an equivalence of groupoids, the diagram

$$\begin{tikzcd}
 & \mathcal{X} \arrow[dl, "P"'] \arrow[dr, "Q"] \arrow[dd, "F"] & \\
\mathcal{A} \arrow[r, phantom, "\Uparrow", near end] & {} \arrow[r, phantom, "\Uparrow", near start] & \mathcal{B} \\
 & \mathcal{X}' \arrow[ul, "PF^{-1}"] \arrow[ur, "QF^{-1}"'] &
\end{tikzcd}$$

is an equivalence of spans. The natural isomorphism $PF^{-1}F \xrightarrow{\sim} P$ is given by $P\eta$, where $\eta$ denotes the natural isomorphism $F^{-1}F \to \mathbb{1}_{\mathcal{X}'}$. The situation is symmetric for the natural isomorphism $QF^{-1}F \xrightarrow{\sim} P$.

We now need to show that the choice of representatives does not matter. Note that it suffices to show this in the case where only one choice representative is changed. Explicitly, we want to show that in the situation

$$\begin{tikzcd}
 & & \mathcal{Z} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{X} \arrow[dl] \arrow[dr] \arrow[dd] \arrow[rr, phantom, "\Longrightarrow"] & & \mathcal{Y} \arrow[dl] \arrow[dr] \arrow[dd, equal] & \\
\mathcal{A} \arrow[r, phantom, "\Uparrow", near end] & {} \arrow[r, phantom, "\Uparrow", near start] & \mathcal{B} & & \mathcal{C} \\
 & \mathcal{X}' \arrow[ul] \arrow[ur] \arrow[rr, phantom, "\Longrightarrow"] & & \mathcal{Y} \arrow[ul] \arrow[ur] & \\
 & & \mathcal{Z}' \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

The two composites $\mathcal{A} \leftarrow \mathcal{Z} \to \mathcal{C}$ and $\mathcal{A} \leftarrow \mathcal{Z}' \to \mathcal{C}$ are equivalent. According to the previous part of the proof, we may ourselves choose 2-pullbacks $\mathcal{Z}, \mathcal{Z}'$. We identify both with $\mathcal{X} \times_{\mathcal{B}}^{(2)} \mathcal{Y}$ as in the following diagram:

$$
\begin{tikzcd}
 & \mathcal{X} \times^{(2)}_{\mathcal{B}} \mathcal{Y} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathcal{X} \arrow[dr, "P"] \arrow[dd, "F"'] \arrow[rr, phantom, "\overset{\varphi}{\Longrightarrow}"] \arrow[dd, phantom, "\alpha\Nearrow" right, shift left=3ex] & & \mathcal{Y} \arrow[dl, "Q"'] \arrow[dd, equal] \\
 & \mathcal{B} & \\
\mathcal{X}' \arrow[ur, "P'"] \arrow[rr, phantom, "\overset{\varphi\alpha}{\Longrightarrow}"] & & \mathcal{Y} \arrow[ul, "Q"'] \\
 & \mathcal{X} \times^{(2)}_{\mathcal{B}} \mathcal{Y} \arrow[ul, "F\pi_1"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}
$$

For this to work, we need that the diagram

$$
\begin{tikzcd}
\mathcal{X} \times^{(2)}_{\mathcal{B}} \mathcal{Y} \arrow[r, "F\pi_1"] \arrow[d, "\pi_2"'] & Y \arrow[d, "Q"] \\
\mathcal{X}' \arrow[r, "P'"'] \arrow[ur, phantom, "\overset{\varphi\alpha}{\Nearrow}"] & \mathcal{B}
\end{tikzcd}
$$

is a 2-pullback diagram. In other words, we need that the functor

$$
\begin{aligned}
\Phi\colon \mathcal{X} \times^{(2)}_{\mathcal{B}} \mathcal{Y} &\to \mathcal{X}' \times^{(2)}_{\mathcal{B}} \mathcal{Y} \\
(x, y, \varphi_x\colon Px \to Qy) &\mapsto (Fx, y, \varphi_x \alpha_x\colon P'Fx \to Qy) \\
(f, g) &\mapsto (Ff, g)
\end{aligned}
$$

is an equivalence. We show that it is fully faithful and essentially surjective. For essential surjectivity, let $(x', y, \varphi') \in \mathcal{X}' \times^{(2)}_{\mathcal{B}} \mathcal{Y}$. By essential surjectivity of $F$, there is an $x \in \mathcal{X}$ with an isomorphism $h\colon Fx \to x'$. Now consider the diagram

$$
\begin{tikzcd}
Px \arrow[r, "\varphi' P' h (\alpha_x)^{-1}"] & Qy \arrow[d, equal] \\
P'Fx \arrow[u, "\alpha_x"] \arrow[r, "\varphi' P' h"] \arrow[d, "P'h"'] & Qy \arrow[d, equal] \\
P'x' \arrow[r, "\varphi'"'] & Qy
\end{tikzcd}
$$

The commutativity of this diagram, and the fact that every arrow involved is an isomorphism, tells us that $(x, y, \varphi' P' h(\alpha_x)^{-1})$ is an object in $\mathcal{X} \times^{(2)}_{\mathcal{B}} \mathcal{Y}$, and that its image under $\Phi$ is isomorphic to $(x', y, \varphi')$ through the map $(h, \mathrm{id}_y)$, which is an isomorphism in $\mathcal{X}' \times^{(2)}_{\mathcal{B}} \mathcal{Y}$.

We now prove full faithfulness. Let $(x, y, \varphi), (a, b, \psi)$ be objects in $\mathcal{X} \times^{(2)}_{\mathcal{B}} \mathcal{Y}$. We want to show that the induced map

$$\begin{aligned} \mathrm{Hom}((x, y, \varphi), (a, b, \psi)) &\to \mathrm{Hom}((Fx, y, \varphi\alpha_x), (Fa, b, \psi\alpha_a)) \\ (f, g) &\mapsto (Ff, g) \end{aligned}$$

is bijective. Since $F$ is fully faithful, the assignment $(f, g) \mapsto (Ff, g)$ is bijective as a function $\mathrm{Hom}(x, a) \times \mathrm{Hom}(y, b) \to \mathrm{Hom}(Fx, Fa) \times \mathrm{Hom}(y, b)$. It remains to show that an ordered pair $(f, g)$ is a morphism in $\mathcal{X} \times^{(2)}_{\mathcal{B}} \mathcal{Y}$ if and only if $(Ff, g)$ is a morphism in $\mathcal{X}' \times^{(2)}_{\mathcal{B}} \mathcal{Y}$. Consider the diagram

$$\begin{tikzcd} P'Fx \arrow[r, "\alpha_x"] \arrow[d, "P'Ff"] & Px \arrow[r, "\varphi"] \arrow[d, "Pf"] & Qy \arrow[d, "g"] \\ P'Fa \arrow[r, "\alpha_a"'] & Pa \arrow[r, "\psi"'] & Qb \end{tikzcd}$$

The left square is commutative since $\alpha$ is a natural transformation. Thus the right square is commutative if and only if the outer square is. Commutativity of the right square means $(f, g)$ is a morphism, and commutativity of the outer square means $(Ff, g)$ is a morphism. □

### 2.4.2 Algebra Objects

*Remark* 2.4.4. The notion of an algebra object requires the notion of a monoidal category. In the interest of space, we do not dwell on the details of monoidal categories. For a resource on the topic we refer to [EGNO16], ch.2.

**Definition 2.4.5.** A *monoid object* or *algebra object* in a monoidal category $\mathcal{C}$ is a triplet $(A, \quad \mu\colon A \otimes A \to A, \quad e\colon I \to A)$. Where $A$ is an object in $\mathcal{C}$, and $I$ is the unit in $\mathcal{C}$. The morphisms $\mu$ and $e$ must make the following diagrams commute:

1. (Associativity).

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd}$$

2. (Unitality).

$$\begin{tikzcd} I \otimes A \arrow[r, "e\otimes \mathrm{id}"] \arrow[dr, "l_a"'] & A \otimes A \arrow[d, "\mu"] & A \otimes I \arrow[l, "\mathrm{id}\otimes e"'] \arrow[dl, "r_a"] \\ & A & \end{tikzcd}$$

where $l_a$ and $r_a$ denote the components of the left and right unitor, respectively.

*Example* 2.4.6. Using the underlying monoidal structure of the tensor product, a monoidal object in the category of abelian groups is a ring. Similarly, a monoidal object in the category of vector spaces over some field is an algebra. Besides the unitality and associativity axioms which correspond to the commutativity of the necessary diagrams, the bilinearity requirements for rings and algebras can be recovered from the requirement that $\mu$ be a morphism from the tensor product.

The category Span(**Grpd**) has a natural monoidal structure. The tensor product is defined on objects as $\mathcal{A} \otimes \mathcal{B} := \mathcal{A} \times \mathcal{B}$ and on morphisms as

$$[\mathcal{A} \leftarrow \mathcal{X} \rightarrow \mathcal{B}] \otimes [\mathcal{A}' \leftarrow \mathcal{X}' \rightarrow \mathcal{B}'] := [\mathcal{A} \times \mathcal{A}' \leftarrow \mathcal{X} \times \mathcal{X}' \rightarrow \mathcal{B} \times \mathcal{B}'].$$

Where the morphisms in the right hand side are given by the usual product of morphisms in **Grpd**. Notice that the unit object is the trivial groupoid, also denoted $\mathcal{S}_0$ in the context of the $\mathcal{S}_\bullet$-construction. The left unitor is given, for each groupoid $\mathcal{A}$, by the span

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "{(0,\mathrm{id})}"'] \arrow[dr, "\mathrm{id}"] & \\ \mathcal{S}_0 \times \mathcal{A} & & \mathcal{A} \end{tikzcd}$$

and the right unitor is defined symmetrically. The associator is induced by the usual associator on the category **Cat**.

*Remark* 2.4.7. As we saw in Example 2.4.6, algebra objects supply the multiplicative structure of groups and rings. Recall that in Chapter 1, the Hall algebra arose as first essentially taking a free functor, then describing the multiplicative structure of the hall algebra. Our strategy here will be to describe an algebra object in Span(**Grpd**), then applying a functor to obtain the Hall algebra.

**Theorem 2.4.8.** [Dyc18, p.23] *Let $\mathcal{C}$ be a proto-abelian category and let $\mathcal{S}_\bullet$ denote the corresponding simplicial groupoid of wings. Then there is an algebra object $(\mathcal{S}_1, \mu, e)$ in the category Span(**Grpd**) given by the morphisms*

$$\mu\colon \quad \begin{tikzcd} & \mathcal{S}_{\{0,1,2\}} \arrow[dl, "(d_2, d_0)"'] \arrow[dr, "d_1"] & \\ \mathcal{S}_{\{0,1\}} \times \mathcal{S}_{\{1,2\}} & & \mathcal{S}_{\{0,2\}} \end{tikzcd}$$

*and*

$$e\colon \quad \begin{tikzcd} & \mathcal{S}_0 \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\sigma"] & \\ \mathcal{S}_0 & & \mathcal{S}_1 \end{tikzcd}$$

*Here, $d_0, d_1$ and $d_2$, are the zeroth, first and second face maps $\mathcal{S}_2 \to \mathcal{S}_1$, respectively, and $\sigma$ is the unique map in $\mathcal{S}_\bullet$ which maps $\mathcal{S}_0 \to \mathcal{S}_1$.*

*This algebra object is called the* Abstract Hall Algebra.

*Remark* 2.4.9. Note that in Span(**Grpd**), we have $\mu\colon \mathcal{S}_1 \otimes \mathcal{S}_1 \to \mathcal{S}_1$, and $e\colon \mathcal{S}_0 \to \mathcal{S}_1$.

The proof is based on the sketch found in [Dyc18].

*Proof.* We need to verify the commutativity of the diagrams in Definition 2.4.5. We first verify the identity $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$. The left-hand side is given by the diagram

$$\begin{tikzcd} \mathcal{S}_{\{0,1,2,3\}} \arrow[r] \arrow[d] & \mathcal{S}_{\{0,2,3\}} \arrow[r, "F"] \arrow[d, "G"] & \mathcal{S}_{\{0,3\}} \\ \mathcal{S}_{\{0,1,2\}} \times \mathcal{S}_{\{2,3\}} \arrow[r, "F \times \mathrm{id}"'] \arrow[d, "G \times \mathrm{id}"'] & \mathcal{S}_{\{0,2\}} \times \mathcal{S}_{\{2,3\}} & \\ \mathcal{S}_{\{0,1\}} \times \mathcal{S}_{\{1,2\}} \times \mathcal{S}_{\{2,3\}} & & \end{tikzcd}$$

Where we claim that the upper left square is 2-pullback.

The functor $F$ is an isofibration by Proposition 2.2.11, so hence the functor $F \times \mathrm{id}$ also is an isofibration. In light of Proposition 2.2.10 we work with ordinary pullbacks instead of 2-pullbacks. There is a natural sequence of functors

$$\mathcal{S}_{\{0,1,2,3\}} \to (\mathcal{S}_{\{0,1,2\}} \times \mathcal{S}_{\{2,3\}}) \times_{(\mathcal{S}_{\{0,2\}} \times \mathcal{S}_{\{2,3\}})} \mathcal{S}_{\{0,2,3\}} \to \mathcal{S}_{\{0,1,2\}} \times_{\mathcal{S}_{\{0,2\}}} \mathcal{S}_{\{0,2,3\}} \tag{2.4.10}$$

(Note that while the functors involved are not labelled, they are implicitly determined by the notation used, see Remark 2.1.6).

The composition is an equivalence since $\mathcal{S}_\bullet$ is 2-segal (Theorem 2.3.5), using the 2-segal condition in the case of the subdivision of a square given by the diagonal $(0,2)$. We claim that the rightmost arrow is an equivalence. Indeed, it is even an isomorphism of categories. The idea is that the $\mathcal{S}_{\{2,3\}}$-component of each object in the middle pullback is redundant, since the information there has to agree with a subset of the information given in the $\mathcal{S}_{\{0,2,3\}}$-component. A similar argument will be made several times throughout this proof, so we do it carefully once and omit the details in the remaining cases. Let $d_i$ denote the $i$th face map $\mathcal{S}_2 \to \mathcal{S}_1$, so we can explicitly write out the cospan

$$\begin{tikzcd} & \mathcal{S}_{\{0,2,3\}} \arrow[d, "(d_2, d_0)"] \\ \mathcal{S}_{\{0,1,2\}} \times \mathcal{S}_{\{2,3\}} \arrow[r, "d_1 \times \mathrm{id}"] & \mathcal{S}_{\{0,2\}} \times \mathcal{S}_{\{2,3\}} \end{tikzcd}$$

An object in $(\mathcal{S}_{\{0,1,2\}} \times \mathcal{S}_{\{2,3\}}) \times_{(\mathcal{S}_{\{0,2\}} \times \mathcal{S}_{\{2,3\}})} \mathcal{S}_{\{0,2,3\}}$ is a tuple $((A^1, A^2), B)$ where $A^1, B \in \mathcal{S}_2$ and $A^2 \in \mathcal{S}_1$, satisfying the relations

$$\begin{cases} d_1(A^1) = d_2(B) \\ A^2 = d_0(B). \end{cases}$$

A morphism is a tuple $((f_1, f_2), g)$ such that $(d_1 f_1, f_2) = (d_2 g, d_0 g)$.

On the other hand, an object in $\mathcal{S}_{\{0,1,2\}} \times_{\mathcal{S}_{\{0,2\}}} \mathcal{S}_{\{0,2,3\}}$ is a pair $(A, B)$, both objects of $\mathcal{S}_2$, such that

$$\begin{cases} d_1(A) = d_2(B). \end{cases}$$

A morphism is a pair $(f, g)$ with $(d_1 f = d_2 g)$. It is now clear that the assignments:

$$\begin{aligned} (\mathcal{S}_{\{0,1,2\}} \times \mathcal{S}_{\{2,3\}}) \times_{\mathcal{S}_{\{0,2\}} \times \mathcal{S}_{\{2,3\}}} \mathcal{S}_{\{0,2,3\}} &\to \mathcal{S}_{\{0,1,2\}} \times_{\mathcal{S}_{\{0,2\}}} \mathcal{S}_{\{0,2,3\}} \\ ((A^1, A^2), B) &\mapsto (A^1, B) \\ ((f_1, f_2), g) &\mapsto (f_1, g) \end{aligned}$$

as well as

$$\begin{gathered}\mathcal{S}_{\{0,1,2\}} \times_{\mathcal{S}_{\{0,2\}}} \mathcal{S}_{\{0,2,3\}} \to (\mathcal{S}_{\{0,1,2\}} \times \mathcal{S}_{\{2,3\}}) \times_{\mathcal{S}_{\{0,2\}} \times \mathcal{S}_{\{2,3\}}} \mathcal{S}_{\{0,2,3\}} \\ (A, B) \mapsto ((A, d_0(B)), B) \\ (f, g) \mapsto ((f, d_0 g), g)\end{gathered}$$

determine mutually inverse functors. Looking back at Eq. (2.4.10), the composition and the rightmost functor are equivalences, so the leftmost functor is also an equivalence, as desired. Now 2.2.10 tells us that the upper left square in the composition $\mu \circ (\mu \otimes \mathrm{id})$ is 2-pullback, so we conclude that this composition is given by the span

$$\begin{tikzcd} & \mathcal{S}_{\{0,1,2,3\}} \arrow[dl] \arrow[dr] & \\ \mathcal{S}_{\{0,1\}} \times \mathcal{S}_{\{1,2\}} \times \mathcal{S}_{\{2,3\}} & & \mathcal{S}_{\{0,3\}} \end{tikzcd}$$

where the functors are the natural ones determined by the universal property of the product as well as the notation described in Remark 2.1.6. We now compute $\mu \circ (\mathrm{id} \times \mu)$, which is given by the diagram

$$\begin{tikzcd} \mathcal{S}_{\{0,1,2,3\}} \arrow[r] \arrow[d] & \mathcal{S}_{\{0,1,3\}} \arrow[r, "F"] \arrow[d, "G"] & \mathcal{S}_{\{0,3\}} \\ \mathcal{S}_{\{0,1\}} \times \mathcal{S}_{\{1,2,3\}} \arrow[r, "\mathrm{id} \times F"'] \arrow[d, "\mathrm{id} \times G"'] & \mathcal{S}_{\{0,1\}} \times \mathcal{S}_{\{1,3\}} & \\ \mathcal{S}_{\{0,1\}} \times \mathcal{S}_{\{1,2\}} \times \mathcal{S}_{\{2,3\}} & & \end{tikzcd}$$

where we again claim that the upper left corner is 2-pullback. As before, we work with normal pullbacks rather than 2-pullbacks since $\mathrm{id} \times F$ is an isofibration. There is a natural sequence of functors

$$\mathcal{S}_{\{0,1,2,3\}} \to (\mathcal{S}_{\{0,1\}} \times \mathcal{S}_{\{1,2,3\}}) \times_{(\mathcal{S}_{\{0,1\}} \times \mathcal{S}_{\{1,3\}})} \mathcal{S}_{\{0,1,3\}} \to \mathcal{S}_{\{1,2,3\}} \times_{\mathcal{S}_{\{1,3\}}} \mathcal{S}_{\{0,1,3\}}$$

again the composition is an equivalence, this time due to the 2-segal condition in the case of the subdivision of a square given by the diagonal $(1,3)$. The rightmost functor is again an equivalence by reasoning analagous to that given above when computing $\mu \circ (\mu \otimes \mathrm{id})$. In this case, it is the information in the $\mathcal{S}_{\{0,1\}}$-component which is redundant. As before, we obtain that the leftmost functor is an equivalence and that the composition $\mu \circ (\mathrm{id} \otimes \mu)$ is given by the span

$$\begin{tikzcd} & \mathcal{S}_{\{0,1,2,3\}} \arrow[dl] \arrow[dr] & \\ \mathcal{S}_{\{0,1\}} \times \mathcal{S}_{\{1,2\}} \times \mathcal{S}_{\{2,3\}} & & \mathcal{S}_{\{0,3\}} \end{tikzcd}$$

In other words, we have $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$, and we have checked associativity of the abstract hall algebra.

Our next step is to show the unitality condition of Definition 2.4.5. We claim that the composition $\mu \circ (e \otimes \mathrm{id})$ is given by the diagram

$$\begin{tikzcd} \mathcal{S}_1 \arrow[r, "\sigma_0"] \arrow[d, "(d, \mathrm{id})"] & \mathcal{S}_2 \arrow[r, "d_1"] \arrow[d, "(d_2, d_0)"] & \mathcal{S}_1 \\ \mathcal{S}_0 \times \mathcal{S}_1 \arrow[r, "\sigma \times \mathrm{id}"'] \arrow[d, "\mathrm{id}"'] & \mathcal{S}_1 \times \mathcal{S}_1 & \\ \mathcal{S}_0 \times \mathcal{S}_1 & & \end{tikzcd}$$

Here $d_i$ denotes the $i$th face map $\mathcal{S}_2 \to \mathcal{S}_1$, $\sigma_0$ denotes the zeroth degeneracy map $\mathcal{S}_1 \to \mathcal{S}_2$, $\sigma$ denotes the unique degeneracy map $\mathcal{S}_0 \to \mathcal{S}_1$, and $d$ denotes any of the two (equal) face maps $\mathcal{S}_1 \to \mathcal{S}_0$. Once again the claim is that the upper left square is pullback. Like before, we consider the composition

$$\mathcal{S}_1 \to (\mathcal{S}_0 \times \mathcal{S}_1) \times_{(\mathcal{S}_1 \times \mathcal{S}_1)} \mathcal{S}_2 \to \mathcal{S}_0 \times_{\mathcal{S}_1} \mathcal{S}_2 \qquad (2.4.11)$$

Again, the rightmost functor is an equivalence by the same argument as with the previous two computations, in this case it is the information in the $\mathcal{S}_1$-component which is redundant. We want to show that the composition is an equivalence. To make things clearer, we explicitly write out the cospan which the rightmost groupoid is a pullback of:

$$\begin{tikzcd} & \mathcal{S}_0 \arrow[d, "\sigma"'] \\ \mathcal{S}_2 \arrow[r, "d_2"] & \mathcal{S}_1 \end{tikzcd}$$

We are given by the second of the 2-segal conditions (diagram 2.3.2) that the square

$$\begin{tikzcd} \mathcal{S}_1 \arrow[r, "d"] \arrow[d, "\sigma_0"] & \mathcal{S}_0 \arrow[d, "\sigma"] \\ \mathcal{S}_2 \arrow[r, "d_2"] & \mathcal{S}_1 \end{tikzcd}$$

is 2-pullback, such that the composite in 2.4.11 is an equivalence. By the same argument as in the previous two cases, we conclude that the composite $\mu \circ (e \otimes \mathrm{id})$ is given by the span

$$\begin{tikzcd} & \mathcal{S}_1 \arrow[dl, "(d{,}\mathrm{id})"'] \arrow[dr, "d_1 \circ \sigma_0"] & \\ \mathcal{S}_0 \times \mathcal{S}_1 & & \mathcal{S}_1 \end{tikzcd}$$

Noting that $d_1 \circ \sigma_0 = \mathrm{id}_{\mathcal{S}_1}$, we have that this is the left unitor of $\mathcal{S}_1$ in $\mathrm{Span}(\mathbf{Grpd})$, so that the first unitality condition is satisfied. The proof of the second unitality condition follows from symmetry. □

## 2.5 Counting with Groupoids

In this section, we develop the theory which allows us to count in groupoids properly, in a way which lets us recover the classical Hall algebra from the Abstract Hall algebra. The terminology in this section is all from [Dyc18]. Throughout this section $\mathcal{A}$ and $\mathcal{B}$ denote groupoids, unless otherwise stated.

**Definition 2.5.1.** A groupoid $\mathcal{A}$ is called *finite* if the set $\pi_0(\mathcal{A})$ of isomorphism classes is finite and for each $a \in \mathcal{A}$, the set of automorphisms $\mathrm{Aut}(a)$ is finite.

**Proposition 2.5.2.** *In a groupoid $\mathcal{A}$, if $a \cong b$ then the sets $\mathrm{Aut}(a)$ and $\mathrm{Aut}(b)$ have the same cardinality.*

*Proof.* Choose an isomorphism $f : a \xrightarrow{\sim} b$. Then conjugation by $f$ forms a bijection $\mathrm{Aut}(a) \xrightarrow{\sim} \mathrm{Aut}(b)$. □

The above proposition implies that the following definition is well-defined

**Definition 2.5.3.** [Dyc18, p. 24] Given a finite groupoid $\mathcal{A}$, the *groupoid cardinality* of $\mathcal{A}$ is defined as

$$|\mathcal{A}| = \sum_{[a] \in \pi_0(\mathcal{A})} \frac{1}{|\mathrm{Aut}(a)|}.$$

*Remark* 2.5.4. Of course, if there are no non-identity morphisms in $\mathcal{A}$, groupoid cardinality agrees with the cardinality of the underlying set.

*Remark* 2.5.5. [Dyc18, Remark 3.10] Groupoid cardinality is invariant under equivalences of finite groupoids. This follows immediately from the fact that equivalences are full and faithful and essentially surjective.

The following motivation is taken from [Dyc18, p. 25].

Given a set $K$ and a function $\varphi\colon K \to \mathbb{Q}$ with finite support, we can use integral notation:

$$\int_K \varphi := \sum_{k\in K} \varphi(k).$$

Notice that this has nothing to do with integrals from analysis. Indeed, the sum is finite. This integral has the property that $\int_K \mathbb{1} = |K|$, where $\mathbb{1}$ denotes the constant 1 function on $K$. We wish to generalize this to groupoids such that this property is preserved. First, we introduce a few more notions.

**Definition 2.5.6.** Given a groupoid $\mathcal{A}$ we denote by $\mathcal{F}(\mathcal{A})$ the $\mathbb{Q}$-vector space of functions $\varphi\colon \operatorname{ob}\mathcal{A} \to \mathbb{Q}$ which are constant on isomorphism classes, and nonzero on only finitely many isomorphism classes. This is also the vector space of formal linear combinations of isomorphism classes of $\mathcal{A}$, and is canonically isomorphic to the vector space

$$\bigoplus_{[a]\in\pi_0(\mathcal{A})} \mathbb{Q}[a]$$

which appears in the definition of the Hall Algebra.

**Definition 2.5.7.** We call a groupoid $\mathcal{A}$ *locally finite* if every connected component $\mathcal{A}(a)$ is finite.

*Remark* 2.5.8. Since $\mathcal{A}$ is a groupoid, the connected components correspond to isomorphism classes.

**Definition 2.5.9.** [Dyc18, p. 25] Given a locally finite groupoid $\mathcal{A}$ and $\varphi \in \mathcal{F}(\mathcal{A})$, we define the *groupoid integral*

$$\int_{\mathcal{A}} \varphi := \sum_{[a]\in\pi_0(\mathcal{A})} \frac{\varphi(a)}{|\mathrm{Aut}(a)|}.$$

*Remark* 2.5.10. The sum in Definition 2.5.9 is finite since $\varphi$ is zero on all but finitely many equivalence classes of $\mathcal{A}$. The locally finite condition on $\mathcal{A}$ ensures that for each $a$, $|\mathrm{Aut}(a)|$ is finite. As desired, if $\mathcal{A}$ is finite then $\int_{\mathcal{A}} \mathbb{1} = |\mathcal{A}|$.

**Definition 2.5.11.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between groupoids, and let $b \in \mathcal{B}$ be an object. The *2-fiber of $F$ at $b$* is the groupoid with elements given by pairs $(a, \varphi)$ where $a \in \mathcal{A}$ and $\varphi\colon Fa \xrightarrow{\sim} b$. A map $f\colon (a, \varphi) \to (a', \psi)$ is a map $f\colon a \to a'$ such that the following diagram

$$\begin{tikzcd} Fa \arrow[r, "\varphi"'] \arrow[d, "Ff"] & b \arrow[d, equal] \\ Fa' \arrow[r, "\psi"] & b \end{tikzcd}$$

commutes. In other words, $\psi \circ Ff = \varphi$. The 2-fiber of $F$ at $b$ is denoted $\mathcal{A}_b$ or, which such notation would be ambiguous, by $\text{2-Fib}(F, b)$. It comes equipped with a canonical functor of groupoids $\pi\colon \mathcal{A}_b \to \mathcal{A}$, $(a, \varphi) \mapsto a$.

*Remark* 2.5.12. The commutative square of groupoids

$$\begin{tikzcd} \mathcal{A}_b \arrow[r, hook] \arrow[d] & \mathcal{A} \arrow[d, "F"'] \\ \{*\} \arrow[r, "* \mapsto b"'] & \mathcal{B} \end{tikzcd}$$

is 2-pullback. This corresponds corresponds to how in for example **Set**, the square

$$\begin{tikzcd} f^{-1}(b) \arrow[r, hook] \arrow[d] & A \arrow[d, "f"] \\ \{*\} \arrow[r, "* \mapsto b"'] & B \end{tikzcd}$$

is pullback. In fact, this property tells us that we ought to define the 2-fiber this way.

**Definition 2.5.13.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor of groupoids and $b \in \mathcal{B}$ be an object. Then we denote by $F|_{\mathcal{A}_b}$ the *restriction of $F$ to $b$*, which is given by the composition $\mathcal{A}_b \xrightarrow{\pi} \mathcal{A} \xrightarrow{\varphi} \mathcal{B}$.

**Definition 2.5.14.** [Dyc18, p. 25] A functor $F\colon \mathcal{A} \to \mathcal{B}$ is called:

- *Finite* if each 2-fiber of $F$ is finite.
- *Locally finite* if, for every $a \in \mathcal{A}$, the restriction of $F$ to $\mathcal{A}(a)$ is finite.
- *$\pi_0$-finite* if the induced map of sets $\pi_0(\mathcal{A}) \to \pi_0(\mathcal{B})$ has finite fibers.

**Proposition 2.5.15.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a locally finite functor. Then, for each object $b \in \mathcal{B}$, the 2-fiber* 2-Fib$(F, b)$ *is locally finite.*

*Proof.* Let $(a, \varphi) \in$ 2-Fib$(F, b)$. We want to show that $(\text{2-Fib}(F, b))(a, \varphi)$ - the connected component of 2-Fib$(F, b)$ which includes $(a, \varphi)$ - is finite. $F$ is locally finite, so $F|_{\mathcal{A}(a)}$ is finite (notice that this restriction is in the usual sense, not that of Definition 2.5.13). This means that 2-Fib$(F|_{\mathcal{A}(a)}, b)$ is finite. However, we have an inclusion

$$(\text{2-Fib}(F, b))(a, \varphi) \subseteq \text{2-Fib}(F|_{\mathcal{A}(a)}, b)$$

so the left-hand side is finite, as desired. □

**Proposition 2.5.16.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a locally finite functor, then for each object $b \in \mathcal{B}$, the canonical map $\pi\colon \mathcal{A}_b \to \mathcal{A}$ is $\pi_0$-finite.*

*Proof.* Choose $[a] \in \pi_0(\mathcal{A})$ and let $\{(a_i, \varphi_i)\}_{i \in I}$ be a complete set of representatives of the isomorphism classes in $\mathcal{A}_b$ which are sent to $[a]$ by $\pi$. We wish to show that $I$ is a finite set. Notice that for each $i$,

$$(a_i, \varphi_i) \in \text{2-Fib}(F|_{\mathcal{A}(a)}, b).$$

Since $F$ is locally finite, this two-fiber is finite, so amongst the $(a_i, \varphi_i)$ there are only finitely many isomorphism classes in 2-Fib$(F|_{\mathcal{A}(a)}, b)$. However, an isomorphism in 2-Fib$(F|_{\mathcal{A}(a)}, b)$ is also an isomorphism in 2-Fib$(F, b) = \mathcal{A}_b$, so each of the $(a_i, \varphi_i)$ are pairwise non-isomorphic in 2-Fib$(F|_{\mathcal{A}(a)}, b)$. As a result $I$ must be finite, as desired. □

**Definition 2.5.17.** Given a locally finite functor $F\colon \mathcal{A} \to \mathcal{B}$ and a function $\varphi \in \mathcal{F}(\mathcal{A})$, we define the *pushforward* $F_!\varphi$ by

$$F_!\varphi(b) := \int_{\mathcal{A}_b} \varphi|_{\mathcal{A}_b}.$$

*Remark* 2.5.18. The integral in Definition 2.5.17 is well-defined since $\mathcal{A}_b$ is locally finite by Proposition 2.5.15. Also notice that $\varphi|_{\mathcal{A}_b} \in \mathcal{F}(\mathcal{A})$ since restriction preserves the properties of being constant on isomorphism classes. That $\varphi|_{\mathcal{A}_b}$ is nonzero on finitely many isomorphism classes follows from the fact that $\varphi$ is, and that the morphism $\pi\colon \mathcal{A}_b \to \mathcal{A}$ is $\pi_0$-finite.

**Proposition 2.5.19.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a locally finite functor, and let $\varphi \in \mathcal{F}(\mathcal{A})$. Then $F_!\varphi \in \mathcal{F}(\mathcal{B})$.*

*Proof.* Unravelling the definitions,

$$F_!\varphi(b) = \int_{\mathcal{A}_b} \varphi|_{\mathcal{A}_b} = \sum_{(a,\varphi)\in\pi_0(\mathcal{A}_b)} \frac{\varphi(a)}{|\mathrm{Aut}((a,\varphi))|}.$$

To show that $F_!\varphi$ is constant on isomorphism classes, we claim that isomorphic objects have isomorphic 2-fibers. Indeed, for an isomorphism $\alpha\colon b \to b'$, we have an isomorphism of groupoids

$$\begin{aligned} \mathcal{A}_b &\to \mathcal{A}_{b'} \\ (a,\varphi) &\mapsto (a,\alpha\varphi), \\ f &\mapsto f. \end{aligned}$$

That this is a valid isomorphism of groupoids is a consequence of the commutativity of the diagram

$$\begin{tikzcd} Fa \arrow[dd, "Ff"'] \arrow[dr, "\varphi"] & & \\ & b \arrow[r, "\alpha"'] & b' \\ Fa' \arrow[ur, "\psi"'] & & \end{tikzcd}$$

and the fact that $\alpha$ is an isomorphism.

Let $\widetilde{F}$ denote the set-function $\pi_0(\mathcal{A}) \to \pi_0(\mathcal{B})$ induced by $F$. Then $\widetilde{F}^{-1}([b])$ for $[b] \in \pi_0(\mathcal{B})$ forms a partition of $\pi_0(\mathcal{A})$. Since $\varphi$ is nonzero on only finitely many elements of $\pi_0(\mathcal{A})$, we have only finitely many values of $[b] \in \pi_0(\mathcal{B})$ for which $\widetilde{F}^{-1}([b])$ contains an $[a] \in \pi_0(\mathcal{A})$ where $\varphi$ is nonzero. As such, $F_!\varphi$ is nonzero on finitely many isomorphism classes of $\mathcal{B}$. □

**Proposition 2.5.20.** *The map $F_!\colon \mathcal{F}(\mathcal{A}) \to \mathcal{F}(\mathcal{B})$ is $\mathbb{Q}$-linear.*

*Proof.* Let $f, g$ in $\mathcal{F}(\mathcal{A})$ and $\lambda \in \mathbb{Q}$. Then,

$$\begin{aligned} F_!(\lambda f + g)(b) &= \int_{\mathcal{A}_b} (\lambda f + g)|_{\mathcal{A}_b} = \int_{\mathcal{A}_b} (\lambda f)|_{\mathcal{A}_b} + g|_{\mathcal{A}_b} \\ &= \int_{\mathcal{A}_b} (\lambda f)|_{\mathcal{A}_b} + \int_{\mathcal{A}_b} g|_{\mathcal{A}_b} = \lambda \int_{\mathcal{A}_b} f|_{\mathcal{A}_b} + \int_{\mathcal{A}_b} g|_{\mathcal{A}_b} \\ &= \lambda F_! f(b) + F_! g(b). \end{aligned}$$

So that $F_!(\lambda f + g) = \lambda F_! f + F_! g$, as desired. □

The 2-fiber is often cumbersome to deal with, so it is useful to have some alternative ways to work with the pushforward operation and local finiteness of functors. This is what we will now try to develop.

**Lemma 2.5.21.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor of groupoids, and let $a \in \mathcal{A}$ be an object. Denote by $f_a$ the function $\mathrm{Aut}(a) \to \mathrm{Aut}(Fa)$ induced by $F$. Then we have the equality*

$$|\pi_0(\text{2-Fib}(F|_{\mathcal{A}(a)}, Fa)| = [\mathrm{Aut}(Fa) : \mathrm{im}(f_a)]$$

*and for each object $(a,\varphi) \in \text{2-Fib}(F|_{\mathcal{A}(a)}, Fa)$*

$$|\mathrm{Aut}((a,\varphi))| = |\ker(f_a)|$$

*In particular, $F$ is locally finite if and only if these values are both finite for each $a \in \mathcal{A}$.*

*Proof.* We claim that each object in the 2-fiber can be represented by a pair $(a,\varphi)$, i.e with first component equal to $a$. Indeed, if $(a',\psi) \in \text{2-Fib}(F|_{\mathcal{A}(a)}, Fa)$, then we have some isomorphism $\alpha\colon a \to a'$. The diagram

$$\begin{tikzcd} Fa \arrow[r, "\psi\circ F\alpha"] \arrow[d, "F\alpha"] & b \\ Fa' \arrow[ur, "\psi"'] & \end{tikzcd}$$

shows that $\alpha$ is an isomorphism $(a, \psi \circ F\alpha) \xrightarrow{\sim} (a',\psi)$

We have that the automorphism class of $(a,\varphi)$ in $\text{2-Fib}(F|_{\mathcal{A}(a)}, Fa)$ is determined by the element $\varphi \in \mathrm{Aut}(Fa)$. Two morphisms in $\varphi,\psi \in \mathrm{Aut}(Fa)$ give rise to isomorphic objects if there is an $\alpha \in \mathrm{Aut}(a)$ such that the diagram

$$\begin{tikzcd} Fa \arrow[r, "\varphi"] \arrow[d, "F\alpha"] & Fa \\ Fa \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes. In other words, if $\psi^{-1} \circ \varphi \in \mathrm{im}(F_0)$. Thus we have

$$|\pi_0(\text{2-Fib}(F|_{\mathcal{A}(a_0)}, F(a_0))| = [\mathrm{Aut}(Fa_0) : \mathrm{im}(F_0)].$$

Now pick an object $(a,\varphi)$ and consider its automorphism group. An element of this group is a map $\alpha \in \mathrm{Aut}(a)$ such that

$$\begin{tikzcd} Fa \arrow[r, "\varphi"] \arrow[d, "F\alpha"'] & Fa \\ Fa \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commutes. This is the same as the equation $F\alpha = \mathrm{id}_{Fa}$, so that $\alpha \in \ker(f_a)$. We conclude that

$$|\mathrm{Aut}((a, \varphi))| = |\ker(f_a)|$$

as desired. □

**Definition 2.5.22.** For a groupoid $\mathcal{A}$ and an object $a \in \mathcal{A}$ we denote by $\overline{a} \in \mathcal{F}(\mathcal{A})$ the element of $\mathcal{F}(\mathcal{A})$ which is 1 on the isomorphism class of $a$ and zero otherwise. These functions form a basis for $\mathcal{F}(\mathcal{A})$.

**Theorem 2.5.23.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a locally finite functor of groupoids, and let $a \in \mathcal{A}$ be an object. Let $f_a$ be the function* $\mathrm{Aut}(a) \to \mathrm{Aut}(Fa)$ *induced by $F$. Then we have*

$$F_!(\overline{a}) = \frac{[\mathrm{Aut}(Fa) : \mathrm{im}(f_a)]}{|\ker(f_a)|}\overline{Fa}.$$

*Furthermore, in the case where either* $|\mathrm{Aut}(Fa)|$ *or* $|\mathrm{Aut}(a)|$ *is finite, this simplifies to*

$$F_!(\overline{a}) = \frac{|\mathrm{Aut}(Fa)|}{|\mathrm{Aut}(a)|}\overline{Fa}.$$

*Proof.* We compute:

$$\begin{aligned} F_!(\overline{a})(b) &= \int_{\text{2-Fib}(F,b)} \overline{a}\big|_{\text{2-Fib}(F,b)} \\ &= \sum_{[(a',\varphi)] \in \pi_0(\text{2-Fib}(F,b))} \frac{\overline{a}(a')}{|\mathrm{Aut}((a,\varphi))|}. \end{aligned}$$

The summand is zero whenever $a'$ is not isomorphic to $a$, so so we can restrict to summing over the iso. classes of 2-Fib$(F|_{\mathcal{A}(a)}, b)$. If $Fa$ is not isomorphic to $b$, this 2-fiber is empty and $F_!(\overline{a})(b) = 0$. Otherwise, since isomorphic objects have isomorphic 2-fibers (see proof of Proposition 2.5.19), we can identify $b$ with $Fa$. As in the proof of Lemma 2.5.21, the isomorphism class of each object

$(a', \psi)$ in 2-Fib$(F|_{\mathcal{A}(a)}, Fa)$ can be represented by an object on the form $(a, \varphi)$, i.e with first component equal to $a$. Combining these insights, we write

$$\begin{aligned} F_!(\overline{a})(b) &= \sum_{[(a,\varphi)]\in\pi_0(\text{2-Fib}(F,Fa))} \frac{\overline{a}(a)}{|\mathrm{Aut}((a,\varphi))|} \\ &= \sum_{[(a,\varphi)]\in\pi_0(\text{2-Fib}(F,Fa))} \frac{1}{|\ker(f_a)|} \\ &= \frac{[\mathrm{Aut}(Fa) : \mathrm{im}(f_a)]}{|\ker(f_a)|}. \end{aligned}$$

where the second and third equalities use Lemma 2.5.21. We conclude that

$$F_!(\overline{a})(b) = \begin{cases} \frac{[\mathrm{Aut}(Fa):\mathrm{im}(f_a)]}{|\ker(f_a)|} & \text{if } Fa \cong b \\ 0 & \text{otherwise.} \end{cases}$$

so that $F_!(\overline{a}) = \frac{[\mathrm{Aut}(Fa):\mathrm{im}(f_a)]}{|\ker(f_a)|}\overline{Fa}$, as desired.

We now consider the case where $|\mathrm{Aut}(Fa)|$ or $|\mathrm{Aut}(a)|$ is finite. Since $[\mathrm{Aut}(Fa) : \mathrm{im}(f_a)]$ and $|\ker(f_a)|$ are finite, it is clear by the first isomorphism theorem that if either of $|\mathrm{Aut}(Fa)|$ or $|\mathrm{Aut}(a)|$ are finite, the other one also is. We compute:

$$\frac{[\mathrm{Aut}(Fa) : \mathrm{im}(f_a)]}{|\ker(f_a)|} = \frac{|\mathrm{Aut}(Fa)|}{|\mathrm{im}(f_a)||\ker(f_a)|} = \frac{|\mathrm{Aut}(Fa)|}{|\mathrm{Aut}(a)|}$$

where the last equality is given by passing to cardinality in the first isomorphism theorem. This concludes the proof. □

**Corollary 2.5.24.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an equivalence of groupoids and let $a \in \mathcal{A}$ be an object. Then $F_!(\overline{a}) = \overline{Fa}$*

*Proof.* In this case, $f_a$ is a bijection $\mathrm{Aut}(a) \xrightarrow{\sim} \mathrm{Aut}(Fa)$. The result is immediate from the previous theorem. □

We now move on from the pushforward to its cousin, the pullback (not to be confused with the categorical pullback). Thankfully, the pullback is less complex and essentially just pre-composition.

**Definition 2.5.25.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a $\pi_0$-finite functor and $\varphi \in \mathcal{F}(\mathcal{B})$ we define the *pullback* $F^*\varphi$ by $F^*\varphi = \varphi \circ F$.

**Proposition 2.5.26.** *In the notation of the above definition, $F^*\varphi \in \mathcal{F}(\mathcal{A})$, and the map $F^*\colon \mathcal{F}(\mathcal{B}) \to \mathcal{F}(\mathcal{A})$ is $\mathbb{Q}$-linear*

*Proof.* That $F^*\varphi$ is constant on isomorphism classes follows from $F$ preserving isomorphisms, and $\varphi$ being constant on isomorphism classes. That $F^*\varphi$ is nonzero on finitely many isomorphism classes follows from that $\varphi$ is, and that $F$ is $\pi_0$-finite so that each isomorphism class where $\varphi$ is nonzero has finitely many isomorphism classes as fiber. It remains to show that $F^*$ is $\mathbb{Q}$-linear. Let $f, g \in \mathcal{F}(\mathcal{B})$ and $\lambda \in \mathbb{Q}$. Then we compute

$$F^*(\lambda f + g) = (\lambda f + g) \circ F = \lambda f \circ F + g \circ F = \lambda F^*(f) + F^*(g).$$

as desired. □

Some properties of these pushforward and pullback operations are summarised in the following theorem:

**Theorem 2.5.27.** [Dyc18, Adapted from prop. 3.14]. *The following statements hold:*

1. *If $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ are $\pi_0$-finite, then $G \circ F$ is $\pi_0$-finite and $(G \circ F)^* = F^* \circ G^*$*

2. *If $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ are locally finite, then $G \circ F$ is locally finite and $(G \circ F)_! = G_! \circ F_!$*

3. *If $F$ and $G$ are naturally isomorphic functors $\mathcal{A} \to \mathcal{B}$, then $F^* = G^*$ if they are $\pi_0$-finite, and $F_! = G_!$ if they are locally finite.*

4. *Given two equivalent spans*

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "P_1"'] \arrow[dr, "P_2"] \arrow[dd, "F"] & \\ \mathcal{A} \arrow[r, phantom, "\Nwarrow"] & & \mathcal{B} \arrow[l, phantom, "\Nearrow"] \\ & \mathcal{X}' \arrow[ul, "Q_1"] \arrow[ur, "Q_2"'] & \end{tikzcd}$$

   *in* $\mathrm{Span}(\mathbf{Grpd})$*, then $(P_2)_! \circ (P_1)^* = (Q_2)_! \circ (Q_1)^*$.*

5. *If*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F'"] \arrow[d, "G'"'] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] \arrow[ur, phantom, "\eta \Nearrow"] & \mathcal{C} \end{tikzcd}$$

   *Is a 2-pullback diagram with $F$ locally finite, and $G$ $\pi_0$-finite, then:*

*a) $F'$ is locally finite.*

*b) $G'$ is $\pi_0$-finite.*

*c) $(F')_! \circ (G')^* = G^* \circ F_!$.*

We separate the proof of Theorem 2.5.27 into several parts, since the combined proof would be very long.

*Proof of Theorem 2.5.27 (1).* It is clear that compositions of $\pi_0$-finite functors are $\pi_0$-finite. For a function $\varphi \in \mathcal{F}(\mathcal{C})$, we compute

$$(G \circ F)^*(\varphi) = \varphi \circ G \circ F = F^*(\varphi \circ G) = (F^* \circ G^*)(\varphi).$$

□

*Proof of Theorem 2.5.27 (2).* Let us first compute $G_! \circ F_!$. Pick an object $a \in \mathcal{A}$ and its corresponding basis element $\overline{a} \in \mathcal{F}(\mathcal{A})$. Let $f_a, g_a$ be the morphisms in the diagram

$$\mathrm{Aut}(a) \xrightarrow{f_a} \mathrm{Aut}(Fa) \xrightarrow{g_a} \mathrm{Aut}(GFa)$$

generated by $F$ and $G$, respectively. We compute:

$$\begin{aligned}(G_! \circ F_!)(\overline{a}) &= G_!\left(\frac{[\mathrm{Aut}(Fa) : \mathrm{im}(f_a)]}{|\ker(f_a)|}\overline{Fa}\right) = \frac{[\mathrm{Aut}(Fa) : \mathrm{im}(f_a)]}{|\ker(f_a)|}G_!(\overline{Fa}) \\ &= \frac{[\mathrm{Aut}(Fa) : \mathrm{im}(f_a)]}{|\ker(f_a)|}\frac{[\mathrm{Aut}(GFa) : \mathrm{im}(g_a)]}{|\ker(g_a)|}\overline{GFa}.\end{aligned}$$

Where we repeatedly use Theorem 2.5.23. By Lemma B.0.4, the values $[\mathrm{Aut}(GFa) : \mathrm{im}(g_a \circ f_a)]$ and $|\ker(g_a \circ f_a)|$ are finite, so that Lemma 2.5.21 implies that $G \circ F$ is locally finite. We conclude that

$$\begin{aligned}&\frac{[\mathrm{Aut}(Fa) : \mathrm{im}(f_a)]}{|\ker(f_a)|}\frac{[\mathrm{Aut}(GFa) : \mathrm{im}(g_a)]}{|\ker(g_a)|}\overline{GFa} = \\ &\qquad = \frac{[\mathrm{Aut}(GFa) : \mathrm{im}(g_a \circ f_a)]}{|\ker(g_a \circ f_a)|}\overline{GFa} = (G \circ F)_!(\overline{a})\end{aligned}$$

We have shown that $(G_! \circ F_!) = (G \circ F)_!$ on the basis elements of $\mathcal{F}(\mathcal{A})$, so by linearity of the pushforward (Proposition 2.5.20), these functions are equal on all of $\mathcal{F}(\mathcal{A})$, as desired. □

*Proof of Theorem 2.5.27(3).* for the pullback case, letting $\varphi \in \mathcal{F}(\mathcal{B})$ and $a \in \mathcal{A}$,

$$F^*\varphi(a) = \varphi(Fa) = \varphi(Ga) = G^*\varphi(a).$$

so $F^* = G^*$. For the pushforward case, we use Theorem 2.5.23. Fix an object $a \in \mathcal{A}$ and letting $f_a, g_a$ be the induced morphisms $\mathrm{Aut}(a) \to \mathrm{Aut}(Fa)$ and $\mathrm{Aut}(a) \to \mathrm{Aut}(Ga)$, respectively. The natural isomorphism $\eta$ between $F$ and $G$ provides an isomorphism between $Fa$ and $Ga$, so that their automorphism groups are isomorphic. The naturality condition ensures that $\eta$ provides a bijection between $\mathrm{im}(f_a)$ and $\mathrm{im}(g_a)$. By the first isomorphism theorem, we have that $|\ker(f_a)| = |\ker(g_a)|$. Combining this information, we conclude by Theorem 2.5.23 that $F_! = G_!$ on the basis elements of $\mathcal{F}(\mathcal{A})$, as desired. □

*Proof of Theorem 2.5.27(4).* We compute:

$$(P_2)_! \circ (P_1)^* = (Q_2 \circ F)_! \circ (Q_1 \circ F)^* = (Q_2)_! \circ F_! \circ F^* \circ (Q_1)^*$$

Where we use the previous parts of the theorem. If we can show that $F_! \circ F^* \colon \mathcal{F}(\mathcal{X}') \to \mathcal{F}(\mathcal{X}')$ is the identity, then we are done. Since $F$ is an equivalence, it induces an isomorphism $\pi_0(\mathcal{X}) \to \pi_0(\mathcal{X}')$. This means that the elements $\overline{Fx}$ for $x \in \mathcal{X}$ span $\mathcal{F}(\mathcal{F}\S)$. Notice also that $F^*(\overline{Fx}) = \overline{x}$. This combined with **??** tells us that

$$(F_! \circ F^*)(\overline{Fx}) = F_1(\overline{x}) = \overline{Fx}.$$

As desired. □

*Proof of Theorem 2.5.27 (5a).* Due to the previous part of this theorem, we can identify $\mathcal{X}$ with $\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}$, so that the 2-pullback diagram in question is

$$\begin{tikzcd} \mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\Nearrow"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

First, we show that if $F$ is locally finite, then $\pi_2$ is locally finite. We choose $(a, b, \varphi) \in \mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}$ and aim to show that $\pi_2|_{(\mathcal{A}\times^{(2)}_{\mathcal{C}}\mathcal{B})(a,b,\varphi)}$ is finite. This is the same as showing that the groupoid

$$\text{2-Fib}(\pi_2|_{(\mathcal{A}\times^{(2)}_{\mathcal{C}}\mathcal{B})(a,b,\varphi)}, b)$$

is finite, since this groupoid is the only nonzero 2-fiber. An object of this 2-fiber is a tuple $((a_0, b_0, \varphi \colon Fa_0 \to Gb_0), \alpha \colon b_0 \to b)$. We claim that each such object is isomorphic to the object $((a_0, b, (G\alpha)\varphi), \mathrm{id}_b)$. Indeed, the isomorphism is given by the tuple $(\mathrm{id}_a, \alpha)$, and that this is a morphism is witnessed by the commutative diagrams

$$\begin{tikzcd} Fa_0 \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}"] & Gb_0 \arrow[d, "G\alpha"] \\ Fa_0 \arrow[r, "(G\alpha)\varphi"] & Gb \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} b \arrow[r, "\alpha"] \arrow[d, "\alpha"] & b \\ b \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

Now, an isomorphism of objects $((a, b, \varphi), \mathrm{id}) \to ((a', b, \psi), \mathrm{id})$ is a pair of maps $(f, g)$ such that the diagrams

$$\begin{tikzcd} Fa \arrow[r, "\varphi"] \arrow[d, "Ff"] & Gb \arrow[d, "Gg"] \\ Fa' \arrow[r, "\psi"] & Gb \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} b \arrow[r, "\mathrm{id}"] \arrow[d, "g"] & b \\ b \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

commute.

This is true if and only if $g = \mathrm{id}$ and $f$ is a morphism $(a, \varphi) \to (a', \psi)$ in $2\text{-Fib}(F|_{\mathcal{A}(a)}, b)$. This fact gives rise to a bijection

$$\pi_0\left(2\text{-Fib}(\pi_2|_{(\mathcal{A}\times^{(2)}_{\mathcal{C}}\mathcal{B})(a,b,\varphi)}, b)\right) \xrightarrow{\sim} \pi_0\left(2\text{-Fib}(F|_{\mathcal{A}(a)}, Gb)\right)$$
$$[((a, b, \varphi), \mathrm{id}_b)] \mapsto [(a, \varphi)]$$

and, for each $((a, b, \varphi), \mathrm{id}) \in 2\text{-Fib}(\pi_2|_{(\mathcal{A}\times^{(2)}_{\mathcal{C}}\mathcal{B})(a,b,\varphi)}, b)$, a bijection

$$\mathrm{Aut}\left((a, b, \varphi), id\right) \xrightarrow{\sim} \mathrm{Aut}((a, \varphi))$$
$$(f, \mathrm{id}) \mapsto f.$$

Since $F$ is locally finite, the right-hand side of these bijections are finite sets. Therefore the left-hand sides also are, and $\pi_2$ is locally finite. □

*Proof of Theorem 2.5.27 (5b).* As in the previous proof, we identify $\mathcal{X}$ with $\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}$, and claim that $\pi_1$ is $\pi_0$-finite. let $\widetilde{\pi}_1 \colon \pi_0(\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}) \to \pi_0(\mathcal{A})$ be the induced map of isomorphism classes. Pick $[a] \in \mathcal{A}$, we will show its preimage under $\widetilde{\pi}_1$ is finite. First, we claim that $\pi_1$ is an isofibration. Indeed, if $(a, b, \varphi) \in \mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}$ is an object and $\alpha\colon a \to a'$ is an isomorphism, then $(\alpha, \mathrm{id})\colon (a, b, \varphi) \to (a', b, \varphi(Ff)^{-1})$ is an isomorphism in $\mathcal{A}\times^{(2)}_{\mathcal{C}}\mathcal{B}$ whose image under $\pi_1$ is $\alpha$, as desired.

With this in mind, let $\{(a, b_i, \varphi_i)\}_{i\in I}$ be a complete set of representatives for the isomorphism classes in $\widetilde{\pi}_1^{-1}([a])$. We aim to show that $I$ is finite. Since $G$

is $\pi_0$-finite, and each $b_i$ satisfies $G(b_i) \cong F(a)$, the $b_i$ all only belong to finitely many isomorphism classes in $\mathcal{B}$. As such, if we can show that for each $b \in \mathcal{B}$, the number of isomorphism classes in $\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}$ on the form $(a, b, -)$ is finite, then we will have shown that $I$ is finite, and the theorem will be proven. We introduce the following equivalence relations on $\mathrm{Hom}(Fa, Gb)$: For $\varphi, \psi \in \mathrm{Hom}(Fa, Gb)$ we say that

1. $\varphi \sim_1 \psi$, if there exist isomorphisms $f\colon a \to a$ and $g\colon b \to b$ such that the diagram

$$\begin{tikzcd} Fa \arrow[r, "\varphi"] \arrow[d, "Ff"'] & Gb \arrow[d, "Gg"] \\ Fa \arrow[r, "\psi"'] & Gb \end{tikzcd}$$

   commutes. This is the same as the objects $(a, b, \varphi)$ and $(a, b, \psi)$ being isomorphic in $\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B}$

2. $\varphi \sim_2 \psi$, if there exists an isomorphism $f\colon a \to a$ such that the diagram

$$\begin{tikzcd} Fa \arrow[r, "\varphi"] \arrow[d, "Ff"'] & Gb \arrow[d, equal] \\ Fa \arrow[r, "\psi"'] & Gb \end{tikzcd}$$

   This is the same as the objects $(a, \varphi)$ and $(a, \psi)$ being isomorphic in 2-Fib$(F|_{\mathcal{A}(a)}, G(b))$.

These are clearly equivalence relations. Note that $\varphi \sim_2 \psi \implies \varphi \sim_1 \psi$, so $\sim_2$ is finer than $\sim_1$. This implies the inequality

$$|\mathrm{Hom}(Fa, Gb)/{\sim_1}| \leq |\mathrm{Hom}(Fa, Gb)/{\sim_2}| = \left|\pi_0\left(\text{2-Fib}(F|_{\mathcal{A}(a)}, b)\right)\right|. \quad (2.5.28)$$

The last equality is given by the bijection

$$\begin{aligned} (\mathrm{Hom}(Fa, Gb)/{\sim_2}) &\xrightarrow{\sim} \pi_0\left(\text{2-Fib}(F|_{\mathcal{A}(a)}, b)\right) \\ [\varphi] &\mapsto [(a, \varphi)]. \end{aligned}$$

By the definition of $\sim_2$, this assignment is well-defined and injective. For it to be surjective we need that each element of $\pi_0\left(\text{2-Fib}(F|_{\mathcal{A}(a)}, b)\right)$ can be written on the form $(a, \varphi)$, i.e with first component equal to $a$. This can be shown in a similar manner to how we proved that $\pi_1$ is an isofibration, using the fact that the first component of an object in $\pi_0\left(\text{2-Fib}(F|_{\mathcal{A}(a)}, b)\right)$ must be isomorphic to $a$.

Consider again the inequality 2.5.28. The rightmost value is finite, so the leftmost value must also be. But the leftmost value corresponds to the number of isomorphism classes on the form $(a, b, -)$, and as previously discussed this value always being finite implies that $\pi_1$ is $\pi_0$-finite. □

*Proof of Theorem 2.5.27 (5c).* Once again, we identify our 2-pullback diagram with the diagram

$$\begin{tikzcd}
\mathcal{A} \times^{(2)}_{\mathcal{C}} \mathcal{B} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\Nearrow"] & \mathcal{B} \arrow[d, "G"] \\
\mathcal{A} \arrow[r, "F"'] & \mathcal{C}
\end{tikzcd}$$

We compute, for some $\rho \in \mathcal{F}(\mathcal{A})$:

$$\begin{aligned}
(G^* \circ F_!)(\rho)(b) = F_!(\rho)(Gb) = \\
&= \int_{\text{2-Fib}(F,Gb)} \rho\big|_{\text{2-Fib}(F,Gb)} \\
&= \sum_{[(a,\varphi\colon Fa \to Gb)]} \frac{\rho(a)}{|\mathrm{Aut}_{\text{2-Fib}(F,Gb)}((a,\varphi))|}.
\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}
((\pi_2)_! \circ \pi_1^*)(\rho)(b) &= (\pi_2)_!(\rho \circ \pi_1) \\
&= \int_{\text{2-Fib}(\pi_2,b)} (\rho \circ \pi_1)\big|_{\text{2-Fib}(\pi_2,b)} \\
&= \sum_{[((a,b_0,\varphi\colon Fa \to Gb_0),\alpha\colon b_0 \to b)]} \frac{\rho \circ \pi_1((a, b_0, \varphi))}{|\mathrm{Aut}(((a, b_0, \varphi), \alpha))|} \\
&= \sum_{[((a,b_0,\varphi),\alpha)]} \frac{\rho(a)}{|\mathrm{Aut}_{\text{2-Fib}(\pi_2,b)}(((a, b_0, \varphi), \alpha))|}.
\end{aligned}$$

As in our proof of Theorem 2.5.27 (5$a$), each object $((a, b_0, \varphi), \alpha)$ is isomorphic to an object on the form $((a, b, \varphi'), \mathrm{id}_b)$, and isomorphisms of such objects correspond naturally to isomorphisms in 2-Fib$(F, Gb)$. This gives rise to a bijection

$$\begin{aligned}\pi_0\,(\text{2-Fib}(\pi_2, b)) &\xrightarrow{\sim} \pi_0\,(\text{2-Fib}(F, Gb))\\ [((a, b, \varphi), \mathrm{id}_b)] &\mapsto [(a, \varphi)].\end{aligned}$$

There is also a bijection

$$\begin{aligned}\mathrm{Aut}_{\text{2-Fib}(\pi_2,b)}((a, b, \varphi), \mathrm{id}_b)) &\xrightarrow{\sim} \mathrm{Aut}_{\text{2-Fib}(F,Gb)}((a, \varphi)).\\ (f, \mathrm{id}) &\mapsto f.\end{aligned}$$

Which allows us to write

$$\begin{aligned}((\pi_2)_! \circ \pi_1^*)(\rho)(b) &= \sum_{[((a,b_0,\varphi),\alpha)]} \frac{\rho(a)}{|\mathrm{Aut}_{\text{2-Fib}(\pi_2,b)}(((a, b_0, \varphi), \alpha))|}\\ &= \sum_{[(a,\varphi\colon Fa\to Gb)]} \frac{\rho(a)}{|\mathrm{Aut}_{\text{2-Fib}(F,Gb)}((a, \varphi))|}\\ &= (G^* \circ F_!)(\rho)(b).\end{aligned}$$

We conclude that

$$(\pi_2)_! \circ \pi_1^* = G^* \circ F_!.$$

As desired. □

*Remark* 2.5.29. Informally, one intesting aspect of Theorem 2.5.27(5) is that it turns diagrams on the form

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{D} \end{tikzcd}$$

into diagrams on the form

$$\begin{tikzcd} \mathcal{F}(\mathcal{A}) \arrow[r] & \mathcal{F}(\mathcal{B}) \\ \mathcal{F}(\mathcal{C}) \arrow[r] \arrow[u] & \mathcal{F}(\mathcal{D}) \arrow[u] \end{tikzcd}$$

As opposed to, say, a contravariant functor which would turn it into a diagram on the form

$$\begin{tikzcd} \mathcal{F}(\mathcal{A}) & \mathcal{F}(\mathcal{B}) \arrow[l] \\ \mathcal{F}(\mathcal{C}) \arrow[u] & \mathcal{F}(\mathcal{D}) \arrow[l] \arrow[u] \end{tikzcd}$$

### 2.5.1 Deriving the Classical Hall Algebra from the Abstract Hall Algebra

In this section, we will finally recover the Hall algebra for proto-abelian categories. First, we prove a comparatively simple lemma.

**Lemma 2.5.30.** *Let $\mathcal{C}$ be a proto-abelian category and let $d_1 = \mathcal{S}_{\{0,1,2\}} \to \mathcal{S}_{\{0,2\}}$ be the first face map. Choose an object $C \in \mathcal{C}$ and consider the two-fiber* 2-Fib$(d_1, C)$. *The following statements hold:*

*a) Each isomorphism class in* 2-Fib$(d_1, C)$ *can be represented by a pair*

$$((A \hookrightarrow \mathcal{C} \twoheadrightarrow B), \mathrm{id}_C),$$

*i.e. an exact sequence with middle object equal to $C$, and the identity of $C$ as its second component.*

*b) Let $\mathcal{P}(C)$ denote the set of subobjects of $C$. Then there is a bijection of sets*

$$\begin{aligned} \pi_0(\text{2-Fib}(d_1, C)) &\xrightarrow{\sim} \mathcal{P}(C) \\ [(A \overset{i}{\hookrightarrow} C \twoheadrightarrow B, \mathrm{id}_C)] &\mapsto [A \overset{i}{\hookrightarrow} C]. \end{aligned}$$

*Proof. a)* For an object $((A \hookrightarrow C' \twoheadrightarrow B), \varphi\colon C' \to C) \in$ 2-Fib$(d_1, C)$ the morphism of exact sequences

$$\begin{tikzcd} A' \arrow[r, "i"] \arrow[d, equal] & C' \arrow[r, "p"] \arrow[d, "\varphi"] & B' \arrow[d, equal] \\ A' \arrow[r, hook, "\varphi\circ i"'] & C \arrow[r, "p\circ\varphi^{-1}"'] & B' \end{tikzcd}$$

is an isomorphism to the desired representative in 2-Fib$(R, C)$. The fact that this is a valid morphism in 2-Fib$(R, C)$ is witnessed by the fact that the diagram

$$\begin{tikzcd} C' \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & C \\ C \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

commutes.

*b)* We claim that if we have two objects $((A \hookrightarrow C \twoheadrightarrow B), \mathrm{id})$ and $((A' \hookrightarrow C \twoheadrightarrow B'), id)$ of the 2-fiber, such that $A \hookrightarrow C$ and $A' \hookrightarrow C$ determine the same subobject, then these two objects are isomorphic as objects of the 2-fiber. Consider the diagram

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "\varphi"'] & C \arrow[r, two heads, "p"] \arrow[d, equal] & B \arrow[d, dotted, "\exists\psi"] \\ A' \arrow[r, hook, "j"] & C \arrow[r, two heads, "q"] & B' \end{tikzcd}$$

where the left square commutes because the two inclusions into $C$ determine the same subobject. We have that $q \circ i = q \circ j \circ \varphi = 0$, so by the universal property of the cokernel there is a unique map $\psi\colon B \longrightarrow B'$ such that $\psi \circ p = q$. We claim that $\psi$ is an isomorphism. Indeed, Since $\phi$ is an isomorphism, we symmetrically obtain $\psi'\colon B' \longrightarrow B$ such that $\psi' \circ q = p$. We compute

$$\psi' \circ \psi \circ p = \psi' \circ q = p,$$

which implies $\psi' \circ \psi = \mathrm{id}_B$ since $p$ is epi. Symmetrically we obtain $\psi \circ \psi' = \mathrm{id}_{B'}$, showing that $\psi$ and $\psi'$ are mutually inverse isomorphisms.

Now we consider the relevant assignment

$$\pi_0(\text{2-Fib}(d_1, C)) \xrightarrow{\sim} \mathcal{P}(C)$$
$$[(A \overset{i}{\hookrightarrow} C \twoheadrightarrow B, \mathrm{id}_C)] \mapsto [A \overset{i}{\hookrightarrow} C].$$

This is well-defined since for any two isomorphic objects of the 2-fiber, the induced subobjects are equal. It is injective by the above discussion. It is surjective since cokernels of monomorphisms exist in proto-abelian categories, so that for any subobject $[A \overset{i}{\hookrightarrow} C]$, $((A \overset{i}{\hookrightarrow} C \twoheadrightarrow \operatorname{coker} i), \mathrm{id})$ is an object of the 2-fiber. □

**Definition 2.5.31.** We introduce the groupoid $\mathrm{Span}^f(\mathbf{Grpd}) \subset \mathrm{Span}(\mathbf{Grpd})$ as the groupoid of spans $\mathcal{A} \xleftarrow{L} \mathcal{X} \xrightarrow{R} \mathcal{B}$ where $L$ is $\pi_0$-finite and $R$ is locally finite. We claim that compositions of such spans are well-defined. Indeed, consider the composition below

$$\begin{tikzcd} & & \mathcal{Z} \arrow[dl, "L_3"'] \arrow[dr, "R_3"] & & \\ & \mathcal{X} \arrow[dl, "L_1"'] \arrow[dr, "R_1"'] \arrow[rr, phantom, "\Longrightarrow"] & & \mathcal{Y} \arrow[dl, "L_2"] \arrow[dr, "R_2"] & \\ \mathcal{A} & & \mathcal{B} & & \mathcal{C} \end{tikzcd}$$

the upper square is 2-pullback, so by Theorem 2.5.27 $L_3$ is $\pi_0$-finite and $R_3$ is locally finite. Since compositions of locally finite maps are locally finite, and compositions of $\pi_0$-finite maps are $\pi_0$-finite (Theorem 2.5.27), the composite span

$$\mathcal{A} \xleftarrow{L_1 \circ L_3} \mathcal{Z} \xrightarrow{R_2 \circ R_3} \mathcal{C}$$

is a morphism in $\mathrm{Span}^f(\mathbf{Grpd})$. It is clear that identity spans are in $\mathrm{Span}^f(\mathbf{Grpd})$. (The locally finite case is easily handled by Lemma 2.5.21).

**Theorem 2.5.32.** [Dyc18, Prop. 3.15(i)] *Let $\mathcal{C}$ be a finitary proto-abelian category. Then the abstract Hall Algebra $(\mathcal{S}_1, \mu, e)$ defines an algebra object in the category* $\mathrm{Span}^f(\mathbf{Grpd})$.

*Proof.* The only thing to prove is that in the spans

$$\mu: \quad \begin{tikzcd} & \mathcal{S}_{\{0,1,2\}} \arrow[dl, "(d_2{,}d_0)"'] \arrow[dr, "d_1"] & \\ \mathcal{S}_{\{0,1\}} \times \mathcal{S}_{\{1,2\}} & & \mathcal{S}_{\{0,2\}} \end{tikzcd}$$

and the morphisms $(d_2, d_0)$ and $\mathrm{id}_{\mathcal{S}_0}$ are $\pi_0$-finite and the morphisms $d_1$ and

$$e: \quad \begin{tikzcd} & \mathcal{S}_0 \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\sigma"] & \\ \mathcal{S}_0 & & \mathcal{S}_1 \end{tikzcd}$$

$\sigma$ are locally finite. It is clear that id is $\pi_0$-finite. We first show that $\sigma$ is locally finite. Since $\mathcal{S}_0$ only has one object, this is the same as showing that $\sigma$ is finite. The only nonzero 2-fiber of $\sigma$ is the groupoid $2\text{-Fib}(\sigma, \{0\})$, which is isomorphic to the trivial groupoid and therefore finite. We have shown that $e$ is a morphism in $\mathrm{Span}^f(\mathbf{Grpd})$ and now claim that the morphism

$(d_2, d_0)\colon \mathcal{S}_{\{0,1,2\}} \longrightarrow \mathcal{S}_{\{0,1\}} \times \mathcal{S}_{\{1,2\}}$ is $\pi_0$-finite. Denote by $h$ the induced map $\pi_0(\mathcal{S}_2) \longrightarrow \pi_0(\mathcal{S}_1 \times \mathcal{S}_1)$, choose an object $[A, C]$ in $\pi_0(\mathcal{S}_1 \times \mathcal{S}_1)$ and consider its preimage under $h$. It consists of the isomorphism classes represented by short exact sequences $(A' \hookrightarrow \mathcal{B} \twoheadrightarrow C')$ with $A' \cong A$, $C' \cong C$. We claim that each such isomorphism class can be represented by an exact sequence $(A \hookrightarrow B \twoheadrightarrow C)$, i.e, with first term $A$ and third term $C$. Indeed, consider the diagram

$$\begin{tikzcd}
A \arrow[dr, "\varphi"] & & & \\
& A' \arrow[r, hook, "i"] \arrow[d, two heads] & B \arrow[d, two heads, "p"] & \\
& 0 \arrow[r, hook] & C' \arrow[dr, "\psi"'] & \\
& & & C
\end{tikzcd}$$

If the middle square is bicartesian, then so is the square

$$\begin{tikzcd}
A \arrow[r, hook, "i\circ\varphi"] \arrow[d, two heads] & B \arrow[d, "p"] \\
0 \arrow[r, hook] & C'
\end{tikzcd}$$

and therefore also the square

$$\begin{tikzcd}
A \arrow[r, hook, "i\circ\varphi"] \arrow[d, two heads] & B \arrow[d, "\psi\circ p"] \\
0 \arrow[r, hook] & C'
\end{tikzcd}$$

These exact sequences are isomorphic as objects in $\mathcal{S}_2$ by the isomorphism

$$\begin{tikzcd}
A' \arrow[r, hook, "i"'] \arrow[d, "\varphi^{-1}"'] & B \arrow[r, two heads, "p"'] \arrow[d, equal] & C' \arrow[d, "\psi"'] \\
A \arrow[r, "i\circ\varphi"] & B \arrow[r, "\psi\circ p"] & C
\end{tikzcd}$$

We therefore lose no generality by considering only the exact sequences in $\mathcal{S}_2$ with first term $A$ and third term $C$. We denote the groupoid of such sequences by $\{A \hookrightarrow \bullet \twoheadrightarrow C\}$. Explicitly, we have that

$$|h^{-1}([(A, C)])| = |\pi_0(\{A \hookrightarrow \bullet \twoheadrightarrow C\})|.$$

Note that for two exact sequences $\xi, \varepsilon \in \{A \hookrightarrow \bullet \twoheadrightarrow C\}$:

$$(\xi \sim \varepsilon) \text{ as elements of } \operatorname{Ext}(C, A) \implies (\xi \cong \varepsilon) \text{ as objects of } \{A \hookrightarrow \bullet \twoheadrightarrow C\}.$$

Passing to equivalence classes, we have

$$|h^{-1}([(A, C)])| = |\pi_0(\{A \hookrightarrow \bullet \twoheadrightarrow C\})| \leq |\operatorname{Ext}(C, A)| < \infty,$$

where the last equality is due to $\mathcal{C}$ being finitary. We conclude that $(d_2, d_0)$ is $\pi_0$-finite.

The final claim to show is that $d_1 : \mathcal{S}_{\{0,1,2\}} \to \mathcal{S}_{\{0,2\}}$ is locally finite. Choose an exact sequence $\xi = (A_0 \hookrightarrow B_0 \twoheadrightarrow C_0) \in \mathcal{S}_2$. We want to check that the functor $d_1|_{\mathcal{S}_2(\xi)}$ is finite. The only nonzero 2-fiber of this functor is the groupoid 2-Fib$(d_1|_{\mathcal{S}_2(\xi)}, B_0)$. Objects of this groupoid are tuples

$$((A \hookrightarrow \mathcal{B} \twoheadrightarrow C), \alpha : B \to B_0),$$

where $A \cong A_0$ and $C \cong C_0$. By Lemma 2.5.30 each such tuple is isomorphic with a tuple on the form $((A \hookrightarrow B_0 \twoheadrightarrow C), \mathrm{id}_{B_0})$.

We first show that the Hom-sets in 2-Fib$(d_1|_{\mathcal{S}_2(\xi)}, B_0)$ are finite. Due to properties of groupoids, it suffices to show that the automorphism groups are finite. An automorphism of an object $((A \hookrightarrow B_0 \twoheadrightarrow C), \mathrm{id})$ is a triple

$$(u, v, w) \in \operatorname{Aut}(A) \times \operatorname{Aut}(B_0) \times \operatorname{Aut}(C),$$

making certain diagrams commute. Without having to consider what those diagrams are, we can immediately conclude that

$$|\operatorname{Aut}(((A \hookrightarrow B_0 \twoheadrightarrow C), \mathrm{id}))| \leq |\operatorname{Aut}(A) \times \operatorname{Aut}(B_0) \times \operatorname{Aut}(C)| < \infty,$$

where the last inequality is due to $\mathcal{C}$ being finitary.

It remains to show that the number of isomorphism classes in 2-Fib$(d_1|_{\mathcal{S}_2(\xi)}, B_0)$ is finite. By Lemma 2.5.30, the isomorphism classes are in bijection with the number of subobjects $A \hookrightarrow B_0$ of $B_0$, where $A \cong A_0$. We claim that each such subobject can be represented by a morphism $A_0 \hookrightarrow C$, i.e with domain equal to $A_0$. Indeed, this is verified by the diagram

$$\begin{tikzcd} A_0 \arrow[r, hook, "i \circ \varphi"] \arrow[d, "\varphi"'] & B_0 \\ A \arrow[ur, "i"'] & \end{tikzcd}$$

in which $\varphi\colon A_0 \to A$ is an isomorphism. Thus we have that each subobject of $B_0$ in $\mathcal{S}_2(\xi)$ can be represented by some morphism $A_0 \hookrightarrow B_0$. This yields the inequality

$$|\pi_0(\text{2-Fib}(d_1|_{\mathcal{S}_2(\xi)}, B_0))| \leq |\mathrm{Hom}_{\mathcal{C}}(A_0, B_0)| < \infty,$$

where the last inequality uses that $\mathcal{C}$ is finitary. Thus the 2-fiber is finite, as desired.

We have now finished showing that $(\mathcal{S}_1, \mu, e)$ determines an algebra object in $\mathrm{Span}^f(\mathbf{Grpd}) \subset \mathrm{Span}(\mathbf{Grpd})$. □

**Theorem 2.5.33.** [Dyc18, Prop. 3.15(ii)] *Let $\mathcal{C}$ be a finitary proto-abelian category. The assignment*

$$\begin{aligned} \mathcal{F}\colon\ \mathrm{Span}^f(\mathbf{Grpd}) &\to \mathbf{Vect}_{\mathbb{Q}} \\ \mathcal{A} &\mapsto \mathcal{F}(\mathcal{A}) \\ [\mathcal{A} \xleftarrow{L} \mathcal{X} \xrightarrow{R} \mathcal{B}] &\mapsto (\mathcal{F}(\mathcal{A}) \xrightarrow{R_! \circ L^*} \mathcal{F}(\mathcal{B})). \end{aligned}$$

*is a functor.*

*Proof.*

- (Well-Definedness) We need that $\mathcal{F}$ is well-defined in the sense that two equivalent spans are sent to the same morphism in $\mathbf{Vect}_{\mathbb{Q}}$. This is one of the statements of Theorem 2.5.27.

- (Unitality) The identity span $\mathcal{A} \xleftarrow{\mathrm{id}} \mathcal{A} \xrightarrow{\mathrm{id}} \mathcal{A}$ is sent to the linear map $(\mathrm{id}_! \circ \mathrm{id}^*)\colon \mathcal{F}(\mathcal{A}) \to \mathcal{F}(\mathcal{A})$. We claim that this is the identity map. $\mathcal{F}(\mathcal{A})$ has a basis given by the maps $\overline{a}\colon \mathcal{A} \to \mathbb{Q}$, for $[a] \in \pi_0(\mathcal{A})$. For these basis maps

  $$(\mathrm{id}_! \circ \mathrm{id}^*)(\overline{a}) = \mathrm{id}_!(\overline{a}) = \overline{a}$$

  Where the last equality is a special case of Corollary 2.5.24. Since $(\mathrm{id}_! \circ \mathrm{id}^*)$ is the identity on the basis vectors, it is the identity on all of $\mathcal{F}(\mathcal{F})$.

- (Functoriality)

  Consider the composition of spans:

$$
\begin{tikzcd}
 & & \mathcal{Z} \arrow[dl, "L_3"'] \arrow[dr, "R_3"] & & \\
 & \mathcal{X} \arrow[dl, "L_1"'] \arrow[dr, "R_1"'] \arrow[rr, phantom, "\Longrightarrow"] & & \mathcal{Y} \arrow[dl, "L_2"] \arrow[dr, "R_2"] & \\
\mathcal{A} & & \mathcal{B} & & \mathcal{C}
\end{tikzcd}
$$

We compute:

$$
\begin{aligned}
\mathcal{F}\left(\mathcal{A} \xleftarrow{L_1\circ L_3} \mathcal{Z} \xrightarrow{R_2\circ R_3} \mathcal{C}\right) &= (R_2 \circ R_3)_! \circ (L_1 \circ \mathrm{L}_3)^* \\
&= (R_2)_! \circ (R_3)_! \circ L_3^* \circ L_1^* \\
&= (R_2)_! \circ (L_2)^* \circ (R_1)_! \circ L_1^* \\
&= \mathcal{F}\left(\mathcal{B} \xleftarrow{L_2} \mathcal{Y} \xrightarrow{R_2} \mathcal{C}\right) \circ \mathcal{F}\left(\mathcal{A} \xleftarrow{L_1} \mathcal{X} \xrightarrow{R_1} \mathcal{B}\right)
\end{aligned}
$$

where we in each step use the properties in Theorem 2.5.27.

□

*Remark* 2.5.34. If $\{v_i\}_{i=1}^n$ is a basis for the vector space $V$ and $\{w_j\}_{j=1}^m$ is a basis for the vector space $W$, then the vector space $V \otimes W$ has a basis given by the vectors $v_i \otimes w_j$ for all $i, j$. This means that, for groupoids $\mathcal{A}$ and $\mathcal{B}$, we have a canonical isomorphism

$$
\begin{aligned}
J_{\mathcal{A},\mathcal{B}} \colon \mathcal{F}(\mathcal{A}) \otimes \mathcal{F}(\mathcal{B}) &\xrightarrow{\sim} \mathcal{F}(\mathcal{A} \times \mathcal{B}) \\
\overline{a} \otimes \overline{b} &\mapsto \overline{(a, b)}
\end{aligned}
$$

We also have a canonical isomorphism

$$
\begin{aligned}
\varphi \colon \mathbb{Q} &\xrightarrow{\sim} \mathcal{F}(\mathcal{S}_0) \\
1 &\mapsto \overline{0}
\end{aligned}
$$

It turns out that the triple $(F, J, \varphi)$ satisfies the requirements for $F$ to be a so-called *monoidal functor*. Again, we do not go into the details of monoidal categories and functors, (Remark 2.5.34). For details about monoidal functors, see ([EGNO16], section 2.4).

It is a property of $(F, J, \varphi)$ being a monoidal functor that if we write

$$
\hat{\mu} = \mathcal{F}(\mathcal{S}_1) \otimes \mathcal{F}(\mathcal{S}_1) \xrightarrow{J_{\mathcal{S}_1,\mathcal{S}_1}} \mathcal{F}(\mathcal{S}_1 \times \mathcal{S}_1) \xrightarrow{\mathcal{F}(\mu)} \mathcal{F}(\mathcal{S}_1)
$$

and

$$\hat{e} = \mathbb{Q} \xrightarrow{\varphi} \mathcal{F}(\mathcal{S}_0) \xrightarrow{\mathcal{F}(e)} \mathcal{F}(\mathcal{S}_1)$$

then the tuple $(\mathcal{F}(\mathcal{S}_1), \hat{\mu}, \hat{e})$ forms a monoidal object in $\mathbf{Vect}_{\mathbb{Q}}$, or in other words an algebra over $\mathbb{Q}$.

**Theorem 2.5.35.** *Let $\mathcal{C}$ be a finitary proto-abelian category. Using the notation of Remark 2.5.34, the algebra object $(\mathcal{F}(\mathcal{S}_1), \hat{\mu}, \hat{e})$ has the same multiplication and underlying vector space as $\mathrm{Hall}_{\mathbb{Q}}(\mathcal{C})^{\mathrm{op}}$. In other words, $\mathrm{Hall}_{\mathbb{Q}}(\mathcal{C})$ is well defined and isomorphic to the opposite of $(\mathcal{F}(\mathcal{S}_1), \hat{\mu}, \hat{e})$.*

*Proof.* Recall from Definition 2.5.6 that $\mathcal{F}(\mathcal{S}_1)$ is the vector space of formal linear combinations of objects in $\mathcal{C}$, through the identification

$$\begin{aligned} \mathcal{F}(\mathcal{S}_1) &\xrightarrow{\sim} \bigoplus_{[C] \in \pi_0(\mathcal{C})} \mathbb{Q}[C] \\ \overline{C} &\mapsto [C]. \end{aligned}$$

Therefore, we wish to show that, for $A, B \in \mathcal{C}$,

$$\hat{\mu}(\overline{A} \otimes \overline{B}) = \sum_{[C] \in \pi_0(\mathcal{C})} g^C_{B,A} \overline{C},$$

where $g^C_{B,A}$ as usual denotes the number of subobjects of $C$ isomorphic to $A$ whose quotient is isomorphic to $B$. Notice the change in order of $A$ and $B$ from $\overline{A} \otimes \overline{B}$ to $g^C_{B,A}$, when this difference did not exist in the definition of the Hall Algebra for proto-abelian categories. This is why $(\mathcal{F}(\mathcal{S}_1), \hat{\mu}, \hat{e})$ is isomorphic to the *opposite* of the Hall algebra. Recall the definition of $\mu$:

$$\mu = \mathcal{S}_{\{0,1\}} \times \mathcal{S}_{\{1,2\}} \xleftarrow{L} \mathcal{S}_{\{0,1,2\}} \xrightarrow{R} \mathcal{S}_{\{0,2\}}.$$

We compute:

$$\begin{aligned} \hat{\mu}(\overline{A} \otimes \overline{B})(C) &= (\mu \circ J_{\mathcal{S}_1, \mathcal{S}_1})(\overline{A} \otimes \overline{B})(C) = \mu\left(\overline{(A, B)}\right)(C) \\ &= (R_! \circ L^*)\left(\overline{(A, B)}\right)(C) = R_!\left(\overline{(A, B)} \circ L\right)(C) \\ &= \int_{\text{2-Fib}(R,C)} \overline{(A, B)} \circ L\big|_{\text{2-Fib}(R,C)} \\ &= \sum_{[(\xi,\varphi)] \in \pi_0(\text{2-Fib}(R,C))} \frac{\overline{(A, B)} \circ L(\xi)}{|\mathrm{Aut}((\xi, \varphi))|}, \end{aligned}$$

where $\xi$ denotes short exact sequences $A' \hookrightarrow C' \twoheadrightarrow B'$, and $\varphi$ is an isomorphism $C' \to C$. Since $R = d_1 \colon \mathcal{S}_{\{0,1,2\}} \to \mathcal{S}_{\{0,2\}}$, Lemma 2.5.30(a) tells us that each isomorphism class in 2-Fib$(R, C)$ can be represented by a pair $((A' \hookrightarrow C, \twoheadrightarrow B), \mathrm{id}_C)$, i.e with middle term equal to $C$, and with the identity in the second component. Also notice that for $\xi = A' \hookrightarrow C' \twoheadrightarrow B'$ if $A'$ is not isomorphic to $A$ or $B'$ is not isomorphic to $B$ then the value

$$\overline{(A, B)} \circ L(\xi) = \overline{(A, B)}(A', B') = 0.$$

Thus, we may sum only over isomorphism classes of exact sequences where the first object is isomorphic to $A$ and the last object is isomorphic to $B$. We have:

$$\hat{\mu}(\overline{A} \otimes \overline{B})(C) = \sum_{\substack{[(\xi = A' \hookrightarrow C \twoheadrightarrow B', \mathrm{id})] \\ A' \cong A, B' \cong B}} \frac{1}{|\mathrm{Aut}((\xi, \mathrm{id}))|}. \tag{2.5.36}$$

We claim that $|\mathrm{Aut}((\xi, id))| = 1$. Indeed, an element of $\mathrm{Aut}((\xi, \mathrm{id}))$ is a triple of morphisms $(u, v, w)$ so that the diagrams

$$\begin{tikzcd} A' \arrow[r, "i"] \arrow[d, "u"] & C \arrow[r, "p"] \arrow[d, "v"] & B' \arrow[d, "w"] \\ A' \arrow[r, "i"] & C \arrow[r, "p"] & B' \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} C \arrow[r, "\mathrm{id}"] \arrow[d, "v"] & C \\ C \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

commute. But the rightmost diagram implies $v = \mathrm{id}_C$, and then the leftmost diagrams imply $u = \mathrm{id}_{A'}, w = \mathrm{id}_{B'}$. Therefore $(\mathrm{id}_{A'}, \mathrm{id}_C, \mathrm{id}_{B'})$ is the only element of $\mathrm{Aut}((\xi, \mathrm{id}))$, as desired.

We now have by Eq. (2.5.36) that $\hat{\mu}(\overline{A} \otimes \overline{B})(C)$ equals the number terms in the sum, which is the number of isomorphism classes

$$((A' \hookrightarrow C \twoheadrightarrow B'), \mathrm{id}) \in \pi_0(\text{2-Fib}(R, C))$$

with $A' \cong A$ and $B' \cong B$. By Lemma 2.5.30(b), this is equal to the number of subobjects $A' \subseteq C$ such that $C/A' \cong B$. In other words,

$$\hat{\mu}(\overline{A} \otimes \overline{B})(C) = g^C_{B,A}.$$

Since our initial choice of object $C$ was arbitrary, we have

$$\hat{\mu}(\overline{A} \otimes \overline{B}) = \sum_{C \in \pi_0(C)} g^C_{B,A} \overline{C},$$

as desired.

We need to show that this algebra object has the same unit as $\mathrm{Hall}_{\mathbb{Q}}(\mathcal{C})^{\mathrm{op}}$. To do this, we need to show that for $\hat{e}\colon \mathbb{Q} \to \mathcal{F}(\mathcal{S}_1)$, we have $\hat{e}(1) = \overline{0}$, where 0 denotes the zero object of $\mathcal{C}$. Recall the definition of $e$:

$$e = \mathcal{S}_0 \xleftarrow{\mathrm{id}} \mathcal{S}_0 \xrightarrow{\sigma} \mathcal{S}_1$$

We compute:

$$\begin{aligned} \hat{e}(1) &= (\sigma_! \circ \mathrm{id}^*)(\varphi(1)) = \sigma_!(\overline{0}) = \\ &= \frac{|\mathrm{Aut}_{\mathcal{S}_1}(0)|}{|\mathrm{Aut}_{\mathcal{S}_0}(0)|}\overline{0} = \overline{0}, \end{aligned}$$

where we use Theorem 2.5.23. This finishes the proof. □

*Remark* 2.5.37. We have finally proven Theorem 1.2.13, establishing the existance of Hall Algebras for proto-abelian categories.

# Appendix A

# Lemmas for Pushouts and Pullbacks

We state and prove some important lemmas for pushouts and pullbacks which are important for the thesis.

**Lemma A.0.1** (Pasting Lemma for Pushouts)**.** *In the diagram below, suppose the leftmost square is pushout. Then the outer rectangle is pushout if and only if the rightmost square is pushout.*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "p"] \arrow[d, "g"'] & C \arrow[d, "h"] \\ A' \arrow[r, "j"'] & B' \arrow[r, "q"'] & C' \end{tikzcd}$$

*Proof.* Assume the rightmost square is pushout. We claim the outer rectangle is pushout. Let $\varphi\colon A' \to X$ and $\psi\colon C \to X$ be arbitrary maps to some object $X$, such that $\varphi \circ f = \psi \circ p \circ i$. Then the pushout property of the left square gives a unique map $\eta\colon B' \to X$ such that $\eta \circ j = \varphi$ and $\eta \circ g = \psi \circ p$. The latter of these equalities allows us to use the pushout property of right square, which gives a unique map $\varepsilon\colon C' \to X$ such that $\varepsilon \circ h = \psi$ and $\varepsilon \circ q = \eta$. Now we use the various commutative squares.

$$\varepsilon \circ q \circ j = \eta \circ j = \varphi$$

and

$$\varepsilon \circ h = \psi.$$

And $\varepsilon$ is the unique map with this property, since any other such map $\varepsilon'$ would generate a map $\eta' = \varepsilon' \circ q\colon B' \to X$ with

$$\eta' \circ j = \varphi \text{ and } \eta' \circ g = \psi \circ p.$$

By uniqueness of the pushout property of the left square, $\eta' = \eta$, implying $\varepsilon' = \varepsilon$.

Conversely, assume the outer rectangle is pushout. We want to show that the right square is pushout. Let $\eta \colon B' \to X$ and $\psi \colon C \to X$ be maps such that $\psi \circ p = \eta \circ g$. Then

$$\psi \circ p \circ i = \eta \circ g \circ i = \eta \circ j \circ f.$$

We may therefore use the universal property of outer rectangle to obtain a unique map $\varepsilon \colon C' \to X$ such that $\varepsilon \circ h = \psi$ and $\varepsilon \circ q \circ j = \eta \circ j$. It remains to show that $\varepsilon \circ q = \eta$. Note that, if we can show that

$$(\varepsilon \circ q) \circ g \circ i = \psi \circ p \circ i.$$

Then the uniqueness part of the pushout property of the left square would imply that $\eta = \varepsilon \circ q$. However, this equality holds since

$$(\varepsilon \circ q) \circ g \circ i = \varepsilon \circ h \circ p \circ i = \psi \circ p \circ i.$$

□

Dually, we obtain the following lemma

**Lemma A.0.2** (Pasting lemma for pullbacks)**.** *In the diagram below, suppose the right square is pullback. Then the left square is pullback if and only if the outer square is pullback.*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "p"] \arrow[d, "g"'] & C \arrow[d, "h"] \\
A' \arrow[r, "j"'] & B' \arrow[r, "q"'] & C'
\end{tikzcd}$$

*Proof.* Dual of Lemma A.0.1. □

# Appendix B

# Group-theoretic Counting Lemmas

**Lemma B.0.1.** *Let $M, N$ be subgroups of a group $G$. Then we have an equality of indices*

$$[N : M \cap N] = [NM : M]$$

*Remark* B.0.2. Note that this lemma contains no assumptions on whether $M$ or $N$ are normal subgroups of $G$, in which case it would immediately follow from the second isomorphism theorem. The point is if we're only interested in the indices, this assumption is unnecessary.

*Proof.* We define a bijection

$$\begin{aligned} \varphi \colon N/(M \cap N) &\xrightarrow{\sim} NM/M \\ n(M \cap N) &\mapsto nM. \end{aligned}$$

Note here that $N/(M \cap N)$ and $NM/M$ denote **sets** of cosets, i.e. without necessarily having a group structure. One can show that this map is well-defined and injective the same way as the usual second isomorphism theorem. To show that it is surjective, let $nm \in NM$ be arbitrary. Then

$$[(nm)M] = [nM] \in \operatorname{im}(\varphi),$$

so that $\varphi$ is surjective. This completes the proof. □

*Remark* B.0.3. Note that if neither $N$ nor $M$ are normal, then we need not have $NM = MN$ in general. Thus the order $NM$ in the lemma is important. The final step of showing surjectivity of $\varphi$ would not have worked if we used $MN$ instead of $NM$. If we wanted to use $MN$, we could have used right cosets instead of left.

**Lemma B.0.4.** *Suppose we have a diagram of groups*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

*Where the quantities* $[C : \mathrm{im}(g)]$, $[B : \mathrm{im}(f)]$, $|\ker(f)|$, *and* $|\ker(g)|$ *are known to be finite. Then the quantities* $[C : \mathrm{im}(gf)]$ *and* $|\ker(gf)|$ *are finite, and we have an equality*

$$\frac{[B : \mathrm{im}(f)]}{|\ker(f)|} \cdot \frac{[C : \mathrm{im}(g)]}{|\ker(g)|} = \frac{[C : \mathrm{im}(gf)]}{|\ker(gf)|}, \tag{B.0.5}$$

*where $gf$, for brevity, denotes $g \circ f$.*

*Proof.* We claim that

$$\frac{[B : \mathrm{im}(f)]}{|\ker(g)|} = \frac{[B : \ker(g)\,\mathrm{im}(f)]}{|\ker(g) \cap \mathrm{im}(f)|}$$

We compute

$$[B : \ker(g)\,\mathrm{im}(f)] = \frac{[B : \mathrm{im}(f)]}{[\ker(g)\,\mathrm{im}(f) : \mathrm{im}(f)]} =$$
$$= \frac{[B : \mathrm{im}(f)]}{[\ker(g) : \ker(g) \cap \mathrm{im}(f)]} = \frac{[B : \mathrm{im}(f)]|\ker(g) \cap \mathrm{im}(f)|}{|\ker(g)|},$$

where the first equality is due to the tower law, the second due to Lemma B.0.1, and the third is true since the sets $\ker(f)$ and $\ker(g)$ are finite. This proves our claim. Let LHS denote the left-hand side in Eq. (B.0.5). We have:

$$\mathrm{LHS} = \frac{[C : \mathrm{im}(g)]}{|\ker(f)|} \cdot \frac{[B : \ker(g)\,\mathrm{im}(f)]}{|\ker(g) \cap \mathrm{im}(f)|}.$$

Now we claim that

$$|\ker(f)||\ker(g) \cap \mathrm{im}(f)| = |\ker(gf)|,$$

and in particular that $\ker(gf)$ is a finite set. Indeed, the first isomorphism theorem yields

$$\frac{\ker(gf)}{\ker(f)} \cong \ker(g) \cap \mathrm{im}(f)$$

as groups. Passing to cardinality yields the desired result. We summarize:

$$\mathrm{LHS} = \frac{[C : \mathrm{im}(g)][B : \ker(g)\,\mathrm{im}(f)]}{|\ker(gf)|}.$$

Thus, it only remains to show

$$[C : \operatorname{im}(g)][B : \ker(g)\operatorname{im}(f)] = [C : \operatorname{im}(gf)]$$

..

Which would, in particular, show that the right-hand side is finite. We continue:

$$\frac{B}{\ker(g)\operatorname{im}(f)} \cong \underbrace{\frac{B/\ker(g)}{(\ker(g)\operatorname{im}(f))/\ker(g)}}_{G} \cong \underbrace{\frac{\operatorname{im}(g)}{\operatorname{im}(gf)}}_{H}.$$

The left isomorphism is given by the third isomorphism theorem. The right isomorphism is induced by the diagram

$$\begin{tikzcd}
\frac{(\ker(g)\operatorname{im}(f))}{\ker(g)} \arrow[r, hook] \arrow[d, "u"] & \frac{B}{\ker(g)} \arrow[r, two heads] \arrow[dd, "w"'] & G \arrow[dd, dashed, "\exists!"] \\
\frac{\operatorname{im}(f)}{\ker(g)\cap\operatorname{im}(f)} \arrow[d, "v"] & & \\
\operatorname{im}(gf) \arrow[r, hook] & \operatorname{im}(g) \arrow[r, two heads] & H
\end{tikzcd}$$

where $v$ and $w$ are given by the first isomorphism theorem, $u$ by the second isomorphism theorem, and the left square is easily seen to commute by direct verification. We conclude that

$$[C : \operatorname{im}(g)][B : \ker(g)\operatorname{im}(f)] = [C : \operatorname{im}(g)][\operatorname{im}(g) : \operatorname{im}(gf)] = [C : \operatorname{im}(gf)].$$

as desired. This finishes the proof of the lemma. $\square$

# Bibliography


[DK12] Tobias Dyckerhoff and Mikhail Kapranov. Higher segal spaces i. 2012. preprint at arXiv:1212.3563.

[DK19] Tobias Dyckerhoff and Mikhail Kapranov. *Higher Segal spaces*, volume 2244 of *Lecture Notes in Mathematics*. Springer, Cham, 2019.

[Dyc18] Tobias Dyckerhoff. *Higher Categorical Aspects of Hall Algebras*, pages 1–61. Springer International Publishing, Cham, 2018.

[EGNO16] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor Categories*. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, October 2016.

[Hal59] Philip Hall. The algebra of partitions. *Proceedings of the 4th Canadian Mathematical Congress*, pages 147–159, 1959.

[Lan10] Saunders Mac Lane. *Categories for the working mathematician*. Graduate texts in mathematics. Springer New York, NY, 2 edition, 2010.

[Lei16] Tom Leinster. Basic category theory, 2016.

[Osb00] M. Scott Osborne. *Basic homological algebra*, volume 196 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[Rie11] Emily Riehl. A leisurely introduction to simplicial sets. 2011.

[Rin90] Claus Michael Ringel. Hall algebras and quantum groups. *Inventiones mathematicae*, 101:583–591, 1990.

[Ste01] E. Steinitz. Zur theorie der abel'schen gruppen. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 9(1):80–85, 1901.

Bachelor's Theses in Mathematical Sciences 2024:K12
ISSN 1654-6229

LUNFMA-4168-2024

Mathematics
Centre for Mathematical Sciences
Lund University
Box 118, SE-221 00 Lund, Sweden

http://www.maths.lu.se/