

INDEX THEOREMS WITH SPECTRAL LOCALIZERS

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Abstract

The Atiyah–Singer index theorem is a foundational result in mathematics, providing a deep connection between the analytical properties of an elliptic differential operator and the topological properties of its underlying manifold. This thesis focuses on a numerical approach to index theory for generalized Toeplitz operators. Building on the work of Loring and Schulz-Baldes, we utilize the spectral localizer method, which relates the analytical index of a Toeplitz operator to the signature of an associated finite-dimensional matrix, the spectral localizer. The contributions of this work are a simplified proof of the key formula, and new, less restrictive parameter bounds.

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1 Introduction

Index theory is a rich field in mathematics which has many applications in both mathematics and physics [8, 9]. For a Fredholm operator $T: X \rightarrow Y$ where X, Y are vector spaces, we define the index of T as

$$\text{ind}(T) = \dim \ker(T) - \dim \text{coker}(T),$$

where $\text{coker}(T) = Y/\text{im}(T)$ is a quotient space. The integer $\text{ind}(T)$ is referred to as the analytical index.

One of the most famous and perhaps the most important result in index theory is the Atiyah–Singer index theorem (AS-index theorem) which was first announced by Atiyah and Singer in 1963 [2]. In a later series of papers, Atiyah and Singer refined their proof of the index theorem, and (among other things) presented a formula for the AS-index theorem in terms of cohomology groups which made the AS-index theorem more accessible for practical calculations [3]. A formulation of the AS-index theorem can be stated as follows: If E and F are complex vector bundles over a closed oriented n -dimensional manifold X and $T: C^\infty(E) \rightarrow C^\infty(F)$ is an elliptic differential operator, then

$$\text{ind}(T) = \int_X (-1)^{n(n+1)/2} \text{ch}(T) \text{td}(T_{\mathbb{C}}X) (e(TX))^{-1}. \quad (1)$$

For (1) we have used the formulation found in [4, pp 16]. The right-hand side (RHS) of (1) is also known as the *topological index*, though its exact meaning will not be explained here. What is important to know is that the topological index only depends on the (local) topological properties of X , which are independent of the (global) analytical properties of T , connecting two seemingly disconnected properties.

In this thesis, we will focus on the analytical index for a special type of operators known as Toeplitz operators which can be calculated via an operator called the spectral localizer.

Let A be an invertible bounded operator on a separable Hilbert space $\mathcal{H}$. Let D be an invertible self-adjoint (Dirac) operator with compact resolvent such that $AD(D) \subset \mathcal{D}(D)$ and the commutator $[A, D] = AD - DA: \mathcal{D}(D) \rightarrow \mathcal{H}$ extends to a bounded operator.

The Toeplitz operator T_A is a bounded Fredholm operator defined by

$$T_A = \Pi A \Pi + (I - \Pi),$$

where $\Pi = \chi_{[0, \infty)}(D)$ and χ is the indicator function.

The spectral localizer $L_\kappa: \mathcal{D}(D) \oplus \mathcal{D}(D) \rightarrow \mathcal{H} \oplus \mathcal{H}$ is defined as

$$\begin{pmatrix} \kappa D & A \\ A^* & -\kappa D \end{pmatrix},$$

where $\kappa > 0$ is a tuning parameter.

This thesis will build on previous research by Loring and Schulz-Baldes [14], in which they proved the following theorem:

Theorem 1.1.1 (Spectral localizer for index-theorem [14]). *For a pair of numerical parameters $\kappa > 0, \rho > 0$ (see below), the index of T_A is given by*

$$\text{ind}(T_A) = \frac{1}{2} \text{Sig}(L_{\kappa, \rho}).$$

Here $L_{\kappa, \rho}$ is the spectral localizer compressed to the finite-dimensional subspace $\mathcal{H}_\rho = \chi_{[-\rho, \rho]}(D)\mathcal{H}$ where $\rho > 0$ is another tuning parameter. $L_{\kappa, \rho}$ is an invertible, self-adjoint, finite-dimensional matrix. The signature of the spectral localizer, $\text{Sig}(L_{\kappa, \rho})$, is the difference of

number of positive eigenvalues and the number of negative eigenvalues of $L_{\kappa,\rho}$, counting multiplicities. Note that since $L_{\kappa,\rho}$ is self-adjoint, the geometric and algebraic multiplicities of the eigenvalues $L_{\kappa,\rho}$ coincide.

The significance of Theorem 1.1.1 is that it allows us to calculate $\text{ind}(T)$ numerically. In [14] the authors showed that for a constant

$$\kappa_0 = \frac{g^3}{12\|A\|\|[A, D]\|},$$

Theorem 1.1.1 applies when the constants κ, ρ fulfill

$$2g\kappa_0^{-1} \leq 2g\kappa^{-1} < \rho. \quad (2)$$

Here $g = \|A^{-1}\|^{-1} > 0$.

The purpose of this thesis is twofold: Firstly, to provide a simplified proof of Theorem 1.1.1. Secondly, to show that for a different value:

$$\kappa_0 = \frac{g^2}{2\|[A, D]\|}, \quad (3)$$

and parameters κ, ρ which satisfy (2), Theorem 1.1.1 still applies.

Since ρ determines the size of $L_{\kappa,\rho}$, the new estimate for κ_0 allows us to do the following: When finding parameters κ, ρ which satisfy 2, we can set

$$\kappa_0 = \max\left(\frac{g^3}{12\|A\|\|[A, D]\|}, \frac{g^2}{2\|[A, D]\|}\right),$$

which may allow us to calculate $\text{Sig}(L_{\kappa,\rho})$ using a smaller matrix, which is computationally more efficient.

This thesis will consist of four sections.

In Section 2: "Theory", we will go over the necessary background theory for this thesis. The background theory includes concepts mainly from Functional Analysis such as Hilbert spaces, Operator Theory, and Spectral Theory. We will finish Section 1 with some necessary Index Theory as well.

In Section 3: "Spectral Flow", we will go over the theory of spectral flow. Intuitively, spectral flow is the net number of eigenvalues crossing 0 along a continuous path of self-adjoint, invertible Fredholm operators. The theory of spectral flow will be important to prove the result of Section 4.

In Section 4: "Spectral Localizers for Index theorems", we will go over the main result of this thesis, namely the simplified proof of Theorem 1.1.1, and the proof for the numerical relaxation of κ_0 given by (3).

Lastly, in Section 5: "Spectral Flow of Dirac Operators in Examples", we will give a few examples of using Theorem 1.1.1 together with (3) in practice for two different Dirac operators.

2 Theory

In this section, we will go over some preliminary definitions and theorems which will be important to understand the content. The theory on Functional Analysis is considered standard material and can be found in Chapter 1-5 and 7 of [18], or alternatively in [7]. The material on Index Theory has been collected from Chapter 1 of [10]. Anything not in these texts has been proven or referenced to another source.

2.1 Normed spaces

For these vector spaces, we will also introduce the concept of scalar products and norms. This information can be found in [18, ch. 1].

Definition 2.1.1. *The mapping $\langle \cdot, \cdot \rangle: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ is a scalar product iff for all $f, g, h \in \mathcal{H}$ and $c \in \mathbb{C}$ we have*

- i. $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$,
- ii. $\langle cf, g \rangle = c\langle f, g \rangle$,
- iii. $\langle f, g \rangle = \overline{\langle g, f \rangle}$,
- iv. $\langle f, f \rangle \geq 0$,
- v. $\langle f, f \rangle = 0$ if and only if $f = 0$.

Definition 2.1.2. *The mapping $\| \cdot \|: \mathcal{H} \rightarrow \mathbb{R}$ is called a norm on $\mathcal{H}$ if for all $f, g \in \mathcal{H}$ and $c \in \mathbb{K}$ we have*

- i. $\|f\| \geq 0$, with $\|f\| = 0$ if and only if $f = 0$.
- ii. $\|cf\| = |c|\|f\|$,
- iii. $\|f + g\| \leq \|f\| + \|g\|$.

Every scalar product on $\mathcal{H}$ induces a norm by $\|f\| = \sqrt{\langle f, f \rangle}$. We then refer to the pair $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ as a pre-Hilbert space. Similarly, if we only have a norm, we refer to the pair $(\mathcal{H}, \| \cdot \|)$ as a normed space.

We have the following useful proposition:

Proposition 2.1.3. *Let $\mathcal{H}$ be a pre-Hilbert space. For every $x, y \in \mathcal{H}$ we have*

- i. $|\langle x, y \rangle| \leq \|x\|\|y\|$ (Cauchy-Schwarz inequality),
- ii. $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality).

Let X be a normed space. A set $N \subseteq X$ is a neighborhood of $x \in X$ if there exists a $\epsilon > 0$ such that the open ball

$$B_\epsilon(x) = \{y \in X \mid \|y - x\| < \epsilon\}$$

is contained in N , i.e. $B_\epsilon(x) \subseteq N$. A point $x \in X$ is referred to as an accumulation point of a set S if every neighborhood of x contains an element in S distinct from x . Informally, if x is an accumulation point, then every neighborhood of x contains infinitely many points of S .

2.2 Banach and Hilbert spaces

We have defined normed and pre-Hilbert spaces already. To continue our discussion, we must define Banach and respective Hilbert spaces. This information can be found in [18, ch. 2].

Let $(\mathcal{H}, \| \cdot \|)$ be a normed space. A sequence (f_n) is said to be convergent if there exists an $f \in \mathcal{H}$ such that $\|f_n - f\| \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, a sequence (f_n) is called Cauchy if the elements get arbitrarily close to each other in a norm-sense, i.e. $\|f_n - f_m\| \rightarrow 0$ as $n, m \rightarrow \infty$. Note that every convergent sequence is Cauchy. We are now ready for our definition:

Definition 2.2.1. *Let $(\mathcal{H}, \| \cdot \|)$ be a pre-Hilbert (normed) space. We call $\mathcal{H}$ a Hilbert (Banach) space if every Cauchy sequence in $\mathcal{H}$ is convergent.*

One can think of Hilbert (Banach) spaces intuitively as saying "a Hilbert (Banach) space is a pre-Hilbert (normed) space that has no gaps in it".

A subspace $M \subseteq \mathcal{H}$ is called *closed* if it is closed in the norm topology induced by the inner product.

Let $\mathcal{H}$ be a Hilbert space, and let $M \subseteq \mathcal{H}$ be a closed subspace. The *orthogonal complement* of M is defined as

$$M^\perp := \{ h \in \mathcal{H} : \langle h, g \rangle = 0 \text{ for all } g \in M \}.$$

If every $f \in \mathcal{H}$ can be uniquely expressed as $f = g + h$ with $g \in M$ and $h \in M^\perp$, then we write

$$\mathcal{H} = M \oplus M^\perp$$

and call this the *orthogonal direct sum*. In this decomposition, the component $g \in M$ is referred to as the *orthogonal projection* of f onto M .

Proposition 2.2.2 (Orthogonal Projection). *Let M be a closed subspace of a Hilbert space $\mathcal{H}$. Then*

$$\mathcal{H} = M \oplus M^\perp.$$

In particular, every $f \in \mathcal{H}$ can be uniquely written as $f = g + h$ with $g \in M$ and $h \in M^\perp$. Moreover,

$$M^{\perp\perp} = M.$$

2.3 Basis in Hilbert Spaces

After defining Hilbert spaces, we can discuss the concept of basis in a Hilbert space. This information can be found in [18, ch. 3]. Let $\mathcal{H}$ be a Hilbert space. A family of elements $M = \{e_\alpha \mid e_\alpha \in \mathcal{H}, \alpha \in A\}$ for some set A is called an orthonormal system (ONS) if we have

$$\langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta} = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta, \end{cases}$$

($\delta_{\alpha\beta}$ is referred to as the Kronecker–Delta function). A relevant proposition is the following:

Proposition 2.3.1. *Let $\mathcal{H}$ be a Hilbert space, and let $M = \{e_\alpha \mid \alpha \in A\}$ be an ONS. For each $f \in \mathcal{H}$ the vector*

$$\sum_{\alpha \in A} \langle f, e_\alpha \rangle e_\alpha$$

is the orthogonal projection of f onto $\overline{L(M)}$.

An ONS B is known as an orthonormal basis (ONB) of $\mathcal{H}$ if B is total in $\mathcal{H}$, i.e. $B \subset \mathcal{H}$ and $\mathcal{H} \subset \overline{L(B)}$. Here $\overline{L(B)}$ is the closed hull of all linear combinations of B . A nice property of any inner-product space S is that $\overline{S} = (S^\perp)^\perp$. If $B = \{e_\alpha \mid \alpha \in A\}$ is a basis in a Hilbert space then for every $f \in \mathcal{H}$ there exists a unique representation

$$f = \sum_{\alpha \in A} c_\alpha e_\alpha.$$

In this thesis we will be working with countable bases. A countable basis is an ONB with a countable set of elements. We have the following proposition:

Proposition 2.3.2. *Every Hilbert space has an ONB.*

Proposition 2.3.2 is a standard result in functional analysis which has many different proofs. One way to prove Proposition 2.3.2 is by using *Zorn's lemma* which is a standard result in set theory, and for such a proof, the interested reader is referred to [1].

Although Proposition 2.3.2 promises us an ONB for every Hilbert space $\mathcal{H}$, it does not mention the requirements for if the ONB of $\mathcal{H}$ is countable or not. There does exist a requirement for when an ONB of $\mathcal{H}$ is countable, but we need some prerequisites.

We say that a Hilbert space $\mathcal{H}$ is separable if there exists a countable, dense subset $B \subset \mathcal{H}$ (dense meaning that $\overline{B} = \mathcal{H}$). We have the following proposition:

Proposition 2.3.3. *A Hilbert space $\mathcal{H}$ is separable if and only if $\mathcal{H}$ has a countable ONB.*

In this thesis, unless otherwise indicated, $\mathcal{H}$ will always refer to a separable Hilbert space. The assumptions of Theorem 1.1.1 cannot be fulfilled unless $\mathcal{H}$ is separable.

2.4 Bounded, compact and self-adjoint operators

We will consider three types of operators: bounded, unbounded and compact operators. We will also define the adjoint of these operators and consider self-adjoint operators. This information can be found in [18, ch. 4–5].

An operator $T: \mathcal{D}(T) \rightarrow \mathcal{R}(T)$, is a mathematical object which sends elements from a vector space known as its *domain* $\mathcal{D}(T)$ to elements in another vector space known as its *range* $\mathcal{R}(T)$. An operator T is said to be linear if it fulfills the following properties:

- i. $T(x + y) = T(x) + T(y), \forall x, y \in \mathcal{D}(T)$,
- ii. $T(cx) = cT(x), \forall x \in \mathcal{D}(T), c \in \mathbb{C}$.

If T is linear we then write $T(x) = Tx$. All operators considered in this thesis are linear.

Let $\mathcal{H}$ be a Hilbert space. We say that an operator $T: \mathcal{H} \rightarrow \mathcal{H}$ is bounded if there exists a $C \geq 0$ s.t.

$$\|Tf\| \leq C\|f\|, \forall f \in \mathcal{H}.$$

Another formulation of bounded operators is the following: A bounded operator is an operator that maps bounded sets to bounded sets.

Furthermore if T is bounded, then we can define the *norm* of T :

Definition 2.4.1 (Operator norm). *Let T be a bounded operator on $\mathcal{H}$. The operator norm of T is*

$$\|T\| = \inf\{C \geq 0 \mid \|Tf\| \leq C\|f\|, \forall f \in \mathcal{H}\}.$$

One can easily check that Definition 2.4.1 is equivalent to

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

We say that an operator is unbounded if it is not necessarily guaranteed to be bounded.

Definition 2.4.2 (Commutator). *Let $T: \mathcal{D}(T) \rightarrow \mathcal{H}$ be an unbounded operator and $S: \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator such that $S\mathcal{D}(T) \subset \mathcal{D}(T)$. We define the commutator $[T, S]: \mathcal{D}(T) \rightarrow \mathcal{H}$ as the operator*

$$[T, S] = TS - ST.$$

Definition 2.4.3 (Inverse operator). *We say that an operator T on $\mathcal{H}$ is injective if $Tx = 0$ if and only if $x = 0$. If T is injective, then we define the inverse T^{-1} of T as $\mathcal{D}(T^{-1}) = \mathcal{R}(T)$ and $T^{-1}y = x$ for every $y = Tx \in \mathcal{R}(T)$.*

An operator T is said to be surjective if $\mathcal{R}(T) = \mathcal{H}$. T is said to be bijective if T is both injective and surjective. The concepts of invertibility of an operator T is a bit different from what we are used to from Linear Algebra, and we will make a distinction between invertibility for bounded and unbounded operators. For bounded operators, we use the standard definition of invertibility:

Definition 2.4.4 (Invertibility of bounded operators). *A bounded operator T is said to be invertible if T is bijective and the inverse T^{-1} is bounded.*

For unbounded operators we define invertibility as the following:

Definition 2.4.5 (Invertibility of unbounded operators). *An unbounded operator $T: \mathcal{D}(T) \rightarrow \mathcal{H}$ is said to be invertible if T is bijective and there exists an everywhere defined bounded operator $S: \mathcal{H} \rightarrow \mathcal{D}(T)$ such that*

$$ST = I_{\mathcal{D}(T)}, TS = I,$$

where $I_{\mathcal{D}(T)}$ is the identity operator restricted to the domain $\mathcal{D}(T)$

For bounded operators have the following well-known theorem:

Theorem 2.4.6 (Bounded inverse theorem). *Every bounded bijective operator $T: \mathcal{H} \rightarrow \mathcal{H}$ has a bounded inverse.*

Theorem 2.4.6 is actually a corollary of the *Open Mapping Theorem*, and for a proof of Theorem 2.4.6, see [7, pp. 62–64].

To define the adjoint of operators we will need the following:

Theorem 2.4.7 (Riesz representation theorem). *Let $\mathcal{H}$ be a Hilbert space. For every bounded linear functional $f: \mathcal{H} \rightarrow \mathbb{C}$, there exists a unique element $z \in \mathcal{H}$ such that f can be represented as $f(x) = \langle x, z \rangle$ for every $x \in \mathcal{H}$. We also have that $\|f\| = \|z\|$.*

Let $T: \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator. Consider the bounded linear functional $\mathcal{D}(T) \ni x \mapsto \langle Tx, y \rangle$. By Theorem 2.4.7 there exists a unique $z \in \mathcal{H}$ such that $\langle Tx, y \rangle = \langle x, z \rangle$ for every $x \in \mathcal{D}(T)$. Since every z is unique, we can then define $z = T^*y$, where T^* is known as the *adjoint* of T .

However, for an unbounded operator $T: \mathcal{D}(T) \rightarrow \mathcal{H}$ in general the case is more complicated, since we can not guarantee that $\langle Tx, y \rangle$ is a bounded functional in x . Firstly, we require $\mathcal{D}(T)$ to be dense, i.e. $\overline{\mathcal{D}(T)} = \mathcal{H}$, we say then that T is densely defined. We can then define

$$\mathcal{D}(T^*) = \{y \in \mathcal{H} \mid \mathcal{D}(T) \ni x \mapsto \langle Tx, y \rangle \text{ is a bounded functional}\}.$$

We then define T^*y as the element $\langle Tx, y \rangle = \langle x, T^*y \rangle$, which is guaranteed to exist by Theorem 2.4.7. For a given x , the element T^*y is uniquely determined, for if there is a $\tilde{z}$ such that $\langle Tx, y \rangle = \langle x, z \rangle = \langle x, \tilde{z} \rangle$, then $\langle x, z - \tilde{z} \rangle = \langle Tx, y \rangle - \langle Tx, y \rangle = 0 \Rightarrow z = \tilde{z}$, since $\mathcal{D}(T)$ is dense. Note that all bounded operators are densely defined, since trivially $\mathcal{D}(T) = \mathcal{H}$ is dense. Let us make an official definition:

Definition 2.4.8. *Let $\mathcal{H}$ be Hilbert spaces. Let $T: \mathcal{D}(T) \rightarrow \mathcal{H}$ be a densely defined operator. The operator $T^*: \mathcal{D}(T^*) \rightarrow \mathcal{H}$ is known as the adjoint of T , which is defined by*

$$\mathcal{D}(T^*) = \{y \in \mathcal{H} \mid \mathcal{D}(T) \ni x \mapsto \langle Tx, y \rangle \text{ is a bounded functional}\}$$

and the relation

$$\langle Tx, y \rangle = \langle x, T^*y \rangle$$

for every $x \in \mathcal{D}(T)$ and every $y \in \mathcal{D}(T^*)$.

If $\langle Tx, y \rangle = \langle x, Ty \rangle$ for every $x, y \in \mathcal{D}(T)$, then we say T is symmetric. If T is symmetric and $\mathcal{D}(T) = \mathcal{D}(T^*)$, then we say T is self-adjoint.

We write $(A^*)^* = A^{**}$. For Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, we will write $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}, \langle \cdot, \cdot \rangle_{\mathcal{H}_2}$ to differentiate between the inner products on $\mathcal{H}_1$ and $\mathcal{H}_2$. For a bounded operator $A: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ we have, by previous discussion, the adjoint $A^*: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ defined by the relation:

$$\langle Ax, y \rangle_{\mathcal{H}_2} = \langle x, A^*y \rangle_{\mathcal{H}_1},$$

for every $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$.

Proposition 2.4.9. *Let $A: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The adjoint operator $A^*: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ is also bounded. The following properties hold:*

- i. $A^{**} = A$.
- ii. $\|A^*\| = \|A\|$.
- iii. $\|A^*A\| = \|A\|^2$.

Proof. i. The adjoint $A^*: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ is defined by the relation:

$$\langle Ax, y \rangle_{\mathcal{H}_2} = \langle x, A^*y \rangle_{\mathcal{H}_1} \quad \text{for all } x \in \mathcal{H}_1, y \in \mathcal{H}_2$$

Similarly, the adjoint of A^* , denoted $A^{**} = (A^*)^*$, is an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ defined by:

$$\langle A^*y, x \rangle_{\mathcal{H}_1} = \langle y, A^{**}x \rangle_{\mathcal{H}_2} \quad \text{for all } x \in \mathcal{H}_1, y \in \mathcal{H}_2$$

By taking the complex conjugate of the second equation, we get $\overline{\langle y, A^{**}x \rangle_{\mathcal{H}_2}} = \overline{\langle A^*y, x \rangle_{\mathcal{H}_1}}$, which simplifies to $\langle A^{**}x, y \rangle_{\mathcal{H}_2} = \langle x, A^*y \rangle_{\mathcal{H}_1}$.

Comparing this with the definition of A^* , we have:

$$\langle Ax, y \rangle_{\mathcal{H}_2} = \langle A^{**}x, y \rangle_{\mathcal{H}_2}$$

This can be rewritten as $\langle (A - A^{**})x, y \rangle_{\mathcal{H}_2} = 0$ for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$. Since this holds for any $y \in \mathcal{H}_2$, we can choose $y = (A - A^{**})x$. This gives:

$$\|(A - A^{**})x\|_{\mathcal{H}_2}^2 = 0$$

This implies that $(A - A^{**})x = 0$ for all $x \in \mathcal{H}_1$, and therefore, $A = A^{**}$.

ii. First, we show that $\|A^*\| \leq \|A\|$. For any $y \in \mathcal{H}_2$, consider:

$$\|A^*y\|_{\mathcal{H}_1}^2 = \langle A^*y, A^*y \rangle_{\mathcal{H}_1} = \langle A(A^*y), y \rangle_{\mathcal{H}_2}$$

By the Cauchy-Schwarz inequality, followed by the definition of the operator norm for A :

$$\langle A(A^*y), y \rangle_{\mathcal{H}_2} \leq \|A(A^*y)\|_{\mathcal{H}_2} \|y\|_{\mathcal{H}_2} \leq \|A\| \|A^*y\|_{\mathcal{H}_1} \|y\|_{\mathcal{H}_2}$$

So, we have $\|A^*y\|_{\mathcal{H}_1}^2 \leq \|A\| \|A^*y\|_{\mathcal{H}_1} \|y\|_{\mathcal{H}_2}$. If $\|A^*y\|_{\mathcal{H}_1} \neq 0$, we can divide by it to get:

$$\|A^*y\|_{\mathcal{H}_1} \leq \|A\| \|y\|_{\mathcal{H}_2}$$

This inequality also holds if $\|A^*y\|_{\mathcal{H}_1} = 0$. Since this is true for all $y \in \mathcal{H}_2$, it implies that A^* is bounded and $\|A^*\| \leq \|A\|$.

To show the reverse inequality, we apply the result we just proved to the operator A^* :

$$\|(A^*)^*\| \leq \|A^*\|$$

Using the result from *i.*, $A^{**} = A$, we get $\|A\| \leq \|A^*\|$. Combining the two inequalities, we conclude that $\|A^*\| = \|A\|$.

iii. First, using the property of operator norms and the result from *ii*:

$$\|A^*A\| \leq \|A^*\|\|A\| = \|A\|\|A\| = \|A\|^2$$

So, we have established that $\|A^*A\| \leq \|A\|^2$.

For the reverse inequality, consider any $x \in \mathcal{H}_1$:

$$\|Ax\|_{\mathcal{H}_2}^2 = \langle Ax, Ax \rangle_{\mathcal{H}_2} = \langle x, A^*Ax \rangle_{\mathcal{H}_1}$$

By the Cauchy-Schwarz inequality:

$$\langle x, A^*Ax \rangle_{\mathcal{H}_1} \leq \|x\|_{\mathcal{H}_1} \|A^*Ax\|_{\mathcal{H}_1}$$

By the definition of the operator norm for the operator A^*A (which maps $\mathcal{H}_1 \rightarrow \mathcal{H}_1$):

$$\|A^*Ax\|_{\mathcal{H}_1} \leq \|A^*A\| \|x\|_{\mathcal{H}_1}$$

Combining these, we get $\|Ax\|_{\mathcal{H}_2}^2 \leq \|A^*A\| \|x\|_{\mathcal{H}_1}^2$. Taking the square root of both sides gives $\|Ax\|_{\mathcal{H}_2} \leq \sqrt{\|A^*A\|} \|x\|_{\mathcal{H}_1}$. This implies that $\|A\| \leq \sqrt{\|A^*A\|}$, or $\|A\|^2 \leq \|A^*A\|$.

Combining the two inequalities, we conclude that $\|A^*A\| = \|A\|^2$. $\square$

A well-known result is that if A is invertible and densely defined then $(A^*)^{-1} = (A^{-1})^*$.

For an operator T we define the graph of T as

$$\mathcal{G}(T) = \{(x, Tx) \mid x \in \mathcal{D}(T)\}.$$

We say that T is a closed operator if $\mathcal{G}(T)$ is closed.

Proposition 2.4.10. *$T: \mathcal{D}(T) \rightarrow \mathcal{H}$ is closed if and only if the following holds. If (x_n) is a convergent sequence in $\mathcal{D}(T)$ and (Tx_n) is convergent in $\mathcal{H}$, then $x := \lim_{n \rightarrow \infty} x_n \in \mathcal{D}(T)$ and $Tx = \lim_{n \rightarrow \infty} Tx_n \in \mathcal{H}$.*

Proof. For the first direction, assume that T is closed operator and (x_n) , (Tx_n) are convergent sequences where $x := \lim_{n \rightarrow \infty} x_n$ and $y := \lim_{n \rightarrow \infty} Tx_n$. By definition $\mathcal{G}(T)$ is closed which means that $\mathcal{G}(T)$ contains its limit points and thus $(x_n, Tx_n) \rightarrow (x, y) \in \mathcal{G}(T)$. In conclusion $x \in \mathcal{D}(T)$ and $y = Tx \in \mathcal{H}$.

For the other direction: Let (x_n, Tx_n) be convergent sequence with limit point (u, v) . By the assumption we have that $x_n \rightarrow x$ and $Tx_n \rightarrow Tx$, which means that $(u, v) = (x, Tx) \in \mathcal{G}(T)$. Since $\mathcal{G}(T)$ contains all of its limit points, $\mathcal{G}(T)$ must be closed. $\square$

Proposition 2.4.11. *Let T be an injective operator. T is closed if and only if T^{-1} is closed.*

Proof. Since T is injective, T is one-to-one, i.e. for every $x \in \mathcal{D}(T)$ there exists exactly one $y = Tx$. We then have that

$$\mathcal{G}(T) = \{(x, y) \mid x \in \mathcal{D}(T), y = Tx\} \text{ and } \mathcal{G}(T^{-1}) = \{(y, x) \mid y \in \mathcal{D}(T^{-1}), x = T^{-1}y\}$$

are homomorphic with a homomorphism given by $S: \mathcal{G}(T) \rightarrow \mathcal{G}(T^{-1})$ given by $S(x, y) = (y, x)$ for every $(x, y) \in \mathcal{G}(T)$, i.e. just switching the order of x and y . Clearly, $\mathcal{G}(T)$ is closed if and only if $\mathcal{G}(T^{-1})$ is closed. $\square$

Definition 2.4.12. Let $T: \mathcal{D}(T) \rightarrow \mathcal{H}$ be an operator. On $\mathcal{D}(T)$ we can define a scalar product by

$$\langle f, g \rangle_T = \langle f, g \rangle_{\mathcal{H}} + \langle Tf, Tg \rangle_{\mathcal{H}}.$$

Proposition 2.4.13. T is closed if and only if $(\mathcal{D}(T), \langle \cdot, \cdot \rangle_T)$ is a Hilbert space.

Theorem 2.4.14. Let T be a bijective operator on $\mathcal{H}$. T has a bounded inverse if and only if T is closed.

Clearly, every bounded operator must also be closed. *Ex.* Let $\mathcal{H}$ be a Hilbert space. The orthogonal projection mentioned in Proposition 2.2.2, $P_M: \mathcal{H} \rightarrow M$ onto a closed subspace $M \subset \mathcal{H}$, is a closed operator.

Proposition 2.4.15. Let T be a densely defined operator on $\mathcal{H}$. Then T^* is closed.

An obvious consequence is that every self-adjoint operator is closed.

For two operators T and S on $\mathcal{H}$, we say that $T \geq S$ if

$$\langle Tx, x \rangle \geq \langle Sx, x \rangle, \forall x \in \mathcal{H}.$$

As a shorthand we write $T \geq \epsilon$ for some $\epsilon > 0$ if we have

$$\langle Tx, x \rangle \geq \langle \epsilon x, x \rangle = \epsilon \|x\|^2, \forall x \in \mathcal{D}(T), \epsilon > 0.$$

Lemma 2.4.16. Let T be an invertible operator with $g = \|T^{-1}\|^{-1} > 0$. Then

$$\|Tx\| \geq g\|x\|, \forall x \in \mathcal{D}(T),$$

and

$$\|T^*x\| \geq g\|x\|, \forall x \in \mathcal{D}(T^*).$$

Proof. Since T is invertible we have $\ker T = \{0\}$ and T^{-1} is bounded so we have for every $x \in \mathcal{D}(T)$ that

$$\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\| \|Tx\| = g^{-1} \|Tx\| \Rightarrow g\|x\| \leq \|Tx\|.$$

Using a similar argument, one can show that $\|T^*x\| \geq g\|x\|, \forall x \in \mathcal{D}(T^*)$ by using the fact that $\|(T^{-1})^*\| = \|T^{-1}\|$. $\square$

We have the following proposition which will prove useful:

Proposition 2.4.17. Let T be a self-adjoint operator on $\mathcal{H}$. T is invertible if and only if $T^2 \geq \epsilon$ for some $\epsilon > 0$.

Proof. For the first direction assume that T is invertible: Since T is invertible, by Lemma 2.4.16 there exists a $g > 0$ such that $\|Tx\| \geq g\|x\| \Rightarrow \|Tx\|^2 \geq g^2\|x\|^2$, with $\epsilon = g^2$.

For the second direction: We have $\|Tx\|^2 \geq \epsilon\|x\|^2$ for some $\epsilon > 0$, which implies $\ker(T) = \{0\}$.

To show that T is surjective, note first that

$$\text{im}(T)^\perp = \ker(T^*) = \ker(T) = \{0\}.$$

And we have $\overline{\text{im}(T)} = (\text{im}(T)^\perp)^\perp = \{0\}^\perp = \mathcal{H}$. We would like to show that $\text{im}(T) = \overline{\text{im}(T)} = \mathcal{H}$ to prove that T is surjective.

First take some $y \in \overline{\text{im}(T)}$. For the first direction we want to show that $y \in \text{im}(T)$. Since T is self-adjoint, it is also closed. We have from closure that there must exist some sequence

(x_n) in $\mathcal{D}(T)$ such that $Tx_n \rightarrow y$ as $n \rightarrow \infty$. What remains is to show is that there exists some $x \in \mathcal{D}(T)$ such that $x_n \rightarrow x$. We have that

$$\|Tx\|^2 \geq \epsilon \|x\|^2,$$

for some $\epsilon > 0$, which implies

$$\|x_n - x_m\| \leq \frac{1}{\sqrt{\epsilon}} \|Tx_n - Tx_m\|.$$

Since (Tx_n) convergent it is also so Cauchy, which implies that (x_n) is also Cauchy. Since every Cauchy sequence is convergent in a Hilbert space, (x_n) is also convergent. Since T is closed, Proposition 2.4.10 if $x_n \rightarrow x$ and $Tx_n \rightarrow y$ then it must be that $x \in \mathcal{D}(T)$ and $Tx = y \Rightarrow y \in \text{im}(T)$. We have now shown that $\overline{\text{im}(T)} \subseteq \text{im}(T)$. It is also trivially true that $\text{im}(T) \subseteq \overline{\text{im}(T)}$. In conclusion $\text{im}(T) = \overline{\text{im}(T)} = \mathcal{H}$ which implies that T is surjective.

Since T is bijective and closed, T^{-1} is bounded by Theorem 2.4.14. Since T is bijective and T^{-1} is bounded, T also be invertible. $\square$

Note: As a shorthand notation some authors will sometimes write for a self-adjoint operator T that $T^2 > 0$ as meaning that T is invertible, note however, that technically $\langle T^2x, x \rangle = \|Tx\|^2 \geq \epsilon \|x\|^2$ for $\epsilon > 0$ is *not* equivalent with $\langle Tx, x \rangle > 0$ for $x \neq 0$. For example for $Te_n = n^{-1}e_n$ for $n \neq 0$ we have that $\langle Tx, x \rangle > 0$ but there exists no $\epsilon > 0$ such that $\langle Tx, x \rangle \geq \epsilon \|x\|^2$, which one can see by letting $n \rightarrow \infty$. So at best T is self-adjoint and $T^2 > 0$ means that T is injective.

We say that an operator is continuous if for every convergent sequence $(f_n) \rightarrow f$ in $\mathcal{D}(T)$ we have $(Tf_n) \rightarrow Tf$. Boundedness and continuity are equivalent by the following proposition:

Proposition 2.4.18. *An operator T is bounded if and only if T is continuous.*

We say an operator T is compact if every bounded sequence (f_n) has a subsequence (f_{n_k}) for which (Tf_{n_k}) is convergent. Similarly to bounded operators, a compact operator maps bounded sets to pre-compact sets. A pre-compact set is a set whose closure is a compact set. The two types of operators are related by the following proposition:

Proposition 2.4.19. *Every compact operator is bounded.*

Proof. Let T be a compact operator. Every pre-compact set is bounded, so T maps bounded sets to bounded sets which means that T is bounded. $\square$

For a Hilbert space $\mathcal{H}$ we write $\mathcal{B} = \mathcal{B}(\mathcal{H})$ as the set of bounded operators on $\mathcal{H}$, and $\mathcal{K} = \mathcal{K}(\mathcal{H})$ as the set of compact operators on $\mathcal{H}$.

Proposition 2.4.20. *Let $\mathcal{H}$ be a Hilbert space. Then the following properties hold:*

- i. *If $S, T \in \mathcal{B}$ and one of them is compact, then $ST \in \mathcal{K}$.*
- ii. *If $S, T \in \mathcal{K}$ and $a, b \in \mathbb{K}$, then $aS + bT \in \mathcal{K}$.*
- iii. *$T \in \mathcal{K}$ if and only if $T^*T \in \mathcal{K}$.*
- iv. *$T \in \mathcal{K}$ if and only if $T^* \in \mathcal{K}$.*
- v. *If (T_n) is a sequence of compact operators and $\|T_n - T\| \rightarrow 0$ as $n \rightarrow \infty$ for some $T \in \mathcal{B}$, then $T \in \mathcal{K}$.*

Proof. Let (x_n) be a bounded sequence in $\mathcal{H}$ with bound M .

i. Assume that $S \in \mathcal{B}$ and $T \in \mathcal{K}$. Since T is compact, there exists a subsequence (x_{n_k}) such that (Tx_{n_k}) is convergent. Since S is bounded and thus continuous, the sequence $S(Tx_{n_k}) = (STx_{n_k})$ is convergent and thus $ST \in \mathcal{K}$.

Now since S is bounded, $(Sx_n) = (y_n)$ must also be a bounded sequence. Since T is compact, there must be a subsequence (y_{n_k}) such that (Ty_{n_k}) is convergent. However, $Ty_n = T(Sx_n) = (TS)x_n$ so (TSx_n) must have a convergent subsequence which means that $TS \in \mathcal{K}$.

ii. Since $S \in \mathcal{K}$, there must be a convergent subsequence (Sx_{n_k}) , and without loss of generality assume that (Tx_{n_k}) is not convergent. However, the subsequence x_{n_k} is still bounded, and since $T \in \mathcal{K}$ there must be further subsequence $(Tx_{n_{k_j}})$ which is convergent. Clearly $(Sx_{n_{k_j}})$ is convergent, so $a(Sx_{n_{k_j}}) + b(Tx_{n_{k_j}}) = ((aS + bT)x_{n_{k_j}})$ is convergent. Therefore $aS + bT \in \mathcal{K}$.

iii. The proof consists of two directions. First, assume that $T: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is compact which implies that T is also bounded. Since T is bounded, by Proposition 2.4.9 the adjoint $T^*: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ is also a bounded. Then by i, $T^*T \in \mathcal{K}$.

Now, assume that $T^*T: \mathcal{H}_1 \rightarrow \mathcal{H}_1$ is compact. We want to show that $T: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is compact.

Let (x_n) be a bounded sequence in $\mathcal{H}_1$. This means there exists a real number $M > 0$ such that $\|x_n\|_{\mathcal{H}_1} \leq M$ for all $n \in \mathbb{N}$. Since T^*T is a compact operator on $\mathcal{H}_1$, there exists a subsequence (x_{n_k}) of (x_n) such that the sequence $(T^*Tx_{n_k})$ is convergent.

Since $(T^*Tx_{n_k})$ is a convergent sequence, it is also a Cauchy sequence. Now, consider the sequence (Tx_{n_k}) in $\mathcal{H}_2$. For any $j, k \in \mathbb{N}$, we have:

$$\begin{aligned} \|Tx_{n_j} - Tx_{n_k}\|_{\mathcal{H}_2}^2 &= \langle T(x_{n_j} - x_{n_k}), T(x_{n_j} - x_{n_k}) \rangle_{\mathcal{H}_2} = \langle x_{n_j} - x_{n_k}, T^*T(x_{n_j} - x_{n_k}) \rangle_{\mathcal{H}_1} \\ &= \langle x_{n_j} - x_{n_k}, T^*Tx_{n_j} - T^*Tx_{n_k} \rangle_{\mathcal{H}_1} \\ &\leq \|x_{n_j} - x_{n_k}\|_{\mathcal{H}_1} \cdot \|T^*Tx_{n_j} - T^*Tx_{n_k}\|_{\mathcal{H}_1} \\ &\leq (\|x_{n_j}\|_{\mathcal{H}_1} + \|x_{n_k}\|_{\mathcal{H}_1}) \cdot \|T^*Tx_{n_j} - T^*Tx_{n_k}\|_{\mathcal{H}_1} \\ &= 2M\|T^*Tx_{n_j} - T^*Tx_{n_k}\|_{\mathcal{H}_1}. \end{aligned}$$

As $j, k \rightarrow \infty$, the term $\|T^*Tx_{n_j} - T^*Tx_{n_k}\|_{\mathcal{H}_1} \rightarrow 0$ because $(T^*Tx_{n_k})$ is a Cauchy sequence. Consequently, $\|Tx_{n_j} - Tx_{n_k}\|_{\mathcal{H}_2}^2 \rightarrow 0$. This shows that (Tx_{n_k}) is a Cauchy sequence in $\mathcal{H}_2$.

Since $\mathcal{H}_2$ is a Hilbert space, every Cauchy sequence is convergent, which means that (Tx_{n_k}) is convergent in $\mathcal{H}_2$.

We have shown that for an arbitrary bounded sequence (x_n) in $\mathcal{H}_1$, we can find a subsequence (x_{n_k}) such that (Tx_{n_k}) converges. Thus $T \in \mathcal{K}$.

iv. Assume $T^* \in \mathcal{K}$. Since T^* is also bounded, by Proposition 2.4.9, T is bounded and so $T^*T \in \mathcal{K}$ which by iii. means that $T \in \mathcal{K}$.

Now assume $T \in \mathcal{K}$. Since T is bounded, by Proposition 2.4.9 T^* is bounded and so $TT^* \in \mathcal{K}$. By iii. this means that $T^* \in \mathcal{K}$ since by Proposition 2.4.9 $T^{**} = T$ and so $TT^* = (T^*)^*T^*$.

v. We will use a diagonalization argument. Since T_1 is compact there exists a subsequence $(x_{1,m})$ of (x_n) such that $(T_1x_{1,m})$ is convergent. Since $(x_{1,m})$ is bounded and T_2 compact, there exists a subsequence $(x_{2,m})$ of $(x_{1,m})$ such that $(T_2x_{2,m})$ is compact. Continue this process for every $k \geq 2$ by creating a subsequence $(x_{k,m})$ of $(x_{k-1,m})$ such that $(T_kx_{k,m})$ is convergent. Now create the sequence (y_m) by taking the m -th term of the m -th subsequence $y_m = x_{m,m}$. Since (y_m) is a subsequence of (x_n) , (y_m) is also bounded by m . For every fixed k , the sequence (T_ky_m) is convergent and thus Cauchy.

We claim that (Ty_m) is Cauchy. Let $\epsilon > 0$. Since $\|T_k - T\| \rightarrow 0$, there exists some $p \in \mathbb{N}$ such that $\|T_p - T\| < \epsilon/3M$. Since (T_py_m) is Cauchy, there is some $N > 0$ such that $\|T_py_j - T_py_k\| <$

$\epsilon/3$ whenever $j, k > N$. Therefore, for $j, k > N$ we have

$$\begin{aligned}\|Ty_j - Ty_k\| &= \|Ty_j - T_p y_j + T_p y_j - T_p y_k + T_p y_k - Ty_k\| \\ &\leq \|Ty_j - T_p y_j\| + \|T_p y_j - T_p y_k\| + \|T_p y_k - Ty_k\| \\ &\leq \|T - T_p\| \|y_j\| + \frac{\epsilon}{3} + \|T - T_p\| \|y_k\| \\ &\leq \frac{\epsilon}{3M} M + \frac{\epsilon}{3} + \frac{\epsilon}{3M} M = \epsilon,\end{aligned}$$

which shows that (Ty_m) is Cauchy. Since $\mathcal{H}$ is a Hilbert space, (Ty_m) is convergent, which implies that $T \in \mathcal{K}$. $\square$

In this thesis we will also be working with operators between finite dimensional vector spaces, for which we have the following proposition:

Proposition 2.4.21. *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$, and let T be an operator on $\mathcal{H}$ whose image $\text{im}(T)$ is finite-dimensional. Then T is compact.*

Proof. Let $\{Te_1, \dots, Te_n\}$ be a basis of $\text{im}(T)$, with $e_1, \dots, e_n \in \mathcal{H}$. Then $\text{im}(T)$ is isomorphic to $\mathbb{C}^n$ via $Te_k \mapsto v_k$, where $\{v_1, \dots, v_n\}$ is the standard basis of $\mathbb{C}^n$.

If (x_k) is a bounded sequence in $\mathcal{H}$, then (Tx_k) is bounded in $\text{im}(T)$, hence corresponds to a bounded sequence in $\mathbb{C}^n$. By the Bolzano–Weierstrass theorem, it has a convergent subsequence. Therefore, T maps bounded sequences to sequences with convergent subsequences, so T is compact. $\square$

Definition 2.4.22. *A bounded linear operator $V \in \mathcal{B}(\mathcal{H})$ is said to be a partial isometry if for every $x \in \ker(V)^\perp$, V is an isometry, i.e. $\|Vx\| = \|x\|$.*

Proposition 2.4.23. *Let $V \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

- i. V is a partial isometry.
- ii. V^* is a partial isometry.
- iii. VV^* is a projection.
- iv. V^*V is a projection.

2.5 Spectral Theory

An important aspect when studying a bounded operator, is its distribution of eigenvalues and its spectrum. For finite-dimensional vector spaces, these concepts are equivalent, but in infinite dimensions things are more complicated. This information can be found in Chapter 5 and 7 of [18].

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and let T be a bounded operator on $\mathcal{H}$. Let $\lambda \in \mathbb{C}$. If for some $x \in \mathcal{H}$ we have

$$Tx = \lambda x$$

then we say x is an eigenvector of T with λ as its eigenvalue, and the set $\ker(T - \lambda)$ is known as the eigenspace of λ which is spanned by the eigenvectors of λ . The multiplicity of λ is equal to $\dim \ker(T - \lambda)$.

We are motivated to investigate the behavior of the operator $T - z$ as z varies in $\mathbb{C}$. If for $z \in \mathbb{C}$ the operator $R(z, T) = (T - z)^{-1}$ is well defined and bounded, then $R(z, T)$ is known as the resolvent. The set

$$\rho(T) = \{z \in \mathbb{C} \mid T - z \text{ is invertible}\},$$

is known as the resolvent set, and $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is the spectrum of T . *Note:* If T is not closed then $T - z$ is not closed and by Proposition 2.4.11 $R(z, T)$ is not closed which by Theorem 2.4.14 means that $R(z, T)$ is not bounded and $\rho(T) = \emptyset$. Therefore, unless otherwise indicated, we will assume that all operators are closed henceforth.

The set $\sigma_p(T)$ of all eigenvalues of T , i.e.

$$\sigma_p(T) = \{z \in \mathbb{C} \mid T - z \text{ is not injective}\},$$

is known as the point spectrum of T . Note that $\sigma_p(T) \subseteq \sigma(T)$, so even if a point $z \in \sigma(T)$, it does not necessarily mean that z is an eigenvalue of T .

The definition of eigenvalues, eigenvectors and eigenspaces is the same for unbounded operators, but the definition of the resolvent set and spectrum is a bit different.

Let $\mathcal{H}$ be a Hilbert space and let $T: \mathcal{D}(T) \rightarrow \mathcal{H}$ be an unbounded, densely defined operator on a subset $\mathcal{D}(T) \subset \mathcal{H}$. A point $\lambda \in \mathbb{C}$ is said to be in the resolvent set $\rho(T)$ if the operator $T - \lambda$ has a bounded everywhere-defined inverse, i.e. there exists an operator $S: \mathcal{H} \rightarrow \mathcal{D}(T)$ such that

$$S(T - \lambda) = I_{\mathcal{D}(T)}, (T - \lambda)S = I.$$

For an unbounded operator we then define its spectrum as the set $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

For self-adjoint operators we have the following proposition:

Proposition 2.5.1. *Let T be a self-adjoint operator on a complex Hilbert space. Then $\sigma(T) \subset \mathbb{R}$.*

An obvious consequence of Proposition 2.5.1 is that all eigenvalues of a self-adjoint operator are real.

Furthermore, we have the following proposition for compact operators:

Proposition 2.5.2. *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$, and let $T \in \mathcal{K}(\mathcal{H})$. If $\mathcal{H}$ is infinite-dimensional, then $\sigma(T) = \sigma_p(T) \cup \{0\}$. The operator T has at most countably many eigenvalues with 0 as the only possible accumulation point. Every non-zero eigenvalue has finite multiplicity.*

Recall that in linear algebra we had the spectral theorem for self-adjoint matrices, i.e. you can always diagonalize a self-adjoint matrix into an orthonormal eigenbasis with real eigenvalues. A similar result exists for self-adjoint compact operators:

Theorem 2.5.3 (Spectral theorem for self-adjoint compact operators). *Let T be a self-adjoint, compact operator on an infinite dimensional Hilbert space $\mathcal{H}$. Then there exists in $\mathcal{H}$ an ONB $\{e_1, e_2, \dots\}$ consisting of eigenvectors of T . For every $x \in \mathcal{H}$,*

$$Tx = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda x,$$

where P_λ is the orthogonal projection onto the eigenspace of λ defined as

$$P_\lambda x = \sum_{\{n \mid \lambda = \lambda_n\}} P_n x = \sum_{\{n \mid \lambda = \lambda_n\}} \lambda_n \langle x, e_n \rangle e_n,$$

where e_n is a normalized eigenvector corresponding to λ_n . [13]

Essentially, Theorem 2.5.3 says that we can diagonalize the operator T into spectral projections, similarly to diagonalization in linear-algebra.

We have the following theorem:

Theorem 2.5.4. *Let T be an invertible, self-adjoint operator with a compact resolvent on a separable, infinite-dimensional Hilbert space $\mathcal{H}$. Then the following holds:*

- i. $\sigma(T) \subset \mathbb{R}$,
- ii. $0 \notin \sigma_p(T)$,
- iii. $\sigma(T) = \sigma_p(T)$,
- iv. T has at most countably many eigenvalues with only possible accumulation points at $\pm\infty$,
- v. $\mathcal{H}$ has a countable ONB consisting of eigenvectors of T .
- vi. For every

$$x \in \mathcal{D}(T) = \{x \in \mathcal{H} \mid \sum_{\lambda \in \sigma(T)} \lambda^2 \|P_\lambda x\|^2 < \infty\}$$

we have that

$$Tx = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda x,$$

where P_λ is the closed orthogonal projection onto the eigenspace of λ .

Proof. i: Follows directly from proposition 2.5.1.

ii: If T is invertible then T must be injective, so $Tx = 0$ iff $x = 0$. If we assume for contradiction that 0 is an eigenvalue of T , then there exists an $x \neq 0$ such that $Tx = 0$, which contradicts our assumption, and so $0 \notin \sigma_p(T)$.

iii, iv: Let μ_n be an eigenvalue of T^{-1} . Since T is invertible, T and T^{-1} shared the same set of eigenvectors and $\mu_n \neq 0$. Since T has a compact resolvent we know there is a countable number of μ_n by Proposition 2.5.2. We write

$$T^{-1}x = \mu_n x \Rightarrow \mu_n^{-1}x = Tx,$$

which means T has eigenvalues μ_n^{-1} . Let μ_n be a sequence of eigenvalues such that $\lim_{n \rightarrow \infty} \mu_n = 0$ with 0 being the only possible accumulation point by Proposition 2.5.2. We see then that $\mu_n^{-1} \rightarrow \pm\infty$, with the only possible accumulation points only at $\pm\infty$, which means that $\sigma(T)$ can consist only of eigenvalues, i.e. $\sigma(T) = \sigma_p(T)$.

v: The resolvent $(T - \lambda)^{-1}$ share the same set of eigenvectors as T . The statement then follows from Theorem 2.5.3.

vi: By i-v., we have that the spectrum of $\sigma(T)$ consists only of a countable number of real eigenvalues with finite multiplicity, and furthermore that there exists an ONB consisting of eigenvectors to T . Then for every $x \in \mathcal{D}(T)$ we can write

$$Tx = \sum_{\lambda \in \sigma(T)} \lambda P_\lambda x,$$

where P_λ is the orthogonal projection onto the eigenspace of λ defined as

$$P_\lambda x = \sum_{\{n \mid \lambda = \lambda_n\}} P_n x = \sum_{\{n \mid \lambda = \lambda_n\}} \lambda_n \langle x, e_n \rangle e_n,$$

where e_n is a normalized eigenvector corresponding to λ_n . □

We can use Theorem 2.5.4 to define a functional calculus for operators with compact resolvent. Let T be an operator with compact resolvent. If we have an arbitrary function $f: \mathbb{C} \rightarrow \mathbb{C}$ then we can define the operator

$$f(T)x = \sum_{\lambda \in \sigma(T)} f(\lambda)P_\lambda x.$$

Here λ will always denote an eigenvalue of the operator T and P_λ will always refer to the orthogonal, closed projection onto the eigenspace of λ .

Note: Strictly speaking, one can define the functional calculus $f(T)$ for an operator T that does not have a compact resolvent, but doing so would require more work, and in general requires more regularity on f . The reader is referred to a more advanced book on Functional Analysis such as [18] for more details. What is important to know is if we have an operator T with $\sigma(T) = \sigma_p(T)$, for each eigenvalue $\lambda \in \sigma_p(T)$ we can still define the eigenvalues of $f(T)$ as $f(\lambda)$.

We say that an operator U is unitary if U satisfies $U^*U = UU^* = I$. In the following proposition we will define polar decomposition, which uses the functional calculus for operators that generally do not have compact resolvent. However, this will not be important for this thesis so we will continue without going into further detail:

Theorem 2.5.5 (Polar decomposition [18]). *Let $T: \mathcal{D}(T) \rightarrow \mathcal{H}$ be a densely defined closed operator in a Hilbert space $\mathcal{H}$. The operator T can be uniquely represented in the form $T = US$, where $S: \mathcal{D}(T) \rightarrow \mathcal{R}(S)$ is a non-negative self-adjoint operator and $U: \overline{\mathcal{R}(S)} \rightarrow \overline{\mathcal{R}(T)}$ is a partial isometry. We have $S = |T| = (T^*T)^{\frac{1}{2}}$.*

Note: In Theorem 2.5.5 we have used the extended functional calculus for $(T^*T)^{\frac{1}{2}}$, which uses something called spectral projections, which we will not go through here.

We have the following special case of Theorem 2.5.5 for bounded operators:

Corollary 2.5.6. *Let $T: \mathcal{H} \rightarrow \mathcal{H}$ be an invertible, bounded operator. In the polar decomposition $T = US$, we have that U is a unitary operator.*

Proof. Since T is bounded, T is clearly densely defined and closed so from Theorem 2.5.5 we can use polar decomposition to write $T = US$. Since T is bounded and invertible we have $\mathcal{D}(T) = \mathcal{H}$ and $\mathcal{R}(T) = \overline{\mathcal{R}(T)} = \mathcal{H}$. Since T is invertible, S must also be invertible since

$$S^{-1} = (T^*T)^{-\frac{1}{2}} = (T^{-1}(T^{-1})^*)^{\frac{1}{2}},$$

so $\mathcal{R}(S) = \mathcal{H}$. Since T and S are invertible, U is invertible since $U = TS^{-1}$. We have $U: \mathcal{H} \rightarrow \mathcal{H}$ and $U = TS^{-1}$ so

$$U^*U = UU^* = S^{-1}T^*TS^{-1} = (T^*T)^{-\frac{1}{2}}T^*T(T^*T)^{-\frac{1}{2}} = I. \quad \square$$

Using the functional calculus, we can now continue with some needed propositions:

Lemma 2.5.7. *Let D be a self-adjoint operator. D has a compact resolvent if and only if $(D^2 + 1)^{-1} \in \mathcal{K}$.*

Proof. Since D is self-adjoint, we have $\sigma(D) \subset \mathbb{R}$. First note that

$$(D^2 + 1)^{-1} = (D + i)^{-1}(D - i)^{-1} = (D - i)^{-1}(D + i)^{-1}.$$

Assume that D has compact resolvent. Since $\pm i \notin \sigma(D)$ we have that $(D + i)^{-1}, (D - i)^{-1} \in \mathcal{K}$ which by Proposition 2.4.20.i. means their product is compact and thus $(D^2 + 1)^{-1} \in \mathcal{K}$.

Now assume that $(D^2 + 1)^{-1} \in \mathcal{K}$. Set $T = (D - i)^{-1}$, and note that $T^* = (D + i)^{-1}$ since D is self-adjoint. Since the product $T^*T = (D^2 + 1)^{-1} \in \mathcal{K}$, we have by Proposition 2.4.20 parts iii. and iv. that T and T^* are compact. Now assume we have some $\lambda \in \rho(D)$ such that $\lambda \neq i$. Then from the resolvent identity

$$R(z, T) - R(w, T) = (z - w)R(z, T)R(w, T),$$

we have

$$(D - \lambda)^{-1} = (\lambda - i)(D - \lambda)^{-1}(D - i)^{-1} + (D - i)^{-1}.$$

Since $(D - i)^{-1} \in \mathcal{K}$ and $(D - \lambda)^{-1} \in \mathcal{B}$, we have from Proposition 2.4.20 parts i. and ii. that $(D - \lambda)^{-1}$. Since λ was chosen arbitrarily, this means that D has compact resolvent. $\square$

Proposition 2.5.8. *Let $D: \mathcal{D}(D) \rightarrow \mathcal{H}$ be a self-adjoint operator with compact resolvent and let A be a bounded operator on $\mathcal{H}$. Then $D + A$ also has a compact resolvent.*

Proof. We have the graph-scalar product:

$$\langle x, y \rangle_D = \langle x, y \rangle_{\mathcal{H}} + \langle Dx, Dy \rangle_{\mathcal{H}}.$$

Since D is self-adjoint, D is closed and thus by Proposition 2.4.13 we have that $(\mathcal{D}(D), \langle \cdot, \cdot \rangle_D)$ is a Hilbert space.

Now let $\hat{i}: \mathcal{D}(D) \rightarrow \mathcal{H}$ be the inclusion operator defined as $\hat{i}x = x, \forall x \in \mathcal{D}(D)$, which is a bounded operator in the graph-norm. We have for every $x \in \mathcal{D}(D)$ and $y \in \mathcal{H}$:

$$\langle \hat{i}x, y \rangle_{\mathcal{H}} = \langle x, y \rangle_{\mathcal{H}} = \langle x, (1 + D^2)^{-1}y \rangle_D,$$

since

$$\begin{aligned} \langle x, (1 + D^2)^{-1}y \rangle_D &= \langle x, (1 + D^2)^{-1}y \rangle_{\mathcal{H}} + \langle Dx, D(1 + D^2)^{-1}y \rangle_{\mathcal{H}} \\ &= \langle x, (1 + D^2)^{-1}y \rangle_{\mathcal{H}} + \langle x, D^2(1 + D^2)^{-1}y \rangle_{\mathcal{H}} \\ &= \langle x, (1 + D^2)(1 + D^2)^{-1}y \rangle_{\mathcal{H}} \\ &= \langle x, y \rangle_{\mathcal{H}}. \end{aligned}$$

Thus the operator $(1 + D^2)^{-1} = \hat{i}^*: \mathcal{H} \rightarrow \mathcal{D}(D)$ is the adjoint of the inclusion operator $\hat{i}$.

Since D has compact resolvent, by Lemma 2.5.7 the operator $(1 + D^2)^{-1} \in \mathcal{K}$ which also means that it is bounded. Note that since $(1 + D^2)^{-1}$ is bounded and $\mathcal{D}(D) \subset \mathcal{H}$, we can extend the range of $(1 + D^2)^{-1}: \mathcal{H} \rightarrow \mathcal{H}$ and the relation

$$\langle x, y \rangle_{\mathcal{H}} = \langle x, (1 + D^2)^{-1}y \rangle_D$$

will still be well-defined.

By Proposition 2.4.20. iv. we have that $\hat{i}$ is compact if and only if $\hat{i}^* = (1 + D^2)^{-1}$ is compact, which means that $\hat{i} \in \mathcal{K}$ if and only if D has compact resolvent.

Now define the operator $\tilde{D} = D + A$ where A is bounded operator, and let $\tilde{i}: \mathcal{D}(\tilde{D}) \rightarrow \mathcal{H}$ be the inclusion operator such that $\tilde{i}x = x$ for every $x \in \mathcal{D}(\tilde{D})$. However, since A is bounded, we have $\mathcal{D}(\tilde{D}) = \mathcal{D}(D)$ and so $\tilde{i} = \hat{i}$.

We have that $\hat{i}$ is compact if and only if D has compact resolvent, and by the same logic $\tilde{i}$ is compact if and only if $\tilde{D}$ has compact resolvent. By assumption D has compact resolvent so $\hat{i}$ is compact. We have $\tilde{i} = \hat{i}$, so $\tilde{i}$ is compact, and thus $\tilde{D} = D + A$ must have compact resolvent. $\square$

Proposition 2.5.9. *Let D be a self-adjoint operator with compact resolvent and let f be a bounded function, i.e. $|f| \leq C$ for some $C \in [0, \infty)$. Then $f(D) \in \mathcal{B}$.*

Proof. Using the functional calculus we have

$$f(D) = \sum_{\lambda \in \sigma(T)} f(\lambda) P_\lambda,$$

and since each $f(\lambda)$ is bounded and P_n is bounded, $f(D)$ must also be bounded since

$$\|f(T)\| = \max_{\lambda \in \sigma(T)} \|f(\lambda)\| < \infty. \quad \square$$

For our further discussion we write $C(\mathbb{R})$ as the space of all continuous functions defined on $\mathbb{R}$. Let then $C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid \lim_{|x| \rightarrow \infty} f(x) = 0\}$ be the space of all continuous functions vanishing at infinity. We also need the following lemma:

Lemma 2.5.10. *Let $f \in C_0(\mathbb{R})$. Then f is bounded.*

Proof. Saying that f vanishes at infinity means there is a number $M > 0$ large enough such that for $|x| > M$ then $|f(x)| < \epsilon$, for $\epsilon > 0$. Consider the compact interval $[-M, M]$. Since f is continuous on $[-M, M]$, by the extreme value theorem, f must be bounded on $[-M, M]$. Let then $K > 0$ be the value such that $|f(x)| \leq K$ for $x \in [-M, M]$. We then have that $|f(x)| \leq \max(K, \epsilon)$ for all $x \in \mathbb{R}$ so f is bounded. $\square$

We have the following proposition:

Proposition 2.5.11. *Let D be a self-adjoint operator with compact resolvent and let $f \in C_0(\mathbb{R})$. Then $f(D) \in \mathcal{K}$.*

Proof. From Lemma 2.5.10 we know that f is bounded and thus by Proposition 2.5.9 we have that $f(D) \in \mathcal{B}$.

Now by functional calculus we can write

$$f(D) = \sum_{\lambda \in \sigma(T)} f(\lambda) P_\lambda = \sum_{n=1}^{\infty} f(\lambda_n) P_n,$$

λ_n is an eigenvalue and P_n is an orthogonal projection. Here we have ordered λ_n in increasing order of absolute value, i.e. $n \geq m$ if and only if $|\lambda_n| \geq |\lambda_m|$.

Let now $K_N = \sum_{n=1}^N f(\lambda_n) P_n$. Since each P_n is a finite-dimensional projection, K_N is a finite-dimensional operator which means that K_N is compact. The sequence (K_N) is a sequence of compact operators.

We have that

$$\|f(D) - K_N\| = \left\| \sum_{n=N}^{\infty} f(\lambda_n) P_n \right\| = \max_{n \geq N} \|f(\lambda_n)\|,$$

where P_n are pair-wise orthogonal projections. Since f vanishes at infinity, we have that $\lim_{N \rightarrow \infty} (\max_{n \geq N} \|f(\lambda_n)\|) = 0$. So in total $\|f(D) - K_N\| \rightarrow 0$ as $N \rightarrow \infty$ and by Proposition 2.4.20.v. $f(D) \in \mathcal{K}$. $\square$

2.6 Integrals

In this subsection, we will define integrals of operators, which will yield some useful results for later sections.

Definition 2.6.1. Let $I = [a, b] \subset \mathbb{R}$. A partition of I is a finite sequence of numbers (x_i) such that $a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$. For each $i \in [1, N]$, we call $[x_{i-1}, x_i]$ a subinterval of the partition (x_i) and $\max_{i \in [1, N]}(x_{i+1} - x_i) = \Delta x$ the mesh of the partition (x_i) . A tagged partition $P = ((\xi_i), (x_i))$ is a partition (x_i) of I and a set of points (ξ_i) such that $\xi_i \in [x_i, x_{i+1}]$ for every subinterval of (x_i) arbitrarily chosen.

Let $I = [a, b] \subset \mathbb{R}$. We say that we refine a tagged partition (x_i) for $i \in [0, N]$ to (x'_i) by increasing N such that $\Delta x' < \Delta x$, and can continue refining I into the limit $\Delta x \rightarrow 0$ which we write as $N \rightarrow \infty$.

Definition 2.6.2. Let $f: [0, a] \rightarrow \mathcal{K}$ be a norm-continuous function and let f also be norm-integrable, i.e. $\int_0^a \|f(x)\| dx < \infty$ for every $a \in [0, \infty)$. Let $P_I = ((\xi_i), (x_i))$ be a tagged partition of $I = [0, a]$. Then we define the integral of f over $[0, a]$ as

$$\int_0^a f(x) dx = \lim_{N \rightarrow \infty} \sum_{i=1}^N f(\xi_i)(x_i - x_{i-1}).$$

We also define the improper integral of f as

$$\int_0^\infty f(x) dx = \lim_{a \rightarrow \infty} \int_0^a f(x) dx.$$

Since we are working with norm-continuous functions, one can use the same methods as in one-variable calculus to check that the integral in Definition 2.6.2 is well-defined and follows the same calculation rules as the Riemann-integral.

Proposition 2.6.3. Let $f: [0, \infty) \rightarrow \mathcal{K}$ be norm-continuous and $\int_0^\infty \|f(x)\| dx < \infty$. Then the operator

$$K = \int_0^\infty f(x) dx \in \mathcal{K}.$$

Proof. Let $P_{[0, a]} = ((x_i), (\xi_i))$ be a tagged partition of $[0, a]$ clear the each partial sum

$$K_N = \sum_{i=1}^N f(\xi_i)(x_i - x_{i-1}) \in \mathcal{K},$$

and since f is norm-continuous:

$$\lim_{N \rightarrow \infty} \|K - K_N\| = 0,$$

which shows that $K \in \mathcal{K}$, which also holds as $a \rightarrow \infty$. □

The following Lemma comes from the article [5], and a more explicit proof was presented in [6] which we present here. What to note is that in [6], the author assumes only that D has a locally compact resolvent which is weaker than compact resolvent. We will not dive into the details here, but just note that since we assume D has compact resolvent, the proof of the following lemma can be simplified:

Lemma 2.6.4. Let D be a self-adjoint, invertible operator with compact resolvent and let A be a bounded, invertible operator such that $AD(D) \subset \mathcal{D}(D)$ and $[A, D]: AD(D) \rightarrow \mathcal{H}$ extends to a bounded operator. The operator $D[(D^2 + 1)^{-1/2}, A]$ is compact.

Proof. Since D is a self-adjoint, invertible operator with compact resolvent, for every $a \geq 0$ the operator $(D^2 + a)^{-1} \in \mathcal{K}$ since $(D^2 + a)^{-1} = (D + i\sqrt{a})^{-1}(D - i\sqrt{a})^{-1} = (D - i\sqrt{a})^{-1}(D + i\sqrt{a})^{-1} \in \mathcal{K}$, since $\pm i\sqrt{a} \notin \sigma(D)$. Furthermore, using the formula

$$(D^2 + 1)^{-1/2} = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (D^2 + 1 + \lambda)^{-1} d\lambda,$$

we have that

$$D[(D^2 + 1)^{-1/2}, A] = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} D[(D^2 + 1 + \lambda)^{-1}, A] d\lambda, \quad (4)$$

which is a uniformly norm convergent integral [6]. We have that

$$D[(D^2 + 1 + \lambda)^{-1}, A] = D(D^2 + 1 + \lambda)^{-1} A - [D, A](D^2 + 1 + \lambda)^{-1} + AD(D^2 + 1 + \lambda)^{-1}.$$

Since $[D, A] \in \mathcal{B}$, and for every $\lambda \geq 0$: $(D^2 + 1 + \lambda)^{-1} \in \mathcal{K}$ and $D(D^2 + 1 + \lambda)^{-1} = D(D^2 + 1 + \lambda)^{-1/2}(D^2 + 1 + \lambda)^{-1/2} \in \mathcal{K}$ since $(D^2 + 1 + \lambda)^{-1/2} \in \mathcal{K}$ and $D(D^2 + 1 + \lambda)^{-1/2} \in \mathcal{B}$. Thus for every term in the integral in (4) we have $D[(D^2 + 1 + \lambda)^{-1}, A] \in \mathcal{K}$, which means that $D[(D^2 + 1)^{-1/2}, A] \in \mathcal{K}$ by Proposition 2.6.3. $\square$

Proposition 2.6.5. *Let D be a self-adjoint, invertible operator with compact resolvent. Let A be a bounded, invertible operator such that $AD(D) \subset \mathcal{D}(D)$ and $[A, D]: AD(D) \rightarrow \mathcal{H}$ extends to a bounded operator. Define the function*

$$\beta(x) := \frac{x}{\sqrt{x^2 + 1}}.$$

The operator

$$[\beta(D), A] = [D(1 + D^2)^{-1/2}, A],$$

is compact.

Proof. We have

$$[\beta(D), A] = D(1 + D^2)^{-1/2} A - AD(1 + D^2)^{-1/2} = [D, A](1 + D^2)^{-1/2} + D[(1 + D^2)^{-1/2}, A].$$

The first term, $[D, A](1 + D^2)^{-1/2}$, is compact since $[D, A] \in \mathcal{B}$ and $(1 + D^2)^{-1/2} \in \mathcal{K}$ since $(1 + x^2)^{-1/2} \in C_0(\mathbb{R})$. The second term $D[(1 + D^2)^{-1/2}, A]$ is compact by Lemma 2.6.4. So in total $[\beta(D), A] \in \mathcal{K}$. $\square$

2.7 Index theory

This section will go over the basics of Index Theory, an essential part of this thesis. Most of this information can be found in [10, ch. 1].

For vector spaces $Y \subset X$, we can define the quotient space $X/Y = X/\sim$ by the equivalence class: $x \sim y$ if and only if $x - y \in Y$. For X/Y to be well-defined, we require Y to be a closed subspace of X .

A special type of bounded operators are known as Fredholm operators and are defined as follows:

Definition 2.7.1 (Fredholm operators). *A bounded operator $T: X \rightarrow Y$ is called Fredholm if $\ker(T)$ and $\text{coker}(T) = Y/\text{im}(T)$ are finite-dimensional.*

Proposition 2.7.2. *An operator T is Fredholm if and only if the following holds:*

- i. $\ker(T)$ is finite-dimensional,
- ii. $\ker(T^*)$ is finite-dimensional,
- iii. T has closed range.

We can actually use Proposition 2.7.2 to define Fredholm operators for unbounded operators:

Definition 2.7.3. Let $T: \mathcal{D}(T) \rightarrow \mathcal{H}$ be a densely defined operator with closed range, finite-dimensional $\ker(T)$ and finite-dimensional $\ker(T^*)$. Then we say T is Fredholm with index

$$\text{ind}(T) = \dim \ker(T) - \dim \ker(T^*).$$

Proposition 2.7.4. Let T be a linear operator. If $\dim \text{coker}(T) < \infty$ then T has closed range and $\dim \text{coker}(T) = \dim \ker(T^*)$. [17, pp. 65–66]

Ex. For a finite-dimensional operator $T: X \rightarrow Y$ of rank r , where $\dim(X) = n$ and $\dim(Y) = m$, the index is trivially $\text{ind}(T) = \dim \ker(T) - \dim \text{coker}(T) = (n - r) - (m - r) = n - m$.

Let $\mathcal{H}$ be a Hilbert space. The set of bounded operators $\mathcal{B}$ form a ring with composition of operators as multiplication and from the properties of $\mathcal{K}$ in Proposition (2.4.20) it is clear that $\mathcal{K}$ is a closed two-sided ideal of $\mathcal{B}$. We can thus define the quotient $\mathcal{Q} = \mathcal{Q}(\mathcal{H}) = \mathcal{B}/\mathcal{K}$ as the Calkin algebra and let $\pi: \mathcal{B} \rightarrow \mathcal{Q}$ be the bounded projection mapping. The projection mapping π is a surjective algebra homomorphism which admits the properties:

- i. $\pi(K) = 0, \forall K \in \mathcal{K}$,
- ii. $\pi(T + S) = \pi(T) + \pi(S), \forall S, T \in \mathcal{B}$,
- iii. $\pi(\lambda T) = \lambda \pi(T), \forall \lambda \in \mathbb{R}, \forall T \in \mathcal{B}$,
- iv. $\pi(ST) = \pi(S)\pi(T), \forall S, T \in \mathcal{B}$,
- v. $\pi(I)$ is the identity element in $\mathcal{Q}$.

We can also define the spectrum for an element $\pi(T) \in \mathcal{Q}$ as

$$\sigma_{\mathcal{Q}}(\pi(T)) = \{\lambda \in \mathbb{C} \mid \pi(T) - \lambda \text{ is not invertible in } \mathcal{Q}\}.$$

We also have the following theorem:

Theorem 2.7.5 (Atkinson). Let $\mathcal{H}$ be a Hilbert space. An operator $T \in \mathcal{B}$ is Fredholm if and only if $\pi(T)$ is invertible in $\mathcal{Q}$.

Definition 2.7.6. Let T be a self-adjoint operator. The essential spectrum $\sigma_e(T)$ is the set of points in $\sigma(T)$ that are either accumulation points of $\sigma(T)$ or isolated eigenvalues of infinite multiplicity.

The definition of $\sigma_e(T)$ is connected to Fredholm operators by the following proposition:

Proposition 2.7.7 (Essential Spectrum of self-adjoint operators [12]). Let T be a self-adjoint (unbounded) operator. Then

$$\sigma_e(T) = \{\lambda \in \mathbb{R} \mid T - \lambda \text{ is not Fredholm}\}.$$

We also have the following proposition:

Proposition 2.7.8. *Let $T \in \mathcal{B}$ be a self-adjoint operator. Then*

$$\sigma_e(T) = \sigma_{\mathcal{Q}}(\pi(T)).$$

Proof. Let $\lambda \in \sigma_e(T)$. We have by Proposition 2.7.7 and Theorem 2.7.5:

$$\begin{aligned} \lambda \in \sigma_e(T) &\Leftrightarrow T - \lambda \text{ is not Fredholm} \Leftrightarrow \pi(T - \lambda) = \pi(T) - \lambda\pi(I) \text{ not invertible in } \mathcal{Q} \\ &\Leftrightarrow \lambda \in \sigma_{\mathcal{Q}}(\pi(T)), \end{aligned}$$

which is true for every $\lambda \in \sigma_e(T)$ since λ was chosen arbitrarily. $\square$

A consequence of Proposition 2.7.8 is that the essential spectrum is stable under compact perturbations, i.e.

$$\sigma_e(T + K) = \sigma_{\mathcal{Q}}(\pi(T + K)) = \sigma_{\mathcal{Q}}(\pi(T)) = \sigma_e(T), \forall K \in \mathcal{K}.$$

In fact, not only is the essential spectrum stable under compact perturbations, the essential spectrum is also stable under relatively compact perturbations. For a self-adjoint operator T , we say that the operator S is relatively compact if the operator $S(T + i)^{-1}$ is compact.

Theorem 2.7.9 (Weyl's theorem [15]). *Let T be a self-adjoint (unbounded) operator and let S be a symmetric, relatively compact operator. Then*

$$\sigma_e(T + S) = \sigma_e(T).$$

For a self-adjoint operator T and a self-adjoint compact operator K , we also have the following proposition:

Proposition 2.7.10. *Let T be a Fredholm operator. For every $K \in \mathcal{K}$ we have*

$$\text{ind}(T + K) = \text{ind}(T).$$

Another nice consequence of Proposition 2.7.8 is the following proposition:

Proposition 2.7.11. *Let $\mathcal{H}$ be a separable Hilbert space and $D: \mathcal{D}(D) \rightarrow \mathcal{H}$ be a self-adjoint operator with compact resolvent. Then $\sigma_e(D) = \emptyset$ and D is Fredholm with $\text{ind}(D) = 0$.*

Proof. Since D has compact resolvent, its spectrum consists only of eigenvalues with finite multiplicity and there are no accumulation points inside $\mathbb{R}$, which means that $\sigma_e(D) = \emptyset$. So for every $\lambda \in \mathbb{R}$ it must be that $D - \lambda$ is Fredholm, and in particular D is Fredholm. Since D is self-adjoint, it is clear that

$$\text{ind}(D) = \dim \ker(D) - \dim \ker(D^*) = \dim \ker(D) - \dim \ker(D) = 0. \quad \square$$

A clear consequence of Propositions 2.5.8 and 2.7.11 is that if D is self-adjoint operator with compact resolvent and A is bounded operator, then the operator $D + A: \mathcal{D}(D) \rightarrow \mathcal{H}$ has compact resolvent and is thus Fredholm.

3 Spectral flow

In our discussion of the method of spectral localizers, we are going to make use of the concept of spectral flow. Intuitively, for a path of self-adjoint, bounded Fredholm operators, the spectral flow is the net number of eigenvalues crossing 0. The material on spectral flow can be found [16], while some of its properties are laid out more explicitly in the Appendix of [14].

3.1 Definition

Let $\mathcal{F} = \{T \in \mathcal{B} \mid \pi(T) \text{ invertible in } \mathcal{Q}\}$ be the set of Fredholm operators, and let $\mathcal{F}^{sa}$ denote the set of self-adjoint, bounded Fredholm operators. The space $\mathcal{F}^{sa}$ has three components determined by the essential spectrum:

$$\begin{aligned}\mathcal{F}_+^{sa} &= \{T \in \mathcal{F}^{sa} \mid \sigma_e(T) \subseteq [0, \infty)\}, \\ \mathcal{F}_-^{sa} &= \{T \in \mathcal{F}^{sa} \mid \sigma_e(T) \subseteq (-\infty, 0]\}, \\ \mathcal{F}_*^{sa} &= \{T \in \mathcal{F}^{sa} \mid \sigma_e(T) \not\subseteq [0, \infty) \text{ and } \sigma_e(T) \not\subseteq (-\infty, 0]\}.\end{aligned}$$

Proposition 3.1.1. *Given $F \in \mathcal{F}_*^{sa}$, there is a positive number a and a neighbourhood N of F in $\mathcal{F}_*^{sa}$ so that $S \mapsto \chi_{[-a,a]}(S)$ is a norm-continuous, finite-rank projection-valued function on N . Here $\chi_{[-a,a]}$ denotes the characteristic function of $[-a, a]$.*

Notation: If E is a finite rank-spectral projection for the self-adjoint operator S , let E^+ denote the projection on the subspace $E(\mathcal{H})$ spanned by those eigenvectors for S in $E(\mathcal{H})$ having nonnegative values.

Definition 3.1.2 (Spectral flow). *Let $F_t: [0, 1] \rightarrow \mathcal{F}_*^{sa}$ be a norm-continuous path. Choose a partition: $0 = t_0 < t_1 < \dots < t_n = 1$ of $[0, 1]$, and positive numbers $a_1, a_2, \dots, a_n$ so that for each $i = 1, 2, \dots, n$ the function $t \mapsto E_i(t) = \chi_{[-a_i, a_i]}(F_t)$ is continuous and finite rank on $[t_{i-1}, t_i]$. The spectral flow of F_t is then defined as*

$$\text{Sf}(F_t) = \sum_{k=1}^n \dim(E_k^+(t_k)) - \dim(E_k^+(t_{k-1})).$$

Proposition 3.1.3. *Spectral flow is well-defined, that is, spectral flow depends only on the continuous function $F_t: [0, 1] \rightarrow \mathcal{F}_*^{sa}$.*

We can also define the spectral flow for self-adjoint, invertible and unbounded Fredholm operators with compact resolvents:

Definition 3.1.4 (β -Spectral flow). *Define the function $\beta: \mathbb{R} \rightarrow [-1, 1]$ as*

$$\beta(x) = x(1 + x^2)^{-1/2}.$$

Let D_t be a path of self-adjoint, invertible Fredholm operators with compact resolvents such that $\beta(D_t)$ is norm-continuous in $t \in [0, 1]$. Then we define β -Spectral flow as

$$\text{Sf}_\beta(D_t) = \text{Sf}(\beta(D_t)).$$

The choice of β in Definition 3.1.4 is motivated by the following reasons: Firstly, β is a bounded function, which by Proposition 2.5.9 means that $\beta(D)$ is bounded. Secondly, $\text{sgn}(\beta(x)) = \text{sgn}(x), \forall x \in \mathbb{R}$, which means intuitively that β should not change the number of positive eigenvalues crossing 0 if applied to a norm-continuous path of bounded self-adjoint Fredholm operators. Thirdly, the choice of β is justified by Proposition 2.6.5, which provides nice properties that will prove useful later in this thesis.

The definition of β -spectral flow is well-defined by the following proposition:

Proposition 3.1.5. *Let D_t for $t \in [0, 1]$ be a norm-continuous path of self-adjoint, bounded Fredholm operators. Then*

$$\text{Sf}(D) = \text{Sf}_\beta(D).$$

Proof. Clearly if D_t is a norm-continuous path, then $\beta(D_t)$ is also norm-continuous since $\beta: \mathbb{R} \rightarrow [0, 1]$ given by

$$\beta(x) = \frac{x}{\sqrt{x^2 + 1}}$$

is a continuous, bounded function.

By Definition 3.1.2 we can choose a certain partition $0 = t_0 < t_1 < \dots < t_n = 1$ of $[0, 1]$ and positive numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ so that for each $i = 1, 2, \dots, n$ the functions $t \mapsto E_i(t) = \chi_{[-a_i, a_i]}(D_t)$ and $t \mapsto B_i(t) = \chi_{[-b_i, b_i]}(\beta(D_t))$ are norm-continuous and finite rank on $[t_{i-1}, t_i]$, since each eigenvalue of D_t has a finite dimensional rank. The spectral flow of D_t is defined as

$$\text{Sf}(D_t) = \sum_{k=1}^n \dim(E_k^+(t_k)) - \dim(E_k^+(t_{k-1})).$$

Note first by the functional calculus that every eigenvalue $\lambda \in \sigma(D_t)$ for $t \in [0, 1]$ corresponds to $\beta(\lambda) \in \sigma(\beta(D_t))$. Note in particular that β does not change the sign, i.e. $\text{sgn}(\beta(\lambda)) = \text{sgn}(\lambda), \forall \lambda \in \sigma(D_t)$. By Proposition 3.1.3, spectral flow does not depend on the choice of positive numbers b_i , so for every a_i we can choose $b_i = a_i(1 + a_i^2)^{-1/2}$ so that $B_i(t) = E_i(t)$ for all $i = 1, 2, \dots, n$. In particular

$$\text{Sf}(\beta(D_t)) = \sum_{k=1}^n \dim(B_k^+(t_k)) - \dim(B_k^+(t_{k-1})) = \text{Sf}(D_t). \quad \square$$

3.2 Properties

Proposition 3.2.1. *The spectral flow fulfills the following properties:*

i. (Homotopy Invariance) Let $s \in [0, 1] \mapsto F_t(s)$ be a homotopy of paths with fixed endpoints $F_0(s)$ and $F_1(s)$. Then

$$\text{Sf}(t \in [0, 1] \mapsto F_t(0)) = \text{Sf}(t \in [0, 1] \mapsto F_t(1)).$$

ii. (Concatenation) For paths $t \in [0, 1] \mapsto F_t$ and $t \in [1, 2] \mapsto F_t$,

$$\text{Sf}(t \in [0, 1] \mapsto F_t) + \text{Sf}(t \in [1, 2] \mapsto F_t) = \text{Sf}(t \in [0, 2] \mapsto F_t).$$

iii. (Unitary invariance) For any unitary U ,

$$\text{Sf}(t \in [0, 1] \mapsto U^* F_t U) = \text{Sf}(t \in [0, 1] \mapsto F_t).$$

iv. (Additivity) For paths $t \in [0, 1] \mapsto F_t$ and $t \in [0, 1] \mapsto F'_t$,

$$\text{Sf}(t \in [0, 1] \mapsto F_t \oplus F'_t) = \text{Sf}(t \in [0, 1] \mapsto F_t) + \text{Sf}(t \in [0, 1] \mapsto F'_t).$$

v. For a path $t \in [0, 1] \mapsto F_t$ with 0 not in the spectrum $\sigma(F_t)$ of F_t for all $t \in [0, 1]$.

$$\text{Sf}(t \in [0, 1] \mapsto F_t) = 0.$$

Note: From the functional calculus, β -Spectral flow inherits the same spectral flow properties listed in Proposition 3.2.1.

Definition 3.2.2 (Spectral flow of straight-line path). *For two self-adjoint, invertible, bounded Fredholm operators F_0, F_1 with compact difference $F_1 - F_0 \in \mathcal{K}$, we define*

$$\text{Sf}(F_0, F_1) = \text{Sf}(t \in [0, 1] \mapsto (1 - t)F_0 + tF_1),$$

as the spectral flow of the straight-line path between F_1, F_0

Note: It is not strictly necessary to $F_1 - F_0 \in \mathcal{K}$ to define the straight-line path in Definition 3.2.2, but the choice is motivated by the following proposition:

Proposition 3.2.3. *Let F_0, F_1 be two self-adjoint, invertible Fredholm operators with compact difference $F_1 - F_0 \in \mathcal{K}$. Let $F = F(t)$ be a norm-continuous path in $\mathcal{F}_{sa}^*$ such that $F(0) = F_0, F(1) = F_1$ and $\pi(F)$ is constant in $t \in [0, 1]$. Then*

$$\text{Sf}(F) = \text{Sf}(F_0, F_1).$$

Proof. Since F has the same endpoints as $(1 - t)F_0 + tF_1$, by homotopy invariance in Proposition 3.2.1. i. we have

$$\text{Sf}(F) = \text{Sf}(F_0, F_1)$$

if both F and $F'_t = (1 - t)F_0 + tF_1$ are norm-continuous paths in $\mathcal{F}_{sa}^*$.

For the first path F , we have $\pi(F) = \pi(F_0) = \pi(F_1)$ by the assumption, so the path F is Fredholm since F_0, F_1 are Fredholm.

The second path $F'_t = (1 - t)F_0 + tF_1$ is clearly norm-continuous. Furthermore, since $\pi((1 - t)F_0 + tF_1) = \pi(F_0) + t\pi(F_1 - F_0) = \pi(F_0) = \pi(F_1)$, the path F'_t is Fredholm since F_0, F_1 are Fredholm. $\square$

The following corollaries follow directly from Proposition 3.2.1:

Corollary 3.2.4. *Let F_0, F_1 be bounded self-adjoint Fredholm operators with compact difference. Then*

$$\text{Sf}(F_0, F_1) = -\text{Sf}(F_1, F_0).$$

Proof. We have by concatenation

$$0 = \text{Sf}(F_0, F_0) = \text{Sf}(F_0, F_1) + \text{Sf}(F_1, F_0). \quad \square$$

Corollary 3.2.5. *Let A, B, C be bounded, self-adjoint Fredholm operators with pair-wise compact differences. If $\text{Sf}(B, C) = 0$ then*

$$\text{Sf}(A, B) = \text{Sf}(A, C).$$

Proof. By concatenation of spectral flow

$$\text{Sf}(A, C) = \text{Sf}(A, B) + \text{Sf}(B, C) = \text{Sf}(A, B). \quad \square$$

We can also define the straight-line path for β -Spectral flow:

Definition 3.2.6 (β -Spectral flow of straight-line path). *Let D_0, D_1 be two self-adjoint, invertible Fredholm operators such that the path $t \mapsto \beta((1 - t)D_0 + tD_1)$ is a path within the space $\mathcal{F}_{sa}^*$. Then we define β -spectral flow of straight-line path as*

$$\text{Sf}_\beta(D_0, D_1) = \text{Sf}(t \in [0, 1] \mapsto \beta((1 - t)D_0 + tD_1)).$$

We say that a topological space S is contractible if S can be continuously deformed to a single point. An important property of $\mathcal{F}^{sa}$ is given by the following proposition:

Proposition 3.2.7. *The spaces $\mathcal{F}_+^{sa}$ and $\mathcal{F}_-^{sa}$ are contractible to I and $-I$, respectively.*

Proof. Let $F \in \mathcal{F}_+^{sa}$. We can then define the homotopy

$$H(s, F) = (1 - s)F + sI, \forall s \in [0, 1].$$

We have that

$$\pi(H(s, F)) = \pi((1 - s)F + sI) = (1 - s)\pi(F) + s\pi(I) \geq 0, \forall s \in [0, 1],$$

since $(1 - s)\pi(F) + s\pi(I)$ is a convex linear combination, which means that $H(s, F) \in \mathcal{F}_+^{sa}, \forall s \in [0, 1]$ and in particular $H(1, F) = I$, which means that the entire space $\mathcal{F}_+^{sa}$ is contractible to I . One can prove that $\mathcal{F}_-^{sa}$ is contractible to $-I$ by similar argument. $\square$

We say that a path is a loop if the start and end points are the same. We say that a space is simply connected if every loop can be homotopically contracted to a point. We say that a space is path-connected if every pair of points can be connected by a path.

Proposition 3.2.8. *Every contractible space is simply connected and path connected.*

Proof. Let S be a contractible space.

Let F_t be a loop in S . For every point $F \in F_t$ we can homotopically contract F_t to a F , so S must also be simply connected.

Let $F_0, F_1 \in S$ be two points. Since S is contractible, we can homotopically contract these to the same point, which can then be continuously reversed and we can draw a path between F_0 and F_1 during the reversal of the process. $\square$

We have the following proposition:

Proposition 3.2.9. *For every closed loop in $F_t \in \mathcal{F}_\pm^{sa}$ we have that*

$$\text{Sf}(t \in [0, 1] \mapsto F_t) = 0.$$

Proof. Let F_t be a loop in $\mathcal{F}_\pm^{sa}$ with a fixed endpoint F_0 . Since F_t is a loop, the endpoint is arbitrary, so we can choose to keep F_0 as a fixed endpoint while homotopically deforming F_t which does not cause spectral flow. Since $\mathcal{F}_\pm^{sa}$ is contractible, $\mathcal{F}_\pm^{sa}$ is also simply connected and we can contract F_t to the constant loop F_0 and we get

$$\text{Sf}(t \in [0, 1] \mapsto F_t) = \text{Sf}(t \in [0, 1] \mapsto F_0) = 0,$$

for every loop $F_t \in \mathcal{F}_\pm^{sa}$. $\square$

For a matrix A we define $\text{Sig}(A) = n_+ - n_-$ where n_+ is the number of positive eigenvalues and n_- is the number of negative eigenvalues (counting multiplicities in both cases). We have the following proposition:

Proposition 3.2.10. *Let A and B be two self-adjoint, invertible matrices of same size. Then*

$$\text{Sf}(A, B) = \frac{1}{2}(\text{Sig}(B) - \text{Sig}(A)). \quad (5)$$

Proof. A matrix is a finite-dimensional operator, so a matrix is always a compact operator. Clearly, the difference $A - B$ is compact. For a path $t \rightarrow F_t$ we have that for every eigenvalue that passes 0 we have change in spectral flow as $\Delta \text{Sf}(F_t) = \pm 1$ which corresponds to a change of signature: $\Delta \text{Sig}(F_t) = (n_+ \pm 1) - (n_- \pm 1) = \text{Sig}(F_t) \pm 2$. Since A, B are self-adjoint (also known as Hermitian), they are both diagonalizable and so the geometric and algebraic multiplicities of each eigenvalue coincide. So it is clear that for every eigenvalue passing 0 we have $2\Delta \text{Sf}(F_t) = \Delta \text{Sig}(F_t)$ and since we are looking at the entire path we get (5) as desired. $\square$

Proposition 3.2.11. *Let $F_0, F_1 \in \mathcal{F}_{sa}^*$ with $F_0^2 - I, F_1^2 - I$ and $F_1 - F_0$ compact. Then*

$$\text{Sf}(F_0, F_1) = \text{Sf}(F_0|F_0|^{-1}, F_1|F_1|^{-1}).$$

Proof. Let F_t be the concatenation of the paths $t \in [0, 1] \mapsto F_0|F_0|^{t-1}$, $t \in [0, 1] \mapsto (1-t)F_0 + tF_1$ and $t \in [0, 1] \mapsto F_1|F_1|^{-t}$. We have that

$$\text{Sf}(t \in [0, 1] \mapsto F_0|F_0|^{t-1}) = \text{Sf}(t \in [0, 1] \mapsto F_1|F_1|^{-t}) = 0,$$

since no eigenvalues cross 0 for these paths. By concatenation, we have then

$$\text{Sf}(t \in [0, 1] \mapsto F_t) = \text{Sf}(t \in [0, 1] \mapsto (1-t)F_0 + tF_1) = \text{Sf}(F_0, F_1).$$

We now want to show by Proposition 3.2.3 that we homotopically deform the path F_t to $F'_t = (1-t)F_0|F_0|^{-1} + tF_1|F_1|^{-1}$. Since F_t and F'_t have the same endpoints, by Proposition 3.2.3 we have

$$\text{Sf}(F_t) = \text{Sf}(F'_t)$$

if we can show that $\pi(F_t)$ is constant.

Since $F_0^2 - I$ is compact we have

$$\pi(F_0^2 - I) = \pi(F_0)^2 - \pi(I) = 0,$$

which shows that $|\pi(F_0)| = \pi(|F_0|) = \pi(I)$, and by similar argument one can show that $\pi(|F_1|) = \pi(I)$. It directly follows that $\pi(F_0|F_0|^{t-1}) = \pi(F_0)|\pi(F_0)|^{t-1} = \pi(F_0)$ and $\pi(F_1|F_1|^{-t}) = \pi(F_1)|\pi(F_1)|^{-t} = \pi(F_1)$.

Since $F_0 - F_1 \in \mathcal{K}$ we have

$$\pi((1-t)F_0 + tF_1) = \pi(F_0) + t\pi(F_1 - F_0) = \pi(F_0) = \pi(F_1).$$

It is clear then that $\pi(F_t) = \pi(F_0) = \pi(F_1)$ for the entire path, and applying Proposition 3.2.3 completes the proof. $\square$

Note: For Proposition 3.2.11, it is not strictly necessary for $F_0^2 - I, F_1^2 - I \in \mathcal{K}$, but rather the assumption helps to simplify the proof. What is necessary is to show that the paths F_t, F'_t are valid paths in $\mathcal{F}_{sa}^*$.

4 Spectral Localizers for Index Theorems

We have arrived at the formulation and proof of the main result of this thesis. Essentially, the result is that for certain numerical parameters, the index of a so-called Toeplitz operator can be calculated as the signature of a finite matrix known as the spectral localizer. The main result is based on the results and proof in [14], but here we offer a slightly different proof of the constancy of the signature and provide a different numerical estimate.

4.1 Definitions and main result

Definition 4.1.1. *Let A be an invertible bounded operator on a separable Hilbert space $\mathcal{H}$. An unbounded odd Fredholm module is a self-adjoint invertible (Dirac) operator D with compact resolvent such that $AD(D) \subset \mathcal{D}(D)$ and $[A, D]: AD(D) \rightarrow \mathcal{H}$ extends to a bounded operator.*

Definition 4.1.2. The spectral localizer $L_\kappa: \mathcal{D}(D) \oplus \mathcal{D}(D) \rightarrow \mathcal{H} \oplus \mathcal{H}$ is a self-adjoint operator defined by

$$L_\kappa = \begin{pmatrix} \kappa D & A \\ A^* & -\kappa D \end{pmatrix}. \quad (6)$$

An important fact is that the spectral localizer has compact resolvent and is thus Fredholm with $\text{ind}(L_\kappa) = 0$:

Proposition 4.1.3. The spectral localizer $L_\kappa: \mathcal{D}(D) \oplus \mathcal{D}(D) \rightarrow \mathcal{H} \oplus \mathcal{H}$ is a self-adjoint operator with compact resolvent. Furthermore, L_κ is Fredholm with index $\text{ind}(L_\kappa) = 0$.

Proof. The spectral localizer L_κ is self-adjoint since L_κ is clearly symmetric and $\mathcal{D}(L_\kappa^*) = \mathcal{D}(D^*) \oplus \mathcal{D}(D^*) = \mathcal{D}(D) \oplus \mathcal{D}(D) = \mathcal{D}(L_\kappa)$ since D is self-adjoint. Furthermore,

$$\kappa D' = \begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix}$$

is self-adjoint and has compact resolvent for $\kappa > 0$, and the operator

$$H = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

is bounded. Then by Proposition 2.5.8, the spectral localizer $L_\kappa = \kappa D' + H$ has a compact resolvent since $\kappa D'$ has a compact resolvent and H is bounded.

Since L_κ is self-adjoint and has a compact resolvent, by Proposition 2.7.11 it must be that L_κ is Fredholm with $\text{ind}(L_\kappa) = 0$. $\square$

Definition 4.1.4. The Toeplitz operator T_A is defined by

$$T_A = \Pi A \Pi + (I - \Pi), \quad (7)$$

where $\Pi = \chi_{[0, \infty)}(D)$ and χ is the indicator function.

Lemma 4.1.5. Let D be a self-adjoint, invertible operator with compact resolvent and let A be a bounded, invertible operator such that $A\mathcal{D}(D) \subset \mathcal{D}(D)$ and $[A, D]: A\mathcal{D}(D) \rightarrow \mathcal{H}$ extends to a bounded operator. Then the operator $[D|D|^{-1}, A]$ is compact. [11, pp. 462]

Proposition 4.1.6. The Toeplitz operator T_A is bounded and Fredholm.

Proof. Since A is bounded and Π is a projection, then $\Pi A \Pi$ and $(I - \Pi)$ are bounded. Since the sum of two bounded operators is bounded then $T_A = \Pi A \Pi + (I - \Pi)$ is bounded.

For the second part of the proof first note that $[\Pi, A] = [\frac{1}{2}(D|D|^{-1} + I), A] = \frac{1}{2}[D|D|^{-1}, A] \in \mathcal{K}$ by Lemma 4.1.5 since $[I, A] = 0$.

$$T_A T_{A^{-1}} = \Pi A \Pi A^{-1} \Pi + (I - \Pi) = \Pi(\Pi A - [\Pi, A])A^{-1} \Pi + (I - \Pi) = I - \Pi[\Pi, A]A^{-1} \Pi,$$

and similarly

$$T_{A^{-1}} T_A = I - \Pi A^{-1} [\Pi, A] \Pi$$

since $\Pi^2 = \Pi$, $(I - \Pi)^2 = (I - \Pi)$ and $(I - \Pi)\Pi = \Pi(I - \Pi) = 0$. Since $[\Pi, A] \in \mathcal{K}$ and $\Pi, A \in \mathcal{B}$, we have $\Pi[\Pi, A]A^{-1} \Pi \in \mathcal{K}$ and $\Pi A^{-1} [\Pi, A] \Pi \in \mathcal{K}$. Therefore $\pi(T_A T_{A^{-1}}) = \pi(T_{A^{-1}} T_A) = \pi(I)$ which means that $\pi(T_A)$ is invertible with inverse $\pi(T_{A^{-1}})$ and by Theorem 2.7.5 T_A is Fredholm. $\square$

Proposition 4.1.7. *The index of the Toeplitz operator T_A is*

$$\text{ind}(T_A) = \text{ind}(\Pi U \Pi + I - \Pi) = \text{Sf}(U^*(2\Pi - I)U, 2\Pi - I),$$

where $U = A|A|^{-1}$ is the polar of A . [14, pp. 6]

Denote the subspace

$$\mathcal{H}_\rho = \chi_{[-\rho, \rho]}(D)\mathcal{H},$$

Since D has a compact resolvent on a separable Hilbert space, each $\mathcal{H}_\rho$ is just the linear span of a finite set of eigenvectors and is thus finite-dimensional. Let B be a set of eigenvectors to D which form an ONB of $\mathcal{H}$. Since D has a countable set eigenvalues λ with finite multiplicity we can order them as $B = \{e_\lambda \mid De_\lambda = \lambda e_\lambda\}$, i.e. the subscript of e_λ refers to its eigenvalue. Now let $\pi_\rho: \mathcal{H} \rightarrow \mathcal{H}_\rho$ be a partial isometry defined by

$$\pi_\rho e_\lambda = \begin{cases} e_\lambda, & |\lambda| \leq \rho, \\ 0, & |\lambda| > \rho. \end{cases}$$

One can check that $I_\rho = \pi_\rho \pi_\rho^*: \mathcal{H}_\rho \rightarrow \mathcal{H}_\rho$ is the identity on $\mathcal{H}_\rho$ while $\pi_\rho^* \pi_\rho: \mathcal{H} \rightarrow \mathcal{H}$ is the projection onto the subspace $\mathcal{H}_\rho \subset \mathcal{H}$. For an arbitrary operator T we will also write $T_\rho = \pi_\rho T \pi_\rho^*$.

Ex. Let $B = \{e_{-3/2}, e_{1/2}, e_{3/2}\}$ be a basis and set $\mathcal{H} = L(B)$. Define $D: \mathcal{H} \rightarrow \mathcal{H}$ as the operator $De_n = ne_n$.

Let $\rho = 1$. We have then

$$\pi_\rho = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad \pi_\rho^* = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \pi_\rho \pi_\rho^* = (1), \quad \pi_\rho^* \pi_\rho = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and we also have that

$$D = \begin{pmatrix} -\frac{3}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}, \quad D_\rho = \pi_\rho D \pi_\rho^* = \begin{pmatrix} \frac{1}{2} \end{pmatrix}.$$

For an invertible operator A we define the spectral gap $g = \|A^{-1}\|^{-1} > 0$.

We are now ready for the main result of this thesis:

Theorem 4.1.8 (Spectral Localizers). *Let A be an invertible bounded operator on a separable Hilbert space and let $g = \|A^{-1}\|^{-1} > 0$ be the spectral gap of A . Let D specify an odd Fredholm module for A . Set*

$$\kappa_0 = \frac{g^2}{2\|[A, D]\|}.$$

Suppose that κ and ρ are such that

$$2g\kappa_0^{-1} \leq 2g\kappa^{-1} < \rho. \tag{8}$$

Then the matrix $L_{\kappa, \rho} = (L_\kappa)_\rho = \pi_\rho L_\kappa \pi_\rho^*: \mathcal{H}_\rho \oplus \mathcal{H}_\rho \rightarrow \mathcal{H}_\rho \oplus \mathcal{H}_\rho$ satisfies the bound

$$L_{\kappa, \rho}^2 \geq \frac{1}{2}g^2 I_\rho. \tag{9}$$

In particular, $L_{\kappa, \rho}$ is invertible and thus has a well-defined signature $\text{Sig}(L_{\kappa, \rho})$. Furthermore,

$$\text{ind}(\Pi A \Pi + (I - \Pi)) = \frac{1}{2} \text{Sig}(L_{\kappa, \rho}). \tag{10}$$

In the following subsections we will go over the proof of Theorem 4.1.8. In subsection 4.2 we will prove that if we have a pair κ, ρ satisfying (8) then $L_{\kappa, \rho}$ is invertible and that $\text{Sig}(L_{\kappa, \rho})$ is well-defined. In subsection 4.3 we will prove that for some pair κ, ρ satisfying (8) and assuming $L_{\kappa, \rho}$ is invertible, then we have (10).

4.2 Proof of constancy of the signature

The purpose of this section is twofold. Firstly, we will show for a pair κ, ρ satisfying (8) then $L_{\kappa, \rho}$ is invertible. Secondly, we will show that $\text{Sig}(L_{\kappa, \rho})$ is well-defined, i.e. for two pairs (κ, ρ) and (κ', ρ') which both satisfy (8), $\text{Sig}(L_{\kappa, \rho}) = \text{Sig}(L_{\kappa', \rho'})$. Note that logically speaking it is not required to prove the constancy of the signature, but since we will be reusing a lot of the proofs in the next section it can be nice to show it anyway.

We start off with a simple lemma:

Lemma 4.2.1. *Let $\mathcal{H}$ be a Hilbert space and let $\pi: \mathcal{H} \rightarrow \mathcal{S}$ be a partial isometry onto $\mathcal{S} \subset \mathcal{H}$. Let $T: \mathcal{H} \rightarrow \mathcal{H}$ be a self-adjoint operator. If*

$$T^2 \geq \epsilon$$

for some $\epsilon > 0$, then

$$\pi T^2 \pi^* \geq \epsilon \pi I \pi^*.$$

Proof. We have for some $\epsilon > 0$:

$$\langle \pi T^2 \pi^* x, x \rangle = \langle T \pi^* x, T \pi^* x \rangle = \|T \pi^* x\|^2 \geq \epsilon \|\pi^* x\|^2 = \epsilon \langle \pi^* x, \pi^* x \rangle = \epsilon \langle \pi \pi^* x, x \rangle. \quad \square$$

Proposition 4.2.2. *Let κ, ρ be a pair satisfying (8). Then $L_{\kappa, \rho}$ satisfies the bound*

$$L_{\kappa, \rho}^2 \geq \frac{1}{2} g^2 I_\rho.$$

In particular, $L_{\kappa, \rho}$ is invertible.

Proof.

$$L_{\kappa, \rho}^2 = \kappa^2 (D'_\rho)^2 + H_\rho^2 + \kappa [D' H + H D']_\rho.$$

We have that

$$\kappa^2 (D')^2 = \kappa^2 \begin{pmatrix} D^2 & 0 \\ 0 & D^2 \end{pmatrix} \geq \mu$$

for some $\mu > 0$ since D is self-adjoint and invertible. Note that H^2 is self-adjoint and we have for $z = (x, y) \in \mathcal{H} \oplus \mathcal{H}$:

$$\begin{aligned} \langle H^2 z, z \rangle &= \langle H z, H z \rangle = \left\langle \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \|A^* x\|^2 + \|A y\|^2 \\ &\geq g^2 (\|x\|^2 + \|y\|^2), \end{aligned}$$

where the last inequality follows from Lemma 2.4.16. We have then that

$$L_{\kappa, \rho}^2 \geq g^2 I_\rho + \kappa [D' H + H D']_\rho.$$

The last term can be seen as an error term. We have the error term

$$D' H + H D' = \begin{pmatrix} 0 & [D, A] \\ [D, A]^* & 0 \end{pmatrix}$$

which one can easily show has an upper bound $\|D'H + HD'\| \leq \|[D, A]\| = \|[A, D]\|$.

In conclusion we have from Theorem 4.1.8

$$L_{\kappa, \rho}^2 \geq (g^2 - \kappa \|[A, D]\|) I_\rho \geq \frac{1}{2} g^2 I_\rho,$$

and invertibility of $L_{\kappa, \rho}$ follows directly from Proposition 2.4.17 with $\epsilon = \frac{1}{2} g^2 > 0$. $\square$

Corollary 4.2.3. *The spectral localizer L_κ is invertible.*

Proof. From Proposition 4.2.2 we know that $L_{\kappa, \rho}$ is invertible. The statement follows directly by letting $\rho \rightarrow \infty$. $\square$

We have now shown that $L_{\kappa, \rho}$ is invertible which was the first part of this proof. The second part is to show that

$$\text{Sig}(L_{\kappa, \rho}) = \text{Sig}(L_{\kappa', \rho'}).$$

for two pairs κ, ρ and κ', ρ' which both fulfill (8). We will treat $\kappa \rightarrow \kappa'$ and $\rho \rightarrow \rho'$ as two separate cases. For the first case we have the following proposition:

Proposition 4.2.4. *Let (κ, ρ) and (κ', ρ) be two pairs of parameters such that they fulfill (8) and $\rho < \infty$. We then have*

$$\text{Sig}(L_{\kappa, \rho}) = \text{Sig}(L_{\kappa', \rho}).$$

Proof. Since the difference $L_{\kappa, \rho} - L_{\kappa', \rho}$ is compact we can write

$$\text{Sig}(L_{\kappa', \rho}) - \text{Sig}(L_{\kappa, \rho}) = 2 \text{Sf}(L_{\kappa, \rho}, L_{\kappa', \rho}).$$

Here we have

$$\text{Sf}(L_{\kappa, \rho}, L_{\kappa', \rho}) = \text{Sf} \left(t \in [0, 1] \mapsto F_t = \begin{pmatrix} [\kappa + t(\kappa' - \kappa)] D_\rho & A_\rho \\ A_\rho^* & -[\kappa + t(\kappa' - \kappa)] D_\rho \end{pmatrix} \right).$$

Without loss of generality, assume that $\kappa \leq \kappa'$. It is clear that $\kappa \leq \kappa + t(\kappa' - \kappa) \leq \kappa'$ for all $t \in [0, 1]$. Since both κ, κ' fulfill (8), so must $\kappa + t(\kappa' - \kappa)$ for all $t \in [0, 1]$. By Proposition 4.2.2 we have that F_t is invertible for all $t \in [0, 1]$ so by Proposition 3.2.1.v. we have that $\text{Sf}(L_{\kappa, \rho}, L_{\kappa', \rho}) = 0$ and thus $\text{Sig}(L_{\kappa, \rho}) = \text{Sig}(L_{\kappa', \rho})$. $\square$

For the case $\rho \rightarrow \rho'$ we would like to do something similar to the last proof: If the difference $L_{\kappa, \rho} - L_{\kappa, \rho'}$ is compact then we write

$$\text{Sig}(L_{\kappa', \rho'}) - \text{Sig}(L_{\kappa, \rho}) = 2 \text{Sf}(L_{\kappa, \rho}, L_{\kappa', \rho'}).$$

However, for certain ρ', ρ , the matrices $L_{\kappa, \rho}, L_{\kappa', \rho'}$ may be of different sizes, so their difference might not even be well-defined. We can resolve this problem by doing the following: Without loss of generality let $\rho' \geq \rho$. Let $\pi_{\rho', \rho} = \pi_{\rho'} \ominus \pi_\rho$ be the partial isometry onto $\mathcal{H}_{\rho', \rho} = \mathcal{H}_{\rho'} \ominus \mathcal{H}_\rho$ defined by

$$\pi_{\rho', \rho} e_\lambda = \begin{cases} e_\lambda, & \rho < |\lambda| \leq \rho', \\ 0, & \text{otherwise.} \end{cases}$$

We can thus "pad" $L_{\kappa, \rho}$ with an identity matrix of appropriate size $I_{\rho', \rho}$ and create $L_{\kappa, \rho} \oplus I_{\rho', \rho}$. Then $L_{\kappa, \rho} \oplus I_{\rho', \rho} - L_{\kappa, \rho'}$ is compact and we can use a spectral flow argument. First however, we will introduce a couple of useful lemmas.

Lemma 4.2.5. *Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{m \times m}$ be two self-adjoint, invertible, finite matrices. We have then*

- i. $\text{Sig}(A \oplus B) = \text{Sig}(A) + \text{Sig}(B)$,
- ii. $\text{Sig}(\alpha A) = \text{sgn}(\alpha) \text{Sig}(A), \forall \alpha \neq 0, \alpha \in \mathbb{R}$.

Proof. i. Since $A \oplus B, A, B$ are all self-adjoint and invertible, the signature of each is well-defined. For a block matrix, we have that

$$\det \begin{pmatrix} A - \lambda I_n & 0 \\ 0 & B - \lambda I_m \end{pmatrix} = \det(A - \lambda I_n) \det(B - \lambda I_m) = p_A(\lambda) p_B(\lambda),$$

so it is clear that $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$. For every $\lambda \in \sigma(A \oplus B)$, we have $\ker(A \oplus B - \lambda I_{n+m}) = \ker(A - \lambda I_n) \oplus \ker(B - \lambda I_m)$ so

$$\dim \ker(A \oplus B - \lambda I_{n+m}) = \dim \ker(A - \lambda I_n) + \dim \ker(B - \lambda I_m).$$

- ii. It is clear that for $\alpha \neq 0$, A is still invertible and self-adjoint. We have

$$\det(\alpha A - \lambda I_n) = \alpha^n \det(A - \frac{\lambda}{\alpha} I_n) = \alpha^n p_A(\frac{\lambda}{\alpha}).$$

There are two cases: If $\alpha > 0$ then all the positive roots of p_A remain positive $\text{Sig}(\alpha A) = \text{Sig}(A)$. If $\alpha < 0$ then the roots of p_A change sign so $\text{Sig}(\alpha A) = -\text{Sig}(A)$. Together we get the desired result. $\square$

Lemma 4.2.6. *Let S be a bounded operator on a Hilbert space $\mathcal{H}$. Then*

$$\pm(\langle Sx, y \rangle + \langle S^*y, x \rangle) \geq -\|S\|(\|x\|^2 + \|y\|^2), \forall x, y \in \mathcal{H}.$$

Proof.

$$\pm(\langle Sx, y \rangle + \langle y, Sx \rangle) = \pm(\langle Sx, y \rangle + \overline{\langle Sx, y \rangle}) = \pm 2\Re \langle Sx, y \rangle \geq -2|\langle Sx, y \rangle|.$$

Since S is bounded we have $-2|\langle Sx, y \rangle| \geq -2\|Sx\| \cdot \|y\| \geq -2\|S\| \cdot \|x\| \cdot \|y\|$ and by the AM-GM inequality we have that $-2\|x\| \cdot \|y\| \geq -(\|x\|^2 + \|y\|^2)$. All in all we get then that

$$\pm(\langle Sx, y \rangle + \langle S^*y, x \rangle) \geq -\|S\|(\|x\|^2 + \|y\|^2). \quad \square$$

Proposition 4.2.7. *Let $\rho' > \rho$ and $\pi_{\rho', \rho} = \pi_{\rho'} \ominus \pi_{\rho}$ be the surjective partial isometry onto $\mathcal{H}_{\rho', \rho} = \mathcal{H}_{\rho'} \ominus \mathcal{H}_{\rho}$. Assume that $\sigma(D_{\rho', \rho}) \neq 0$, i.e. there exists at least one $\lambda \in \sigma(D)$ such that $\rho < |\lambda| \leq \rho'$. Let $Q_{\kappa, \rho', \rho}(s): \mathcal{H}_{\rho', \rho} \oplus \mathcal{H}_{\rho', \rho} \rightarrow \mathcal{H}_{\rho', \rho} \oplus \mathcal{H}_{\rho', \rho}$ be the operator given by*

$$Q_{\kappa, \rho', \rho}(s) = (1-s)\kappa D'_{\rho', \rho} + sL_{\kappa, \rho', \rho},$$

for $s \in [0, 1]$ (note that $Q_{\kappa, \rho', \rho}(0) = \kappa D'_{\rho', \rho}$ and $Q_{\kappa, \rho', \rho}(1) = L_{\kappa, \rho', \rho}$). Then the following inequality is true:

$$\begin{aligned} Q_{\kappa, \rho', \rho}^2(s) &= \begin{pmatrix} \kappa^2 D_{\rho', \rho}^2 + s^2 A_{\rho', \rho} A_{\rho', \rho}^* & s\kappa [D_{\rho', \rho}, A_{\rho', \rho}] \\ s\kappa [D_{\rho', \rho}, A_{\rho', \rho}]^* & \kappa^2 D_{\rho', \rho}^2 + s^2 A_{\rho', \rho}^* A_{\rho', \rho} \end{pmatrix} \\ &\geq (\kappa^2 \rho^2 + s^2 g^2 - s\kappa \|[D, A]\|) I_{\rho', \rho}. \end{aligned} \quad (11)$$

Furthermore, if (8) holds then $Q_{\kappa, \rho', \rho}(s)$ is invertible for all $s \in [0, 1]$, and in particular $Q_{\kappa, \rho', \rho}^2(1) = L_{\kappa, \rho', \rho}^2 \geq \frac{9}{2} g^2 I_{\rho', \rho}$ for $g > 0$.

Proof. Let us write $z = (x, y) \in \mathcal{H}_{\rho', \rho} \oplus \mathcal{H}_{\rho', \rho}$ and calculate

$$\begin{aligned} \langle Q_{\kappa, \rho', \rho}(s)z, z \rangle &= \langle (\kappa^2 D_{\rho', \rho}^2 + s^2 A_{\rho', \rho}^* A_{\rho', \rho})x, x \rangle + s\kappa \langle [D_{\rho', \rho}, A_{\rho', \rho}]x, y \rangle \\ &\quad + \langle (\kappa^2 D_{\rho', \rho}^2 + s^2 A_{\rho', \rho}^* A_{\rho', \rho})y, y \rangle + s\kappa \langle [D_{\rho', \rho}, A_{\rho', \rho}]^* y, x \rangle. \end{aligned}$$

Starting with the term

$$\langle (\kappa^2 D_{\rho', \rho}^2 + A_{\rho', \rho}^* A_{\rho', \rho})x, x \rangle = \kappa^2 \langle D_{\rho', \rho}^2 x, x \rangle + \langle A_{\rho', \rho}^* A_{\rho', \rho} x, x \rangle,$$

we can use Theorem 2.5.3 for the first term. Since we have a countable set of eigenvalues, let λ_n be an eigenvalue, and let $S \subset \mathbb{N}$ be the set of lower indices such that $n \in S \Leftrightarrow \rho < \lambda_n \leq \rho'$. For every $n \in S$, let P_n denote the orthogonal projection onto the eigenspace of λ_n and let e_n be a normalized eigenvector associated to λ_n .

$$\begin{aligned} \kappa^2 \langle D_{\rho', \rho}^2 x, x \rangle &= \kappa^2 \langle D_{\rho', \rho} x, D_{\rho', \rho} x \rangle = \kappa^2 \langle \sum_{n \in S} \lambda_n P_n x, \sum_{m \in S} \lambda_m P_m x \rangle = \\ &= \kappa^2 \sum_{n \in S} \sum_{m \in S} \lambda_n \lambda_m x_n \bar{x}_m \langle e_n, e_m \rangle = \kappa^2 \sum_{n \in S} \lambda_n^2 |x_n|^2 = \kappa^2 \lambda^2 \|x\|_{\rho', \rho}^2 \geq \kappa^2 \rho^2 \|x\|_{\rho', \rho}^2, \end{aligned}$$

where the last inequality follows from the definition of $D_{\rho', \rho}$. For the second term notice that

$$\langle A_{\rho, \rho'} A_{\rho, \rho'}^* x, x \rangle = \langle A_{\rho, \rho'}^* x, A_{\rho, \rho'} x \rangle = \|A_{\rho, \rho'} x\|^2 \geq g^2 \|x\|_{\rho, \rho'}^2.$$

Thus we have $\langle (\kappa^2 D_{\rho', \rho}^2 + A_{\rho', \rho}^* A_{\rho', \rho})x, x \rangle \geq (\kappa^2 \rho^2 + g^2) \|\pi \pi^* x\|^2$ and $\langle (\kappa^2 D_{\rho', \rho}^2 + A_{\rho', \rho}^* A_{\rho', \rho})y, y \rangle \geq (\kappa^2 \rho^2 + g^2) \|\pi \pi^* y\|^2$ by the same argument.

For the mixed terms we have

$$\kappa \langle [D_{\rho', \rho}, A_{\rho', \rho}]x, y \rangle + \kappa \langle [D_{\rho', \rho}, A_{\rho', \rho}]^* y, x \rangle \geq -\kappa \| [A, D] \| (\|\pi \pi^* x\|^2 + \|\pi \pi^* y\|^2).$$

Putting all of it together, we have shown that

$$\begin{aligned} \langle Q_{\kappa, \rho', \rho}^2(s)z, z \rangle &\geq (\kappa^2 \rho^2 + s^2 g^2 - s\kappa \| [D_{\rho', \rho}, A_{\rho', \rho}] \|) \|\pi \pi^* x\|^2 \\ &\quad + (\kappa^2 \rho^2 + s^2 g^2 - s\kappa \| [D_{\rho', \rho}, A_{\rho', \rho}] \|) \|\pi \pi^* y\|^2 \\ &= \langle (\kappa^2 \rho^2 + s^2 g^2 - s\kappa \| [D, A] \|) I_{\rho', \rho} z, z \rangle. \end{aligned}$$

We now would like to show that if (8) holds then $Q_{\kappa, \rho', \rho}^2(s)$ is invertible for all $s \in [0, 1]$. From (8) we have that $\kappa^2 \rho^2 > 4g^2$ and $\kappa \| [D, A] \| \leq \frac{1}{2} g^2$ so clearly $\kappa^2 \rho^2 + s^2 g^2 - s\kappa \| [D, A] \| > \kappa^2 \rho^2 - \kappa \| [D, A] \| \geq 4g^2 - \frac{1}{2} g^2 = \frac{7}{2} g^2$ where $g > 0$. In particular, we have the better estimate $Q_{\kappa, \rho', \rho}^2(1) = L_{\kappa, \rho', \rho}^2 \geq 4g^2 + g^2 - \frac{1}{2} g^2 = \frac{9}{2} g^2$. Then by Proposition 2.4.17, $L_{\kappa, \rho', \rho}^2$ is invertible. $\square$

We have the following Corollary from Proposition 4.2.7:

Corollary 4.2.8.

$$\text{Sf}_\beta(L_{\kappa, \rho', \rho}, \kappa D'_{\rho', \rho}) = 0.$$

Proof. Note that

$$\text{Sf}(L_{\kappa, \rho', \rho}, \kappa D'_{\rho', \rho}) = \text{Sf}(t \in [0, 1] \mapsto Q_{\kappa, \rho', \rho}((1 - t))).$$

Since $L_{\kappa, \rho', \rho}(1 - t)$ is invertible for all $t \in [0, 1]$ by Proposition 4.2.7, by Proposition 3.2.1.v we have $\text{Sf}(t \in [0, 1] \mapsto Q_{\kappa, \rho', \rho}((1 - t))) = 0$. $\square$

Proposition 4.2.9 (Neumann series). *Let T be a bounded operator on a Hilbert space $\mathcal{H}$. If $\|T\| < 1$ then $I - T$ has a unique bounded inverse given by*

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

[10, pp. 11]

Lemma 4.2.10. *Let $\rho' > \rho$ and $\pi_{\rho',\rho} = \pi_{\rho'} \ominus \pi_{\rho}$ be the surjective partial isometry onto $\mathcal{H}_{\rho',\rho} = \mathcal{H}_{\rho'} \ominus \mathcal{H}_{\rho}$. Assume that $\sigma(D_{\rho',\rho}) \neq 0$, i.e. there exists at least one $\lambda \in \sigma(D)$ such that $\rho < |\lambda| \leq \rho'$. Then*

$$\text{Sf}_{\beta}(L_{\kappa,\rho'}, L_{\kappa,\rho} \oplus L_{\kappa,\rho',\rho}) = 0.$$

Proof. The operator $D_{\rho'}$ can be diagonalized into $D = D_{\rho} \oplus D_{\rho',\rho}$ simply because the partial isometries used are defined from the eigenvalues of D . We cannot diagonalize $L_{\kappa,\rho}$ though, but we can homotopically turn it into something diagonal. For an arbitrary operator A we can write

$$A = A_{\rho} + A_{\rho',\rho} + \pi_{\rho} A \pi_{\rho',\rho}^* + \pi_{\rho',\rho} A \pi_{\rho}^*.$$

So we can rewrite $L_{\kappa,\rho'}$ as

$$L_{\kappa,\rho'}(s) = L_{\kappa,\rho} \oplus L_{\kappa,\rho',\rho} + s \begin{pmatrix} 0 & \pi_{\rho} H \pi_{\rho',\rho}^* \\ \pi_{\rho',\rho} H \pi_{\rho}^* & 0 \end{pmatrix},$$

for some $s \in [0, 1]$. Note that $L_{\kappa,\rho'}(1) = L_{\kappa,\rho'}$ and $L_{\kappa,\rho'}(0) = L_{\kappa,\rho} \oplus L_{\kappa,\rho',\rho}$. From Proposition 4.2.2 we know that $L_{\kappa,\rho}$ is invertible and from Proposition 4.2.7 we know that $L_{\kappa,\rho',\rho}$ is invertible. We can use polar decomposition and rewrite:

$$L_{\kappa,\rho'}(s) = |L_{\kappa,\rho} \oplus L_{\kappa,\rho',\rho}|^{\frac{1}{2}} (S + sK) |L_{\kappa,\rho} \oplus L_{\kappa,\rho',\rho}|^{\frac{1}{2}},$$

where

$$K = \begin{pmatrix} 0 & |L_{\kappa,\rho}|^{-\frac{1}{2}} \pi_{\rho} H \pi_{\rho',\rho}^* |L_{\kappa,\rho',\rho}|^{-\frac{1}{2}} \\ |L_{\kappa,\rho',\rho}|^{-\frac{1}{2}} \pi_{\rho',\rho} H \pi_{\rho}^* |L_{\kappa,\rho}|^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

Firstly we rewrite $S + sK = S(I - (-sS^{-1}K))$ and by Proposition 4.2.9 we have that $S + sK$ is invertible if we show that $\|S^{-1}sK\| = s\|S^*K\| \leq \|K\| < 1$. Note that polar decomposing $A \mapsto A|A|^{-1} = U$ does not cause spectral flow since it does not change the sign of eigenvalues close to 0, so we do that first and we have

$$H = \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix}$$

which has an operator norm $\|H\| = \sqrt{\|UU^*\|} = 1$, and note that $g = \|U^{-1}\|^{-1} = \|U^*\|^{-1} = 1$. In total we have then

$$\begin{aligned} \||L_{\kappa,\rho}|^{-\frac{1}{2}} \pi_{\rho} H \pi_{\rho',\rho}^* |L_{\kappa,\rho',\rho}|^{-\frac{1}{2}}\| &\leq \|L_{\kappa,\rho}\|^{-\frac{1}{2}} \|\pi_{\rho}\| \|H\| \|\pi_{\rho',\rho}^*\| \|L_{\kappa,\rho',\rho}\|^{-\frac{1}{2}} \\ &\leq \frac{2^{1/4}}{\sqrt{g}} \cdot 1 \cdot 1 \cdot 1 \cdot \frac{2^{1/4}}{\sqrt{3g}} = \sqrt{\frac{2}{3}} < 1. \end{aligned}$$

Since $L_{\kappa,\rho'}(s)$ remains invertible for all $s \in [0, 1]$ and we have by Proposition 3.2.1.v that

$$\text{Sf}_{\beta}(L_{\kappa}, L_{\kappa,\rho} \oplus L_{\kappa,\rho',\rho}) = 0. \quad \square$$

We are now ready to handle the case $\rho' \rightarrow \rho$:

Proposition 4.2.11. *Let ρ, ρ' fulfill (8) such that $L_{\kappa, \rho}$ and $L_{\kappa, \rho'}$ are finite-dimensional (self-adjoint) matrices. We then have*

$$\text{Sig}(L_{\kappa, \rho}) = \text{Sig}(L_{\kappa, \rho'}).$$

Proof. If $\rho = \rho'$ then the statement is trivially true. Without loss of generality assume that $\rho' > \rho$. There are two cases to consider:

Case 1: Assume that there does *not* exist a $\lambda \in \sigma(D)$ such that $\rho < |\lambda| \leq \rho'$. In that case we have the same subspace $\mathcal{H}_\rho = \mathcal{H}_{\rho'}$ so $L_{\kappa, \rho} = L_{\kappa, \rho'}$ so the statement is trivially true.

Case 2: Assume that there does exist a $\lambda \in \sigma(D)$ such that $\rho < |\lambda| \leq \rho'$. Then $L_{\kappa, \rho}$ and $L_{\kappa, \rho'}$ have different dimensions. We can do the following trick using Lemma 4.2.5:

$$\text{Sig}(L_{\kappa, \rho}) = \text{Sig}(L_{\kappa, \rho} \oplus I_{\rho', \rho}) - \text{Sig}(I_{\rho', \rho}).$$

The difference $L_{\kappa, \rho} \oplus I_{\rho', \rho} - L_{\kappa, \rho'}$ is compact and thus

$$\begin{aligned} \text{Sig}(L_{\kappa, \rho'}) - \text{Sig}(L_{\kappa, \rho}) &= \text{Sig}(L_{\kappa, \rho'}) - (\text{Sig}(L_{\kappa, \rho} \oplus I_{\rho', \rho}) - \text{Sig}(I_{\rho', \rho})) \\ &= 2 \text{Sf}(L_{\kappa, \rho} \oplus I_{\rho', \rho}, L_{\kappa, \rho'}) + \text{Sig}(I_{\rho', \rho}). \end{aligned}$$

By Lemma 4.2.10 we have that

$$\text{Sf}(L_{\kappa, \rho'}, L_{\kappa, \rho} \oplus L_{\kappa, \rho', \rho}) = 0,$$

and by concatenation and additivity of spectral flow we get

$$\begin{aligned} \text{Sf}(L_{\kappa, \rho} \oplus I_{\rho', \rho}, L_{\kappa, \rho'}) &= \text{Sf}(L_{\kappa, \rho} \oplus I_{\rho', \rho}, L_{\kappa, \rho} \oplus L_{\kappa, \rho', \rho}) \\ &= \text{Sf}(L_{\kappa, \rho}, L_{\kappa, \rho}) + \text{Sf}(I_{\rho', \rho}, L_{\kappa, \rho', \rho}) \\ &= \text{Sf}(I_{\rho', \rho}, L_{\kappa, \rho', \rho}). \end{aligned}$$

By Corollary 4.2.8 and concatenation of spectral flow we have

$$\text{Sf}(I_{\rho', \rho}, L_{\kappa, \rho', \rho}) = \text{Sf}(I_{\rho', \rho}, \kappa D'_{\rho', \rho}) = \frac{1}{2}(\text{Sig}(\kappa D'_{\rho', \rho}) - \text{Sig}(I_{\rho', \rho})).$$

Note that

$$\text{Sig}(\kappa D'_{\rho', \rho}) = \text{Sig}(\kappa D_{\rho', \rho} \oplus -\kappa D_{\rho', \rho}) = \text{Sig}(\kappa D_{\rho', \rho}) - \text{Sig}(\kappa D_{\rho', \rho}) = 0.$$

In total we have then

$$\begin{aligned} \text{Sig}(L_{\kappa, \rho'}) - \text{Sig}(L_{\kappa, \rho}) &= 2 \text{Sf}(L_{\kappa, \rho} \oplus I_{\rho', \rho}, L_{\kappa, \rho'}) + \text{Sig}(I_{\rho', \rho}) \\ &= -\text{Sig}(I_{\rho', \rho}) + \text{Sig}(I_{\rho', \rho}) = 0. \end{aligned}$$

□

4.3 Proof that the index equals the half-signature

The purpose of this section is to show that if for some κ, ρ such that (8) holds, then (10) is true. We start with the fundamental connection from Proposition 4.1.7:

$$\text{ind}(T_A) = \text{ind}(\Pi U \Pi + I - \Pi) = \text{Sf}(U^*(2\Pi - I)U, 2\Pi - I),$$

where $U = A|A|^{-1}$ is the polar of A . Note that $2\Pi - I = \beta(D)|\beta(D)|^{-1}$, which we can deform to $\beta(D)$ without causing spectral flow by the following proposition:

Proposition 4.3.1.

$$\text{Sf}(U^*\beta(D)|\beta(D)|^{-1}U, \beta(D)|\beta(D)|^{-1}) = \text{Sf}_\beta(U^*DU, D).$$

Proof. For this proof we will use Proposition 3.2.11. From the functional calculus we have that $F_0 = \beta(U^*DU) = U^*\beta(D)U$ and $F_1 = \beta(D)$. We have $F_0 = U^*F_1U = F_1 + [U, F_1]U^*$. From Proposition 2.6.5 we know that $[U, F_1] = [U, \beta(D)] \in \mathcal{K}$ and since U^* is clearly bounded we have $[U, F_1]U^* \in \mathcal{K}$ which means that

$$\pi(F_0 - F_1) = 0.$$

Furthermore, $\beta(x)^2 = 1 - (1 + x^2)^{-1}$ and since $(1 + x^2)^{-1} \in C_0(\mathbb{R})$ implies that $(1 + D^2)^{-1} \in \mathcal{K}$ by Proposition 2.5.11 we have

$$\pi(F_1^2 - I) = 0.$$

In turn we have $\pi(F_1^2 - I) = \pi(F_1)^2 - \pi(I) = \pi(F_0)^2 - \pi(I) = \pi(F_0^2 - I) = 0$. In total we have that $F_0 - F_1$, $F_0^2 - I$ and $F_1^2 - I$ are all compact so by Proposition 3.2.11 we have

$$\text{Sf}(U^*\beta(D)|\beta(D)|^{-1}U, \beta(D)|\beta(D)|^{-1}) = \text{Sf}(U^*\beta(D)U, \beta(D)).$$

We would like to use Proposition 3.2.3 to show that

$$\text{Sf}(U^*\beta(D)U) = \text{Sf}_\beta(U^*DU, D).$$

We already know by a previous argument that $\beta(D) - U^*\beta(D)U \in \mathcal{K}$. What remains is to show that for the path $D_t = (1 - t)U^*DU + tD$, we have that $\pi(\beta(D_t)) = \pi(U^*\beta(D)U) = \pi(\beta(D))$ is constant for all $t \in [0, 1]$.

The following fact will be stated without a formal proof: For a self-adjoint, invertible operator D with compact resolvent and a bounded operator A , we have that

$$\beta(D + A) - \beta(D) \in \mathcal{K}. \quad (12)$$

A proof for (12) can be sketched in the following manner: By Proposition 2.5.8 we have that both D and $D + A$ have compact resolvents. Furthermore, one can find a suitable integral that relates to $\beta(D + A) - \beta(D)$, for instance

$$\beta(D + A) - \beta(D) = \frac{1}{\pi} \int_0^\infty (D + A)((D + A)^2 + 1 + \lambda)^{-1} - D(D^2 + 1 + \lambda)^{-1} d\lambda = \frac{1}{\pi} \int_0^\infty f(\lambda) d\lambda,$$

is a uniformly convergent integral, and that each term $f(\lambda) \in \mathcal{K}$. Then by Proposition 2.6.3, the operator $\beta(D + A) - \beta(D) \in \mathcal{K}$.

We have that $D_t = (1 - t)U^*DU + tD = D + (1 - t)[U^*, D]U$. By assumption $[U^*, D] \in \mathcal{B}$, which clearly implies $(1 - t)[U^*, D]U \in \mathcal{B}$. So by (12) we have

$$\pi(\beta(D_t) - \beta(D)) = 0 \Rightarrow \pi(\beta(D_t)) = \pi(\beta(D)) = \pi(U^*\beta(D)U).$$

Since $\pi(D_t)$ is constant in $t \in [0, 1]$, we have by Proposition 3.2.3:

$$\text{Sf}(U^*\beta(D)U) = \text{Sf}_\beta(U^*DU, D). \quad \square$$

We thus have $\text{ind}(T_A) = \text{Sf}_\beta(U^*DU, D) = \text{Sf}_\beta(\kappa U^*DU, \kappa D)$, where the last equality follows since multiplying both operators by $\kappa > 0$ does not change the spectral flow. Clearly

$$\text{Sf}_\beta(-\kappa D, -\kappa D) = 0,$$

so by additivity of spectral flow

$$\begin{aligned}\text{ind}(T_A) &= \text{Sf}_\beta(\kappa U^* DU, \kappa D) = \text{Sf}_\beta(\kappa U^* DU, \kappa D) + \text{Sf}_\beta(-\kappa D, -\kappa D) \\ &= \text{Sf}_\beta(\kappa U^* DU \oplus -\kappa D, \kappa D \oplus -\kappa D).\end{aligned}$$

Here we have that

$$\kappa U^* DU \oplus -\kappa D = \begin{pmatrix} \kappa U^* DU & 0 \\ 0 & -\kappa D \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}^* \begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}.$$

Denote

$$\hat{U} = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix},$$

and note that $\hat{U}\hat{U}^* = \hat{U}^*\hat{U} = I$ so these matrices are also unitary.

The next step is to add a "mass term" $m = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, i.e. we would like to show that

$$\begin{aligned}\text{ind}(T_A) &= \text{Sf}_\beta \left(\hat{U}^* \begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix} \hat{U}, \begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix} \right) \\ &= \text{Sf}_\beta \left(\hat{U}^* \begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix} \hat{U}, \begin{pmatrix} \kappa D & I \\ I & -\kappa D \end{pmatrix} \right)\end{aligned}\tag{13}$$

We start with the following proposition:

Proposition 4.3.2. *For $\kappa > 0$*

$$\text{Sf}_\beta \left(\begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix}, \begin{pmatrix} \kappa D & I \\ I & -\kappa D \end{pmatrix} \right) = 0.$$

Proof. First note that

$$(1-t) \begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix} + t \begin{pmatrix} \kappa D & I \\ I & -\kappa D \end{pmatrix} = \begin{pmatrix} \kappa D & t \\ t & -\kappa D \end{pmatrix} := K.$$

If we can show that K is invertible for every $t \in [0, 1]$ then by proposition 3.2.1 (v) we have proved the statement.

K is self-adjoint and D has a compact resolvent, so we can use the Theorem 2.5.4 to decompose $Dx = \sum_j \lambda_j P_j x$, where each P_j is an orthogonal projection onto a closed subspace of $\mathcal{H}$. Since $\mathcal{R}(D) = \mathcal{H}$ and we have that $\mathcal{H} = \oplus_j \mathcal{R}(P_j)$, each x can be decomposed as $x = \sum_\lambda P_\lambda x = \sum_\lambda x_\lambda$.

$$\begin{aligned}Kz &= \begin{pmatrix} \kappa D & t \\ t & -\kappa D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \bigoplus_j \begin{pmatrix} \kappa \lambda_j P_j & t P_j \\ t P_j & -\kappa \lambda_j P_j \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \bigoplus_j \begin{pmatrix} \kappa \lambda_j & t \\ t & -\kappa \lambda_j \end{pmatrix} \begin{pmatrix} x_j \\ y_j \end{pmatrix} = \bigoplus_j K_j z_j.\end{aligned}$$

Solving for the eigenvalues $\oplus (K_j - \lambda) z_j = 0$ we have

$$\bigoplus_j \begin{pmatrix} \kappa \lambda_j - \lambda & t \\ t & -\kappa \lambda_j - \lambda \end{pmatrix} \begin{pmatrix} x_j \\ y_j \end{pmatrix} = 0.$$

If the elements in each matrix commute, one can use the normal determinant formula, which one can check easily is true for the matrix above and we get

$$\begin{aligned}\det \begin{pmatrix} \kappa\lambda_j - \lambda & t \\ t & -\kappa\lambda_j - \lambda \end{pmatrix} &= (\kappa\lambda_j - \lambda')(-\kappa\lambda_j - \lambda') - t^2 \\ &= -\kappa^2\lambda_j^2 + (\lambda')^2 - t^2 = 0.\end{aligned}$$

So we see then $\lambda = \pm\sqrt{(\kappa\lambda_j)^2 + t^2}$, and the only thing that has happened is an increase in the spectral gap. Since D is invertible, $\lambda_j \neq 0$ for all j , so there is no $t \in [0, 1]$ such that $\lambda = 0$, which means that K is invertible for every $t \in [0, 1]$. $\square$

By proposition 4.3.2 and by concatenation of paths, we have

$$\text{ind}(T_A) = \text{Sf}_\beta \left(\hat{U}^* \begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix} \hat{U}, \begin{pmatrix} \kappa D & I \\ I & -\kappa D \end{pmatrix} \right).$$

We then perform the following steps

$$\begin{aligned}\text{ind}(T_A) &= \text{Sf}_\beta \left(\hat{U}^* \begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix} \hat{U}, \begin{pmatrix} \kappa D & I \\ I & -\kappa D \end{pmatrix} \right) \\ (3.2.1.\text{iii}) &= \text{Sf}_\beta \left(\begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix}, \hat{U} \begin{pmatrix} \kappa D & I \\ I & -\kappa D \end{pmatrix} \hat{U}^* \right) \\ &= \text{Sf}_\beta \left(\begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix}, \begin{pmatrix} \kappa U D U^* & U \\ U^* & -\kappa D \end{pmatrix} \right).\end{aligned}$$

We can see already that it is beginning to resemble the spectral localizer. For the next step in the proof we will use the following Lemma:

Lemma 4.3.3. *For $\kappa > 0$*

$$\text{Sf}_\beta \left(\begin{pmatrix} \kappa U D U^* & U \\ U^* & -\kappa D \end{pmatrix}, \begin{pmatrix} \kappa D & U \\ U^* & -\kappa D \end{pmatrix} \right) = 0.$$

Proof. Note that $U D U^* = D - [D, U]U^*$ so

$$\begin{aligned}& \text{Sf}_\beta \left(\begin{pmatrix} \kappa U D U^* & U \\ U^* & -\kappa D \end{pmatrix}, \begin{pmatrix} \kappa D & U \\ U^* & -\kappa D \end{pmatrix} \right) \\ &= \text{Sf}_\beta \left(t \in [0, 1] \mapsto (1-t) \begin{pmatrix} \kappa(D - [D, U]U^*) & U \\ U^* & -\kappa D \end{pmatrix} + t \begin{pmatrix} \kappa D & U \\ U^* & -\kappa D \end{pmatrix} \right) \\ &= \text{Sf}_\beta \left(t \in [0, 1] \mapsto \begin{pmatrix} \kappa D - (1-t)\kappa[D, U]U^* & U \\ U^* & -\kappa D \end{pmatrix} \right).\end{aligned}$$

Set

$$M = \begin{pmatrix} \kappa D - (1-t)\kappa[D, U]U^* & U \\ U^* & -\kappa D \end{pmatrix}.$$

Since M is self-adjoint we can use Proposition 2.4.17 and prove that M is invertible by showing that $M^2 \geq \epsilon$ for some $\epsilon > 0$. Set $s = (1-t)$ and let $\mu = \kappa[D, U]U^*$

$$M^2 = \begin{pmatrix} (\kappa D - s\mu)^2 + I & \kappa[D, U] - s\mu U \\ \kappa[D, U]^* - sU^*\mu & \kappa^2 D^2 + I \end{pmatrix}$$

For $s \in [0, 1]$ and $z = (x, y)$ we have

$$\begin{aligned}\langle M^2 z, z \rangle &= \langle (\kappa D - \mu)^2 x, x \rangle + \kappa^2 \langle D^2 y, y \rangle + \|x\|^2 + \|y\|^2 \\ &\quad + \kappa \langle [D, U] y, x \rangle + \kappa \langle [D, U]^* x, y \rangle - s \langle \mu U y, x \rangle - s \langle U^* \mu x, y \rangle.\end{aligned}$$

From Lemma 4.2.6 we have

$$\begin{aligned}&\kappa \langle [D, U] y, x \rangle + \kappa \langle [D, U]^* x, y \rangle - s \langle \mu U y, x \rangle - s \langle U^* \mu x, y \rangle \\ &= \kappa(1-s)(\langle [D, U] y, x \rangle + \langle [D, U]^* x, y \rangle) \\ &\geq -\kappa(1-s)\|[D, U]\|(\|x\|^2 + \|y\|^2).\end{aligned}$$

In total we have

$$\begin{aligned}\langle M^2 z, z \rangle &\geq \langle (\kappa D - \mu)^2 x, x \rangle + \kappa^2 \langle D^2 y, y \rangle + \|x\|^2 + \|y\|^2 \\ &\quad - (1-s)\kappa\|[D, U]\|(\|x\|^2 + \|y\|^2) \\ &\geq \|(\kappa D - \mu)x\|^2 + \kappa^2 \|Dy\|^2 + \|x\|^2 + \|y\|^2 - \frac{1}{2}(\|x\|^2 + \|y\|^2) \\ &\geq \frac{1}{2}(\|x\|^2 + \|y\|^2).\end{aligned}$$

Here we have used the fact that D^2 is invertible and self-adjoint, $\kappa D - \mu$ is self-adjoint and $(1-s)\kappa\|[D, U]\| \leq \kappa\|[D, U]\| \leq \frac{1}{2}g^2 = \frac{1}{2}\|U^*\|^{-2} = \frac{1}{2}$, for $s \in [0, 1]$.

By Proposition 2.4.17, M is invertible for all $t \in [0, 1]$ so by Proposition 3.2.1.v the statement is proven. $\square$

By Lemma 4.3.3 and by concatenation

$$\text{ind}(T_A) = \text{Sf}_\beta \left(\begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix}, \begin{pmatrix} \kappa D & U \\ U^* & -\kappa D \end{pmatrix} \right).$$

Deforming $U = A|A|^{-1}$ to A does not cause spectral flow [14]. We have

$$\text{ind}(T_A) = \text{Sf}_\beta \left(\begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix}, \begin{pmatrix} \kappa D & A \\ A^* & -\kappa D \end{pmatrix} \right).$$

On the right side we have L_κ and we would like to diagonalize it in $\mathcal{H} = (\mathcal{H}_\rho \oplus \mathcal{H}_\rho) \oplus (\mathcal{H}_{\rho^c} \oplus \mathcal{H}_{\rho^c})$, where $\mathcal{H}_{\rho^c}$ is the complement to $\mathcal{H}_\rho$. D' is already diagonalized from the definition of $\mathcal{H}_\rho$, but L_κ is not. However, from Lemma 4.2.10 we have

$$\text{Sf}_\beta(L_\kappa, L_{\kappa, \rho} \oplus L_{\kappa, \rho^c}) = 0,$$

by letting $\rho' \rightarrow \infty$, i.e. $\rho', \rho = \rho^c$. By concatenation and additivity we have then

$$\begin{aligned}\text{ind}(T_A) &= \text{Sf}_\beta \left(\begin{pmatrix} \kappa D_\rho & 0 \\ 0 & -\kappa D_\rho \end{pmatrix} \oplus \begin{pmatrix} \kappa D_{\rho^c} & 0 \\ 0 & -\kappa D_{\rho^c} \end{pmatrix}, L_{\kappa, \rho} \oplus L_{\kappa, \rho^c} \right) \\ &= \text{Sf}_\beta \left(\begin{pmatrix} \kappa D_\rho & 0 \\ 0 & -\kappa D_\rho \end{pmatrix}, L_{\kappa, \rho} \right) + \text{Sf}_\beta \left(\begin{pmatrix} \kappa D_{\rho^c} & 0 \\ 0 & -\kappa D_{\rho^c} \end{pmatrix}, L_{\kappa, \rho^c} \right).\end{aligned}$$

For the term

$$\text{Sf}_\beta \left(\begin{pmatrix} \kappa D_{\rho^c} & 0 \\ 0 & -\kappa D_{\rho^c} \end{pmatrix}, L_{\kappa, \rho^c} \right) = \text{Sf}_\beta(\kappa D'_{\rho^c}, L_{\kappa, \rho^c}),$$

note that $(1-t)\kappa D'_{\rho^c} + tL_{\kappa,\rho^c} = Q_{\kappa,\rho^c}(t)$. From Corollary 4.2.8, we have that (by letting $\rho' \rightarrow \infty$):

$$\text{Sf}_\beta \left(\begin{pmatrix} \kappa D_{\rho^c} & 0 \\ 0 & -\kappa D_{\rho^c} \end{pmatrix}, L_{\kappa,\rho^c} \right) = 0.$$

All that remains then is

$$\text{ind}(T_A) = \text{Sf}_\beta \left(\begin{pmatrix} \kappa D_\rho & 0 \\ 0 & -\kappa D_\rho \end{pmatrix}, L_{\kappa,\rho} \right) = \text{Sf} \left(\begin{pmatrix} \kappa D_\rho & 0 \\ 0 & -\kappa D_\rho \end{pmatrix}, L_{\kappa,\rho} \right),$$

where the last equality follows from the fact that $L_{\kappa,\rho}$ and $\begin{pmatrix} \kappa D_\rho & 0 \\ 0 & -\kappa D_\rho \end{pmatrix}$ are finite-dimensional operators and so clearly the path $t \in [0, 1] \mapsto (1-t)D'_\rho + tL_{\kappa,\rho}$ is bounded. We thus get from Lemma 4.2.5:

$$\text{ind}(T_A) = \frac{1}{2} \left(\text{Sig}(L_{\kappa,\rho}) - \text{Sig} \begin{pmatrix} \kappa D_\rho & 0 \\ 0 & -\kappa D_\rho \end{pmatrix} \right),$$

where by Lemma 4.2.5 we have

$$\text{Sig} \begin{pmatrix} \kappa D_\rho & 0 \\ 0 & -\kappa D_\rho \end{pmatrix} = \text{Sig}(\kappa D) - \text{Sig}(\kappa D) = 0.$$

In total we get

$$\text{ind}(T_A) = \frac{1}{2} \text{Sig}(L_{\kappa,\rho}).$$

This completes the proof of our main result, Theorem 4.1.8.

5 Spectral Flow of Dirac Operators in Examples

In this section we will provide some examples of using Theorem 4.1.8 in practice, using two Dirac operators.

5.1 Dirac Operator on 1-Torus

The 1-Torus or the 1-Sphere (otherwise known as the unit-circle) is defined by $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z} \cong S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Let our Hilbert space $\mathcal{H}$ be defined as

$$\mathcal{H} = L^2(S^1) = \left\{ f : S^1 \rightarrow \mathbb{C} \mid \int_0^{2\pi} |f(\theta)|^2 d\theta < \infty \right\}.$$

$\mathcal{H}$ has a countable basis $e_n(\theta) = e^{in\theta}$, $\theta \in [0, 2\pi]$ and is thus separable. Any function $f \in L^2(S^1)$ can be represented as a Fourier series:

$$f = \sum_{n=-\infty}^{\infty} c_n e^{in\theta},$$

for $\theta \in [0, 2\pi]$ and $c_n \in \mathbb{C}$.

Let us try to calculate the index of the Toeplitz operator T_A where A is the (invertible) multiplication operator $Af(\theta) = e^{i\theta}f(\theta)$. We see that $Ae_n = e_{n+1}$. For $m \in (0, 1)$ let

$$D = -i \frac{d}{d\theta} + m,$$

be the Dirac operator on D with domain

$$\mathcal{D}(D) = \left\{ f \in L^2(S^1) \mid \int_0^{2\pi} |Df(\theta)|^2 d\theta < \infty \right\}.$$

$$\langle Df, g \rangle = \int_0^{2\pi} (-if'(\theta) + m)g(\theta)d\theta = \int_0^{2\pi} f(\theta)(-ig'(\theta) + m)d\theta = \langle f, Dg \rangle,$$

for every $f \in \mathcal{D}(D), g \in \mathcal{D}(D^*)$. Here we have used partial integration and $f(0) = f(2\pi)$. For the domain $\mathcal{D}(D^*)$ note that

$$\langle Dx, y \rangle = \langle x, Dy \rangle \leq \|x\| \|Dy\|,$$

so $\langle Dx, y \rangle$ is bounded for $x \in L^2(S^1)$ and $y \in \mathcal{D}(D)$ which means that $\mathcal{D}(D^*) \subset \mathcal{D}(D)$. On the other hand,

$$\langle Dx, y \rangle \leq \|Dx\| \|y\|,$$

so $\langle Dx, y \rangle$ is bounded for $x \in \mathcal{D}(D)$ and $y \in L^2(S^1)$ so $\mathcal{D}(D) \subset \mathcal{D}(D^*)$. Thus $\mathcal{D}(D) = \mathcal{D}(D^*)$ and D is self-adjoint.

D has the eigenvectors $De_n = (n+m)e_n$. D is invertible since D is bijective and the operator $Se_n = (n+m)^{-1}e_n$ is bounded. Clearly $DS = I$ and $SD = I_{\mathcal{D}(D)}$.

We will now show that the resolvent $R(\lambda, D) = (D - \lambda)^{-1}e_n = (n+m-\lambda)^{-1}e_n$ is compact: Define for every $N \in [0, \infty)$ the operator

$$R_N(\lambda, D)e_n = \begin{cases} (n+m-\lambda)^{-1}e_n, & |n| \leq N \\ 0, & |n| > N. \end{cases}$$

Since $R_N(\lambda, D)$ is a finite-dimensional operator, $R_N(\lambda, D)$ is also compact. For each $N \in [0, \infty)$ one can show that:

$$\|R(\lambda, D) - R_N(\lambda, D)\| \leq |N+m-\lambda|^{-1}.$$

Since $\lim_{N \rightarrow \infty} \|R(\lambda, D) - R_N(\lambda, D)\| = 0$ and $(R_N(\lambda, D))$ is a sequence of compact operators, then by Theorem 2.4.20 $R(\lambda, D)$ is also compact.

We can also check that $[D, A]e_n = DAe_n - AD e_n = De_{n+1} - A(n+m)e_n = (n+m+1)e_{n+1} - (n+m)e_{n+1} = e_{n+1}$ so $[D, A]$ is bounded.

So we have an odd Fredholm module that fulfills all conditions. Now we can calculate $T_A = \Pi A \Pi + (1 - \Pi)$. First, note that

$$\Pi e_n = \chi_{[0, +\infty)}(D)e_n = \begin{cases} e_n, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

So we get:

$$T_A e_n = \begin{cases} e_{n+1}, & n \geq 0, \\ e_n, & n < 0. \end{cases}$$

Now note that $\ker(T_A) = 0$ and $\text{coker}(T_A) = \ker(T_A^*) = \{e_0\}$, so we have $\text{ind}(T_A) = 0 - 1 = -1$.

Let us calculate the index numerically: it is clear that $\|A\| = \|A^{-1}\| = 1$ and $\|[A, D]\| = 1$. So $g = 1$ and

$$\kappa_0 = \frac{g^2}{2\|[A, D]\|} = \frac{1}{2}.$$

Note: with the other formula presented in [14], we would have

$$\kappa_0 = \frac{g^3}{12\|A\|\|[A, D]\|} = \frac{1}{12},$$

and we would need for example $\kappa = 0.08$ and $\rho = 26$.

If we set $\kappa = 0.5$ and $\rho = 5$ we get $2g\kappa_0^{-1} = 4 \leq 2g\kappa^{-1} = 4 < 5$ and so we can calculate the signature of $L_{0.5,5}$.

$\mathcal{H}_\rho = \mathcal{R}(\chi[-5, 5](D))$ which is just $\{e_4, \dots, e_{-5}\}$ (note that eigenvalue of e_5 is $5 + m > 5$). We get then that

$$D_5 = \begin{pmatrix} 4+m & 0 & 0 & \dots & 0 \\ 0 & 3+m & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -5+m \end{pmatrix}$$

and

$$A_5 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

where A is of size 10×10 . Now putting A_5, D_5 into the matrix

$$L_{0.5,5} = \begin{pmatrix} 0.5D_5 & A_5 \\ A_5^* & -0.5D_5 \end{pmatrix}$$

and calculating the signature we get $\text{Sig}(L_{0.5,5}) = -2$ so $\text{ind}(T_A) = -1$.

5.2 Dirac Operator on 3-Torus

The 3-Torus is defined $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3 = S^3$. Define

$$L^2(S^3) = \left\{ f : S^3 \rightarrow \mathbb{C} \mid \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \|f(x_1, x_2, x_3)\|^2 dx_1 dx_2 dx_3 < \infty \right\}.$$

Define our Hilbert space $\mathcal{H} = L^2(S^1) \times \mathbb{C}^2$. For $m \in (0, 1)$ let

$$D = \begin{pmatrix} i\partial_3 & \partial_1 + i\partial_2 \\ -\partial_1 + i\partial_2 & -i\partial_3 \end{pmatrix} + mI$$

be the Dirac operator acting on $\mathcal{H}$ with domain

$$\mathcal{D}(D) = \{f \in \mathcal{H} \mid \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \|Df(x_1, x_2, x_3)\|^2 dx_1 dx_2 dx_3 < \infty\},$$

where $\partial_j = \partial/\partial x_j$. Every $f \in \mathcal{H}$ has the Fourier series

$$f = \sum_{\mathbf{n} \in \mathbb{Z}^3} a_{\mathbf{n}} z^{\mathbf{n}} v = \sum_{(n_1, n_2, n_3) \in \mathbb{Z}^3} a_1 a_2 a_3 z_1^{n_1} z_2^{n_2} z_3^{n_3} v,$$

where $z_j^{n_j} = e^{in_j x_j}$ and $v \in \mathbb{C}^2$. One can check by similar methods as in Section 5.1 that D is self-adjoint and invertible.

To find the eigenvectors of D we can make the ansatz $w = z^{\mathbf{n}}v$ for $v \in \mathbb{C}^2$,

$$(D - mI)z^{\mathbf{n}}v = \begin{pmatrix} -n_3 & in_1 - n_2 \\ -in_1 - n_2 & n_3 \end{pmatrix} z^{\mathbf{n}}v = \lambda' z^{\mathbf{n}}v.$$

So we have the equation:

$$(D - mI)w = M_{\mathbf{n}}w = \lambda'w \quad (14)$$

We can solve (14) with techniques from linear algebra:

$$\det \begin{pmatrix} -n_3 - \lambda' & in_1 - n_2 \\ -in_1 - n_2 & n_3 - \lambda' \end{pmatrix} = (-n_3 - \lambda')(n_3 - \lambda') - (in_1 - n_2)(-in_1 - n_2) = 0,$$

and we get

$$\lambda' = \lambda'_{\mathbf{n}} = \pm \sqrt{n_1^2 + n_2^2 + n_3^2} = \pm |\mathbf{n}|,$$

So D has eigenvalues $\lambda_{\mathbf{n}} = m + \lambda'_{\mathbf{n}} = m \pm |\mathbf{n}|$. For each eigenvalue λ we have the eigenvectors $z^{\mathbf{n}}v_{\mathbf{n}}^{\pm}$, where

$$v_n^+ = \begin{pmatrix} 1 \\ \frac{n_3 + |\mathbf{n}|}{in_1 - n_2} \end{pmatrix}, \quad v_n^- = \begin{pmatrix} 1 \\ \frac{n_3 - |\mathbf{n}|}{in_1 - n_2} \end{pmatrix}.$$

So if we denote the eigenbasis to $L^2(\mathbb{T}^3)$ as

$$e_{\mathbf{n}}(\mathbf{x}) = e^{i(n_1x_1 + n_2x_2 + n_3x_3)},$$

and write $s_{\mathbf{n}}^{\pm} = v_{\mathbf{n}}^{\pm}/|v_{\mathbf{n}}^{\pm}|$ as the normalized eigenvectors of $M_{\mathbf{n}}$, we can write the basis to D as

$$\psi_{\mathbf{n}}^{\pm}(\mathbf{x}) = \begin{cases} s_{\mathbf{n}}^+ e_{\mathbf{n}}(\mathbf{x}), & \lambda_{\mathbf{n}} > 0, \\ s_{\mathbf{n}}^- e_{\mathbf{n}}(\mathbf{x}), & \lambda_{\mathbf{n}} \leq 0. \end{cases}$$

One can verify that the resolvent $R(\lambda, D) = (D - \lambda)^{-1}$ is compact. Thus all conditions for D are met.

As an example we can calculate the index of the Toeplitz operator T_A where $A = e^{i(x_1 + x_2 + x_3)}$ and for $T = \Pi A \Pi + (1 - \Pi)$ we get

$$T_A \psi_{\mathbf{n}}^{\pm}(\mathbf{x}) = \begin{cases} s_{\mathbf{n}}^+ e_{\mathbf{n}+\mathbf{1}}(\mathbf{x}), & \lambda'_{\mathbf{n}} > 0 \\ s_{\mathbf{n}}^- e_{\mathbf{n}}(\mathbf{x}), & \lambda'_{\mathbf{n}} \leq 0. \end{cases}$$

In this case we always have $\text{ind}(T_A) = 0$, since $\ker(T_A) = \text{coker}(T_A) = \{0\}$.

We have

$$[D, A]\psi_{\mathbf{n}}^{\pm} = DA\psi_{\mathbf{n}}^{\pm} - AD\psi_{\mathbf{n}}^{\pm} = D\psi_{\mathbf{n}+\mathbf{1}}^{\pm} \mp A|\mathbf{n}|\psi_{\mathbf{n}}^{\pm} = \pm(|\mathbf{n} + \mathbf{1}| + m - (|\mathbf{n}| + m))\psi_{\mathbf{n}+\mathbf{1}}^{\pm}.$$

Using the triangle inequality, $|\mathbf{n} + \mathbf{1}| - |\mathbf{n}| \leq |\mathbf{n} + \mathbf{1} - \mathbf{n}| = |\mathbf{1}| = \sqrt{3}$. So $[D, A]$ is bounded with norm $\|[D, A]\| \leq \sqrt{3}$. We have $A^{-1} = e^{-i(x_1 + x_2 + x_3)}$ and so $\|A^{-1}\| = g = 1$. We then have $\kappa_0 = (2\sqrt{3})^{-1}$. Setting $\kappa = \kappa_0$ and $\rho = 7$ we have $2g\kappa^{-1} = 4\sqrt{3} \approx 6.93 < \rho = 7$.

Set $m = 0.5$. Now $\mathcal{H}_7 = \mathcal{R}(\chi_{[-7, 7]}(D))$ corresponds to all the eigenvectors $\psi_{\mathbf{n}}^{\pm}$ fulfilling

$$|m \pm \sqrt{n_1^2 + n_2^2 + n_3^2}| \leq 7 \quad (15)$$

which is a tedious problem to solve analytically, but by doing it numerically we see that there are 2980 eigenvalues fulfilling (15). Writing out D_7, A_7 is very tedious and not helpful in this case, so we will just go ahead and write

$$L_{(2\sqrt{3})^{-1},7} = \begin{pmatrix} (2\sqrt{3})^{-1}D_7 & A_7 \\ A_7^* & -(2\sqrt{3})^{-1}D_7 \end{pmatrix},$$

which is a sparse 5960×5960 matrix. In MATLAB, there exist built-in tools for sparse matrices that can be used to calculate the eigenvalues of $L_{(2\sqrt{3})^{-1},7}$ efficiently. Since calculating the eigenvalues for this example is still relatively simple we won't dive into such optimizations here.

In total we get $\text{ind}(T_A) = \frac{1}{2} \text{Sig}(L_{(2\sqrt{3})^{-1},7}) = 0$ as expected.

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