LU TP 16-30
June 2016

U(1) Froggatt-Nielsen model for quark mass hierarchy in 2HDM

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Abstract

One approach to understanding the observed hierarchy in quark masses is by using the Froggatt-Nielssen mechanism. In this model new exotic (and presumable yet undetected) particles are introduced. By forcing the particles of the Standard Model to interact with the postulated particles it is possible to create a mass suppression mechanism which can explain the observed quark mass relations.

The mechanism is studied in a 2-Higgs-doublet model without any restricting symmetries other than CP-conservation. A danger with such an approach is flavor changing neutral currents at tree level in the Higgs sector - which yet are unobserved. Whether the mechanism can be used to explain mass ratios, the CKM-matrix for quark mixing and the absence of flavor changing currents is investigated with focus on U(1)-symmetries for the interaction between particles of the Standard Model and the new particles.

Populärvetenskaplig beskrivning

Universum består utav mycket små byggstenar vilka kallas elementarpartiklar. Varje sådan partikel har speciella egenskaper. Ett exempel är ljus, vilket egentligen är ljuspartiklar som kallas *fotoner*, vilka färdas med ljushastigheten. Ett annat är elektricitet vilket är *elektroner* i rörelse. En av elektronernas egenskaper är att de har en negativ elektrisk laddning.

En fråga som många fysiker ställer är varför har vissa partiklar massa; varför är alla inte masslösa som ljuset är? Den vedertagna lösningen är att det finns ytterligare en partikel, Higgspartikeln, vilken har som egenskap att den ger vissa andra partiklar massa.

Några av de vanligaste elementarpartiklarna är kvarkarna, vilka tillsammans med elektronerna bygger upp atomerna - det som vi huvudsakligen består utav. Det finns sex olika sorters kvarkar. Förhållandet mellan de olika kvarkarnas massor skiljer sig kraftigt ifrån varandra - vilket ännu inte har någon vedertagen förklaring. Ett förslag till en förklaring av dessa förhållanden är *Froggatt-Nielsen mekanismen*. Tanken bakom denna mekanism är att man antar att det existerar ett flertal oupptäckta partiklar vilka interagerar på ganska specifika sätt, till exempel med kvarkarna.

Det finns många förslag på teorier som går bortom Standardmodellen. Ett sådant förslag är supersymmetri vilken säger att för varje partikel vi känner till finns det ytterligare en oupptäckt partikel; dess supersymmetriska partner. För att supersymmetri ska kunna fungera krävs flera nya Higgspartiklar. Ett problem med att införa fler Higgspartiklar är det kan uppstå *smakändrande strömmar* vilket betyder att en kvark kan övergå till en annan kvark på ett sätt som ännu inte har observerats.

Detta arbete undersöker olika sätt att konstruera en Froggatt-Nielsen mekanism och hur stora de smakändrande strömmarna blir i modellen. Modellen jämförs med experimentella data.

Contents

Acknowledgements

A great thanks to J. Rathsman for supervising this thesis and his support and interest in this work.

I would also like to thank my colleagues F. Tellander and N. Hermansson Truedsson for mathematical insight.

Abbreviations

2HDM	Two-Higgs-doublet model
FCNC	Flavor Changing Neutral Currents
FNM	Froggatt-Nielsen Mechanism
RD	Ratio of Determinants
SM	Standard Model
VEV	Vacuum Expectation Value

1 Introduction

The Standard Model of particle physics (SM) [1] describes the interaction between the most fundamental constituents of matter. It has been most successful with describing experimental particle physics data. However, it does leave several phenomena unexplained.

- There are more than 20 free parameters which can only be determined from experiments.
- The relevant gauge group is SU(3)×SU(2)×U(1) but there is no explanation to why this particular group is utilized.
- Dimensionless parameters are expected to be of order 1. However, mass hierarchies are examples where this type of naturalness is not fulfilled.

A particular interesting issue is the quark mass ratios, as mentioned in the last point. An example of this relation is shown in the equations below [2]:

$$m_u : m_c : m_t \approx 1 : 550 : 75000 \qquad m_d : m_s : m_b \approx 1 : 20 : 870.$$

This is problematic since it implies major differences in the Yukawa couplings for the respective quarks - which are dimensionless parameters and hence are expected to be of roughly the same order. As is evident, this is far from the case. One explanation is the one Froggatt and Nielsen [3] introduced in their Froggatt-Nielsen mechanism. In said mechanism it is postulated that not all particles couple directly to the Higgs field, but do so via new exotic particles. It offers an explanation of the quark mass hierarchy while allowing the naturalness to be fulfilled.

Many theories going beyond the Standard Model, e.g. supersymmetry, require the existence of more than one Higgs doublet [4]. The simplest extension is to introduce one additional Higgs doublet. Such a model is called a Two-Higgs-Doublet Model (2HDM). One of the dangers of 2HDM is the possibility of Flavor Changing Neutral Currents (FCNC) at tree level in the Higgs sector being present - which have yet not been observed. Hence these currents need to be suppressed in order for the model to be able to fit to experimental data.

There are several approaches in order to minimize FCNC:s, e.g. models with postulated symmetries between the doublets. Another possibility might be the Froggatt-Nielsen mechanism which does not rely on such symmetries. Although the model originally was introduced in order to explain mass hierarchies in the SM it will here be investigated to what extent it at the same time can give FCNC:s in accordance with experiment in a general 2HDM. This study will consider U(1) as the gauge group of the intermediate particles responsible for the mechanism, but the group U(1)×...×U(1) is also discussed.

In the original article by C. D. Froggatt and H. B. Nielsen [3], where the Froggatt-Nielsen mechanism was originally proposed, the utilized symmetry group was U(1). They worked in the SM with only one Higgs doublet and the article is, from an experimental point of view, outdated.

Some earlier work on the Froggatt-Nielsen mechanism in a 2HDM has been presented in [5] but imposing a $\mathbb{Z}_2$-symmetry on the two doublets. In [6] the model was extended to U(1)×U(1) symmetry but again limiting to the case with a $\mathbb{Z}_2$-symmetry on the two doublets.

This thesis is organized as follows: section 2 contains a discussion of 2HDM and its concepts. In section 3 the Froggatt-Nielsen mechanism is presented, firstly using U(1) in SM but is later extended to 2HDM using U(1)×...×U(1). Section 4 considers phenomenology and a U(1) model is used to explain the experimental quark mass relations. Finally, in section 5 this work is concluded with a discussion of the validity of the results and the assumptions on which the theory is based.

2 Two-Higgs-doublet model

This section will in a compact manner discuss the established Two-Higgs-Doublet Model (2HDM). A more thorough discussion is featured in [4]. The concept of mass and current matrices will be introduced followed by a derivation of the CKM-matrix in the presence of two Higgs doublets. Additionally, one can find a summary of the SM Higgs mechanism in appendix A.

2.1 Introducing the doublets

Consider two hypercharge $\frac{1}{2}$ complex $\mathrm{SU}(2)_L$ doublets, Φ_1 and Φ_2, expressed in unitary gauge [4]. In order to be able to generate both up and down quark masses their conjugate doubles, $\Phi^c := \varepsilon\Phi^*$, are required. Here ε denotes a two-dimensional Levi-Civita tensor. Further, by utilizing a Higgs flavor index $a \in \{1, 2\}$ said doublets can be formulated as

$$\Phi_a = \begin{pmatrix} \phi_a^+ \\ v_a + \frac{1}{\sqrt{2}}(\xi_a + i\eta_a) \end{pmatrix}. \tag{2.1}$$

In order to be able to write down the Higgs Lagrangian for this system in a compact way the covariant derivative, D_μ, is defined in eq. 2.2. Here the weak hypercharge is denoted as Y_W, the gauge field of $\mathrm{U}(1)_{Y_W}$ as B_μ, the Pauli matrices as $\boldsymbol{\sigma}$ and the $\mathrm{SU}(2)_L$ gauge fields as $\mathbf{W}_\mu$.

$$D_\mu = \partial_\mu + i\frac{g_1}{2}Y_W B_\mu + i\frac{g_2}{2}\boldsymbol{\sigma}\cdot\mathbf{W}_\mu \tag{2.2}$$

If terms with kinetic mixing are neglected the Lagrangian for the model can be expressed as below using the covariant derivative.

$$\mathcal{L} = (D_\mu\Phi_a)^\dagger D^\mu\Phi_a - V \tag{2.3}$$

The potential V can be formulated using the same approach as in [7] by utilizing implicit sums over the Higgs flavor indexes a, b, c and d.

$$V = \mu_{ab}\Phi_a^\dagger\Phi_b + \lambda_{abcd}\left(\Phi_a^\dagger\Phi_b\right)\left(\Phi_c^\dagger\Phi_d\right) \tag{2.4}$$

For some values on the parameters μ the symmetry group $\mathrm{SU}(2)_L$ is spontaneously broken which gives rise to non-zero Vacuum Expectation Value:s (VEV:s) - which in turn allows for mass generation. For the gauge fields B and $\mathbf{W}$ mass generation follow in the same manner as in the SM case.

One interesting remark is that any other set of doublets capable of spanning the same space constitutes an equally valid basis, i.e. all basis that relate to the initial basis via a transform $\mathcal{A} \in \mathrm{U}(2)$. Such a transformation is shown in eq. 2.5 for a new basis $\boldsymbol{\Phi}'$.

$$\begin{pmatrix} \Phi_1' \\ \Phi_2' \end{pmatrix} = \mathcal{A}\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \tag{2.5}$$

As discussed in [7], the physical observables are independent of such a change of basis, although the parameters of the potential are not.

2.2 Higgs basis

Since one can choose a new basis, as mentioned previously, it might be possible to choose a basis such that one VEV vanishes [4]. This choice is known as the Higgs basis and is the frame easiest to compare to the SM.

Consider two doublets H_1 and H_2 that are related to Φ_1 and Φ_2 via a two-dimensional rotation $\mathcal{R}_2$ with an angle[1] $-\beta$, i.e.

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \mathcal{R}_2(-\beta) \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \tag{2.6}$$

Now, assume that this rotation results in that the VEV of H_2 vanishes and H_1 acquires a VEV $v \approx 174$ GeV [2]. From this assumption the linear system of equations in eq. 2.7 arises. This system has unique non-trivial solutions and hence this rotation is possible.

$$\begin{cases} v = v_1 \cos\beta + v_2 \sin\beta \\ 0 = -v_1 \sin\beta + v_2 \cos\beta \end{cases} \Leftrightarrow \begin{cases} v_1 = v\cos\beta \\ v_2 = v\sin\beta \end{cases} \tag{2.7}$$

A result of eq. 2.7 is the property $v^2 = v_1^2 + v_2^2$, which alternatively can be found by considering v the radial distance in the space spanned by Φ_1 and Φ_2.

2.3 Fermion masses and coupling matrices

In order to obtain masses for the fermions one must construct a Yukawa Lagrangian, $\mathcal{L}_\mathrm{Y}$. Said Lagrangian can be divided into two terms, one from each doublet. The Lagrangian for one doublet has the same form as $\mathcal{L}_\mathrm{Y}$ of the SM.

Define[2]

$$Q_L := \left(\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right)^\mathrm{T} \qquad U_R := (u_R, c_R, t_R)^\mathrm{T} \qquad D_R := (d_R, s_R, b_R)^\mathrm{T}$$

and let Y be the Yukawa coupling matrices then

$$\mathcal{L}_\mathrm{Y}^\mathrm{2HDM} = \underbrace{\overline{Q}_L Y_1^u \Phi_1^c U_R - \overline{Q}_L Y_1^d \Phi_1 D_R}_{\text{First doublet}} + \underbrace{\overline{Q}_L Y_2^u \Phi_2^c U_R - \overline{Q}_L Y_2^d \Phi_2 D_R}_{\text{Second doublet}} + h.c. \tag{2.8}$$

As the doublets get nonzero VEV:s one can identify current terms of the form $-\overline{q}_L^i M_{ij} q_R^j$ when a left-handed quark q^i turns into a right-handed quark q^j. The matrix M, often labeled mass matrix, contains the strength of each current. For example, from $\mathcal{L}_\mathrm{Y}^\mathrm{2HDM}$ the 3×3 mass matrices M_u and M_d in eq. 2.9 can be identified[3].

$$M_u = v\left(\frac{v_1}{v} Y_1^u + \frac{v_2}{v} Y_2^u\right) \qquad M_d = v\left(\frac{v_1}{v} Y_1^d + \frac{v_2}{v} Y_2^d\right) \tag{2.9}$$

[1] The fact that $-\beta$ is used instead of β is a matter of convention.

[2] Here the transpose of a vector/matrix does not transpose its elements.

[3] For the remainder of this work the factor v will not be included in the mass matrices.

2.3.1 Fermion masses in the Higgs basis

The derivation of the fermion masses could equally well have been done in the Higgs basis with some benefits that soon will be clear. Analogously with eq. 2.8 the Yukawa Lagrangian can be constructed as

$$\mathcal{L}_{\mathrm{Y}}^{\mathrm{Higgs}} = \overline{Q}_L(\kappa_u H_1^c + \rho_u H_2^c)U_R - \overline{Q}_K(\kappa_d H_1 + \rho_d H_2)D_R + h.c. \tag{2.10}$$

Since both the Higgs basis and the one of Φ_1 and Φ_2 are equally valid the Lagrangian must be unchanged, i.e. $\mathcal{L}_{\mathrm{Y}}^{\mathrm{2HDM}} = \mathcal{L}_{\mathrm{Y}}^{\mathrm{Higgs}}$. Using a flavor index $f \in \{u, d\}$ this gives

$$Y_1^f\Phi_1 + Y_2^f\Phi_2 = \kappa_f H_1 + \rho_f H_2 = \kappa_f(\Phi_1\cos\beta + \Phi_2\sin\beta) + \rho_f(-\Phi_1\sin\beta + \Phi_2\cos\beta)$$

which implies that

$$\begin{pmatrix}\kappa_f\\ \rho_f\end{pmatrix} = \mathcal{R}_2(-\beta)\begin{pmatrix}Y_1^f\\ Y_2^f\end{pmatrix}. \tag{2.11}$$

Since the VEV of H_2 is vanishing, the only contribution to the mass is from H_1, i.e. $M_f = v\kappa_f$ - which can be diagonalized with a biunitary transform - see appendix B. However, the same transform will also apply to ρ, which does not have to be become diagonal. The off-diagonal elements of the transformed ρ correspond to tree level Flavor Changing Neutral Currents (FCNC:s) in the Higgs sector - which are severely limited by experimental bounds. Note that M will also contain neutral currents but those will not be flavor changing and are therefore dismissed in this analysis.

2.3.2 Different 2HDM:s and minimizing FCNC:s

There are several versions of the Two-Higgs-doublet model utilizing different approaches to minimize FCNC:s. Below the most common ones are briefly discussed.

One approach is to impose a $\mathbb{Z}_2$ symmetry[4] on the system forcing each fermion to only couple to a single doublet. Hence, by the Glashow-Weinberg theorem [2], there exists no FCNC:s in such a model. Although eliminating FCNC:s, a direct consequence of postulating such symmetries is a loss of generality. However, in the most general model, 2HDM III, FCNC:s are allowed to be non-zero, but have to be small enough to satisfy experimental bounds.

2HDM I In 2HDM I a $\mathbb{Z}_2$ symmetry is postulated such that all fermions couple only to Φ_2. The relevant particles transform as

$$\Phi_1 \to -\Phi_1 \qquad \Phi_2 \to \Phi_2 \qquad Q_L \to Q_L \qquad U_R \to U_R \qquad D_R \to D_R. \tag{2.12}$$

2HDM II In the 2HDM II model a $\mathbb{Z}_2$ symmetry is imposed on the system such that up quarks couple only to Φ_2 and down quarks couple only to Φ_1. The relevant particles transform as

$$\Phi_1 \to -\Phi_1 \qquad \Phi_2 \to \Phi_2 \qquad Q_L \to Q_L \qquad U_R \to U_R \qquad D_R \to -D_R. \tag{2.13}$$

[4] $x \to -x$ for some x and $x \to x$ for all other x.

2HDM III If none of the restrictions above are introduced, one is left with 2HDM III. Potentially, one can minimize FCNC:s by e.g. using a Froggatt-Nielsen mechanism, discussed in sec. 3. The FCNC:s that might occur can be parametrized as

$$\lambda_{ij}\frac{\sqrt{m_i m_j}}{v}. \tag{2.14}$$

This is known as the Cheng-Sher Ansatz where λ_{ij} are assumed to be parameters of order one [4]. Note that if m_i and m_j already are dimensionless then v is not included.

2.4 The CKM-matrix

The mass eigenstates $\widetilde{q}$ are obtained by rotating the electroweak states q via unitary transformations. Using a set of unitary matrices, V, said transformations can be described as

$$q_{L,R} \to \widetilde{q}_{L,R} := V^q_{L,R}\, q_{L,R} \qquad \overline{q}_{L,R} \to \overline{\widetilde{q}}_{L,R} := \overline{q}_{L,R} V^{q\dagger}_{L,R}. \tag{2.15}$$

where $q \in \{u, d\}$.

In order for the mass terms in the Lagrangian to be unchanged the coupling matrices must transform via biunitary transforms in eq. 2.16, which are discussed further in appendix B. The transformation matrices are chosen such that the resulting mass matrices of the system are diagonal with positive elements.

$$\kappa_q \to \widetilde{\kappa}_q := V^q_L \kappa_q V^{q\dagger}_R \qquad \rho_q \to \widetilde{\rho}_q := V^q_L \rho_q V^{q\dagger}_R \tag{2.16}$$

Now, consider the part of Yukawa Lagrangian for the down sector in the Higgs basis (eq. 2.10). For simplicity, temporary declare $H_i = (A_i, B_i)^{\mathrm{T}}$, then

$$\mathcal{L}_{\mathrm{D}} = \overline{u}_L(\kappa^d A_1 + \rho^d A_2)d_R + \overline{d}_L(\kappa^d B_1 + \rho^d B_2)d_R + h.c. \tag{2.17}$$

By insertion of $\mathbb{I} = V^\dagger V$ between quark and coupling matrices one can show the equality

$$\mathcal{L}_{\mathrm{D}} = \overline{u}_L V^{u\dagger}_L V^u_L V^{d\dagger}_L V^d_L(\kappa^d A_1 + \rho^d A_2)V^{d\dagger}_R V^d_R d_R + \overline{d}_L V^{d\dagger}_L V^d_L(\kappa^d B_1 + \rho^d B_2)V^{d\dagger}_R V^d_R d_R. \tag{2.18}$$

Since V commutes with the components of Φ eq. 2.18 can be reduced to

$$\mathcal{L}_{\mathrm{D}} = \overline{\widetilde{u}}_L V^u_L V^{d\dagger}_L(\widetilde{\kappa}^d A_1 + \widetilde{\rho}^d A_2)\widetilde{d}_R + \overline{\widetilde{d}}_L(\widetilde{\kappa}^d B_1 + \widetilde{\rho}^d B_2)\widetilde{d}_R. \tag{2.19}$$

Hence, the neutral currents transform as $\overline{d}_L\kappa^d d_R \to \overline{\widetilde{d}}_L\widetilde{\kappa}^d\widetilde{d}_R$ while charged currents transform as $\overline{u}_L\kappa^d d_R \to \overline{\widetilde{u}}_L V^u_L V^{d\dagger}_L\widetilde{\kappa}^d\widetilde{d}_R$. The matrix included in the last term is called the CKM-matrix. This experimentally well-known matrix is by convention defined as

$$V_{\mathrm{CKM}} := V^u_L V^{d\dagger}_L. \tag{2.20}$$

The result for the up sector is analogous but instead utilizes the Hermitian conjugate of this matrix.

The CKM matrix can be parametrized using three successive Euler rotations around the Cartesian axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ with the angles θ_3, θ_2 and θ_1. Using a 3×3 rotation matrix $\mathcal{R}_3$ and for simplicity assuming the elements of V_{CKM} are real gives

$$V_{\mathrm{CKM}} = \mathcal{R}_3(\hat{\mathbf{x}}, \theta_3)\, \mathcal{R}_3(\hat{\mathbf{y}}, \theta_2)\, \mathcal{R}_3(\hat{\mathbf{z}}, \theta_1). \tag{2.21}$$

Said angles can be obtained from the elements of the matrix [2]. The results are shown in eq. 2.22.

$$\sin\theta_1 = \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}} \quad \sin\theta_2 = |V_{ub}| \quad \sin\theta_3 = \sin\theta_1 \left|\frac{V_{cb}}{V_{us}}\right| \tag{2.22}$$

This parametrization will henceforth be referred to as the *standard parametrization*.

To conclude this subsection one can state that the CKM-matrix can be calculated directly from the transformation matrices diagonalizing the mass matrices. Experimental values on the entries of V_{CKM} are examined in sec. 4.

3 Froggatt-Nielsen mechanism

In the Standard Model fermions acquire mass by swapping chirality via interacting with the Higgs VEV. One approach for explaining differences in fermion masses is by prohibiting such a direct interaction and forcing the mass generation to go via some postulated secondary particles. This introduces a natural mass suppression mechanism known as the Froggatt-Nielsen mechanism [3], abbreviated FNM.

For the sake of simplicity only the quark sector will be treated in this work but it is straightforward to extend the procedure to the lepton sector.

The machinery of how the mechanism works in practice is developed by considering a single U(1)-symmetry in the SM. This is then generalized into a 2HDM III framework. Lastly, the techniques are extended to U(1)×...×U(1).

3.1 Concept

By following the approach in [3], a new scalar particle called the *flavon*, denoted S, with no colour charge, no weak hypercharge and singlet under $\mathrm{SU}(2)_L$ is introduced. The flavon is associated with some new symmetry and has an associated charge called *flavon charge*.

Further, in the same way as with the Higgs mechanism of the SM, the *flavon symmetry* can be spontaneously broken with a nonzero VEV $\langle S \rangle$, allowing for expanding S around said VEV. The Lagrangian of the flavon (eq. 3.1) is almost identically to $\mathcal{L}_{\mathrm{SM}}^{\mathrm{Higgs}}$. However, the covariant derivative does not include the gauge fields of the SM.

$$\mathcal{L}^{\mathrm{Flavon}} = (D_\mu S)^* \, D^\mu S - V \tag{3.1}$$

Since one is free to assign flavon charges to all particles of the model one can choose charge configurations such that the flavon cannot interact with two SM particles. As will be shown later this is a crucial step for the functionality of the mechanism. The simplest of such configurations is to set the flavon charge to zero for all particles which are not supposed to be affected by FNM, i.e. in this case all non-quark particles.

3.1.1 Froggatt-Nielsen fermions

As a next step, the FNM assumes the existence of a set of super-heavy and yet undetected fermions, labelled *Froggatt-Nielsen fermions*. For each SM particle with non-trivial flavon charge a set of Froggatt-Nielsen fermions exist with identical quantum numbers, except for mass and flavon charge. Utilizing the approach developed by Froggatt and Nielsen [3] the left-handed state of the quark q_j has the charge $c + b_j$ and the right-handed has $c - a_j$ where $a, b, c \in \mathbb{N}$. This gives a charge difference of $b_j + a_j$. Further, it is assumed that the charges are ordered according to $b_i \leq b_{i+1}$ and $a_j \leq a_{j+1}$.

The introduced superheavy quarks acquire mass only via some presumably undiscovered Higgs particle $\widehat{H}$ with zero flavon charge. As the mass of the Froggatt-Nielsen fermions is generated in the same way as in the SM Higgs case, one can identify the relation $m = \hat{v}\hat{g}$ where $\hat{v}$ is the VEV of $\widehat{H}$ and $\hat{g}$ is the coupling between a quark and $\widehat{H}$ as $\hat{g}$.

The process in which the Froggatt-Nielsen fermions acquire mass is illustrated in fig 1. Here it is assumed that this is an amputated diagram from some more complicated process, indicated by the dotted lines.

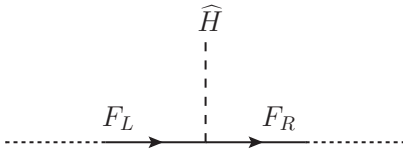


Figure 1: An incoming particle Froggatt-Nielsen Fermion switches chirality under $\hat{H}$.

In the limit $|p|/m \to 0$ the Feynman amplitude of fig. 1 can be approximated with a process-independent constant, shown in eq. 3.2.

$$|\mathcal{M}| = \left|\frac{\not{p} + m}{p^2 - m^2}\right| |\hat{v}\hat{g}| \left|\frac{\not{p} + m}{p^2 - m^2}\right| \overset{|p|/m \to 0}{\longrightarrow} \frac{1}{m} \tag{3.2}$$

From now on, assume that all of the introduced Froggatt-Nielsen fermions have approximately the same mass m. The limit $|p|/m \to 0$ will be implicit for the remainder of this work.

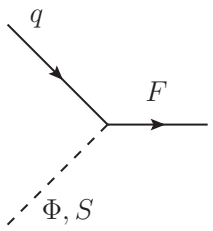


Figure 2: A SM quark interacts with a Froggatt-Nielsen fermion.

Further, it is assumed that the Froggatt-Nielsen fermions can only interact with the SM quarks via the doublets and the flavon as illustrated in fig. 2.

3.2 U(1) model in the SM

The simplest form of the mechanism is in the case where the associated symmetry group of the flavon is U(1) and the framework is the one of the SM.

Firstly, an interaction incorporating the Higgs particle $\hat{H}$, responsible for the Froggatt-Nielsen fermion masses, and the flavon S is considered as illustrated in fig. 3.

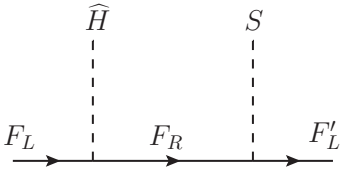


Figure 3: An incoming particle F switches chirality once under $\hat{H}$ and once under S. The flavon charge of F'_L differs from F_L by one unit.

Consider the Feynman amplitude for the process in fig. 3. As the flavon symmetry is spontaneously broken the flavon interaction factor can be replaced with $\langle S \rangle$ and the contribution from $\widehat{H}$ can be replaced with $1/m$, as discussed in sec. 3.1.1. Combining these two one can approximate the amplitude of the process with a constant ε, also known as the *symmetry breaking parameter*. This parameter is fundamental to the mechanism and from its definition below it can be seen that is strictly positive.

$$\varepsilon := \frac{\langle S \rangle}{m} \tag{3.3}$$

The result of the process is that the outgoing particle differs in flavon charge by one unit[5] from the incoming one. For an incoming quark q_i with a charge $c + b_i$ this process can be repeated $b_i + a_j$ times until a quark q_j with a charge $c - a_j$ is outgoing.

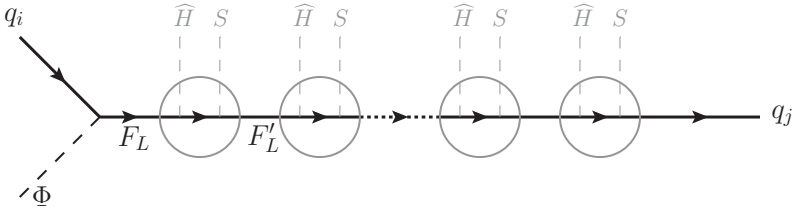


Figure 4: A quark q_L with charge b_i is incoming, interacts with a Higgs doublet and turns into an Froggatt-Nielsen fermion. The process from fig. 3 is repeated until a quark q_R with charge a_j is outgoing.

Consider the process depicted in fig. 4. If the process from fig. 3 is repeated n_{ij} times then the outgoing particle will have a charge $b_i + R - n_{ij}$ where R is the flavon charge of the

[5]One unit means the flavon charge of S, which here will be taken to be -1.

Higgs field. Since this must be equal to a_j the amount of steps required is $n_{ij} = b_i + a_j + R$. For each step the Feynman amplitude is suppressed by the symmetry breaking parameter resulting in a total suppression of $\varepsilon^{b_i+a_j+R}$. The amplitude for the total process, considering the down quark sector, is (without any implicit summation) given by

$$\mathcal{M}_{ij} \to \varepsilon^{b_i+a_j+R}\, \overline{Q}_i Y_{ij} \Phi q_j. \tag{3.4}$$

The Yukawa matrices here include the couplings between Froggatt-Nielsen fermions and the flavon, as shown in fig. 3, i.e.

$$Y_{ij} = g_{q_i \Phi F_1} g_{F_n S q_j} \prod_{k=1}^{n-1} g_{F_k S F_{k+1}}. \tag{3.5}$$

Worth to notice is that if by naturalness all these couplings are of order unity then Y_{ij} is also of order unity.

The amplitude $|\mathcal{M}_{ij}|$ corresponds to the effective Lagrangian of the process as the intermediate Froggatt-Nielsen fermion propagators and reactions are replaced by powers of ε, i.e. the process in fig. 4 is replaced by a single point-like vertex. The obtained effective Lagrangian for the entire down sector becomes

$$\mathcal{L}_{\text{eff}}^{\text{Down}} = -\sum_{ij} \varepsilon^{b_i+a_j+R}\, \overline{Q}_i Y_{ij} \Phi D_j. \tag{3.6}$$

After symmetry breaking of the Higgs field the mass matrix in eq. 3.7 can be identified in the same manner as in sec. 2.3. This type of matrix will also be called *weighted coupling matrix*.

$$M_{ij} = Y_{ij} \varepsilon^{b_i+a_j+R} \tag{3.7}$$

Now the next step is to find the eigenvalues to this mass matrix, i.e. the mass eigenstates. But first a mathematical detour. Begin by considering a 2×2 matrix M where the elements are products of couplings from a coupling matrix G and powers of ε. Using the element matrix multiplication $\bullet$ the matrix M can be written as[6]

$$M = G \bullet \begin{pmatrix} \varepsilon^{b_1+a_1} & \varepsilon^{b_1+a_2} \\ \varepsilon^{b_2+a_1} & \varepsilon^{b_2+a_2} \end{pmatrix}. \tag{3.8}$$

For simplicity, let $G_{n\times n}$ be the $n \times n$ sub matrix of G starting in the upper left corner with $|G_{0\times 0}| = 1$. This gives e.g. $G_{1\times 1} = g_{11}$.

The eigenvalues λ of M are derived from its characteristic equation. One can approximate λ by utilizing a Taylor series. Considering only the leading order in ε gives

$$2\lambda = |G_{1\times 1}|\varepsilon^{b_1+a_1} + g_{22}\varepsilon^{b_2+a_2} \pm \left(|G_{1\times 1}|\varepsilon^{b_1+a_1} + g_{22}\varepsilon^{b_2+a_2} - 2\frac{|G_{2\times 2}|}{|G_{1\times 1}|}\varepsilon^{a+b} \right).$$

[6] $(A \bullet B)_{ij} = A_{ij} B_{ij}$.

Thus, the eigenvalues can be expressed in the following form (again a leading order approximation)

$$\lambda = \begin{cases} |G_{1\times 1}|\varepsilon^{b_1+a_1} \\ \frac{|G_{2\times 2}|}{|G_{1\times 1}|}\varepsilon^{b_2+a_2} \end{cases} . \tag{3.9}$$

This reasoning can be extended to 3×3 matrices, giving eq. 3.10 - which is done in [3]. Note that the eigenvalues are here given in a decreasing order. One should keep in mind that b_i and a_j are ordered.

$$\lambda_i \approx \frac{|G_{i\times i}|}{|G_{i-1\times i-1}|}\varepsilon^{b_i+a_i} \tag{3.10}$$

Further, label the ratio of determinants - which will be abbreviated *RD* - as $F(i)$ where i is the dimension of the matrix in the numerator. The diagonalized mass-matrix is then given by

$$\boxed{\widetilde{M}_{ij} = |F(i)|\varepsilon^{b_i+a_j}\delta_{ij}.} \tag{3.11}$$

In order to find the left-handed bi-unitary transformation matrices V_L (which are needed to calculate the CKM-matrix) one can construct a product of mass matrices eliminating V_R;

$$\widetilde{M} = V_L M V_R^\dagger \Rightarrow \widetilde{M}\widetilde{M}^\dagger = V_L M M^\dagger V_L^\dagger. \tag{3.12}$$

Hence, finding V_L corresponds to finding the eigenvectors of the product $MM^\dagger$, which behaves as

$$\left(MM^\dagger\right)_{ij} = \varepsilon^{b_i+b_j}\sum_k g_{ik}g_{jk}\,\varepsilon^{2a_k} := h_{ij}\varepsilon^{b_i+b_j}. \tag{3.13}$$

Assume that the eigenvectors of $MM^\dagger$ are powers of ε. Now, presume that the coefficients h are of order unity in order to get an understanding of how the matrices behave as powers of ε. Further, since $MM^\dagger$ has the same structure as M the eigenvalues λ of $MM^\dagger$ are known to be approximately the diagonal elements. Considering the eigenvector equation gives the following system of equations:

$$\begin{cases} \varepsilon^{b_1+b_1+x} + \varepsilon^{b_1+b_2+y} + \varepsilon^{b_1+b_3+z} = \lambda\varepsilon^x, \\ \varepsilon^{b_2+b_1+x} + \varepsilon^{b_2+b_2+y} + \varepsilon^{b_2+b_3+z} = \lambda\varepsilon^y, \\ \varepsilon^{b_3+b_1+x} + \varepsilon^{b_3+b_2+y} + \varepsilon^{b_3+b_3+z} = \lambda\varepsilon^z. \end{cases} \tag{3.14}$$

Since the transformation matrices are unitary their diagonal elements should be approximately one. Using this fact a leading order solution in ε to eq. 3.14 can be constructed as

$$\lambda = \varepsilon^{b_1+b_1} : \begin{pmatrix} \varepsilon^0 \\ \varepsilon^{b_2-b_1} \\ \varepsilon^{b_3-b_1} \end{pmatrix}, \quad \lambda = \varepsilon^{b_2+b_2} : \begin{pmatrix} \varepsilon^{b_2-b_1} \\ \varepsilon^0 \\ \varepsilon^{b_3-b_2} \end{pmatrix}, \quad \lambda = \varepsilon^{b_3+b_3} : \begin{pmatrix} \varepsilon^{b_3-b_1} \\ \varepsilon^{b_3-b_2} \\ \varepsilon^0 \end{pmatrix}. \tag{3.15}$$

Combining said eigenvectors to a block matrix gives $(V_L)_{ij} = \varepsilon^{|b_j-b_i|}$.

This can be treated in a more exact manner, by not neglecting h, as is done in appendix C where the same powers in ε are obtained. The exact treatment will be used later for numerical calculations.

Finally, since no assumption was made on working in the up or down sector when deriving V_L the transformation matrices must be the same in both cases (in this approximation). Hence, the CKM-matrix is to leading order given by

$$\boxed{(V_{\text{CKM}})_{ij} = \varepsilon^{|b_i - b_j|}.} \tag{3.16}$$

In this convention the CKM matrix has V_{tb} as its first element and V_{ud} as its last. Following the same convention[7] gives

$$\boxed{\sin\theta_1 \approx \varepsilon^{b_3-b_2} \qquad \sin\theta_2 \approx \varepsilon^{b_3-b_1} \qquad \sin\theta_3 \approx \varepsilon^{b_2-b_1}} \tag{3.17}$$

in the standard parametrization in a leading order approximation. For reference, $\sin\theta_1 \approx 0.2$.

[7]The index mapping is $\{t, b\} \to 1$, $\{c, s\} \to 2$ and $\{u, d\} \to 3$.

3.3 Extension to 2HDM

The mechanism can be incorporated into a Two-Higgs-Doublet Model (2HDM III). However, since there are now two doublets instead of one there are also two contributions. Not only does this complicate the mass matrix but it also introduces the possibility of FCNC:s at tree level.

The effective Lagrangian derived in the previous subsection takes on the form in eq. 3.18 for the down sector where R_1 and R_2 are the flavon charges of the two doublets.

$$\mathcal{L}_{\text{eff}}^{\text{Down}} = \sum_{ij} \varepsilon^{b_i+a_j} \overline{Q}_i \left[Y_{ij}^1 \varepsilon^{R_1} \Phi_1 + Y_{ij}^2 \varepsilon^{R_2} \Phi_2 \right] D_j \tag{3.18}$$

In order to find separate mass currents and FCNC:s change to the Higgs basis;

$$\mathcal{L}_{\text{eff}}^{\text{Down}} = \sum_{ij} \varepsilon^{b_i+a_j} \overline{Q}_i \left[Y_{ij}^1 \varepsilon^{R_1} (H_1 \cos\beta - H_2 \sin\beta) + Y_{ij}^2 \varepsilon^{R_2} (H_1 \sin\beta + H_2 \cos\beta) \right] D_j.$$

As discussed in sec. 2.3.1 the coupling matrix to H_1 is diagonal while H_2 necessarily is not. Hence the mass matrix M and the FCNC matrix K can be identified as

$$\begin{aligned} M_{ij} &= \varepsilon^{b_i+a_j} \left[Y_{ij}^1 \varepsilon^{R_1} \cos\beta + Y_{ij}^2 \varepsilon^{R_2} \sin\beta \right], \\ K_{ij} &= \varepsilon^{b_i+a_j} \left[-Y_{ij}^1 \varepsilon^{R_1} \sin\beta + Y_{ij}^2 \varepsilon^{R_2} \cos\beta \right]. \end{aligned} \tag{3.19}$$

In a similar manner to sec. 3.2 the eigenvalues of the mass matrix can be calculated in an easier way than the usual eigenvalue decomposition. As was shown earlier, the approximate eigenvalues depend on the determinant of $Y_{ij}^1 \varepsilon^{R_1} \cos\beta + Y_{ij}^2 \varepsilon^{R_2} \sin\beta$. Said determinant can be expressed in terms of Γ which is the sum of the determinants of all row permutations of two matrices [8]. Introduce the function $T(i)$ as done in:

$$\begin{aligned} T(i) &:= \left| Y_{ij}^1 \varepsilon^{R_1} \cos\beta + Y_{ij}^2 \varepsilon^{R_2} \sin\beta \right| = \sum_{k=1}^{i} \Gamma_i^k (Y_{ij}^1 \varepsilon^{R_1} \cos\beta, Y_{ij}^2 \varepsilon^{R_2} \sin\beta) \\ &\overset{Y_{ij} \approx 1}{\approx\approx} \sum_{k+l=i} \cos^k\beta \sin^l\beta \, \varepsilon^{kR_1+lR_2} \, \Gamma_i^k (Y_{ij}^1, Y_{ij}^2). \end{aligned}$$

The RD then becomes

$$\frac{|G_{i\times i}|}{|G_{i-1\times i-1}|} = \frac{T(i)}{T(i-1)} =: F(i). \tag{3.20}$$

Using F the diagonalized mass matrix $\widetilde{M}$ is can be expressed as[8]

$$\boxed{\widetilde{M}_{ij} = |F(i)| \, \varepsilon^{b_i+a_i} \delta_{ij}.} \tag{3.21}$$

Since $\widetilde{M}$ should be roughly the same as the diagonal of M it is reasonable to expect $|F(i)| \sim \left[Y_{ij}^1 \varepsilon^{R_1} \cos\beta + Y_{ij}^2 \varepsilon^{R_2} \sin\beta \right]$.

[8]The absolute value of F is added by choice to guarantee non-negative masses

For sake of interest, if $Y_{ij} \approx 1$ and assuming[9] that $\Gamma_i^k(Y_{ij}^1, Y_{ij}^2) \approx 1$ then F takes on the following form:

$$F(i) = \sum_{k+l=i} \varepsilon^{kR_1+lR_2} \cos^k\beta \sin^l\beta \Bigg/ \sum_{k+l=i-1} \varepsilon^{kR_1+lR_2} \cos^k\beta \sin^l\beta \,. \tag{3.22}$$

At this point it is difficult to continue analytically without further assumptions. Hence two cases will be discussed; $Y \approx \mathbb{I}$ and $Y_{ij} \approx 1$.

3.3.1 Coupling matrices are identity matrices

In the case of $Y \approx \mathbb{I}$ finding the transformation matrices are trivial, since M and K are diagonal to start with. This implies that

$$\begin{aligned} \widetilde{M}_{ij} &= \varepsilon^{b_i+a_j} \left[\varepsilon^{R_1}\cos\beta + \varepsilon^{R_2}\sin\beta\right] \delta_{ij}, \\ \widetilde{K}_{ij} &= \varepsilon^{b_i+a_j} \left[-\varepsilon^{R_1}\sin\beta + \varepsilon^{R_2}\cos\beta\right] \delta_{ij}. \end{aligned} \tag{3.23}$$

Since $V = \mathbb{I}$ it follows that $V_{\text{CKM}} = \mathbb{I}$, further, since $\widetilde{K}$ is diagonal this approach has no FCNC:s. Hence the only constraint on possible charges in this model comes from the mass ratios (eq. 3.24) and depend only on ε, b_i and a_j.

$$\frac{m_i}{m_j} = \varepsilon^{(b_i-b_j)+(a_i-a_j)} \tag{3.24}$$

For reference, consider the Cheng-Sher Ansatz from eq. 2.14 gives

$$\lambda_{ij} = \varepsilon^{\frac{1}{2}(b_i+a_j)-\frac{1}{2}(b_j+a_i)} \frac{-\varepsilon^{R_1}\sin\beta + \varepsilon^{R_2}\cos\beta}{\varepsilon^{R_1}\cos\beta + \varepsilon^{R_2}\sin\beta}\delta_{ij}. \tag{3.25}$$

3.3.2 Couplings of order unity

Given the diagonalized mass matrices from eq. 3.21 the transformation matrices can be calculated. The main behaviour of V_L should be unchanged from the SM case, i.e. $V_L \sim \varepsilon^{|b_i-b_j|}$. Define the matrix h such that $\left(MM^\dagger\right)_{ij} = h_{ij}\varepsilon^{b_i+b_j}$.

Under the assumption that all couplings are of order unity, i.e. $Y_{ij} \approx 1$, it is reasonable to expect that all elements of h are roughly the same. Using the results derived in appendix C the coefficients of the powers of ε vanish under this assumption. Hence the resulting left-handed transformation matrices are the same as in the SM-case,

$$\left(V_L\right)_{ij} \approx \varepsilon^{|b_i-b_j|}. \tag{3.26}$$

[9] A rough assumption. The determinant of matrix with normal random numbers ($\mu = 1$, $\sigma = 1/2$) is typically confined to $[-5, 5]$.

The right-handed transformation matrices can be found using the inverse diagonalized mass matrix, $\tilde{M}^{-1} = \text{diag}\left\{\varepsilon^{-b_i-a_i}/|F(i)|\right\}$, and the mass matrix itself (see appendix B). Since M transforms biunitary it follows that

$$V_R = \widetilde{M}^{-1} V_L M. \tag{3.27}$$

Given V_L and V_R one can also calculate the FCNC:s of the model;

$$\widetilde{K} = V_L K V_R^\dagger. \tag{3.28}$$

Doing a leading order approximation results (eq. 3.29) in that the right-handed transformation matrices only depend on the right-handed charges[10].

$$(V_R)_{ij} \approx \frac{1}{|F(i)|}\varepsilon^{|a_j-a_i|}\left[\varepsilon^{R_1}\cos\beta + \varepsilon^{R_2}\sin\beta\right] \tag{3.29}$$

However, if our assumption on the behaviour on $|F|$ from eq. 3.21 is correct then $(V_R)_{ij}$ reduces to $\varepsilon^{|a_j-a_i|}$. This seems reasonable since if one would calculate V_R from the eigenvectors of $M^\dagger M$ and let V_L be a function of V_R then this result does seem plausible. Note that the product $M^\dagger M$ does only depend on the right-handed charges but is otherwise identical to $MM^\dagger$.

Further, this allows for a leading order approximate calculation of the current matrix;

$$\boxed{\widetilde{K}_{ij} \approx \varepsilon^{b_i+a_j}\left[-\varepsilon^{R_1}\sin\beta + \varepsilon^{R_2}\cos\beta\right].} \tag{3.30}$$

This behaviour in ε is expected since $K_{ij} \sim \varepsilon^{b_i+a_j}$ and the transformation matrices become roughly the identity matrix when examining only the leading order in ε.

Now, consider the Cheng-Sher ansatz discussed in sec. 2.3.2 for eq. 3.30. Solving for the power in ε gives

$$\lambda_{ij} = \varepsilon^{\frac{1}{2}(b_i+a_j)-\frac{1}{2}(a_i+b_j)}\frac{-\varepsilon^{R_1}\sin\beta + \varepsilon^{R_2}\cos\beta}{\varepsilon^{R_1}\cos\beta + \varepsilon^{R_2}\sin\beta}|F(i)|^{-1/2}|F(j)|^{-1/2}. \tag{3.31}$$

This is almost the same result as obtained in sec. 3.3.1, but the Kronecker delta have been replaced with $|F(i)|^{-1/2}|F(j)|^{-1/2}$.

Finally, since the up and down quarks share left-handed flavon charge the left-handed transformation matrices should be approximately the same (due to the unitarity property) and hence V_{CKM}. The only difference is in the unweighed coupling matrices. If one performs the multiplication $V_L V_L^\dagger$ (using $(V_L)_{ij} \approx \varepsilon^{|b_i-b_j|}$) the same result as in sec. 3.2 is obtained, i.e.

$$\boxed{(V_{\text{CKM}})_{ij} \approx \varepsilon^{|b_i-b_j|}.} \tag{3.32}$$

The requirement on the Euler angles of the standard parametrization from eq. 3.17 still holds.

[10]The absolute value is as earlier added for physical reasons.

3.4 Speculations on U(1)×...× U(1)

So far only a single flavon with a U(1) symmetry have been considered. Now, assume that there in fact exist not only one flavon but several and their total symmetry group is $\mathrm{U}(1)^N$ for some $N \in \mathbb{N}$. Each flavon S_l has its own type of flavon charge (b^l for left-handed quarks and a^l for right-handed quarks) and associated symmetry breaking parameter $\varepsilon(S_l)$.

The number of exchanges n_{ij}^l of the flavon S_l required for the process $q_i \to q_j$ is given by the solutions to the linear system of equations in eq. 3.33.

$$\sum_l n_{ij}^l \mathbf{R}_l = \sum_l (b_i^l + a_j^l + R_k^l)\hat{\mathbf{e}}_l \tag{3.33}$$

Here $\mathbf{R}_l$ denotes the charge of the lth flavon in the total flavon charge space spanned by $\hat{\mathbf{e}}_l$ and R_k^l the lth component of the flavon charge of doublet k. Note that n_{ij}^l is different depending on which doublet is considered and whether one is working in the up or down sector.

If n_{ij}^l is written as $B_i^l + A_j^l + r_k^l$ for some new unknown constants B_i^l, A_j^l and r_k^l then eq. 3.33 has solutions if eq. 3.34 has solutions.

$$\begin{cases} \sum_l B_i^l \mathbf{R}_l = \sum b_i^l \hat{\mathbf{e}}_l \\ \sum_l A_j^l \mathbf{R}_l = \sum a_j^l \hat{\mathbf{e}}_l \\ \sum_l r_k^l \mathbf{R}_l = \sum R_k^l \hat{\mathbf{e}}_l \end{cases} \tag{3.34}$$

The total suppression from all symmetry groups is described by the amplitude in eq. 3.35 for the down sector. Note that the same formalism also applies to the up sector.

$$\mathcal{M}_{ij} = -\sum_k \overline{Q}_i Y_{ij}^k \Phi_k q_j \prod_l \varepsilon^{B_i^l + A_j^l + r_k^l}(S_l) \tag{3.35}$$

If $\varepsilon(S_1) = ... = \varepsilon(S_n)$ then the ordering of $\sum_l B_i^l$ and $\sum_l A_j^l$ is important. Since if $\sum_l B_i^l \leq \sum_l B_{i+1}^l$ and $\sum_l A_j^l \leq \sum_l A_{j+1}^l$ then the the same machinery from sec. 3.3 can be used where the summed parameters take the roles of b_i and a_j.

Further, if $\mathbf{R}_l = \hat{\mathbf{e}}_l$ then $B_i^l = b_i^l$, $A_j^l = a_j^l$ and $r_k^l = R_k^l$. If also $\varepsilon(S_1) = ... = \varepsilon(S_n)$ then the amplitude reduces to

$$\mathcal{M}_{ij} = -\varepsilon^{\sum_l b_i^l + \sum_l a_j^l} \sum_k \overline{Q}_i Y_{ij}^k \Phi_k q_j \varepsilon^{\sum_l R_k^l}. \tag{3.36}$$

Denote $\sum_l b_i^l = \hat{b}_i$, $\sum_l a_j^l = \hat{a}_j$ and $\sum_l r_k^l = \hat{r}_k$. The theory of sec. 3.3 applies if b_i is replaced with $\hat{b}_i$, a_j with $\hat{a}_j$ and r_k with $\hat{r}_k$. Since the requirement of ordering holds the same results is obtained. However, this is in principle no difference to a single U(1) symmetry. Phenomenologically, more degrees of freedom are introduced and hence such a model is better suited for fitting experimental data.

To conclude, considering a $\mathrm{U}(1)^n$ model is interesting if the different flavons have different VEVs or if $\mathbf{R}_l \neq \hat{\mathbf{e}}_l$ for some flavon.

3.5 Relation between $\cos\beta$ and $\sin\beta$

Further study the mixing angle β introduced in sec. 2.2. Since both v_1 and v_2 are non-negative it follows that $\beta \in \mathbb{R}[0, \pi/2]$. Further, there should be some relation between the quantities $\cos\beta$ and $\sin\beta$. Assume that

$$\cos\beta \approx \varepsilon^{\eta} \sin\beta \qquad \eta \in \mathbb{Z}. \tag{3.37}$$

Applying this assumption gives the rotation matrix

$$\mathcal{R}_2 = \sin\beta \begin{pmatrix} \varepsilon^{\eta} & 1 \\ -1 & \varepsilon^{\eta} \end{pmatrix}. \tag{3.38}$$

Preferred bounds are $\tan\beta \in [1, 50]$, as discussed by [9], which are equivalent with $\cos\beta \leq \sin\beta \leq 50\cos\beta$. Using the approximation introduced in eq. 3.37 gives

$$0 \leq \eta \leq |\log_{\varepsilon} 50|. \tag{3.39}$$

For example, if $\varepsilon = 1/5$ is chosen then $\eta \in \{0, 1, 2\}$.

Since $\sin\beta$ is of order ε^0 it will be ignored in $\mathcal{R}_2$ henceforth[11]. In this assumption the mass and current matrices are given by

$$\begin{aligned} M_{ij} &= \varepsilon^{b_i + a_j} \left[(Y_1)_{ij} \, \varepsilon^{R_1 + \eta} + (Y_2)_{ij} \, \varepsilon^{R_2} \right] \\ K_{ij} &= \varepsilon^{b_i + a_j} \left[- (Y_1)_{ij} \, \varepsilon^{R_1} + (Y_2)_{ij} \, \varepsilon^{R_2 + \eta} \right]. \end{aligned} \tag{3.40}$$

The leading order approximations for different cases where M and K behave differently relative to each other are presented in 1. Worth to notice is that due to our requirement of $\eta \geq 0$ case 2 is not possible.

Table 1: Different cases depending on R_1, R_2 and η.

Case	Currents
1: $R_1 \leq R_2 - \eta$	$M_{ij} \approx \varepsilon^{b_i + a_j + R_1 + \eta} (Y_1)_{ij}$ $K_{ij} \approx -\varepsilon^{b_i + a_j + R_1} (Y_1)_{ij}$
2: $R_1 + \eta \leq R_2 \leq R_1 - \eta$	$M_{ij} \approx \varepsilon^{b_i + a_j + R_1 + \eta} (Y_1)_{ij}$ $K_{ij} \approx \varepsilon^{b_i + a_j + R_2 + \eta} (Y_2)_{ij}$
3: $R_2 - \eta \leq R_1 \leq R_2 + \eta$	$M_{ij} \approx \varepsilon^{b_i + a_j + R_2} (Y_2)_{ij}$ $K_{ij} \approx -\varepsilon^{b_i + a_j + R_1} (Y_1)_{ij}$
4: $R_2 \leq R_1 - \eta$	$M_{ij} \approx \varepsilon^{b_i + a_j + R_2} (Y_2)_{ij}$ $K_{ij} \approx \varepsilon^{b_i + a_j + R_1 + \eta} (Y_2)_{ij}$

[11] If not neglected this factor vanishes when calculating λ.

4 Phenomenology

In this section some experimental data is introduced followed by a U(1) model with designated charges trying mimic said data.

Firstly, the quark masses are needed. Since the models discussed only treat relations between the different quarks only those are considered. Said relations are presented in table 2 where the relations are derived from individual quark masses in the $\overline{\text{MS}}$ scheme from Particle Data Group (2014) [2]. Calculation of the errors were carried out using $\text{cov}[m_i, m_j] \propto \delta_{ij}$ and the standard formula for error propagation presented in eq 4.1 for two quantities A and B, their errors σ_A and σ_B and the resulting multiplication/division, f, of A and B.

$$\sigma_C^2 \overset{\text{cov}[A,B]=0}{=\!=} |f|^2 \left(\sigma_A^2/A^2 + \sigma_B^2/B^2\right) \tag{4.1}$$

Table 2: Quark mass relations for the up and down sector.

Relation	Value	Relation	Value
m_c/m_u	$5 \cdot 10^2 \pm 2 \cdot 10^2$	m_s/m_d	$2.0 \cdot 10^1 \pm 2 \cdot 10^0$
m_t/m_u	$7 \cdot 10^4 \pm 2 \cdot 10^4$	m_b/m_d	$8.7 \cdot 10^2 \pm 9 \cdot 10^1$
m_t/m_c	$1.25 \cdot 10^2 \pm 5 \cdot 10^0$	m_b/m_s	$4.4 \cdot 10^1 \pm 2 \cdot 10^0$

An additional mass ratio that is interesting is

$$\frac{m_t}{m_b} = 3.8 \cdot 10^1 \pm 1 \cdot 10^0. \tag{4.2}$$

Secondly, the standard parametrization of the CKM-matrix will be used to evaluate the performance of the constructed model. Measurements of the Euler angles from the parametrization are displayed in table 3 and can be calculated using eq. 2.22 and the CKM elements from Particle Data Group (2014) [2]. This was done using eq. 4.1 and under the assumption that all elements of the CKM-matrix are real.

Table 3: Measurements of the angles in the standard parametrization of the CKM-matrix.

Parameter	Value [Degrees]
$\sin\theta_1$	$2.254 \cdot 10^{-1} \pm 6 \cdot 10^{-4}$
$\sin\theta_2$	$4.1 \cdot 10^{-3} \pm 5 \cdot 10^{-4}$
$\sin\theta_3$	$4.1 \cdot 10^{-2} \pm 1 \cdot 10^{-3}$

One ratio that will be interesting later is

$$\frac{\sin\theta_2}{\sin\theta_3} = 1.0 \cdot 10^{-1} \pm 1 \cdot 10^{-2}. \tag{4.3}$$

4.1 U(1) model

One can incorporate the experimental constraints into the charges. This is done below for the mass ratios presented on the previous page and the standard parametrization.

From the approximation in eq. 3.17 the following system of equations below arise;

$$\begin{cases} \log_\varepsilon(\sin\theta_1) = b_3 - b_2, \\ \log_\varepsilon(\sin\theta_2) = b_3 - b_1, \\ \log_\varepsilon(\sin\theta_3) = b_2 - b_1. \end{cases} \tag{4.4}$$

If $\sin\theta_1 = \sin\theta_2/\sin\theta_3$ the system has an infinite amount of solutions and if not there exists no solution. From eq. 4.3 it is clear that they are roughly of the same order. Hence the following solutions are obtained;

$$\begin{cases} b_1 = s, \\ b_2 = s + \log_\varepsilon(\sin\theta_3), \\ b_3 = s + \log_\varepsilon(\sin\theta_2). \end{cases} \tag{4.5}$$

From the experimental mass ratios one can introduce constraints on the right-handed charges. Here it is shown for the up sector with $R_1 = 0$ but the same applies for the down sector. If the RD for two different masses cancel then;

$$\begin{aligned} \frac{m_i}{m_j} \approx \varepsilon^{b_i - b_j + a_i - a_j} \Rightarrow & \begin{cases} \log_\varepsilon(m_c/m_u) + b_3 - b_2 = a_2^u - a_3^u, \\ \log_\varepsilon(m_t/m_u) + b_3 - b_1 = a_1^u - a_3^u, \\ \log_\varepsilon(m_t/m_c) + b_2 - b_1 = a_1^u - a_2^u, \end{cases} \\ \Rightarrow & \begin{cases} a_1^u = s' \\ a_2^u = s' + b_1 - b_2 - \log_\varepsilon(m_t/m_c), \\ a_3^u = s' + b_1 - b_3 - \log_\varepsilon(m_t/m_u). \end{cases} \end{aligned} \tag{4.6}$$

One can consider e.g. the mass ratio m_t/m_b in order to relate a^u to a^d;

$$\log_\varepsilon\left(\frac{m_t}{m_b}\right) = a_1^u - a_1^d. \tag{4.7}$$

Now on to the flavon charges of the doublets. Let $\varepsilon = 1/5$ (inspired by [5]). If one wants the FCNC:s to be more suppressed than the masses then $R_2 \leq R_1 - \eta$ is a natural choice (from table. 1). If $R_1 = 0$ - which was assumed earlier - then $R_2 = -2$ guarantees this relation independently of $\eta \in \{0, 1, 2\}$ since $-2 \leq -\eta$.

A proposal of charges fulfilling eq. 4.4 - 4.7, given table 2 and 3, is presented in table 4.

Table 4: Proposed flavon charges for relevant particles from constraints.

Charge	Value	Origin
R_1	0	Choice
R_2	-2	To guarantee $R_2 \leq R_1 - \eta$
b_1	0	No suppression for top quark
b_2	2	Eq. 4.5
b_3	3	Eq. 4.5
a_1^u	0	No suppression for top quark
a_2^u	2	Mass ratio
a_3^u	4	Mass ratio
a_1^d	2	Eq. 4.7
a_2^d	2	Mass ratio
a_3^d	3	Mass ratio

One of the difficulties of the model is knowing what values the coupling matrices have. In this study two different approaches are considered.

1. Approximate identity; $Y \approx \mathbb{I}$. In this regime the results of sec. 3.3.1 are valid. It follows that

$$\widetilde{K}_u \approx \begin{pmatrix} \varepsilon^0 & 0 & 0 \\ 0 & \varepsilon^4 & 0 \\ 0 & 0 & \varepsilon^7 \end{pmatrix} \left[-1 + \varepsilon^{\eta-2}\right] \qquad \widetilde{K}_d \approx \begin{pmatrix} \varepsilon^2 & 0 & 0 \\ 0 & \varepsilon^4 & 0 \\ 0 & 0 & \varepsilon^6 \end{pmatrix} \left[-1 + \varepsilon^{\eta-2}\right]. \tag{4.8}$$

   The Cheng-Sher parametrization in this approach should be approximately diagonal.

2. Natural couplings; $Y_{ij} \approx 1$. Here the results of sec. 3.3.2 are valid. It follows that

$$\widetilde{K}_u \approx \begin{pmatrix} \varepsilon^0 & \varepsilon^2 & \varepsilon^3 \\ \varepsilon^3 & \varepsilon^4 & \varepsilon^5 \\ \varepsilon^4 & \varepsilon^6 & \varepsilon^7 \end{pmatrix} \left[-1 + \varepsilon^{\eta-2}\right] \qquad \widetilde{K}_d \approx \begin{pmatrix} \varepsilon^2 & \varepsilon^4 & \varepsilon^5 \\ \varepsilon^2 & \varepsilon^4 & \varepsilon^5 \\ \varepsilon^3 & \varepsilon^5 & \varepsilon^6 \end{pmatrix} \left[-1 + \varepsilon^{\eta-2}\right] \tag{4.9}$$

   which gives the following Cheng-Sher parametrization

$$\lambda_u \approx \begin{pmatrix} 1 & 1 & \varepsilon^{-1/2} \\ \varepsilon^1 & 1 & \varepsilon^{-1/2} \\ \varepsilon^{1/2} & \varepsilon^{1/2} & 1 \end{pmatrix} \left[-1 + \varepsilon^{\eta-2}\right] \qquad \lambda_d \approx \begin{pmatrix} 1 & \varepsilon^1 & \varepsilon^1 \\ \varepsilon^{-1} & 1 & 1 \\ \varepsilon^{-1} & 1 & 1 \end{pmatrix} \left[-1 + \varepsilon^{\eta-2}\right]. \tag{4.10}$$

4.1.1 Computation

In order to calculate the mass ratios, the CKM angles and the resulting currents numerical computations were used. The advantage of a numerical approach is that the RD and the coefficients of the ε powers in V_L can be treated more formally. Since the coupling matrices of the model are unknown the computations were performed using random numbers generated according to the following two approaches.

1. Approximate identity; $Y = \mathbb{I} + \xi_1$ where ξ_1 is a matrix of normal random numbers with zero average and standard deviation $1/2$.

2. Natural couplings; $Y = \xi_2$ where ξ_2 is a matrix of normal random numbers with average 1 and standard deviation $1/2$.

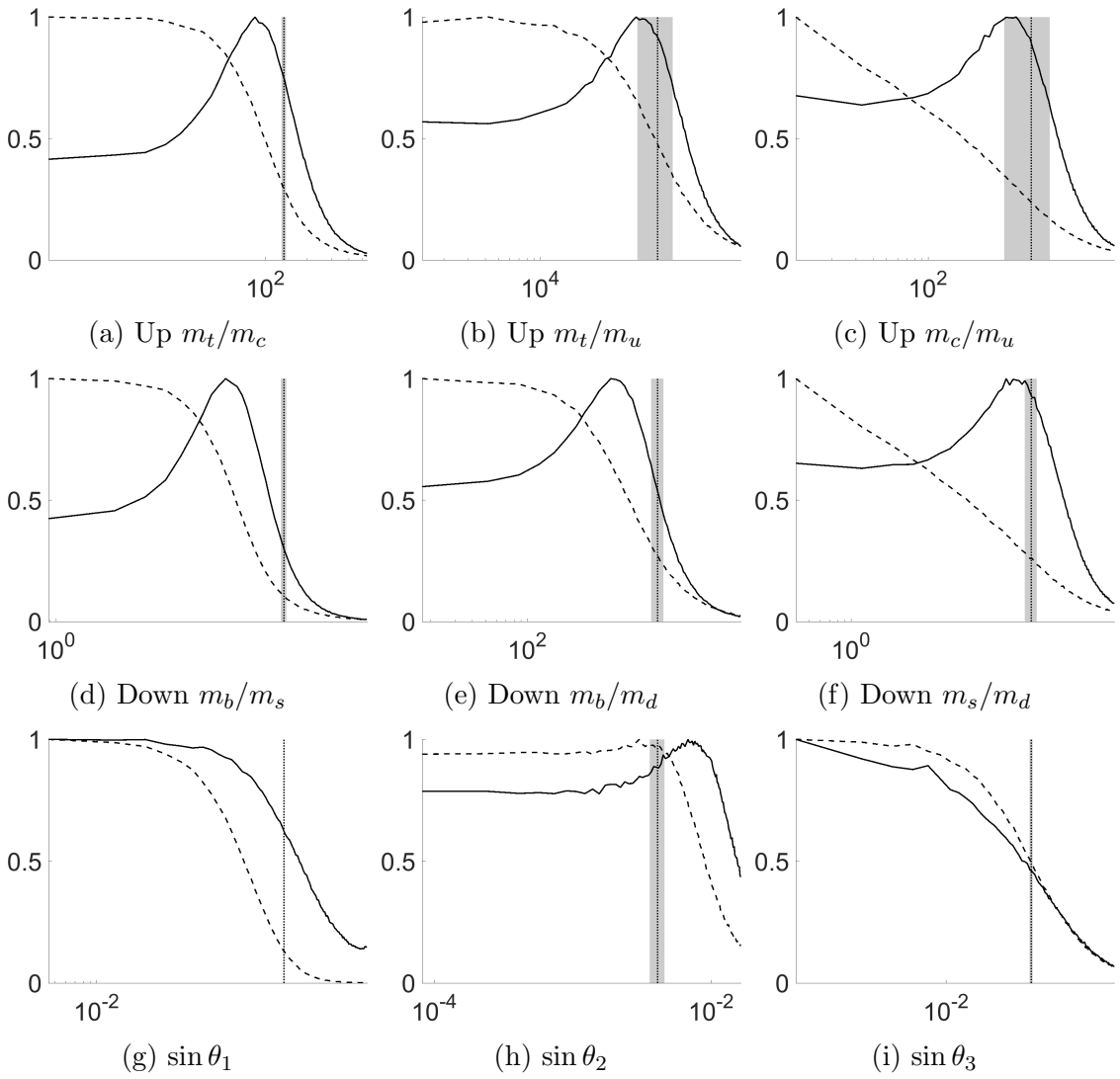
The matrices M and K from eq. 3.40 were used with $\eta \in \{0, 1, 2\}$. The coupling matrices in said equation were generated from one of the two approaches above. From M the mass eigenstates were calculated using eq. 3.21 but without any assumptions on the determinant of the coupling matrices. Using the mass matrix and the mass eigenstates the transformation matrices were constructed in accordance with appendix B and C - which allowed for calculation of $\widetilde{K}$ and V_{CKM}.

For each approach 10^6 computations were done in MATLAB. Each computation was based on a new set of coupling matrices that were generated using approach 1 or 2. In order to evaluate the accuracy of the results (and to decide the optimal value on η) a version of a χ^2 test was utilized (see eq. 4.11). Denote the data (i.e. m_2/m_1, m_3/m_1, ...) as $\mathbf{x}^1, ..., \mathbf{x}^n$ which are measures of the experimental values $x^1_{\text{exp}}, ..., x^n_{\text{exp}}$ with a experimental error $\sigma_1, ..., \sigma_n$.

The index j sums over the different probed parameters while k spans over the different computations ($k \in \mathbb{N}[0, 10^6]$).

$$\chi^2(\mathbf{x}) = \sum_{jk} \frac{1}{\sigma_j^2} \left| x_k^j - x_{\text{exp}}^j \right|^2 \tag{4.11}$$

A value of $\varepsilon = 1/5$ (inspired by [5]) was used. For approach 1 $\eta = 2$ gave a minimal χ^2-value, while in approach 2 $\eta = 0$ gave a minimal χ^2-value. The distributions for the different probed parameters with $\eta = 2$ in approach 1 and with $\eta = 0$ in approach 2 are featured in fig. 5.



(a) Up m_t/m_c (b) Up m_t/m_u (c) Up m_c/m_u

(d) Down m_b/m_s (e) Down m_b/m_d (f) Down m_s/m_d

(g) $\sin\theta_1$ (h) $\sin\theta_2$ (i) $\sin\theta_3$

Figure 5: Distributions for different parameters. The figures illustrate approach 1 (solid curve) and approach 2 (dashed curve). The experimental values are marked by horizontal lines and the shaded area is points inside range of experimental error.

The results from the calculations of points in parameter space are displayed in fig. 5. The maximal element of each distribution have been set to 1 in order to display curve behaviour. Hence the vertical axis is of arbitrary units.

One can see that approach 1 peaks close to the experimental mass ratios. However, for the standard parametrization angles both approaches fail.

In terms of naturalness approach 1 seems unlikely and might require an underlying theoretical structure - which the FNM was meant to be for the quark mass hierarchy problem. From the same perspective approach 2 seems most natural. However, in contrast to approach 1 the determinants of the Yukawa matrices are not centred around one but zero with a very wide distribution.

Choosing the randomized coupling matrices that are within 9σ from the experimental values[12] for the probed parameters (fig. 5) give 2 surviving points in approach 1 and 1 point in approach 2. The FCNC parameters λ_{12}, λ_{13} and λ_{23} for the surviving points are given in tables[13] 5 and 6 below. From the numerical computations one could see that the currents are the same for both the up and the down sector, e.g. $\lambda_{12} = \lambda_{tc} = \lambda_{bs}$. This contradicts the analytic results in 4.8, 4.9 and 4.10.

Table 5: Resulting λ parameters for the FCNC of surviving points in approach 1 with order estimations.

Point	$\lvert\lambda_{12}\rvert$	$\lvert\lambda_{13}\rvert$	$\lvert\lambda_{23}\rvert$
1	1950	2.65	0.90
2	3500	6.90	6.04
Order	ε^{-5}	ε^{-1}	ε^{-1}

Table 6: Resulting λ parameters for the FCNC of surviving points in approach 2 with order estimations.

Point	$\lvert\lambda_{12}\rvert$	$\lvert\lambda_{13}\rvert$	$\lvert\lambda_{23}\rvert$
1	7810	15.3	5.67
Order	ε^{-6}	ε^{-2}	ε^{-1}

It is clear that the results in table 5 and 6 differ greatly from the ε power matrices derived earlier where the maximal element was ε^{-1}. In additions, there is no distinction between the up and down sector in these numerical results - which the analytic results indicate that there should be.

One can analytically see that the Cheng-Sher parametrization gives rise to $\lambda \geq 1$ in approach 2 (e.g. λ_{31}). Currents of that size should already have been discovered.

Numerically, the points in table 5 and 6 failed at predicting λ_{12} of order unity. Hence, if this model is to be correct then Higgs mediated $t \to c$ and $b \to s$ decays would be observed frequently.

One interesting aspect is that, as was shown in sec. 4.1, approach 1 should have almost no FCNC:s (eq. 4.8). The same applies to the Cheng-Sher parametrization. However, this is in contradiction with the numerical results in table 5. Since the theory of approach 1 is rather trivial this can signal for some numerical error which might be in the code used for the actual calculations.

[12] σ is the experimental error for respective parameter.

[13] The parameters λ_{21}, λ_{31} and λ_{32} are not included in order to simplify this analysis.

5 Conclusions

The derived models used only real parameters. The coupling matrices could equally well be complex which would introduce further degrees of freedom. In order to incorporate complex couplings the majority of the theory would be unchanged. For a more general model one could also omit the requirement of CP-conservation that have been implicit throughout this work.

The distributions in fig. 5 for the two approaches differed greatly where approach 2, motivated by naturalness, failed at predicting any parameter. However, approach 1 was remotely successful predicting the mass ratios but approach 1 does not constitute a satisfying theoretical framework since there is no motivation for it. Although intentions were to include mass ratios and standard parametrization angles as constraints on the charges those constraints were only approximate. The choosing of charges is likely more successful if there are more free parameters in the model to fixate. If more free parameters better can predict the CKM-matrix can be investigated further.

There exists points in at least 9σ from all of the experimental values but those points do not satisfy experimental bounds. Throughout this work it has been assumed that $\lambda_{ij} \approx 1$ but from experiments it is shown that λ is in fact smaller - which invalidates the results even further.

The examined model had a minimal number of degrees of freedom and hence models with symmetry group $\mathrm{U}(1)^{n>1}$ might be better suited for the task of explaining experimental data. One interesting question which can be studied further is: what is required of the flavon charges of the flavons for the parameters A and B from sec. 3.4 to be ordered?

To conclude, this model is not interesting in its current form since it fails at making accurate predictions. Models using higher dimensional symmetry groups might be better suited for such predictions but would also introduce further unknown free parameters.

A Standard Model Higgs Mechanism

In the Standard Model particles acquire mass via the Higgs mechanism. Said mechanism is constructed via introducing a $\mathrm{SU}(2)_L$ doublet Φ, see eq. A.1. For reasons that will be clear later a conjugate doublet, Φ^c, is also constructed.

$$\Phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} \qquad \Phi^c = \begin{pmatrix} -\phi_0^* \\ \phi_+^* \end{pmatrix} \tag{A.1}$$

In order to be able to compactly write down the Lagrangian for Φ one can utilize the covariant derivative D_μ (eq. A.2). Let Y_W, $\boldsymbol{\sigma}$ and $\boldsymbol{\lambda}$ be the generators of $\mathrm{U}(1)_Y$, $\mathrm{SU}(2)_L$ and $\mathrm{SU}(3)_C$ respectively.

$$D_\mu = \partial_\mu + i\frac{g_1}{2} Y_W B_\mu + i\frac{g_2}{2}\boldsymbol{\sigma}\cdot\mathbf{W}_\mu + \left(i\frac{g_3}{2}\boldsymbol{\lambda}\cdot\mathbf{G}_\mu\right) \tag{A.2}$$

Further, assume Φ is singlet under $\mathrm{SU}(3)_C$ which allows us to dismiss the gluon fields from D_μ. This is a choice, since massive gluons are not desired. The Lagrangian becomes

$$\mathcal{L}_{\text{Higgs}} = (D_\mu\Phi)^\dagger D^\mu\Phi - V(\Phi). \tag{A.3}$$

The Higgs potential V is given by

$$V(\Phi) = \mu^2\Phi^2 + \lambda\Phi^4. \tag{A.4}$$

where $\mu^2 \in \mathbb{R}$ and $0 \leq \lambda \in \mathbb{R}$.

If $\mu^2 < 0$ then $\mathrm{SU}(2)_L$ is broken giving rise to a non-zero vacuum expectation value v. This allows for expanding $\Phi = (0, v)^{\mathrm{T}} + \mathbf{h}$. If one for a moment considers a covariant derivative including only one field X then one can see that $(D_\mu\Phi)^\dagger D^\mu\Phi$ contains the following terms;

$$(\partial_\mu - iX_\mu)\Phi^\dagger(\partial^\mu + iX^\mu)\Phi = \partial_\mu\Phi^\dagger\partial^\mu\Phi + X_\mu\Phi^\dagger X^\mu\Phi + ... \tag{A.5}$$

The term $X_\mu\Phi^\dagger X^\mu\Phi$ can, as Φ take upon non-zero VEV, be regarded as a mass term and X has acquired mass.

Worth to notice is that the $\lambda\Phi^4$-term allows for a mass term for the Higgs field itself. This terms will take on the form v^2h^2.

Fermion masses

Fermion masses on the other hand can be generated by introducing the Yukawa Lagrangian, $\mathcal{L}_{\mathrm{Y}}$. Mark Q as a vector containing all left-handed doublets. Mark Y as a Yukawa coupling matrix. The conjugate doublet is required in order to generate masses for the up sector. The Yukawa Lagrangian is given by

$$\mathcal{L}_{\mathrm{Y}} = \overline{Q} Y_u \Phi^c U - \overline{Q} Y_d \Phi D + h.c. \tag{A.6}$$

The minus sign on the second term is due to the fact that mass terms in a Lagrangian must be negative, while the minus sign in the first term arises in Φ^c.

As $\mathrm{SU}(2)_L$ is spontaneously broken the mass matrix M relates linearly to the Yukawa matrix Y;

$$M = vY. \tag{A.7}$$

In order to find the physical mass states M must be diagonalized via a matrix transform. Since $M \propto Y$ the Yukawa matrix will also be diagonalized in this process.

B Diagonalization by biunitary transforms

One type of diagonalization is the one of binunitary transforms [10].

Let A be a square matrix. There exist $n \times n$ unitary matrices V_1 and V_2 such that a biunitary transform

$$\tilde{A} = V_1 A V_2^\dagger \tag{B.1}$$

results in a diagonal matrix $\tilde{A}$. In order to find said transformation matrices consider

$$\tilde{A}\tilde{A}^\dagger = V_1 A A^\dagger V_1^\dagger. \tag{B.2}$$

Assume $AA^\dagger \mathbf{x}_i = X_i \mathbf{x}_i$ for a set of eigenvectors $\{\mathbf{x}_1, ...\mathbf{x}_n\}$ and eigenvalues $\{X_1, ...X_n\}$. Then V_1 can be constructed as a block matrix out of the vectors $\mathbf{x}$, i.e. $V_1 = (\mathbf{x}_1, ..., \mathbf{x}_n)$.

The matrix V_2 can be found by using the same technique, but considering $\tilde{A}^\dagger\tilde{A}$ instead of $\tilde{A}\tilde{A}^\dagger$. Alternatively, for numerical purposes it is better to solve eq. B.1 for V_2. Denote the eigenvalues of $\tilde{A}\tilde{A}^\dagger$ as $\tilde{X}_i$. These can easily be found when V_1 have been calculated. Due the properties of diagonal matrices one can quickly establish that

$$\tilde{A} = \text{diag}\left\{\sqrt{\tilde{X}_i}\right\} \overset{\tilde{X}_i \neq 0}{\Longrightarrow} \tilde{A}^{-1} = \text{diag}\left\{1/\sqrt{\tilde{X}_i}\right\}. \tag{B.3}$$

Given $\tilde{X}_i$ it is straightforward to construct an explicit relation between V_1 and V_2 - which allows for calculation of V_2;

$$V_2 = \tilde{A}^{-1} V_1 A. \tag{B.4}$$

If one is only interested in finding $\tilde{A}$ and dismissing the actual values on V_1 and V_2 one can instead utilize singular value decomposition.

C Left-handed transformation matrices

Here the left-handed transformation matrices will be derived. This is also done in [3].

Assume that left-handed transformation matrices can be written as

$$V_L = \begin{pmatrix} 1 & v_{12}\varepsilon^{n_{12}} & v_{13}\varepsilon^{n_{13}} \\ v_{21}\varepsilon^{n_{21}} & 1 & v_{23}\varepsilon^{n_{23}} \\ v_{31}\varepsilon^{n_{31}} & v_{32}\varepsilon^{n_{32}} & 1 \end{pmatrix}. \tag{C.1}$$

Evaluating the product $V_L V_L^\dagger$ - which must be identity since V_L is unitary - to leading order gives

$$\begin{cases} v_{12}\varepsilon^{n_{12}} + v_{21}\varepsilon^{n_{21}} = 0, \\ v_{13}\varepsilon^{n_{13}} + v_{31}\varepsilon^{n_{31}} = 0, \\ v_{23}\varepsilon^{n_{23}} + v_{32}\varepsilon^{n_{32}} = 0, \end{cases} \Rightarrow \begin{cases} n_{ij} = n_{ji}, \\ v_{ij} = -v_{ji}. \end{cases} \tag{C.2}$$

The product $V_L H V_L^\dagger$, where $H = MM^\dagger$, should be a diagonal matrix $\tilde{H}$. The matrix H can be conveniently written if a set of constants, h_{ij}, are introduced;

$$H_{ij} = \varepsilon^{b_i+b_j} \sum_k g_{ik} g_{jk} \varepsilon^{2a_k} =: h_{ij}\varepsilon^{b_i+b_j}. \tag{C.3}$$

The diagonal matrix $\widetilde{H}$ can be evaluated to leading order, as done[14] in eq. C.4. Note that different threshold for the truncation is used for the different elements. This behaviour is a result of the fact that this diagonalization is only approximate.

$$\widetilde{H} = \begin{cases} \tilde{h}_{11} = h_{11}\varepsilon^{b_1+b_1} \\ \tilde{h}_{22} = h_{22}\varepsilon^{b_2+b_2} \\ \tilde{h}_{33} = h_{33}\varepsilon^{b_3+b_3} \\ \tilde{h}_{12} = h_{12}\varepsilon^{b_1+b_2} - h_{11}v_{12}\varepsilon^{b_1+b_1+n_{12}} \\ \tilde{h}_{21} = h_{21}\varepsilon^{b_2+b_1} - h_{11}v_{12}\varepsilon^{b_1+b_1+n_{12}} \\ \tilde{h}_{13} = h_{13}\varepsilon^{b_1+b_3} - h_{11}v_{13}\varepsilon^{b_1+b_1+n_{13}} \\ \tilde{h}_{31} = h_{31}\varepsilon^{b_3+b_1} - h_{11}v_{13}\varepsilon^{b_1+b_1+n_{13}} \\ \tilde{h}_{23} = h_{23}\varepsilon^{b_2+b_3} - h_{13}v_{12}\varepsilon^{b_1+b_3+n_{12}} - h_{21}v_{13}\varepsilon^{b_1+b_2+n_{13}} - h_{22}v_{23}\varepsilon^{2b_2+n_{23}} \\ +h_{11}v_{12}v_{13}\varepsilon^{2b_1+n_{12}+n_{13}} + h_{12}v_{12}v_{23}\varepsilon^{b_1+b_2+n_{12}+n_{23}} \\ \tilde{h}_{32} = h_{32}\varepsilon^{b_2+b_3} - h_{31}v_{12}\varepsilon^{b_1+b_3+n_{12}} - h_{12}v_{13}\varepsilon^{b_1+b_2+n_{13}} - h_{22}v_{23}\varepsilon^{2b_2+n_{23}} \\ +h_{11}v_{12}v_{13}\varepsilon^{2b_1+n_{12}+n_{13}} + h_{21}v_{12}v_{23}\varepsilon^{b_1+b_2+n_{12}+n_{23}} \end{cases} \tag{C.4}$$

The elements $\tilde{h}_{12}$, $\tilde{h}_{21}$, $\tilde{h}_{13}$ and $\tilde{h}_{31}$ are zero if

$$\boxed{\begin{cases} n_{12} = b_2 - b_1 & v_{12} = h_{12}/h_{11}, \\ n_{13} = b_3 - b_1 & v_{13} = h_{13}/h_{11}. \end{cases}} \tag{C.5}$$

[14] This calculation was carried out with Mathematica but truncated by hand.

To obtain n_{23} and v_{23} the higher orders in $\tilde{h}_{23}$ and $\tilde{h}_{32}$ must be considered. In order for the two quantities to be zero it is required that $n_{23} = b_3 - b_2$. Due to $h_{ij} = h_{ji}$ the two expressions result in the same equation;

$$h_{23} - h_{13}v_{12} - h_{21}v_{13} - h_{22}v_{23} + h_{11}v_{12}v_{13} + h_{12}v_{12}v_{23} = 0. \tag{C.6}$$

Solving for v_{23} gives

$$\boxed{v_{23} = \frac{h_{11}h_{23} - h_{21}h_{13}}{h_{11}h_{22} - h_{21}h_{12}}, \quad n_{23} = b_3 - b_2.} \tag{C.7}$$

The most important conclusion to be drawn is that transformation matrices behave as

$$\boxed{(V_L)_{ij} \sim \varepsilon^{|b_i - b_j|}.} \tag{C.8}$$

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