

Photon and electron transport in a resonator-coupled double quantum dot

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Abstract

The following thesis investigates the electron and photon transport in a microwave resonator-coupled double quantum dot (DQD). The investigation and description of this system are done mathematically. The DQD is also treated uncoupled from the resonator, for a description without photon interaction.

Mathematically the system is analyzed using a density matrix as well as a master equation approach.

The results were shown in the form of electric and photonic currents through the DQD that were plotted against the detuning δ of the energy levels in the dots.

It was found that the resonator-coupled DQD had electrical current peaks at two δ values, while the un-coupled DQD had only one, a difference that arises from enhanced transport through the DQD at the resonator's resonance frequency. The photon current is also plotted and shows a peak at the resonator's resonance frequency.

Contents

1	Introduction	4
2	Theory and method	5
2.1	Dirac notation	5
2.2	Density matrix	5
2.3	Quantum Master Equation	6
3	System and Quantum Model	6
3.1	Double Quantum Dot	6
3.2	System without resonator	9
3.3	Effective classical model	12
3.4	Calculating the tunneling rate γ	13
3.5	Resonator-coupled DQD	14
3.6	Finding the photon emission rate	19
4	Conclusion and Outlook	20
A	Appendix: Calculation for P_R (no photons):	24
B	Appendix: Calculations for resonator-coupled DQD	27
C	Appendix: Python Code for Current calculation	31

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Both Peter and Drilon showed me what the life of a researcher could look like and were very inclusive, letting me be part of weekly meetings with other researchers and students. All of this while always keeping things light and fun and not being too strict with me. I had a great time working on this thesis, all thanks to this team.

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1 Introduction

Quantum physics seems to be a term that captures the curiosity of many people and of course physicists above all. The idea that at small enough scales, our understanding of classical physics breaks down and all of a sudden probabilities and uncertainties take over is simply fascinating. Quantum physics is an immensely large field, and will by no means be completely talked about in this thesis.

In the following pages of this bachelor thesis, I will investigate an interesting system that consists of a double quantum dot (DQD) placed in an electric circuit within a microwave resonator. The primary focus of the research is to study the transport of single electrons through the double quantum dot. The system will be analyzed mathematically using a master equation approach and a density matrix approach.

A large bias is applied across the DQD such that the electron flow is uni-directional. An electron can tunnel from one dot to the other either sequentially or through the emission of a microwave photon. It should be noted, that because of this large bias and the way we treat the system, there will be no photon absorption.

The transmission of electrons through the DQD will be interpreted as electric current flowing in the circuit. The photons emitted by the electron will be seen as a photonic current.

This system could have several interesting real-world applications and serve as a single photon source or detector. In the field of quantum technologies such as quantum cryptography and quantum computing, reliable sources of single photons are essential. These technologies have the potential to impact various sectors of society such as healthcare, defense, financial modeling, meteorological prediction, and cybersecurity. [1]

However, there are some challenges in this setup. Achieving a reliable source of single microwave photons is difficult, as these processes happen at energy scales much lower than visible light photon sources. The entire system must be cooled down to extremely cold temperatures just above absolute zero, making it impractical for commercial use.

Nevertheless, these experiments can be valuable for proof of concept and finding models or ideas that could be useful in helping progress in other areas of the industry. Here in Lund, a lot of work is being done on this system with a main focus on single microwave photodetection rather than creation, and experiments are executed as well.

2 Theory and method

2.1 Dirac notation

Before exploring the theory and especially the mathematics used in this project, it is important to introduce the notation that was used, namely the Dirac notation.

In this notation, a vector α is represented by a *ket*, $|\alpha\rangle$. In quantum mechanics, the state of a system can be described by a state vector, which lies in an abstract mathematical space called Hilbert space. A state vector contains all the relevant information needed to describe a system in this state. So it is common to denote the state A of a system as $|A\rangle$. The complex conjugate of a state vector $|A\rangle$ is denoted by a *bra*, $\langle A|$.

This notation makes calculations convenient to write.

An inner product (or dot product) between two states $|A\rangle$ and $|B\rangle$ is written as $\langle A|B\rangle$ and as we know gives a number. This inner product also satisfies the condition $\langle a|b\rangle = \overline{\langle b|a\rangle}$, where $\overline{\langle b|a\rangle}$ is the complex conjugate of $\langle b|a\rangle$.

An outer product on the other hand results in a matrix and is written as $|A\rangle\langle B|$. The outer product is defined by $(|A\rangle\langle B|)|C\rangle = \langle B|C\rangle|A\rangle$.

For most calculations in this project, it is essential to remember, that the inner product between two states equals zero if they are perpendicular to one another. Similarly, $\langle a|\mathbf{0}|b\rangle = 0$, because the zero operator maps every vector to the zero vector.

Finally, we will use the following notation for the tensor product $|A\rangle \otimes |B\rangle = |A, B\rangle$, which is a way of combining two vector spaces, allowing us to describe two systems living in different Hilbert spaces.[2]

2.2 Density matrix

When a given system can be found in more than one state it is often convenient to approach it mathematically using the density matrix formalism.

In quantum mechanics, a system is in a pure state when it can be described by a single state vector.

A pure state $|\psi\rangle$ is defined by the following density matrix ρ

$$\rho = |\psi\rangle\langle\psi|. \tag{1}$$

A mixed state on the other hand arises when there is some uncertainty about the state of the system. Here the density matrix is a combination of all possible states and their probabilities

and can generally be written as [3]

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|, \quad (2)$$

where p_i is the probability to be in state $|\phi_i\rangle$.

A density matrix should adhere to a few principles. Firstly it is considered normalized if the trace of the matrix is equal to one, $\text{Tr}\{\rho\} = 1$; secondly, a density matrix should be self-adjointed, $\rho^\dagger = \rho$ and finally the density matrix is a positive operator. This is an important condition, connecting to the probability interpretation of the density matrix which cannot be negative.[3][2]

2.3 Quantum Master Equation

Now that the density matrix defines the state of our system, we are interested in the dynamics and time evolution of this quantum system. The equations of motion in such systems are given by the *quantum master equation* (QME). One of the most commonly used QME for density matrices is given by the Lindblad master equation[3][4]

$$\frac{d}{dt}\rho = -i[H, \rho] + \sum_\alpha \Gamma_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2} \{L_\alpha^\dagger L_\alpha, \rho\} \right) = \mathcal{L}\rho. \quad (3)$$

This differential equation has two terms. The first term is the Von-Neumann equation, which describes the evolution of the system governed by the inherent Hamiltonian H and disregards outside influences. The second term is often referred to as the Lindblad term and describes dissipative effects due to interaction with the environment via the Lindblad operators L_α . Γ_α is a rate that applies to each term of the Lindblad term. Another way of writing this differential equation is with the Liouvillian super operator \mathcal{L} . This will be interesting to consider later on.

3 System and Quantum Model

3.1 Double Quantum Dot

Quantum dots are nanoparticles that display fascinating properties. They can be considered as quasi-zero-dimensional crystals only a few nanometers in size. QDs are most commonly made of semiconductor material and find applications across various technologies, including QLED TVs, solar cells, and biomedicine. [5]

Quantum dots convey a similar image as a simple one-dimensional potential well, a key difference being, that QDs confine electrons in three dimensions, thus acting and often referred to as artificial atoms.[6] Similar to atoms, QDs also have quantized energy levels.

QDs differ from atoms in the possibility of manipulating their quantized energy levels. By adjusting the size of a quantum dot or applying external fields, one can tune the QD to and shift its energy levels, without having to find new elements as one would for atoms. This makes them extremely versatile and useful.

In this project, we will consider two quantum dots in series, forming a double quantum dot. This is an interesting setup because now there are two adjustable energy levels next to one another, which means that one can engineer the size of an energy gap. This engineered gap ensures that the energy of photon emissions resulting from transport across this barrier precisely matches the requirements for a given application.

QDs can be manufactured in a variety of ways: the Nobel Prize in chemistry in 2023 was awarded for the "discovery and synthesis of quantum dots"[7], where a new method was presented to create quantum dots very accurately by injecting substances into a high-temperature solvent.

Here at Lund University in the research center NanoLund, one way DQDs are engineered is by means of InAs nanowires (see Fig. 1) through epitaxial growth, which is a way of growing films onto substrates via condensation. The exact workings of this are not part of this thesis and will not be explored further.[8]

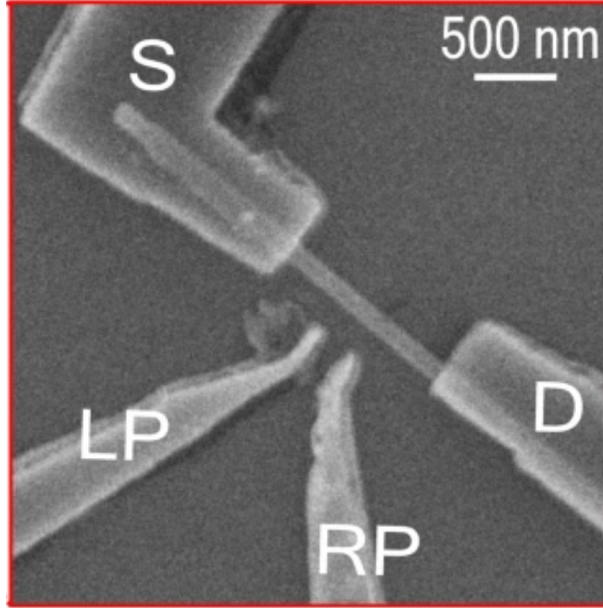


Figure 1: Scanning electron micrograph image of a system featuring a double quantum dot (DQD) on an InAs nanowire. [9]

The above image shows a DQD connected to an electron source and drain labeled S and D respectively. The gates labeled LP and RP control the voltages applied to the left and right dots, effectively controlling the energy levels and thus the energy gap between them. In the image the DQD is located on the nanowire right in front of the gates LP and RP.

For an electron to tunnel into the DQD it must overcome some charging energy. Charging energy plays a big role in determining transport properties in these kinds of systems.[10]

Consider the case where an electron is already in the dot. Should another electron try to enter the QD, it would interact electromagnetically with the electron already occupying the dot. This means that the charging energy for the QD will increase. When the charging energy is larger than the thermal energy of the electron, no more electrons enter the system. This phenomenon is called the Coulomb blockade. [11]

To simplify our model we make three assumptions. i), we assume coulomb blockade after just one electron enters the DQD so that at any time there will only be a maximum of one electron present. ii), we say that an electron only makes single jumps, meaning it cannot jump from the left of the double dot straight to the right dot. iii) for the sake of simplicity electron spin will be neglected throughout the project.

3.2 System without resonator

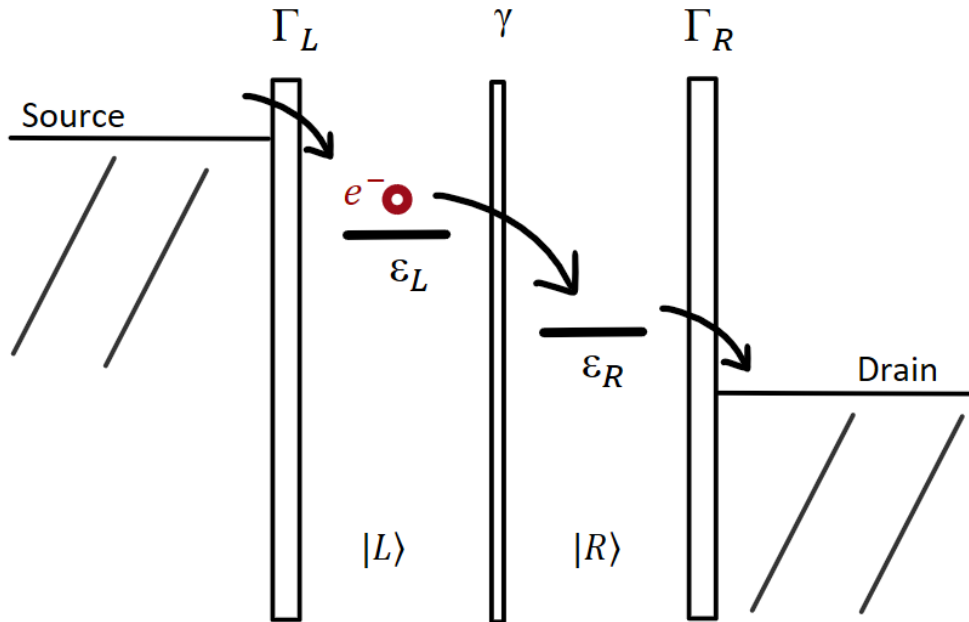


Figure 2: Schematic illustration of DQD system.

We will first look at the DQD outside of the resonator, meaning we disregard photon-electron interactions. The system can be visualized schematically as in Fig. 2.

The DQD is connected to leads on both sides, which serve as electron source and drain. An electron must tunnel through the tunneling barriers, labeled in the figure by their tunneling rates, Γ_L , γ and Γ_R . The system can be in three states: $|L\rangle$ and $|R\rangle$ represent the state when an electron is present in the left or right dot respectively. $|0\rangle$ represents the state where the DQD is empty.

There is a large bias applied between the source and the drain, which is illustrated in the figure by their difference in height. Similarly, the energy level ϵ_L in the left dot is larger than ϵ_R in the right dot, both are assumed to be within the bias window. This window is the energy range in which electron transport is possible.[12]

To start describing the dynamics of the system via the Lindblad equation, we first must formulate a density matrix and Hamiltonian for the system.

The density matrix ρ for a system such as our DQD looks as follows,

$$\rho = P_L|L\rangle\langle L| + P_R|R\rangle\langle R| + C_{LR}|L\rangle\langle R| + C_{RL}|R\rangle\langle L| + P_0|0\rangle\langle 0|. \quad (4)$$

Here P_L denotes the probability of being in state $|L\rangle$, P_R the probability of being in state $|R\rangle$ and P_0 denotes the probability of being in state $|0\rangle$. The coefficients C_{LR} and C_{RL} describe the quantum coherence between the states $|L\rangle$ and $|R\rangle$ due to superposition.

The Hamiltonian H describes the dynamics of the system without considering environmental effects, so two states need considering, $|L\rangle$ and $|R\rangle$, [12]

$$H = \epsilon_L |L\rangle\langle L| + \epsilon_R |R\rangle\langle R| + t |L\rangle\langle R| + t |R\rangle\langle L|. \quad (5)$$

As we saw in Fig. 2, ϵ_L and ϵ_R represent the energy levels of the left and right dot respectively. The coefficient t describes the coupling between the left and the right dot.

Now we shift our focus to the QME. To simplify dealing with the Lindblad QME in Eq.(3), we rename the Lindblad term as $\mathcal{D}(L_\alpha)$,

$$\mathcal{D}(L_\alpha) = L_\alpha \rho L_\alpha^\dagger - \frac{1}{2} \{L_\alpha^\dagger L_\alpha, \rho\}. \quad (6)$$

We find the Lindblad operators, by examining the interaction that the system has with the environment.

We start with the case where an electron enters the dot through the left tunneling barrier,

$$L_1 = |L\rangle\langle 0| \quad (7)$$

The operator for an electron tunneling out of the right dot is given by

$$L_2 = |0\rangle\langle R| \quad (8)$$

In the final QME the Lindblad terms $\mathcal{D}(L_\alpha)$ are multiplied by the tunneling rates corresponding to their processes. The Lindbladian from Eq.(3) now looks something like

$$\dot{\rho} = -i[H, \rho] + \Gamma_L \mathcal{D}(|L\rangle\langle 0|) + \Gamma_R \mathcal{D}(|0\rangle\langle R|). \quad (9)$$

The Lindblad equation was calculated in Appendix A as

$$\begin{aligned} \dot{\rho} &= -i[H, \rho] + \sum_{\alpha} \left(L_{\alpha} \rho L_{\alpha}^{\dagger} - \frac{1}{2} \{L_{\alpha}^{\dagger} L_{\alpha}, \rho\} \right) \\ &= -i(|L\rangle\langle R|(\epsilon_L C_{LR} + t P_R - P_L t - C_{LR} \epsilon_R) + |R\rangle\langle L|(\epsilon_R C_{RL} + t P_L - P_R t - C_{RL} \epsilon_L) \\ &\quad + |L\rangle\langle L|(C_{RL} t - C_{LR} t) + |R\rangle\langle R|(t C_{LR} - C_{RL} t)) \\ &\quad + \Gamma_L P_0 (|L\rangle\langle L| - |0\rangle\langle 0|) + \Gamma_R P_R (|0\rangle\langle 0| - |R\rangle\langle R|) - \Gamma_R \frac{1}{2} (C_{RL} |R\rangle\langle L| + C_{LR} |L\rangle\langle R|). \end{aligned} \quad (10)$$

We are interested in finding the electrical current through the DQD. We came up with the following expression, which gives the electron flow out of the dot and thus gives us an equation for the current.

$$I_e = P_R \cdot \Gamma_R \cdot e, \quad (11)$$

where e is the electron charge.

Note that we will work in the steady state regime for the remainder of the thesis work, which can be found by setting the left-hand side of Eq.(10) to zero. In the steady state, the probability P_R that we need to calculate the current is found by taking the expectation value of the Lindblad equation with respect to the state $|R\rangle$, which is given by

$$\langle R|\dot{\rho}|R\rangle = \langle R|(-i[H, \rho])|R\rangle + \langle R|\Gamma_L\mathcal{D}(s_L^\dagger)|R\rangle + \langle R|\Gamma_R\mathcal{D}(s_R)|R\rangle = 0. \quad (12)$$

After some calculations (see Appendix A), we get

$$P_R = \frac{\Gamma_L t^2}{t^2\Gamma_R + \Gamma_L(2t^2 + 1/4\Gamma_R^2 + \delta^2)}, \quad \text{with } \delta = \epsilon_L - \epsilon_R. \quad (13)$$

(Fig. 3 shows the current I_e plotted against δ . The current was calculated using Eq.(11) and the expression for P_R just calculated in Eq.(13). For this graph, we chose some arbitrary values for the tunneling rates Γ_L and Γ_R as well as the coupling coefficient t .

There is a clear peak at $\delta = 0$, which represents the situation where the energy levels ϵ_L and ϵ_R are the same. At $\delta = 0$ the electrons feel the smallest resistance to tunnel from one dot to the other, which explains why the current peaks here.

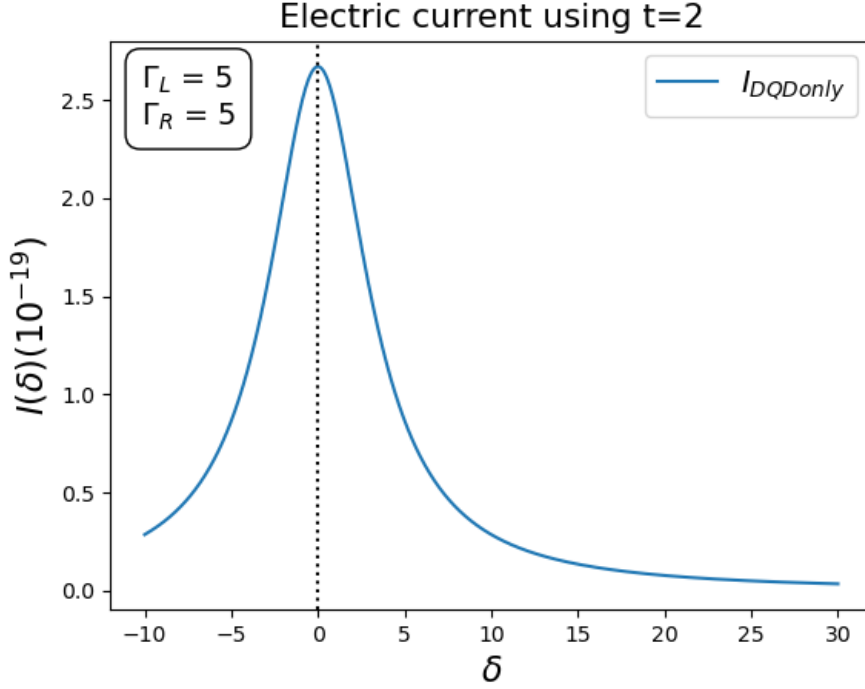


Figure 3: Electrical current through DQD.

3.3 Effective classical model

We are also interested in finding the effective tunneling rate between the left and right dots. To that end, we relate the quantum model outlined in the previous section to a classical rate equation model of the DQD.

Electrons tunneling through a DQD are described by the following coupled rate equations [10]

$$\dot{P}_0 = -\Gamma_L P_0 + \Gamma_R P_R \quad (14)$$

$$\dot{P}_L = \Gamma_L P_0 - \gamma P_L + \gamma P_R \quad (15)$$

$$\dot{P}_R = -\gamma P_L + (-\Gamma_R - \gamma) P_R \quad (16)$$

We rewrite these equations in matrix form as

$$\frac{d}{dt} \begin{pmatrix} P_0 \\ P_L \\ P_R \end{pmatrix} = \begin{pmatrix} -\Gamma_L & 0 & \Gamma_R \\ \Gamma_L & -\gamma & \gamma \\ 0 & \gamma & (-\Gamma_R - \gamma) \end{pmatrix} \begin{pmatrix} P_0 \\ P_L \\ P_R \end{pmatrix}. \quad (17)$$

3.4 Calculating the tunneling rate γ

To find the tunneling rate γ we revisit the quantum model and try to find a similarity to the effective classical one from the previous section.

We can rewrite our density matrix as a vector in the Fock-Liouville space [4] as

$$|\rho\rangle\rangle = \begin{pmatrix} P_{00} \\ P_L \\ P_R \\ C_{LR} \\ C_{RL} \end{pmatrix} \quad (18)$$

This notation for a vector in the Fock-Liouville space comes from [4]. The time evolution of the system, as seen in Eq.(3), is described by

$$\dot{\rho} = \mathcal{L}\rho. \quad (19)$$

The Liouvillian can be obtained by taking the expectation values of the Lindbladian (Eq(10)) with respect to the probabilities and coherence coefficients and identifying the system of equations. Ultimately we get the following 5x5 matrix

$$\frac{d}{dt} \begin{pmatrix} P_0 \\ P_L \\ P_R \\ C_{LR} \\ C_{RL} \end{pmatrix} = \begin{pmatrix} -\Gamma_L & 0 & \Gamma_R & 0 & 0 \\ \Gamma_L & 0 & 0 & it & -it \\ 0 & 0 & \Gamma_R & -it & it \\ 0 & it & -it & A & 0 \\ 0 & -it & it & 0 & B \end{pmatrix} \begin{pmatrix} P_0 \\ P_L \\ P_R \\ C_{LR} \\ C_{RL} \end{pmatrix}. \quad (20)$$

Here, $A = -i\epsilon_L + i\epsilon_R - 1/2\Gamma_R$ and $B = -i\epsilon_R + i\epsilon_L - 1/2\Gamma_R$.

As a trick to simplify calculations along the line, we rewrite Eq.(20) as a block matrix with four entries containing all the information from the previous matrix. M_{11} will correspond to the 3x3 matrix in the top left corner, M_{22} will be the 2x2 matrix in the bottom right corner, and the other two follow.

We also compress our vector into two entries. The top three entries will now correspond to P , standing for probabilities, and the bottom two entries will be Q , for quantum coherence.

$$\frac{d}{dt} \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}. \quad (21)$$

We manipulate this system of equations from to get rid of the coherence term Q ,

$$\frac{d}{dt}P = M_{11}P + M_{12}Q = 0 \quad (22)$$

$$\frac{d}{dt}Q = M_{21}P + M_{22}Q = 0 \quad (23)$$

$$\implies 0 = M_{11}P - (M_{12}M_{22}^{-1}M_{21})P. \quad (24)$$

Now we calculate this matrix

$$(M_{12}M_{22}^{-1}M_{21}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-t^2/A - t^2/B) & (t^2/A + t^2/B) \\ 0 & (t^2/A + t^2/B) & (-t^2/A - t^2/B) \end{pmatrix} \quad (25)$$

$$P(M_{11} - M_{12}M_{22}^{-1}M_{21}) = \begin{pmatrix} -\Gamma_L & 0 & \Gamma_R \\ 0 & (t^2/A + t^2/B) & (-t^2/A - t^2/B) \\ \Gamma_L & (-t^2/A - t^2/B) & (t^2/A + t^2/B - \Gamma_R) \end{pmatrix} \begin{pmatrix} P_0 \\ P_L \\ P_R \end{pmatrix}. \quad (26)$$

To find γ , we examine the above matrix and identify the entries corresponding to interactions between $|L\rangle$ and $|R\rangle$, so the entries P_L and P_R .

We can see the clear correlation to the effective classical matrix (Eq.(17)), and how the entries line up with γ .

We get the following result

$$\gamma = -\frac{t^2}{A} - \frac{t^2}{B} = -\frac{t^2}{-i(\epsilon_L - \epsilon_R) - 1/2\Gamma_R} - \frac{t^2}{i(\epsilon_L - \epsilon_R) - 1/2\Gamma_R} = \frac{t^2\Gamma_R}{(\epsilon_L - \epsilon_R)^2 + \frac{\Gamma_R^2}{4}}. \quad (27)$$

This gamma represents the rate of electrons tunneling from the left to the right dot and thus contributes to the overall electrical current, but the rate itself will not be used for current calculations.

3.5 Resonator-coupled DQD

In the following section, we will place the DQD in a microwave cavity, thus introducing photons in our model.

Before finding expressions for a density matrix and Hamiltonian for this system we must introduce a change of basis to the system, which will simplify calculations along the way.

Instead of a base spanned by $|L\rangle$ and $|R\rangle$, it will be spanned by $|g\rangle$ and $|e\rangle$ representing the ground state and excited state of the DQD respectively and which will be the eigenstates of the Hamiltonian H_{DQD} . This idea is sketched in Fig. 4.

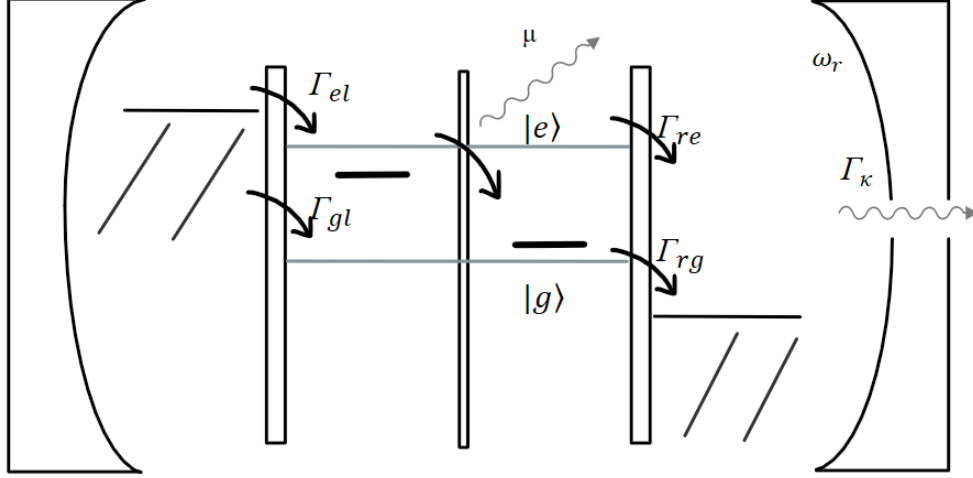


Figure 4: Schematic illustration of a resonator-coupled DQD, showcasing all tunneling possibilities with a different basis and the resonator frequency ω_r .

Note how now there are two ways to tunnel into the dot, into the excited and the ground state. The same holds for exiting the DQD. The rates shown in the figure will appear in our density matrix.

The change of basis is given by the following rotation matrix[12]

$$\begin{pmatrix} |g\rangle \\ |e\rangle \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} |L\rangle \\ |R\rangle \end{pmatrix}, \quad (28)$$

with the mixing angle $\cos(\theta) = -\delta/\Omega$, where $\Omega = \sqrt{4t^2 + \delta^2}$ represents the energy splitting between $|g\rangle$ and $|e\rangle$.

The point of difference to the previous section, is that now we must consider the existence of photons in our model. To make the model attainable for this bachelor's project, we make the simplification that at any time there can be at most one photon in the cavity. This means we have two photonic states in the cavity, $|0_p\rangle$ and $|1_p\rangle$ for zero and one photon present in the system respectively. One builds the resonator in a way in which it leaks photons at a fast enough rate to create this condition.

We now define the Hamiltonian for our system, which has two additional terms compared to our previous one, as

$$H = H_{DQD} + H_{photon} + H_{coupling}. \quad (29)$$

The Hamiltonian describing the DQD itself does not change much except for the change of basis which yields

$$H_{DQD} = \frac{\Omega}{2}(|e\rangle\langle e| - |g\rangle\langle g|). \quad (30)$$

H_{photon} describes the creation and annihilation of photons in the resonator and lies only in the Hilbert space spanned by $|0_p\rangle$ and $|1_p\rangle$. We introduce the creation and annihilation operators a^\dagger and a for a photon in the cavity. The Hamiltonian for the cavity includes the number operator $a^\dagger a$ and becomes

$$H_{photon} = \hbar\omega a^\dagger a. \quad (31)$$

Finally, the Hamiltonian for the coupling is given by a Jaynes-Cummings interaction Hamiltonian

$$H_{coupling} = \hbar g(a^\dagger |g\rangle\langle e| + a |e\rangle\langle g|), \quad (32)$$

where g is a constant that scales the coupling strength between the DQD and the photons. Now we have the total Hamiltonian [12]

$$H = \frac{\Omega}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \hbar\omega a^\dagger a + \hbar g(a^\dagger |g\rangle\langle e| + a |e\rangle\langle g|). \quad (33)$$

To start calculating the Lindblad equation we formulate the new density matrix

$$\begin{aligned} \rho = & P_{e,0} |e, 0_p\rangle\langle e, 0_p| + P_{g,0} |g, 0_p\rangle\langle g, 0_p| + P_{g,1} |g, 1_p\rangle\langle g, 1_p| \\ & + P_{0,0} |0, 0_p\rangle\langle 0, 0_p| + C_{eg,01} |e, 0_p\rangle\langle g, 1_p| + C_{ge,10} |g, 1_p\rangle\langle e, 0_p|. \end{aligned} \quad (34)$$

The tensor products couple the photon states to the states of the DQD. As an example for the notation, the probability $P_{e,0}$ corresponds to the state where an electron is in the excited state $|e\rangle$ and no photon is in the resonator $|0_p\rangle$.

Now we think about the Lindblad terms for the QME

$$\dot{\rho} = -i[H, \rho] + \sum_{\alpha} \mathcal{D}(L_{\alpha}). \quad (35)$$

We construct the Lindblad operators based on the dynamics in the system, they are prefaced by the tunneling rates Γ_{ab} , where a represents the position of the electron after the tunneling and b shows its origin. a and b can be l, r, e, g for the left and right of the dot, excited and ground state respectively with the exception of Γ_{κ} which symbolizes photons escaping the resonator, see Fig 4.

- $\Gamma_{el}\mathcal{D}(|e\rangle\langle 0|)$
- $\Gamma_{gl}\mathcal{D}(|g\rangle\langle 0|)$
- $\Gamma_{re}\mathcal{D}(|0\rangle\langle e|)$

- $\Gamma_{rg}\mathcal{D}(|0\rangle\langle g|)$
- $\Gamma_{\kappa}\mathcal{D}(a)$

Finally, the Lindblad QME can be calculated and the same matrix calculations as in the previous section are applied. We get the following expression for the QME

$$\begin{aligned}
\dot{\rho} &= -i[H, \rho] + \sum_{\alpha} \left(L_{\alpha}\rho L_{\alpha}^{\dagger} - \frac{1}{2}\{L_{\alpha}^{\dagger}L_{\alpha}, \rho\} \right) \\
&= -i(\Omega C_{eg,01} |e, 0_p\rangle\langle g, 1_p| - \Omega C_{ge,10} |g, 1_p\rangle\langle e, 0_p| + \hbar\omega(C_{ge,10} |g, 1_p\rangle\langle e, 0_p| - C_{eg,01} |e, 0_p\rangle\langle g, 1_p|) \\
&\quad + \hbar g(P_{e0} |g, 1_p\rangle\langle e, 0_p| + P_{g1} |e, 0_p\rangle\langle g, 1_p| + C_{eg,01} |g, 1_p\rangle\langle g, 1_p| + C_{ge,10} |e, 0_p\rangle\langle e, 0_p| \\
&\quad - P_{e0} |e, 0_p\rangle\langle g, 1_p| - P_{g1} |g, 1_p\rangle\langle e, 0_p| - C_{eg,01} |e, 0_p\rangle\langle e, 0_p| - C_{ge,10} |g, 1_p\rangle\langle g, 1_p|) \\
&\quad + \Gamma_{el}(P_{00}(|e, 0_p\rangle\langle e, 0_p| - |0, 0_p\rangle\langle 0, 0_p|)) + \Gamma_{gl}(P_{00}(|g, 0_p\rangle\langle g, 0_p| - |0, 0_p\rangle\langle 0, 0_p|)) \\
&\quad + \Gamma_{re}(P_{e0}(|0, 0_p\rangle\langle 0, 0_p| - |e, 0_p\rangle\langle e, 0_p|) - \frac{1}{2}(C_{eg,01} |e, 0_p\rangle\langle g, 1_p| + C_{ge,10} |g, 1_p\rangle\langle e, 0_p|)) \\
&\quad + \Gamma_{rg}(P_{g0}(|0, 0_p\rangle\langle 0, 0_p| - |g, 0_p\rangle\langle g, 0_p|) + P_{g1}(|0, 1_p\rangle\langle 0, 1_p| - |g, 1_p\rangle\langle g, 1_p|) \\
&\quad - \frac{1}{2}(C_{eg,01} |e, 0_p\rangle\langle g, 1_p| + C_{ge,10} |g, 1_p\rangle\langle e, 0_p|)) \\
&\quad + \Gamma_{\kappa}(P_{g1}(|g, 0_p\rangle\langle g, 0_p| - |g, 1_p\rangle\langle g, 1_p|) - \frac{1}{2}(C_{ge,10} |g, 1_p\rangle\langle e, 0_p| + C_{eg,01} |e, 0_p\rangle\langle g, 1_p|)).
\end{aligned}$$

As in the section above, we construct a matrix version of the QME,

$$\frac{d}{dt} \begin{pmatrix} P_{e0} \\ P_{g0} \\ P_{g1} \\ P_{00} \\ C_{eg,01} \\ C_{ge,10} \end{pmatrix} = \begin{pmatrix} -\Gamma_{re} & 0 & 0 & \Gamma_{el} & i\hbar g & -i\hbar g \\ 0 & -\Gamma_{rg} & \Gamma_{\kappa} & \Gamma_{gl} & 0 & 0 \\ 0 & 0 & (-\Gamma_{rg} - \Gamma_{\kappa}) & 0 & -i\hbar g & i\hbar g \\ \Gamma_{re} & \Gamma_{rg} & 0 & (-\Gamma_{gl} - \Gamma_{el}) & 0 & 0 \\ i\hbar g & 0 & -i\hbar g & 0 & A & 0 \\ -i\hbar g & 0 & i\hbar g & 0 & 0 & B \end{pmatrix} \begin{pmatrix} P_{e0} \\ P_{g0} \\ P_{g1} \\ P_{00} \\ C_{eg,01} \\ C_{ge,10} \end{pmatrix}, \quad (36)$$

where

$$\begin{aligned}
A &= -i\Omega + i\hbar\omega - \frac{1}{2}\Gamma_{re} - \frac{1}{2}\Gamma_{rg} - \frac{1}{2}\Gamma_{\kappa} \\
B &= i\Omega - i\hbar\omega - \frac{1}{2}\Gamma_{re} - \frac{1}{2}\Gamma_{rg} - \frac{1}{2}\Gamma_{\kappa}.
\end{aligned}$$

We try to calculate a δ -dependant current to compare this model to the previous. Finding an analytical solution to this 6x6 matrix is quite demanding, so we opted for calculating numerical solutions for the interesting probabilities using the programming language *Python*. Here it was possible to create a linear system of equations, while also implementing the

condition $P_{e0} + P_{g0} + P_{g1} + P_{00} = 1$, which makes it possible to find the non-zero solutions. In our new basis, there is more than one way for an electron to escape the DQD, so we must update our equation for the current. The current in this basis will look as follows

$$I_e = P_{e,0} \cdot \Gamma_{re} + (P_{g,0} + P_{g,1}) \cdot \Gamma_{rg}. \quad (37)$$

To get the right dimensions for plotting, we must express the tunneling rates in terms of the simpler Γ_L and Γ_R . We have the following expressions [12]

$$\Gamma_{el} = \Gamma_L \cdot \sin^2(\theta/2), \quad (38)$$

$$\Gamma_{gl} = \Gamma_L \cdot \cos^2(\theta/2), \quad (39)$$

$$\Gamma_{re} = \Gamma_R \cdot \cos^2(\theta/2), \quad (40)$$

$$\Gamma_{rg} = \Gamma_R \cdot \sin^2(\theta/2). \quad (41)$$

With these expressions for the tunneling rates, we can now solve for the relevant probabilities and plot the resulting current in the same basis as our previous result.

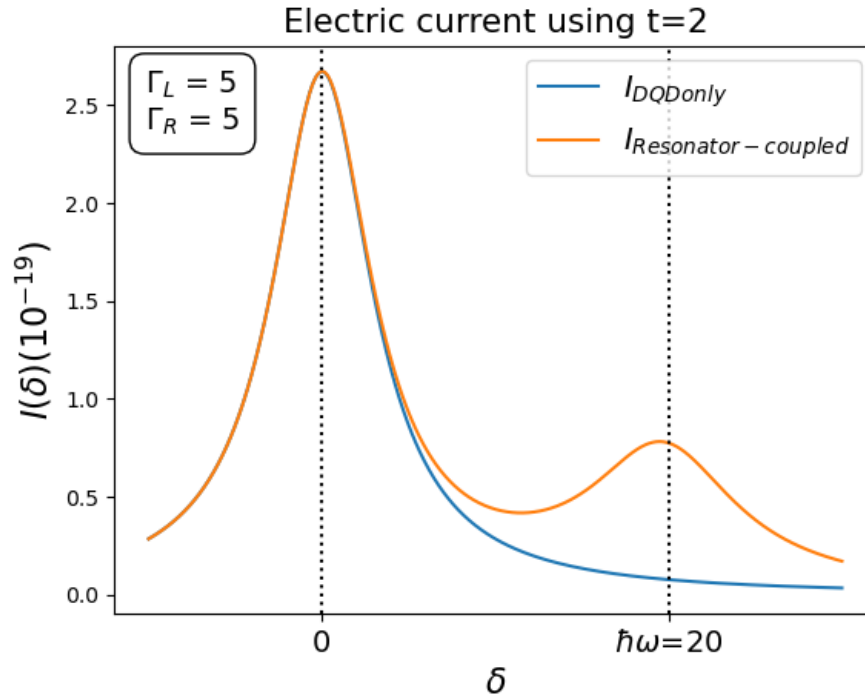


Figure 5: Current through DQD coupled to the resonator (orange) compared to previously calculated current through DQD outside of resonator.

The figure above shows some interesting features to make comparisons in the current

between the DQD in and out of a microwave cavity. The most obvious difference is that now we have a second current peak at $\delta = \hbar\omega$ corresponding to the resonance frequency of the cavity. This makes sense since there will be increased photon emission and thus increased electron transport at this frequency, which was not the case in the previous system.

We can see, that there is a very nice agreement in the current between the two curves at the main peak at $\delta = 0$, which confirms the calculated result.

Now that we found the electric current through our DQD system, it makes sense to investigate the photonic current coming from it. This current will be given by the photon emission rate μ .

3.6 Finding the photon emission rate

We do the same matrix manipulation as was done for finding γ . We rewrite Eq.(36) as a block matrix, separating the probabilities from the quantum coherence terms. Finally, we get the following matrix

$$0 = M_{11}P - (M_{12}M_{22}^{-1}M_{21})P \quad (42)$$

$$= \begin{pmatrix} -\Gamma_{re} + \frac{\hbar^2 g^2 (A+B)}{AB} & 0 & -\frac{\hbar^2 g^2 (A+B)}{AB} & \Gamma_{el} \\ 0 & -\Gamma_{rg} & \Gamma_{\kappa} & \Gamma_{gl} \\ -\frac{\hbar^2 g^2 (A+B)}{AB} & 0 & (-\Gamma_{rg} - \Gamma_{\kappa}) + \frac{\hbar^2 g^2 (A+B)}{AB} & 0 \\ \Gamma_{re} & \Gamma_{rg} & 0 & -\Gamma_{gl} - \Gamma_{el} \end{pmatrix} \begin{pmatrix} P_{e0} \\ P_{g0} \\ P_{g1} \\ P_{00} \end{pmatrix}. \quad (43)$$

We are interested in the rate between states $|e, 0\rangle$ and $|g, 1\rangle$ because this is the situation where a photon would be emitted. In the above matrix, the entries connecting P_{e0} and P_{g1} should represent that very rate μ , as a photon is only created after an electron jumps from the excited to ground state, and only if after this jump, the system is in the photonic state $|1_p\rangle$.

$$\mu = -\frac{\hbar^2 g^2 (A+B)}{AB} = \frac{\hbar^2 g^2 (\Gamma_{re} + \Gamma_{rg} + \Gamma_{\kappa})}{(\Omega - \omega\hbar)^2 + \frac{1}{4}(\Gamma_{re}^2 + \Gamma_{rg}^2 + \Gamma_{\kappa}^2) + \frac{1}{2}(\Gamma_{re}\Gamma_{rg} + \Gamma_{re}\Gamma_{\kappa} + \Gamma_{rg}\Gamma_{\kappa})} \quad (44)$$

$$= \frac{\hbar^2 g^2 (\Gamma_{re} + \Gamma_{rg} + \Gamma_{\kappa})}{(\Omega - \omega\hbar)^2 + \left(\frac{\Gamma_{re} + \Gamma_{rg} + \Gamma_{\kappa}}{2}\right)^2}. \quad (45)$$

Now, we can put this photon current into context with our electrical current. The following plot shows the photonic current added to the plot from Fig. 5.

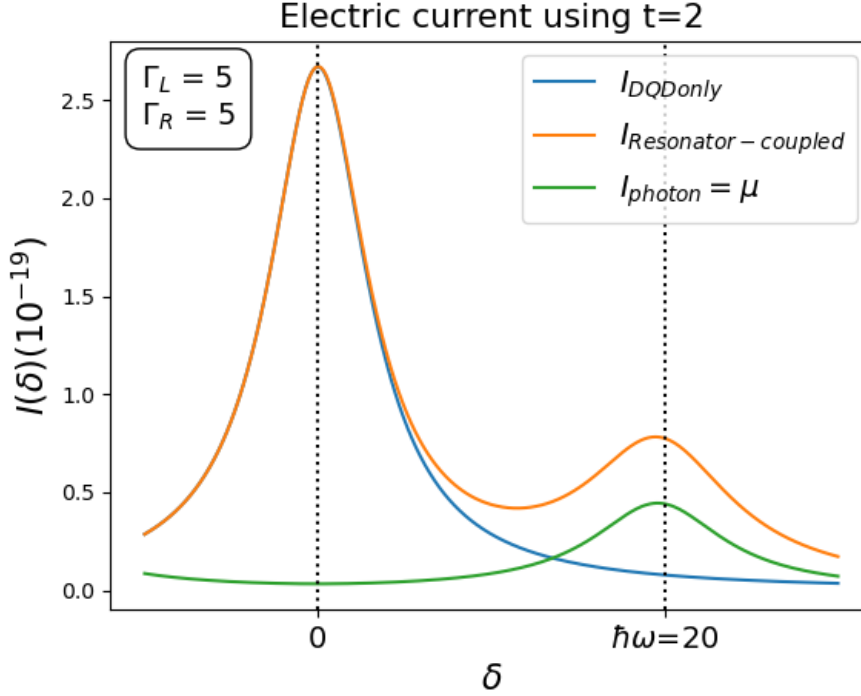


Figure 6: Same plot as Fig. 5 with the addition of photonic current shown in green.

The above figure shows the results of our system from three different perspectives. The blue line represents just a DQD not connected to any resonator, where photon interactions are disregarded. The orange line represents the electronic current through a resonator-coupled DQD, showing two peaks, one at $\delta = 0$, and the other at $\delta = \hbar\omega_r$. Finally, the green line shows only the rate of photon emissions, which also has the peak at the cavity's resonance frequency ω_r .

4 Conclusion and Outlook

In conclusion, this thesis explored the behavior of a resonator-coupled double quantum dot system.

We started our mathematical approach by first uncoupling the DQD from the resonator, which made it possible for us to analyze it without taking photon interaction into consideration. This also allowed us to make an interesting comparison of our mathematical model to an effective classical model, which usually facilitates a more intuitive understanding.

This approach allowed us to build up the necessary mathematical tools to calculate the final electric and photonic current.

This research investigates this resonator-DQD system on a fundamental level, it aims to

contribute at least a fragment to the overall understanding of such nanosystems. I believe we examined this particular system, with a mathematical approach that has not been done before on such a simplified version of the system.

If there were more time for this work, it could be interesting to look at more complex situations, for example, a resonator in which there are two or more photons at a time. A simple addition of a photon can have a lot of implications and add a lot of complications.

As was briefly mentioned, microwave nanosystems such as ours are commercially not very viable at this point due to the constraints that come with working on such low-energy systems. But with further research and commitment to this area of physics, single-photon sources and detectors at this scale could have valuable applications in the increasingly technological future.

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A Appendix: Calculation for P_R (no photons):

Here we do the calculation for the probability P_R of state $|R\rangle$. We first note again what we had for the Hamiltonian H , the density matrix ρ , and the Lindblad QME.

$$\rho = P_L|L\rangle\langle L| + P_R|R\rangle\langle R| + c_{LR}|L\rangle\langle R| + c_{RL}|R\rangle\langle L| + P_0|0\rangle\langle 0|. \quad (46)$$

$$H = \epsilon_L|L\rangle\langle L| + \epsilon_R|R\rangle\langle R| + t|L\rangle\langle R| + t|R\rangle\langle L| \quad (47)$$

$$\frac{d}{dt}\rho = -i[H, \rho] + \sum_{\alpha} \left(L_{\alpha}\rho L_{\alpha}^{\dagger} - \frac{1}{2}\{L_{\alpha}^{\dagger}L_{\alpha}, \rho\} \right). \quad (48)$$

$$\sum_{\alpha} \mathcal{D}(L_{\alpha}) = \sum_{\alpha} \left(L_{\alpha}\rho L_{\alpha}^{\dagger} - \frac{1}{2}\{L_{\alpha}^{\dagger}L_{\alpha}, \rho\} \right). \quad (49)$$

We start by calculating the commutator $[H, \rho]$

$$\begin{aligned} [H, \rho] &= \epsilon_L c_{LR}|L\rangle\langle R| + \epsilon_R c_{RL}|R\rangle\langle L| + tP_R|L\rangle\langle R| + C_{RL}t|L\rangle\langle L| + tP_L|R\rangle\langle L| + tC_{LR}|R\rangle\langle R| \\ &\quad - P_L t|L\rangle\langle R| - P_R t|R\rangle\langle L| - C_{LR}\epsilon_R|L\rangle\langle R| - C_{LR}t|L\rangle\langle L| - C_{RL}\epsilon_L|R\rangle\langle L| - C_{RL}t|R\rangle\langle R| \end{aligned} \quad (50)$$

$$\begin{aligned} &= |L\rangle\langle R|(\epsilon_L C_{LR} + tP_R - P_L t - C_{LR}\epsilon_R) + |R\rangle\langle L|(\epsilon_R C_{RL} + tP_L - P_R t - C_{RL}\epsilon_L) \\ &\quad + |L\rangle\langle L|(C_{RL}t - C_{LR}t) + |R\rangle\langle R|(tC_{LR} - C_{RL}t) \end{aligned} \quad (51)$$

Now we calculate $\mathcal{D}(L_{\alpha})$,

$$\begin{aligned} \mathcal{D}(L_1) &= \mathcal{D}(s_L^{\dagger}) = \mathcal{D}(|L\rangle\langle 0|) \\ s_L^{\dagger}\rho s_L &= P_0|L\rangle\langle L| \\ \frac{1}{2}\{s_L s_L^{\dagger}, \rho\} &= P_0|0\rangle\langle 0| \\ \mathcal{D}(s_L^{\dagger}) &= P_0|L\rangle\langle L| - P_0|0\rangle\langle 0| \end{aligned} \quad (52)$$

$$\begin{aligned} \mathcal{D}(L_2) &= \mathcal{D}(s_R) = \mathcal{D}(|0\rangle\langle R|) \\ s_R\rho s_R^{\dagger} &= P_R|0\rangle\langle 0| \\ \frac{1}{2}\{s_R^{\dagger}s_R, \rho\} &= P_R|R\rangle\langle R| + \frac{1}{2}C_{RL}|R\rangle\langle L| + \frac{1}{2}C_{LR}|L\rangle\langle R| \\ \mathcal{D}(s_R) &= P_R(|0\rangle\langle 0| - |R\rangle\langle R|) - \frac{1}{2}(C_{RL}|R\rangle\langle L| + C_{LR}|L\rangle\langle R|) \end{aligned} \quad (53)$$

So we get the final Lindblad QME looking as follows,

$$\begin{aligned}
\dot{\rho} &= -i[H, \rho] + \sum_{\alpha} \left(L_{\alpha} \rho L_{\alpha}^{\dagger} - \frac{1}{2} \{ L_{\alpha}^{\dagger} L_{\alpha}, \rho \} \right) \\
&= -i(|L\rangle\langle R|(\epsilon_L C_{LR} + tP_R - P_L t - C_{LR}\epsilon_R) + |R\rangle\langle L|(\epsilon_R C_{RL} + tP_L - P_R t - C_{RL}\epsilon_L) \\
&\quad + |L\rangle\langle L|(C_{RL}t - C_{LR}) + |R\rangle\langle R|(tC_{LR} - C_{RL}t)) \\
&\quad + \Gamma_L P_0(|L\rangle\langle L| - |0\rangle\langle 0|) + \Gamma_R P_R(|0\rangle\langle 0| - |R\rangle\langle R|) - \Gamma_R \frac{1}{2}(C_{RL}|R\rangle\langle L| + C_{LR}|L\rangle\langle R|) \quad (54)
\end{aligned}$$

Now we take the expectation value of this QME with respect to the states $|R\rangle, |L\rangle$, and $|0\rangle$, and we set the equations to 0 to evaluate the steady state solution. Note also the condition that all probabilities must add up to 1,

$$\langle R|\dot{\rho}|R\rangle = -i(tC_{LR} - tC_{RL}) - \Gamma_R P_R = 0 \quad (55)$$

$$\langle L|\dot{\rho}|L\rangle = -i(tC_{RL} - tC_{LR}) + \Gamma_L P_0 = 0 \quad (56)$$

$$\langle 0|\dot{\rho}|0\rangle = -\Gamma_L P_0 + \Gamma_R P_R = 0 \quad (57)$$

$$1 = P_0 + P_L + P_R \quad (58)$$

To find C_{LR} and C_{RL} we need to do the following,

$$\langle L|\dot{\rho}|R\rangle = -i(\epsilon_L C_{LR} + tP_R - tP_L - C_{LR}\epsilon_R) - 1/2\Gamma_R C_{LR} \quad (59)$$

$$\langle R|\dot{\rho}|L\rangle = -i(\epsilon_R C_{RL} + tP_L - tP_R - C_{RL}\epsilon_L) - 1/2\Gamma_R C_{RL} \quad (60)$$

$$\implies C_{LR} = \frac{it(P_L - P_R)}{1/2\Gamma_R + i\epsilon_L - i\epsilon_R} \quad (61)$$

$$\implies C_{RL} = \frac{it(P_R - P_L)}{1/2\Gamma_R + i\epsilon_R - i\epsilon_L}. \quad (62)$$

We plug these expressions into our expression for $\langle R|\dot{\rho}|R\rangle$,

$$\langle R|\dot{\rho}|R\rangle = -it \left(\frac{it(P_L - P_R)\Gamma_R}{1/4\Gamma_R^2 + (\epsilon_L - \epsilon_R)^2} \right) - \Gamma_R P_R = 0. \quad (63)$$

Solving for P_R ,

$$P_R = \frac{t^2 P_L}{t^2 + 1/4\Gamma_R^2 + (\epsilon_L - \epsilon_R)^2}. \quad (64)$$

Now we use Eq.(58) and the expectation value with respect to 0 to find an expression for P_L .

$$P_0 = \frac{\Gamma_R P_R}{\Gamma_L} \quad (65)$$

$$P_L = 1 - P_0 - P_R = 1 - \frac{\Gamma_R P_R - P_R \Gamma_L}{\Gamma_L}. \quad (66)$$

Finally, we can plug this into our expression for P_R ,

$$P_R = \frac{\Gamma_L t^2}{t^2 \Gamma_R + \Gamma_L (2t^2 + 1/4\Gamma_R^2 + \delta^2)}, \quad \text{with } \delta = \epsilon_L - \epsilon_R. \quad (67)$$

Using the previous two equations, we also get P_0 and P_L ,

$$P_L = \frac{\Gamma_R t^2}{t^2 \Gamma_R + \Gamma_L (2t^2 + 1/4\Gamma_R^2 + \delta^2)} \quad (68)$$

$$P_0 = \frac{\Gamma_L (t^2 + 1/4\Gamma_R^2 + \delta^2)}{t^2 \Gamma_R + \Gamma_L (2t^2 + 1/4\Gamma_R^2 + \delta^2)} \quad (69)$$

B Appendix: Calculations for resonator-coupled DQD

$$H = \frac{\Omega}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \hbar\omega a^\dagger a + \hbar g(a^\dagger |g\rangle\langle e| + a |e\rangle\langle g|) \quad (70)$$

$$\begin{aligned} \rho = & P_{e,0} |e, 0_p\rangle\langle e, 0_p| + P_{g,0} |g, 0_p\rangle\langle g, 0_p| + P_{g,1} |g, 1_p\rangle\langle g, 1_p| \\ & + P_{0,0} |0, 0_p\rangle\langle 0, 0_p| + C_{eg,01} |e, 0_p\rangle\langle g, 1_p| + C_{ge,10} |g, 1_p\rangle\langle e, 0_p| \end{aligned} \quad (71)$$

We calculate the Lindblad equation.

First the Commutator with the Hamiltonian and density matrix,

$$\begin{aligned} & [H, \rho] : \\ H\rho = & \frac{\Omega}{2} P_{e0} |e, 0_p\rangle\langle e, 0_p| + \frac{\Omega}{2} C_{eg,01} |e, 0_p\rangle\langle g, 1_p| - \frac{\Omega}{2} P_{g0} |g, 0_p\rangle\langle g, 0_p| - \frac{\Omega}{2} P_{g1} |g, 1_p\rangle\langle g, 1_p| \\ & - \frac{\Omega}{2} C_{ge,10} |g, 1_p\rangle\langle e, 0_p| \\ & + \hbar\omega P_{g1} |g, 1_p\rangle\langle g, 1_p| + \hbar\omega C_{ge,10} |g, 1_p\rangle\langle e, 0_p| \\ & + \hbar g P_{e0} |g, 1_p\rangle\langle e, 0_p| + \hbar g P_{g1} |e, 0_p\rangle\langle g, 1_p| + \hbar g C_{eg,01} |g, 1_p\rangle\langle g, 1_p| \\ & + \hbar g C_{ge,10} |e, 0_p\rangle\langle e, 0_p| \\ \rho H = & \frac{\Omega}{2} P_{e0} |e, 0_p\rangle\langle e, 0_p| - \frac{\Omega}{2} C_{eg,01} |e, 0_p\rangle\langle g, 1_p| - \frac{\Omega}{2} P_{g0} |g, 0_p\rangle\langle g, 0_p| - \frac{\Omega}{2} P_{g1} |g, 1_p\rangle\langle g, 1_p| \\ & + \frac{\Omega}{2} C_{ge,10} |g, 1_p\rangle\langle e, 0_p| \\ & + \hbar\omega P_{g1} |g, 1_p\rangle\langle g, 1_p| + \hbar\omega C_{eg,01} |e, 0_p\rangle\langle g, 1_p| \\ & + \hbar g P_{e0} |e, 0_p\rangle\langle g, 1_p| + \hbar g P_{g1} |g, 1_p\rangle\langle e, 0_p| + \hbar g C_{eg,01} |e, 0_p\rangle\langle e, 0_p| \\ & + \hbar g C_{ge,10} |g, 1_p\rangle\langle g, 1_p| \end{aligned}$$

$$\begin{aligned} H\rho - \rho H = & \Omega C_{eg,01} |e, 0_p\rangle\langle g, 1_p| - \Omega C_{ge,10} |g, 1_p\rangle\langle e, 0_p| + \hbar\omega (C_{ge,10} |g, 1_p\rangle\langle e, 0_p| - C_{eg,01} |e, 0_p\rangle\langle g, 1_p|) \\ & + \hbar g (P_{e0} |g, 1_p\rangle\langle e, 0_p| + P_{g1} |e, 0_p\rangle\langle g, 1_p| + C_{eg,01} |g, 1_p\rangle\langle g, 1_p| + C_{ge,10} |e, 0_p\rangle\langle e, 0_p| \\ & - P_{e0} |e, 0_p\rangle\langle g, 1_p| - P_{g1} |g, 1_p\rangle\langle e, 0_p| - C_{eg,01} |e, 0_p\rangle\langle e, 0_p| - C_{ge,10} |g, 1_p\rangle\langle g, 1_p|) \end{aligned}$$

Now we calculate the Lindblad terms,

$$\mathcal{D}(|e\rangle\langle 0|) :$$

$$L_\alpha \rho L_\alpha^\dagger = P_{00} |e, 0_p\rangle\langle e, 0_p|$$

$$\frac{1}{2}\{|0\rangle\langle 0|, \rho\} = P_{00} |0, 0_p\rangle\langle 0, 0_p|$$

$$\mathcal{D}(|e\rangle\langle 0|) = P_{00}(|e, 0_p\rangle\langle e, 0_p| - |0, 0_p\rangle\langle 0, 0_p|)$$

$$\mathcal{D}(|g\rangle\langle 0|) :$$

$$L_\alpha \rho L_\alpha^\dagger = P_{00} |g, 0_p\rangle\langle g, 0_p|$$

$$\frac{1}{2}\{|0\rangle\langle 0|, \rho\} = P_{00} |0, 0_p\rangle\langle 0, 0_p|$$

$$\mathcal{D}(|g\rangle\langle 0|) = P_{00}(|g, 0_p\rangle\langle g, 0_p| - |0, 0_p\rangle\langle 0, 0_p|)$$

$$\mathcal{D}(|0\rangle\langle e|) :$$

$$L_\alpha \rho L_\alpha^\dagger = P_{e0} |0, 0_p\rangle\langle 0, 0_p|$$

$$\frac{1}{2}\{|e\rangle\langle e|, \rho\} = P_{e0} |e, 0_p\rangle\langle e, 0_p| + \frac{1}{2}(C_{eg,01} |e, 0_p\rangle\langle g, 1_p| + C_{ge,10} |g, 1_p\rangle\langle e, 0_p|)$$

$$\mathcal{D}(|0\rangle\langle e|) = P_{e0}(|0, 0_p\rangle\langle 0, 0_p| - |e, 0_p\rangle\langle e, 0_p|) - \frac{1}{2}(C_{eg,01} |e, 0_p\rangle\langle g, 1_p| + C_{ge,10} |g, 1_p\rangle\langle e, 0_p|)$$

$$\mathcal{D}(|0\rangle\langle g|) :$$

$$L_\alpha \rho L_\alpha^\dagger = P_{g0} |0, 0_p\rangle\langle 0, 0_p| + P_{g1} |0, 1_p\rangle\langle 0, 1_p|$$

$$\frac{1}{2}\{|g\rangle\langle g|, \rho\} = P_{g0} |g, 0_p\rangle\langle g, 0_p| + P_{g1} |g, 1_p\rangle\langle g, 1_p| + \frac{1}{2}(C_{eg,01} |e, 0_p\rangle\langle g, 1_p| + C_{ge,10} |g, 1_p\rangle\langle e, 0_p|)$$

$$\mathcal{D}(|0\rangle\langle g|) = P_{g0}(|0, 0_p\rangle\langle 0, 0_p| - |g, 0_p\rangle\langle g, 0_p|) + P_{g1}(|0, 1_p\rangle\langle 0, 1_p| - |g, 1_p\rangle\langle g, 1_p|) - \frac{1}{2}(C_{eg,01} |e, 0_p\rangle\langle g, 1_p| + C_{ge,10} |g, 1_p\rangle\langle e, 0_p|)$$

$$\mathcal{D}(a) :$$

$$L_\alpha \rho L_\alpha^\dagger = P_{g1} |g, 0_p\rangle\langle g, 0_p|$$

$$\frac{1}{2}\{a^\dagger a, \rho\} = P_{g1} |g, 1_p\rangle\langle g, 1_p| + \frac{1}{2}(C_{ge,10} |g, 1_p\rangle\langle e, 0_p| + C_{eg,01} |e, 0_p\rangle\langle g, 1_p|)$$

$$\mathcal{D}(a) = P_{g1}(|g, 0_p\rangle\langle g, 0_p| - |g, 1_p\rangle\langle g, 1_p|) - \frac{1}{2}(C_{ge,10} |g, 1_p\rangle\langle e, 0_p| + C_{eg,01} |e, 0_p\rangle\langle g, 1_p|)$$

Now we can write the entire Lindblad QME,

$$\begin{aligned}
\dot{\rho} &= -i[H, \rho] + \sum_{\alpha} \left(L_{\alpha} \rho L_{\alpha}^{\dagger} - \frac{1}{2} \{ L_{\alpha}^{\dagger} L_{\alpha}, \rho \} \right) \\
&= -i(\Omega C_{eg,01} |e, 0_p\rangle\langle g, 1_p| - \Omega C_{ge,10} |g, 1_p\rangle\langle e, 0_p| + \hbar\omega(C_{ge,10} |g, 1_p\rangle\langle e, 0_p| - C_{eg,01} |e, 0_p\rangle\langle g, 1_p|) \\
&\quad + \hbar g(P_{e0} |g, 1_p\rangle\langle e, 0_p| + P_{g1} |e, 0_p\rangle\langle g, 1_p| + C_{eg,01} |g, 1_p\rangle\langle g, 1_p| + C_{ge,10} |e, 0_p\rangle\langle e, 0_p| \\
&\quad - P_{e0} |e, 0_p\rangle\langle g, 1_p| - P_{g1} |g, 1_p\rangle\langle e, 0_p| - C_{eg,01} |e, 0_p\rangle\langle e, 0_p| - C_{ge,10} |g, 1_p\rangle\langle g, 1_p|) \\
&\quad + \Gamma_{el}(P_{00}(|e, 0_p\rangle\langle e, 0_p| - |0, 0_p\rangle\langle 0, 0_p|)) + \Gamma_{gl}(P_{00}(|g, 0_p\rangle\langle g, 0_p| - |0, 0_p\rangle\langle 0, 0_p|)) \\
&\quad + \Gamma_{re}(P_{e0}(|0, 0_p\rangle\langle 0, 0_p| - |e, 0_p\rangle\langle e, 0_p|) - \frac{1}{2}(C_{eg,01} |e, 0_p\rangle\langle g, 1_p| + C_{ge,10} |g, 1_p\rangle\langle e, 0_p|)) \\
&\quad + \Gamma_{rg}(P_{g0}(|0, 0_p\rangle\langle 0, 0_p| - |g, 0_p\rangle\langle g, 0_p|) + P_{g1}(|0, 1_p\rangle\langle 0, 1_p| - |g, 1_p\rangle\langle g, 1_p|) \\
&\quad - \frac{1}{2}(C_{eg,01} |e, 0_p\rangle\langle g, 1_p| + C_{ge,10} |g, 1_p\rangle\langle e, 0_p|)) \\
&\quad + \Gamma_{\kappa}(P_{g1}(|g, 0_p\rangle\langle g, 0_p| - |g, 1_p\rangle\langle g, 1_p|) - \frac{1}{2}(C_{ge,10} |g, 1_p\rangle\langle e, 0_p| + C_{eg,01} |e, 0_p\rangle\langle g, 1_p|))
\end{aligned}$$

Take the expectation value of QME to get probabilities,

$$\begin{aligned}
P_{e0} : \langle e, 0_p | \dot{\rho} | e, 0_p \rangle &= -i\hbar g(C_{ge,10} - C_{eg,01}) + \Gamma_{el}P_{00} - \Gamma_{re}P_{e0} \\
P_{g0} : \langle g, 0_p | \dot{\rho} | g, 0_p \rangle &= \Gamma_{gl}P_{00} - \Gamma_{rg}P_{g0} + \Gamma_{\kappa}P_{g1} \\
P_{g1} : \langle g, 1_p | \dot{\rho} | g, 1_p \rangle &= -i\hbar g(C_{eg,01} - C_{ge,10}) - \Gamma_{rg}P_{g1} - \Gamma_{\kappa}P_{g1} \\
P_{00} : \langle 0, 0_p | \dot{\rho} | 0, 0_p \rangle &= -\Gamma_{el}P_{00} - \Gamma_{gl}P_{00} + \Gamma_{re}P_{e0} + \Gamma_{rg}P_{g0} \\
C_{eg,01} : \langle e, 0_p | \dot{\rho} | g, 1_p \rangle &= -i\Omega C_{eg,01} + i\hbar\omega C_{eg,01} - i\hbar g(P_{g1} - P_{e0}) - \Gamma_{re}\frac{1}{2}C_{eg,01} \\
&\quad - \Gamma_{rg}\frac{1}{2}C_{eg,01} - \Gamma_{\kappa}\frac{1}{2}C_{eg,01} \\
C_{ge,10} : \langle g, 1_p | \dot{\rho} | e, 0_p \rangle &= i\Omega C_{ge,10} - i\hbar\omega C_{ge,10} - i\hbar g(P_{e0} - P_{g1}) - \Gamma_{re}\frac{1}{2}C_{ge,10} \\
&\quad - \Gamma_{rg}\frac{1}{2}C_{ge,10} - \Gamma_{\kappa}\frac{1}{2}C_{ge,10}
\end{aligned}$$

We again construct a matrix from this,

$$\frac{d}{dt} \begin{pmatrix} P_{e0} \\ P_{g0} \\ P_{g1} \\ P_{00} \\ C_{eg,01} \\ C_{ge,10} \end{pmatrix} = \begin{pmatrix} -\Gamma_{re} & 0 & 0 & \Gamma_{el} & i\hbar g & -i\hbar g \\ 0 & -\Gamma_{rg} & \Gamma_{\kappa} & \Gamma_{gl} & 0 & 0 \\ 0 & 0 & (-\Gamma_{rg} - \Gamma_{\kappa}) & 0 & -i\hbar g & i\hbar g \\ \Gamma_{re} & \Gamma_{rg} & 0 & (-\Gamma_{gl} - \Gamma_{el}) & 0 & 0 \\ i\hbar g & 0 & -i\hbar g & 0 & A & 0 \\ -i\hbar g & 0 & i\hbar g & 0 & 0 & B \end{pmatrix} \begin{pmatrix} P_{e0} \\ P_{g0} \\ P_{g1} \\ P_{00} \\ C_{eg,01} \\ C_{ge,10} \end{pmatrix}. \quad (72)$$

Where

$$A = -i\Omega + i\hbar\omega - \frac{1}{2}\Gamma_{re} - \frac{1}{2}\Gamma_{rg} - \frac{1}{2}\Gamma_{\kappa}$$

$$B = i\Omega - i\hbar\omega - \frac{1}{2}\Gamma_{re} - \frac{1}{2}\Gamma_{rg} - \frac{1}{2}\Gamma_{\kappa}$$

Again we make a block matrix,

$$\frac{d}{dt} \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}. \quad (73)$$

Where M_{11} is the 4x4 matrix in the top left corner.

And again we do the following,

$$\frac{d}{dt}P = M_{11}P + M_{12}Q = 0 \quad (74)$$

$$\frac{d}{dt}Q = M_{21}P + M_{22}Q = 0 \quad (75)$$

$$\implies 0 = M_{11}P - (M_{12}M_{22}^{-1}M_{21})P. \quad (76)$$

$$0 = M_{11}P - (M_{12}M_{22}^{-1}M_{21})P \quad (77)$$

$$= \begin{pmatrix} -\Gamma_{re} + \frac{\hbar^2 g^2(A+B)}{AB} & 0 & -\frac{\hbar^2 g^2(A+B)}{AB} & \Gamma_{el} \\ 0 & -\Gamma_{rg} & \Gamma_{\kappa} & \Gamma_{gl} \\ -\frac{\hbar^2 g^2(A+B)}{AB} & 0 & (-\Gamma_{rg} - \Gamma_{\kappa}) + \frac{\hbar^2 g^2(A+B)}{AB} & 0 \\ \Gamma_{re} & \Gamma_{rg} & 0 & -\Gamma_{gl} - \Gamma_{el} \end{pmatrix} \begin{pmatrix} P_{e0} \\ P_{g0} \\ P_{g1} \\ P_{00} \end{pmatrix}. \quad (78)$$

We are interested in the rate between states $|e, 0\rangle$ and $|g, 1\rangle$ because this is the situation where an electron would be emitted. In the above matrix, the entries connecting P_{e0} and P_{g1} should represent that very rate μ .

$$\mu = -\frac{\hbar^2 g^2(A+B)}{AB} = \frac{\hbar^2 g^2(\Gamma_{re} + \Gamma_{rg} + \Gamma_{\kappa})}{(\Omega - \omega\hbar)^2 + \frac{1}{4}(\Gamma_{re}^2 + \Gamma_{rg}^2 + \Gamma_{\kappa}^2) + \frac{1}{2}(\Gamma_{re}\Gamma_{rg} + \Gamma_{re}\Gamma_{\kappa} + \Gamma_{rg}\Gamma_{\kappa})} \quad (79)$$

$$= \frac{\hbar^2 g^2(\Gamma_{re} + \Gamma_{rg} + \Gamma_{\kappa})}{(\Omega - \omega\hbar)^2 + \left(\frac{\Gamma_{re} + \Gamma_{rg} + \Gamma_{\kappa}}{2}\right)^2}. \quad (80)$$

C Appendix: Python Code for Current calculation

```
import numpy as np
import matplotlib.pyplot as plt

a = gamma_l = 5 # Gamma_L
b = gamma_r = 5 # Gamma_R
t = 2
g = 1
omega = 20 #hbaromega
# Define the time points for plotting
points = 1000
delta_points = np.linspace(-10, 30, points)

def Gamma(delta):
    GAMMA = (t**2*b)/(delta**2+(b)**2/4)
    return GAMMA
## Quantum version
def P_r(delta):
    P_r = (a*t**2)/(t**2*b+a*(2*t**2+1/4*b**2+delta**2))
    return P_r

gamma = Gamma(delta_points)#+mu
NEW_Pr = P_r(delta_points)

e = 1.60217663* 10**(-19) #Electron charge
#Static solution
Pr_static = (gamma*a)/(b*gamma+2*gamma*a+a*b)
I_static_electron = Pr_static*b*e*10**19

I_NEW = NEW_Pr*b*e*10**19 # Current for no.photon

##%% Calculating current in Resonator-coupled DQD
```

```

def current(delta):
    Omega = (4*t**2+delta**2)**(1/2)
    theta= np.arccos(-delta/Omega)
    a = Gamma_re = gamma_r * np.cos(theta/2)**2
    b = Gamma_el = gamma_l * np.sin(theta/2)**2
    c = Gamma_rg = gamma_r * np.sin(theta/2)**2
    e = Gamma_gl = gamma_l * np.cos(theta/2)**2
    d = Gamma_k = 4
    A = -1j*Omega+1j*omega -1/2*Gamma_re-1/2*Gamma_rg-1/2*Gamma_k
    B = 1j*Omega-1j*omega -1/2*Gamma_re-1/2*Gamma_rg-1/2*Gamma_k

    M = np.array([
        [-a, 0, 0, b, g*1j, -g*1j],
        [0, -c, d, e, 0, 0],
        [0, 0, -c-d, 0, -g*1j, g*1j],
        [a, c, 0, -b-e, 0, 0],
        [g*1j, 0,-g*1j , 0, A, 0],
        [-g*1j, 0, g*1j, 0, 0, B]])

    # Compute the SVD of M
    U, s, Vt = np.linalg.svd(M)

    # Find the null space of M by selecting the last column of V (correspondi
    null_space_vector = Vt[-1, :]

    # Optionally, you may normalize the null space vector
    null_space_vector /= np.linalg.norm(null_space_vector)
    # Define the modified null space vector with the condition  $x_1 + x_2 + x_3 +$ 
    modified_null_space_vector = null_space_vector[:4]
    modified_null_space_vector /= np.sum(modified_null_space_vector)

    steady_state_solution = null_space_vector.real
    P_e0 = steady_state_solution[0]

```

```

P_g0 = steady_state_solution [1]
P_g1 = steady_state_solution [2]
P_00 = steady_state_solution [3]
mu = (g**2*(Gamma_re+Gamma_rg+Gamma_k))/((Omega-omega)**2+(Gamma_re+Gamma

    return P_e0, P_g0, P_g1, P_00, Gamma_re, Gamma_rg, mu
Pe0=[]
Pg0=[]
Pg1=[]
P00=[]
Gamma_re = []
Gamma_rg = []
mu = []
for i in delta_points:
    Pe0.append(current(i)[0])
    Pg0.append(current(i)[1])
    Pg1.append(current(i)[2])
    P00.append(current(i)[3])
    Gamma_re.append(current(i)[4])
    Gamma_rg.append(current(i)[5])
    mu.append(current(i)[6])

Pe0=np.array(Pe0)
Pg0=np.array(Pg0)
Pg1=np.array(Pg1)
P00=np.array(P00)
Gamma_re=np.array(Gamma_re)
Gamma_rg=np.array(Gamma_rg)

Current = (Pe0 * Gamma_re + (Pg0 + Pg1) * Gamma_rg )* e*10**19

#%%
# Plot the result

```

```

plt.plot(delta_points, I_NEW, label="$I_{DQD\text{-}only}$")
plt.plot(delta_points, Current, label = '$I_{\text{Resonator-coupled}}$')
plt.plot(delta_points, mu, label = '$I_{\text{photon}}=\mu$')

plt.text(0.04, 0.85, f'$\Gamma_L=\{a\}\n\Gamma_R=\{b}$', fontsize=14, color='black')
plt.axvline(omega, color = 'black', linestyle = ':')
plt.axvline(0, color = 'black', linestyle = ':')
plt.xticks([0, omega],[0, f'$\hbar\omega=\{\omega}$'], fontsize = 14)
plt.xlabel('$\delta$', fontsize = 16)
plt.ylabel('$I(\delta)$($10^{-19}$)', fontsize = 16)
plt.legend(fontsize = 14, loc = 'upper-right')
plt.title(f'Electric current using  $t=\{t\}$ ', fontsize=15.5)

```