RECURRENCE OF RANDOM WALKS ON RANDOM ENVIRONMENTS

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Abstract

Showing when random walks are transient or recurrent is the central topic we investigate in this thesis. To be able to find the conditions of a random walk, we develop techniques gained from knowledge about electrical networks developed from Snell and Doyle's *"Random Walks and Electrical Networks"* (1984). We aim to use these techniques from Snell and Doyle's work to be able to find conditions of recurrence for random walks in increasingly complex environments. The culmination of this thesis will be finding the necessary conditions for both recurrence and transience for random walks where the transition probabilities are independent and identically distributed random variables.

Popular Scientific Introduction

Föreställ dig att du går ut ur ditt hus och slår ett mynt. Får du huvuden går du ett steg åt vänster och får du svansar går du ett steg åt höger. Om du fortsätter denna process, är det garanterat att du kommer tillbaka hem? Det är möjligt att du bara vänder på huvudet och därför aldrig återvänder hem.

Det visar sig att även om detta är teoretiskt möjligt är sannolikheten för att det händer noll, och detsamma gäller för alla liknande serier. Det är hundra procents sannolikhet att återvända hem. Men är det alltid så? Tänk om du hade fyra olika riktningar som du kunde gå i? Eller vad skulle hända om chansen att få huvuden i sig valdes slumpmässigt varje gång? Kan vi garantera att vi kommer hem?

Denna fråga, om vi garanterat kommer tillbaka från där vi började, är den centrala frågan som denna avhandling kommer att ägna sig åt att besvara. För att kunna svara på denna fråga i alltmer komplexa situationer kommer vi att utveckla matematiska tekniker som gör att vi kan använda metoder utvecklade för att förenkla elektriska nätverk.

Acknowledgments

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Contents

Chapter 1

Introduction

In this thesis, we will explain the use of electrical networks to find conditions for the recurrence of random walks in random environments. We will first introduce a mathematical background for all the concepts discussed. Then we will go through the mathematics needed to show that we can use electrical networks to solve for recurrence, and then explain the importance of these techniques. Lastly, we will show recurrence for increasingly more complex random walks using the techniques constructed.

This thesis relies heavily on the concept of *electrical networks*, and their use in solving recurrence for Markov chains. Therefore, while electrical networks are defined as purely mathematical objects in this thesis, it is important to note that how we define them comes from physical circuits. For example, how we define current in this thesis is such that the network will satisfy *Ohm's Law* and *Kirchhoff's Law*. Thus, while these laws are not explicitly stated, they are the underlying reason we define things the way we do.

Chapter 2

Mathematical Preliminaries

To begin, we go through some mathematical preliminaries. This section contains basic definitions and theories regarding graph theory, probability, and Markov chains. In this section, no theorems or lemmas will be proved, as they are outside the scope of this thesis.

Additionally, if the reader is sufficiently versed in any of these areas they may be able to skip reading this chapter, however things we prove later will rely on this foundation. For proofs of each statement as well as complimentary information, the reader may refer to the books cited in each section.

2.1 Graph Theory

Definition 2.1.1 (Graph) *A* Graph *is the ordered pair $G = (V, E)$. Where V is the set of points or vertexes, and $E = \{(x, y) : x, y \in V\}$ is the set of edges connecting the vertices.*

Definition 2.1.2 (Weighted Graph) *A* weighted graph *is a graph where each edge has an assigned weight. These weights are notated as $E_{x,y}$.*

Definition 2.1.3 (Multigraph) *A* multigraph *is a graph where there can exist more than one edge between any two points. These edges will be notated as $(x, y)^{(1)}$, $(x, y)^{(2)}$, and so on. When we have a weighted multigraph, we will denote the weights as $E^{(1)}_{x,y}$, $E^{(2)}_{x,y}$ and so on.*

For any graph, the distance between two points is assumed to be the *geodesic distance*, i.e., the shortest path distance, between those two points.

Definition 2.1.4 (Harmonic Function) *A function on a weighted graph* $G = (V, E)$ *is called* harmonic *on the set* $I \subseteq V$ *if we have that:*

$$f(x) = \sum_{y \in V} E_{x,y} f(y) \quad \forall x \in I$$

2.2 Probability

Definition 2.2.1 (Sigma Algebra) *Let* Ω *be any set often called the* sample space*, then we define a* Sigma Algebra $\mathcal{F}$*, as any collection of subsets of* Ω *that satisfy the following three properties:*

$$\begin{aligned} &1) \quad \Omega \in \mathcal{F} \\ &2) \quad \textit{If } A \in \mathcal{F} \implies \bar{A} \in \mathcal{F} \\ &3) \quad \textit{If } A_0, A_1, ... \in \mathcal{F} \implies \bigcup_{i=0}^{\infty} A_i \in \mathcal{F} \quad \forall A_i \in \mathcal{F} \end{aligned}$$

Definition 2.2.2 (Probability Measure) *Then we define* probability $I\!P$*, as a finitely additive measure from* $\mathcal{F}$ *to the interval [0,1] that satisfies the following three conditions:.*

$$\begin{aligned} &1) \quad I\!P(\Omega) = 1 \\ &2) \quad I\!P(A) = 1 - I\!P(\bar{A}) \quad \forall A \in \mathcal{F} \\ &3) \quad I\!P(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=0}^{\infty} I\!P(A_i) \quad \forall A_i \in \mathcal{F} \end{aligned}$$

Definition 2.2.3 (Random Variable) *A* Random Variable*, X, is a measurable function that maps from* Ω *to the real line.*

$$X : \Omega \to \mathbb{R}$$

Definition 2.2.4 (Stochastic Process) *A* Stochastic Process X_n*, is a collection of random variables indexed by time n.*

Theorem 2.2.5 (Law of Total Probability) *[4] Let* $A_1, A_2, \ldots$ *be events that form a partition of the sample space* Ω*. Let* B *be any event in* $\mathcal{F}$*. Then*

$$I\!P(B) = I\!P(A_1 \cap B) + P(A_2 \cap B) + ...$$

Remark 2.2.6 *Formally the Law of Total Probability is a property of how we define the probability function. However, we call it a theorem here because it is defined as such in the book from which we cite the theorem.*

Lemma 2.2.7 (Borel-Cantelli) *[3] Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of events in some probability space, then:*

$$\text{If, } \sum_{n=1}^{\infty} I\!P(A_n) < \infty \implies I\!P(\{A_n \ i.o.\}) = 0$$

$$\text{If, } \sum_{n=1}^{\infty} I\!P(A_n) = \infty, \text{ and all } A_n \text{ are independent} \implies I\!P(\{A_n \ i.o.\}) = 1$$

Where $\{A_n \ i.o.\}$ denotes the event A_n happening "infinitely often", and is formally defined as:

$$\{A_N \ i.o.\} := \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k$$

Definition 2.2.8 (Moment Generating Function) *[5] The moment generating function of a random variable X is:*

$$\phi_X(t) = \mathbb{E}[e^{tX}]$$

Provided there exists $h > 0$ such that the expectation exists and is finite for $|t| < h$.

Theorem 2.2.9 *[5] Let X be a random variable whose moment generating function $\phi_X(t)$, exists for $|t| < h$ for some $h > 0$. Then:*

a) all moments exist, that is $\mathbb{E}[|X|^r] < \infty \quad \forall r > 0$

b) $\mathbb{E}[X^r] = \phi_X^{(n)}(0)$ *for,* $n = 1, 2, ...$

Theorem 2.2.10 (Jensen's Inequality) *[3] Let the Borel function $g = g(x)$ be convex downward and $\mathbb{E}[|\xi|] < \infty$. Then:*

$$g(\mathbb{E}[\xi]) \leq \mathbb{E}[g(\xi)]$$

Theorem 2.2.11 *Let ξ be a random variable that has a defined moment generating function $\phi_\xi(t)$. Then:*

$$e^{\mu t} \leq \phi_\xi(t) \tag{2.2.1}$$

Proof: This proof follows directly from *Jensen's Inequality*. Let ξ be a random variable with expectation $\mathbb{E}[\xi] = \mu$. Then since the function $g(x) = e^{xt}$ is convex for all t, by *Jensen's Inequality* we have the following:

$$g(\mathbb{E}[\xi]) = e^{\mu t} \leq \mathbb{E}[g(\xi)] = \mathbb{E}[e^{t\xi}] = \phi_\xi(t)$$

Which proves the statement above.

Theorem 2.2.12 (Chernoff-Cremier Bound) *Let S_n be the sum of n independent random variables with identical distribution ξ. Additionally, let ϕ_ξ be the moment-generating function of ξ, and let a be any real number. Then the following inequalities hold:*

$$\mathbb{P}(S_n \geq an) \leq e^{n \ln(\phi_\xi(\lambda) - \lambda an)} \qquad \forall \lambda > 0 \tag{2.2.2}$$

$$\mathbb{P}(S_n \leq an) \leq e^{n \ln(\phi_\xi(-\lambda) + \lambda an)} \qquad \forall \lambda > 0 \tag{2.2.3}$$

Proof: To begin, we first prove this is true for the first inequality (2.2.2):

$$\begin{aligned} \mathbb{P}(S_n \geq an) &= \mathbb{P}(\lambda S_n \geq \lambda an) \quad \forall \lambda > 0 \\ &= \mathbb{P}(e^{\lambda S_n} \geq e^{\lambda an}) \quad \forall \lambda > 0 \end{aligned}$$

Then, by the *Markov Inequality* we have that:

$$\begin{aligned} \mathbb{P}(e^{\lambda S_n} \geq e^{\lambda an}) &\leq \frac{\mathbb{E}[e^{\lambda S_n}]}{e^{\lambda an}} \qquad \forall \lambda > 0 \\ &\leq \frac{\mathbb{E}[e^{\lambda \xi}]^n}{e^{\lambda an}} \qquad \forall \lambda > 0 \\ &\leq e^{n \ln(\phi_\xi(\lambda)) - \lambda an} \quad \forall \lambda > 0 \end{aligned}$$

This proves (2.2.2) is true for all positive lambdas. Then the second inequality follows directly by substituting λ for $-\lambda$.

2.3 Markov Chains

Definition 2.3.1 (Markov chain) *A* Markov chain *is a Stochastic process, X_n, on a countable or finite state space $S = \{i_0, i_1, i_2, ...\}$ such that the probability of moving to another state is only determined by the current state:*

$$\mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1}, X_{n-2} = i_{i-2}, ...) = \mathbb{P}(X_n = i_n | X_{n-1} = i_{n-1})$$

This is called the "Memory-less property" of Markov chains, and it is the foundation for everything that will follow. Because of this propriety for Markov chains we define the *transition probability*, as the probability from going from any one state to any other:

$$p_{j,i} := \mathbb{P}(X_{k+1} = i | X_k = j)$$

Using these probabilities we can then construct a *transition matrix*, P, for any Markov chain with a finite state space:

$$P := \begin{pmatrix} p_{0,0} & \cdots & p_{0,r} \\ \cdots & \cdots & \cdots \\ p_{r,0} & \cdots & p_{r,r} \end{pmatrix}$$

Definition 2.3.2 (Distribution of a Markov chain) *Let X_n be a Markov chain. We define a* distribution *for that Markov chain as a row vector, where each column defines the probability of being in the corresponding state:*

$$\pi = (\pi_{i_0}, \pi_{i_1}, ...)$$

$$\pi_{i_0} = I\!P(X_n = i_0)$$

Definition 2.3.3 (Stationarity) *A distribution of a Markov chain is said to be stationary if it does not change when multiplied with the transition matrix:*

$$\pi P = \pi$$

Definition 2.3.4 (Reversibilitity) *A Markov chain is called* reversible *if the probability of going from a state $i \in S$ to another state $j \in S$ is equivalent to the probability of going from state $j \in S$ to $i \in S$, when in the stationary distribution:*

$$\pi_i p_{i,j} = \pi_j p_{j,i} \quad \forall i, j \in S$$

Definition 2.3.5 (Connected Markov chains) *A Markov chain is called connected if the probability of going from any state to any other is positive:*

$$\exists n \in \mathbb{N} \ s.t. \ I\!P(X_{k+n} = i | X_k = j) > 0 \quad \forall i, j \in S \quad \forall k \in \mathbb{N}$$

Chapter 3

Theoretical Background

In this section, we will lay the groundwork for what we aim to prove in Chapter 4. We will construct the necessary definitions and statements for proving the theorems in later chapters.

In this chapter, we will first go over the definition of a random walk on a graph, talk about the return probability of random walks on finite graphs, and finally make the connection between the return probability for finite and infinite graphs and relate that to the concept of recurrence. Here it is important to note that the definition of recurrence used in this section comes directly from an intuitive understanding of recurrence, and it can be shown to correspond with the more conventional definition of recurrence; however, that is outside the scope of this thesis.

In the next section, we will go over how we mathematically define an "electrical network" and the properties that electrical networks have. It is critical to note that some definitions used in this section were created by the author for convenience and therefore do not use standard notation. This is especially true for the definition of an "electrical network.". This was done both for convenience of notation and to emphasize the connection between the mathematical object described and "real-world" electrical networks.

Finally, we will show both the benefits we gain through the use of electrical networks and how the effective resistance in a network is related to the recurrence of a random walk on that network.

3.1 Random Walks

Definition 3.1.1 *Let $G = (V, E)$ be a connected weighted multigraph, then we define the* random walk on G *as the Markov chain with state space equal to the set of vertexes, where the transition probabilities are determined by the weights of each edge.*

For all random walks on a graph G, we designate one state as the *origin*, from here forward this state will be notated as s_0. Additionally, we designate one state as the *end state* notated as s_d.

Definition 3.1.2 *Let X_n $n \in \mathbb{Z}$, be a random walk on some graph G. Then we define a function r, called the* return probability*, that signifies the probability that the origin is visited before the end state, given that the chain starts at state x.*

From our definition, we see that the return probability should be one and zero at the origin and end points. This is because the probability of going to state s_0 before s_d while at state s_0 is one. And inversely the probability of going to state s_0 before s_d must be zero specifically. Then by the *Law of Total Probability*, we know that the return probability of all the states that are not the origin or the end state is equal to the sum of the probability of moving to that state multiplied by the return probability. This gives us the following function:

$$r(x) = \begin{cases} r(x) = 1 & \text{if } x = s_0 \\ r(x) = 0 & \text{if } x = s_d \\ \sum_{y \in S} p_{x,y} v(y) & \text{if } x \neq s_0, s_d \end{cases}$$

Definition 3.1.3 *Let X_n be a random walk on some finite graph G. Then we define* escape probability, p_{esc}, *of that random walk as the probability that given we start at the origin we visit the end point before returning to the origin:*

$$p_{esc} = \sum_{y \in S} p_{i_0,y}(1 - r(y))$$

Definition 3.1.4 *Let $G = (V, E)$ be a weighted connected multigraph, and let s_0 denote an arbitrary point in, V called the origin. Then let $\{G^{(d)}\} = \{(V^{(d)}, E^{(d)})\}$ be a sequence of graphs generated by G, such that the limit of this sequence approaches G as d goes to infinity:*

$$G^{(d)} \to G \text{ as, } d \to \infty$$

Where for each $G^{(d)}$ its set of points, $V^{(d)}$, is equal to V for all points at most distance d from the origin, and all points greater than distance d are put together into one point notated s_d. Hence, for $G^{(d)}$, any edge $(x, y) \in E$ where both the points x, y are at most distance d from the origin in G are in $E^{(d)}$. And any edge $(x, y) \in E$ where x is at most distance d from the origin, but y is greater than distance d is replaced with the edge $(x, s_d) \in E^{(d)}$ with weight equal to (x, y).

Definition 3.1.5 *Let G be some graph, and let $\{G^{(d)}\}$ be the sequence of subgraphs of G. Then we can define $\{X_n^{(d)}\}_{d=1}^{\infty}$ as the* sequence of random walks*, where each element in the sequence $X_n^{(d)}$ is the random walk on the graph $G^{(d)}$.*

Definition 3.1.6 *Let G be some graph, with the sequence of subgraphs $\{G^{(d)}\}$, and $\{X_n^{(d)}\}$ be the sequence of random walks corresponding to each subgraph. Then we can define $\{p_{esc}^{(d)}\}$ as the* sequence of escape probabilities*, where each element in the sequence $p_{esc}^{(d)}$ is the escape probability of the random walk $X_n^{(d)}$.*

Definition 3.1.7 *Let X_n be a random walk on an **infinite** graph G, then we define the* escape probability of the random walk *as equal to the limit of the sequence of escape probabilities.*

$$p_{esc} \coloneqq lim_{d \to \infty} p_{esc}^{(d)}$$

Remark 3.1.8 *Here we are extending Definition (3.1.3) such that the escape probability is defined also for Random Walks with infinite state spaces.*

Remark 3.1.9 *We know the limit of the sequences of escape probabilities exists. This is because as d increases, the set of points greater than distance d from the origin can only decrease. Therefore, the probability of visiting s_d before the origin also cannot increase. Thus, since the sequence of escape probabilities is naturally bounded by 0, we have that the sequence is monotonic and bounded, and therefore a limit must exist.*

Definition 3.1.10 *Let X_n be a random walk on an infinite graph G, then we call the random walk **recurrent** if the escape probability of X_n is zero.*

Definition 3.1.11 *Let X_n be a random walk on an infinite graph G, then we call the random walk **transient** if the escape probability of X_n is greater than zero.*

3.2 Electrical Networks

Definition 3.2.1 *We define an* Electrical Network*, $G = (S, R)$, as a weighted, connected, multigraph. Here, S is a finite set of points which we divided into two disjoint subsets B and I, where $B = \{0, n\}$ signifies the points connected by a battery and I represent the interior points of the network that are not connected directly to a battery. R is the set of edges connecting the points in S to each other, we will call this the set of resistors. Let $(x, y) \in R$ be a resistor which connects two points in S, then we write the weight of that resistor as $R_{x,y}$.*

Remark 3.2.2 *To stay consistent with the physical interpretation of electrical networks, we impose the following two requirements for all resistors in an electrical network.*

$$R_{x,y} = R_{y,x} \quad \forall (x, y) \in R$$
$$R_{x,y} > 0 \quad \forall (x, y) \in R$$

Definition 3.2.3 *We can define the resistance at each point in the network as the sum of resistors connected to that point:*

$$R_x = \sum_{y\colon (x,y)\in R} R_{y,x} \quad \forall x \in S$$

Definition 3.2.4 *For each edge in a given electrical network, we define its* conductivity *as:*

$$C_{x,y} = \begin{cases} \frac{1}{R_{x,y}} & \text{if } (x, y) \in R \\ 0 & \text{if } (x, y) \notin R \end{cases}$$

Definition 3.2.5 *As before, we can also define the conductivity at each point $x \in S$ equal to the sum of conductivity attached to that point:*

$$C_x = \sum_{y\in S} C_{y,x} \quad \forall x \in S$$

Definition 3.2.6 *Let G be an electrical network. Then we define a function v, on the points of an electrical network, which we call the* voltage*:*

$$v(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = n \\ \sum_{y\in S} \frac{C_{x,y}}{C_x} v(y) & \text{if } x \in I \end{cases}$$

Remark 3.2.7 *We define voltage this way such that our electrical network will follow Kirchhoff's law. Additionally, here $v(0)$ signifies the voltage of the battery attached to the network, which we assume to be one for convenience.*

Definition 3.2.8 *Let G be an electrical network. Then let $(x, y) \in R$ be two points connected by a resistor. We define the* current *between x and y as a function such that:*

$$i_{x,y} = \frac{v(x) - v(y)}{R_{x,y}} = (v(x) - v(y))C_{x,y} \quad \forall (x, y) \in R$$

Definition 3.2.9 *We can also define the current at a point as the sum of currents going out from that point:*

$$i_x = \sum_{y:\ (x,y) \in R} i_{y,x} \quad \forall x \in S$$

Definition 3.2.10 *For any electrical network G, we define a flow on G as any function on the set of resistors R with the following properties:*

$$\begin{aligned} &1)\ j_{x,y} = -j_{y,x} \\ &2)\ j_x = \sum_{y:\ (x,y) \in R} j_{x,y} = 0 \ \textit{if } x \in I \\ &3)\ j_{x,y} = 0 \ \textit{if } (x, y) \notin R \end{aligned}$$

Remark 3.2.11 *We also define the flow at the end points as the sum of the flow connected to those end points. However, unlike above, this is not necessarily zero.*

$$j_x := \sum_{y:\ (x,y) \in R} j_{x,y} > 0 \ \textit{if } x \in B$$

Here we note that for any flow on an electrical network G we have that $j_0 = -j_n$ since:

$$j_0 + j_n = \sum_{x \in S} j_x = \sum_{x \in S} \sum_{y \in S} j_{x,y} = \frac{1}{2} \sum_{x \in S} \sum_{y \in S} (j_{x,y} + j_{y,x}) = 0$$

Theorem 3.2.12 *[1] For any electrical network G the current on that network is a flow.*

Proof: Given how we defined current, we can see that the first and third properties are met; thus, all we need to do is prove the second. However, this also comes from our definition (3.2.7), since given $x \in I$:

$$i_x = \sum_{y \in S}(v(x) - v(y))C_{x,y} = v(x)\sum_{y \in S} C_{x,y} - \sum_{y \in S} v(y)C_{x,y}$$

$$= C_x(\sum_{y \in S} \frac{C_{x,y}}{C_x} v(y) - \sum_{y \in S} v(y)C_{x,y} = 0$$

Definition 3.2.13 *Let G be an electrical network, and let j be a flow on G. Then we define the* total energy dispersion*, E_j of the network G through the flow j.*

$$E_j := \frac{1}{2} \sum_{(x,y) \in R} j_{x,y}^2 R_{x,y}$$

Lemma 3.2.14 (Conservation of energy) *[1] Let $G = (S, R)$ be an electrical network and let w be any function defined on S, and let j be any flow defined on R. Then:*

$$(w(0) - w(n))j_0 = \frac{1}{2} \sum_{x \in S} \sum_{y \in S} (w(x) - w(y))j_{x,y}$$

Proof: To prove this, we expand the left-hand side.

$$\sum_{x \in S}\sum_{y \in S}(w(x) - w(y))j_{x,y} = \sum_{x \in S}(w(x)\sum_{y \in S} j_{x,y}) - \sum_{y \in S}(w(y)\sum_{x \in S} j_{x,y})$$

$$= (w(0)(j_0) + w(n)(j_n)) - (w(0)(-j_0) + w(n)(-j_n))$$

$$= 2(w(0) - w(n))j_0$$

Definition 3.2.15 *For any electrical network, define the effective resistance of that network as:*

$$R_{\text{eff}} := \frac{1}{i_0}$$

In the same manner, we can also define the *effective conductance* of an electrical network:

$$C_{\text{eff}} = \frac{1}{R_{\text{eff}}} = i_0$$

Theorem 3.2.16 *[1]: The effective conductance of an electrical network is equal to the total energy dispersion through the current.*

Proof: To prove this we use the conservation of energy and set $w = v$ and $j = i$ which gives us:

$$E_i = \frac{1}{2} \sum_{(x,y) \in R} i_{x,y}^2 R_{x,y} = \frac{1}{2} \sum_{x \in S}\sum_{y \in S}(v(x) - v(y))i_{x,y} = (v(0) - v(n))i_0 = C_{\text{eff}}$$

3.3 Properties of Electric Networks

Theorem 3.3.1 (Thomson's Principle) *[1] Current is the flow which uniquely minimizes the total energy dissipation for any electrical network.*

Proof: [1] Let j be any unit flow on an electrical network $G = (S, R)$ from the origin to the end point. Then we define a new function, d, on G such that $d_{x,y} := j_{x,y} - i_{x,y}$, then by definition, d is a flow and $d_0 = \sum_x (j_{0,x} - i_{0,x}) = 0$. Then we have that:

$$\sum_{x,y} j_{x,y}^2 R_{x,y} = \sum_{x,y} (i_{x,y} + d_{x,y})_{x,y}^2 R_{x,y}$$

$$\sum_{x,y} j_{x,y}^2 R_{x,y} = \sum_{x,y} j_{x,y}^2 R_{x,y} + 2\sum_{x,y} i_{x,y} d_{x,y} R_{x,y} + \sum_{x,y} d_{x,y}^2 R_{x,y}$$
$$= \sum_{x,y} j_{x,y}^2 R_{x,y} + 2\sum_{x,y} (v_x - v_y) d_{x,y} + \sum_{x,y} d_{x,y}^2 R_{x,y}$$

Then, by the law of conservation of energy, we have that the middle term is equal to 0 since, $2\sum_{x,y}(v_x - v_y)d_{x,y} = 4(v_0 - v_n)d_0 = 0$, thus this gives us:

$$\sum_{x,y} j_{x,y}^2 R_{x,y} = \sum_{x,y} j_{x,y}^2 R_{x,y} + \sum_{x,y} d_{x,y}^2 R_{x,y} \geq \sum_{x,y} i_{x,y}^2 R_{x,y}$$

Thus we have proved the statement.

Theorem 3.3.2 (Rayleigh's Monotonicity Law) *[1] Given an electrical network, if the weights of the resistors are increased, then the effective resistance can only ever increase. And if the weights of the resistors are decreased, the effective resistance can only ever decrease.*

Proof: [1] Let G be an electrical network and let i be the current on G. Then we let $\hat{R}_{x,y}$ be a new set of resistors on G such that $\hat{R}_{x,y} \geq R_{x,y}$ for all $x, y \in S$. Then this new set of resistors will have a new current, which we will denote j. Then, for our network, we have the following equation:

$$\hat{R}_{\text{eff}} = \frac{1}{2}\sum_{x,y} j_{x,y}^2 \hat{R}_{x,y} \geq \frac{1}{2}\sum_{x,y} j_{x,y}^2 R_{x,y}$$

However, due to Thomson's principle, we have that the current reduces the energy lost, which gives us:

$$\frac{1}{2}\sum_{x,y} j_{x,y}^2 R_{x,y} \geq \frac{1}{2}\sum_{x,y} i_{x,y}^2 R_{x,y} = R_{\text{eff}}$$

Thus increasing the weight of resistors in an electrical network can never decrease the effective resistance. The proof for the decreasing of the weight of the resistors is identical, therefore we have proved the theorem.

Corollary 3.3.3 (Shorting Law) *[1] Given an electrical network, adding a resistor can only increase the total resistance.*

Proof: By *Rayleigh's Monotonicity Law*, we know that if the total resistance of a network is increased, then the effective resistance can only increase; thus, all we need to show is that adding a resistor decreases the total resistance. This, however, is clear to see because we do not allow resistors with negative weight; thus, removing a resistor cannot ever increase the total resistance and therefore can never increase the effective resistance.

Corollary 3.3.4 *If the voltages between two points in an electrical network are equal, one can short them together without increasing the effective resistance.*

Proof: This statement follows directly from how we define the weight of a resistor. Since if the voltage of two points is the same, then the resistance of a resistor between those points will be zero. Thus, adding a resistor of weight zero to our network cannot increase nor decrease the effective resistance, and by *Rayleigh's Monotonicity Law*, this means that the effective resistance can neither increase nor decrease.

Corollary 3.3.5 (Cutting Law) *[1] Given an electrical network, removing a resistor can only decrease the total resistance.*

Proof: By *Rayleigh's Monotonicity Law*, we know that if the total resistance of a network is decreased then the effective resistance can only decrease; thus, all we need to show is that removing a resistor decreases the total resistance. This, however, is clear to see because we do not allow resistors with negative weight; thus, removing a resistor cannot ever increase the total resistance and therefore can never increase the effective resistance.

Theorem 3.3.6 (Series Law) *[2] Let $G = (S, R)$ be an electric network, then if $y \in I$ is a point of degree two with neighbors x,z and we replace the edges (x, y), (y, z) by a single edge (x, z) having resistance $R_{x,z} := R_{x,y} + R_{y,z}$ then the effective resistance of G is unchanged.*

Proof: Let G be an electrical network and let x, y, z be three points in G such that y only has degree two and is directly connected to both points x and z.

Then we create a new network G' which is identical except there is no point y and there is only one resistor between points x and z with weight $R'_{x,z} = R_{x,y} + R_{y,z}$. In our new network G' the sum of the weights of all resistors neither increased nor decreased; thus, by *Rayleigh's Monotonicity Law* we have that the effective resistance also can neither increase nor decrease, thus it must stay the same.

Theorem 3.3.7 (Parallel Law) *[2] Let $G = (S, R)$ be an electric network, then if two edges $(x, y)^{(1)}$ and $(x, y)^{(2)}$ that both join points $x, y \in S$ are replaced by a single edge (x, y) joining x, y of conductance $C_{x,y} \coloneqq C^{(1)}_{x,y} + C^{(2)}_{x,y}$, then the effective current of G is unchanged.*

Proof: The proof for this is the same for the theorem above, and thus is proved analogously.

3.4 Equivalence of random walks to electrical networks

Definition 3.4.1 *Let $G = (S, R)$ be an electrical network, and let then we define a* random walk on G *as the Markov chain with state space equal to S, and the transition probabilities are determined by:*

$$p_{x,y} = \frac{C_{x,y}}{C_x} \quad \forall x, y \in S$$

Theorem 3.4.2 *[1] For a random walk on an electrical network, the return probability of any state in the Markov chain is equivalent to the voltage at the corresponding electrical network at all points.*

Proof: Let $G = (S, R)$ be the electrical network, and let X_n be a random walk on G such that the origin is the point $0 \in S$ and the end point is the point $n \in S$. Then we see that the values at these two points are identical, thus all we need to show is that $v(x) = r(x)$ on all the interior points of the network. To be able to prove this, we first need to prove two theorems.

Theorem 3.4.3 (Maximal Principle) *[2] Let $G = (S, R)$ be any electrical network. If f is a harmonic function on G, and the supremum of f on S is achieved at some element $x \in I$, then f is constant on all of S.*

Proof: [2] Let $K := \{y \in S; f(y) = sup f\}$. Note that if $x \in I \cap K$ and $(x, y) \in R$, then $y \in K$ because f is harmonic at x. Hence, the conclusion follows.

Theorem 3.4.4 (Uniqueness of Harmonic Functions) *[2] Let* $G = (S, R)$ *be an electrical network. If f,g are two functions that are both harmonic on G and agree on all the boundary points (that is,* $f(x) = g(x)$ *for all* $x \in B$*), then* $f = g$*.*

Proof: [2] Let $h \coloneqq f - g$, we aim to show that $h \leq 0$. This suffices to establish the proof, since then $h \geq 0$ by symmetry. Since I is finite, h achieves its overall supremum at some point $x \in S$. If $x \in B$, then by definition $h(x) \leq 0$, as desired. On the other hand, if $x \in I$, then by the maximum principle, $h(x) \leq sup\ h = 0$, which again shows that $h \leq 0$.

Then since it is clear that both r and v are harmonic on G by the *Uniqueness of Harmonic functions*, we have that $r(x) = v(x)$ for all $x \in S$ since they are equal on their boundaries.

Theorem 3.4.5 *[1] For a random walk on an electrical network, the probability of escape is equal to the effective conductivity of the electrical network generated by that random walk divided by the conductance at the origin.*

Proof: Let $\{G^{(d)}\}$ be a sequence of electrical networks generated by some electrical network G, and let $X_n^{(d)}$ be the corresponding random walk on each $G^{(d)}$. Then, for each electrical network in the series $\{G^{(d)}\}$ we have that the effective conductance of $G^{(d)}$ is:

$$C_{\text{eff}}^{(d)} = i_0 = \sum_{y \in S} (v(0) - v(y))C_{0,y} = \sum_{y \in S} C_{0,y} - v(y)C_{0,y} = C_0 - \sum_{y \in S} v(y)C_{0,y}$$

Then we divide everything by C_0, which gives us:

$$C_{\text{eff}}^{(d)} = \frac{1}{C_0}(1 - \sum_{y \in S} v(y)\frac{C_{0,y}}{C_0})$$

Which, by [Theorem 3.4.2] gives us:

$$C_{\text{eff}}^{(d)} = \frac{1}{C_0}(1 - \sum_{y \in S} v(y)\frac{C_{0,y}}{C_0}) = \frac{1}{C_0}(1 - \sum_{y \in S} r(y)p_{0,y}) = \frac{1}{C_0}(p_{esc}^{(d)})$$

Thus, we have that:

$$p_{esc}^{(d)} = \frac{C_{\text{eff}}^{(d)}}{C_0} = \frac{1}{C_0 R_{\text{eff}}^{(d)}}$$

Corollary 3.4.6 *[1] A random walk is only recurrent if and only if the effective resistance of the electric network generated by the random walk is infinity. Additionally, a random walk is only transient if and only if the effective resistance is finite.*

Proof: First, we note that as d increases the amount of resistors can only increase, thus by *Rayleigh's Monotonicity Law* the effective resistance can only increase if we have more resistors. Therefore, the limit of effective resistance must exist, although it can be infinity. Then, by definition [3.1.8] we have that a random walk is recurrent if and only the limit of the probability of escape is zero. This gives us:

$$\lim_{d\to\infty} p_{esc} = \lim_{d\to\infty} \frac{1}{C_0 R_{\text{eff}}^{(d)}} = 0 \iff \lim_{d\to\infty} R_{\text{eff}}^{(d)} = \infty$$

Therefore, a random walk is recurrent if and only if the effective resistance goes to infinity. We use the same logic to prove the statement about transience:

$$\lim_{d\to\infty} p_{esc} = \lim_{d\to\infty} \frac{1}{C_0 R_{\text{eff}}^{(d)}} > 0 \iff \lim_{d\to\infty} R_{\text{eff}}^{(d)} < \infty$$

Chapter 4

Showing Recurrence of Random Walks

Now we will show recurrence for random walks on different electrical networks. We will use the tools developed in the previous section to prove each of the following theorems involving the recurrence of random walks.

To begin, we will show that a simple random walk in both one and two dimensions is recurrent. Normally, showing that simple random walks exist in both one and two dimensions is not a trivial matter. However, with the use of electrical networks, as we will show, the proof becomes quite straightforward. This allows us to demonstrate the power of the technique created.

After this, we will find conditions for recurrence for two different electrical networks. The first will be a so-called "highly connected" random walk. Where each point is connected to each other point in the network by a resistor of a weight determined by the distance between the points. (It is important to note that the distance will be defined as the Euclidean distance, since the shortest path is always one between two points.) Our goal here will be to find a bound on the function that determines the weight such that a random walk on the network will be recurrent.

Next we will have a network in which the transition matrix for the random walk is determined randomly, such that each transition probability is some i.i.d. observation of a random variable. Such a random walk is called a "Random Walk in a Random Environment" or RWRE. Our goal will be to show conditions for that underlying random variable such that a random walk on the graph will be recurrent on $\mathbb{N}$.

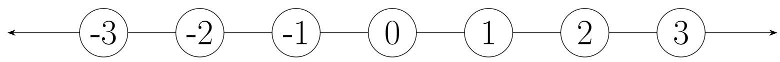
Finally, we will show both conditions for recurrence and transience for a RWRE on a tree where each node has two “offspring”. This will be the most challenging proof in this work and will serve as the culmination of the thesis. Additionally, the reader may note that each proof in this section, excluding those for simple random walks, is provided by the author, although the results are by no means novel.

4.1 Simple Random Walks

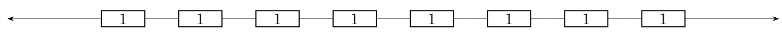
Definition 4.1.1 *A* simple random walk on $\mathbb{Z}^d$ *is a random walk with state space, $S = \mathbb{Z}^d$ where the probability of moving from any state $x \in S$ to any other state $y \in S$ is equal if are distance one from each other, and zero otherwise.*

Theorem 4.1.2 *A simple random walk on $\mathbb{Z}^d$ is recurrent if $d = 1, 2$.*

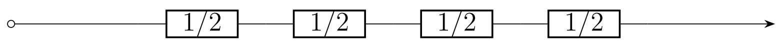
Proof if d=1: To begin this proof, first we draw the graph on which the random walk will take place:



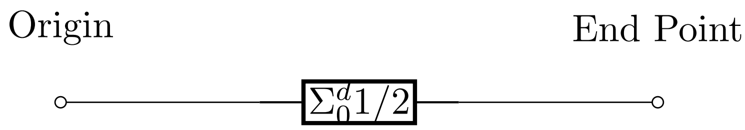
Then we convert this graph into an electrical network. Let $G = (S, R)$ be an electrical network such that $S = \mathbb{Z}^1$, and $R = \{(x, x+1) : x \in \mathbb{Z}^1\}$ with $R_{x,x+1} = R_{x,x-1} = 1$. Below is a drawing of our electrical network:



Then we start by shorting all points of equal distance from each other together. We can do this because shorting our network can only decrease effective resistance, thus if we show that the shorted electrical network has an infinite effective resistance, we also show that our original electrical network does as well. Then by the *Parallel Law*, we can replace each of the resistors $R_{x,x+1}, R_{-x,-x-1} = 1$ with one resistor $R^{'}_{x,x+1} = \frac{1}{2}$:



Then we notice that all the resistors are now in series, which by the series law allows us to replace them all with a signal resistor with weight $R_{0,n} = \sum_{i=0}^{n} R^{'}_{x,x+1}$:



Thus, since we have reduced our network to a single resistor we can easily find the effective resistance for each sub electrical network at distance d:

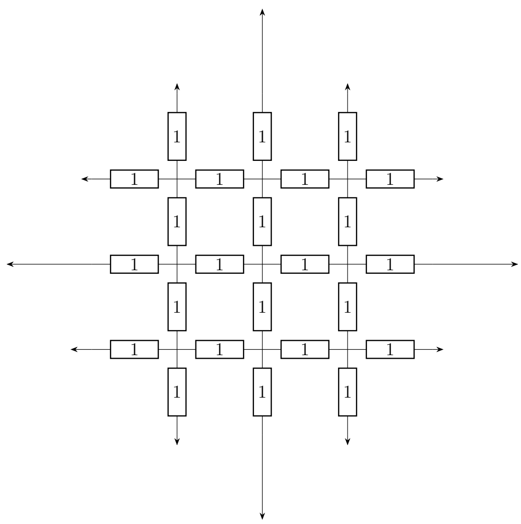
$$R_{\text{eff}}^{(d)} = \frac{1}{i_0} = R_{0,d} \geq \sum_{i=0}^{d} R^{'}_{x,x+1} = \sum_{i=0}^{d} \frac{1}{2}$$

Thus it is clear to see that:

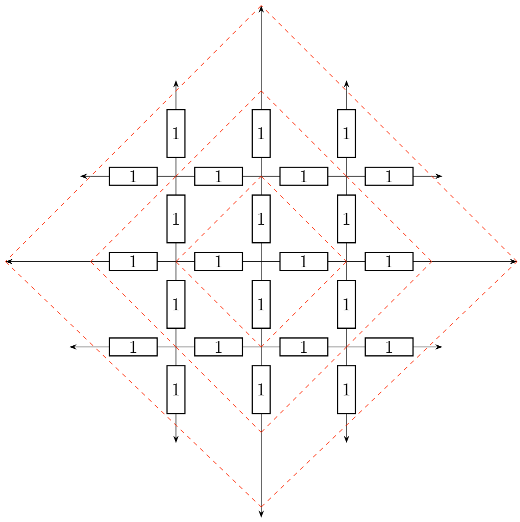
$$\lim_{d \to \infty} R_{\text{eff}}^{(d)} \to \infty$$

Therefore by Corollary 3.4.6. we have that X_n is recurrent.

Proof if d= 2: We start this proof in the same way as the one dimensional case, skipping drawing the original graph. Thus, to start we let $G = (S, R)$ be an electrical network with $S = \{(x, y) : x, y \in \mathbb{Z}\}$ $(= \mathbb{Z}^2)$, and $R = \{((x, y), (x, y + 1)) : x, y \in \mathbb{Z}^1\} \cup \{((x, y), (x + 1, y)) : x, y \in \mathbb{Z}^1\}$ where all resistors have equal weight which for simplicity set equal to one.



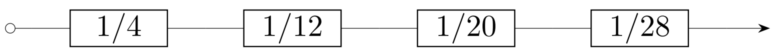
We start like in the last proof by shorting all points of equal distance from the origin to each other together. Visually, we depict this shorting below:



Then by the parallel law we can replace all the resistors between any two points with a single one with weight:

$$R'_{x,x+1} = \frac{1}{8x+4}$$

We get this sum as the new weight because as we increase the distance from the origin by one the amount of resistors increases by eight, and there are four resistors from the origin to its neighboring points. This gives us the new electrical network:



Then we notice that all the resistors are now in series, which by the series law allows us to replace them all with a signal resistor with weight $R_{0,n} = \sum_{i=0}^{n} R'_{x,x+1}$. Thus, since we have reduced our network to a single resistor we can easily find the effective resistance for each sub electrical network at distance d:

$$R_{\text{eff}}^{(d)} = R_{0,d} \geq \sum_{i=0}^{d} R'_{x,x+1} = \sum_{i=0}^{d} \frac{1}{8i+4}$$

Then since we have that:

$$R_{\text{eff}}^{(d)} \geq \sum_{i=0}^{d} \frac{1}{8(i+1)} = \frac{1}{8} \sum_{i=0}^{d} \frac{1}{(i+1)}$$

Then since the right-hand side of this equation is a harmonic series we know the limit of it goes to infinity we get.

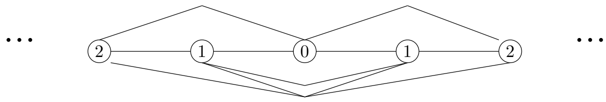
$$\lim_{d \to \infty} R_{\text{eff}}^{(d)} \to \infty$$

Therefore, by Corollary 3.4.6. we have that X_n is recurrent.

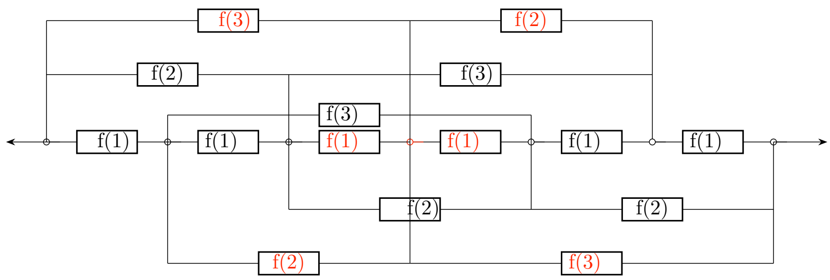
4.2 Highly Connected Random Walk

Theorem 4.2.1 *Let $G = (S, R)$ be a graph where $S = \mathbb{Z}$, and $R = \{(x, y) : \forall x, y \in S\}$ where the weight of each resistor is determined by some function $R_{x,y} = f^{-1}(x - y)$. Then a random walk on G is recurrent if $f(x)$ is bounded from above by x^{-3}.*

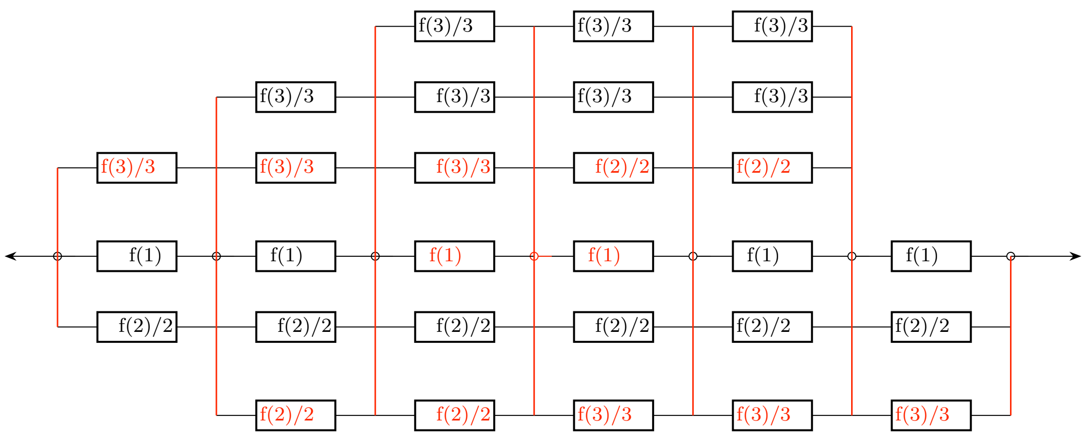
Proof: We begin this proof by first drawing a picture of the graph.



Then we translate this into an electrical network $G = (S, R)$, where $S = \mathbb{Z}$ and $R = \{R_{x,y} : \forall x, y \in \mathbb{Z}\}$. This gives us the following electrical network, (here we highlight each of the resistors going to or from the origin):



Then because the series law does not change effective resistance, we can split each resistor with weight $f(n)$ into n resistors with weight $\frac{f(n)}{n}$. Additionally, after doing this, we short along each node, which decreases the total resistance, which gives us the following electrical network:



Thus, we can use the parallel rule to replace all the resistors connecting each node with just one, which we will denote $R^{'}_{x,x+1}$. To find the weight of these new resistors, we first look at the one that will connect zero to one, which gives us the following equation:

$$R^{'}_{0,1} = (\sum_{k=1}^{\infty} k \cdot \frac{1}{R_{0,k}})^{-1}$$

We get the k factor at the beginning because for each weight $f(k)$, there are k of those that pass through zero and one. This is true because for the resistor between the points 0 and k, a resistor of the same weight exists between the points $(-1, k-1), ..., (k-1, 1)$, and thus there are a total of k resistors of that weight between zero and one. Therefore, we return to our sum:

$$R^{'}_{0,1} = (\sum_{k=1}^{\infty} k \cdot \frac{1}{R_{0,k}})^{-1} = (\sum_{k=1}^{\infty} k \cdot f(k))^{-1}$$

Next if we let f be such that $f(k) < k^{-3}$, we get the following sum:

$$R^{'}_{0,1} = (\sum_{k=1}^{\infty} k \cdot f(k))^{-1} < (\sum_{k=1}^{\infty} k^{-2})^{-1} = \frac{1}{2}$$

Then by symmetry we know that all the new resistors have the same weight, thus we have that the total recurrence is equal to the sum of all resistors which is:

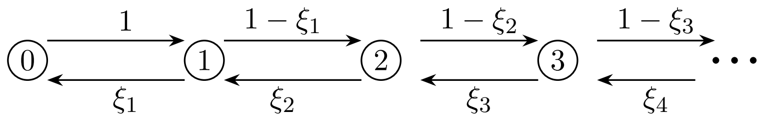
$$R^{(d)}_{\text{eff}} = \sum_{k=-d}^{d} R_{k,k+1} > \sum_{k=-d}^{d} \frac{1}{2} = d$$

Therefore the effective resistance clearly goes to infinity as d goes to infinity, and therefore by Corollary 3.4.6 we have that any random walk on G is recurrent.

4.3 RWRE on $\mathbb{N}$

Theorem 4.3.1 *Let X_n be a random walk on a graph with state space $S = \mathbb{N}$, and such that $I\!P(X_{n+1} = i - 1 | X_n = i) = \xi_i$, where each ξ_i is an independent random variable identically distributed as some random variable $\xi \in [0, 1)$. Then the random walk is recurrent if $\mathbb{E}(\ln(\frac{\xi}{1-\xi})) > 0$.*

Proof: To begin this proof, we first draw a model of our graph.



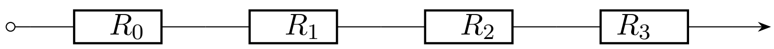
Then we can translate this graph into an electrical network $G = (S, R)$, with $S = \mathbb{N}$ and the weight of each resistor can be found by making sure the ratio of the resistor to the left and right of each point is equal to the ratio of the probability of going left and right. To make sure this is true, we need the following to hold for all resistors in G:

$$\frac{R_n}{R_{n-1}} = \frac{\xi_n}{1 - \xi_n} \quad \forall n \in \mathbb{N}$$

Then this will give us the following recursive formula for determining the weight of all the resistors in G:

$$R_n = R_{n-1}\frac{\xi_n}{1 - \xi_n} \quad \forall n \in \mathbb{N}$$

Then we can draw our newly constructed electrical network:



Next by the rules of logarithms we can rewrite each resistor as:

$$R_n = \frac{(\xi_1)...(\xi_n)}{(1-\xi_1)...(1-\xi_n)} = \exp\{\ln\left(\frac{\xi_1}{1-\xi_1}\right) + ... + \ln\left(\frac{\xi_n}{1-\xi_n}\right)\}$$

Then we let η_i be a random variable defined by:

$$\eta_i := \ln\left(\frac{\xi_i}{1-\xi_i}\right)$$

Additionally we define μ as the expectation of this random variable:

$$\mu := \mathbb{E}(\eta) = \mathbb{E}[\ln\left(\frac{\xi}{1-\xi}\right)]$$

And we let S_n be equal to the sum of the first n of these random variables which gives us:

$$R_n = \exp\{S_n\}$$

Here we will call a resistor "bad" if it has weight less than one. And we define the following event, B_n, that a resistor is bad:

$$B_n = \{R_n < 1\}$$

Then we let $\delta > 0$, and we assume the following inequality is true for all resistors in our network:

$$\mathbb{P}(R_n \leq 1) \leq e^{-\delta n} \tag{4.3.1}$$

Then if the equation above holds we have that:

$$\sum_n B_n = \sum_n \mathbb{P}(S_n \leq 0) \leq \sum_n e^{-\delta n} < \infty$$

Therefore by the *Borel-Cantelli Lemma* the event B_n happens only finitely often, and thus after some time N, all S_n where $n > N$ we have that $S_n \geq 0$. Thus, after this point, we can replace all resistors with ones of weight equal to one and only decrease the resistance. After doing this we can use the *series law* which gives us:

$$R_{\text{eff}}^d \geq \sum_{n \geq N+1}^{d} R_n \geq \sum_{n \geq N+1}^{d} 1$$

Thus it is clear to see that:

$$R_{\text{eff}} \geq \sum_{n \geq N+1}^{d} 1 = (d - N - 1) \to \infty \text{ as, } d \to \infty$$

Thus by corollary 3.4.6 we have that if equation (4.3.1) holds, then a random walk on our network is recurrent. Thus, all we need to show is that equation (4.3.1) holds for all S_n in our network. To do this we use the *Chernoff Bound* which says that:

$$\mathbb{P}(S_n \leq an) \leq e^{n \ln(\phi_\eta(-\lambda)) - \lambda an} \quad \forall \lambda > 0$$

Then we let $\epsilon > 0$, and set $a = 0$, which gives us:

$$\mathbb{P}(S_n \leq 0) \leq e^{n \ln(\phi_\eta(-\lambda)) n} \quad \forall \lambda > 0$$

Thus if the right-hand side of this equation is less than the right-hand side of (4.3.1) we have finished the proof. Thus, all we need to find is some $\lambda > 0$ for which the inequality below holds:

$$e^{n\ln(\phi_\eta(-\lambda))} \leq e^{-\delta n}$$

Then we take the logarithm of both sides and divide by n, which gives us:

$$\ln(\phi_\eta(-\lambda)) \leq \delta$$

Then since δ is just some arbitrarily small positive real number we get the following inequality:

$$\ln(\phi_\eta(-\lambda)) < 0$$

Then we define a function, J, equal to the left-hand side of this equation:

$$J(\lambda) = \ln(\phi_\eta(-\lambda))$$

Whereby definition of our function we have the two following properties

$$1)\ J(0) = 0$$

$$2)\ J^{'}(\lambda) = \frac{-\phi_\eta^{'}(-\lambda)}{\phi_\eta(-\lambda)}$$

Thus since the moment generating function is log convex we have that a sufficient condition for recurrence is that:

$$J^{'}(0) = \frac{-\mu}{1} < 0$$

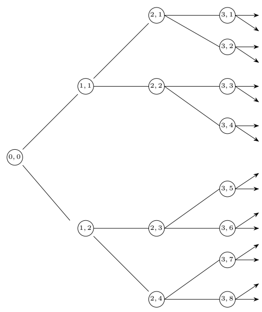
Which can only happen if $\mu > 0$. Therefore, we have shown that if $\mathbb{E}[\frac{\xi}{1-\xi}] > 0$ that a random walk on this electrical network will be recurrent.

4.4 RWRE on a Binary Tree

Definition 4.4.1 *A* Binary Tree *is a graph $G = (V, E)$ such that each point, except for the first point which is only connected to the two points in the next level, in the graph at level n is connected to two points on the next level, $n+1$, and one point on the previous level $n-1$.*

Theorem 4.4.2 *Let X_n be a Markov Chain on a binary tree where the transition probabilities $I\!P(X_n = i_{n-1,\lceil \frac{k}{2} \rceil} | X_{n-1} = i_{n,k}) = \xi_{n,k}$, where each $\xi_{n,k}$ is an independent random variable identically distributed as some random variable $\xi \in [0, 1)$. Then X_n is recurrent if $\mathbb{E}[\ln(\frac{2\xi}{1-\xi})] > \ln(2)$.*

Proof: To begin, we first look at a drawing of our graph: Then we translate



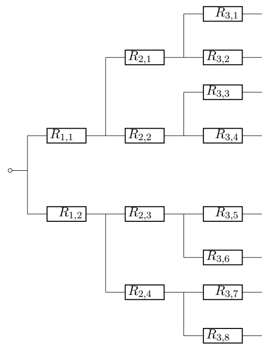
this into an electrical network $G = (S, R)$ where S is a binary tree, and the weight of each resistor is determined by making the ratio of resistors equal to the ratio of probabilities for the Markov Chain:

$$\frac{R_{n,k}}{R_{n-1,\lceil k/2 \rceil}} = \frac{2\xi_{n,k}}{1 - \xi_{n,k}}$$

Which gives us a recursive formula for the value of each resistor:

$$R_{n,k} = \frac{2\xi_{n,k}}{1 - \xi_{n,k}} R_{n-1,\lceil k/2 \rceil}$$

Then we can draw our newly constructed electrical network:



Next we define a random variable equal to the logarithm of the ratio of probabilities:

$$\eta_{n,k} := \ln(\frac{2\xi_{n,k}}{1-\xi_{n,k}})$$

Then we use this definition, and by the logarithm rules we can rewrite each resistor in the following way:

$$R_{n,k} = \exp\left(\eta_{n,k} + \eta_{n-1,\lceil k/2 \rceil} + ... + \eta_{1,\lceil k/2^{n-1} \rceil}\right) = \exp\left(\sum_{i=0}^{n-1} \eta_{n-i,\lceil k/2^i \rceil}\right)$$

And for notational simplicity we can define the inner sum as a random variable.

$$S_{n,k} := \left(\sum_{i=0}^{n-1} \eta_{n-i,\lceil k/2^i \rceil}\right)$$

This gives us the following equation for the weight of each resistor in our network.

$$R_{n,k} = \exp\left(S_{n,k}\right)$$

Then in our network we call a resistor a "bad resistor" if it has weight smaller than 2^n. And we define the event that a resistor, $R_{n,k}$, is bad:

$$\mathrm{B}_{n,k} = \{R_{n,k} < 2^n\}$$

Next also we define the event that on some level n there exists at least one bad resistor.

$$B_n = \bigcup_{k=1}^{2^n} \{R_{n,k} < 2^n\}$$

Next we *assume* that the following equation holds:

$$\mathbb{P}(\mathrm{B}_{n,k}) = \mathbb{P}(R_{n,k} \leq 2^n) \leq 2^{-(1+\delta)n} \tag{4.4.1}$$

We can reformulate this condition as:

$$\mathbb{P}(S_{n,k} \leq n\ln(2)) \leq e^{-n(1+\delta)\ln(2)}$$

Then we use this to find a bound for the probability at least any one resistor at level n is bad:

$$\mathbb{P}(\{\mathrm{B}_n\}) \leq \sum_{k=1}^{2^n} \mathbb{P}(\mathrm{B}_{n,k})$$

But because of (4.4.1) we have that:

$$\sum_{k=1}^{2^n} \mathbb{P}(\mathrm{B}_{n,k}) \leq \sum_{k=1}^{2^n} 2^{-(1+\delta)n} = 2^{-\delta n}$$

From this we can use the *Borel-Cantelli Lemma* to show that the event that a level has a bad resistor happens only a finite amount of times because $2^{-\delta n}$ is summable:

$$\sum_{n=1}^{\infty} \mathbb{P}(\mathrm{B}_n) \leq \sum_{n=1}^{\infty} 2^{-\delta n} < \infty$$

Thus we know that after some random level N each resistor past that level has a minimum weight of 2^n. Then we can short across each level, because this only decreases the effective resistance, and then we can use the *Parallel Law* to replace all resistors at each level by a new resistor $\tilde{R}_n$, which as value:

$$\tilde{R}_n = (\sum_{k=1}^{2^n} R_{n,k}^{-1})^{-1}$$

Then we can replace all resistors after level N by 2^n, and because this can never increase the weight of any individual resistor, and by *Rayleigh's Monotonicity Law* the effective resistance of the whole network can only decrease. This gives us the following inequality.

$$\tilde{R}_n \geq (\sum_{k=1}^{2^n} 2^{-n})^{-1} = (2^n 2^{-n})^{-1} = 1$$

Now, since our network only consists of resistors in series, we can replace them all by one resistor equal to their sum. This new resistor will be equal to our effective resistance, thus we can say the following about the effective resistance of our network:

$$R_{\text{eff}}^d = \sum_{n=1}^{d} \tilde{R}_n \geq \sum_{n=N+1}^{d} \tilde{R}_n = \sum_{n=N+1}^{d} 1 = (d - N - 1) \to \infty \text{ as, } d \to \infty$$

Therefore we have that if our equation (4.4.1) holds that the effective resistance is infinite and thus the Markov chain is recurrent. Then all we need to find is the condition for which (4.4.1) holds. We do this using the *Chernoff-Cramer Bound* proved in Chapter 2.

Proof of (4.4.1): We know that by the *Chernoff-Cramer Bound* the following inequality holds for any a.

$$\mathbb{P}(S_n \leq an) \leq e^{n\ln(\phi_\eta(-\lambda)+\lambda an)} \quad \forall \lambda > 0$$

Thus, if we let $a = \ln(2)$, all we need to show is that the right-hand side of this equation is less than that of (4.4.1), i.e., we need that there exists a $\lambda > 0$ such that:

$$e^{n(\ln(\phi_\eta(-\lambda))+\lambda\ln(2))} \leq e^{-n(1+\delta)\ln(2)}$$

Taking the logarithm of both sides and dividing by n gives us:

$$\ln(\phi_\eta(-\lambda)) + \lambda\ln(2) \leq -(1+\delta)\ln(2)$$

Since δ is just some arbitrarily positive small real number then the previous expression is true if and only if the following inequality holds:

$$\ln(\phi_\eta(-\lambda)) + \lambda\ln(2) < -\ln(2)$$

Then if we move all the terms to one side we get a necessary and sufficient condition for recurrence.

$$\exists\lambda > 0 \implies \ln(\phi_\eta(-\lambda)) + (\lambda+1)\ln(2) < 0$$

Thus, if there exists a λ such that this inequality holds we have found a necessary and sufficient condition for recurrence, since if such a λ exists then (4.4.1) holds and therefor any random walk on our electrical network is recurrent. To be able to relate this too, μ we find a necessary condition for this to hold. To do this we isolate the moment generating function to the left-hand side, which gives us:

$$\phi_\eta(-\lambda) < e^{-(\lambda+1)\ln(2)}$$

And from Chapter 2 we have the following property of moment generating functions:

$$e^{\mu t} \leq \phi_\eta(t) \quad \forall t \in \mathbb{R}$$

Thus this gives us another bound for our equation:

$$e^{\mu(-\lambda)} \leq \phi_\eta(-\lambda) < e^{-(\lambda+1)\ln(2)}$$

Then taking the logarithm of the left and right-hand sides of the equation we get:

$$-\mu(\lambda) < -(\lambda + 1)\ln(2)$$

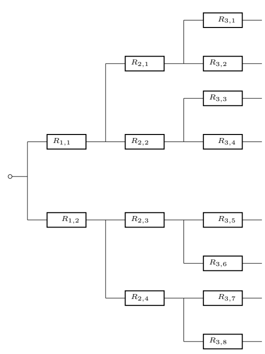
Then since λ cannot be zero we divide both sides by $-\lambda$ which gives us:

$$\mu > \ln(2) + \frac{\ln(2)}{\lambda}$$

And since λ is some cam be any arbitrarily large real number, we get that if $\mu > \ln(2)$ that there exists a lambda for which this equation holds. Thus, we have shown that $\mu > \ln(2)$ is a necessary condition that equation (4.4.1) holds.

Theorem 4.4.3 *Let X_n be a Markov Chain on a binary tree where the transition probabilities $I\!P(X_n = i_{n-1,\lceil \frac{k}{2} \rceil} | X_{n-1} = i_{n,k}) = \xi_{n,k}$, where each $\xi_{n,k}$ is an independent random variable identically distributed as some random variable ξ. Where ξ is a random variable who takes values inside $(0,1)$ and whose moment generating function that exits. Then X_n is transient if $\mathbb{E}[\ln(\frac{2\xi}{1-\xi})] < \ln(2)$.*

Proof: To prove this, we start similarly to the proof of the condition for recurrence. As the graph itself is identical, we begin with the electrical network pictured below:



And as before we get that the weight each resistor in this network can be determined by the following equation:

$$R_{n,k} = \exp\left(\eta_{n,k} + \eta_{n-1,\lceil k/2 \rceil} + ... + \eta_{1,\lceil k/2^{n-1} \rceil}\right) = \exp\left(\sum_{i=0}^{n-1} \eta_{n-i,\lceil k/2^i \rceil}\right) = \exp\left(S_{n,k}\right)$$

Then if we fix some $\epsilon > 0$ we can define the event that resistor is a bad resistor as one which is greater than $2^{(1-\epsilon)n}$:

$$\mathrm{B}_{n,k} = \{R_{n,k} > 2^{(1-\epsilon)n}\}$$

We can also define the event that there exists at least one resistor on level n which is "bad".

$$B_n = \bigcup_{k=1}^{2^n} \{R_{n,k} > 2^{(1-\epsilon)n}\}$$

Then by the axioms of probability we can find a bound on the probability for the event $\{B_n\}$ to occur:

$$I\!P(B_n) \leq \sum_{k=1}^{2^n} I\!P(\mathrm{B}_{n,k})$$

Then we let, $\delta, \epsilon > 0$, and assume the following equation holds:

$$\mathbb{P}(S_{n,k} > 2^{(1-\epsilon)n}) \leq 2^{-(1+\delta)n} \tag{4.4.2}$$

This gives us:

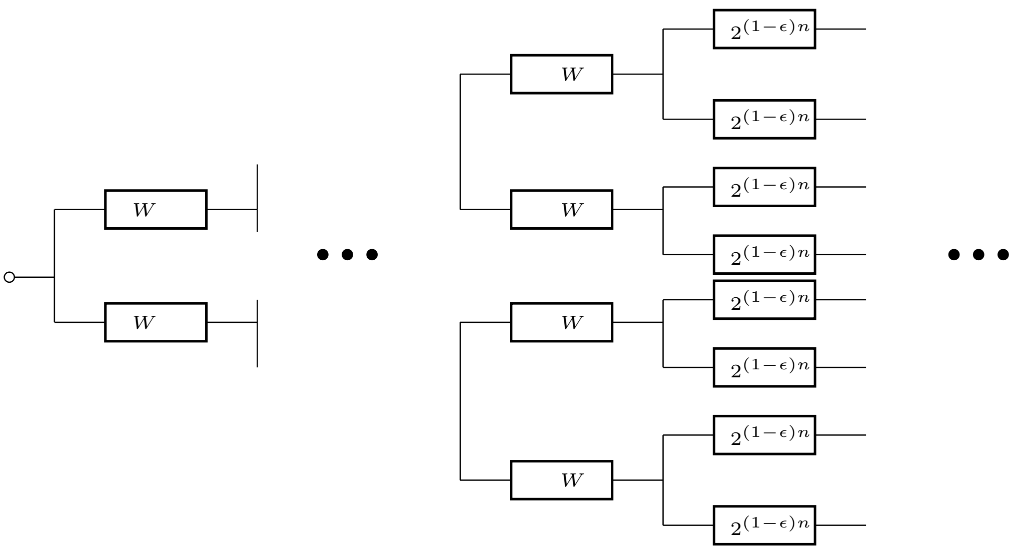
$$\mathbb{P}(B_n) \leq \sum_{k=1}^{2^n} 2^{-(1+\delta)n} = 2^{-\delta n}$$

Then since δ is some positive number, that the right-hand side is summable over n. Thus, we obtain the following equation:

$$\sum_{n=1}^{\infty} \mathbb{P}(B_n) \leq \sum_{n=1}^{\infty} 2^{-(\delta)n} < \infty \quad \forall \delta > 0$$

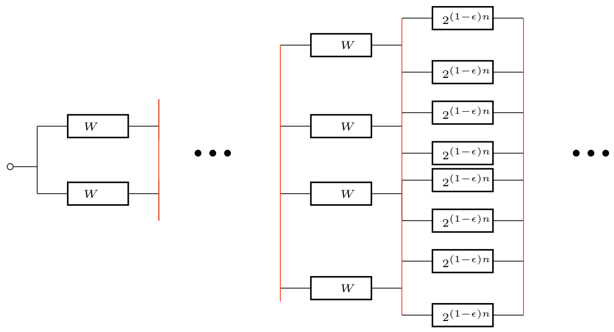
Then by the *Borel-Cantelli Lemma*, we get that the event that there is a single bad resistor on level n happens only finitely often. Therefore, we know that after some level N all resistors on each level $n > N$ have weight less than or equal to $2^{(1-\epsilon)n}$.

And by *Rayleigh's Monotonicity Law* for all resistors on level $n > N$ we can set their weight to $2^{(1-\epsilon)n}$, and this can only ever increase the effective resistance. Then we let W be the largest weight of all the resistors on levels at or below level N, and then we set the weight of all resistors at or below level N equal to W. Doing this will give us the completely symmetrical network pictured below:



And thus, since our new network is completely symmetrical, the voltage of all points on each level are equal. Then by *Corollary 3.3.4* we know that if all points on each level have identical voltage, we can add resistors between them without changing the effective resistance.

Thus, after adding these resistors, we are allowed to short our network without changing the effective resistance. This allows us to short our network as pictured below:



Then we can use the *Parallel Law*, and replace all resistors on each level above N with one resistor with weight:

$$\tilde{R}_n = (\sum_{k=1}^{2^n} R_{n,k}^{-1})^{-1} = (\sum_{k=1}^{2^n} 2^{(1-\epsilon)n})^{-1} = (2^n 2^{-(1-\epsilon)n})^{-1} = 2^{-\epsilon n}$$

And all resistors at or below level N with one of weight equal to:

$$\tilde{R}_n \leq (\sum_{k=1}^{2^n} R_{n,k}^{-1})^{-1} = (\sum_{k=1}^{2^n} W^{-1})^{-1} = (2^n W^{-1})^{-1} = \frac{W}{2^n}$$

Thus we get that the effective resistance of our sum can be calculated by:

$$R_{\text{eff}}^d \leq \sum_{n=1}^{N} (\frac{W}{2^n}) + \sum_{n=N+1}^{d} 2^{-\epsilon n}$$

And since both sums on the right-hand side are summable we have that the limit of the effective resistance of our network is finite as d goes to infinity, and thus by *Corollary 3.4.6* assuming that (4.4.2) holds we have that a random walk on G is transient. Therefore, all we have left to do is show when equation (4.4.2) holds.

Proof of (4.4.2): To show that (4.4.2) holds we show that we use the *Chernoff-Cramer* bound:

$$\mathbb{P}(S_n \geq an) \leq e^{n \ln(\phi_\eta(\lambda) - \lambda an)} \quad \forall \lambda \geq 0$$

Then if we let $a = (1-\epsilon)\ln(2)$, for some $\epsilon > 0$, we get the following inequalities:

$$\mathbb{P}(S_n \geq (1-\epsilon)\ln(2)n) \leq e^{n\ln(\phi_\eta(\lambda)-\lambda(1-\epsilon)\ln(2)n)}$$

Thus if we can find a $\lambda \geq 0$ such that the right-hand side of the equation above is less than the bound assumed in (4.4.2), we have found the needed condition for transience. Thus, all we need is to find the condition when the following equation holds:

$$e^{n\ln(\phi_\eta(\lambda)-\lambda(1-\epsilon)\ln(2)n)} \leq e^{-(1+\delta)\ln(2)n}$$

Then we take the logarithm of each side and divide both sides by $-n$ which gives us:

$$\lambda(1-\epsilon)\ln(2) - \ln(\phi_\eta(\lambda)) \geq (1+\delta)\ln(2)$$

Then because both δ and ϵ are just some arbitrarily small real numbers we get the following strict inequality

$$\lambda\ln(2) - \ln(\phi_\eta(\lambda)) > \ln(2)$$

Then if we move all the terms to one side we get our condition for transience.

$$\exists\lambda > 0 \text{ such that, } (\lambda-1)\ln(2) - \ln(\phi_\eta(\lambda)) > 0 \tag{4.4.3}$$

Thus if there exists a λ such that this inequality holds we have found a necessary and sufficient condition for transience, since if such a λ exists then (4.4.2) holds and therefor any random walk on our electrical network is transience. However, we want to formulate this condition in terms of μ. To be able to do this we can only find a necessary condition for (4.4.2) to hold, to find such a necessary condition we first isolate the moment generating function:

$$\phi_\eta(\lambda) < e^{(\lambda-1)\ln(2)}$$

And from section 2 we have the following property of moment generating functions:

$$e^{\mu t} \leq \phi_\eta(t) \quad \forall t \in \mathbb{R}$$

Thus this gives us another bound for our equation:

$$e^{\mu\lambda} \leq \phi_\eta(\lambda) < e^{(\lambda-1)\ln(2)}$$

Then taking the logarithm of the left and right-hand sides of the equation we get:

$$\mu(\lambda) < (\lambda-1)\ln(2)$$

Then since λ cannot be zero we divide both sides by λ which gives us:

$$\mu < \ln(2) - \frac{\ln(2)}{\lambda}$$

And since λ is some cam be any arbitrarily large real number, we get that if μ is some number less than $\ln(2)$ that there exists a lambda for which this equation holds. Thus, we have shown that $\mu < \ln(2)$ is necessary for (4.4.3) to hold.

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