

On the Non-Existence of Compact Lorentzian Manifolds with Constant Positive Curvature

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Abstract

In this Master's Thesis we study Semi-Riemannian Geometry. More specifically, this is done by discussing some well-known classification theorems from Riemannian geometry and then showing some counter-examples to these in the semi-Riemannian setting. In particular, we briefly discuss the theorem by Hopf and Rinow, and give a counter-example to this in the semi-Riemannian setting.

Moreover, we study a well-known and important result by Klingler, which asserts that a compact Lorentzian manifold of constant sectional curvature must be geodesically complete. We also discuss a theorem by Calabi and Markus, which together with the theorem by Klingler asserts that in the Lorentzian setting there can be no compact manifolds of positive constant curvature.

Throughout this work it has been my firm intention to give reference to the stated results and credit to the work of others. All theorems, propositions, lemmas and examples left unmarked are assumed to be too well known for a reference to be given.

"Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. You go into the first room and it's dark, completely dark. You stumble around, bumping into the furniture. Gradually, you learn where each piece of furniture is. And finally, after six months or so, you find the light switch and turn it on. Suddenly, it's all illuminated and you can see exactly where you were. Then you enter the next dark room..."

- Andrew Wiles

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Chapter 1

Introduction

In this work we set out to study the subject of semi-Riemannian Geometry. This is different from the usual Riemannian Geometry in that the metric in this setting need not be positive definite. Moreover, many well-known classification theorems do no longer hold. To study these examples, we build up necessary pre-requisites and then give some examples or results illustrating this.

The text is written assuming that the reader has a good understanding of introductory Riemannian geometry, corresponding to the level e.g. in [15]. Furthermore a good understanding of basic group theory is assumed.

A review article, relevant to the current work, can be found in [21].

1.1 Completeness and the Hopf-Rinow Theorem

Let us start by reminding the reader of the notion of (geodesic) completeness.

Definition 1.1. Let (M, g) be a Riemannian manifold. A geodesic $\gamma : [0, b) \rightarrow M$ is said to be **complete** if it can be extended to all $t \in \mathbb{R}$. A Riemannian manifold (M, g) is called **complete** if every geodesic is complete.

We introduce some examples concerning completeness and incompleteness.

Example 1.2. As is well known, the geodesics in E^m (\mathbb{R}^m equipped with the standard Euclidean metric) are the straight lines of the form $\gamma(t) = p + t \cdot v$. As this obviously is defined for all $t \in \mathbb{R}$ this makes E^m into a (geodesically) complete Riemannian manifold.

Example 1.3. As a less trivial example, we consider the upper half-plane equipped with the Lobachevsky metric. It is the set

$$\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \quad (1.1)$$

equipped with the Riemannian metric

$$g(X, Y) = \frac{1}{y^2} \langle X, Y \rangle_{\mathbb{R}^2}. \quad (1.2)$$

We now solve for the unique geodesic γ with $\gamma(0) = p = (0, 1)$ and $\dot{\gamma}(0) = (0, -1)$. We begin by simply stating the Christoffel symbols in the given metric. They are

$$\begin{aligned} \Gamma_{11}^1 &= 0, & \Gamma_{12}^1 &= \Gamma_{21}^1 = -\frac{1}{y}, & \Gamma_{22}^1 &= 0, \\ \Gamma_{11}^2 &= \frac{1}{y}, & \Gamma_{21}^2 &= \Gamma_{21}^2 = 0, & \Gamma_{22}^2 &= -\frac{1}{y}. \end{aligned}$$

These we use to solve the geodesic equation, which we state for recollection:

$$\frac{d^2 \gamma_k(t)}{dt^2} + \sum_{i,j=1,2} \dot{\gamma}_i(t) \dot{\gamma}_j(t) \Gamma_{ij}^k(\gamma(t)) = 0. \quad (1.3)$$

We assume that $\dot{\gamma}_1(t) \equiv 0$, and so we solve for $\gamma_2(t)$ only. We have

$$\begin{aligned} 0 &= \frac{d^2\gamma_2(t)}{dt^2} + \sum_{i,j=1,2} \dot{\gamma}_i(t)\dot{\gamma}_j(t)\Gamma_{ij}^2(\gamma(t)) \\ &= \frac{d^2\gamma_2(t)}{dt^2} + \dot{\gamma}_2(t)\dot{\gamma}_2(t)\Gamma_{22}^2(\gamma(t)) \\ &= \ddot{\gamma}_2(t) + \left(-\frac{\dot{\gamma}_2(t)^2}{\gamma_2(t)}\right). \end{aligned} \tag{1.4}$$

Given initial conditions on γ and $\dot{\gamma}$, the general solution is

$$\gamma_2(t) = \exp(-t) \quad \gamma(t) = (0, \exp(-t)) \tag{1.5}$$

and so we see that the geodesic is defined for all t .

It can be shown, (see [8], p. 175), that the isometry group of the hyperbolic space is the group

$$GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \text{ and } ad - bc \neq 0 \right\}. \tag{1.6}$$

For any two points p and q and any vector $v \in T_p\mathbb{R}_+^2$ there is an element $f \in GL_2(\mathbb{R})$ and a vector $u \in T_q\mathbb{R}_+^2$ such that $f(q) = p$ and $df(u) = v$. Hence any geodesic may be viewed as the image of a geodesic on the above form and so must be complete.

Example 1.4. To show an example of incompleteness, we just take the punctured plane $\mathbb{R}^2 - \{0\}$. The geodesic $\gamma(t) = (1, 0) - t(1, 0)$ is defined on the interval $(-\infty, 1)$ but cannot be extended. Hence the punctured plane is not complete.

Having discussed some examples of geodesic completeness, we are now ready to state an important classification result, regarding this property. The result was initially proven in [16].

Theorem 1.5. ([8], p. 146, "The Hopf-Rinow Theorem")

Let (M, g) be a Riemannian manifold and $p \in M$. Then the following assertions are equivalent:

1. M is complete as a metric space (or topologically complete).
2. M is geodesically complete.
3. For any $q \in M$ there exists a geodesic γ joining p and q with $\ell(\gamma) = d(p, q)$.

We get the following important corollaries.

Corollary 1.6. ([8], p. 149) If M is compact, then M is (geodesically) complete.

Proof. By the compactness we have metric completeness by standard topological results. By Theorem 1.5 this is equivalent to (geodesic) completeness. \square

Corollary 1.7. ([8], p. 149) Closed submanifolds of complete manifolds are complete in the induced metric.

We will show, in Example 2.40, that there are compact semi-Riemannian manifolds equipped with metrics that are not geodesically complete.

1.2 Historical background

The study of semi-Riemannian geometry dates far back and much of its interest has stemmed from Physics, more specifically the General Theory of Relativity. This theory is mathematically founded on the sub-branch called Lorentzian Geometry - a notion that will be made precise in Chapter 2.

As is intuitively clear from the classical Gaussian geometry, it can be shown that the (geodesically) complete Riemannian manifolds of constant positive curvature must be compact. The simplest example of this is surely the standard sphere S^n as a subset of the familiar Euclidean space $E^{n+1} = (\mathbb{R}^{n+1}, \delta = \text{diag}(1, \dots, 1))$. In the Riemannian setting, these spaces will be on the form

$$M^n \simeq S^n / \Gamma, \quad n > 2, \quad (1.7)$$

where Γ is a special type of group acting on S^n . The exact meaning of Γ will be made precise in Section 2.36 of Chapter 2.

In the Lorentzian setting, there is an analogous classification. The Lorentzian (geodesically) complete manifolds of positive constant curvature are on the form

$$M^n \simeq dS^n / \Gamma, \quad (1.8)$$

where dS^n is the so-called **de-Sitter space**, which will be defined in Chapter 2, Section 2.3. However, the striking result of Theorem 4.19 by Calabi and Markus, that was proven 1962, shows that the order of Γ in this case is finite. As an immediate consequence of this, (which is stated as Corollary 4.50), there can be no compact (geodesically) complete Lorentzian manifolds of constant positive curvature for dimension $n > 2$.

The situation for dimension $n = 2$ is relatively straight-forward to discuss. In Section 4.1 we will prove the Lorentzian analogue of the well-known Gauss-Bonnet theorem. The result in the Lorentzian setting will have a slightly different form and the class of surfaces of interest for our discussion is narrowed down quite severely - they are two-dimensional tori. The theorem will then be applied in the standard way: disjoining the surface M into components and then applying the Gauss-Bonnet formula to each one individually. The boundary integrals cancel out and what remains is the integral

$$\int_M \kappa = 0 \quad (1.9)$$

from which we immediately get the remarkable result that compact Lorentzian surfaces must be flat. In particular, they cannot have constant positive curvature.

The question then remains, whether a compact Lorentzian manifold with constant positive curvature can exist. From Corollary 4.50 it could not be complete. This question was open until the early 90s, where it was finally answered by Bruno Klingler in Theorem 4.21, which asserts that a compact Lorentzian manifold of constant curvature is (geodesically) complete. This theorem shows the assertion in the three relevant curvature cases, that is $\kappa = +1$, $\kappa = 0$ and $\kappa = -1$.

It was previously known that completeness would follow in the flat case, i.e. the case $\kappa = 0$. This was proven by Yves Carrière in 1989. Moreover, it was shown (with different methods) in the spherical case ($\kappa = +1$) to be true as well by Morrill 1996.

The work [17] by Klingler is very involved, and it is certainly the bulk part of the work that is this thesis. It relies on the so-called (G, X) -structures, that were introduced by William Goldman in [11] and [13] and many of the concepts in the proof are generalizations of some technical tools used in [6]. The theorem is stated and proven in Section 4.3.

Chapter 2

Semi-Riemannian Manifolds

In the following we introduce the formalism used for manifolds with an indefinite metric. Such manifolds are called semi-Riemannian (pseudo-Riemannian). We prepare this by discussing some properties of the tangent space T_pM .

2.1 Indefinite tangent spaces

We now introduce the necessary formalism needed to understand the tangent space T_pM of a semi-Riemannian manifold. Indeed, as has been mentioned, the topological structure is quite different. We follow the discussion in [20].

Definition 2.1. Let V be a real vector space. A **symmetric bilinear form** b is a mapping $b : V \times V \rightarrow \mathbb{R}$ such that $b(v, w) = b(w, v)$ (symmetry) and b is linear in both its arguments. The form is called **non-degenerate** if $b(v, w) = 0$ for all v implies $w = 0$.

Definition 2.2. Let V be a real vector space. A **scalar product** g is a non-degenerate symmetric bilinear form on V . If g is positive definite, it is called an **inner product**.

Furthermore a scalar product induces a quadratic form by $q(v) = g(v, v)$. This can be negative on some non-zero v , and so the span $\mathbb{R}v$ generated by this vector is a **negative definite** vector subspace. This motivates the following definition.

Definition 2.3. Let V be a real vector space and g be a scalar product on V . The **index** ν of g is the maximal dimension of a vector subspace W in V such that the restriction $g|_W$ is negative definite. If $\nu = 1$ the vector space is said to be **Lorentzian**.

Example 2.4. Let \mathbb{R}^2 be equipped with the metric $g = \text{diag}(-1, 1)$. The set of vectors $v = (v_1, v_2)$ such that $v_1^2 > v_2^2$ will span negative definite vector subspaces. Hence the index is at least $\nu \geq 1$. But as the space is two-dimensional, and for instance the vector $w = (0, 1)$ satisfies $g(w, w) = 1 > 0$, the index ν can be at most 1. Hence it is Lorentzian. Furthermore the subspaces $\mathbb{R}(1, 1)$ and $\mathbb{R}(1, -1)$ are degenerate as subspaces as the basis vectors are orthogonal to themselves.

Definition 2.5. Let V be a real vector space, equipped with a scalar product g . A vector v in V is said to be

1. *timelike* if $g(v, v) < 0$,
2. *spacelike* if $g(v, v) > 0$ and
3. *null* if $g(v, v) = 0$.

We prove the analogue to the dimension theorem in linear algebra. First we need to specify what the orthogonal complement means.

Definition 2.6. Let V be a real vector space, equipped with a scalar product g , and let W be a vector subspace. The set

$$W^\perp = \{v \in V \mid g(v, w) = 0, w \in W\} \quad (2.1)$$

is called the **orthogonal complement** of W .

The only non-trivial difference to the Euclidean case is that although we assume the vector space to be non-degenerate, it can still have degenerate vector subspaces, as seen above. This allows for $W = W^\perp$ to happen. The following two lemmas will still hold, though.

Lemma 2.7. ([20], p. 49) *Let V be a real vector space, equipped with a scalar product g and W a vector subspace. Then*

1. $\dim W + \dim W^\perp = \dim V = n$,
2. $(W^\perp)^\perp = W$.

Proof. ([20], p. 49) Let $\{e_1, \dots, e_n\}$ be a basis for V such that $\{e_1, \dots, e_k\}$ is a basis for W .

We first show (1.). The elements $w \in W^\perp$ are precisely those that satisfy $g(w, e_i) = 0$ for $i = 1, \dots, k$. Let w be expressed in terms of the basis vectors as $w = w_1 e_1 + \dots + w_n e_n$. In terms of the components of g , this yields the following equations:

$$0 = g(w, e_i) = \sum_{j=1}^n g_{ij} w_j, \quad i = 1, \dots, k. \quad (2.2)$$

Suppose the rows of the matrix g are linearly independent. Then we know from linear algebra that the equation system is invertible and solutions exist. It follows also that the space of solutions must be $(n - k)$ -dimensional. Hence the assertion follows this from assumption. But the linear independence follows naturally for any non-degenerate bilinear form: $g(v, w) = 0$ for all w if and only if $g(v, e_i) = 0$ for all e_i . Hence

$$0 = g(v, e_i) = \sum_j^n g(v_j e_j, e_i) = \sum_j^n v_j b_{ij}, \quad i = 1, \dots, n. \quad (2.3)$$

This means that for some non-zero v_j the system is singular and hence not invertible. Therefore it is equivalent that the bilinear form is non-degenerate and that the coefficient matrix is invertible. This proves (1.).

Obviously $W \subseteq (W^\perp)^\perp$. But from the first assertion, it follows that they must have the same dimension. Hence we have equality. This shows (2.). \square

Lemma 2.8. ([20], p. 49) *A vector subspace W of V is non-degenerate if and only if V is the direct sum $V = W \oplus W^\perp$.*

Proof. Regardless of the scalar product, one has for finite dimensional vector spaces that

$$\dim(W \oplus W^\perp) + \dim(W \cap W^\perp) = \dim W + \dim W^\perp. \quad (2.4)$$

If W is non-degenerate, then by the Lemma 2.7 it must hold that $\dim W + \dim W^\perp = \dim V$ and so V is the direct sum. If V is the direct sum, the left hand side of the above implies

$$\dim(W \cap W^\perp) = 0, \quad (2.5)$$

which is equivalent to W being non-degenerate. \square

Definition 2.9. A vector space V with a scalar product g has a **norm** defined through

$$\|v\| = \sqrt{|g(v, v)|}. \quad (2.6)$$

Definition 2.10. Let V be a Lorentzian vector space and $z \in V$. The **orthogonal complement** z^\perp to z is the orthogonal complement of the vector subspace $\mathbb{R}z$, as in Definition 2.6.

Lemma 2.11. ([20], p. 141) *Let V be a Lorentzian vector space and suppose that z is time-like. Then*

1. z^\perp is spacelike,
2. $V = \mathbb{R}z \oplus z^\perp$.

Proof. ([20], p. 141) As $\mathbb{R}z$ is non-degenerate, it follows from Lemma 2.8 that, as $\mathbb{R}z$ and z^\perp intersects trivially, and so V is the direct sum. This shows $V = \mathbb{R}z \oplus z^\perp$.

As the index of $\mathbb{R}z$ is 1, and V is Lorentz, z^\perp cannot have a positive index. If so, there would be a time-like vector $w \in z^\perp$, linearly independent of z and the vector subspace $W = \text{span}\{z, w\}$ would be strictly time-like and have dimension 2. \square

Lemma 2.12. ([20], p. 52) *Two scalar product spaces V and W have the same dimension and index if and only if there exists a linear isometry between them.*

Proof. See ([20], p. 52). \square

2.2 Semi-Riemannian manifolds

We are now equipped with some understanding of indefinite vector spaces. Hence we can construct the manifolds of indefinite metrics.

Definition 2.13. Let M be a smooth topological manifold. A **metric** g is a non-degenerate symmetric $(0,2)$ -tensor field, of constant index ν , defined on M .

The metric assigns to each point p a non-degenerate scalar product to the tangent space T_pM . The following conventions are necessary.

Definition 2.14. Let M be a smooth topological manifold equipped with a metric g . Then M is called a **semi-Riemannian manifold**. In the case of $\nu = 0$, M is naturally called **Riemannian**. In the case of $\nu = 1$ and $\dim M = n \geq 2$, M is called a **Lorentzian manifold**.

Theorem 2.15. ([20], p. 61, "Levi-Civita") *Let (M, g) be a semi-Riemannian manifold. Then there exists a unique metric and torsion-free connection ∇ .*

Proof. See ([20], p. 61). \square

Definition 2.16. A complete and connected semi-Riemannian manifold of constant (sectional) curvature is called a **space form**. If, furthermore, the manifold is Lorentzian, it is said to be a **relativistic** space form. Space forms of positive constant curvature $\kappa > 0$ are called **spherical**, those of negative constant curvature $\kappa < 0$ are called **hyperbolic** and those with $\kappa = 0$ are called **flat**.

Lorentz manifolds are of particular interest in semi-Riemannian geometry. Hence we study some examples of Lorentz manifolds. The three Lorentzian manifolds defined in the following are all space forms.

Definition 2.17. Let \mathbb{R}^n be equipped with the metric $g = \text{diag}(-1, 1, \dots, 1)$. By convention, this metric is denoted by $g = \eta$. The space (\mathbb{R}^n, η) is called the **Minkowski space** and is denoted by \mathcal{M}^n .

Definition 2.18. Consider the $(n + 1)$ -dimensional Minkowski space \mathcal{M}^{n+1} and the quadratic form $Q_1^{n+1}(x, y)$, defined by the equation

$$Q_1^{n+1}(x, y) = -x_1y_1 + x_2y_2 + \dots + x_{n+1}y_{n+1}, \quad (2.7)$$

for two points $x = (x_1, \dots, x_{n+1})$ and $y = (y_1, \dots, y_{n+1})$ in \mathcal{M}^{n+1} . The subset defined through $Q_1^{n+1}(x, x) = 1$, that is the set of points $x \in \mathcal{M}^{n+1}$ that satisfies

$$1 + x_1^2 = x_2^2 + \dots + x_{n+1}^2, \quad (2.8)$$

gives a semi-Riemannian submanifold equipped with the induced metric. This submanifold is called the **de-Sitter space**, or the dS-space for short.

Definition 2.19. Consider the set \mathbb{R}^{n+1} and the quadratic form

$$Q_{-1}^{n+1}(x, y) = +x_1y_1 + x_2y_2 - (x_3y_3 + \dots + x_ny_n + x_{n+1}y_{n+1}). \quad (2.9)$$

The submanifold given by $Q_{-1}^{n+1}(x, x) = 1$, i.e. the set of points $x \in \mathbb{R}^{n+1}$ satisfying the equation

$$x_1^2 + x_2^2 = 1 + x_3^2 + \dots + x_{n+1}^2, \quad (2.10)$$

combined with the induced metric from (\mathbb{R}^{n+1}, g) , where $g = \text{diag}(-1, -1, 1, \dots, 1)$, is called the **anti de-Sitter space**, or the AdS-space for short.

Proposition 2.20. *The Minkowski \mathcal{M}^n space has the following properties:*

1. \mathcal{M}^n is simply connected,
2. \mathcal{M}^n is complete and
3. \mathcal{M}^n has constant curvature $\kappa = 0$.

Proof. Since \mathcal{M}^n is convex Example A.10 shows that the simple connectivity follows. This shows (1.).

Since $\eta = \text{diag}(-1, 1, \dots, 1)$ is constant, the Christoffel symbols vanish. This means both that the sectional curvature vanishes and that the geodesic equation reduces to $\frac{d^2\gamma}{d\gamma^2} = 0$. The solutions to this differential equation are the straight lines, which are defined for all parameter values. This shows both (2.) and (3.). \square

Proposition 2.21. *For $n \geq 2$ consider dS^n . Then*

1. dS^n is diffeomorphic to $\mathbb{R} \times S^{n-1}$,
2. dS^n is simply connected for $n \geq 3$,
3. dS^n has constant curvature $\kappa = +1$.

Proof. We denote the coordinates (x_1, \dots, x_{n+1}) of the ambient Minkowski space by (t, \vec{x}) .

It is easily seen, that the mapping

$$\psi(t, \vec{x}) = \left(t, \frac{\vec{x}}{\sqrt{1+t^2}}\right) \quad (2.11)$$

is a diffeomorphism onto the space $\mathbb{R} \times S^{n-1}$. This shows (1.).

For the first fundamental group, we have $\pi_1(dS^n) \simeq \pi_1(\mathbb{R} \times S^{n-1}) \simeq \pi_1(\mathbb{R}) \times \pi_1(S^{n-1}) \simeq \{e\}$, where we used Propositions A.14, A.13 and A.25. By the diffeomorphism we just showed, dS^n is path connected and the fundamental group is trivial. Hence dS^n is simply connected.

For (3.), we show the case to be true in the $n = 2$ case first. Consider the parametrization

$$\sigma(t, \phi) = \left(t, \sqrt{1+t^2} \cos(\phi), \sqrt{1+t^2} \sin(\phi)\right). \quad (2.12)$$

The pull-back metric via σ satisfies

$$g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)(t, \phi) = -\frac{1}{1+t^2}, \quad g\left(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}\right)(t, \phi) = (1+t^2), \quad g\left(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial t}\right)(t, \phi) = 0. \quad (2.13)$$

The corresponding Christoffel symbols are

$$\begin{aligned} \Gamma_{tt}^t &= -\frac{t}{1+t^2}, & \Gamma_{t\phi}^t &= \Gamma_{\phi t}^t = 0, & \Gamma_{\phi\phi}^t &= t(1+t^2), \\ \Gamma_{\phi\phi}^\phi &= 0, & \Gamma_{\phi t}^\phi &= \Gamma_{t\phi}^\phi = \frac{t}{1+t^2}, & \Gamma_{tt}^\phi &= 0. \end{aligned}$$

The **Gaussian curvature** κ becomes, (after using Proposition 8.11 in [15]),

$$\begin{aligned}
\kappa &= \frac{g\left(R\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}, \frac{\partial}{\partial\phi}\right)}{g_{\phi\phi}g_{tt} - g_{t\phi}^2} \\
&= \frac{1}{g_{\phi\phi}g_{tt}}g_{\phi\phi}\left(\frac{\partial\Gamma_{tt}^\phi}{\partial\phi} - \frac{\partial\Gamma_{\phi t}^\phi}{\partial t} + (\Gamma_{tt}^\phi\Gamma_{\phi\phi}^\phi - \Gamma_{\phi t}^\phi\Gamma_{t\phi}^\phi) + (\Gamma_{tt}^t\Gamma_{t\phi}^\phi - \Gamma_{\phi t}^t\Gamma_{tt}^\phi)\right) \\
&= -(1+t^2)\left(0 - \frac{1-t^2}{(1+t^2)^2} + \left(0 - \frac{t^2}{(1+t^2)^2}\right) + \left(-\frac{t^2}{(1+t^2)^2} - 0\right)\right) \\
&= 1.
\end{aligned} \tag{2.14}$$

To prove the general case, we observe that if $m < n$, the mapping

$$\begin{aligned}
\rho: dS^n &\rightarrow dS^n, & \text{by } \rho(x, y) &= (x, -y), \\
\text{where } x &= (x_1, \dots, x_{m+1}) & \text{and } y &= (x_{m+2}, \dots, x_{n+1}),
\end{aligned} \tag{2.15}$$

is an isometry with dS^m as fix-point set. Hence $dS^m \subset dS^n$ is totally geodesic (by Proposition 7.25 in [15], modified to the semi-Riemannian setting). The metric η restricted to y is space-like and so the Gauss-equation implies the curvature must be the same. Thus, the statement follows by induction. \square

Proposition 2.22. *Let $n \geq 2$ and consider dS^n . Then*

1. *If, for two points x and y of dS^n , $Q_1^{n+1}(x, y) = 1$, then there is a null geodesic in dS^n , i.e. a straight line in \mathcal{M}^{n+1} , connecting x and y .*
2. *If, for two points x and y of dS^n , $-1 < Q_1^{n+1}(x, y) < 1$, then there is a space-like geodesic ellipse in dS^n , connecting x and y .*
3. *If, for two non-antipodal points x and y of dS^n , we have $Q_1^{n+1}(x, y) > 1$, there is a time-like geodesic hyperbola connecting x and y .*
4. *If, for two non-antipodal points x and y of dS^n , $Q_1^{n+1}(x, y) \leq -1$, then there is no geodesic in dS^n connecting x and y .*
5. *The de-Sitter space is complete.*

Proof. We denote the coordinates (x_1, \dots, x_{n+1}) of the ambient Minkowski space by (t, \vec{x}) .

We now discuss (1.). We have, as both x and y are in dS^n , that $Q_1^{n+1}(x, x) = Q_1^{n+1}(y, y) = 1$. If we can show that the line $\ell(s) = x + s(y - x)$ is both contained in dS^n and is *null* seen in \mathcal{M}^{n+1} , we are done, as the line then would be a null geodesic. As the space-like restriction of the metric has a rotational symmetry, we may assume that $x = (u, \sqrt{1+u^2}, 0, \dots, 0)$. Hence, we have $Q_1^{n+1}(y, y) = -y_1^2 + y_2^2 + \dots + y_{n+1}^2 = 1$ and $Q_1^{n+1}(x, y) = -uy_1 + \sqrt{1+u^2}y_2 = 1$. The following

shows that the line is contained in dS^n :

$$\begin{aligned}
Q_1^{n+1}(\ell, \ell)(s) &= -\left(u + s(y_1 - u)\right)^2 + \left(\sqrt{1+u^2} + s(y_2 - \sqrt{1+u^2})\right)^2 \\
&\quad + \cdots + \left(sy_n\right)^2 + \left(sy_{n+1}\right)^2 \\
&= -\left(u^2 + 2us(y_1 - u) + s^2(y_1 - u)^2\right) \\
&\quad + \left((1+u^2) + 2\sqrt{1+u^2}s(y_2 - \sqrt{1+u^2}) + s^2(y_2 - \sqrt{1+u^2})^2\right) \\
&\quad + \cdots + s^2y_n^2 + s^2y_{n+1}^2 \\
&= -\left(u^2 + 2usy_1 - 2u^2s + s^2y_1^2 - 2s^2y_1u + s^2u^2\right) \\
&\quad + \left((1+u^2) + 2\sqrt{1+u^2}sy_2 - 2s(1+u^2)\right) \\
&\quad + \left(s^2y_2^2 - 2s^2y_2\sqrt{1+u^2} + s^2(1+u^2)\right) + \cdots + s^2y_n^2 + s^2y_{n+1}^2 \\
&= 2s\left(\underbrace{-uy_1 + \sqrt{1+u^2}}_{=1}\right) - 2s^2\left(\underbrace{-uy_1 + \sqrt{1+u^2}}_{=1}\right) \\
&\quad + s^2\left(\underbrace{-y_1^2 + y_2^2 + \cdots + y_n^2 + y_{n+1}^2}_{=1}\right) \\
&\quad + \left(-u^2(1-s)^2 + (1+u^2)(1-s)^2\right) \\
&= 2s - 2s^2 + s^2 + (1-s)^2 \\
&= 1.
\end{aligned} \tag{2.16}$$

Furthermore, the tangent vector of ℓ is *null*:

$$\begin{aligned}
\eta(x - y, x - y) &= -(u - y_1)^2 + (\sqrt{1+u^2} - y_2)^2 + \cdots + y_n^2 + y_{n+1}^2 \\
&= -\left(u^2 - 2uy_1 + y_1^2\right) + \left((1+u^2) - 2\sqrt{1+u^2}y_2 + y_2^2\right) + \cdots + y_n^2 + y_{n+1}^2 \\
&= 1 + \left(\underbrace{-y_1^2 + y_2^2 + \cdots + y_n^2 + y_{n+1}^2}_{=1}\right) + 2\left(\underbrace{uy_1 - \sqrt{1+u^2}y_2}_{=-1}\right) \\
&= 0.
\end{aligned} \tag{2.17}$$

This shows that the curve ℓ is a geodesic in \mathcal{M}^{n+1} . The definition of the induced Levi-Civita connection to dS^n guarantees that it is also a geodesic in the submanifold: $\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = (\nabla_{\dot{\gamma}}\dot{\gamma})^\top = 0$.

Consider now the case $-1 < Q_1^{n+1}(x, y) < 1$. For the sake of brevity, we let $Q_1^{n+1}(x, y) \equiv Q(x, y)$. We consider the plane Π that contains the origin, x and y , i.e. $\Pi = \text{span}\{x, y\}$. We find two orthogonal normal elements e_1 and e_2 in Π , so as to parametrize the intersection. We claim that the curve

$$\begin{aligned}
e(t) &= \sin(t)e_1 + \cos(t)e_2 \\
&= \sin(t)x + \cos(t)\left(\frac{Q(x, y)}{\sqrt{1-Q^2(x, y)}}x - \frac{1}{\sqrt{1-Q^2(x, y)}}y\right).
\end{aligned} \tag{2.18}$$

is the desired parametrization. Clearly $e(t) \in \Pi$ as $\Pi = \text{span}\{x, y\}$. The following shows that the

basis vectors e_1 and e_2 are orthonormal. Clearly $Q(e_1, e_1) = Q(x, x) = 1$ and we have

$$\begin{aligned} Q(e_1, e_2) &= Q\left(x, \frac{Q(x, y)}{\sqrt{1 - Q^2(x, y)}}x - \frac{1}{\sqrt{1 - Q^2(x, y)}}y\right) \\ &= Q(x, x) \frac{Q(x, y)}{\sqrt{1 - Q^2(x, y)}} - Q(x, y) \frac{Q(y, y)}{\sqrt{1 - Q^2(x, y)}} \\ &= 0. \end{aligned} \quad (2.19)$$

Furthermore, there is

$$\begin{aligned} Q(e_2, e_2) &= Q\left(\frac{Q(x, y)}{\sqrt{1 - Q^2(x, y)}}x - \frac{1}{\sqrt{1 - Q^2(x, y)}}y, \frac{Q(x, y)}{\sqrt{1 - Q^2(x, y)}}x - \frac{1}{\sqrt{1 - Q^2(x, y)}}y\right) \\ &= Q(x, x) \frac{Q^2(x, y)}{1 - Q^2(x, y)} - 2Q(x, y) \frac{Q(x, y)}{1 - Q^2(x, y)} + Q(y, y) \frac{1}{1 - Q^2(x, y)} \\ &= \frac{1 + Q^2(x, y)}{1 - Q^2(x, y)} \\ &= 1 \end{aligned} \quad (2.20)$$

so that e_2 is a spacelike vector of unit norm. With this we have

$$\begin{aligned} Q(e(t), e(t)) &= \sin(t)e_1 + \cos(t)e_2 \\ &= Q(e_1, e_1) \sin^2(t) + 2Q(e_1, e_2) \sin(t) \cos(t) + Q(e_2, e_2) \cos^2(t) \\ &= 1 \end{aligned} \quad (2.21)$$

so that $e(t) \in dS^n$.

We only need to show that $e(t)$ is a geodesic. To do this, we show $\ddot{e}(t) \in N_{e(t)}dS^n$. This follows, as for any $e(t)$, we have that $\ddot{e}(t) = -e(t)$. We choose coordinates so that, for some fixed t , $e(t)$ is on the form $e(t) = (s, \sqrt{1 + s^2}, 0, \dots, 0)$. We compute the derivative in the s -direction:

$$\frac{de(t)}{ds} = \left(1, \frac{s}{\sqrt{1 + s^2}}, 0, \dots, 0\right), \quad (2.22)$$

from which it follows that

$$\eta(\ddot{e}(t), \frac{de(t)}{ds}) = -(1 \cdot s) + (\sqrt{1 + s^2} \cdot \frac{s}{\sqrt{1 + s^2}}) = 0. \quad (2.23)$$

Hence $\ddot{e}(t) \in N_{e(t)}dS^n$ and so e is a geodesic. Since the parametrization σ is global, the geodesic connecting x and y must be unique. This proves (2).

We again consider the plane Π spanned by x and y and denote $Q_{-1}^{n+1}(x, y) \equiv Q(x, y)$. We claim that the curve

$$\begin{aligned} h(t) &= \cosh(t)e_1 + \sinh(t)e_2 \\ &= \cosh(t)y + \sinh(t) \left(\frac{1}{\sqrt{Q^2(x, y) - 1}}x - \frac{Q(x, y)}{\sqrt{Q^2(x, y) - 1}}y \right) \end{aligned} \quad (2.24)$$

is the geodesic that satisfies the claimed properties. Obviously $h(t) \in \Pi$. We have $h(0) = y$ and for some t_x we have $\sinh(t_x) = \sqrt{Q^2(x, y) - 1}$, so that $\cosh(t_x) = Q(x, y)$ and therefore $h(t_x) = x$. The vectors e_1 and e_2 form an orthonormal basis for Π :

$$\begin{aligned} Q(e_1, e_2) &= Q\left(y, \frac{1}{\sqrt{Q^2(x, y) - 1}}x - \frac{Q(x, y)}{\sqrt{Q^2(x, y) - 1}}y\right) \\ &= Q(y, x) \frac{1}{\sqrt{Q^2(x, y) - 1}} - Q(y, y) \frac{Q(x, y)}{\sqrt{Q^2(x, y) - 1}} \\ &= 0 \end{aligned} \quad (2.25)$$

and

$$\begin{aligned}
Q(e_2, e_2) &= Q\left(\frac{1}{\sqrt{Q^2(x, y) - 1}}x - \frac{Q(x, y)}{\sqrt{Q^2(x, y) - 1}}y, \frac{1}{\sqrt{Q^2(x, y) - 1}}x - \frac{Q(x, y)}{\sqrt{Q^2(x, y) - 1}}y\right) \\
&= Q(x, x)\frac{1}{Q^2(x, y) - 1} - 2Q(x, y)\frac{Q^2(x, y)}{Q^2(x, y) - 1} + Q(y, y)\frac{Q(x, y)^2}{Q^2(x, y) - 1} \\
&= \frac{1 - Q^2(x, y)}{Q^2(x, y) - 1} \\
&= -1
\end{aligned} \tag{2.26}$$

so that e_2 is a timelike unit vector orthogonal to e_1 .

We can now show that $h(t) \in dS^n$. We have

$$\begin{aligned}
Q(h(t), h(t)) &= Q\left(\cosh(t)e_1 + \sinh(t)e_2, \cosh(t)e_1 + \sinh(t)e_2\right) \\
&= Q(e_1, e_1)\cosh^2(t) + 2Q(e_1, e_2)\sinh(t)\cosh(t) + Q(e_2, e_2)\sinh^2(t) \\
&= \cosh^2(t) - \sinh^2(t) \\
&= 1
\end{aligned} \tag{2.27}$$

so that $h(t)$ is a curve in $dS^n \cap \Pi$.

Furthermore, we have $\dot{h}(t) = h(t)$ and so by the calculation for the ellipse we already know that such a curve is a geodesic. This shows (3.).

For a geodesic the acceleration vector must be in the normal space. When we investigated the properties of the elliptic geodesic, we showed that $N_x dS^n = \mathbb{R}x$. It follows that this happens for every point along a geodesic if and only if it is the intersection $dS^n \cap \Pi$ for some plane Π (except in the case of the geodesic being a straight line, which we have already dealt with). Hence all geodesics must be on the forms we have covered. This means on the one hand that for two (non-antipodal) points x and y on some geodesic, we must have $Q_1^{n+1}(x, y) > -1$ and on the other that every geodesic is complete. This shows (4.) and (5.). \square

Proposition 2.23. *Let $n \geq 2$ and consider AdS^n . Then*

1. AdS^n is diffeomorphic to $\mathbb{R}^{n-1} \times S^1$,
2. AdS^n is **not** simply connected,
3. AdS^n has constant curvature $\kappa = -1$.

Proof. As in the proof of Proposition 2.21, we denote the coordinates $(x_1, x_2, \dots, x_{n+1}) = (t_1, t_2, \vec{x})$.

One easily sees that the mapping

$$\psi(t_1, t_2, \vec{x}) = \left(\frac{t_1}{\sqrt{1 + \vec{x} \cdot \vec{x}}}, \frac{t_2}{\sqrt{1 + \vec{x} \cdot \vec{x}}}, \vec{x}\right) \tag{2.28}$$

is a diffeomorphism onto $S^1 \times \mathbb{R}^{n-1}$.

We have $\pi_1(AdS^n) \simeq \pi_1(\mathbb{R}^{n-1} \times S^1) \simeq \pi_1(\mathbb{R}^{n-1}) \times \pi_1(S^1) \simeq \langle \mathbb{Z}, + \rangle$, where we used Propositions A.14, A.13 and A.25. As the fundamental group is not trivial, AdS^n is not simply connected.

We proceed as in the proof of (3.) in Proposition 2.21 and first work out the case of $n = 2$. The space has the global parametrization

$$\sigma(s, \phi) = (\cos(\phi)\sqrt{1 + s^2}, \sin(\phi)\sqrt{1 + s^2}, s). \tag{2.29}$$

The metric, calculated through pull-back is

$$\begin{aligned}
g\left(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}\right)(s, \phi) &= -(1 + s^2), & g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial \phi}\right)(s, \phi) &= 0, \\
g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right)(s, \phi) &= \frac{1}{1 + s^2}.
\end{aligned} \tag{2.30}$$

We get the Christoffel symbols

$$\begin{aligned}\Gamma_{ss}^s &= \frac{s}{1+s^2}, & \Gamma_{s\phi}^s &= \Gamma_{\phi s}^s = 0, & \Gamma_{\phi\phi}^s &= s(1+s^2), \\ \Gamma_{\phi\phi}^\phi &= 0, & \Gamma_{\phi s}^\phi &= \Gamma_{s\phi}^\phi = -\frac{s}{1+s^2}, & \Gamma_{ss}^\phi &= 0.\end{aligned}$$

The **Gaussian curvature** κ becomes, (after using Proposition 8.11 in [15]),

$$\begin{aligned}\kappa &= \frac{g\left(R\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial \phi}\right)\frac{\partial}{\partial \phi}, \frac{\partial}{\partial s}\right)}{g_{\phi\phi}g_{ss} - g_{s\phi}^2} \\ &= \frac{1}{g_{\phi\phi}g_{ss}}g_{\phi\phi}\left(\frac{\partial\Gamma_{\phi\phi}^s}{\partial s} - \frac{\partial\Gamma_{\phi s}^s}{\partial \phi} + (\Gamma_{\phi\phi}^s\Gamma_{ss}^s - \Gamma_{\phi s}^s\Gamma_{s\phi}^s) + (\Gamma_{\phi\phi}^\phi\Gamma_{s\phi}^s - \Gamma_{\phi s}^\phi\Gamma_{\phi\phi}^s)\right) \\ &= \frac{1}{1+s^2}\left(- (1+3s^2) + 0 + (s^2-0) + (0 - (-s^2))\right) \\ &= \frac{1}{1+s^2}\left(- (1+s^2)\right) \\ &= -1.\end{aligned}\tag{2.31}$$

To show for the general case, we observe that if $m < n$, the mapping

$$\rho : AdS^n \rightarrow AdS^n, \quad \text{by} \quad \rho(x, y) = (x, -y),\tag{2.32}$$

is an isometry with AdS^m as fix-point set. Hence $AdS^m \subset AdS^n$ is totally geodesic. The metric η restricted to y is space-like and so the Gauss-equation implies the curvature must be the same. Thus, the statement follows by induction. \square

Proposition 2.24. *Let $n \geq 2$ and consider AdS^n . Then*

1. *if, for two non-antipodal points x and y of AdS^n , $Q_{-1}^{n+1}(x, y) = 1$, then there is a null geodesic in AdS^n , i.e. a straight line in \mathcal{M}^{n+1} , connecting x and y .*
2. *If, for two non-antipodal points x and y of AdS^n , $-1 < Q_{-1}^{n+1}(x, y) < 1$, then there is a timelike geodesic ellipse in AdS^n , connecting x and y .*
3. *If, for two non-antipodal points x and y of AdS^n , we have $Q_{-1}^{n+1}(x, y) > 1$, there is a spacelike geodesic hyperbola connecting x and y .*
4. *If, for two non-antipodal points x and y of AdS^n , $Q_{-1}^{n+1}(x, y) \leq -1$, then there is no geodesic in AdS^n connecting x and y .*
5. *The anti de-Sitter space is complete.*

Proof. We do the analogous calculations to those we did in the dS^n case. For the sake of brevity, we write $Q_{-1}^{n+1}(x, y) = Q(x, y)$.

For two non-antipodal points x and y that satisfies $Q(x, y) = 1$ we claim that the curve $\ell(t) = x + t(y - x)$ is a null geodesic. From the $O(2)$ -symmetry of the timelike part and the $O(n-1)$ -symmetry of the spacelike part, we may assume that $x = (0, \sqrt{1+u^2}, u, 0, \dots, 0)$. We denote $y = (y_1, y_2, \dots, y_{n+1})$. The condition $Q(x, y) = 1$ gives $\sqrt{1+u^2}y_2 - uy_3 = 1$. The following

shows that the curve ℓ is contained in AdS^n :

$$\begin{aligned}
Q(\ell, \ell) &= t^2 y_1^2 + \left(\sqrt{1+u^2} + t(y_2 - \sqrt{1+u^2}) \right)^2 - \left(u + t(y_3 - u) \right)^2 \\
&\quad - t^2 y_4^2 - \dots - t^2 y_{n+1}^2 \\
&= t^2 y_1^2 + \left((1+u^2)(1-t)^2 + 2\sqrt{1+u^2}(1-t)ty_2 + t^2 y_2^2 \right) \\
&\quad - \left(u^2(1-t)^2 + 2u(1-t)ty_3 + t^2 y_3^2 \right) - \dots - t^2 y_{n+1}^2 \\
&= t^2(y_1^2 + y_2^2 - y_3^2 - \dots - y_{n+1}^2) + (1+u^2)(1-t)^2 - u^2(1-t)^2 \\
&\quad + 2(1-t)t \left(\sqrt{1+u^2}y_2 - uy_3 \right) \\
&= t^2 + (1-t)^2 + 2(1-t)t \\
&= 1.
\end{aligned} \tag{2.33}$$

Furthermore the tangent vector $\dot{\ell} = x - y$ is *null*, as is shown in the following. With the AdS^n -metric $g = \text{diag}(-1, -1, 1, \dots, 1)$, we get

$$\begin{aligned}
g(x - y, x - y) &= -(-y_1)^2 - \left(\sqrt{1+u^2} - y_2 \right)^2 + (u - y_3)^2 + \dots + (-y_{n+1})^2 \\
&= -y_1^2 - y_2^2 + y_3^2 + \dots + y_{n+1}^2 - (1+u^2) + u^2 + 2 \left(\sqrt{1+u^2}y_2 - uy_3 \right) \\
&= -1 - 1 + 2 \\
&= 0.
\end{aligned} \tag{2.34}$$

This shows that the curve ℓ is a null straight line in AdS^n . Since the metric g is constant, straight lines are geodesics in (\mathbb{R}^{n+1}, g) . The definition of the induced Levi-Civita connection to AdS^n guarantees that it is also a geodesic in the submanifold: $\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} = (\nabla_{\dot{\gamma}} \dot{\gamma})^\top = 0$. This shows (1).

We suppose now that $-1 < Q(x, y) < 1$. We claim that the curve

$$\begin{aligned}
e(t) &= \cos(t)e_1 + \sin(t)e_2 \\
&= \cos(t)x + \sin(t) \left(\frac{1}{\sqrt{1-Q^2(x, y)}}y - \frac{Q(x, y)}{\sqrt{1-Q^2(x, y)}}x \right)
\end{aligned} \tag{2.35}$$

is a geodesic. We have $Q(x, x) = Q(y, y) = 1$ and since Q is the metric g but with reverse sign, it follows that $\Pi = \text{span}x, y$ is a timelike plane. The basis vectors e_1 and e_2 form an orthonormal basis:

$$\begin{aligned}
Q(e_1, e_2) &= Q\left(x, \frac{1}{\sqrt{1-Q^2(x, y)}}y - \frac{Q(x, y)}{\sqrt{1-Q^2(x, y)}}x\right) \\
&= Q(x, y) \frac{1}{\sqrt{1-Q^2(x, y)}} - Q(x, x) \frac{Q(x, y)}{\sqrt{1-Q^2(x, y)}} \\
&= 0
\end{aligned} \tag{2.36}$$

and

$$\begin{aligned}
Q(e_2, e_2) &= Q\left(\frac{1}{\sqrt{1-Q^2(x, y)}}y - \frac{Q(x, y)}{\sqrt{1-Q^2(x, y)}}x, \frac{1}{\sqrt{1-Q^2(x, y)}}y - \frac{Q(x, y)}{\sqrt{1-Q^2(x, y)}}x\right) \\
&= Q(y, y) \frac{1}{1-Q^2(x, y)} - 2Q(x, y) \frac{Q(x, y)}{1-Q^2(x, y)} + Q(x, x) \frac{Q^2(x, y)}{1-Q^2(x, y)} \\
&= \frac{1-Q^2(x, y)}{1-Q^2(x, y)} \\
&= 1.
\end{aligned} \tag{2.37}$$

Furthermore, we show that $e(t) \in dS^n$. We have

$$\begin{aligned} Q(e(t), e(t)) &= Q\left(\cos(t)e_1 + \sin(t)e_2, \cos(t)e_1 + \sin(t)e_2\right) \\ &= \cos^2(t)Q(e_1, e_1) + 2\sin(t)\cos(t)Q(e_1, e_2) + \sin^2(t)Q(e_2, e_2) \\ &= \cos^2(t) + \sin^2(t) \\ &= 1. \end{aligned} \quad (2.38)$$

Clearly, we have $\ddot{e}(t) = -e(t)$. To show that e is a geodesic, we need only to show that the acceleration $\ddot{e}(t) \in N_{e(t)}AdS^n$. If $x = (0, \sqrt{1+u^2}, u, 0, \dots, 0)$ we get the vertical part of the tangent space to be

$$\frac{\partial x}{\partial u} = \left(0, \frac{u}{\sqrt{1+u^2}}, 1, \dots, 0\right) \quad (2.39)$$

and so

$$Q\left(x, \frac{\partial x}{\partial u}\right) = \sqrt{1+u^2} \frac{u}{\sqrt{1+u^2}} - u \cdot 1 = 0. \quad (2.40)$$

Therefore $N_x AdS^n = \mathbb{R}x$ and so e is a geodesic. This shows (2.).

We now suppose $Q(x, y) > 1$. We claim that the curve

$$\begin{aligned} h(t) &= \cosh(t)e_1 + \sinh(t)e_2 \\ &= \cosh(t)y + \sinh(t)\left(\frac{1}{\sqrt{Q^2(x, y) - 1}}x - \frac{Q(x, y)}{\sqrt{Q^2(x, y) - 1}}y\right) \end{aligned} \quad (2.41)$$

is a spacelike geodesic. We show that e_1 and e_2 form an orthonormal basis:

$$\begin{aligned} Q(e_1, e_2) &= Q\left(y, \frac{1}{\sqrt{Q^2(x, y) - 1}}x - \frac{Q(x, y)}{\sqrt{Q^2(x, y) - 1}}y\right) \\ &= Q(y, x) \frac{1}{\sqrt{Q^2(x, y) - 1}} - Q(y, y) \frac{Q(x, y)}{\sqrt{Q^2(x, y) - 1}} \\ &= 0 \end{aligned} \quad (2.42)$$

and

$$\begin{aligned} Q(e_2, e_2) &= Q\left(\frac{1}{\sqrt{Q^2(x, y) - 1}}x - \frac{Q(x, y)}{\sqrt{Q^2(x, y) - 1}}y, \frac{1}{\sqrt{Q^2(x, y) - 1}}x - \frac{Q(x, y)}{\sqrt{Q^2(x, y) - 1}}y\right) \\ &= Q(x, x) \frac{1}{Q^2(x, y) - 1} - 2Q(x, y) \frac{Q(x, y)}{Q^2(x, y) - 1} + Q(y, y) \frac{Q^2(x, y)}{Q^2(x, y) - 1} \\ &= \frac{1 - Q^2(x, y)}{Q^2(x, y) - 1} \\ &= -1. \end{aligned} \quad (2.43)$$

Finally we show that $h(t) \in AdS^n$. We have

$$\begin{aligned} Q(h(t), h(t)) &= Q\left(\cosh(t)e_1 + \sinh(t)e_2, \cosh(t)e_1 + \sinh(t)e_2\right) \\ &= \cosh^2(t)Q(e_1, e_1) + 2\sinh(t)\cosh(t)Q(e_1, e_2) + \sinh^2(t)Q(e_2, e_2) \\ &= \cosh^2(t) - \sinh^2(t) \\ &= 1. \end{aligned} \quad (2.44)$$

We clearly have $\ddot{h}(t) = h(t)$ and by the previous this means that $\ddot{h}(t) \in N_{h(t)}AdS^n$. Hence h is a geodesic. This shows (3.).

All these geodesics are clearly complete and since we know that $N_x AdS^n = \mathbb{R}x$ it follows that all geodesics must be intersections $AdS^n \cap \Pi$, where Π is some plane. This shows that the geodesics we have parametrized are all geodesics. Thus we have shown (4.) and (5.). \square

The following definitions will be of use.

Definition 2.25. On an n -dimensional semi-Riemannian manifold (M, g) we say that a smooth n -form ω is a **volume element** if for each orthonormal frame on M , we have $\omega(e_1, \dots, e_n) = \pm 1$.

Definition 2.26. For a smooth vector field $V \in C^\infty(TM)$ we define

$$\begin{aligned} L_V(f) &= V(f), & \text{for all } f \in C^\infty(M) \\ L_V(X) &= [V, X], & \text{for all } X \in C^\infty(TM). \end{aligned} \quad (2.45)$$

The tensor derivation L_V is called the **Lie derivative** with respect to V .

Lemma 2.27. ([20], p. 195) *If ω is a local volume element on a semi-Riemannian manifold (M, g) and X is a smooth vector field, then $L_X\omega = (\operatorname{div}X)\omega$.*

Proof. ([20], p. 195) Let e_1, \dots, e_n be a local orthonormal frame such that $\omega(e_1, \dots, e_n) = 1$. We have

$$\begin{aligned} (L_X\omega)(e_1, \dots, e_n) &= X\omega(e_1, \dots, e_n) - \sum_i^n \omega(e_1, \dots, L_X e_i, \dots, e_n) \\ &= - \sum_i^n \omega(e_1, \dots, L_X e_i, \dots, e_n), \end{aligned} \quad (2.46)$$

as $X\omega(e_1, \dots, e_n) = 0$. If we write $L_X e_i = [X, e_i] = \sum_j f_{ij} e_j$ we get

$$\begin{aligned} (L_X\omega)(e_1, \dots, e_n) &= - \sum_i^n \omega(e_1, \dots, \sum_j f_{ij} e_j, \dots, e_n) \\ &= - \sum_i^n \omega(e_1, \dots, f_{ii} e_i, \dots, e_n) \\ &= - \sum_i^n f_{ii} \omega(e_1, \dots, e_n), \end{aligned} \quad (2.47)$$

where only the diagonal terms f_{ii} survive the summand, as the volume element is skew-symmetric.

On the other hand we have

$$\begin{aligned} \operatorname{div}X &= \sum_i g(\nabla_{e_i} X, e_i) \\ &= \sum_i g([e_i, X], e_i) + \underbrace{\sum_i g(\nabla_X e_i, e_i)}_{=0} \\ &= - \sum_i f_{ii}, \end{aligned} \quad (2.48)$$

where the second term vanishes as $g(e_i, e_i)$ is constant and the connection ∇ is skew. \square

We finally briefly discuss the notion of a manifold with boundary. We denote the **half-space** in \mathbb{R}^n by

$$\mathcal{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\} \quad (2.49)$$

and equip it with the relative topology from the standard metric Euclidean topology in \mathbb{R} , i.e. an open set V in \mathcal{H}^n is the intersection $V = \mathcal{H}^n \cap U$, where U is open in \mathbb{R} .

Definition 2.28. An **n -dimensional smooth manifold with boundary** is a set M and a set of charts (U_α, ϕ_α) such that

1. $\bigcup_\alpha U_\alpha = M$,

2. for all pairs α and β with $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) = W \neq \emptyset$ the sets $\phi_\alpha(W)$ and $\phi_\beta(W)$ are open sets in \mathcal{H}^n and the transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ and $\phi_\beta \circ \phi_\alpha^{-1}$ are smooth, and
3. the family $\{(U_\alpha, \phi_\alpha)\}$ is maximal relative the conditions (1.) and (2).

A point p in M is said to be a **boundary point** if for some chart $p \in U_\alpha$ and $f_\alpha = (x_1, \dots, 0)$. The set of boundary points is called the **boundary** and is denoted by ∂M . We make the following assertion about manifolds with boundary.

Proposition 2.29. ([7], p. 62) *The boundary ∂M of a smooth manifold with boundary is an n -dimensional manifold without boundary, i.e. $\partial(\partial M) = \emptyset$.*

Proof. See [7], p. 62. □

2.3 Orbit manifolds

Definition 2.30. Let M be a topological manifold and Γ a group of diffeomorphisms on M . For a fixed $p \in M$, the set $\{\mu(p) \mid \mu \in \Gamma\}$ is called a Γ -**orbit** of p . The sets of orbits is denoted M/Γ .

Definition 2.31. Let Γ be a group of diffeomorphisms on a differentiable manifold M . Then Γ is **properly discontinuous** if:

1. for any point $p \in M$ there is a neighbourhood U such that $p \in U$ and $\mu(U) \cap U \neq \emptyset$, for $\mu \in \Gamma$, if and only if $\mu = id_M$,
2. if $p, q \in M$ are not in the same orbit, there are neighbourhoods $U \ni p, V \ni q$ that are disjoint and for all $\mu \in \Gamma$ we have $\mu(U) \cap V = \emptyset$.

We get the following direct consequence.

Lemma 2.32. *Let Γ be a properly discontinuous group of diffeomorphisms on a differentiable manifold M . Let p be a point in M and let U_p be a neighbourhood of p , satisfying (1.) of Definition 2.31. Then $\mu_1(U_p) \cap \mu_2(U_p) \neq \emptyset$ if and only if $\mu_1 = \mu_2$.*

Proof. We assume the converse. Then $\mu_1(U_p) \cap \mu_2(U_p) \neq \emptyset$ and we denote by q some element in the intersection. Then we have $\mu_1(p_1) = q = \mu_2(p_2)$, for $p_1, p_2 \in U_p$. But then $\mu_2^{-1} \circ \mu_1 \equiv \tilde{\mu}$ maps p_1 to p_2 and so $\tilde{\mu}(U_p) \cap U_p \neq \emptyset$. Thus, $\tilde{\mu} = id$ which is the assertion. □

Proposition 2.33. *Let $\varphi : M \rightarrow N$ be a covering map between topological manifolds. Any deck transformation group \mathbb{D} on M is properly discontinuous.*

Proof. We first show (1.). Suppose that $p \in N$ is arbitrary and consider the fiber $\varphi^{-1}(p)$. Since φ covers N evenly, there is a neighbourhood $U \ni p$ such that $\varphi^{-1}(U)$ decomposes as a union of disjoint sheets. As φ , restricted to a sheet, is a homeomorphism, any sheet contains precisely one element in $\varphi^{-1}(p)$. Since the deck transformations are diffeomorphisms, they must be bijective between sheets. Hence the sheets around each element in M has the property in (1.).

Suppose p, q are not in the same orbit. We consider $\varphi(p)$ and $\varphi(q)$ in N . Not being in the same orbit guarantees that $\varphi(p) \neq \varphi(q)$. N is Hausdorff, and so there are open disjoint sets $U \ni \varphi(p)$ and $V \ni \varphi(q)$. Since φ is an open cover, there are two evenly covered neighbourhoods W_p and W_q containing $\varphi(p)$ and $\varphi(q)$, respectively. Consider further the intersections $W_p \cap U$ and $W_q \cap V$. As U and V are disjoint, so must their inverse images $\varphi^{-1}(U)$ and $\varphi^{-1}(V)$ be. As the deck transformations acts bijectively between sheets, so will they on the inverse intersections. Hence we have (2.). □

Definition 2.34. A map $\varphi : M \rightarrow N$ is called a **local isometry** if, for each $p \in M$, there is a neighbourhood U , that contains p , such that $\varphi|_U$ is an isometry onto $\varphi(U)$.

Definition 2.35. A covering map $\varphi : \tilde{M} \rightarrow M$ between a semi-Riemannian manifold \tilde{M} and a connected semi-Riemannian manifold M is called a **semi-Riemannian covering map** if it also is a local isometry.

The following theorem is of great importance.

Proposition 2.36. ([20], p. 188 and p. 191) *Let M be a semi-Riemannian manifold and Γ be a properly discontinuous group of isometries acting on M . Then there is a unique way to make M/Γ into a manifold such that the natural map $\pi : M \rightarrow M/\Gamma$ is a semi-Riemannian covering. If furthermore M is connected, the deck transformation group \mathbb{D} is Γ .*

Proof. See [20], p. 188 and p. 191. □

Some care is required, for the notion of a properly discontinuous group of isometries in the semi-Riemannian setting. The metric now allows for a broader class of isometries on the manifold and it can happen that the standard metric results fail. The first comment regarding this we make is the following proposition, which is not proven in the referred source.

Proposition 2.37. ([20], p. 214) *Let (M, g) be a Riemannian manifold and Γ be a properly discontinuous group of isometries acting on M . Then (1.) from Definition 2.31 implies (2.).*

Proof. We begin by showing that the Γ -orbits are closed sets. Since each point $\mu(p) \in \Gamma(p)$ has a neighbourhood from (1.) of Definition 2.31 we know that they are isolated points. We must show that no sequence in $\Gamma(p)$ converges. We therefore suppose that a convergent sequence exists, and denote it by $\{p_m\}$. Since $\{p_m\}$ is in one Γ -orbit, we may identify $p_m = \mu_m(p_1)$ and pass to a subsequence so that all the μ_m are distinct. Denote the limit by q , i.e. we have $\mu_m(p_1) \rightarrow q$. From (1.) of Definition 2.31 we have some neighbourhood U_q such that $\mu_m(U_q) \cap U_q = \emptyset$ for all m . Then there exists some N such that $\mu_N(p_1) \in U_q$ and $\mu_{N+1}(p_1) \in U_q$. Then on the one hand, there must be some element $\mu \in \Gamma$ such that $\mu \circ \mu_N = \mu_{N+1}$ and so $\mu(p_N) = p_{N+1}$. On the other hand, $\mu(U_q) \cap U_q = \emptyset$ and we have a contradiction.

We now show that two distinct orbits must have positive distance, i.e. that

$$d(\Gamma(p), \Gamma(q)) = \inf_{p \in \Gamma(p), q \in \Gamma(q)} d(p, q) = r > 0. \quad (2.50)$$

We assume not. Then there must be two sequences $\{p_n\} \subset \Gamma(p)$ and $\{q_n\} \subset \Gamma(q)$ such that $d(p_n, q_n) \rightarrow 0$ as $n \rightarrow \infty$. But we have $p_n = \mu_n(p_1)$. Since Γ is a group of isomorphisms, they must preserve distances and so it must follow that $d(p_1, \mu_n^{-1}(q_n)) \rightarrow 0$, as $n \rightarrow \infty$. This means that there would be a sequence $\mu_n^{-1}(q_n) \subset \Gamma(q)$ that converges to p_1 , an element outside of the closed set $\Gamma(q)$, which is a contradiction. Therefore the distance between orbits must be positive.

For a positive distance $r > 0$ between two distinct orbits $\Gamma(p)$ and $\Gamma(q)$ we let $\epsilon = r/3$. Since every $\mu \in \Gamma$ is a Riemannian isometry, we have $\mu(B_\epsilon(p)) = B_\epsilon(\mu(p))$. Hence the sets $U = B_\epsilon(p)$ and $V = B_\epsilon(q)$ satisfies the required properties of (2.). □

The following example shows that Proposition 2.37 is not true in general, i.e. that (2.) of Definition 2.31 is necessary in the general semi-Riemannian setting.

Example 2.38. ([20], p. 214) We consider the punctured plane $M = \mathbb{R}^2 - \{0\}$ equipped with the Lorentzian metric defined through

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.51)$$

We show that the mapping $\mu(u, v) = (2u, v/2)$ is an isometry. Let $\gamma = (\gamma_u, \gamma_v)$ be a curve in M . Then

$$g(d\mu(\dot{\gamma}(t)), d\mu(\dot{\gamma}(t))) = 2\mu(\dot{\gamma}_u(t))\mu(\dot{\gamma}_v(t)) = 2(2\dot{\gamma}_u(t)\dot{\gamma}_v(t)/2) = g(\dot{\gamma}(t), \dot{\gamma}(t)). \quad (2.52)$$

Let us now show that the cyclic isometry group Γ generated by μ is not properly discontinuous. We begin by making the following observation about how Γ behaves in the right- and left half-planes. Since no $\mu \in \Gamma$ changes sign it is clear that the quadrants are mapped into themselves. Furthermore, the right half-plane can be thought of as generated by the half-open strip

$$Q_R = \{(x, y) \in \mathbb{R}^2 \mid x \in [1, 2)\} \quad (2.53)$$

under Γ . The image $\mu^n(Q_R)$ is the set

$$\mu^n(Q_R) = \{(x, y) \in \mathbb{R}^2 \mid x \in [2^n, 2^{n+1})\}. \quad (2.54)$$

For $m \neq n$, the sets $\mu^n(Q_R)$ and $\mu^m(Q_R)$ are disjoint and homeomorphic.

Any point p in the right half-plane will be contained in some $\mu^n(Q_R)$. If p is in the interior, any open ball $B_\epsilon(p)$ that is contained in $\mu^n(Q_R)$ will satisfy (1.) of Definition 2.31. If p were to be contained in the left boundary of $\mu^n(Q_R)$, the ball $B_{2^{n-1}}(p)$ will satisfy (1.) of Definition 2.31.

Clearly, the positive y -axis is mapped into itself and may be thought of as generated by $\{0\} \times [1, 2)$ under Γ in a similar fashion. The same argument holds for both the left half-plane as well as the negative y -axis. Hence Γ satisfies (1.) of Definition 2.31.

However Γ does not fulfill (2.). This may be seen by the following. Let p and q be any two points on the positive y - and x -axes, respectively. Let $B_{\epsilon_p}(p)$ be any ball centered at $p = (0, y_p)$ on the y -axis. Let similarly $B_{\epsilon_q}(q)$ be some ball centered at $q = (x_q, 0)$ on the x -axis. The image $\mu^n(B_{\epsilon_q}(q))$ is an ellipse with semi-minor axis length $\epsilon_q/2^n$ and semi-major axis length $\epsilon_q 2^n$. Hence, we may choose n big enough so that both $x_q/2^n < \epsilon_p$ and $\epsilon_q 2^n > y_p$, and thus the balls will overlap. Hence we do not have (2.) and Γ is not properly discontinuous. This is illustrated in Figure 2.3.

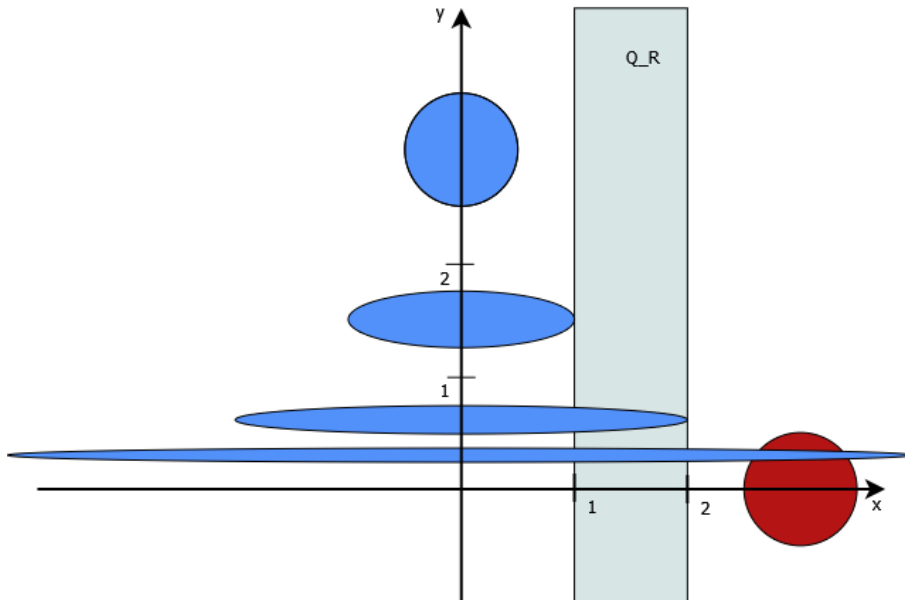


Figure 2.1: Depiction of the base set Q_R together with $B_\epsilon(p)$ and $B_\epsilon(q)$ and the image of $B_\epsilon(q)$ under μ^{-1} .

We now show equivalence of completeness of a semi-Riemannian manifold and that of its cover.

Proposition 2.39. *A semi-Riemannian manifold (M, g) is (geodesically) complete if and only if its universal covering manifold \tilde{M} is (geodesically) complete (via the pull-back metric).*

Proof. Suppose \tilde{M} is complete and $\gamma : [0, b) \rightarrow M$ is a geodesic. Pick any point $e \in \varphi^{-1}(\gamma(b))$ and let $\tilde{\gamma}$ be the lifted geodesic with $\varphi(\tilde{\gamma}) = \gamma$. $\tilde{\gamma}$ extends beyond b and by continuity we may define $\gamma(b) = \varphi(\tilde{\gamma}(b))$. The point $\gamma(b)$ has an evenly covered neighbourhood U that lifts to \tilde{M} such that precisely one sheet in the inverse set $\varphi^{-1}(\gamma(b))$ contains $\tilde{\gamma}(b)$. Since φ is a local diffeomorphism, the extension of $\tilde{\gamma}$ on the sheet maps down isometrically to U and all is shown.

To show the converse, let M be complete and $\tilde{\gamma} : [0, b) \rightarrow \tilde{M}$ be a geodesic. We map it down to M via φ and get the new geodesic $\varphi(\tilde{\gamma}) \equiv \gamma : [0, b) \rightarrow M$. Since M is complete, we may extend it beyond b . Let U be evenly covered in M and contain $\gamma(b)$. Since U is open it must contain some point $\gamma(b - \epsilon)$ that is the image of $\tilde{\gamma}(b - \epsilon)$. Let e be such that $\varphi(e) = \gamma(b)$ and \tilde{U}_e the unique sheet of U containing e . The geodesic $\varphi^{-1}(\gamma) : [b - \epsilon, b + \epsilon]$ lies in \tilde{U}_e and is mapped down to γ in U . There must be an element \tilde{p} in \tilde{U}_e such that $\varphi(\tilde{p}) = \gamma(b - \epsilon)$. In turn, there must be $k \in \mathbb{D}$ such

that $k(\tilde{p}) = \tilde{\gamma}(b - \epsilon)$. Since k is a diffeomorphism it will map \tilde{U}_ϵ to the sheet that contains $\tilde{\gamma}(b - \epsilon)$ but this sheet will contain $k(e) \equiv \tilde{\gamma}(b)$. Since k is an isometry it preserves geodesics. Hence $\tilde{\gamma}$ extends to b and since the sheet $k(\tilde{U}_\epsilon)$ maps isometrically down to U $\tilde{\gamma}$ extends beyond b as well. Hence \tilde{M} is complete. \square

Below we give the example that shows that the Hopf-Rinow Theorem, (Theorem 1.5), does not hold in the semi-Riemannian setting.

Example 2.40. ([20], p. 193, "The Clifton-Pohl Torus") Consider $M = \mathbb{R}^2 - \{0\}$ equipped with the metric

$$g = \frac{1}{u^2 + v^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.55)$$

Define the mapping $\mu_2(u, v) = (2u, 2v)$. The following shows that μ is an isometry. Let $\gamma(t) = (\gamma_u(t), \gamma_v(t))$ be a curve through M . Then

$$\begin{aligned} g(d\mu_2(\dot{\gamma}(t)), d\mu_2(\dot{\gamma}(t)))_{\mu_2(\gamma(t))} &= \frac{2d\mu_2(\gamma_u(t))d\mu_2(\gamma_v(t))}{\mu_2(\gamma_u(t))^2 + \mu_2(\gamma_v(t))^2} \\ &= \frac{2(2\dot{\gamma}_u(t))(2\dot{\gamma}_v(t))}{(2\gamma_u(t))^2 + (2\gamma_v(t))^2} \\ &= \frac{2\dot{\gamma}_u(t)\dot{\gamma}_v(t)}{\gamma_u(t)^2 + \gamma_v(t)^2} \\ &= g(\dot{\gamma}(t), \dot{\gamma}(t))_{\gamma(t)}. \end{aligned} \quad (2.56)$$

Hence the mapping μ_2 generates a cyclic isometry group $\Gamma = \{\mu_2^n \mid n \in \mathbb{Z}\}$, and the isometries are obviously properly discontinuous, from the discussion in Example 2.38. The orbits under Γ are the sets defined through the congruence relation

$$(u, v) \equiv (x, y) \pmod{\Gamma} \Leftrightarrow \frac{u}{x} = \frac{v}{y} = 2^n, \quad (2.57)$$

for some integer n . Hence the quotient set M/Γ can be identified with the set

$$M/\Gamma = \{(u, v) \in M \mid 1 \leq (u^2 + v^2) < 2\}, \quad (2.58)$$

which, after identifying the boundary circles, is a torus and therefore also compact.

We now want to show that M/Γ is not complete. From Proposition 2.36 we know that the map $\pi : M \rightarrow M/\Gamma$ is a local isometry and therefore we need only to show that (M, g) is not complete. This we do by solving for the geodesic equations. The Christoffel symbols for the given metric are

$$\begin{aligned} \Gamma_{uu}^u &= \frac{\partial_u g_{uv}}{g_{uv}} = -\frac{2u}{u^2 + v^2}, & \Gamma_{uu}^v &= 0, & \Gamma_{uv}^u &= 0, \\ \Gamma_{vv}^v &= \frac{\partial_v g_{uv}}{g_{uv}} = -\frac{2v}{u^2 + v^2}, & \Gamma_{vv}^u &= 0, & \Gamma_{uv}^v &= 0, \end{aligned} \quad (2.59)$$

from which it follows that the geodesic equations are

$$\begin{aligned} \ddot{u} - \frac{2u}{u^2 + v^2} \dot{u}^2 &= 0, \\ \ddot{v} - \frac{2v}{u^2 + v^2} \dot{v}^2 &= 0. \end{aligned} \quad (2.60)$$

The curve $\gamma(t)$ is a solution:

$$\gamma(t) = \left(\frac{1}{1-t}, 0 \right), \quad (2.61)$$

but cannot be extended beyonds its domain $(-\infty, 1)$. The projected geodesic $\pi(\gamma)$ is a geodesic in M/Γ that is not complete. Hence M/Γ is a compact Lorentzian manifold that is not complete.

We end this section by stating an important result.

Corollary 2.41. *Suppose (M, g) is a complete semi-Riemannian manifold of constant (sectional) curvature $\kappa = \pm 1$ or 0. Then*

$$M \simeq \tilde{M}/\Gamma, \tag{2.62}$$

where Γ is a properly discontinuous group of isometries and \tilde{M} is the simply connected universal covering space, which is unique up to isometry. If furthermore the curvature is positive, i.e. $\kappa = 1$, then $\tilde{M} = dS^n$.

Proof. From Theorem A.33 we know that M has some universal cover and we use the notation $\varphi : \tilde{M} \rightarrow M$. By Theorem 2.39 we know, that as M is complete, so is also \tilde{M} . We know from Theorem A.24 that there is a bijection between $\Gamma(p)$ and $\varphi^{-1}(p)$, for each $p \in M$. But then $\varphi^{-1}(p) = \Gamma(p)$. If we consider the orbit manifold M/Γ it is obvious that M and \tilde{M}/Γ are in bijection, through $\Gamma(p) \simeq p$. By Theorem 2.36, the quotient \tilde{M}/Γ is a semi-Riemannian manifold, and since the metric on \tilde{M}/Γ is the pull-back metric through a local isometry, it must have the same curvature.

From Proposition A.35 we have that since the curvature is positive, the covering space must be dS^n . This follows as it has constant curvature $\kappa = 1$, is complete and simply connected by Propositions 2.21 and 2.22. \square

Chapter 3

Homogeneous spaces and (G,X)-manifolds

In this chapter we begin by discussing the concept of a smooth homogeneous space and a Lie-group action on a manifold. This is a necessity as we need some properties of special isometry groups to the spaces \mathcal{M}^n , dS^n and AdS^n . Moreover we discuss the important notion of the so-called (G, X) -manifolds, which is a tool that will be essential for the proof of Theorem 4.21 in Chapter 6.

3.1 Homogeneous spaces

The following discussion stems from [1], [20] and [22]. We begin by reminding the reader of the notion of a Lie group.

Definition 3.1. Let G be a smooth manifold and suppose the following holds:

1. G is a group, and
2. the mapping $\rho : G \times G \rightarrow G$ by $\rho(x, y) = xy^{-1}$ is smooth.

Then G is called a **Lie group**.

Definition 3.2. Let G be a Lie group and H be a subgroup of G under the given group structure, such that H is an immersed submanifold of G . Then H is said to be a **Lie subgroup**.

Definition 3.3. Let H be a Lie subgroup of a Lie group G . If H is a (topologically) closed subset of G H is called a **closed subgroup**.

A priori, a closed subgroup and a Lie subgroup are distinct things. It turns out, however, that being a closed subgroup implies being a Lie subgroup.

Theorem 3.4. ([22], p. 110 Theorem 3.42) *Let G be a Lie group and H be a closed subgroup. Then H is a Lie subgroup and the submanifold structure of H is unique.*

Proof. See ([22], p. 110 Theorem 3.42). □

We make the following definition:

Definition 3.5. Let G be a Lie group and K be a closed subgroup. The set G/K , consisting of all left cosets gK , $g \in G$, is called the **coset space**.

The coset space may be turned into a manifold as is shown in the following theorem:

Theorem 3.6. ([1], [20], [22]) *Let G be a Lie group and K a closed subgroup. Then there is a unique way to make G/K into a manifold, so that the natural projection $\pi : G \rightarrow G/K$ with $g \rightarrow gK$ is a smooth submersion.*

Proof. See [22]. □

Definition 3.7. Let M be a manifold and G be a Lie group. A smooth map

$$\lambda : G \times M \rightarrow M, \tag{3.1}$$

such that $\lambda(e, x) = x$ and $\lambda(a, \lambda(b, x)) = \lambda(a \cdot b, x)$, is called a **left action** by G on M . The Lie group G is said to act on M through λ .

Definition 3.8. Let G be a Lie group acting on M . If for each $x, y \in M$ there is an element $\mu \in G$ such that $\mu(x) = y$, we say that G acts **transitively** on M .

Definition 3.9. Let G be a Lie group acting on a smooth manifold M . The subgroup K_x of G , defined through

$$K_x = \{g \in G \mid \lambda(g, x) = x\}, \tag{3.2}$$

is called the **isotropy subgroup** of G at x .

With the above at hand, we may state the following:

Proposition 3.10. [1] *Let G be a Lie group acting transitively on M and K_x be the isotropy group at x . Then we have the following:*

1. K_x is closed in G , and
2. the map $\rho : G/K_x \rightarrow M$ with $\rho(gK_x) = \lambda(g, x)$ is a diffeomorphism.

Proof. See [20]. □

We are now ready to define the notion of a homogeneous space.

Definition 3.11. A (smooth) manifold M is said to be a **homogeneous space** if there exists a group G acting transitively on it.

With the above, we may identify any homogeneous space M with some (not necessarily unique) coset manifold G/K , where G is a Lie group and K is a closed subgroup of G .

In the above we only assume that M is a smooth manifold, but now we continue assuming that M is equipped with some metric.

Definition 3.12. Let (M, g) be a semi-Riemannian manifold. The set of isometric diffeomorphisms form a group under composition. The **isometry group** on (M, g) is denoted by $I(M)$.

Definition 3.13. Let (M, g) be a semi-Riemannian manifold with isometry group $I(M)$. If $I(M)$ acts transitively, then (M, g) is said to be a **semi-Riemannian homogeneous manifold**.

The following theorem shows that our previous discussion of this chapter may be applied to a semi-Riemannian homogeneous space.

Theorem 3.14. ([20], p. 255) *Let (M, g) be a semi-Riemannian homogeneous space together with an isometry group $I(M)$. Then there is a unique way to make $I(M)$ into a manifold such that:*

1. $I(M)$ is a Lie group, and
2. the natural action $I(M) \times M \rightarrow M$ is smooth.

Definition 3.13 is made for the general semi-Riemannian setting, but there are some results that hold only in the Riemannian case. The following theorem is such an example.

Theorem 3.15. ([20], p. 257) *Let (M, g) be a homogeneous Riemannian manifold. Then (M, g) is complete.*

Proof. ([20], p. 257) We suppose we have a geodesic $\gamma : [0, b) \rightarrow M$ and we want to show that it can be extended beyond $[0, b)$. As (M, g) is Riemannian, every point p is contained in some normal neighbourhood U . Since U is open, there is an open ball $B_\epsilon(p) \subset U$ with $\epsilon < b/2$. Let ϕ be some element mapping $\gamma(0)$ to the point $\gamma(b - \epsilon/2)$. Denote by \tilde{v} the tangent vector $\dot{\gamma}(b - \epsilon/2)$. Since G is transitive any differential is surjective, and there is an element $v \in T_{\gamma(0)}M$ such that $d\phi(v) = \tilde{v}$. The unique geodesic $\tilde{\gamma}$ with $\tilde{\gamma}(0) = \gamma(0)$ and $\dot{\tilde{\gamma}}(b - \epsilon/2) = v$ extends γ by $\phi \circ \tilde{\gamma}$. Since the $B_\epsilon(p)$ contains $\gamma(b)$ as an interior point, the geodesic extends beyond $\gamma(b)$ and (M, g) is complete. \square

Theorem 3.15 does not hold in the semi-Riemannian setting, as the following example shows.

Example 3.16. [20] We consider the set

$$\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}, \quad (3.3)$$

equipped with the metric defined through

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.4)$$

This space has isometries on the form $\mu_a(u, v) = (u/a, av)$, $a > 0$, and $\phi_a(u, v) = (u, v - a)$. Letting (a, b) be arbitrary, we have $\phi_{ab} \circ \mu_a(a, b) = \phi_{ab}(1, ab) = (1, 0)$. Conversely, if (c, d) is arbitrary, then $\mu_{1/c} \circ \phi_{-cd}(1, 0) = \mu_{1/c}(1, cd) = (c, d)$ and so any two points can be joined by isometries. This makes (M, g) a semi-Riemannian homogeneous manifold. As (M, g) is flat, the geodesics are straight lines, and so the line $\gamma(t) = (1, 0) - t(1, 0)$ is a geodesic. γ is defined on $(-\infty, 1)$ but cannot be extended. Hence (M, g) is a homogeneous semi-Riemannian manifold that is not geodesically complete.

We want to state and prove Theorem 3.21 of Marsden, that tells us that a compact and homogeneous semi-Riemannian manifold is geodesically complete. For this we need the following lemmas.

Lemma 3.17. ([20], p. 30) *Let $\gamma : [0, b) \rightarrow M$, $b < \infty$ be an integral curve of some vector field X in $C^\infty(TM)$. Then the following are equivalent:*

1. γ is extendible as an integral curve of X to $[0, b + \epsilon)$,
2. γ is extendible,
3. γ lies in a compact subset of M , and
4. there exists a sequence $\{t_n\} \rightarrow b$ such that $\gamma(t_n)$ converges.

Proof. See ([20], p. 30). \square

Definition 3.18. A **Killing vector field** X on a semi-Riemannian manifold is a vector field such that $L_X g = 0$.

Lemma 3.19. ([20], p. 252, "The conservation lemma") *Let X be a Killing vector field on a semi-Riemannian manifold M and let γ be a geodesic in M . Then the restriction of X to γ is a Jacobi field and $g(\dot{\gamma}, X)$ is constant along γ .*

Proof. See ([20], p. 252). \square

Lemma 3.20. ([20], p. 258) *Each tangent vector to a homogeneous semi-Riemannian manifold M extends to a Killing vector field on M .*

Proof. See ([20], p. 258). \square

We may now prove Marsdens Theorem.

Theorem 3.21. ([20], p. 258, "Marsdens Theorem") *Let (M, g) be a compact homogeneous semi-Riemannian manifold. Then M is (geodesically) complete.*

Proof. [20] One needs to show extendability for a geodesic $\gamma(t)$ defined on $[0, b)$. Let $\{t_n\}$ be a sequence in $[0, b)$ that converges increasingly towards b . Then, by the compactness, there is a subsequence $\gamma(t_{n_k})$ that converges to $p := \gamma(b)$. By continuity of γ , the limit is independent of subsequence and so γ can be extended to the closed interval $[0, b]$. Let $\{v_1, \dots, v_n\}$ be a basis for $T_p M$. By Lemma 3.20, each vector v_i extends to a Killing vector field V_i on M . Furthermore, by Lemma 3.19, each component $c_i = g(\dot{\gamma}, X_i)$ is constant along γ . From this it follows that the sequence $\dot{\gamma}(t_n)$ converges in TM to a vector $v \in T_p M$.

It can be shown, that the vector field $\dot{\gamma}$ may be extended to a vector field on TM . From Lemma 3.17 it follows that since the sequence $\dot{\gamma}(t_n)$ converges as in (4.), we have that the integral curve extends and so completeness follows. \square

3.2 Isometry groups and their properties

We continue by discussing isometry groups. Example 3.22 shows that for a space with indefinite scalar product, i.e. a space with non-zero index ν , the isometry group will be different from the standard $O(n)$ isometry group of Euclidean space.

Example 3.22. Let us consider the two-dimensional Lorentzian space \mathcal{M}^2 . The metric η of this space is, as we are familiar with from above, the tensor $\eta = \text{diag}(-1, 1)$. For two vectors $v = (v_1, v_2)$ and $u = (u_1, u_2)$, we thus have

$$g(u, v) = \eta(u, v) = -u_1 v_1 + u_2 v_2. \quad (3.5)$$

This is obviously not invariant under $SO(2)$ -rotations ($u \rightarrow A(\phi)u$, $v \rightarrow A(\phi)v$ where $A(\phi) \in SO(2)$), as the following calculation shows:

$$\begin{aligned} (A(\phi)u)^T \eta A(\phi)v &= (u_1, u_2)^T \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= (u_1, u_2)^T \begin{pmatrix} \sin^2(\phi) - \cos^2(\phi) & 2 \cos(\phi) \sin(\phi) \\ 2 \cos(\phi) \sin(\phi) & \cos^2(\phi) - \sin^2(\phi) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &\neq (u_1, u_2)^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= u^T \eta v. \end{aligned} \quad (3.6)$$

However, the failure for this symmetry might be remedied - the diagonal terms look like the hyperbolic identity. Hence, one might guess that hyperbolic functions might preserve the scalar product. Indeed, they do:

$$\begin{aligned} (B(\phi)u)^T \eta B(\phi)v &= (u_1, u_2)^T \begin{pmatrix} \cosh(\phi) & \sinh(\phi) \\ -\sinh(\phi) & \cosh(\phi) \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh(\phi) & -\sinh(\phi) \\ \sinh(\phi) & \cosh(\phi) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= (u_1, u_2)^T \begin{pmatrix} \sinh^2(\phi) - \cosh^2(\phi) & \cosh(\phi) \sinh(\phi) - \cosh(\phi) \sinh(\phi) \\ \cosh(\phi) \sinh(\phi) - \cosh(\phi) \sinh(\phi) & \cosh^2(\phi) - \sinh^2(\phi) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= (u_1, u_2)^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= u^T \eta v. \end{aligned} \quad (3.7)$$

Definition 3.23. The group of linear isometries on (\mathbb{R}^n, g) , where $g = \text{diag}(-1, \dots, -1, 1, \dots, 1)$ is the diagonal metric with ν negative signs and $n - \nu$ positive signs, is called the **semi-orthogonal group**. The group of linear isometries on $\mathcal{M}^n = (\mathbb{R}^n, \eta)$, where $\eta = \text{diag}(-1, 1, \dots, 1)$, is denoted by $O_1(n)$, and is called the **Lorentz group**.

From Example 3.22 we know that $O_1(n)$ is not compact. In order to better understand its structure, let us calculate its Lie algebra. The discussion is similar to that of ([15], Example 3.11).

Lemma 3.24. *The tangent space $T_e O_1(n)$ at the neutral element is given by*

$$T_e O_1(n) = \{X \in \mathbb{R}^{n \times n} \mid X^T \eta + \eta X = 0\}, \quad (3.8)$$

where $\eta = \text{diag}(-1, 1, \dots, 1)$.

Proof. Let $\gamma : I \rightarrow O_1(n)$ be a curve in $O_1(n)$ with $\gamma(0) = e$. Then $\gamma(t)^T \eta \gamma(t) = \eta$ for all $t \in I$. We differentiate and get

$$\dot{\gamma}(0)^T \eta + \eta \dot{\gamma}(0) = 0, \quad (3.9)$$

after evaluation at $t = 0$.

We also need to show the reverse inclusion. Suppose that X satisfies $X^T + \eta X \eta = 0$. Then $X^T = -\eta X \eta$ and $(X^T)^n = (-\eta X \eta)^n = \eta (-1)^n X^n \eta$. It follows that, for $A = \text{Exp}(X)$, we get

$$\begin{aligned} A^T &= \text{Exp}(X)^T \\ &= \text{Exp}(X^T) \\ &= E + X^T + \frac{1}{2}(X^T)^2 + \frac{1}{3!}(X^T)^3 + \dots + \frac{1}{n!}(X^T)^n + \dots \\ &= \eta E \eta + (-\eta X \eta) + \frac{1}{2}(\eta X^2 \eta) - \frac{1}{3}(\eta X^3 \eta) + \dots + \frac{(-1)^n}{n!}(\eta X^n \eta) + \dots \\ &= \eta \left(E - X + \frac{1}{2}(X^2 \eta)^2 - \frac{1}{3}(\eta X^3 \eta) + \dots + \frac{(-1)^n}{n!}(\eta X^n \eta) + \dots \right) \eta \\ &= \eta \text{Exp}(-X) \eta, \end{aligned} \quad (3.10)$$

from which it follows that

$$\begin{aligned} A^T \eta A &= \text{Exp}(X)^T \eta \text{Exp}(X) \\ &= \text{Exp}(X^T) \eta \text{Exp}(X) \\ &= \eta \text{Exp}(-X) \eta \text{Exp}(X) \\ &= \eta \text{Exp}(X - X) \\ &= \eta. \end{aligned} \quad (3.11)$$

Hence $A = \text{Exp}(X) \in O_1(n)$ and we have shown the assertion. \square

Lemma 3.25. *Let $X \in O_1(n)$ be an element of the Lie algebra of $O_1(n)$. Then X has the (unique) decomposition*

$$X = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} 0 & u \\ u^T & 0 \end{pmatrix}, \quad (3.12)$$

where $B \in O(n-1)$ and $u = (u_2, \dots, u_n)$ is a row-vector.

Proof. From Lemma 3.24 we know that $X^T = -\eta X \eta$. Since $\eta = \text{diag}(-1, 1, \dots, 1)$ the right-lower $(n-1) \times (n-1)$ components Y of X satisfy $\{Y + Y^T = 0\}$ and from ([15], Proposition 3.12) we know that this is precisely $T_e O(n-1)$.

By simply looking at the rows u and u^T we see that they make X satisfy $X^T = -\eta X \eta$. \square

Let $X, Y \in \mathbb{R}^{n \times n}$. We let $\langle \cdot, \cdot \rangle$ denote the standard Euclidean scalar product on the space $\mathbb{R}^{n \times n}$, which is explicitly given by $\langle X, Y \rangle = \text{Re} \sum_{ij} X_{ij} Y_{ij}$. The following Lemma will be of use.

Lemma 3.26. ([20], p. 303) *Let G be a Lie subgroup of $GL_n(\mathbb{C})$. Then*

1. $Ad_a(X) = a X a^{-1}$ for all $a \in G$ and $X \in \mathfrak{g}_0$, and
2. if $X \in \mathfrak{g}_0$ implies that $X^T \in \mathfrak{g}_0$, then $B(X, Y) = \text{Re Trace}(XY) = \langle X^T, Y \rangle$ is an Ad -invariant scalar product.

Proof. See [20], p. 303. \square

The following definitions, stemming from ([18], p. 446), will be useful.

Definition 3.27. A bijective mapping θ from a set into itself is called an **involution** if it is its own inverse.

Definition 3.28. Let $M = G/H$ be a homogeneous space. Then M is said to be **reductive** if the Lie algebra \mathfrak{g}_0 of G has a decomposition $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{m}_0$ where \mathfrak{h}_0 is the Lie algebra of H and \mathfrak{m}_0 is $Ad(H)$ -invariant.

Definition 3.29. Let \mathfrak{g}_0 be a Lie algebra. Then the set

$$(\mathfrak{g}_0)^{\mathbb{C}} = \{u + iv \mid u, v \in \mathfrak{g}_0\} \quad (3.13)$$

is called the **complexification** of \mathfrak{g}_0 .

We remind ourselves that an **automorphism** on a Lie algebra is an invertible linear map L that preserves Lie brackets, i.e.

$$L([X, Y]) = [L(X), L(Y)]. \quad (3.14)$$

We suppose for simplicity that \mathfrak{g}_0 is real. It follows that the Ad-mapping in Proposition 3.26 is a Lie algebra-automorphism and may thus be viewed as a Lie group homomorphism

$$Ad : G \rightarrow \text{Aut}_{\mathbb{R}}(\mathfrak{g}_0) \subseteq \text{GL}_{\mathbb{R}}(\mathfrak{g}_0) \quad \text{by} \quad Ad(g) : X \rightarrow gXg^{-1}. \quad (3.15)$$

Since Ad is a map between Lie groups, it has a differential between the associated Lie algebras:

$$ad : \mathfrak{g}_0 \rightarrow \text{End}_{\mathbb{R}}(\mathfrak{g}_0) \quad \text{by} \quad ad : X \rightarrow [X, \cdot], \quad (3.16)$$

where $X \in \mathfrak{g}_0$ and $\text{End}_{\mathbb{R}}(\mathfrak{g}_0)$ is the Lie algebra of $\text{GL}_{\mathbb{R}}(\mathfrak{g}_0)$. From the Jacobi-identity of Lie-brackets, it follows that the ad-map is a Lie-algebra homomorphism. Hence we may view the image $ad(\mathfrak{g}_0)$ as a Lie subalgebra. We may now make the following definition.

Definition 3.30. Let \mathfrak{g}_0 be a Lie algebra. By $\text{Int } \mathfrak{g}_0$ we mean the subgroup that has $ad(\mathfrak{g}_0)$ as Lie algebra. An automorphism L is called **inner** if $L \in \text{Int } \mathfrak{g}_0$.

Clearly $\text{Int } \mathfrak{g}_0 \subseteq \text{Aut}_{\mathbb{R}}$ with equality if $\text{Aut}_{\mathbb{R}}$ is connected.

Definition 3.31. The bilinear form $B(X, Y) = \text{Trace}(Ad(X) \circ Ad(Y))$ is called the **Killing form**.

Definition 3.32. ([18], p. 446) A **reductive Lie group** is a 4-tuple (G, K, θ, B) , where G is a Lie group, K is a compact Lie subgroup, θ is an involution of the Lie algebra \mathfrak{g}_0 of G and B is a non-degenerate, $Ad(G)$ -invariant, θ -invariant bilinear form on \mathfrak{g}_0 , such that

1. \mathfrak{g}_0 is a reductive Lie algebra,
2. the decomposition of \mathfrak{g}_0 into θ -eigenspaces with eigenvalues ± 1 is $\mathfrak{g}_0 = \mathfrak{l}_0 \oplus \mathfrak{p}_0$ where \mathfrak{l}_0 is the Lie algebra of K ,
3. \mathfrak{l}_0 and \mathfrak{p}_0 are orthogonal under B and B is positive definite on \mathfrak{p}_0 and negative definite on \mathfrak{l}_0 (i.e. $B(P, P) > 0$ and $B(L, L) < 0$ for $P \in \mathfrak{p}_0$ and $L \in \mathfrak{l}_0$),
4. multiplication as a map from $K \times \exp \mathfrak{p}_0$ into G , is a diffeomorphism onto, and
5. every automorphism $Ad(g)$ of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ is **inner** for $g \in G$.

With this, θ is called the **Cartan involution**, $\mathfrak{g}_0 = \mathfrak{l}_0 \oplus \mathfrak{p}_0$ is called the **Cartan decomposition**, K is called the **maximal compact subgroup** and B is called the **invariant bilinear form**.

Observe that \mathfrak{p}_0 need not be a Lie subalgebra of \mathfrak{g}_0 .

We show that the isometry group $O_1(n+1)$ naturally has the structure of a reductive Lie group.

Proposition 3.33. *The 4-tuple $(O_1(n+1), O(n), \theta, B)$, where $\theta(X) = -X^T$ and B is the Killing form, is a reductive Lie group.*

Proof. That $O_1(n)$ is a Lie group follows from Theorem 3.14. $O(n)$ is compact and hence a closed subgroup of $O_1(n+1)$. It follows that $O(n)$ is also a Lie subgroup and from construction, it is obvious that θ is an involution.

We want to show (1.). From Lemma 3.25 we get the decomposition $\mathfrak{g}_0 = \mathfrak{l}_0 \oplus \mathfrak{p}_0$, where

$$\begin{aligned} \mathfrak{l}_0 &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \mid B + B^T = 0 \right\}, \quad \text{and} \\ \mathfrak{p}_0 &= \left\{ \begin{pmatrix} 0 & u \\ u^T & 0 \end{pmatrix} \mid u = (u_2, \dots, u_n) \right\} \end{aligned} \quad (3.17)$$

and that \mathfrak{l}_0 is precisely the Lie algebra (at the neutral element) of $K = O(n-1)$. In order for \mathfrak{g}_0 to be reductive, we must therefore show that \mathfrak{p}_0 is $Ad(K)$ -invariant. Let $P \in \mathfrak{p}_0$ and $L \in \mathfrak{l}_0$. Then

$$Ad_L(P) = [P, L] = \begin{pmatrix} 0 & uB \\ Bu^T & 0 \end{pmatrix} \in \mathfrak{p}_0 \quad (3.18)$$

and so \mathfrak{g}_0 is reductive. This shows (1.).

Let $P \in \mathfrak{p}_0$ and $L \in \mathfrak{l}_0$ be on the form in Lemma 3.25. We have

$$\begin{aligned} \theta(L) &= - \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \\ 0 & -B^T \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} = L, \quad \text{and} \\ \theta(P) &= - \begin{pmatrix} 0 & u \\ u^T & 0 \end{pmatrix}^T = - \begin{pmatrix} 0 & u \\ u^T & 0 \end{pmatrix} = -P. \end{aligned} \quad (3.19)$$

Thus we see that \mathfrak{l}_0 and \mathfrak{p}_0 are the corresponding eigenspaces of θ . This shows (2.).

We need to show the properties of B in (3.). We start by showing that B is symmetric. Let $\{e_1, \dots, e_{n+1}\}$ be a basis for \mathfrak{g}_0 . There is $[e_i, e_j] = \sum_k c_k^{ij} e_k$, where c_k^{ij} are the structure constants. We have

$$\begin{aligned} [e_i, [e_j, e_k]] &= [e_i, \sum_m c_m^{jk} e_m] \\ &= \sum_m c_m^{jk} [e_i, e_m] \\ &= \sum_m c_m^{jk} \sum_l c_l^{im} e_l \\ &= \sum_m \sum_l c_m^{jk} c_l^{im} e_l, \end{aligned} \quad (3.20)$$

from which we see that $B_{ij} = \sum_k \sum_m c_m^{jk} c_k^{im}$ and thus obviously $B_{ij} = B_{ji}$. Hence B is symmetric.

We now show (3.). From (2.) of Lemma 3.26 we know that $B(X, Y) = X^T \cdot Y$. From this we immediately get

$$\begin{aligned} B(P, P) &= P^T \cdot P = P \cdot P = \sum_{ij} |P_{ij}|^2 > 0, \\ B(L, L) &= L^T \cdot L = -L \cdot L = - \sum_{ij} |L_{ij}|^2 < 0 \quad \text{and} \\ B(L, P) &= L^T \cdot P = \sum_{ij} P_{ij} L_{ij} = 0, \end{aligned} \quad (3.21)$$

where the last line follows as every term $P_{ij} L_{ij} = 0$. From this we also get the θ -invariance of B , and from Lemma 3.26 we already know that B is Ad -invariant. Hence we have (3.).

Property (4.) follows from Proposition 7.14 in [18].

Property (5.) follows as a special case of Example (4) in [18] p. 474. \square

Proposition 3.34. ([18], p. 459, "KAK-decomposition") *Suppose G is a reductive Lie group and \mathfrak{a}_0 is a maximal abelian vector subspace of \mathfrak{p}_0 . Then $G = KAK$, where K is the maximal compact subgroup in the 4-tuple (G, K, θ, B) , and A is the subgroup of G with \mathfrak{a}_0 as Lie algebra.*

Proof. See ([18], p. 459). □

Proposition 3.35. [17] *Let g be an element of $O_1(n+1)$ and B be a closed Euclidean ball. Then gB is an ellipsoid with axis lengths $e^{-t}, 1, \dots, 1, e^t$, for some real $t \geq 0$.*

Proof. [17] We consider \mathfrak{p}_0 and define the vector subspace \mathfrak{a}_0 to be the matrices $a_t \in \mathfrak{p}_0$ where $u = (u_2 = t, 0, \dots, 0)$. It follows that allowing any other u_i to be non-zero ruins the closure under the Lie bracket. Moreover the matrices in \mathfrak{a}_0 obviously commute. Hence \mathfrak{a}_0 is a maximal abelian vector subspace of \mathfrak{l}_0 . The exponentiation

$$\begin{aligned} \text{Exp} \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}^2 + \frac{1}{3!} \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}^3 + \dots + \frac{1}{n!} \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}^n + \dots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 0 & \alpha^3 \\ \alpha^3 & 0 \end{pmatrix} + \dots \\ &= \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \end{aligned} \tag{3.22}$$

shows that

$$A_t = \text{Exp}(a_t) = \begin{pmatrix} \cosh(t) & \sinh(t) & 0 & \dots & 0 \\ \sinh(t) & \cosh(t) & 0 & \dots & \vdots \\ 0 & 0 & 1 & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}. \tag{3.23}$$

We therefore define $A = \text{Exp}(\mathfrak{a}_0)$ and from Proposition 3.34 get that

$$O_1(n+1) = O(n)AO(n), \tag{3.24}$$

in the sense that any $g \in G$ can be written as $g = k_1 A_t k_2$, where $A_t \in A$ and

$$k_i = \begin{pmatrix} 1 & 0 \\ 0 & B_i \end{pmatrix}, \quad B_i \in O(n). \tag{3.25}$$

To investigate the geometrical interpretation of a rotation of such a matrix, we consider the two-dimensional case and let B be the closed unit ball centered around the origin. The boundary, which is identified with S^1 , may be parametrized as $(t, x) = (\sin(\theta), \cos(\theta))$. We get, after the hyperbolic rotation, that

$$\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cosh(t) \sin(\theta) + \sinh(t) \cos(\theta) \\ \sinh(t) \sin(\theta) + \cosh(t) \cos(\theta) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{3.26}$$

Suppose for simplicity that t is positive. A straightforward calculation shows that

$$y_1^2 + y_2^2 = \cosh(2t) + \sin(2\theta) \sinh(2t) \tag{3.27}$$

and so we see that the Euclidean distance is at most $a^2 = \cosh(2t) + \sinh(2t) = e^{2t}$ and at least $b^2 = \cosh(2t) - \sinh(2t) = e^{-2t}$, for the values $\theta = k\pi + \pi/4$ and $\theta = k\pi - \pi/4$, respectively. It remains only to show that the rotated ball is indeed an ellipse with these axes. We claim that the new semi-major and semi-minor axes are the lines $t = x$ and $t = -x$, respectively. The elliptic equation is

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1, \tag{3.28}$$

where a and b are the semi-major and semi-minor axes. We change basis into mentioned axes $u \leftrightarrow x = t$ and $v \leftrightarrow x = -t$ by rotating counter clock-wise by $\phi = \pi/4$. The new elliptic equation

is

$$\begin{aligned}
\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 &= \left(\frac{y_1+y_2}{\sqrt{2}e^t}\right)^2 + \left(\frac{y_1-y_2}{\sqrt{2}e^{-t}}\right)^2 \\
&= \frac{1}{2}\left(\frac{y_1^2 + 2y_1y_2 + y_2^2}{e^{2t}}\right) + \frac{1}{2}\left(\frac{y_1^2 - 2y_1y_2 + y_2^2}{e^{-2t}}\right) \\
&= \frac{1}{2}\left(y_1^2 + y_2^2\right)\left(\frac{1}{e^{2t}} + \frac{1}{e^{-2t}}\right) + \left(y_1y_2\right)\left(\frac{1}{e^{2t}} - \frac{1}{e^{-2t}}\right) \\
&= \left(\cosh(2t) + \sin(2\theta)\sinh(2t)\right)\cosh(2t) \\
&\quad - \left(\sinh(2t) + \sin(2\theta)\cosh(2t)\right)\sinh(2t) \\
&= 1.
\end{aligned} \tag{3.29}$$

So we see that the image of the rotated unit circle $A_t(S^1)$ is an ellipse with semi-major and semi-minor axis lengths e^t and e^{-t} .

The result now follows as for higher dimensions, the $O(n)$ group rotates the spatial parts isometrically and hence axis lengths are preserved. Furthermore, since $O_1(n+1)$ is a linear isometry group, we may use the decomposition

$$g(B_\epsilon(p)) = g(p + B_\epsilon(0)) = g(p) + g(B_\epsilon(0)) \tag{3.30}$$

to see that the action on any Euclidean closed ball centered at p is an ellipsoid with mentioned principal axes, centered at p . \square

We now state the isometry groups of the three Lorentzian space forms.

Proposition 3.36. ([20], p. 239-240) *The Lorentzian space forms have the following isometry groups:*

1. *The Minkowski space \mathcal{M}^n has as isometry group the semi-direct product $O_1(n) \rtimes T^n$, where T^n is the group of translations on \mathcal{M}^n ,*
2. *dS^n has the isometry group $I(dS^n) = O_1(n+1)$, and*
3. *AdS^n has the isometry group $I(AdS^n) = O_2(n+1)$.*

Proof. See [20], p. 239-240. \square

The group $I(\mathcal{M}^n)$ is called the **Poincaré group**. From the proof to Proposition 3.35 we get the following Corollary:

Corollary 3.37. *The isotropy subgroups K_p of $O_1(n+1)$ is $O_1(n)$.*

Proof. Suppose $p = (0, \dots, 0, 1)$. We know that $O_1(n+1) = O(n)A_tO(n)$. Clearly, the isometry subgroup preserving p must be $O(n-1)A_tO(n-1)$, but this is just $O_1(n)$.

We must show that any two isotropy groups are conjugate. Let p and q be two points in \mathcal{M}^{n+1} and $h(p) = q$ for some $h \in O_1(n+1)$. Let $k_p \in K_p$ be an element in the isotropy subgroup at p , and similarly $k_q \in K_q$. Since both $h^{-1}k_ph \in K_q$ and $hk_qh^{-1} \in K_p$ it follows that $hK_ph^{-1} = K_q$ and so two arbitrary isotropy subgroups are conjugate. The result now follows. \square

Let (X, d) be some metric space and let \mathcal{F} denote collection of the closed and bounded sets in this metric. The following definition equips \mathcal{F} with a metric.

Definition 3.38. Let (X, d) be a metric space and let X and Y be two closed and bounded subsets of X . The **Hausdorff distance** $d_H(X, Y)$ between X and Y is defined to be

$$d_H(X, Y) = \max \left\{ \sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y) \right\}. \tag{3.31}$$

The 2-tuple (\mathcal{F}, d_H) is called the **Hausdorff metric space**. A sequence of sets $\{A_n\}$ is said to be **Hausdorff convergent** to A if $d_H(A_n, A) \rightarrow 0$ as $n \rightarrow \infty$. We show the following immediate property.

Lemma 3.39. *The metric space (\mathcal{F}, d_H) is complete if and only if (X, d) is complete. Furthermore, for a Cauchy sequence $\{A_n\}$ in (\mathcal{F}, d_H) let A be the set of limit points of all Cauchy sequences $\{x_n\}$ in (X, d) with $x_n \in A_n$ and let \bar{A} be its closure. Then $d_H(A_n, \bar{A}) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. We suppose first that (\mathcal{F}, d_H) is complete. Then a Cauchy-sequence $\{x_n\}$ in (X, d) may be viewed as a sequence in (\mathcal{F}, d_H) and clearly the limit $\{x\} \in \mathcal{F}$ serves as limit in (X, d) . Hence we have shown that (X, d) is complete.

Conversely, we suppose that (X, d) is complete. We let $\{A_n\}$ be a Cauchy sequence in (\mathcal{F}, d_H) and let A be the set as stated in the assertion. We suppose that $A_n \not\rightarrow A$. Then we may assume, by perhaps passing to a subsequence, that $d_H(A_n, A) \geq c > 0$. By again passing to a subsequence, we may assume that $d(A_n, A_{n+1}) < 1/2^n$. From the definition of d_H the first condition means that, for each n , there exists either

$$\begin{aligned} (1) \quad x_n \in A \quad \text{s.t.} \quad \inf_{y \in A_n} d(x_n, y) \geq c, \quad \text{or} \\ (2) \quad x_n \in A_n \quad \text{s.t.} \quad \inf_{y \in A} d(x_n, y) \geq c. \end{aligned} \tag{3.32}$$

Possibility (1) is ruled out immediately by definition. Any element $y \in A$ is by definition a limit of some Cauchy sequence and so $\inf_{y \in A_n} d(x_n, y)$ can only be bounded below by 0 over n . Hence (1) is ruled out.

We now consider (2). Let n be big enough so that $d_H(A_n, A_{n+1}) < c/2^2$ and let $y \in A_n$ be an element as in (2). We denote y by x_1 . We know that

$$\inf_{z \in A_{n+1}} d(x_1, z) \leq \max_{x \in A_n} \inf_{z \in A_{n+1}} d(x, z) \leq d_H(A_n, A_{n+1}) < c/2^2. \tag{3.33}$$

Hence there exists $x_2 \in A_{n+1}$ such that $d(x_1, x_2) < c/2^2$. For each k we get, by iterating this procedure, that $d(x_k, x_{k+1}) < c/2^{k+1}$. This defines a sequence $\{x_k\}$ in (X, d) with $x_k \in A_{n+k}$. This sequence is on the one hand clearly Cauchy and as (X, d) is complete, it has a limit $x_k \rightarrow x$. But

$$d(x_1, x) \leq d(x_1, x_2) + d(x_2, x_3) + \dots < c \left(\frac{1}{2^2} + \frac{1}{2^3} + \dots \right) = c \frac{1}{2} < c \tag{3.34}$$

so that $x \notin A$. This is a contradiction. Therefore we must have $d_H(A_n, A) \rightarrow 0$. By taking the closure of A and observing that $d_H(A, \bar{A}) = 0$ it follows that $d_H(A_n, \bar{A}) \rightarrow 0$ and so (\mathcal{F}, d_H) is complete. \square

With the constructions in this work, we need to modify this definition to the following:

Definition 3.40. A sequence $\{F_n\}$ of closed sets is said to **converge** to F if, for each closed Euclidean ball B , the sequence $F_n \cap B$ converges to F in the Hausdorff sense.

Definition 3.41. In \mathcal{M}^n , a **hyperplane** H is a subset $H = v + W$, where v is some vector in \mathcal{M}^n and W is a $(n-1)$ -dimensional vector subspace. If $v \in W$, H is said to be **vectorial**, otherwise H is said to be **affine**. If the subspace W contains its orthogonal complement, i.e. $W^\perp \subset W$, H is said to be **coisotropic**.

With these definitions at hand, we get the following corollary to Proposition 3.35.

Corollary 3.42. [17] *Suppose $\{g_n\}$ is a sequence in $O_1(n+1)$, $B_\epsilon(p)$ is a closed Euclidean ball and the sequence $\{g_n B_\epsilon(p)\}$ converges with limit \mathcal{E} . Then \mathcal{E} is either of codimension 0 or 1, and in the latter case is contained in a coisotropic hyperplane. If $\{g_n\}$ lie in the isotropy group K_p , the hyperplane is vectorial and contains p .*

Proof. Since $\{g_n\} \subset O_1(n+1)$, every g_n is a linear mapping. Hence we may consider the decomposition

$$g_n(B_\epsilon(p)) = g_n(p + B_\epsilon(0)) = g_n(p) + g_n(B_\epsilon(0)). \tag{3.35}$$

From Proposition 3.35 it is clear that the limit may have codimension 1 or 0. If the sequence $\{g_n\}$ is in the isotropy group K_p we have $g_n(p) = p$ and moreover as g_n is linear, it must fix the line $\mathbb{R}p$. The segment of this line is contained in $B_\epsilon(p)$ must therefore also be contained in the limit \mathcal{E} . Thus \mathcal{E} is a vector subspace. \square

3.3 (G,X)-structures

In this section we briefly describe the so-called (G,X)-structures. This construction will be crucial, when proving Theorem 4.21. We follow [11], [12] and [13].

Definition 3.43. Let X be a real analytic manifold and G be a Lie-group acting transitively on X . A smooth manifold M is called a **(G,X)-manifold** if there exists a maximal atlas of local charts $\{(\phi_\alpha, U_\alpha)\}$ such that $\phi_\alpha : U_\alpha \rightarrow X$ where $U_\alpha \subset M$, with the following properties:

1. $M = \bigcup_{\alpha \in A} U_\alpha$,
2. ϕ_α is a diffeomorphism onto its image $\phi_\alpha(U_\alpha)$ and
3. whenever $U_\alpha \cap U_\beta \neq \emptyset$ there is, for each connected component C of the intersection, an element $g_{C,\alpha,\beta} \in G$ such that $g_{C,\alpha,\beta} \circ \phi_\alpha = \phi_\beta$.

The atlas is denoted by (Φ, U) , where $\Phi = \{\phi_\alpha \mid \alpha \in A\}$ and $U = \{U_\alpha \mid \alpha \in A\}$. Figure 3.3 illustrates the construction. The following result is of enormous importance.

Theorem 3.44. ([12], p. 69, "The Development Theorem") Let M be a (G,X)-manifold with universal covering space \tilde{M} and deck group $\mathbb{D} \simeq \pi_1(M)$. Then there exists a pair (D, h) where D is a local diffeomorphism $D : \tilde{M} \rightarrow X$, called the **developing map**, and h is a group homomorphism $h : \mathbb{D} \rightarrow G$, such that the following diagram is commutative:

$$\begin{array}{ccc}
 \tilde{M} & \xrightarrow[k]{\mathbb{D}} & \tilde{M} \\
 \downarrow D & & \downarrow D \\
 X & \xrightarrow[h(k)]{\Gamma} & X
 \end{array} \tag{3.36}$$

or equivalently, that the developing map is equivariant:

$$D \circ k = h(k) \circ D, \quad \text{for all } k \in \mathbb{D}. \tag{3.37}$$

The image $h(\mathbb{D}) = \Gamma$ is called the **holonomy subgroup** of G .

Furthermore, the pair (D, h) is unique up to holonomy-conjugation, i.e. if (D', h') is another pair, then there exists $g \in \Gamma$ such that

$$\begin{aligned}
 D' &= g \circ D, & g &\in \Gamma, & \text{and} \\
 h'(k) &= \text{Inn}(g) h(k).
 \end{aligned} \tag{3.38}$$

Proof. See [12], p. 69. \square

Conversely, it can be shown that the existence of a developing pair (D, h) is equivalent to existence of a (G,X)-structure on M . See e.g. [11] for details.

It will become apparent that it is easier to work with the definition in Theorem 3.44 as this definition is, in some sense, more concrete.

The following insight is of great importance.

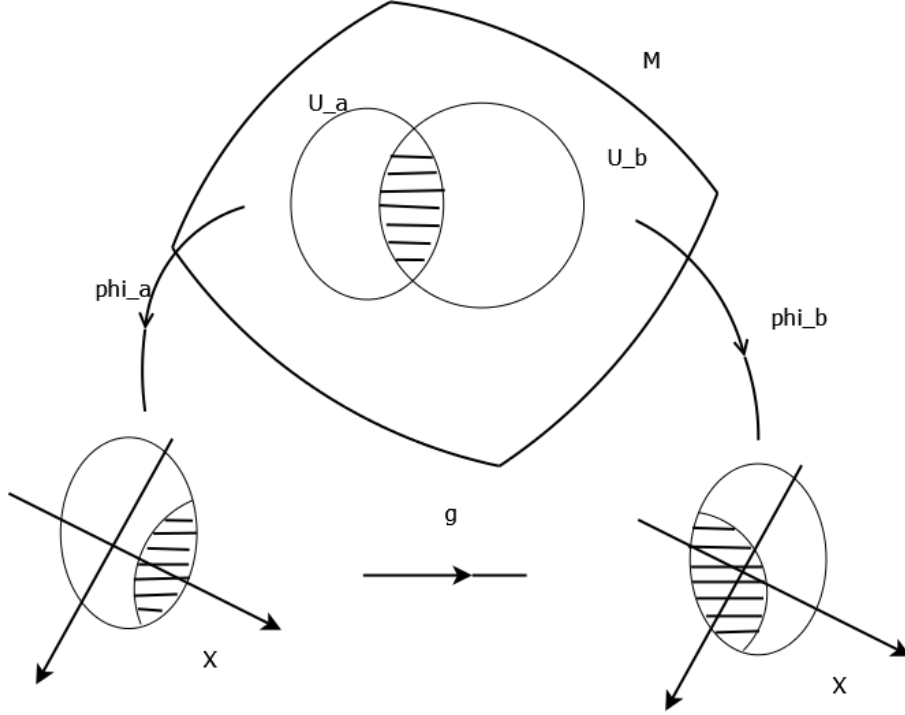


Figure 3.1: Depiction of the action of two charts (U_α, ϕ_α) and (U_β, ϕ_β) on a (G, X) -manifold M .

Theorem 3.45. ([12], p. 72) Let M be a (\tilde{G}, \tilde{X}) manifold and let (G, X) the pair where \tilde{X} is the universal cover of X and \tilde{G} is the lift of G to \tilde{X} with group homomorphism $\psi: \tilde{G} \rightarrow G$ such that the following diagram commutes:

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow[\tilde{g}]{\tilde{G}} & \tilde{X} \\
 \downarrow \tilde{\varphi} & & \downarrow \tilde{\varphi} \quad \text{i.e.} \quad \psi(\tilde{g}) \circ \tilde{\varphi} = \tilde{\varphi} \circ \tilde{g}. \\
 X & \xrightarrow[\psi(\tilde{g})]{G} & X
 \end{array} \tag{3.39}$$

Let the pair (\tilde{D}, \tilde{h}) be the development pair for (\tilde{G}, \tilde{X}) . Then the pair $(\tilde{\varphi} \circ \tilde{D}, \psi \circ \tilde{h})$ is a development pair for (G, X) .

Proof. Since \tilde{D} and $\tilde{\varphi}$ are both local diffeomorphisms their composition is also a local diffeomorphism. Similarly the composition $\psi \circ \tilde{h}$ is a group homomorphism. We need only to show the equivariance condition. For any $\tilde{p} \in \tilde{M}$ we have

$$\tilde{\varphi} \circ \tilde{D}(\tilde{g}(\tilde{p})) = \tilde{\varphi}(\tilde{h}(\tilde{g})) \circ \tilde{D}(\tilde{p}) = \psi \circ \tilde{h}(\tilde{g}) \circ \tilde{\varphi}(\tilde{D}(\tilde{p})) \tag{3.40}$$

so that the development pair $(\tilde{\varphi} \circ \tilde{D}, \psi \circ \tilde{h})$ is also holonomy-equivariant. \square

Theorem 3.45 shows that if we model M on some space X that is not simply connected we know from Theorem 3.44 that the development pair (D, h) on (G, X) will be "holonomy-equivalent" to the pair $(\tilde{\varphi} \circ \tilde{D}, \psi \circ \tilde{h})$ and hence this pair may be the one of choice.

Theorem 3.46. [12] Let (M, g) be a semi-Riemannian manifold of constant (sectional) curvature. Then there exists a (G, X) -structure on M .

Proof. See [11]. \square

We get the following properties for the (G, X) -manifolds.

Lemma 3.47. *Suppose (M, g) is a (G, X) -manifold and $\Gamma = h(\mathbb{D})$ is the holonomy subgroup of G . Then Γ acts properly discontinuously on the image $D(\tilde{M})$ in X .*

Proof. We first show property (1.) from Definition 2.31. Suppose x is in $D(\tilde{M})$ and it is the image of $D(\tilde{x})$. Then there is a neighbourhood U_x in X , containing x such that D^{-1} , restricted to U_x is a diffeomorphism. If $x = D(\tilde{x})$, we denote the lift $\tilde{U}_{\tilde{x}}$ to the neighbourhood that contains \tilde{x} . But \mathbb{D} acts properly discontinuously on \tilde{M} and so there is a neighbourhood $\tilde{V}_{\tilde{x}}$ around \tilde{x} that satisfies (1.) of Definition 2.31. Let $\tilde{W}_{\tilde{x}} = \tilde{U}_{\tilde{x}} \cap \tilde{V}_{\tilde{x}}$ denote the intersection.

We claim that the image $D(\tilde{W}_{\tilde{x}}) = W_x$ of the intersection satisfies (1.) of Definition 2.31. We suppose not. The converse is that for any open ball B_n there must exist some $\mu_n \in \Gamma$ such that $\mu_n \circ B_n \cap B_n \neq \emptyset$. We let $\{B_n\}$ be a sequence of balls, of decreasing radius and successive inclusion, centered at x and contained in W_x . By hypothesis, for each n we have some $z_n = \mu_n(y_n)$ in the intersection. The respective sequences $\{z_n\}$ and $\{y_n\}$ must obviously converge to x , as $n \rightarrow \infty$. Since D is a diffeomorphism on W_x , we may lift both sequences up to \tilde{M} . We denote the lifted sequences by $\{\tilde{y}_n\}$ and $\{\tilde{z}_n\}$ and they converge to the limit \tilde{x} . On the one hand, they are two sequences in $\tilde{W}_{\tilde{x}}$ that converges to \tilde{x} . On the other hand there must for each n exist a $k_n \in \mathbb{D}$ such that $h(k_n) = \mu_n$ and therefore also $k_n(\tilde{y}_n) = \tilde{z}_n$. This contradicts the proper discontinuity of \mathbb{D} if the k_n are non-trivial.

We now show property (2.) of Definition 2.31. Again we assume this not to be true and derive a contradiction. The converse is that for some two points x and y that are not congruent, i.e. $x \not\equiv y \pmod{\Gamma}$, there are no two neighbourhoods $V_x \ni x$ and $V_y \ni y$ satisfying $\mu(V_x) \cap V_y = \emptyset$. We let $B_{n,x}$ and $B_{n,y}$ be decreasing, nested sequences of balls centered around x and y , respectively. By hypothesis we have $\mu_n(B_{n,x}) \cap B_{n,y} \neq \emptyset$ for each n . We let $y_n = \mu_n(x_n)$ be in the intersections. Clearly $\{y_n\}$ converges to y and $\{x_n\}$ converge to x .

We let \tilde{x} and \tilde{y} be elements in \tilde{M} such that $x = D(\tilde{x})$ and $y = D(\tilde{y})$. We must have $\tilde{x} \not\equiv \tilde{y} \pmod{\mathbb{D}}$, as otherwise

$$x = D(\tilde{x}) = D(k(\tilde{y})) = h(k) \circ D(\tilde{y}) = h(k)(y), \quad (3.41)$$

which would contradict our hypothesis that x and y were not congruent under Γ . Since \mathbb{D} acts properly discontinuously we therefore have $\tilde{V}_{\tilde{x}}$ and $\tilde{V}_{\tilde{y}}$ such that $k(\tilde{V}_{\tilde{x}}) \cap \tilde{V}_{\tilde{y}} = \emptyset$ for all non-trivial $k \in \mathbb{D}$. We lift the sequences $\{x_n\}$ to $\tilde{V}_{\tilde{x}}$ and denote the lifted sequence by $\{\tilde{x}_n\}$. This is done by the same procedure as the discussion on proving (1.) - we assume D is a diffeomorphism here as otherwise we take intersections. Similarly, $\{y_n\}$ is lifted to $\{\tilde{y}_n\}$ in $\tilde{V}_{\tilde{y}}$.

For each n there must exist $k_n \in \mathbb{D}$ such that $h(k_n) = \mu_n$. It now follows that $k_n(\tilde{y}_n) = \tilde{x}_n$. We have

$$y_n = D(\tilde{y}_n) = D(k_n(\tilde{x}_n)) = h(k_n)D(\tilde{x}_n) = \mu_n(x_n). \quad (3.42)$$

This contradicts the proper discontinuity of \mathbb{D} acting on \tilde{M} .

Hence Γ must act in a properly discontinuous manner on $D(\tilde{M}) \subset X$. \square

We now want to show that if M is compact the developing map D is a covering onto its image.

Lemma 3.48. *Let (M, g) be a compact (G, X) -manifold. Then every fibre $D^{-1}(x)$ is finite.*

Proof. We assume that the assertion is false, i.e. that there exists a sequence $\{\tilde{x}_n\}$ where we have $D(\tilde{x}_n) = x$ for each n . We first demonstrate that this set is closed. In fact we demonstrate that it is the union of isolated points, i.e. that there is no convergent subsequence in $D^{-1}(x)$. If such a convergent subsequence existed, we would have $\tilde{y}_n \rightarrow \tilde{y}$. Then any neighbourhood containing \tilde{y} would contain some \tilde{y}_N and this breaks the local injectivity.

Now we use the compactness of M . $\{\tilde{x}_n\}$ has no convergent subsequence by what we just deduced. Let $\{u_n\}$ be the projected sequence in M , i.e. $u_n = \varphi(\tilde{x}_n)$ for each n . By compactness of M , there is a convergent subsequence, which for notational convenience, we assume is $\{u_n\}$. The sequence converges to some limit u and this point has an evenly covered neighbourhood U_u . We pick any of the sheets $\varphi^{-1}(U_u)$ and denote it by \tilde{U}_u . We may assume that all the u_n are contained in the neighbourhood U_u as we otherwise discard the first N elements and keep the "tail". Denote by $\tilde{y}_n = \varphi^{-1}(u_n) \cap \tilde{U}_u$. Since φ is a local diffeomorphism, this sequence must converge, i.e. $\tilde{y}_n \rightarrow \tilde{y} = \varphi^{-1}(u) \cap \tilde{U}_u$.

We now consider how the sequences $\{\tilde{y}_n\}$ and $\{\tilde{x}_n\}$ are related. If, for each n , we were to have $k(\tilde{x}_n) = \tilde{y}_n$, where $k \in \mathbb{D}$, it would follow that \tilde{x}_n would be a convergent sequence. Hence no

$k \in \mathbb{D}$ may relate any \tilde{y}_n to any \tilde{x}_n for more than finitely many n . From this it follows that, as we can always pass to a subsequence, we may assume that there is $k_n(\tilde{x}_n) = \tilde{y}_n$ where all $k_n \in \mathbb{D}$ are distinct.

From the developing map equation it now follows that

$$D(\tilde{y}_n) = D(k_n(\tilde{x}_n)) = h(k_n)D(\tilde{x}_n) = h(k_n)(x) \quad (3.43)$$

and hence that the image sequence $\{D(\tilde{y}_n)\}$ must both lie in the Γ -orbit Γx and be convergent. By the proof in Proposition 2.37 this cannot happen. Hence $D^{-1}(x)$ is finite. \square

In particular, Lemma 3.48 shows that if M is compact, then $\text{Ker}(h)$ is a finite subgroup of \mathbb{D} . The following theorem is crucial in the proof of Theorem 4.21.

Theorem 3.49. *Let M be a compact (G, X) -manifold. Then the developing map D is a covering map onto the image $\Omega = D(\tilde{M})$.*

Proof. From Lemma 3.48 we know that the fibers $D^{-1}(x)$ have finite cardinality. If the fibres have the same cardinality N , then for each $x \in X$ we may construct an evenly covered neighbourhood U_x in the following way. Since \tilde{M} is Hausdorff we may take finitely many intersections so as to have disjoint neighbourhoods $\tilde{U}_{\tilde{x}_i}$ around each element \tilde{x}_i in $D^{-1}(x)$ such that D , restricted to $\tilde{U}_{\tilde{x}_i}$, is a diffeomorphism. Then we just take the intersection

$$W = \bigcap_i^N D(\tilde{U}_{\tilde{x}_i}) \quad (3.44)$$

and in turn let $D^{-1}(W) \cap \tilde{U}_{\tilde{x}_i}$ be the sheet containing \tilde{x}_i . This makes D into a covering map onto the image $\Omega = D(\tilde{M})$.

However, the cardinalities need not be the same. We consider the sets

$$\mathcal{O}_n = \{x \in \Omega \mid |D^{-1}(x)| = n\}. \quad (3.45)$$

Clearly, we have

$$\Omega = \bigcup_{n=1}^{\infty} \mathcal{O}_n \quad (3.46)$$

and in principle infinitely many \mathcal{O}_n may be non-empty. We now want to show that the path-lifting property will still hold in this setting. We show both existence and uniqueness of such a lift¹.

We suppose that $\gamma : [0, 1] \rightarrow X$ is a curve and that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are two lifts with starting points $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) = \tilde{x}_0$. We define the set

$$I = \{t \in [0, 1] \mid \tilde{\gamma}_1(t) = \tilde{\gamma}_2(t)\}. \quad (3.47)$$

Since $[0, 1]$ is connected and $\{0\} \in I$, so that I is non-empty, it is sufficient to show that $I \subset [0, 1]$ is both open and closed.

We show that I is open by considering any element $t_0 \in I$. We need to show that there exists an $\epsilon > 0$ such that the interval $(t_0 - \epsilon, t_0 + \epsilon) \subset I$. The developing map D is a local diffeomorphism and so there is some neighbourhood \tilde{U} around $\tilde{\gamma}_1(t_0) = \tilde{\gamma}_2(t_0)$ such that D , restricted to \tilde{U} , is a diffeomorphism. Since γ is continuous there exists an $\epsilon > 0$ such that $\gamma((t_0 - \epsilon, t_0 + \epsilon)) \subset D(\tilde{U})$. This shows that I is open.

To show that I is closed we consider the set

$$\Delta X = \{(x, x) \in X \times X \mid x \in X\}. \quad (3.48)$$

The function $\tilde{\gamma}_1 \times \tilde{\gamma}_2 : [0, 1] \times [0, 1] \rightarrow \tilde{M} \times \tilde{M}$ is continuous, and as ΔX is closed we have that $(\tilde{\gamma}_1 \times \tilde{\gamma}_2)^{-1}(\Delta X) = I$ is also closed. Hence $I = [0, 1)$ and by continuity of the lifts this extends to $I = [0, 1]$. We have now showed that $\tilde{\gamma}_1 = \tilde{\gamma}_2$ so that any lift is unique.

It remains to show that the lift exists. We define

$$t = \sup\{s \in [0, 1] \mid \text{there exists a lift } \tilde{\gamma} : [0, s] \rightarrow \tilde{M}\}. \quad (3.49)$$

¹The following arguments are essentially due to Arne Meurman.

Since D is a local homeomorphism we must have $t > 0$. By the preceding any such lift will necessarily be unique. We suppose now that $t < 1$. Let $\{t_n\} \subset [0, 1]$ be a monotonically increasing sequence with $t_n \rightarrow t$. We define the curves $\tilde{\gamma}_n : [0, t_n] \rightarrow \tilde{M}$ as the unique lifts of $\gamma : [0, t_n] \rightarrow X$. This defines a lift $\tilde{\gamma} : [0, t) \rightarrow \tilde{M}$ and we define the sequence $\{\tilde{x}_n\}$ via $\tilde{x}_n = \tilde{\gamma}(t_n)$.

Now either $\{\tilde{x}_n\}$ contains a convergent subsequence or not. If not, then by the proof of Lemma 3.48 we may assume that there exists a sequence $\{k_n\} \subset \mathbb{D}$ of distinct elements such that $k_n \tilde{x}_n = \tilde{y}_n \rightarrow \tilde{y}$. But $D(\tilde{x}_n) = \gamma(t_n) \rightarrow \gamma(t)$ as γ is continuous. Hence $\{\tilde{x}_n\}$ must have a convergent subsequence and by the local continuity of D we must have $D(\tilde{\gamma}(t)) = \gamma(t)$. Again by local continuity D is homeomorphic on some $\tilde{U} \ni \tilde{\gamma}(t)$ and there exists some $\epsilon > 0$ such that $\tilde{\gamma}$ may be extended to be defined on $[0, t + \epsilon]$. Hence we are in contradiction to the assumption that $t < 1$ and the path lifting exists.

We can now show the covering property. We suppose that some $x \in \mathcal{O}_n$ and for W as above, containing x , there is some element $y \in \mathcal{O}_m$, where $m > n$. Now on the one hand as by Lemma A.30 X is locally path-connected, we have that there is a path $\gamma : [0, 1] \rightarrow W$ with $\gamma(0) = y$ and $\gamma(1) = x$. Let \tilde{x} denote the lift of x to some of the sheets \tilde{U}_i constructed above, with image W , and similarly \tilde{y} . We let \tilde{z} be some elements not in a sheet but with $D(\tilde{z}) = y$. Now on the one hand, we have just deduced existence of a unique path-lift of γ that connects \tilde{z} and \tilde{x} . But on the other hand D is locally diffeomorphic onto W and so this lift must lie in the lifted sheet. This is a contradiction to the assumption that \tilde{z} does not. Hence $y \in \mathcal{O}_n$ and D is a covering onto its image. \square

It follows that when the developing map D is a covering map onto the image $D(\tilde{M})$ the number of sheets equal $\text{Ker } h$. This, together with Proposition 2.36, gives the manifold diffeomorphism

$$\tilde{M}/\text{Ker}(h) \simeq \Omega, \quad \Omega = D(\tilde{M}). \quad (3.50)$$

The special case when X is simply connected restricts the possibilities even further.

Proposition 3.50. *Let (M, g) be a (G, X) -manifold where X is simply connected. If D is a covering, then D is injective.*

Proof. We assume that $D(\tilde{x}) = D(\tilde{y}) = x$. Since \tilde{M} is path-connected by Lemma A.28 there is a curve $\gamma : [0, 1] \rightarrow \tilde{M}$ such that $\gamma(0) = \tilde{x}$ and $\gamma(1) = \tilde{y}$. This would mean that the curve $D(\gamma)$ is a loop in X and as X is simply connected, it is contractible to the trivial one-point curve. We denote the associated homotopy map by $F(s, t)$. By Lemma A.21 there must exist a homotopy map $G(s, t)$ in \tilde{M} with $G(0, 0) = \gamma(0) = \tilde{x}$ and $D(G) = F$. Since G is not a loop we are in contradiction. Hence $\tilde{x} = \tilde{y}$. \square

An immediate consequence of this insight is that when D is a covering and X is simply connected, the group homomorphism $h : \mathbb{D} \rightarrow G$ is an isomorphism onto its image $h(\mathbb{D}) = \Gamma$. With this we get the following diffeomorphisms:

$$M \simeq \tilde{M}/\mathbb{D} \simeq \Omega/\Gamma, \quad \Omega = D(\tilde{M}). \quad (3.51)$$

We invoke the following definition of completeness.

Definition 3.51. A (G, X) -manifold M is **complete** if the developing map D is a covering.

Proposition 3.52. *Let (M, g) be a (G, X) -manifold, where X is a (geodesically) complete semi-Riemannian manifold. If M is complete in the sense of Definition 3.51, it is (geodesically) complete.*

Proof. In light of Proposition 2.39, we need only to show that any geodesic $\gamma : [0, b) \rightarrow \tilde{M}$ extends beyond b . But this follows since D is a diffeomorphism and X is (geodesically) complete, the image $D(\gamma)$ is defined for all parameters and so geodesic completeness of \tilde{M} follows. \square

Chapter 4

Curvature in Lorentzian spaces

This chapter is devoted to the understanding of constant curvature in Lorentzian spaces. Indeed, such an understanding is quite different from the Riemannian case, as will be shown. We begin by stating the well-known Gauss-Bonnet theorem for two-dimensional Lorentzian manifolds. We proceed by proving the Theorem of Calabi and Markus and proceed by studying the Theorem of Klingler.

4.1 The Lorentzian Gauss-Bonnet Theorem

In this section we prove the Gauss-Bonnet Theorem for Lorentzian surfaces. In order to do so, we must make precise the notion of an angle. This was initially done in [3]. The same authors then showed the Gauss-Bonnet Theorem in the Lorentzian setting in [2].

We begin by defining the notion of orientability. Let \mathbb{R}^n be equipped with two bases $\{e_1, \dots, e_n\}$ and $\{\tilde{e}_1, \dots, \tilde{e}_n\}$. If there is a matrix A such that $e_i = A_{ij}\tilde{e}_j$ with $\det(A) > 0$ these bases are said to have the **same orientation**. If $\det A < 0$ they are said to have **opposite orientation**. It follows that the set of bases with the same orientation form an equivalence class and that precisely two such equivalence classes exist. We denote by

$$\lambda = [e_1, \dots, e_n] \tag{4.1}$$

the orientation of a given basis.

We take this definition to the manifold setting in the following:

Definition 4.1. Let (M, g) be an n -dimensional semi-Riemannian manifold and p any point in M . Suppose (U_p, ϕ_p) is a local chart containing p . Then let λ_{ϕ_p} be an orientation to each basis in T_pM , i.e.

$$\lambda_{\phi_p} = \left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right]. \tag{4.2}$$

An **orientation** of M is a mapping λ that assigns an orientation at each point p in M and is smooth in the sense that for each coordinate system λ_{ϕ_p} we have $\lambda = \lambda_{\phi_p}$. A manifold with an orientation is said to be **oriented**.

With this definition, we may discuss the vector space geometry of the Lorentzian surfaces. We will denote the two-dimensional Minkowski space by \mathcal{M}^2 , and assume it to be oriented. For any timelike unit vector z in $T_p\mathcal{M}^2$, we will always denote by z^\perp the unique spacelike unit vector that both satisfies $\eta(z, z^\perp) = 0$ and is such that the ordered basis $\{z, z^\perp\}$ has positive orientation. We will say that a coordinate system (x_1, x_2) in $T_p\mathcal{M}^2$ is **allowable** if $(0, 1)$ is a future-pointing timelike unit vector and $(0, 1)^\perp = (1, 0)$.

The concept of orientability extends to a more elaborate notion in Lorentzian geometry, namely that of *time-orientability*. To make this notion precise, we invoke the following definition.

Definition 4.2. Let V be a Lorentzian vector space with scalar product g and consider a timelike vector $v \in V$. The set

$$C(v) = \{u \in V \mid g(u, v) < 0, \text{ and } u \text{ is timelike}\} \tag{4.3}$$

is called the **timecone** containing v . The **opposite timecone** is denoted by $C(-v) = -C(v)$.

It follows straightforwardly that if v and u lie in the same timecone, then $C(v) = C(u)$. With this definition we may discuss the following.

Definition 4.3. Let τ be a function on M that for each point p in M assigns a timecone τ_p in T_pM . τ is **smooth** if for each p in M there is a smooth vector field V on some neighbourhood U_p of p such that $V(q) \in \tau_q$ for all $q \in U_p$. τ is called a **time-orientation** of M . If M admits a time-orientation, M is said to be **time-orientable**. If such an orientation is chosen, M is said to be **time-oriented**.

Clearly this definition is very analogous to Definition 4.1. We state an immediate consequence.

Lemma 4.4. ([20], p. 145) *A connected Lorentzian manifold M is time-orientable if and only if there exists a timelike vector field $X \in C^\infty(TM)$.*

Proof. ([20], p. 145) If such an X exists, it will for each point p assign a timecone τ_p . Since X is globally defined and M is connected this gives the time-orientation.

We suppose now that M is time-oriented. From Definition 4.3 we know that every point p has a neighbourhood U_p and a vector field V such that $V|_q$ lies in τ_q for all $q \in U_p$. We now let $\{f_\alpha | \alpha \in A\}$ be a smooth partition of unity subordinate to the covering of neighbourhoods U_p . All f_α are non-negative and each f_α is contained in some U_{p_α} . Since timecones are convex any finite sum $\sum_\alpha v_\alpha$ of timelike vectors v_α must be timelike and contained within the same timecone. It follows that the sum

$$\sum_\alpha f_\alpha V_{q_\alpha} \quad (4.4)$$

is a globally defined timelike vector field. \square

Depending on which timecone the vector field V lies in, V is called either **future-pointing** or **past-pointing** in the natural sense.

With this preparation, we are ready to impose the following definition.

Definition 4.5. Let (M, g) be an oriented Lorentzian surface and let $\gamma : I \rightarrow M$ be a smooth curve in M . Then

$$k_g = g(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}^\perp) \quad (4.5)$$

is called the **geodesic curvature** along γ .

One might say, that the geodesic curvature k_g measures the deviation of γ from being a geodesic. For a geodesic, $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ and so $k_g = 0$.

We can now state the Gauss-Bonnet theorem for a Riemannian manifold.

Theorem 4.6. (*"The Gauss-Bonnet Theorem"*) *Let (M, g) be a Riemannian surface and D be a domain in M with piece-wise smooth boundary ∂D . Then*

$$\int_{\partial D} k_g ds + \iint_D K dA + \sum_i \theta_i = 2\pi \quad (4.6)$$

where θ_i are the inner angles made by the "polygon" ∂D .

We now wish to generalise this to the Lorentzian setting, following [2]. We make the following definition.

Definition 4.7. Let (M, g) be a connected, oriented and time-oriented Lorentzian manifold and X, Y be two unit time-like future-pointing (or past-pointing) vectors at T_pM . Then the **angle** α between X and Y is defined by

$$\begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (4.7)$$

where $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ in any allowable coordinate system. We will use the notation $\alpha(X, Y)$.

If Y is past-pointing time-like and X is future-pointing time-like, we make the following definition. Observe that $(-Y)$ is future-pointing. Hence $\alpha(X, -Y)$ is

$$\begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -y_1 \\ -y_2 \end{pmatrix} \quad (4.8)$$

from which it follows that

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh(-\alpha) & \sinh(-\alpha) \\ \sinh(-\alpha) & \cosh(-\alpha) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (4.9)$$

We now ignore the $(-E)$ matrix and define the angle as $-\alpha$, i.e. $\alpha(X, Y) = -\alpha(X, -Y)$.

Before stating any properties, we make the following remark.

Remark 4.8. The angle α is independent of the choice of the allowable coordinate system.

The angle has the following properties.

Lemma 4.9. [2] *Let X and Y be unit time-like vectors on a \mathcal{M}^2 . Then*

1. $\alpha(X, X) = 0$,
2. $\alpha(X, -X) = 0$,
3. $\alpha(X, Y) + \alpha(Y, Z) = \alpha(X, Z)$,
4. $\alpha(X, Y) = -\alpha(Y, X)$,
5. $\alpha(X, -Y) = \alpha(X, Y)$.

Proof. We denote the matrix in Definition 4.7 by $B(\alpha)$. (1.) follows immediately, as the $B(\alpha) = E$ if and only if $\alpha = 0$.

If X is future-pointing, $-X$ is past-pointing. From the definition we have

$$\alpha(-X, X) = -\alpha(-(-X), X) = -\alpha(X, X) = 0. \quad (4.10)$$

This shows (2.).

Suppose first that X, Y and Z have the same time-orientation, say future-pointing. Denote the angles by $\theta_{XZ} = \alpha(X, Z)$, $\theta_{XY} = \alpha(X, Y)$ and $\theta_{YZ} = \alpha(Y, Z)$. Then we have

$$Z = B(\theta_{YZ})Y = B(\theta_{YZ})B(\theta_{XY})X. \quad (4.11)$$

The matrix product satisfies $B(\theta_{YZ})B(\theta_{XY}) = B(\theta_{YZ} + \theta_{XY})$, which can be seen, using standard trigonometric formulas:

$$\begin{aligned} & \begin{pmatrix} \cosh(\theta_{YZ}) & \sinh(\theta_{YZ}) \\ \sinh(\theta_{YZ}) & \cosh(\theta_{YZ}) \end{pmatrix} \begin{pmatrix} \cosh(\theta_{XY}) & \sinh(\theta_{XY}) \\ \sinh(\theta_{XY}) & \cosh(\theta_{XY}) \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\theta_{YZ} + \theta_{XY}) & \sinh(\theta_{YZ} + \theta_{XY}) \\ \sinh(\theta_{YZ} + \theta_{XY}) & \cosh(\theta_{YZ} + \theta_{XY}) \end{pmatrix}. \end{aligned} \quad (4.12)$$

But the product matrix can also be identified with $B(\theta_{XZ})$ and so the assertion follows.

Suppose that X is past-pointing, the others future-pointing. Then (4.), whose proof is independent of this, may be used and the same argument follows *mutatis mutandis*. The same goes for the case of two past-pointing vectors.

From the matrix identity $B(\alpha)B(\beta) = B(\alpha+\beta)$ that we just showed, it follows that $B(\theta_{XY})^{-1} = B(-\theta_{XY})$. From this, in turn, it follows that since $B(\theta_{XY})X = Y$ we have $B(\theta_{XY})^{-1}Y = X$ and thus $B(-\theta_{XY})Y = X$, which is (4.).

Using (3.) and (2.), we get $\alpha(X, -Y) = \alpha(X, Y) + \underbrace{\alpha(Y, -Y)}_{=0} = \alpha(X, Y)$. and so (5.) follows. \square

The following lemma will be useful.

Lemma 4.10. *Suppose X is a future-pointing timelike vector in the oriented manifold \mathcal{M}^2 that makes an angle $\alpha(X, z)$ with the timelike basis vector in some allowable coordinate system $\{z, z^\perp\}$. Then*

$$X = \cosh(\alpha)z - \sinh(\alpha)z^\perp. \quad (4.13)$$

Proof. We let $X = c_1z + c_2z^\perp$ in the allowable coordinate system. That X is unit implies $c_2^2 - c_1^2 = 1$. The angle assumption implies

$$\begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (4.14)$$

In total, we have the three conditions

$$\begin{aligned} 1 &= c_2^2 - c_1^2 \\ 0 &= \cosh(\alpha)c_2 + \sinh(\alpha)c_1 \\ 1 &= \sinh(\alpha)c_2 + \cosh(\alpha)c_1, \end{aligned} \quad (4.15)$$

from which it immediately follows that $c_1 = \cosh(\alpha)$ and $c_2 = -\sinh(\alpha)$. \square

With Example 4.11 we give an illustration of Lorentzian trigonometry.

Example 4.11. Let $v_1, v_2 \in T_p\mathcal{M}^2$ be time-like and future-pointing. Then $v_1 + v_2$ is also time-like and future-pointing. We consider the triangle spanned by these three vectors (v_1 at the origin, v_2 at the tip of v_1 and $-(v_1 + v_2)$ starts at the tip of v_2 and takes us back to the origin). From (3.) and (4.) above, the following holds:

$$\begin{aligned} \alpha(v_1, v_2) + \alpha(v_2, v_1 + v_2) + \alpha(v_1 + v_2, v_1) &= \alpha(v_1, v_2) + \alpha(v_2, v_1) \\ &= \alpha(v_1, v_2) - \alpha(v_1, v_2) \\ &= 0. \end{aligned} \quad (4.16)$$

This has an interesting geometric interpretation, as we try to illustrate in Figure 4.1.

The following lemma generalizes this example.

Lemma 4.12. *Let D be a region in \mathcal{M}^2 , bounded by a polygon ∂D of piece-wise time-like straight lines, denoted γ_i , $i = 1, 2, \dots, k$. Then*

$$\alpha(v_1, v_2) + \dots + \alpha(v_{n-1}, v_n) + \alpha(v_n, v_1) = 0. \quad (4.17)$$

Proof. We use induction over n corners. We have already shown the case to be true for $n = 3$. Suppose the statement holds for polygons with n corners. We have

$$\begin{aligned} \sum_i \alpha &= \alpha(v_1, v_2) + \dots + \alpha(v_{n-1}, v_n) + \alpha(v_n, v_{n+1}) + \alpha(v_{n+1}, v_1) \\ &= \underbrace{\alpha(v_1, v_2) + \dots + \alpha(v_{n-1}, v_n) + \alpha(v_n, v_1)}_{=0} \\ &\quad - \alpha(v_n, v_1) + \alpha(v_n, v_{n+1}) + \alpha(v_{n+1}, v_1) \\ &= -\alpha(v_n, v_1) + \alpha(v_n, v_{n+1}) + \alpha(v_{n+1}, v_1) \\ &= \alpha(v_1, v_n) + \alpha(v_n, v_{n+1}) + \alpha(v_{n+1}, v_1) \\ &= 0. \end{aligned} \quad (4.18)$$

Here the sum of the first n terms vanishes by the induction hypothesis and the last three by Example 4.11. The assertion now follows by induction. \square

Lemma 4.13. [2] *Let $\gamma(t)$ be a smooth time-like curve on \mathcal{M}^2 and Z be a parallel future-pointing vector field on γ . Then we have*

$$\frac{d}{dt}\alpha(\dot{\gamma}(t), Z(t)) = -k_g. \quad (4.19)$$

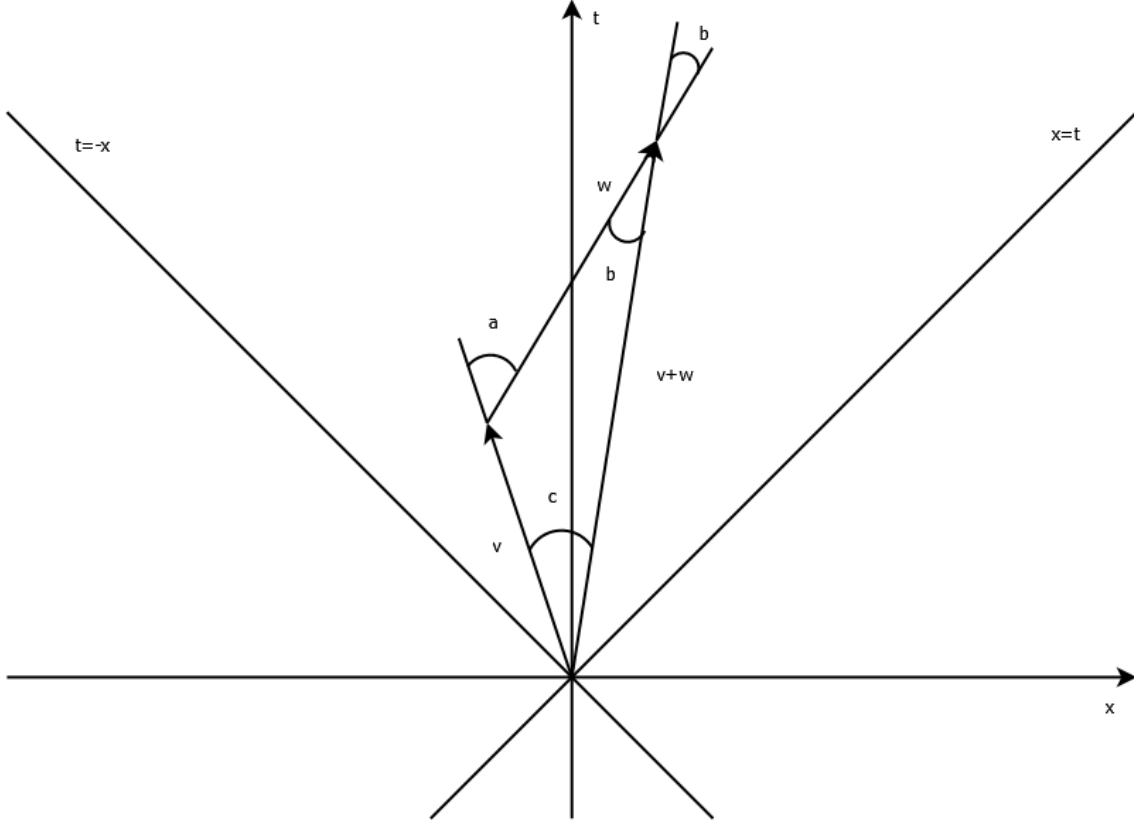


Figure 4.1: The addition of two future pointing time-like vectors in \mathcal{M}^2 . The angle $\alpha(v, w)$ is denoted by a , $\alpha(w, v + w)$ is denoted by b and $\alpha(v + w, v)$ is denoted by c .

Proof. [2] We first observe that if Z is parallel then so is Z^\perp . This follows from

$$0 = \frac{d}{dt}g(Z, Z^\perp) = g(\underbrace{\nabla_{\dot{\gamma}}Z}_{=0}, Z^\perp) + g(Z, \nabla_{\dot{\gamma}}Z^\perp) = g(Z, \nabla_{\dot{\gamma}}Z^\perp) \quad (4.20)$$

together with

$$0 = \frac{d}{dt}g(Z^\perp, Z^\perp) = 2g(Z^\perp, \nabla_{\dot{\gamma}}Z^\perp). \quad (4.21)$$

Therefore $\nabla_{\dot{\gamma}}Z^\perp$ is orthogonal to both Z and Z^\perp and by non-degeneracy of the vector space we must have $\nabla_{\dot{\gamma}}Z^\perp = 0$.

In the case of $\dot{\gamma}(t)$ being future-pointing, we have from Lemma 4.10 that

$$\dot{\gamma} = \cosh(\alpha)Z - \sinh(\alpha)Z^\perp, \quad \dot{\gamma}^\perp = -\sinh(\alpha)Z + \cosh(\alpha)Z^\perp. \quad (4.22)$$

If $\dot{\gamma}(t)$ is a past-pointing vector, $-\dot{\gamma}(t)$ is future-pointing, and we have from Lemma 4.10 that

$$\dot{\gamma} = -\cosh(\alpha)Z + \sinh(\alpha)Z^\perp, \quad \dot{\gamma}^\perp = \sinh(\alpha)Z - \cosh(\alpha)Z^\perp. \quad (4.23)$$

We differentiate the angle α with respect to the time, t . We get, in the future pointing case, that

$$\begin{aligned} \nabla_{\dot{\gamma}}\dot{\gamma} &= \nabla_{\dot{\gamma}}(\cosh(\alpha)Z - \sinh(\alpha)Z^\perp) \\ &= \cosh(\alpha)\nabla_{\dot{\gamma}}Z + \sinh(\alpha)\frac{d\alpha}{dt}Z \\ &\quad - \sinh(\alpha)\nabla_{\dot{\gamma}}Z^\perp - \cosh(\alpha)\frac{d\alpha}{dt}Z^\perp \\ &= \sinh(\alpha)\frac{d\alpha}{dt}Z - \cosh(\alpha)\frac{d\alpha}{dt}Z^\perp \end{aligned} \quad (4.24)$$

and in the past-pointing case, by similar calculation, that

$$\nabla_{\dot{\gamma}}\dot{\gamma} = -\sinh(\alpha)\frac{d\alpha}{dt}Z + \cosh(\alpha)\frac{d\alpha}{dt}Z^\perp. \quad (4.25)$$

In either case, we get

$$k_g = g(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}^\perp) = \left(\sinh^2(\alpha) - \cosh^2(\alpha) \right) \frac{d\alpha}{dt} = -\frac{d\alpha}{dt} \quad (4.26)$$

which was the assertion. \square

Before stating and proving the Lorentzian Gauss-Bonnet Theorem, we need the following lemma. As \mathcal{M}^2 is time-orientable there exists by Lemma 4.4 a unit, time-like and future pointing vector field X on \mathcal{M}^2 . Hence, we can define the **connection form**:

$$\omega_X(V) = g(\nabla_V X, X^\perp). \quad (4.27)$$

Remark 4.14. Since we have the formula $0 = Vg(X, X^\perp) = g(\nabla_V X, X^\perp) + g(X, \nabla_V X^\perp)$, it follows that $\omega_X(V) = -g(X, \nabla_V X^\perp)$.

Lemma 4.15. [2] *The connection form satisfies*

$$d\omega = \kappa dA, \quad (4.28)$$

where κ is the **Gaussian curvature**.

Proof. [2] Let V and W be arbitrary vector fields. Then

$$\begin{aligned} d\omega_X(V, W) &= V\omega_X(W) - W\omega_X(V) - \omega_X([V, W]) = \\ &= Vg(\nabla_W X, X^\perp) - Wg(\nabla_V X, X^\perp) - g(\nabla_{[V, W]} X, X^\perp) \\ &= g(\nabla_V \nabla_W X, X^\perp) + g(\nabla_W X, \nabla_V X^\perp) - g(\nabla_W \nabla_V X, X^\perp) \\ &\quad - g(\nabla_V X, \nabla_W X^\perp) - g(\nabla_{[V, W]} X, X^\perp) \\ &= g(R(V, W)X, X^\perp) + g(\nabla_W X, \nabla_V X^\perp) - g(\nabla_V X, \nabla_W X^\perp). \end{aligned} \quad (4.29)$$

From the metric compatibility of the Levi-Civita connection, it follows that $0 = Vg(X, X) = 2g(\nabla_V X, X)$ and so $\nabla_V X$ is orthogonal to X . Similarly one gets that $\nabla_V X^\perp$ is orthogonal to X^\perp . Thus the last two terms vanish, and for $V = X$ and $W = X^\perp$, we get

$$d\omega_X(X, X^\perp) = g(R(X, X^\perp)X, X^\perp) = \kappa. \quad (4.30)$$

The corresponding volume element satisfies $dA(X, X^\perp) = 1$ and so the assertion follows. \square

We can now state and prove the famous Gauss-Bonnet Theorem in the Lorentzian setting.

Theorem 4.16. [2] *Let (M, g) be an oriented Lorentzian surface. Let D be a domain in M with piece-wise smooth boundary curve ∂D consisting of time-like curve segments. Then*

$$\int_{\partial D} k_g dt + \sum_i \theta_i - \iint_D K dA = 0. \quad (4.31)$$

Proof. [2] Let Z be a unit and timelike vector field parallel along the boundary γ . As γ was only piece-wise smooth, we denote the partial smooth curves γ_i and the start- and endpoints $\gamma(t_0), \gamma(t_1), \dots, \gamma(t_k)$ for the k curve-segments, so that $\theta_i = \alpha(\dot{\gamma}_i(t_i), \dot{\gamma}_{i+1}(t_i))$. Again $k_g = g(\dot{\gamma}(t), Z)$. Along the first curve γ_1 , we have by Lemma 4.13

$$-\int_{\gamma_1} k_g dt = \int_{\gamma_1} \left(\frac{d\alpha}{dt} \right) = \alpha(\dot{\gamma}_1(t_1), Z(t_1)) - \alpha(\dot{\gamma}_1(t_0), Z(t_0)) \quad (4.32)$$

so that

$$\begin{aligned}
-\int_{\gamma_1} k_g dt - \theta_1 &= \alpha(\dot{\gamma}_1(t_1), Z(t_1)) - \alpha(\dot{\gamma}_1(t_0), Z(t_0)) \\
&\quad - \alpha(\dot{\gamma}_1(t_1), \dot{\gamma}_2(t_1)) \\
&= \alpha(\dot{\gamma}_1(t_1), Z(t_1)) - \alpha(\dot{\gamma}_1(t_0), Z(t_0)) \\
&\quad + \alpha(\dot{\gamma}_2(t_1), \dot{\gamma}_1(t_1)) \\
&= \alpha(\dot{\gamma}_2(t_1), Z(t_1)) - \alpha(\dot{\gamma}_1(t_0), Z(t_0))
\end{aligned} \tag{4.33}$$

where we first used (4.) and then (3.) of Lemma 4.9. With the same argument, we get for the general curve segment γ_i that

$$-\int_{\gamma_i} k_g dt - \theta_i = \alpha(\dot{\gamma}_{i+1}(t_i), Z(t_i)) - \alpha(\dot{\gamma}_k(t_{i-1}), Z(t_{i-1})) \tag{4.34}$$

for the i :th curve. For the final segment, we identify $\theta_k = \alpha(\dot{\gamma}_k(t_k), \dot{\gamma}_1(t_k)) = \alpha(\dot{\gamma}_k(t_0), \dot{\gamma}_1(t_0))$. We now sum up our k terms:

$$\sum_i \left(-\int_{\gamma_i} k_g dt - \theta_i \right) = -\int_{\gamma} k_g dt - \sum_i \theta_i = \alpha(Z(t_k), Z(t_0)), \tag{4.35}$$

where we used the notation $\alpha(Z(t_0), Z(t_k))$ for the angle between Z before and after the parallel transportation along γ . Let X be some future-pointing time-like vector field defined on \mathcal{M}^2 and denote $\alpha(Z, X) = \beta$. Then Z can be decomposed, following Lemma 4.10, as

$$Z = \cosh(\beta)X - \sinh(\beta)X^\perp. \tag{4.36}$$

With this decomposition we get

$$\begin{aligned}
0 &= \nabla_{\dot{\gamma}} Z = \nabla_{\dot{\gamma}} \left(\cosh(\beta)X - \sinh(\beta)X^\perp \right) \\
&= \sinh(\beta) \frac{d\beta}{dt} X + \cosh(\beta) \nabla_{\dot{\gamma}} X \\
&\quad - \cosh(\beta) \frac{d\beta}{dt} X^\perp - \sinh(\beta) \nabla_{\dot{\gamma}} X^\perp \\
&= \sinh(\beta) \frac{d\beta}{dt} X + \cosh(\beta) \underbrace{g(\nabla_{\dot{\gamma}} X, X)}_{=0} X \\
&\quad + \cosh(\beta) g(\nabla_{\dot{\gamma}} X, X^\perp) X^\perp - \cosh(\beta) \frac{d\beta}{dt} X^\perp \\
&\quad - \sinh(\beta) \underbrace{g(\nabla_{\dot{\gamma}} X^\perp, X^\perp)}_{=0} X^\perp - \sinh(\beta) g(\nabla_{\dot{\gamma}} X^\perp, X) X \\
&= \sinh(\beta) \left(\frac{d\beta}{dt} - \omega(\dot{\gamma}) \right) X + \cosh(\beta) \left(\omega(\dot{\gamma}) - \frac{d\beta}{dt} \right) X^\perp,
\end{aligned} \tag{4.37}$$

where the terms are zero as $0 = \frac{d}{dt} g(X, X) = 2g(\nabla_{\dot{\gamma}} X, X)$ and similar for X^\perp . The above calculation implies, as X and X^\perp are linearly independent, that $\omega(\dot{\gamma}) = \frac{d\beta}{dt}$.

Now we can relate this to the angle:

$$\begin{aligned}
\alpha(Z(t_k), Z(t_0)) &= \alpha(Z(t_k), X) + \alpha(X, Z(t_0)) \\
&= \alpha(Z(t_k), X) - \alpha(Z(t_0), X) \\
&= \int_{\gamma} \left(\frac{d\beta}{dt} \right) dt \\
&= \int_{\gamma} \omega(\dot{\gamma}) dt \\
&= \iint_D d\omega \\
&= \iint_D \kappa dA,
\end{aligned} \tag{4.38}$$

where we use Greens theorem in the next-to-last line. This completes the proof. \square

Before stating Corollary 4.18, we need the following result.

Proposition 4.17. ([20], p. 149) *Let M be a smooth manifold. Then the following is equivalent:*

1. M admits a Lorentzian metric,
2. M admits a time-orientable metric,
3. there exists a non-vanishing vector field on M , and
4. either M is non-compact, or M is compact with Euler characteristic $\chi(M) = 0$.

Proof. See [20], p. 149. \square

With this at hand, we are ready to state the following remarkable result.

Corollary 4.18. *A compact, oriented Lorentzian surface with constant curvature κ must be flat.*

Proof. From Proposition 4.17 it follows that the only oriented, connected and compact Lorentzian surfaces are tori, T_2 . On T_2 we apply the standard procedure of dividing T_2 into disjoint regions bounded by time-like curves. The Gauss-Bonnet formula is applied to each of these. The curve integrals will go in reverse directions and so must add up to zero. The angle sum must add up to zero by Lemma 4.12. What remains is the integral

$$\int_{T_2} \kappa dA = 0 \tag{4.39}$$

from which the result follows immediately. \square

4.2 The Calabi-Markus Theorem

In this section we prove Theorem 4.19 of Calabi and Markus. This is important when classifying spherical Lorentzian space forms.

Let (M^n, g) be an n -dimensional Lorentzian space form with positive constant (sectional) curvature. Let furthermore Γ be a properly discontinuous group of isometries on M^n . From Proposition 2.36 we know that the quotient M^n/Γ admits a manifold structure and the "push-forward" metric on M^n/Γ preserves curvature, so that M^n/Γ has positive constant curvature as well. From Corollary 2.41 we know that such a space form must be on the form

$$M^n \simeq dS^n/\Gamma. \tag{4.40}$$

The following theorem shows finiteness of Γ .

Theorem 4.19. ([5], "The Calabi-Markus Theorem") *Let Γ be a properly discontinuous group of isometries on dS^n . Then Γ is finite.*

Proof. We follow the discussions in ([5], p. 67), and ([20], p. 245).

From Proposition 3.36 we know that $I(dS^n) = O_1(n+1)$, which is a group of linear isometries. Therefore Γ is also a group of linear isometries. We denote by S^{n-1} the sphere

$$S^{n-1} = \{(x_1, x_2, \dots, x_{n+1}) \in \mathcal{M}^{n+1} \mid x_1 = 0, \quad x_2^2 + \dots + x_{n+1}^2 = 1\}. \tag{4.41}$$

We assume Γ is infinite. Let p be a point on the unit sphere S^{n-1} , i.e. $p = (0, x_2, \dots, x_{n+1})$. We claim that there is a neighbourhood U_p of p such that $\mu(U) \cap S^{n-1} \neq \emptyset$ only for finitely many $\mu \in \Gamma$. If not, then for every neighbourhood V that contains p , there would exist an infinite sequence $\{\mu_m\}$ such that $\mu_m(V) \cap S^{n-1} \neq \emptyset$. We may assume the $\{\mu_m\}$ all to be distinct.

We now claim that it is possible to construct a sequence $\{p_m\}$ in S^{n-1} that both converges to p and for each m satisfies $\mu_m(p_m) \in S^{n-1}$. Let N be such that the open ball $B_{1/N}(p)$ satisfies (1.) of Definition 2.31. By hypothesis, we must for each $m > N$, have $\mu_m(B_{1/m}(p)) \cap S^{n-1} \neq \emptyset$. Let

p_m be an element in this intersection. Clearly $p_m \rightarrow p$ and $\{\mu_m(p_m)\} \subset S^{n-1}$.

Since S^{n-1} is compact, the sequence $\{\mu_m(p_m)\}$ must have a convergent subsequence, which we for the sake of notational convenience assume to be the same. We denote the limit by q . Now we can have either $q \equiv p \pmod{\Gamma}$ or not. We show that in both cases this will contradict the proper discontinuity of Γ .

We first suppose not. Then, by (2.) of Definition 2.31 there are open balls $B_\epsilon(p)$ and $B_\epsilon(q)$ that satisfy

$$\mu(B_\epsilon(p)) \cap B_\epsilon(q) = \emptyset, \quad \mu \in \Gamma. \quad (4.42)$$

The ball $B_\epsilon(p)$ must contain some tail of $\{p_m\}$. Similarly, the ball $B_\epsilon(q)$ must contain some tail of the sequence $\{\mu_m(p_m)\}$. But this is a contradiction to (2.), as then there is some N such that both $p_N \in B_\epsilon(q)$ and $\mu_N(p_N) \in B_\epsilon(q)$. Hence, we are contradicted.

Therefore, we assume that $q \equiv p \pmod{\Gamma}$, i.e. that $\mu(q) = p$. By construction, we must have $\mu \circ \mu_m(p_m) \rightarrow p$. By (1.) of Definition 2.31 there is an open ball $B_\epsilon(p)$ such that $\mu(B_\epsilon(p)) \cap B_\epsilon(p) = \emptyset$. Now, for some N , there must be $p_N \in B_\epsilon(p)$, but then on the other hand $\mu \circ \mu_N(p_N) \in B_\epsilon(p)$ as well and we are contradicted.

Therefore, we may assume existence of some neighbourhood U containing p , such that only finitely many $\mu \in \Gamma$ satisfy $\mu(U) \cap S^{n-1} \neq \emptyset$.

We now want to demonstrate a consequence of the linearity of Γ . Let W be the vector subspace defined by $x_1 = 0$. Then $S^{n-1} \subset W$. Since Γ is linear, the image $\mu(W)$ must, for any $\mu \in \Gamma$ be a vector subspace of the same dimension. By Lemma 2.7 we have

$$\dim(W \cap \mu(W)) + \dim(W + \mu(W)) = \underbrace{\dim W}_n + \underbrace{\dim \mu(W)}_n = 2n. \quad (4.43)$$

Furthermore, $\mu(W)$ must be spacelike. Since $(W + \mu(W))$ is at most the full ambient space, we have $\dim(W + \mu(W)) \leq n + 1$ from which it follows that $\dim(W \cap \mu(W)) \geq n - 1 > 0$. Therefore, any $\mu \in \Gamma$ must satisfy $\mu(S^{n-1}) \cap S^{n-1} \neq \emptyset$.

The neighbourhoods U_p form an open cover of S^{n-1} which is a compact set. Therefore we have

$$S^{n-1} \subset \bigcup_i^N U_{p_i}. \quad (4.44)$$

For each U_{p_i} we have only finitely many μ_1, \dots, μ_{K_i} such that U_{p_i} intersects S^{n-1} non-trivially. For each N such neighbourhoods we remove these finitely many elements. As Γ is infinite there must remain elements in Γ that would satisfy $\mu(S^{n-1}) \cap S^{n-1} = \emptyset$. This is a contradiction and so Γ must be finite. \square

Corollary 4.20. [5] *A spherical relativistic space form M^n is not compact.*

Proof. By Corollary 2.41, any spherical space form M^n must be on the form dS^n/Γ . The space dS^n is not compact by Proposition 2.21. By Theorem 4.19, Γ is a finite group.

Denote by $\varphi : dS^n \rightarrow M$ the covering map. Any point $p \in M$ has an evenly covered neighbourhood U_p through φ , and the finiteness of \mathbb{D} implies there are k disjoint sheets in $\varphi^{-1}(U_p)$.

The sets $\bigcup_{p \in M^n} U_p$ forms an open cover, and compactness implies $M^n \subset \bigcup_i^N U_{p_i}$. It follows that

$\tilde{M}^n \subset \bigcup_i^N \varphi^{-1}(U_{p_i})$, i.e. that \tilde{M}^n is contained within $k \cdot N$ open neighbourhoods. Since dS^n is

not compact, there is a sequence $\{x_n\} \subset dS^n$ with no convergent subsequence. Since dS^n , by hypothesis, is supposed to be finitely covered, some sheet V must contain infinitely many points of $\{x_n\}$. Since $\varphi(V) \subset M^n$ and M^n is compact, there is a convergent subsequence. We lift this through φ^{-1} and as φ is a local diffeomorphism, this must be mapped onto a convergent subsequence $\{y_n\}$ of $\{x_n\}$, which is a contradiction and so $M^n \simeq dS^n/\mathbb{D}$ cannot be compact. \square

4.3 The Klingler Theorem

In this section we prove Theorem 4.21 of Klingler. It shows that a compact Lorentzian manifold with constant sectional curvature must be (geodesically) complete:

Theorem 4.21. ([17], "Klinglers Theorem") *Every compact Lorentzian manifold of constant curvature, with dimension $n > 2$, must be geodesically complete.*

This will, together with Calabis and Markus' Theorem 4.19 and its consequence, Corollary 4.20, show our main classification result Corollary 4.50.

Since the proof of Theorem 4.21 is rather considerable, we discuss it in parts below. Throughout, (M, g) is assumed to be a smooth, compact Lorentzian manifold of constant (sectional) curvature.

For the respective cases $\kappa = \pm 1, 0$, we identify (M, g) with the spaces

$$\begin{aligned} \kappa = +1 : & \quad M^n = (O_1(n+1), dS^n) \\ \kappa = 0 : & \quad M^n = (O_1(n), \mathcal{M}^n) \\ \kappa = -1 : & \quad M^n = (O_2(n+1), AdS^n). \end{aligned} \tag{4.45}$$

From Propositions 2.20, 2.21, 2.22, 2.23 and 2.24 we know that these spaces are (geodesically) complete spaces with the desired sectional curvatures. From the discussion in Section 3.3 we already know that in the $\kappa = 1$ and $\kappa = 0$ cases, as X is simply connected, D is injective.

Before moving on to discuss the proof, we first describe the method of attack. By Theorem 3.49 we know that $D : \tilde{M} \rightarrow X$ will be a covering map onto its image. The overall idea is then to show that it is a surjection. As, in all three cases the space X is (geodesically) complete, this would imply (geodesic) completeness of \tilde{M} . Proposition 2.39 then would imply that (M, g) is complete.

We will assume that D is not surjective so as to derive a contradiction. If \tilde{M} were not complete, the image $D(\tilde{M}) = \Omega$ would be a strict subset of X . It is possible to equip \tilde{M} with a Riemannian metric. This is explicitly done by equipping X , in each three cases, with the standard Euclidean metric and pulling it back via D^{-1} . Via this metric we would be able to discuss the metric completion of \tilde{M} .

If $D : \tilde{M} \rightarrow X$ were not surjective, it can be shown that the boundary of $D(\tilde{M})$ in X must identify with a hyperplane. This is Corollary 4.46. This is shown more specifically by studying how the holonomy group $\Gamma \subset G$ treats a sequence of Euclidean balls that converge. It will be shown that these "collapse" into subsets of hyperplanes in X from which Corollary 4.46 is rather direct.

Having established that the "new boundary" in the completion of \tilde{M} must identify with hyperplanes in X via D we arrive at a concrete problem in the well-known spaces \mathcal{M}^n , dS^n and AdS^n . In Propositions 4.47, 4.48 and 4.49 it is shown that there can be at most one such hyperplane for the flat, spherical and hyperbolic cases respectively. In the proof of Theorem 4.21 it is finally shown that existence of one such component results in contradiction and so the metric completion of $D(\tilde{M})$ is empty. This is to say that D is surjective onto the (geodesically) complete space X which in turn implies (geodesic) completeness and all is shown.

We start by making the following definition.

Definition 4.22. Let X be any of the space forms \mathcal{M}^n , dS^n or AdS^n . For \mathcal{M}^n we define a **segment** between two points x and y to be the (closed) geodesic (a straight line) that connects the two points. For dS^n and AdS^n we define a **segment** to be a geodesic connecting the two points x and y that does not contain any anti-podal points. If a segment connecting x and y exists, we say that x **sees** y and it is denoted by $[x, y]$.

For a fixed point $x \in X$, there is a subset of X consisting of the set of points connected to x by some segment. This stimulates for the following definition:

Definition 4.23. Let x be a point in X . The set of points seen by x is denoted X_x and is called the **star** of X .

From the definition of a segment in X , we get the following definition for a segment in \tilde{M} :

Definition 4.24. A **segment** in \tilde{M} is a curve γ such that identifies locally through $D(\gamma)$ with a segment in X . As D is a local isometry via the pull-back metric, the segment in \tilde{M} is also a geodesic. If it exists, it is denoted by $[\tilde{p}, \tilde{q}]$.

The stars $\tilde{M}_{\tilde{p}}$ and M_p are defined in a similar manner. The following shows what the stars are, in the three (G, X) -manifolds:

Example 4.25. For an arbitrary point p , the following sets are the stars in the corresponding relativistic space forms:

1. $\mathcal{M}_x^{n+1} = \mathcal{M}^{n+1}$
2. $dS_x^n = \{y \in dS^n \mid Q_1^{n+1}(x, y) > -1\}$
3. $AdS_x^n = \{y \in AdS^n \mid Q_{-1}^{n+1}(x, y) > -1\}$.

The first case is obvious, as the straight line between two points is a segment. The other two follow from Propositions 2.22 and 2.24.

Corollary 4.26. *The sets X_x , $\tilde{M}_{\tilde{p}}$ and M_p are open.*

Proof. For the case of X_x with $\kappa = \pm 1$, Example 4.25 shows that X_x is the inverse image of an open set under the continuous functions $Q_1^{n+1}(p, \cdot)$ and $Q_{-1}^{n+1}(p, \cdot)$ and thus is open. Since both $D : \tilde{M} \rightarrow X$ and $\varphi : \tilde{M} \rightarrow M$ are continuous it follows that also $\tilde{M}_{\tilde{p}}$ and M_p are open. \square

Definition 4.27. A subset C of M (or \tilde{M}) is said to be **simply convex** if for any two of its points p and q the segment $[p, q]$ exists and is contained within C .

Definition 4.28. Suppose C_1 and C_2 are two subsets of X such that $C_1 \subset C_2$. Then C_1 is said to be **relatively convex** in C_2 if, when two points $x, y \in C_1$ are connected by a segment in C_2 , the segment is contained in C_1 .

A simply convex set must be relatively convex subset of any set containing it. Note however, that a relatively convex subset $C_1 \subset C_2$ of X , can still have two points seeing one another but the segment is outside of C_2 .

Lemma 4.29. *Let x be a point in X . Then there exists a simply convex neighbourhood in X , containing x .*

Proof. The assertion is, by Example 4.25, trivial in the case $\kappa = 0$. For the cases $\kappa = \pm 1$, we assume not and derive a contradiction.

We start by showing that there is some neighbourhood containing x where every point sees one another. To show this, we construct Euclidean balls $B_{1/n}(x)$ containing z_n and y_n that are points in X that satisfy $Q_{\pm 1}^{n+1}(z_n, y_n) \leq -1$. Clearly both $\{z_n\}$ and $\{y_n\}$ converge to x . But $Q^{n+1}(x, x) = 1$ as $x \in X$. This contradicts the continuity of $Q_{\pm 1}^{n+1}$.

Now let $x = (0, 1, \dots, 0)$ in dS^n or $x = (1, 0, \dots, 0)$ in AdS^n . By what we just deduced, there must be some neighbourhood U_x where all elements see one another. If ϵ is small enough, the intersection $B_\epsilon(x) \cap X = W_x$ will both be contained in U_x and must contain all segments $[y, z]$ where $y, z \in W_x$.

Since $G = O_1(n+1)$ or $G = O_2(n+1)$ acts transitively and continuously on X , we may for any element $y \in X$ find the element $g \in G$ that maps $gx = y$ and then $g(W_x)$ is simply convex around y . \square

The following two properties are immediate:

Lemma 4.30. *Let $D(\tilde{x}) = x$. Then $D(\tilde{M}_{\tilde{x}}) \subset X_x$.*

Proof. If \tilde{x} sees \tilde{y} there is a segment in $\tilde{M}_{\tilde{x}}$ joining them. But this segment is the image of $D^{-1}([x, y])$ by definition, so there is a segment $[x, y]$ joining x and y , (with $x = D(\tilde{x})$ and $y = D(\tilde{y})$) and so $y \in X_x$. \square

Lemma 4.31. *D restricted to $\tilde{M}_{\tilde{x}}$ is injective.*

Proof. We assume that D restricted to $\tilde{M}_{\tilde{x}}$ is not injective, i.e. that there are two points in $\tilde{M}_{\tilde{x}}$ such that $D(\tilde{y}) = y$, $D(\tilde{z}) = z$ and $y = z$. Hence the segments $[x, y]$ and $[x, z]$ have the same endpoints. From Propositions 2.23 and 2.21, it follows that the segments are unique. So since D is a local diffeomorphism, it follows that $\tilde{z} = \tilde{y}$. \square

The following proposition is of great importance.

Proposition 4.32. [17] *The image $D(\tilde{M}_{\tilde{x}})$ is relatively convex in X_x .*

Proof. ([17]) We do proof by contradiction and suppose that so is not the case. The converse to the statement of $D(\tilde{M}_{\tilde{x}})$ being relatively convex in X_x is that there is a segment in X_x , joining two points y and z of $D(\tilde{M}_{\tilde{x}})$ not contained within $D(\tilde{M}_{\tilde{x}})$. We suppose that these exist and that the segment $[y, z]$ is contained in X_x .

We now consider some of the geometry induced by x, y and z . The idea is to construct a "triangle" in X_x with end-points being x, y, z , which is done separately for the zero and non-zero curvature cases.

For $\kappa = 0$, we consider the affine (or vectorial) plane S spanned by x, y and z . The triangle is denoted by T_{xyz} and is the set contained by the straight lines (segments) connecting x, y and z . T_{xyz} is clearly well-defined, i.e. it is a triangle and not a straight line; if that were the case, then the segment $[y, z]$ would be contained in $D(\tilde{M}_{\tilde{x}})$, which we assumed not to be the case.

For the case of $\kappa = \pm 1$, we define it as follows: we define S to be the intersection $S = X \cap \text{span}\{x, y, z\}$. The span must be three-dimensional; if it were not, it would be a plane $\text{span}\{x, y, z\} = \Pi$. By Propositions 2.22 and 2.24 the intersection S would be a geodesic and as this geodesic contains x, y and z it would follow that y sees z . This is contrary to our hypothesis that $[y, z] \not\subset D(\tilde{M}_{\tilde{x}})$. The constraint of being an intersection with X reduces the dimensionality so that S is two-dimensional.

Given now what S is, we construct T_{xyz} by considering the three planes spanned by the individual segments and the origin. The segments are intersections of X with some plane that intersects the origin, and the three such planes will define a "half-cone". The planes, being pair-wise intersecting in precisely one line (spanned between the origin and one of the three points) will produce a cone. The intersection of the half of the cone containing x, y, z with X and the interior of this closed and bounded set will be T_{xyz} .

It follows immediately that T_{xyz} is simply convex: any two points u, v can be joined by a segment contained within; two points and the origin span a plane that must intersect two other planes and so the full geodesic containing the segment $[u, v]$ must intersect the boundary of T_{xyz} in two distinct segments. Hence T_{xyz} is simply convex.

Another property of T_{xyz} is that it contains no antipodal points. If it were so, there would be a plane containing both the origin and u and $-u$. By the simple convexity that we just showed this segment $[u, -u]$ will be contained in T_{xyz} . On the other hand $Q_{\pm 1}^n(u, -u) = -1$ and by the discussion of Example 4.25 these elements cannot see one another. Hence T_{xyz} contains no anti-podal points.

We now consider the segment $[y, z]$ in X_x . By hypothesis this was not included in $D(\tilde{M}_{\tilde{x}})$. Since $D(\tilde{M}_{\tilde{x}})$ is open and contains y , there must be some half-open segment $[y, v)$ in $D(\tilde{M}_{\tilde{x}})$. Moreover, the segment $[x, v)$ will be contained in $D(\tilde{M}_{\tilde{x}})$.

Let now $\{v_n\}$ be a sequence in $[x, v)$ that converges to v . We denote by $\{\tilde{v}_n\}$ the lift through D of this sequence to \tilde{M} , which by Lemma 4.31 exists uniquely. A depiction of this is given in Figure 4.3. We make use of the compactness of M . Let $\{u_n\}$ be the projected sequence in M , i.e. $u_n = \varphi(\tilde{v}_n)$ for each n . By compactness of M , there is a convergent subsequence, which for notational convenience, we assume is $\{u_n\}$. The sequence converges to some limit u and this point has an evenly covered neighbourhood U_u . We pick any of the sheets $\varphi^{-1}(U_u)$ and denote it by \tilde{U}_u . We may assume that all the u_n are contained in the neighbourhood U_u , otherwise we discard the first N elements and keep the "tail". Denote by \tilde{w}_n the set $\varphi^{-1}(u_n) \cap \tilde{U}_u$. Since φ is a local diffeomorphism, this sequence must converge, i.e. $\tilde{w}_n \rightarrow \tilde{w} = \varphi^{-1}(u) \cap \tilde{U}_u$. We now consider how the sequences $\{\tilde{w}_n\}$ and $\{\tilde{v}_n\}$ are related. If, for each n , we were to have $k(\tilde{v}_n) = \tilde{w}_n$, where $k \in \mathbb{D}$, it would follow that \tilde{v}_n would be a convergent sequence. This in turn would mean that the geodesic $[\tilde{x}, \tilde{v})$ extends to $[\tilde{x}, \tilde{v}]$. But this contradicts the inclusion of v in $D(\tilde{M}_{\tilde{x}})$. Hence no $k \in \mathbb{D}$ may relate any \tilde{w}_n to any \tilde{v}_n for more than finitely many n . From this it follows that, as we can always pass to a subsequence, we may assume that there is $k_n(\tilde{v}_n) = \tilde{w}_n$ where all the $k_n \in \mathbb{D}$ are distinct.

We now consider Euclidean balls in \mathbb{R}^{n+1} centered at $w_n = D(\tilde{w}_n)$. We define the following projection

$$\phi : x \rightarrow \begin{cases} x & \text{if } \kappa = 0, \\ x / \sqrt{Q_{\pm 1}^{n+1}(x, x)} & \text{if } \kappa = \pm 1. \end{cases} \quad (4.46)$$

If $X = dS^n$ ϕ is, restricted to the spacelike part of \mathbb{R}^{n+1} , clearly a projection onto dS^n that maps straight lines to geodesics. If $X = AdS^n$ then ϕ is, restricted to the timelike part of \mathbb{R}^{n+1} , a projection onto AdS^n . Furthermore, if $\kappa = \pm 1$, then we have

$$\phi(g(x)) = g(x)/\sqrt{Q_{\pm 1}^{n+1}(g(x), g(x))} = g(x)/\sqrt{Q_{\pm 1}^{n+1}(x, x)} = g(\phi(x)), \quad (4.47)$$

where the G -invariance of Q follows as $G = O_1(n+1)$ is a linear isometry. Therefore ϕ is G -equivariant for all three cases $\kappa = 1, \pm 1$.

We now choose a compact neighbourhood \tilde{B} , containing \tilde{w} in \tilde{M} , such that $k(\tilde{B}) \cap \tilde{B} = \emptyset$ for non-trivial $k \in \mathbb{D}$. Such a ball will exist, as by (1.) in Definition 2.31 the neighbourhood U will contain \tilde{w} . We take any coordinate chart and restrict to its intersection with U , map down to \mathbb{R}^n , take a closed epsilon-ball centered at the image of w and contained in the image of the intersection, and map it back up. Clearly this definition of \tilde{B} will have the property. By local continuity of D , the image $D(\tilde{B})$ will also be compact in X .

We now choose r small enough, so that the projected image $\phi(\overline{B_{2r}(w)}) \subset D(\tilde{B})$, where $B_{2r}(w)$ is the Euclidean ball of radius $2r$ centered at w in \mathbb{R}^{n+1} . We let $B_n = \overline{B_r(w_n)}$. When n is big enough, we have $B_n \subset \overline{B_{2r}(w)}$ and therefore also $\phi(B_n) \subset \phi(\overline{B_{2r}(w)}) \subset D(\tilde{B})$. Denote further $\tilde{B}_n = \tilde{B} \cap D^{-1}(\phi(B_n))$ and $\tilde{C}_n = k_n^{-1}(\tilde{B}_n)$. Clearly the \tilde{C}_n are compact and by Lemma 2.32 they are also disjoint.

We define $g_n = h(k_n^{-1})$. With this, it follows from the equivariance equation $D \circ k = h(k) \circ D$ that

$$v_n = D(\tilde{v}_n) = D(k_n^{-1}(\tilde{w}_n)) = h(k_n^{-1})(D(\tilde{w}_n)) = h(k_n^{-1})(w_n) = g_n(w_n) \quad (4.48)$$

and

$$D(\tilde{C}_n) = D(k_n^{-1}(\tilde{B}_n)) = h(k_n^{-1})(D(\tilde{B}_n)) = g_n(\phi(B_n)) = \phi(g_n(B_n)), \quad (4.49)$$

where we used the G -equivariance of ϕ in the last line.

We now define three sequences in G . The sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are defined so that

$$w_n = a_n^{-1}(w), \quad v_n = b_n(v) \quad \text{and} \quad c_n = b_n g, \quad (4.50)$$

where $g \in G$ is an element such that $v = g(w)$. With the equality $w_n = g_n(v_n)$ we get the successive computations

$$\begin{aligned} v_n &= g_n(w_n) = g_n a_n^{-1}(w) \\ v &= b_n^{-1}(v_n) = b_n^{-1} g_n a_n^{-1}(w) \\ w &= g^{-1}(v) = g^{-1} b_n^{-1} g_n a_n^{-1}(w) = c_n^{-1} g_n a_n^{-1}(w) \end{aligned} \quad (4.51)$$

so that $c_n^{-1} g_n a_n^{-1} \in K_w \simeq O_1(n)$ (or $K_w \simeq O_2(n)$). Thus $g_n = c_n o_n a_n$, where $o_n \in K_w$.

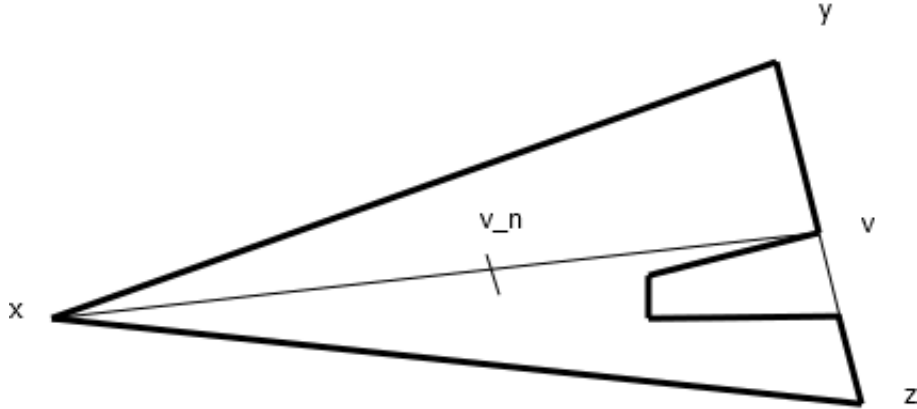


Figure 4.2: Depiction of the triangle T_{xyz} and an element of the convergent sequence $\{v_n\}$. The boldface denotes the intersection $T_{xyz} \cap D(\tilde{M}_x)$. The picture is taken from [17].

We now consider the sequence of closed sets $\{g_n(B_n)\}$. As B_n is a Euclidean ball and $g_n \in O_1(n+1)$, Proposition 3.35 shows that $g_n(B_n)$ is an ellipsoid for each n . Furthermore, as ϕ is G -invariant, the $\{\tilde{C}_n\}$ are disjoint and $v_n \rightarrow v$, it must follow that the $\{\phi(g_n(B_n))\}$ are disjoint. The $\{\phi(g_n(B_n))\}$ are disjoint and so it follows that the $\{g_n(B_n)\}$ are disjoint and so the limit set $g_n(B_n) \rightarrow \mathcal{E}$ must be degenerate. By Corollary 3.42 we have that \mathcal{E} is a coisotropic hyperplane of codimension 1. Clearly $\phi(g_n(B_n)) \rightarrow \phi(\mathcal{E})$ and since the $\{\phi(g_n(B_n))\}$ are disjoint in $\text{span}\{x, y, z\}$ it follows that $\phi(\mathcal{E}) \cap \text{span}\{x, y, z\}$ must be two-dimensional.

We consider now the intersections of $\{g_n(B_n)\}$ and T_{xyz} . Since S is a surface, the tangent spaces $\{T_{v_n}S\}$ must be affine planes in \mathbb{R}^{n+1} , more specifically in $\text{span}\{x, y, z\}$. Since the $\{a_n\}$ and the $\{c_n\}$ lie in compact subsets, there is a real number r' such that $r > r' > 0$ and r' is smaller than every principal axis but one. Therefore, it follows that each $g_n(B_n)$ must contain some affine segment σ_n of length $2r'$, centered at v_n . We define $\delta_n = \phi(\sigma_n)$. Clearly, as the σ_n are all straight lines the projected images δ_n must be contained in vectorial planes and by Propositions 2.20, 2.22 and 2.24 therefore be geodesics. Since ϕ maps $\text{span}\{x, y, z\}$ to itself the geodesics δ_n must be contained in S .

Let us decompose $\sigma_n = v_n + \sigma_n^0$. Then the closed ball $\overline{B_{2r'}(0)}$ will contain all segments σ_n^0 . Since $\sigma_n \in g_n(B_n) \cap T_{xyz}$, the segments σ_n^0 must converge to some segment σ^0 that in turn is contained in some limit hyperplane \mathcal{E}' . As furthermore $v_n \rightarrow v$, we must have $\sigma_n = v_n + \sigma_n^0 \rightarrow v + \sigma^0 = \sigma$, which is an affine segment of length $2r'$ that is centered at v and contained in T_vS . We let $\delta = \phi(\sigma)$ and clearly $\delta_n \rightarrow \delta$.

Since every δ_n intersects T_{xyz} , the limit segment δ must intersect $\overline{T_{xyz}}$. This can happen in two distinct ways; either $\delta \cap [y, v] \neq \emptyset$, or $\delta \cap \text{int}(T_{xyz}) \neq \emptyset$. At any case, let s be an element in this intersection. By Lemma 3.39 there exists a sequence $\{s_n\}$ of elements converging to s , and with $s_n \in \delta_n$. We may (uniquely) lift the sequence $\{s_n\}$ to $\tilde{M}_{\tilde{x}}$, and we denote this sequence $\{\tilde{s}_n\}$. Since also $s \in D(\tilde{M}_{\tilde{x}})$ there is a (unique) corresponding lifted point $\tilde{s} \in \tilde{M}_{\tilde{x}} \cap D^{-1}(s)$ and $\tilde{s}_n \rightarrow \tilde{s}$.

Each element in the sequence $\{\tilde{s}_n\}$ is contained in \tilde{C}_n . By construction, $\tilde{C}_n = k_n^{-1}(\tilde{B}_n)$ and so $k_n(\tilde{s}_n) \in \tilde{B}_n$ for each n . Since $\tilde{B}_n \subset \tilde{B}$ we have $\{k_n(\tilde{s}_n)\} \subset \tilde{B}$. As \tilde{B} was compact, there is a convergent subsequence in $\{k_n(\tilde{s}_n)\}$ and the limit \tilde{t} is contained in \tilde{B} .

We have now reached a contradiction to the proper discontinuity of \mathbb{D} ; there must on the one hand be some element $k \in \mathbb{D}$ such that $k(s) = \tilde{t}$. Since k is a diffeomorphism, it maps convergent sequences to convergent sequences, and so $k(\tilde{s}_n) \rightarrow \tilde{t}$. But on the other hand, this cannot happen as by construction $k(\tilde{B}) \cap \tilde{B} = \emptyset$ and so we have a contradiction.

Hence, $D(\tilde{M}_{\tilde{x}})$ is relatively convex in X_x .

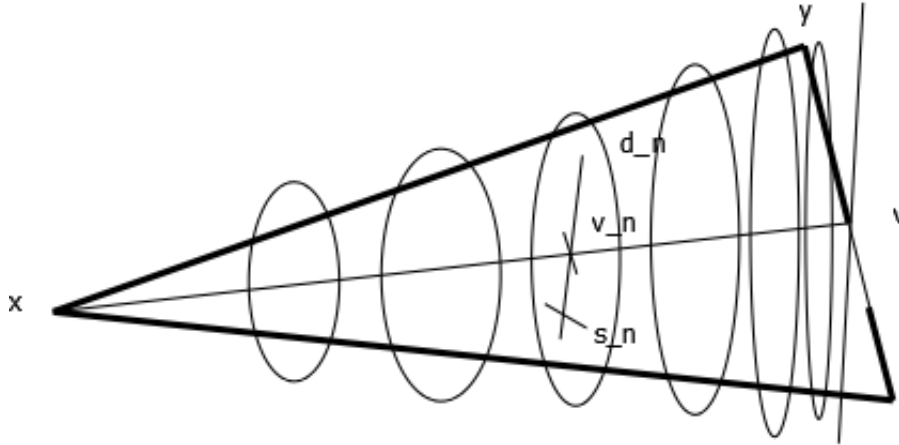


Figure 4.3: Depiction of the triangle T_{xyz} and the ellipsoids, containing the segments $\{\delta_n\}$ converging along the convergent sequence $\{v_n\}$. The boldface denotes the intersection $T_{xyz} \cap D(\tilde{M}_{\tilde{x}})$. The picture is taken from [17].

□

Before stating the important consequence of Proposition 4.32, we state and prove Lemma 4.33:

Lemma 4.33. [17] Let \tilde{C}_1 and \tilde{C}_2 be two subsets of \tilde{M} and C be a subset of X such that $D(C_1) \subset C$ and $D(C_2) \subset C$. Suppose

1. D restricted to \tilde{C}_1 is injective,
2. $D(\tilde{C}_1) \subset C$ is relatively convex,
3. \tilde{C}_2 is simply convex,
4. $\tilde{C}_2 \subset D^{-1}(C)$,
5. $\tilde{C}_1 \cap \tilde{C}_2 \neq \emptyset$.

Then D , restricted to $\tilde{C}_1 \cup \tilde{C}_2$ is injective.

Proof. [17] By (1.) D is injective on \tilde{C}_1 . We show that it follows from (3.) that D restricted to \tilde{C}_2 is also injective. If any two points \tilde{x} and \tilde{y} in \tilde{C}_2 can be joined by a segment $[\tilde{x}, \tilde{y}]$ included in \tilde{C}_2 the segment $D([\tilde{x}, \tilde{y}]) = [x, y]$ exists in X . If $x = y$ the corresponding segment $[x, y]$ is trivially one point and therefore also the lift $D^{-1}([x, y]) = [\tilde{x}, \tilde{y}]$. Hence we have injectivity also on \tilde{C}_2 .

We now suppose $\tilde{x} \in \tilde{C}_1$ and $\tilde{y} \in \tilde{C}_2$ such that $D(\tilde{x}) = D(\tilde{y})$ and that neither lies in the intersection $\tilde{C}_1 \cap \tilde{C}_2$. Since by (5.) the intersection $\tilde{C}_1 \cap \tilde{C}_2$ is not empty, there exists $\tilde{z} \in \tilde{C}_1 \cap \tilde{C}_2$. Since \tilde{C}_2 is simply convex, there exists a segment $[\tilde{y}, \tilde{z}] \subset \tilde{C}_2$, and since by (4.) $\tilde{C}_2 \subset D^{-1}(C)$ the mapped segment satisfies $[x, z] \subset C$. By the hypothesis that $x = y$, the segment $[x, z]$ exists and must coincide with $[y, z]$ as segments are unique. Now by (2.) $D(\tilde{C}_1)$ is relatively convex, so that $[x, z] \subset D(\tilde{C}_1)$. Hence $D^{-1}([x, z])$ is a segment in \tilde{C}_1 and \tilde{z} sees both \tilde{x} and \tilde{y} . This means that both \tilde{x} and \tilde{y} lie in $\tilde{M}_{\tilde{z}}$. By Lemma 4.31 it follows that $\tilde{x} = \tilde{y}$. Hence D restricted to the union $\tilde{C}_1 \cap \tilde{C}_2$ is injective. \square

Corollary 4.34. [17] Denote by $\tilde{\partial}\tilde{M}_{\tilde{x}}$ the boundary of the star of \tilde{x} in \tilde{M} , and similarly $\partial X_x = \overline{X_x} - X_x$. Then $D(\tilde{\partial}\tilde{M}_{\tilde{x}}) \subset \partial X_x$.

Proof. [17] Let $\tilde{y} \in \tilde{\partial}\tilde{M}_{\tilde{x}}$ and let $\tilde{B}_{\tilde{y}}$ be a simply convex open neighbourhood of \tilde{y} in \tilde{M} that must exist by Lemma 4.29. The point $y = D(\tilde{y})$ is in the closure of X_x as $D(\tilde{M}_{\tilde{x}}) \subset X_x$ and D is continuous. We need to show that y is not in the interior of X_x , so we assume the opposite, i.e. that $y \in X_x$. Since X_x is open by Corollary 4.26 we may assume $D(\tilde{B}_{\tilde{y}}) \subset X_x$.

Now $\tilde{B}_{\tilde{y}} \cap \tilde{M}_{\tilde{x}} \neq \emptyset$, $\tilde{B}_{\tilde{y}}$ is simply convex and by Proposition 4.32 $D(\tilde{M}_{\tilde{x}})$ is relatively convex in X_x . Lemma 4.33 implies that D restricted to the union $\tilde{M}_{\tilde{x}} \cup \tilde{B}_{\tilde{y}}$ is injective.

Since we assumed that $y \in X_x$ the segment $[x, y]$ must exist in X_x . Therefore we may lift $[x, y]$ to $[\tilde{x}, \tilde{y}]$ in \tilde{M} and thus $\tilde{y} \in \tilde{M}_{\tilde{x}}$, i.e. \tilde{x} sees \tilde{y} and we have a contradiction. Hence $D(\tilde{\partial}\tilde{M}_{\tilde{x}}) \subset \partial X_x$. \square

We now equip X with a Riemannian metric as follows. If $X = \mathcal{M}^n$, we simply change metric from η , the Minkowski metric, to the Euclidean metric g_0 . If $X = dS^n$, or $X = AdS^n$, we equip its ambient space \mathbb{R}^{n+1} with the standard Euclidean metric and let the Riemannian metric g_0 on X be the induced metric. With this (X, g_0) becomes a Riemannian metric space in all three $\kappa = 0, \pm 1$ cases. Furthermore (X, g_0) will be complete as a metric space. This follows trivially in the flat case, as then $(X, g_0) = (\mathbb{R}^n, d_E)$ which is just the standard Euclidean space. In the curved cases (X, g_0) is merely a closed metric subspace of a complete space and will thus also be metrically complete. (3.) from Hopf's and Rinow's Theorem 1.5 now implies that any two points $x, y \in X$ may be connected with a geodesic γ such that $\ell(\gamma) = d_0(x, y)$.

By pullback, we get a Riemannian metric $\tilde{g}_0 = D^{-1}(g_0)$ on \tilde{M} . We can thereby discuss the metric completion of \tilde{M} , which we denote by \hat{M} . We make the following definitions:

Definition 4.35. Let $\tilde{x} \in \tilde{M}_{\tilde{x}}$ and $\hat{y} \in \hat{M}_{\tilde{x}}$ and suppose $\gamma : [0, 1] \rightarrow \hat{M}$ is some curve with endpoints \tilde{x} and \hat{y} . γ is called a **segment** if $\gamma([0, 1]) \subset \hat{M}_{\tilde{x}}$ and $D(\gamma([0, 1]))$ is a segment in X . As before, if it exists, it is denoted by $[\tilde{x}, \hat{y}]$.

The definition of a star generalizes in the natural manner, with notation $\hat{M}_{\tilde{x}}$. The following, however, is new:

Definition 4.36. Let \tilde{N} be a subset of \tilde{M} . By $\partial\tilde{N}$ we denote the boundary of \tilde{N} in \hat{M} , and it is the disjoint union of the sets

$$\hat{\partial}\tilde{N} = \partial\tilde{N} \cap (\hat{M} - \tilde{M}), \quad \tilde{\partial}\tilde{N} = \partial\tilde{N} \cap \tilde{M}. \quad (4.52)$$

We state some properties of this construction with the following lemmas.

Lemma 4.37. [17] *The stars $\hat{M}_{\tilde{x}}$ in the completion \hat{M} are the disjoint union*

$$\hat{M}_{\tilde{x}} = \tilde{M}_{\tilde{x}} \cup \left(\hat{\partial}\tilde{M}_{\tilde{x}} \cap D^{-1}(X_x) \right). \quad (4.53)$$

Proof. [17] We show both inclusions. On the one hand, we have the obvious inclusion $\tilde{M}_{\tilde{x}} \subset \hat{M}_{\tilde{x}}$. We suppose $\hat{y} \in \hat{M}_{\tilde{x}}$, but $\hat{y} \notin \tilde{M}_{\tilde{x}}$. Then there must be $\hat{y} \in \hat{\partial}\tilde{M}_{\tilde{x}}$. From the definition there must be a segment $[x, y]$ in X_x such that its lift through D is $[\tilde{x}, \hat{y}] \subset \tilde{M}_{\tilde{x}}$. Therefore $\hat{y} \in D^{-1}(X_x)$. Hence we have the inclusion $\hat{M}_{\tilde{x}} \subset \tilde{M}_{\tilde{x}} \cup (\hat{\partial}\tilde{M}_{\tilde{x}} \cap D^{-1}(X_x))$.

To show the other inclusion, we must show $\hat{\partial}\tilde{M}_{\tilde{x}} \cap D^{-1}(X_x) \subset \hat{M}_{\tilde{x}}$. We suppose $\hat{y} \in \hat{\partial}\tilde{M}_{\tilde{x}} \cap D^{-1}(X_x)$ and let $\{\tilde{y}_n\}$ be a sequence in $\tilde{M}_{\tilde{x}}$ converging to \hat{y} . Since the points \tilde{y}_n are seen by \tilde{x} there are associated segments $[x, y_n] \subset X_x$. Since $y_n \rightarrow y$, and by hypothesis $y \in X_x$, these segments converge to $[x, y] \subset X_x$. By Proposition 4.32, $D(\tilde{M}_{\tilde{x}})$ is relatively convex in X_x . Hence, if $z \in [x, y]$ is an interior point in the segment, the segment $[x, z] \subset D(\tilde{M}_{\tilde{x}})$. Therefore the half-open segment must satisfy $[\tilde{x}, \hat{y}] \subset \tilde{M}_{\tilde{x}}$.

This shows the assertion. \square

Lemma 4.38. *D restricted to $\hat{M}_{\tilde{x}}$ is injective.*

Proof. The proof is much similar to Proposition 4.31. We suppose $D(\hat{y}) = D(\hat{z})$ for $\hat{y}, \hat{z} \in \hat{M}_{\tilde{x}}$. Then there exist segments in \hat{M} that join the points: $[\tilde{x}, \hat{y}]$ and $[\tilde{x}, \hat{z}]$. By construction, we have that the segments $[x, y]$ and $[x, z]$ exist in X , (with usual notation $D(\tilde{x}) = x$, $D(\hat{y}) = y$ and $D(\hat{z}) = z$). In X we have that, as $D(\hat{y}) = D(\hat{z})$, the segments $[x, z]$ and $[x, y]$ are the same. Since, by Lemma 4.31 we have D is injective restricted to $\tilde{M}_{\tilde{x}}$ and so the segments $[\tilde{x}, \hat{z}]$ and $[\tilde{x}, \hat{y}]$ are the same, and by the metric completion $[\tilde{x}, \hat{z}] = [\tilde{x}, \hat{y}]$ from which it follows that $\hat{z} = \hat{y}$. \square

Lemma 4.39. [17] *The stars $\hat{M}_{\tilde{x}}$ are open sets.*

Proof. [17] From Corollary 4.26 and Proposition 4.37, it suffices to show that if $\hat{v} \in \hat{\partial}\tilde{M}_{\tilde{x}} \cap D^{-1}(X_x)$ then \hat{v} is contained in some open ball in $\hat{M}_{\tilde{x}}$.

Let \hat{V}_1 and \hat{V}_2 be two open balls in \hat{M} , centered at \hat{v} , and with radii $r_1 = \epsilon$ and $r_2 = 3\epsilon$, respectively. The distance is that of the Riemannian metric d induced by the pull-back metric $D^{-1}(g_0)$ and ϵ is chosen so that the ball \hat{V}_2 is contained within the open set $D^{-1}(X_x)$. Since $\hat{v} \in \hat{\partial}\tilde{M}$, the segment $[\tilde{x}, \hat{v}]$ lies in $\tilde{M}_{\tilde{x}}$. The ball \hat{V}_1 intersects the segment $[\tilde{x}, \hat{v}]$ with some point \tilde{v}_0 .

Now for each element \hat{y} in \hat{V}_1 we construct a curve $\beta_{\hat{y}}$, lying in the intersection $\hat{V}_2 \cap \tilde{M}$, such that $\beta_{\hat{y}}(0) = \tilde{v}_0$ and $\beta_{\hat{y}}(1) = \hat{y}$. To do this, we let $\{\tilde{y}_n\}$ be a sequence in \tilde{M} such that $\tilde{y}_n \rightarrow \hat{y}$. We may assume that both $\{\tilde{y}_n\}$ and $d(\tilde{y}_n, \tilde{y}_{n+1}) < \epsilon/2^n$ so that the sum

$$\ell(\beta) = \sum_n^\infty d(\tilde{y}_n, \tilde{y}_{n+1}) < \sum_n^\infty \frac{\epsilon}{2^n} = \epsilon < \infty \quad (4.54)$$

is convergent. The latter statement is easy to see: let $\{\tilde{z}_n\}$ be any sequence tending towards \hat{y} . We first pick $N_1 > 0$ so that $d(\tilde{z}_n, \tilde{z}_m) < \epsilon/2$ for all $m, n > N_1$ and let $\tilde{y}_1 = \tilde{z}_{N_1}$. We proceed by choosing N_2 so that $d(\tilde{z}_n, \tilde{z}_m) < \epsilon/2^2$ for all $m, n > N_2$ and let $\tilde{y}_2 = \tilde{z}_{N_2}$. Proceeding in this fashion we choose N_k such that $d(\tilde{z}_n, \tilde{z}_m) < \epsilon/2^k$ for $m, n > N_k$ and let $\tilde{y}_k = \tilde{z}_{N_k}$. Clearly this procedure is well-defined and we have $d(\tilde{y}_n, \tilde{y}_{n+1}) < \epsilon/2^n$ for all n .

With this at hand, we construct a curve β , contained in \hat{V}_1 such that it has end-points \tilde{y}_1 and \hat{y} . This we do by pair-wise path-connecting the points in $\{\tilde{y}_n\}$ and then define $\beta(1) = \hat{y}$. We now claim that the successive pair of elements \tilde{y}_n and \tilde{y}_{n+1} may in fact be geodesically connected. We let U be an open neighbourhood of $D(\hat{y})$ such that D , restricted to $U \cap D(\hat{M})$, is a local isometry.

We may assume that the full sequence $\{\tilde{y}_n\}$ is contained in $D^{-1}(U \cap D(\hat{M}))$. By the Hopf-Rinow Theorem there must exist a geodesic joining $D(\tilde{y}_n)$ to $D(\tilde{y}_{n+1})$ and D is isometric between these two points we may by continuity lift this geodesic to \hat{M} . Geodesics can be parametrized by arc-length and so we have already a parametrization for β . Clearly it follows that $\ell(\beta) < \epsilon$.

Since furthermore

$$d(\tilde{y}_1, \tilde{v}_0) \leq d(\tilde{y}_1, \hat{v}) + d(\hat{v}, \tilde{v}_0) < \epsilon + \epsilon = 2\epsilon \quad (4.55)$$

we may construct a curve δ with $\delta(0) = \tilde{v}_0$ and $\delta(1) = \tilde{y}_1$. Clearly, as both $\tilde{v}_0, \tilde{y}_0 \in \hat{V}_1$ we have Euclidean length $\ell(\delta) < 2\epsilon$. Now the product $\beta \circ \delta$ suffices as $\beta_{\hat{y}}$. Clearly we have $\beta_{\hat{y}}([0, 1]) \subset \hat{M}$.

We consider the product $\alpha_{\hat{y}} = \beta_{\hat{y}} \circ [\tilde{x}, \hat{v}]$ that for each $\hat{y} \in \hat{V}_1$ defines a mapping $\alpha_y : [0, 1] \rightarrow \hat{M}$. Now we have $D(\alpha_y) \subset X_x$; this follows as $[\tilde{x}, \tilde{v}_0]$ is already in $\tilde{M}_{\tilde{x}}$ and β and δ are both contained within \hat{V}_2 that, in turn, was chosen so that $D(\hat{V}_2) \subset X_x$. Clearly we have $\alpha_y([0, 1]) \subset \hat{M}$ and we need only to verify that $\alpha_y([0, 1]) \subset \tilde{M}_{\tilde{x}}$.

By construction we have $\alpha_{\hat{y}}([0, 1]) \subset \tilde{M}$. Therefore, if the element $\alpha_{\hat{y}}(t_0) \notin \tilde{M}_{\tilde{x}}$ for some $t_0 \in [0, 1]$ then $\alpha_{\hat{y}}(t_0) \in \partial\tilde{M}_{\tilde{x}}$. This is a contradiction to Corollary 4.34. Hence $\hat{V}_1 \subset \tilde{M}_{\tilde{x}}$ so that $\hat{M}_{\tilde{x}}$ is open. \square

Lemma 4.40. [17] *The stars $\hat{M}_{\tilde{x}}$ form an open cover of \hat{M} i.e. we have*

$$\hat{M} = \bigcup_{\tilde{x} \in \tilde{M}} \hat{M}_{\tilde{x}}. \quad (4.56)$$

Proof. [17] We need to show, that every $\hat{v} \in \hat{M}$ is seen by some $\tilde{x} \in \tilde{M}$. Again, we define \hat{V}_1 and \hat{V}_2 to be two open balls of radius $r_1 = \epsilon$ and $r_2 = 3\epsilon$, such that they are centered at \hat{v} . Again, the metric is induced from the pull-back $D^{-1}(g_0)$. ϵ is chosen, such that the open image $D(\hat{V}_2)$ is simply convex in X_x . This implies, that for every $\hat{y} \in \hat{V}_2$ we have $D(\hat{V}_2) \subset X_y$, where $y = D(\hat{y})$. Now, we take some element \tilde{x} from the intersection $\hat{V}_1 \cap \tilde{M}$ and construct a curve $\alpha : [0, 1] \rightarrow \hat{V}_2$ with endpoint \tilde{x} and \hat{v} . Again, it must follow that $\alpha([0, 1]) \subset \tilde{M}_{\tilde{x}}$ as otherwise we would have a contradiction to Proposition 4.34. Hence the assertion follows. \square

In the two curved Lorentzian space forms in our discussion, we will define a hyperplane. In order to do so we first need the following insight.

Lemma 4.41. *Let H be a coisotropic vectorial hyperplane in \mathbb{R}^{n+1} and $n > 2$. Then*

1. $\dim H^\perp = 1$,
2. $dS^n \cap H$ is path-connected, and
3. $AdS^n \cap H$ consists of two path connected components.

Proof. Statement (1.) follows immediately from Lemma 2.7, as $\dim H = n - 1$.

We now show (2.). We may always use the $O(n)$ -symmetry to rotate the spatial vectors so that the null vector of H lies in the (x_1, x_2) -plane. That x lies in the intersection $dS^n \cap H$ is equivalent to it satisfying the equation system

$$\left. \begin{array}{l} 1 = -x_1^2 + x_2^2 + \dots + x_{n+1}^2 \\ x_1 = x_2 \end{array} \right\} \Rightarrow 1 = x_3^2 + \dots + x_{n+1}^2, \quad (4.57)$$

from which it follows that the intersection is path-connected.

For the intersection $AdS^n \cap H$, we may use the rotational symmetry so that H is given by the equation $x_2 = x_3$. Hence, we have

$$\left. \begin{array}{l} 1 = x_1^2 + x_2^2 - (x_3^2 + \dots + x_{n+1}^2) \\ x_2 = x_3 \end{array} \right\} \Rightarrow 1 + x_4^2 + \dots + x_{n+1}^2 = x_1^2. \quad (4.58)$$

Here we see that the intersection $AdS^n \cap H$ will have two connected components - one for each sign of x_1 . \square

With Lemma 4.41 at hand, we may state the following.

Definition 4.42. Consider dS^n and AdS^n as subsets of \mathbb{R}^{n+1} . In dS^n a **coisotropic hyperplane** is a connected component of the intersection $dS^n \cap H$ with some coisotropic vectorial hyperplane H in \mathbb{R}^{n+1} . In AdS^n we call one of the two connected components in $AdS^n \cap H$ a **coisotropic half-hyperplane**.

Clearly, a coisotropic hyperplane H in \mathcal{M}^{n+1} must contain one null vector but no timelike vectors in the \mathcal{M}^n and dS^n cases, and two null orthogonal vectors but no timelike vectors in the AdS^n case. For convenience with our next result, we first make the following definition.

Definition 4.43. A subset $C \subset \hat{M}$ is called **totally geodesically coisotropic** if $D(C)$ identifies locally with a coisotropic hyperplane in X .

We state the following important result:

Proposition 4.44. [17] *The boundary $\hat{\partial}\tilde{M}$ is totally geodesically coisotropic, or empty.*

Proof. [17] Let \hat{v} be an element in $\hat{\partial}\tilde{M}$. By Lemma 4.40 and its proof, we know that there is an element \tilde{x} that sees \hat{v} and some neighbourhood \hat{V} , centered at \hat{v} such that its image $D(\hat{V})$ is simply convex. We denote the intersection $\hat{V} \cap \hat{M}$ by \hat{V} .

Let $\alpha : [0, 1] \rightarrow \hat{M}$ be the parametrization of the segment $[\tilde{x}, \hat{v}]$. By construction, $\alpha([0, 1)) \subset \hat{M}$. We let $\{t_n\}$ be a sequence in $[0, 1)$ tending towards 1. The corresponding sequence $\{\alpha(t_n)\}$ tends to \hat{v} . We denote $\alpha(t_n)$ by \tilde{v}_n . From the proof of Proposition 4.32 we have existence of a sequence of distinct elements $\{k_n\} \subset \mathbb{D}$ such that the sequence $\{\tilde{w}_n\} = \{k_n(\tilde{v}_n)\}$ converges to \tilde{w} in \hat{M} . By the same procedure as in the proof of Proposition 4.32, we construct the compact balls \tilde{B}_n , centered at \tilde{w}_n and $\tilde{C}_n = k_n^{-1}(\tilde{B}_n)$, centered at \tilde{v}_n . The $\{\tilde{C}_n\}$ are compact and as they are all subsets of $\hat{M}_{\tilde{x}}$ they are disjoint.

The balls \tilde{C}_n are contained within \hat{V} and thus the intersection $\hat{V} \cap \tilde{C}_n$ is closed in $\hat{M}_{\tilde{x}}$. The sequence of sets is identified with $\phi(g_n(B_n)) \cap D(\hat{V})$ in X , where B_n is the Euclidean closed ball of radius r , centered at w_n in \mathbb{R}^{n+1} . The sequence $\{v_n\}$ converges to v , and so we must have convergence of $\phi(g_n(B_n)) \cap D(\hat{V})$ to some limit ellipsoid \mathcal{E} , centered at v . Furthermore, as in the proof of Proposition 4.32, we may identify $g_n = c_n o_n a_n$, where $o_n \in K_w \simeq O_1(n)$ and $\{a_n\}$ and $\{c_n\}$ are contained in compact subsets of G . Since the $\phi(g_n(B_n))$ are disjoint, the limit must be degenerate. By Corollary 3.42 it follows that \mathcal{E} must have codimension 1 and thus be contained in a coisotropic hyperplane H .

By this it follows that $\phi(g_n(B_n)) \cap D(\hat{V})$ converges, and the limit \mathcal{E} has $\phi(\mathcal{E}) \subset D(\hat{V})$.

We need to show that $\phi(\mathcal{E}) \subset D(\hat{V} - \hat{V})$, i.e. that $D(\hat{V}) \cap \phi(\mathcal{E}) = \emptyset$. Let \hat{u} be any point in $\phi(\mathcal{E})$. From the definition of convergence, we know that there is a sequence $\{u_n\}$ in X such that $u_n \rightarrow \hat{u}$ and for each n we have $u_n \in g_n(B_n)$. Since the $D^{-1}(\phi(g_n(B_n))) \subset \hat{M}_{\tilde{x}}$ we know that there is a uniquely lifted sequence $\{\tilde{u}_n\}$ in $\hat{M}_{\tilde{x}}$ and that for each n we have $\tilde{u}_n \in \tilde{C}_n$.

Now, we must on the one hand have $k_n(\tilde{u}_n) \subset \tilde{B}$ for each n . This means in particular, as \tilde{B} is compact, that there must be a convergent subsequence of $\{k_n(\tilde{u}_n)\}$ that converges to \tilde{t} in \tilde{B} .

On the other hand, the sequences $\{\tilde{u}_n\}$ and $\{k_n(\tilde{u}_n)\}$ are projected down to M to the same sequence, and so if $\tilde{u}_n \rightarrow \tilde{u}$, we must have $k(\tilde{u}) = \tilde{t}$. This means that $\tilde{u} \in \tilde{t}(\tilde{B})$ but the $k(\tilde{B})$ are all disjoint and so there can be no convergence of $\{\tilde{u}_n\}$ within \hat{M} .

By our deduction it now follows that every element in $\phi(\mathcal{E})$ is a limit point to $D(\hat{M})$, which is contained in some coisotropic hyperplane H in \mathbb{R}^{n+1} .

In the case $\kappa = 0$, this is already the assertion. In the case of $\kappa = \pm 1$, we need to show that the hyperplane H is vectorial. In order to show this, we consider the metric boundary $\hat{V} - \hat{V}$ in \hat{M} . Clearly, this set has no interior point. Since D is a local diffeomorphism it must follow that $D(\hat{V} - \hat{V})$ has no interior point either. But $D(\hat{V} - \hat{V}) = \phi(\mathcal{E})$ and if \mathcal{E} were not vectorial, the projected image $\phi(\mathcal{E})$ would have an interior point. Hence H must be vectorial and so $X \cap H$ is a coisotropic hyperplane. \square

In the space \hat{M} we make the following definitions.

Definition 4.45. In \hat{M} we define a **hyperplane** \hat{H} to be a connected subset of \hat{M} such that $D(\hat{H})$ is a hyperplane in X .

We get the following result.

Corollary 4.46. [17] *If the boundary $\hat{\partial}\tilde{M}$ is non-empty, then every connected component is a hyperplane in \hat{M} .*

Proof. This assertion follows, as from Proposition 2.29 we know that every connected component in $\hat{\partial}M$ is an $n-1$ -dimensional manifold without boundary. But on the other hand we know that the image $D(\hat{M}) = \overline{D(\tilde{M})}$ is closed and from this the only remaining possibility is that the connected components must identify with hyperplanes. \square

With this in hand, we formulate a very important result. For convenience, we state and prove it for the three cases $\kappa = 0, \pm 1$ respectively. For the sake of simplicity, we start with the simplest case, namely that of $\kappa = 0$.

Proposition 4.47. [17] *Suppose $D : \tilde{M} \rightarrow \mathcal{M}^n$ is not surjective. Then $D(\tilde{M})$ will be identified with one of the two connected components in $\mathcal{M}^n - H$, for some coisotropic hyperplane H .*

Proof. [17] By Corollary 4.46 the boundary $\hat{\partial}\tilde{M}$ must contain a hyperplane \hat{H} . We denote the image $D(\hat{H}) = H$ the hyperplane in \mathcal{M}^n . Let \hat{h} be a point in \hat{H} . For each $\hat{y} \in \hat{H}$ we define, in \mathcal{M}^n , the curve

$$\alpha_y(t) = y + t(x - h), \quad (4.59)$$

where $y = D(\hat{y})$, $h = D(\hat{h})$ and $x = D(\tilde{x})$. Since these curves are straight lines in \mathcal{M}^n they are geodesics and may be lifted through D to segments in \hat{M} . The curves may be defined for all $t \in [0, \infty)$ or not. If not, they must terminate at some hyperplane $\hat{H}_{\hat{y}}$. Clearly $\hat{H}_{\hat{y}}$ is distinct from H . This happens if and only if they are parallel.

In this manner, if the geodesic α_y is not defined for all positive t , the points \hat{y} are all mapped to some plane H_y . This defines a mapping $\varphi : \hat{y} \rightarrow H_y$, where we take H_y to be the empty set \emptyset if the curve α_y is defined for all positive t . Since this is locally constant and the hyperplanes are all connected components, the mapping φ is globally constant, i.e. $\varphi(\hat{y}) \equiv H_{\hat{y}}$.

Since \tilde{M} is connected and D is a local homeomorphism it follows that $D(\tilde{M}) = \Omega$ is the enclosed interior if H_y is not empty and the open affine half-space in \mathcal{M}^n if H_y is empty. Theorem 2 in [9] p. 771 asserts that $D(\tilde{M})$ cannot lie between two hyperplanes and hence the former possibility is ruled out. The assertion follows. \square

Proposition 4.48. [17] *Suppose $D : \tilde{M} \rightarrow dS^n$ is not surjective. Then $D(\tilde{M})$ will be identified with one of the two connected components in $dS^n - H$, for some coisotropic hyperplane H .*

Proof. [17] We suppose $D : \tilde{M} \rightarrow dS^n$ is not surjective and so the boundary $\hat{\partial}\tilde{M}$ is non-empty, containing some element \hat{h} . Again, by Corollary 4.46 the connected components of $\hat{\partial}\tilde{M}$ must be hyperplanes, and we denote $D(\hat{H}) = H$. From Lemma 4.41 we know that there is an orthonormal basis $\{e_1, \dots, e_{n+1}\}$ where $H \perp e_1$ and in these coordinates the metric is represented by

$$B(x, y) = x_1y_2 + y_1x_2 + x_3y_3 + \dots + x_{n+1}y_{n+1}. \quad (4.60)$$

Again, by Lemma 4.37, there is some element $\tilde{x} \in \tilde{M}$ that sees \hat{h} . We let e_1 be such that connected component Ω of $dS^n - H$ containing $x = D(\tilde{x})$ identifies with the set of points y such that $B(e_1, y) > 0$.

We define the curve

$$\gamma_y(t) = \begin{cases} \frac{\sinh(t)}{B(y, e_2)} e_2 + \exp(-t)y, & \text{if } B(y, e_2) \neq 0 \text{ and} \\ y + te_2, & \text{if } B(y, e_2) = 0. \end{cases} \quad (4.61)$$

If $B(y, e_2) \neq 0$, we have

$$\begin{aligned}
B(\gamma_y, \gamma_y) &= B\left(\frac{\sinh(t)}{B(y, e_2)}e_2 + \exp(-t)y, \frac{\sinh(t)}{B(y, e_2)}e_2 + \exp(-t)y\right) \\
&= \left(\frac{\sinh(t)}{B(y, e_2)}\right)^2 B(e_2, e_2) + 2\exp(-t)\frac{\sinh(t)}{B(y, e_2)}B(e_2, y) \\
&\quad + \exp^2(-t)B(y, y) \\
&= 2\exp(-t)\sinh(t) + \exp(-2t) \\
&= 1,
\end{aligned} \tag{4.62}$$

and if $B(e_2, y) = 0$ we have

$$\begin{aligned}
B(\gamma_y, \gamma_y) &= B(y + te_2, y + te_2) \\
&= B(y, y) + 2tB(y, e_2) + t^2B(e_2, e_2) \\
&= 1
\end{aligned} \tag{4.63}$$

so that in both cases $\gamma_y \subset dS^n$.

Furthermore γ is a geodesic: from Proposition 2.22 we know that any geodesic is on the form $dS^n \cap \Pi$, where Π is a vector sub-plane. Clearly $\gamma_y \subset \text{span}\{e_2, y\}$. Since, in the case $B(e_2, y) \neq 0$, we have $\ddot{\gamma}_y = \gamma_y$ it follows that γ_y is a correctly parametrized geodesic. If $B(e_2, y) = 0$ it is immediately obvious that γ_y is a straight line and so since Π is a degenerate plane with index $\nu = 0$ it is also a geodesic.

We now consider the image of the mapping $\varphi : y \rightarrow H$, just as in the proof of Proposition 4.47; for each fixed y we have that if γ_y is not defined for all t , it must hit the boundary $\partial\tilde{M}$, which by Proposition 4.46 identifies with a hyperplane H_y . (Otherwise, just as before, we take the image to be the empty set). We again denote this mapping $\varphi : y \rightarrow H_y$. Since φ is locally constant and the hyperplanes are connected components, it is constant, i.e. $\varphi \equiv H_y$.

If H_y is non-empty it is a full coisotropic hyperplane in dS^n . But by construction a hyperplane in dS^n is the intersection $H \cap dS^n$, for a vectorial coisotropic hyperplane H in \mathcal{M}^{n+1} . Two such hyperplanes have the origin in common and their intersection must be a vectorial hyperplane of dimension $n - 1$. The intersection will obviously intersect dS^n and hence $H \cap H_y \neq \emptyset$. This contradicts Proposition 4.46 and therefore H_y must be empty.

We need only to verify that $D(\tilde{M})$ is surjective onto Ω . To do this we consider the curve $\tilde{\gamma}(t) = \exp(t)e_1 + \frac{1}{2\exp(t)}e_2$. We have $B(\tilde{\gamma}(t), \tilde{\gamma}(t)) = 2\exp(t)\frac{1}{2\exp(t)} = 1$, so that $\tilde{\gamma} \subset dS^n$. Furthermore, $\tilde{\gamma}$ is also on the form $dS^n \cap \Pi$, where $\Pi = \text{span}\{e_1, e_2\}$ is a Lorentzian plane. By Proposition 2.22 its image is a geodesic. Clearly we also have $\ddot{\tilde{\gamma}} = \tilde{\gamma}$, and by the proof in Proposition 2.22 we then know that $\tilde{\gamma} \in N_\gamma dS^n$ so that γ has the correct parametrization. Clearly, if $y \in H$ is a boundary element the corresponding curve γ_y must be defined for all parameter values $t > 0$. The image of all γ_y is $\Omega - \tilde{\gamma}$. Since $D(\tilde{M})$ is closed and D is continuous it follows that $\tilde{\gamma}$ must necessarily also be contained in the image Ω . Hence the assertion must follow. \square

Proposition 4.49. [17] *Suppose $D : \tilde{M} \rightarrow AdS^n$ is not surjective. Then $D(\tilde{M})$ will be identified with one of the two connected components in $AdS^n - H$, for some coisotropic hyperplane H .*

Proof. [17] Since AdS^n is not simply connected, we consider its universal covering space, which is $\widetilde{AdS^n} = \mathbb{R}^n$. We denote the covering map $\tilde{\varphi} : \mathbb{R}^n \rightarrow AdS^n$ and it is given explicitly

$$\tilde{\varphi}(\theta, x_1, \dots, x_{n-1}) = (\sqrt{x^2 + 1} \cos(\theta), \sqrt{x^2 + 1} \sin(\theta), x_1, \dots, x_{n-1}), \tag{4.64}$$

where we use $x^2 = \sum_k^{n-1} x_k^2$. By Theorem 3.45 we have that the following diagram commutes:

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{D}} & \widetilde{AdS^n} \\
\varphi \downarrow & \searrow D & \downarrow \tilde{\varphi} \\
M & & AdS^n
\end{array} \tag{4.65}$$

i.e. there is $\tilde{\varphi} \circ \tilde{D} = D$. \tilde{D} will be a covering map so that by Proposition 3.50 it will be an isometry and hence we have the diffeomorphisms $\tilde{M}/\mathbb{D} \simeq \Omega$. We denote by $\tilde{\Gamma}$ the image $\tilde{h}(\mathbb{D})$, i.e. the holonomy group on $\widetilde{\text{AdS}^n}$, and Γ the holonomy group on AdS^n .

Let H be a hyperplane in AdS^n and $e = (f_1, f_2, e_1, \dots, e_{n-1})$ be a vector orthogonal to it, (where $f_1 = \sqrt{e^2} \cos(\theta_e)$ and $f_2 = \sqrt{e^2} \sin(\theta_e)$). We solve for an equation for $\varphi^{-1}(H)$: suppose $\varphi(\theta, x) \in H$. Then

$$\begin{aligned} 0 &= Q_{-1}^{n+1}(e, x) = f_1 \sqrt{x^2 + 1} \sin(\theta) + f_2 \sqrt{x^2 + 1} \cos(\theta) - (e_1 x_1 + \dots + e_{n-1} x_{n-1}) \\ &= \sqrt{e^2} \cos(\theta_e) \sqrt{x^2 + 1} \cos(\theta) + \sqrt{e^2} \sin(\theta_e) \sqrt{x^2 + 1} \sin(\theta) - \langle e, x \rangle \\ &= \sqrt{x^2 + 1} \sqrt{e^2} \cos(\theta - \theta_e) - \langle e, x \rangle \\ &\Rightarrow \cos(\theta - \theta_e) = \frac{\langle e, x \rangle}{\sqrt{x^2 + 1} \sqrt{e^2}}, \end{aligned} \quad (4.66)$$

where we defined $\langle e, x \rangle = \sum_i^{n-1} e_i x_i$.

The lift $\varphi^{-1}(H)$ will consist of disjoint connected components, which we denote

$$\tilde{H}_k = \{(\theta, x) \in \tilde{\varphi}^{-1}(H) \mid (\theta - \theta_e) \in (k\pi, (k+1)\pi)\}, \quad k \in \mathbb{Z}. \quad (4.67)$$

These subsets are called **coisotropic half-hyperplanes**. From the definition it follows that the disunion $\widetilde{\text{AdS}^n} - \tilde{H}_k$ consists of two disjoint open sets; one contains \tilde{H}_{k+1} and the other \tilde{H}_{k-1} .

The set \tilde{H}_k is furthermore diffeomorphic to \mathbb{R}^{n-1} . To show this we observe first that as $\langle x, e \rangle$ is an $n-1$ -dimensional Euclidean scalar product we have

$$-\sqrt{x^2} \sqrt{e^2} \leq \langle x, e \rangle \leq \sqrt{x^2} \sqrt{e^2}, \quad (4.68)$$

from which it follows that $-1 < \cos(\theta - \theta_e) < 1$. Hence we may define the following maps:

$$\begin{aligned} \psi_{2k}(x_1, \dots, x_{n-1}) &= \left(\theta = \left(\arccos \left(\frac{\langle e, x \rangle}{\sqrt{x^2 + 1} \sqrt{e^2}} \right) + \theta_e \right) + 2k \cdot \pi, x_1, \dots, x_{n-1} \right) \\ \psi_{2k+1}(x_1, \dots, x_{n-1}) &= \left(2\pi - \left(\theta = \arccos \left(\frac{\langle e, x \rangle}{\sqrt{x^2 + 1} \sqrt{e^2}} \right) + \theta_e \right) + 2k \cdot \pi, x_1, \dots, x_{n-1} \right) \end{aligned} \quad (4.69)$$

that clearly are diffeomorphisms from \mathbb{R}^{n-1} onto \tilde{H}_k .

Clearly we have $\tilde{H}_k = \tilde{H}_{k+2}$ and for even and odd k \tilde{H}_k is mapped down to one of the two associated half-hyperplanes in $\text{AdS}^n - H$.

Suppose \hat{h} belongs to $\hat{H}_0 \subset \hat{\partial}\tilde{M}$, with image $\tilde{D}(\hat{H}_0) = \tilde{H}_0$. By Lemma 4.40 it is seen by some element \tilde{x} . If $\tilde{D}(\tilde{x})$ belongs to the connected component containing \tilde{H}_1 we say that \hat{H}_0 is **positive** and otherwise **negative**.

We now proceed with the actual proof. We suppose that $D : \tilde{M} \rightarrow \text{AdS}^n$, so that Proposition 4.46 implies that there must be a hyperplane \hat{H}_0 in $\hat{\partial}\tilde{M}$, which identifies with $\tilde{D}(\hat{H}_0) = \tilde{H}_0$. We suppose \hat{H}_0 is positive.

For each element $\hat{y} \in \hat{H}_0$ we define the curve $\alpha_y(\theta) = \tilde{y} + (\theta, 0, \dots, 0)$. If α_y is not lifted to \hat{M} through \tilde{D} to a curve that is defined for all $\theta \in \hat{M}$, it is because it hits a hyperplane $\hat{H}_{\tilde{y}} \subset \hat{\partial}\tilde{M}$. This defines a mapping $\hat{y} \rightarrow \hat{H}_y$, where \hat{H}_y is taken to be the empty set if the lifted curve of α_y is defined for all positive t . Furthermore, this mapping is locally constant. Since the connected components of $\hat{\partial}\tilde{M}$ are all connected, the mapping is constant. It must equal either the empty set or one half-hyperplane.

In the case of the image \hat{H}_y being empty, we construct a subset $\tilde{\Omega} \subset \tilde{M}$ that is diffeomorphic through \tilde{D} to the connected component Ω of $\widetilde{\text{AdS}^n} - \tilde{H}_0$, containing \tilde{H}_1 (as \tilde{H}_0 is positive).

If \hat{H}_y is not empty, we let $D(\hat{H}_y) = \tilde{H}_y$ denote the half-hyperplane. Clearly \hat{H}_0 and \hat{H}_y will be the boundary of some connected open set Ω and as such there is a subset $\tilde{\Omega} \subset \tilde{M}$ such that $\tilde{D}(\tilde{\Omega}) = \Omega$.

In both cases, as $\widetilde{\text{AdS}^n}$ is simply connected, the discrete, infinite subgroup $h(\mathbb{D}) = \Gamma \subset G$ acts in a properly discontinuous manner by Lemma 3.47. Since $\Omega/\Gamma \simeq M$ is compact the **cohomological**

dimension $cd \Gamma$ of Γ , must satisfy $cd \Gamma = \dim M = n$, following ([4], p. 209).

If \tilde{H}_y is not empty but $\tilde{H}_y \neq \tilde{H}_1$, we must have non-empty intersection $\tilde{\varphi}^{-1}(H_0) \cap \Omega \neq \emptyset$. This intersection will contain a half-hyperplane, of dimension $n - 1$, which we denote by Ω' . We denote by Γ' the subgroup of Γ that fixes Ω' and we have $[\Gamma : \Gamma'] = N < \infty$.

We now show that the subquotient Ω'/Γ' is also compact. We let $\{\Gamma'x'_n\}$ be a sequence in Ω'/Γ' . Clearly $\{x'_n\} \subset \Omega'$. We consider the sequence $\{\Gamma x'_n\}$ in Ω/Γ that by the compactness has a convergent subsequence, which for notational convenience we assume is itself and the limit is $\Gamma x'$. By Proposition 2.36 there exists a covering map $\psi : \Omega \rightarrow \Omega/\Gamma$ and so we may choose a sequence $\{z_n\} \subset \Omega$ such that $\psi(z_n) = \Gamma x'_n$ and $z_n \rightarrow z$ with $\psi(z) = \Gamma x'$. Furthermore, as $[\Gamma : \Gamma'] = N < \infty$ we have the disjoint union

$$\Gamma = \Gamma' \cup g_1\Gamma' \cup \dots \cup g_{N-1}\Gamma', \quad g_i \in \Gamma. \quad (4.70)$$

Clearly z_n and x'_n are congruent mod Γ , i.e. we have $z_n = g_n x'_n$. But as we have only finitely many cosets $g_i\Gamma'$ in Γ , one of these must contain infinitely many elements of the $\{g_n\}$. We may pass to this subsequence, or assume that already $\{g_n\} \subset g_K\Gamma'$ for notational convenience. But then $g_n = g_K g'_n$, where $g'_n \in \Gamma'$ and so

$$z_n = g_n(x'_n) = g_K g'_n(x'_n). \quad (4.71)$$

Since $z_n \rightarrow z$ and g_K^{-1} is a diffeomorphism, we must have $g_K^{-1}(z_n) \rightarrow x = g_K^{-1}(z)$. On the other hand we have $g_K^{-1}(z_n) = g_K^{-1}g_K g'_n(x'_n) = g'_n(x'_n) \in \Omega'$, as $g'_n \in \Gamma'$ that fixes Ω' . Since $\Omega' \subset \Omega$ was closed it follows that $x \in \Omega'$. Hence we may project down the convergent sequence x'_n into $\Gamma'x'_n \in \Omega'/\Gamma'$ and it converges to $\Gamma'x$. Thus Ω'/Γ' is compact.

Now on the other hand, we have from ([4], p. 187) that $cd\Gamma' = cd\Gamma$. But on the other we have that as Ω'/Γ' is compact we must have $cd\Gamma' = \dim \Omega' = n - 1$ by ([4], p. 209). This is a contradiction. Hence we have $\tilde{H}_y = \tilde{H}_1$, which is the assertion. \square

With the above results in hand, we are finally ready to deduce the full Theorem 4.21.

Proof. ([6], [17]) We deal with the flat and non-flat cases separately.

For the case of $\kappa = 0$, Proposition 4.46 implies that $D(\tilde{M}) = \Omega$, a connected component of $\mathcal{M}^n - H$, for a hyperplane H . The holonomy group Γ must leave a hyperplane parallel to H invariant. This contradicts ([14], Theorem 2.8 p. 644). Hence $D(\tilde{M}) = \mathcal{M}^n$ and as \mathcal{M}^n is complete by Proposition 2.20 it follows that (M, g) must be complete. This argument is initially due to [6].

The following is due to [17]. Suppose $\kappa = \pm 1$. Let H be the coisotropic hyperplane in the ambient space \mathbb{R}^{n+1} and e_1 a vector orthogonal to H . Clearly then e_1 is null. Let furthermore G_I denote the isometry subgroup of G fixing Ω . We denote H_I the isotropy subgroup, so that we may identify $\Omega = G_I/H_I$. We define the vector field

$$Y_y = \frac{e_1}{Q_{\pm 1}^{n+1}(e_1, y)} - y. \quad (4.72)$$

It follows that $Y_y \in T_y\Omega$ as

$$\begin{aligned} Q_{\pm 1}^{n+1}(Y_y, y) &= Q_{\pm 1}^{n+1}\left(\frac{e_1}{Q_{\pm 1}^{n+1}(e_1, y)} - y, y\right) \\ &= \frac{Q_{\pm 1}^{n+1}(e_1, y)}{Q_{\pm 1}^{n+1}(e_1, y)} - Q_{\pm 1}^{n+1}(y, y) \\ &= 0, \end{aligned} \quad (4.73)$$

where we used the fact that if y belongs to dS^n or AdS^n , we have $Q_{\pm 1}^{n+1}(y, y) = 1$.

Furthermore we show that Y_y is G_I -invariant, in the sense that if $g \in G_I$, then $gY_y = Y_{gy}$. Clearly G_I must act on e_1 by scalar multiplication, i.e. $ge_1 = c \cdot e_1$. Furthermore, as it is a group of isometries, it must preserve the scalar product, i.e. $Q_{\pm 1}^{n+1}(ge_1, gy) = Q_{\pm 1}^{n+1}(e_1, y)$. We therefore

get

$$\begin{aligned}
gY_y &= g\left(\frac{e_1}{Q_{\pm 1}^{n+1}(e_1, y)} - y\right) \\
&= g\frac{e_1}{Q_{\pm 1}^{n+1}(e_1, y)} - gy \\
&= \frac{ge_1}{Q_{\pm 1}^{n+1}(ge_1, gy)} - gy \\
&= \frac{c \cdot e_1}{Q_{\pm 1}^{n+1}(c \cdot e_1, gy)} - gy \\
&= \frac{e_1}{Q_{\pm 1}^{n+1}(e_1, gy)} - gy \\
&= Y_{gy},
\end{aligned} \tag{4.74}$$

where we used linearity and scalar product invariance.

We now calculate the divergence of Y_y for both the de-Sitter space and the anti de-Sitter separately. For the de-Sitter space, we let (e_1, \dots, e_{n+1}) in \mathbb{R}^{n+1} be a basis such that e_1 is the vector orthogonal to H and so that the quadratic form Q_{+1}^{n+1} is given by

$$B_{+1}(x, y) = x_1 y_2 + y_1 x_2 + x_3^2 + \dots + x_{n+1}^2. \tag{4.75}$$

in this basis. We get that Ω is the set of points x in \mathbb{R}^{n+1} such that both $B_{+1}(x, x) = 1$ and $x_2 > 0$ and the hyperplane H is orthogonal to $\mathbb{R}e_1$. A parametrization $\rho_{\kappa=1} : \mathbb{R}_+^n \rightarrow \Omega$ for Ω in the dS^n is given by

$$\rho_{\kappa=1}(x_2, \dots, x_{n+1}) = \left(\frac{1 - x_3^2 - \dots - x_{n+1}^2}{2x_2}, x_2, \dots, x_{n+1}\right), \tag{4.76}$$

where \mathbb{R}_+^n is the open half-space in \mathbb{R}^n defined by $x_2 > 0$. We have

$$B_{+1}(\rho, \rho) = 2\frac{1 - x_3^2 - \dots - x_{n+1}^2}{2x_2} \cdot x_2 + x_3^2 + \dots + x_{n+1}^2 = 1, \tag{4.77}$$

so that $\rho \in \Omega$. The metric components in this basis are

$$g_{\kappa=1} = \begin{pmatrix} -\frac{1 - x_3^2 - \dots - x_{n+1}^2}{x_2^2} & -\frac{x_3}{x_2} & \dots & -\frac{x_{n+1}}{x_2} \\ -\frac{x_3}{x_2} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -\frac{x_{n+1}}{x_2} & 0 & 0 & 1 \end{pmatrix} \tag{4.78}$$

so that the determinant becomes

$$g = \sqrt{|\det g|} = \frac{1}{x_2}. \tag{4.79}$$

We have

$$Y(x_2, \dots, x_{n+1}) = -(x_2, \dots, x_{n+1}) \tag{4.80}$$

and with this we may calculate the divergence $\operatorname{div} Y$ following ([20], p. 213):

$$\begin{aligned}
\operatorname{div} Y &= \frac{1}{g} \sum_i \frac{\partial}{\partial x_i} \left(g Y_i \right) \\
&= x_2 \left(\frac{\partial}{\partial x_2} \frac{-x_2}{x_2} + \sum_{i=3}^{n+1} \frac{\partial}{\partial x_i} \left(\frac{-x_i}{x_2} \right) \right) \\
&= x_2 \left(0 + \sum_{i=3}^{n+1} \frac{-1}{x_2} \right) \\
&= -(n-1).
\end{aligned} \tag{4.81}$$

With this we may, together with Lemma 2.27, interpret $L_Y\omega = -(n-1)\omega$.

We now do same calculation in the anti de-Sitter space. The quadratic form Q_{-1}^{n+1} becomes

$$B_{-1}(x, y) = 2x_1x_3 + x_2^2 - (x_4^2 + \dots + x_{n+1}^2) \quad (4.82)$$

in the new basis. A parametrization $\rho_{\kappa=-1} : \mathbb{R}_+^n \rightarrow \Omega$ is

$$\rho_{\kappa=-1}(x_2, \dots, x_{n+1}) = \left(\frac{1 - x_2^2 + (x_4^2 + \dots + x_{n+1}^2)}{2x_3}, x_2, \dots, x_{n+1} \right). \quad (4.83)$$

Clearly, $\rho \in AdS^n$ as

$$B_{-1}(\rho, \rho) = 2 \cdot \frac{1 - x_2^2 + (x_4^2 + \dots + x_{n+1}^2)}{2x_3} \cdot x_3 + x_2^2 - (x_4^2 + \dots + x_{n+1}^2) = 1 \quad (4.84)$$

and similarly if $x \in AdS^n$ it must be on the form $\rho(x_2, \dots, x_{n+1})$. The metric components are

$$g_{\kappa=-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & -\frac{1-x_2^2+(x_4^2+\dots+x_{n+1}^2)}{x_3^2} & -\frac{x_3}{x_3} & \dots & -\frac{x_{n+1}}{x_3} \\ 0 & -\frac{x_3}{x_3} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & -\frac{x_{n+1}}{x_3} & 0 & 0 & 1 \end{pmatrix} \quad (4.85)$$

so that the determinant becomes

$$g = \sqrt{|\det g|} = \frac{1}{x_3}. \quad (4.86)$$

We have

$$Y(x_2, \dots, x_{n+1}) = -(x_2, \dots, x_{n+1}) \quad (4.87)$$

so that we may calculate the divergence $\operatorname{div} Y$:

$$\begin{aligned} \operatorname{div} Y &= \frac{1}{g} \sum_i \frac{\partial}{\partial x_i} \left(g Y_i \right) \\ &= x_3 \left(\sum_{i=2}^{n+1} \frac{\partial}{\partial x_i} \left(\frac{-x_i}{x_3} \right) \right) \\ &= -(n-1). \end{aligned} \quad (4.88)$$

With this we may, together with Lemma 2.27, interpret $L_Y\omega = -(n-1)\omega$.

We now look on this from another angle. The volume element ω_I is also G_I -invariant as G_I is an isometry group. From the proof of Lemma 2.27 we know that $L_Y\omega_I = \lambda\omega_I$ pointwise. The G_I -invariance implies that the λ is constant on dS^n or AdS^n . Furthermore, Y induces a vector field Y_M and a volume element ω_M on M . We may therefore apply Stokes Theorem ([7], p. 63):

$$\int_M L_{Y_M}\omega_M = \int_M \lambda\omega_M = \lambda \int_M \omega_M = \lambda \int_{\partial M} d\omega_M = 0, \quad (4.89)$$

where the integral must vanish as M has no boundary. Hence, $\lambda = 0 \neq -(n-1)$.

We have now showed that there can be no boundary of $D(\tilde{M})$ in X and hence D is surjective. Completeness follows. \square

We get the the following consequence.

Corollary 4.50. *There are no compact Lorentzian manifolds of constant positive curvature.*

Proof. Suppose (M^n, g) is an n -dimensional, compact Lorentzian manifold with positive constant curvature. If $n = 2$, this would be a contradiction to Corollary 4.18. If $n > 2$, Theorem 4.21 shows that (M, g) is geodesically complete. But then it would be a compact spherical space form. This is a contradiction to Corollary 4.20. Hence (M^n, g) cannot exist. \square

Appendix A

Covering spaces

In this Appendix we discuss what is called covering spaces. It will be shown that every topological space, and hence any manifold, has such an associated space. The covering space may be viewed as a simplified version of the original space, where the information of the original space is stored in the corresponding allowable symmetry groups to the covering space. This is of fundamental importance when proving Theorem 4.21.

A.1 Homotopy theory

We now introduce what is known as homotopy theory. We follow the discussion in [10].

Definition A.1. Let (M, τ) be a topological manifold. Suppose $\gamma_1 : [0, 1] \rightarrow M$ and $\gamma_2 : [0, 1] \rightarrow M$ are two curves with $\gamma_1(0) = \gamma_2(0) = a$ and $\gamma_1(1) = \gamma_2(1) = b$. Then they are said to be **homotopic with endpoints fixed** if there is a map $F : [0, 1] \times [0, 1] \rightarrow M$ such that

$$\begin{aligned} F(0, t) &= \gamma_1(t), & \forall t, \\ F(1, t) &= \gamma_2(t), & \forall t, \\ F(s, 0) &= a, & \forall s, \\ F(s, 1) &= b, & \forall s. \end{aligned} \tag{A.1}$$

F is called the **homotopy map**. In this case we write $\gamma_1 \simeq \gamma_2$.

The above definition shows, that if $\gamma_1 \simeq \gamma_2$ the former curve may be continuously deformed into the latter.

Lemma A.2. ([10], p. 114) *Let (M, τ) be a topological space. Then the relation " \simeq ", defined above, is an equivalence relation.*

Proof. ([10], p. 114) We need to prove reflexivity, symmetry and transitivity. For reflexivity, we observe that if γ is a curve through M , then the map $F(s, t) = \gamma(t)$ for all s , gives a map with the necessary properties.

For symmetry, we observe that if $\gamma_1 \simeq \gamma_2$ through F , i.e. $F(0, t) = \gamma_1(t)$ and $F(1, t) = \gamma_2(t)$, then $\hat{F}(s, t) = F(1 - s, t)$ has $\hat{F}(1, t) = \gamma_2(t)$ and $\hat{F}(0, t) = \gamma_1(t)$. Hence we have symmetry.

For reflexivity, suppose $\gamma_1 \simeq \gamma_2$ and $\gamma_2 \simeq \gamma_3$. Let F_{12} and F_{23} be the corresponding homotopy maps. Then the map

$$\hat{F}(s, t) = \begin{cases} F_{12}(2s, t) & 0 \leq s \leq 1/2, \\ F_{23}(2s - 1, t) & 1/2 \leq s \leq 1 \end{cases} \tag{A.2}$$

is a homotopy map with the fixed endpoints a and b and has the properties $\hat{F}(0, t) = \gamma_1(t)$ and $\hat{F}(1, t) = \gamma_3(t)$. Hence $\gamma_1 \simeq \gamma_3$ and " \simeq " is an equivalence relation. \square

The equivalence classes of γ in the above definition are denoted $[\gamma]$, i.e. if $\gamma_1 \simeq \gamma_2$ then $[\gamma_1] = [\gamma_2]$. We need the following lemmas.

Lemma A.3. ([10], p. 114) Let $f : [0, 1] \rightarrow [0, 1]$ be a map such that $f(0) = 0$ and $f(1) = 1$. Then $[\gamma \circ f] = [\gamma]$.

Proof. ([10], p. 114) The explicit map

$$F(s, t) = \gamma((1 - s)t + sf(t)) \quad (\text{A.3})$$

is a homotopy map with $F(0, t) = \gamma(t)$ and $F(1, t) = \gamma(f(t))$. \square

We now want to introduce something akin towards a product of equivalence classes. First, we define the product of two curves.

Definition A.4. Let (M, τ) be a topological manifold, $a, b, c \in M$ and suppose $\alpha(t)$ and $\beta(t)$ are curves in M with $\alpha(0) = a$, $\alpha(1) = \beta(0) = b$ and $\beta(1) = c$. Then the **curve product** is

$$(\alpha\beta)(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2, \\ \beta(1 - 2t) & 1/2 \leq t \leq 1. \end{cases} \quad (\text{A.4})$$

Lemma A.5. ([10], p. 115) The curve product is well-defined and associative up to reparametrization, where the former is meant in the sense that if $\alpha_1 \simeq \alpha_2$ and $\beta_1 \simeq \beta_2$ then $[\alpha_1\beta_1] = [\alpha_2\beta_2]$, and the latter in the sense of $(\alpha\beta)\gamma(t) = \alpha(\beta\gamma)(f(t))$.

Proof. ([10], p. 115) The construction is obvious from the definition.

Let α , β and γ be curves such that the endpoints of the former is the starting point of the latter, as above. We look at the different combinations explicitly:

$$((\alpha\beta)\gamma) = \begin{cases} \alpha(4s), & 0 \leq s \leq 1/4, \\ \beta(4s - 1), & 1/4 \leq s \leq 1/2, \\ \gamma(2s - 1), & 1/2 \leq s \leq 1. \end{cases} \quad (\text{A.5})$$

and

$$(\alpha(\beta\gamma)) = \begin{cases} \alpha(2s), & 0 \leq s \leq 1/2, \\ \beta(4s - 2), & 1/2 \leq s \leq 3/4, \\ \gamma(4s - 3), & 3/4 \leq s \leq 1. \end{cases} \quad (\text{A.6})$$

Comparing we see that with the reparametrization

$$f(t) = \begin{cases} s/2, & 0 \leq s \leq 1/2, \\ s - 1/4, & 1/2 \leq s \leq 3/4, \\ 2s - 1, & 3/4 \leq s \leq 1. \end{cases} \quad (\text{A.7})$$

the curves are homotopic with endpoints fixed. \square

Definition A.6. Let $[\gamma]$ be a homotopy class. The **inverse class** is defined as $[\gamma(t)]^{-1} = [\gamma(t)^{-1}] = [\gamma(1 - t)]$.

Lemma A.7. ([10], p. 117) Let α be a curve that starts at a and ends at b . Then

$$[\alpha][\alpha]^{-1} = [a], \quad (\text{A.8})$$

where a is the constant curve at a .

Proof. ([10], p. 117) The homotopy map

$$F(s, t) = \begin{cases} \alpha(2t), & 0 \leq t \leq s/2, \\ \alpha(s), & s/2 \leq t \leq 1 - s/2, \\ \alpha(1 - 2t), & 1 - s/2 \leq t \leq 1 \end{cases} \quad (\text{A.9})$$

has $F(1, t)$ running along α at double speed and then back, whereas the curve $F(0, t)$ is constant at a . \square

Proposition A.8. ([10], p. 118) Let (M, τ) be a topological manifold and $p \in M$. Then the set of equivalence classes of curves, that start and end at p , form a group under the above composition rule.

Proof. ([10], p. 118) The trivial curve $\gamma(t) \equiv a$ serves as an identity element. We saw in Lemma A.5 that the product composition rule is well-defined and associative. Lemma A.7 shows that every equivalence class $[\gamma]$ has an inverse element $[\gamma]^{-1}$ such that $[\gamma][\gamma]^{-1}$ becomes the identity element. Hence we have group structure. \square

Definition A.9. Let $p \in M$ be any point. The group defined in Proposition A.8, is denoted $\pi_1(M, p)$ and is called the **fundamental group**.

Let us study an example.

Example A.10. We take any convex subset $A \subset \mathbb{R}^2$ and let $p \in A$. Let furthermore γ be any loop from p to p in A . As A is star shaped, any point on γ can be joined to p by a straight line, and thus γ is homotopic to the trivial curve, constant at p . The fundamental group will thus be **trivial**, i.e. $\pi_1(\mathbb{R}^2, p) \simeq \{e\}$.

The following results show that for path-connected sets the fundamental group does not depend on p .

Lemma A.11. ([10], p. 119) Let p and q be points in M and let γ be a curve such that $\gamma(0) = q$ and $\gamma(1) = p$. The mapping

$$\phi_\gamma([\alpha]) = [\gamma][\alpha][\gamma]^{-1} \quad (\text{A.10})$$

is an isomorphism from $\pi_1(M, p)$ to $\pi_1(M, q)$.

Proof. ([10], p. 119) ϕ_γ is a homomorphism as

$$\begin{aligned} \phi_\gamma([\alpha])\phi_\gamma([\beta]) &= [\gamma][\alpha] \underbrace{[\gamma]^{-1}[\gamma]}_{id} [\beta][\gamma]^{-1} \\ &= [\gamma][\alpha][\beta][\gamma]^{-1} \\ &= [\gamma][\alpha\beta][\gamma]^{-1} \\ &= \phi_\gamma([\alpha\beta]), \end{aligned} \quad (\text{A.11})$$

where we used Lemma A.7 and Lemma A.5.

If $\phi_\gamma([\alpha]) = \phi_\gamma([\beta])$, then $[\gamma][\alpha][\gamma]^{-1} = [\gamma][\beta][\gamma]^{-1}$ and by left- and right multiplication with $[\gamma]^{-1}$ and $[\gamma]$ respectively, it follows that $[\alpha] = [\beta]$ and so ϕ_γ is injective.

If $[\alpha] \in \pi_1(M, q)$, then the curve class $[\beta] = [\gamma]^{-1}[\alpha][\gamma]$ is mapped $\phi_\gamma([\beta]) = [\alpha]$ and so ϕ_γ is surjective. Hence ϕ_γ is an isomorphism. \square

If M is path connected any two points can be joined with curves and hence any two fundamental groups $\pi_1(M, p)$ and $\pi_1(M, q)$ are isomorphic by Lemma A.1. Hence we need not include the point in the notation, i.e. $\pi_1(M, p) = \pi_1(M, q) = \pi_1(M)$.

Definition A.12. Let M be a path-connected manifold. If the fundamental group $\pi_1(M)$ is trivial, then M is called **simply connected**.

Proposition A.13. Let M and N be path-connected manifolds. Then

$$\pi_1(M \times N) \simeq \pi_1(M) \times \pi_1(N). \quad (\text{A.12})$$

Proof. A curve in $M \times N$ is on the form $\gamma(t) = (\alpha(t), \beta(t))$. Hence the mapping

$$\phi([\gamma]) = [\alpha] \times [\beta] \quad (\text{A.13})$$

maps $\pi_1(M \times N)$ into $\pi_1(M) \times \pi_1(N)$. That ϕ is onto is obvious from construction. The following shows that ϕ is a homomorphism:

$$\begin{aligned} \phi([\gamma_1]) * \phi([\gamma_2]) &= [\alpha_1] \times [\beta_1] * [\alpha_2] \times [\beta_2] \\ &= [\alpha_1\alpha_2] \times [\beta_1\beta_2] \\ &= \phi([\gamma_1\gamma_2]). \end{aligned} \quad (\text{A.14})$$

It remains only to show that ϕ is injective. But this is clear as if $\phi([\gamma_1]) = \phi([\gamma_2])$ then there must be $[\alpha_1] \times [\beta_1] = [\alpha_2] \times [\beta_2]$. This happens if and only if $\alpha_1 \simeq \alpha_2$ and $\beta_1 \simeq \beta_2$. But then it follows that $\gamma_1 \simeq \gamma_2$ and so $[\gamma_1] = [\gamma_2]$. Thus ϕ is an isomorphism. \square

Proposition A.14. *If two path-connected manifolds M and N are homeomorphic, then $\pi_1(M) \simeq \pi_1(N)$.*

Proof. Let $\phi : M \rightarrow N$ be the homeomorphism. Suppose we have $\gamma_1 \simeq \gamma_2$ in M . Denote the homotopy map $F(s, t)$ with $F(0, t) = \gamma_1(t)$ and $F(1, t) = \gamma_2(t)$. Then the map $\hat{F}(s, t) = \phi(F(s, t))$ is a homotopy in N that shows that the curves $\phi(\gamma_1)$ and $\phi(\gamma_2)$ are homotopic with end-points fixed. Thus the mapping $\phi_*([\gamma]) = [\phi \circ \gamma]$ is a well-defined mapping from $\pi_1(M)$ to $\pi_1(N)$. The following shows that ϕ_* is a group homomorphism:

$$\begin{aligned} \phi_*([\gamma_1]) * \phi_*([\gamma_2]) &= [\phi \circ \gamma_1] * [\phi \circ \gamma_2] \\ &= [(\phi \circ \gamma_1)(\phi \circ \gamma_2)] \\ &= [\phi \circ (\gamma_1 \gamma_2)] \\ &= \phi_*([\gamma_1][\gamma_2]). \end{aligned} \tag{A.15}$$

It remains to show that ϕ_* is bijective. The surjectivity follows from the fact that ϕ is a homeomorphism, and so the mapping ϕ_*^{-1} is just as well-defined as ϕ_* , so that ϕ_* indeed is onto. The injectivity follows in the same manner, i.e. if $\phi_*([\gamma_1]) = \phi_*([\gamma_2]) = [\tilde{\gamma}]$, then $\phi_*^{-1}([\tilde{\gamma}]) = [\gamma_1] = [\gamma_2]$ and so the injectivity follows. Hence ϕ_* is a group isomorphism. \square

A.2 Covering maps

Definition A.15. Let $\varphi : M \rightarrow N$ be a map between two topological manifolds. An open set $U \subset N$ is said to be **evenly covered** by φ if $\varphi^{-1}(U)$ is a number of disjoint sets such that φ , restricted to any of them, is a homeomorphism onto U . If φ is onto N and every $p \in N$ has an evenly covered neighbourhood U , then φ is said to be a **covering map**. The disjoint open sets in $\varphi^{-1}(U)$ are called **sheets**.

Example A.16. We consider the standard exponential map $\exp : \mathbb{R} \rightarrow S^1$. For an arbitrary point $p = \exp(s_0 i 2\pi) \in S^1$ the open set

$$U_m = \{t \in \mathbb{R} \mid |t - (s_0 + m)| < 1/2, m \in \mathbb{Z}\} \tag{A.16}$$

is a union $\bigcup_{m \in \mathbb{Z}} U_m$ in \mathbb{R} of disjoint sheets $U_m = (m - \epsilon, m + \epsilon)$ and the exponential map restricted to each U_m is a homeomorphism. Since s_0 was arbitrary, S^1 is evenly covered by \exp .

Example A.17. We consider the mapping $\phi_n : (\mathbb{C} - \{0\}) \rightarrow (\mathbb{C} - \{0\})$ by $\phi_n(z) = z^n$, $n \in \mathbb{Z}$. For each $z = r \exp(is) \in \mathbb{C}$, the inverse image is

$$\phi_n^{-1}(z) = \underbrace{\{\sqrt[n]{r} \exp(is/n), \sqrt[n]{r} \exp(is/n) \exp(2\pi i/n), \dots, \sqrt[n]{r} \exp(is/n) \exp((n-1)\pi i/n)\}}_n. \tag{A.17}$$

The open balls B_k , centered at $z_0 = \sqrt[n]{r} \exp(is/n) \exp(k\pi i/n)$, $k = \{0, 1, \dots, n-1\}$, and with radius $\epsilon = \sqrt[n]{r} \sin(\pi/n)$, are the corresponding sheets. They are mapped homeomorphically onto their common image. Since z is arbitrary, ϕ_n is a covering map.

For later purposes the following definition is useful.

Definition A.18. Let $\varphi : M \rightarrow N$ be a covering map between topological manifolds, and suppose, for L a topological manifold, $\psi : L \rightarrow N$ is a diffeomorphism. If there is a map $\pi : L \rightarrow M$ such that $\pi = \psi \circ \phi$, such a map is called a **lift**.

The situation is depicted in Figure A.2. Lemma A.19 shows that such a lift is determined by a single point.

Lemma A.19. ([10], p. 125) *Let $\varphi : M \rightarrow N$ be a covering map between topological manifolds, $\psi : L \rightarrow N$ be a diffeomorphism from a connected manifold L and $\pi_1 : L \rightarrow M$ and $\pi_2 : L \rightarrow M$ be two lifts. If $\pi_1(y) = \pi_2(y)$ for some $y \in L$, then $\pi_1 = \pi_2$.*

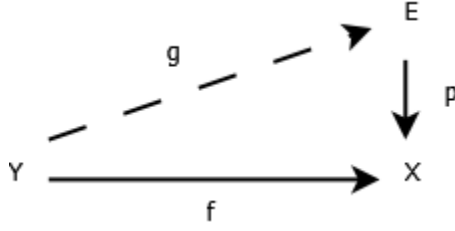


Figure A.1: Depiction of path lift.

Proof. ([10], p. 126) We define the following sets:

$$\begin{aligned} T &= \{y \in L \mid \pi_1(y) = \pi_2(y)\}, \\ S &= \{y \in L \mid \pi_1(y) \neq \pi_2(y)\}. \end{aligned} \tag{A.18}$$

Since both $T \cup S = L$ and $T \cap S = \emptyset$ we have that if both T and S are open, the connectedness of L implies either $T = L$ or $S = L$. Hence we show that both sets are open.

Let $x \in T$ and U be an open evenly covered neighbourhood of $\psi(x)$. Then $\psi^{-1}(U)$ will consist of disjoint sheets, containing $\pi_1(x)$ and $\pi_2(x)$. Since $x \in T$, $\pi_1(x) = \pi_2(x)$ and $\pi_1(x) = \pi_2(x)$ is in some sheet, which we denote V_x . As π_1 and π_2 are continuous, the inverse images of open sets are open. The inverse open sets of the sheet V_x intersects to form an open set: $\pi_1^{-1}(V_x) \cap \pi_2^{-1}(V_x) = W$ and $y \in W$. Furthermore, π_1 and π_2 restricted to W must coincide, i.e. $\pi_1 = \pi_2|_W$. Indeed it must be so, as otherwise there would be $z \in W$ such that $\pi_1(z) \neq \pi_2(z)$ but still $\pi_1(z), \pi_2(z) \in V_x$. Restricted to each sheet, φ is a homeomorphism, and so $\varphi \circ \pi_1(z) \neq \varphi \circ \pi_2(z)$ from the bijectivity of φ . But π_1 and π_2 are lifts, and so must satisfy $\varphi \circ \pi_1(z) = \varphi \circ \pi_2(z) = \psi(z)$ and so we would have a contradiction. Hence $W \subset T$ so that T is open.

To show that S is open, we let $x \in S$. We again let U contain x and be evenly covered. As $\pi_1(x) \neq \pi_2(x)$ they must be in distinct sheets, V_1 and V_2 . Again, π_1 and π_2 are continuous, and so have open inverse images. Both $\pi_1^{-1}(V_1)$ and $\pi_2^{-1}(V_2)$ contain x and their intersection $\pi_1^{-1}(V_1) \cap \pi_2^{-1}(V_2) = W$ is open. Now it must be so that $W \subset S$. Otherwise, for some $z \in W$ we would have $\pi_1(z) = \pi_2(z)$. But then this element would be in the intersection of V_1 and V_2 . These are two disjoint sets by the properties of the covering map, and so we have contradiction. Hence also S is open. \square

Theorem A.20. ([10], p. 126, "The path-lifting Theorem") Let $\varphi : M \rightarrow N$ be a covering map between topological manifolds and suppose $\gamma : [0, 1] \rightarrow N$ is a continuous curve with $\gamma(0) = p$. Then, if $q \in \varphi^{-1}(p)$, there is a unique path lift curve $\tilde{\gamma} : [0, 1] \rightarrow M$ such that $\tilde{\gamma}(0) = q$ and $\varphi(\tilde{\gamma}) = \gamma$.

Proof. ([10], p. 126) Let $x \in X$ be a point and U_x be an evenly covered neighbourhood of x . Then the sets $\{U_x\}_{x \in X}$ form an open covering of X . As γ is continuous, the inverse images $\gamma^{-1}(U_x)$ are also open and form an open covering of $[0, 1]$. The interval $[0, 1]$ is compact, and hence we may choose finitely many $\gamma^{-1}(U_{x_i})$ to cover it. We choose points $0 = s_0 < s_1 < \dots < s_N = 1$ such that each s_i lie in the intersection $\gamma^{-1}(U_{x_{i-1}}) \cap \gamma^{-1}(U_{x_i})$. This means that $\gamma([s_{i-1}, s_i]) \subset U_{x_i}$.

We now lift the curve piece wise. We start at $\gamma(0) = \gamma(s_0)$. Since $\gamma(s_0)$ is evenly covered by U_{x_0} we consider the sheet $\varphi^{-1}(U_{x_0})$ that contains q and denote it W_q . Since φ is a homeomorphism on W_q , the curve segment $\gamma([s_0, s_1])$ is mapped to a curve in W_q , starting at q . Inductively, we continue with the next curve segment $\gamma([s_1, s_2])$. Again, it is contained in the evenly covered neighbourhood U_{x_1} . We consider the sheet that meets the endpoint of $\varphi^{-1}(\gamma(s_0, s_1))$. Again φ is a homeomorphism on the sheet and we may extend the curve to be defined on the interval $[s_0, s_2]$. This process is repeated N times, and hence the curve is lifted to the topological space M .

To show uniqueness we simply apply Lemma A.19. The curve $\tilde{\gamma}$ may be seen as the lift, and $\gamma = \varphi \circ \tilde{\gamma}$ and hence if the lifts were ever to coincide, e.g. in the starting point, they would be the same. \square

Not only paths may be uniquely lifted, as the above theorem showed. In fact, entire homotopy maps may be lifted.

Theorem A.21. ([10], p. 127) Let $\varphi : M \rightarrow N$ be a covering map between topological manifolds. Suppose $F(s, t) : [0, 1] \times [0, 1] \rightarrow N$ is a homotopy map with $F(0, 0) = p$. Then, for $q \in \varphi^{-1}(p)$, there exists a unique homotopy map $G(s, t) : [0, 1] \times [0, 1] \rightarrow M$ such that $G(0, 0) = q$ and $\varphi(G) = F$.

Proof. ([10], p. 127) The uniqueness follows from Lemma A.19.

By Theorem A.20 there is a unique path $\tilde{\gamma} : [0, 1] \rightarrow M$ with $\tilde{\gamma}(0) = e$ such that $\varphi(\tilde{\gamma})(t) = F(0, t)$. In the same manner, for each t , we may lift F in the other variable, i.e. there is a map $G(s, t)$ such that $G(0, t) = \tilde{\gamma}(t)$ and $\varphi(G)(s, t) = F(s, t)$. This defines the desired lift. We need only to demonstrate continuity.

We define the curve $\hat{\gamma}(s) = F(s, 0)$. We use the same notation as in the proof of Theorem A.20. We define U_x to be some evenly covered neighbourhood, for each $x \in M$. We know that $\hat{\gamma}^{-1}(U_x)$ is open in $[0, 1]$, and more precisely an open neighbourhood of $[s_{j-1}, s_j]$. It follows that $F^{-1}(U_j)$ must include $[s_{j-1}, s_j] \times [0, \epsilon]$. That is, we have $F([s_{j-1}, s_j] \times [0, \epsilon]) \subset U_j$. We have $e \in V_1$ and therefore $\hat{\gamma}(s) \in V_1$ for $s \in [0, \epsilon]$. From what we have constructed we now must have

$$G = (\varphi|_{V_1})^{-1} \circ F, \quad \text{on } [0, s_1] \times [0, \epsilon] \quad (\text{A.19})$$

with G continuous on $[0, s_1] \times [0, \epsilon]$. We proceed in this fashion to find that G is continuous on every rectangle $[s_{j-1}, s_j] \times [0, \epsilon]$.

The same argument holds for any $t_0 \in (0, 1]$, so that G is continuous both on any $[s_{j-1}, s_j] \times [t_0 - \epsilon, t_0 + \epsilon]$ and on any $[s_{j-1}, s_j] \times [1 - \epsilon, 1]$ and so G is continuous on $[0, 1] \times [0, 1]$. \square

Theorems A.20 and A.21 have important consequences. Suppose that $\varphi : M \rightarrow N$ is a covering map between connected topological manifolds. Let $\gamma : [0, 1] \rightarrow N$ be a loop with $\gamma(0) = \gamma(1) = p$ and consider a lifted curve $\tilde{\gamma} : [0, 1] \rightarrow M$ with $\tilde{\gamma}(0) = q$ and $\varphi(q) = p$. Although the curve may start at q , in order for it to be mapped into a loop through φ it must not end where it started, only somewhere in $\varphi^{-1}(p)$. If such a curve ends at q , then the loop will be in the trivial equivalence class of $\pi_1(M)$, and so by Theorem A.21 will be mapped down into the trivial equivalence class of $\pi_1(N)$. This suggests the following mapping:

$$\varphi_q : \pi_1(N, p) \rightarrow \phi^{-1}(p), \quad \text{by} \quad \varphi_q([\gamma]) = \tilde{\gamma}(1), \quad (\text{A.20})$$

i.e. φ_q maps the equivalence class $[\gamma]$ to the end-point of the lifted curve $\tilde{\gamma}$ that starts at q . We get the following result:

Lemma A.22. ([10], p. 128) If M is simply connected, the map φ_q is a bijection.

Proof. ([10], p. 128) For any $y \in \varphi^{-1}(p)$ we let $\tilde{\gamma}$ be a curve from q to y . Then the composition $\gamma = \varphi \circ \tilde{\gamma}$ is a curve in N that starts and ends at p , i.e. it is a loop. Therefore $\varphi_q([\gamma]) = \tilde{\gamma}(1) = y$ and so φ_q is onto.

To show injectivity, we suppose $\varphi_q([\gamma_1]) = \varphi_q([\gamma_2])$, where γ_1 and γ_2 are loops based at p . We let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be lifts of γ_1 and γ_2 respectively, that start at q . Since $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have the same terminal point, we may construct the curve $\tilde{\gamma}_1\tilde{\gamma}_2^{-1}$, which is a loop based at q . Since M is simply connected, there is a map $F : [0, 1] \times [0, 1]$ that contracts it down to the trivial curve. Therefore we have $[\tilde{\gamma}_1][\tilde{\gamma}_2]^{-1} = [q]$ and so $[\tilde{\gamma}_1] = [\tilde{\gamma}_2]$ so that φ_q is injective.

Hence φ_q is a bijection. \square

Definition A.23. Let $\varphi : M \rightarrow N$ be a covering map between topological manifolds. A diffeomorphism $k : M \rightarrow M$ is called a **deck transformation** if $\varphi \circ k = \varphi$. If, for any two points p and q in $\varphi^{-1}(x)$, there exists some $k \in \mathbb{D}$ such that $k(p) = q$, \mathbb{D} is called **normal**.

Since the identity mapping is trivially a deck transformation. If k is a deck transformation, then clearly k^{-1} is also. Compositions are also deck transformations and so it follows that the set of deck transformations form a group under composition.

The following is an immediate consequence of Theorem A.24.

Corollary A.24. Let $\varphi : M \rightarrow N$ be a covering map between two topological manifolds and suppose there exists a normal deck group \mathbb{D} on M . If M is simply connected, then $\pi_1(N) \simeq \mathbb{D}$.

Proof. In light of Theorem A.24, we need only to show that the map φ_q may be extended to become a homomorphism. We define

$$\begin{aligned} \psi_q : \pi_1(N, p) &\rightarrow \mathbb{D}, & \text{by } \psi_q : [\gamma] &\rightarrow \varphi_q([\gamma]) = \gamma(1) \rightarrow k, \\ \text{where } k \in \mathbb{D} & & \text{such that } k(q) &= \gamma(1). \end{aligned} \quad (\text{A.21})$$

Since \mathbb{D} is normal, the map ψ_q is well-defined. We consider two curve classes $[\alpha]$ and $[\beta]$ of $\pi_1(N, p)$. Denote the associated lifts, starting at q , by $\tilde{\alpha}$ and $\tilde{\beta}$ respectively. The curve product $[\alpha][\beta] = [\alpha\beta]$ has representative $\alpha\beta$ whose lift $\tilde{\alpha}\tilde{\beta}$ starts at q and ends up at $\alpha\beta(1)$. We denote by $k_\alpha(q) = \tilde{\alpha}(1)$ and $k_\beta(q) = \tilde{\beta}(1)$ the elements of the deck group \mathbb{D} that maps the starting point to the respective endpoints. We must only show that $k_\alpha \circ k_\beta = k_{\alpha\beta}$, where $k_{\alpha\beta}(q) = \alpha\beta(1)$.

The curve $\alpha\beta$ is α for $0 \leq t < 1/2$ and β for $1/2 < t \leq 1$. Any lifted representative $\tilde{\gamma}$ remains projected into the same equivalence class when shifted through \mathbb{D} . Hence, we must have $\varphi(k_\alpha(\beta)) = \beta$. But then the lifted curve $\tilde{\alpha}\tilde{\beta}$ starts at q , passes through $\tilde{\alpha}(1)$ and moves through $k_\alpha(\beta)$ to $k_\alpha(\beta(1))$. This end point is both $\tilde{\alpha}\tilde{\beta}(1)$ and $k_\alpha \circ k_\beta(q)$. Hence there must be

$$\psi_q([\alpha][\beta]) = \begin{cases} \tilde{\alpha}\tilde{\beta}(1) = k_{\alpha\beta}(q), \\ k_\alpha \circ k_\beta(1) = k_\alpha \circ k_\beta(q) \end{cases} \quad (\text{A.22})$$

and so ψ_q is a homomorphism.

Since, by Lemma A.22, ψ_q is also bijective, it must be an isomorphism. \square

Proposition A.25. *The fundamental group of S^1 is $\pi_1(S^1) \simeq \langle \mathbb{Z}, + \rangle$, i.e. the infinite cyclic group.*

Proof. We have already shown that the map $\exp : \mathbb{R} \rightarrow S^1$ is a covering map. In order for a deck transformation to satisfy $\exp \circ k = \exp$ we must have $\exp(2\pi i s_0) = \exp(2\pi i k(s_0))$. This happens if and only if $k(s_0) = s_0 + n$, $n \in \mathbb{Z}$. Denote $k_n(s_0) = s_0 + n$. Compositions of such mappings have $k_n \circ k_m(s_0) = s_0 + (m+n)$. Thus, the mapping $\varphi(n) = k_n$ is an isomorphism from $\langle \mathbb{Z}, + \rangle \rightarrow \mathbb{D}$. Since \mathbb{R} is simply connected, the fundamental group $\pi_1(S^1)$ must be isomorphic to the deck transformation group, which was just the infinite cyclic group. \square

Proposition A.26. *The fundamental group of S^n , $n > 1$, is trivial, i.e. S^n is simply connected.*

Proof. We let $\gamma : [0, 1] \rightarrow S^n$ be a loop based at $x_0 \in S^n$. Let x be a point in S^n that is not in the image of γ . From rotational symmetry we may assume that $x = (0, \dots, 0, 1)$, i.e. the north pole. We use the stereographic projection $P : S^n - \{x\} \rightarrow \mathbb{R}^n$, explicitly given by

$$P : (x_1, \dots, x_{n+1}) \rightarrow \left(\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}} \right). \quad (\text{A.23})$$

We show that P is a map onto \mathbb{R}^n . Clearly the components are continuous. We show that P is injective. Let $y = (y_1, \dots, y_{n+1})$ and $z = (z_1, \dots, z_{n+1})$. If $P(y) = P(z)$, we have

$$\left(\frac{z_1}{1-z_{n+1}}, \frac{z_2}{1-z_{n+1}}, \dots, \frac{z_n}{1-z_{n+1}} \right) = \left(\frac{y_1}{1-y_{n+1}}, \frac{y_2}{1-y_{n+1}}, \dots, \frac{y_n}{1-y_{n+1}} \right). \quad (\text{A.24})$$

Furthermore, $P(y)$ and $P(z)$ have the same norm:

$$\sum_i^n \frac{y_i^2}{(1-y_{n+1})^2} = \sum_i^n \frac{z_i^2}{(1-z_{n+1})^2} \quad (\text{A.25})$$

and since both $y, z \in S^n$ we have

$$\sum_i^{n+1} x_i^2 = 1 \quad \text{and} \quad \sum_i^{n+1} y_i^2 = 1. \quad (\text{A.26})$$

With this we get

$$\sum_i^n \frac{y_i^2}{(1-y_{n+1})^2} = \frac{1-y_{n+1}^2}{(1-y_{n+1})^2} \quad (\text{A.27})$$

and similarly for z , which in turn gives the equality

$$\frac{1 - y_{n+1}^2}{(1 - y_{n+1})^2} = \frac{1 - z_{n+1}^2}{(1 - z_{n+1})^2} \quad (\text{A.28})$$

from which it follows that $y_{n+1} = z_{n+1}$. Since we have equality of components

$$\frac{z_i}{1 - z_{n+1}} = \frac{y_i}{1 - y_{n+1}} \quad (\text{A.29})$$

the injectivity now follows.

To show surjectivity, we suppose $z = (z_1, \dots, z_n) \in \mathbb{R}^n$. Then

$$x = \left(2 \frac{z_1}{z_1^2 + \dots + z_n^2 + 1}, \dots, 2 \frac{z_n}{z_1^2 + \dots + z_n^2 + 1}, \frac{z_1^2 + \dots + z_n^2 - 1}{z_1^2 + \dots + z_n^2 + 1} \right) \quad (\text{A.30})$$

belongs to S^n and satisfies $P(x) = z$. The first assertion follows, as

$$\begin{aligned} \|x\|^2 &= \sum_i^n \frac{4z_i^2}{(z_1^2 + \dots + z_n^2 + 1)^2} + \frac{(z_1^2 + \dots + z_n^2 - 1)^2}{(z_1^2 + \dots + z_n^2 + 1)^2} \\ &= \sum_i^n \frac{4z_i^2}{(z_1^2 + \dots + z_n^2 + 1)^2} + \frac{((z_1^2 + \dots + z_n^2)^2 - 2 \sum_i^n 2z_i^2 + 1)}{(z_1^2 + \dots + z_n^2 + 1)^2} \\ &= \frac{((z_1^2 + \dots + z_n^2)^2 + 2 \sum_i^n 2z_i^2 + 1)}{(z_1^2 + \dots + z_n^2 + 1)^2} \\ &= \frac{(z_1^2 + \dots + z_n^2 + 1)^2}{(z_1^2 + \dots + z_n^2 + 1)^2} \\ &= 1. \end{aligned} \quad (\text{A.31})$$

The second follows as, component-wise, we have

$$1 - x_{n+1} = 1 - \frac{z_1^2 + \dots + z_n^2 - 1}{z_1^2 + \dots + z_n^2 + 1} = \frac{2}{z_1^2 + \dots + z_n^2 + 1}, \quad (\text{A.32})$$

so that

$$P(x) \cdot e_i = \frac{x_i}{1 - x_{n+1}} = 2 \frac{z_i}{z_1^2 + \dots + z_n^2 + 1} \frac{z_1^2 + \dots + z_n^2 + 1}{2} = z_i \quad (\text{A.33})$$

and so P is surjective.

With this we have the homomorphism $S^n - \{x\} \simeq \mathbb{R}^n$ and Lemma A.14 shows that $\pi_1(S^n - \{x\}) = \pi_1(\mathbb{R}^n)$. Example A.10 shows that $\pi_1(\mathbb{R}^n) = \{e\}$ and so it follows that the loop γ is continuously contractible to the trivial loop based at x_0 . Hence S^n is simply connected. \square

Proposition A.27. *The fundamental group of $\mathbb{R}^2 - \{0\}$ is $\pi_1(\mathbb{R}^2 - \{0\}) \simeq \langle \mathbb{Z}, + \rangle$.*

Proof. We work with the topologically equivalent space $\mathbb{C} - \{0\}$ instead and consider the mapping $\exp : z \rightarrow e^z$. For an arbitrary complex number w , we have $\exp^{-1}(w) = \ln(w) + 2\pi in$, $n \in \mathbb{Z}$. This shows, that if we choose the sheets

$$U = \bigcup_{n \in \mathbb{Z}} U_n, \quad U_n = B_\pi(\ln(w) + 2\pi in), \quad (\text{A.34})$$

the exponential becomes a covering map from $\mathbb{C} \rightarrow \mathbb{C} - \{0\}$. A deck transformation k must satisfy $\exp \circ k = \exp$ and so k is a deck transformation if and only if $k_n(z) = z + 2\pi in$. Composition is additive, i.e. $k_m \circ k_n(z) = z + 2\pi i(m + n)$. The full complex plane is simply connected and hence the deck transformation group is isomorphic to $\langle \mathbb{Z}, + \rangle$. \square

With the following lemmas we aim to prove existence of a universal cover.

Lemma A.28. *Every connected topological manifold M is path-connected.*

Proof. We pick a point $p \in M$ and consider the set subset $N \subset M$ consisting of the points that are path-connected to p . We want to show that this set is the full of M , and to do so it would suffice to show that N is both open and closed.

We first show that N is open: We pick an arbitrary point q in N and take a chart (U_q, ϕ) containing it. Then $\phi(U_q) \subset \mathbb{R}^n$ is open and \mathbb{R}^n is path connected. Hence, for some ball $B_\epsilon(\phi(q)) \subset \phi(U_q)$, all points contained can be joined to q with a straight line. Since $\phi^{-1}(B_\epsilon(\phi(q)) \subset \phi(U_q)) \subset U_q$ it follows that this set is now connected to q , and since q is path connected to p , so is the point. Hence q is an interior point of N and so N is open.

We show that N is closed: We show the complement $M - N$ is open. Let $q \notin N$. Then, again, there is a chart (U_q, ϕ) containing q and the open set $\phi(U_q) \subset \mathbb{R}^n$ contains some open ball $B_\epsilon(\phi(q))$. This open ball is of course path connected in \mathbb{R}^n , and hence so is the open ball $\phi^{-1}(B_\epsilon(\phi(q)))$. Since q was not path connected to p , nor can any element in the ball $\phi^{-1}(B_\epsilon(\phi(q)))$ be, and hence $\phi^{-1}(B_\epsilon(\phi(q))) \not\subset N$. Thus the complement of N is open. \square

Definition A.29. For a topological space (M, τ) , we say that it is **locally path-connected** if, for each $p \in M$ and neighbourhood U containing p , there is a subset $V \subset U$ that is path-connected.

Lemma A.30. *Every topological manifold M is locally path-connected.*

Proof. It suffices to show that for $p \in M$ and neighbourhood U containing p , U is path-connected. This follows by similar arguments to those in the proof of Lemma A.28. Every point in U is contained in some chart (U_α, ϕ_α) . Since any open and connected subset of \mathbb{R}^n is also path-connected, we know that $\phi_\alpha(U_\alpha)$ is path-connected, and hence that U_α is. Since U is the union of intersections of the type $U \cap U_\alpha$, that are path-connected, and since U is connected, it follows that also U is path-connected. \square

Definition A.31. A topological space (M, τ) is called **semi-locally 1-connected** if every point $p \in M$ has some neighbourhood U such that any loop, contained in U , starting at p , may be continuously deformed (within U) to the trivial map.

Lemma A.32. *A topological manifold (M, τ) is semi-locally 1-connected.*

Proof. We use the fact that open sets in \mathbb{R}^n have this property. Indeed, if U is open in \mathbb{R}^n and $x \in U$, then there is some ball $B_\epsilon(x) \subset U$. Let $\gamma(t)$ be a loop in $B_\epsilon(x)$ starting and ending at x . Then the homotopical map $F(s, t) = x + s(\gamma(t) - x)$ continuously deforms γ down to the trivial curve at x .

This follows naturally to the manifold setting. For $p \in M$ we take any chart (U_α, ϕ_α) that contains p , and consider the image. The image $\phi_\alpha(U_\alpha)$ is an open subset in \mathbb{R}^n and so there is some ball $B_\epsilon(\phi_\alpha(p))$ that deforms loops within it to the trivial curve. The inverse image $\phi_\alpha^{-1}(B_\epsilon(\phi_\alpha(p)))$ has this property since ϕ_α is a homeomorphism. \square

We are now ready to prove existence of a universal covering space. The following theorem is stated, but not proven in the sources.

Theorem A.33. *([20],[22]) Every connected topological manifold M has a simply connected cover.*

Proof. We consider all curves on M and their equivalence classes. Denote the space of such equivalence classes by \tilde{M} . We now show that this space has a naturally inherited topology, is simply connected and there exists a covering map.

We show existence of a topology. For a curve class $[\gamma]$ in \tilde{M} and neighbourhood U containing $\gamma(1)$, we define $U([\gamma])$ as the elements $[\alpha]$, that are homotopically equivalent to some curve that follows γ and then stays in U . We denote the topology of such open sets $\tilde{\tau}$.

This is first of all well defined; if $U([\gamma])$ contains $[\alpha]$, then $U([\gamma]) = U([\alpha])$. This follows, as by construction α , (which we pick as representative of $[\alpha]$), first follows γ and then some curve $\tilde{\gamma}$ that lies in U . But then conversely $[\gamma]$ has a representative γ that follows α and then $\tilde{\gamma}^{-1}$ and so the definition is well defined.

We show that this space becomes a topological space by showing that these neighbourhoods form a basis for a topology. By construction, they form a cover. We need only to show, that for each two non-trivially intersecting sets $U_1([\gamma_1])$ and $U_2([\gamma_2])$ we have some element $W([\alpha])$ in the

intersection. We now show that the full intersection $U_1([\gamma_1]) \cap U_2([\gamma_2])$ already has this property: suppose $[\alpha] \in U_1([\gamma_1]) \cap U_2([\gamma_2])$. Then on the one hand, $[\alpha]$ has a representative that follows γ_1 and then stays in U_1 , but on the other hand some representative follows γ_2 and then stays in U_2 . Hence the endpoint $\alpha(1) \in U_1 \cap U_2 \equiv W$. We need only to show, that any other element $[\beta] \in W([\alpha])$ has a representative that follows α and then stays in W . But if $[\beta]$ lies in $U_2([\gamma_2])$, then it has a representative β that follows γ_2 and then stays in U_2 , i.e. $\beta = \tilde{\beta} \circ \gamma_2$. Meanwhile the same goes for α ; $\alpha = \tilde{\alpha} \circ \gamma_2$. Finally since Lemma A.30 there is a curve δ in W with $\delta(0) = \alpha(1)$ and $\delta(1) = \beta(1)$. We want to show that $\delta \circ \alpha = \beta$. But this follows as the loops $\alpha^{-1} \circ \tilde{\alpha} \circ \gamma_2$ and $\beta^{-1} \circ \tilde{\beta} \circ \gamma_2$ are both homotopically trivial, from which it follows that the loop $\tilde{\beta}^{-1} \circ \delta \circ \tilde{\alpha}$ is homotopically trivial and therefore $\delta \circ \alpha = \beta$.

Hence we have a basis for a topology and so $(\tilde{M}, \tilde{\tau})$ is a topological space.

We now show that the mapping $\varphi : \tilde{M} \rightarrow M$, $\varphi([\gamma]) = \gamma(1)$ is a covering map. We show existence of evenly covered neighbourhood of any $p \in M$. By Lemma A.32 there is a neighbourhood U_p for which any loop at p in U_p is homotopically trivial. Let $[\gamma]$ be some class with $\gamma(1) = p$. Then we claim that φ restricted to $U_p([\gamma])$ is bijective. The surjectivity follows, as by Lemma A.30 any open connected neighbourhood is path connected. To show injectivity, we suppose $\varphi([\alpha]) = \varphi([\beta])$ in $U_p([\gamma])$. Then each class has a representative α and β respectively, that first follow γ and then $\tilde{\alpha}$ and $\tilde{\beta}$ respectively. Since these classes are not the same, the compositions $\tilde{\alpha} \circ \gamma$ and $\tilde{\beta} \circ \gamma$ cannot be equivalent. Hence $\tilde{\beta}$ and $\tilde{\alpha}$ cannot be equivalent. But they share both end-points and starting points (p) and so the composition $\tilde{\beta}^{-1} \circ \tilde{\alpha}$ is a loop in $U_p([\gamma])$. By hypothesis any loop could be deformed down to the trivial one, and this is equivalent to $\tilde{\alpha} \simeq \tilde{\beta}$ and we are contradicted. Hence U_p is evenly covered.

It remains only to show simple connectivity of \tilde{M} . To do so we let $[\sigma] \in \pi_1(\tilde{M}, \tilde{p})$ be an equivalence class of loops with base \tilde{p} in \tilde{M} . We let $\varphi(\tilde{p}) = p$ and let \tilde{p} be the trivial curve at p in M . We want to show that $[\sigma]$ is trivial. We let σ represent $[\sigma]$ and since σ is a curve in \tilde{M} , we have that $\varphi(\sigma)$ is a curve in M . $\varphi(\sigma)$ starts and ends at p . Both σ and the lift $\varphi^{-1}(\varphi \circ \sigma)$ that starts at \tilde{p} are curves in \tilde{M} . By Lemma A.21, they are the same.

On the one hand, we have $\sigma \in \pi_1(\tilde{M}, \tilde{p})$ so that $\sigma(1) = \tilde{p}$, the constant curve at \tilde{p} . But on the other we have $\sigma(1) = [\varphi(\sigma)]$. Then $\varphi(\sigma)$ must be trivial in $\pi_1(M, p)$ and so $[\sigma]$ must also be trivial. Hence $\pi_1(\tilde{M}, \tilde{p}) \simeq \{e\}$ and \tilde{M} is simply connected. \square

The simply connected covering space is usually called the **universal cover**. By taking compositions of the covering map with the local charts of M , we naturally equip \tilde{M} with an atlas so that it becomes a manifold. The following properties are immediate.

Proposition A.34. *The deck group of the universal covering space \tilde{M} is normal.*

Proof. ([20], p. 444) Let \tilde{p} and \tilde{q} be in \tilde{M} such that $\varphi(\tilde{p}) = \varphi(\tilde{q})$ and let \tilde{x} be any point not \tilde{p} or \tilde{q} such that $\varphi(\tilde{x}) \neq \varphi(\tilde{p})$. Then there is a curve γ that travels from \tilde{p} to \tilde{x} , by path-connectedness. Clearly $\varphi(\gamma)$ is a curve that travels from p to $\varphi(\tilde{x}) = x$. Since \tilde{q} is in $\varphi^{-1}(p)$ we may by Lemma A.20 lift this curve back up to \tilde{M} but with starting point \tilde{q} . We denote this curve by β . We define $k(p) = \beta(1)$. Since this procedure is continuous in each step and \tilde{M} is simply connected, it follows that k is a diffeomorphism and the assertion is shown. \square

Proposition A.35. *The universal cover is unique up to diffeomorphism.*

Proof. See [20], p. 444. \square

We state a constraint given by topological properties of this universal cover.

Lemma A.36. *Suppose M is a manifold and \tilde{M} its universal cover. If \tilde{M} is compact, then the deck group \mathbb{D} is finite.*

Proof. We let $\varphi : \tilde{M} \rightarrow M$ and suppose the assertion is false. Then, for $p \in M$, we have that $\varphi^{-1}(p) \subset \tilde{M}$ would consist of infinitely many points. Since \tilde{M} is compact, there is a convergent subsequence $\{\tilde{p}_n\}$ converging to some $\tilde{p} \in \tilde{M}$. This contradicts the fact that the orbit $\varphi^{-1}(p)$ is the closed union of isolated points. \square

Bibliography

- [1] A. Arvanitoyeorgos, *An Introduction to Lie Groups and Geometry of Homogeneous Spaces*, AMS (2003).
- [2] G. S. Birman, K. Nomizu, *The Gauss-Bonnet theorem for 2-dimensional spacetimes*, Michigan Math. J. **31** (1984), 77-81.
- [3] G. S. Birman, K. Nomizu, *Trigonometry in Lorentzian geometry*, Am. Math. Mon. **91**, (1984), 543-549.
- [4] K. Brown, *Cohomology of Groups*, Springer-Verlag (1982).
- [5] E. Calabi, L. Markus, *Relativistic space forms*, Ann. of Math. **75** (1962), 63-76.
- [6] Y. Carrière, *Autour de la conjecture de L. Markus sur les variétés affines*, Invent. Math. **95** (1989), 615-628.
- [7] M. do Carmo, *Differential forms and applications*, Springer-Verlag (1994).
- [8] M. do Carmo, *Riemannian Geometry*, Birkhäuser (1992).
- [9] D. Fried, *Polynomials on affine manifolds*, Trans. Am. Math. Soc. **274**, 709-719 (1982).
- [10] T. W. Gamelin, R. E. Greene, *Introduction to Topology*, Dover Publications (1999).
- [11] W. Goldman, *Geometric structures on manifolds and varieties of representations*, Contemp. Math. **74** (1988), 169-198.
- [12] W. Goldman, *Geometric structures on manifolds*, <http://www.math.umd.edu/~wmg/gstom.pdf>
- [13] W. Goldman, *Locally homogeneous geometric manifolds*, Hindustan Book Agency, New Delhi (2010), 717-744.
- [14] W. Goldman, M. Hirsch, *The radiance obstruction and parallel forms on affine manifolds*, Trans. Amer. Math. Soc. **286** (1984), 629-649.
- [15] S. Gudmundsson, *An Introduction to Riemannian Geometry*, <http://www.matematikk.lu.se/matematikklu/personal/sigma/Riemann.pdf>
- [16] H. Hopf, W. Rinow, *Über den Begriff der vollständigen differentialgeometrischen Fläche*, Comment. Math. Helv. **3**, (1931) 209-225.
- [17] B. Klingler, *Complétude des variétés lorentziennes à courbure constante*, Math. Ann. **306** (1996), 353-370.
- [18] A. Knapp, *Lie groups beyond an introduction. 2nd ed.*, Progress in Mathematics **140**. Birkhäuser Boston, MA (2002).
- [19] M. Morill, *Nonexistence of compact de Sitter manifolds*, PhD dissertation, University of California, Los Angeles, (1996).
- [20] B. O'Neill, *Semi-Riemannian Geometry*, Pure and Applied Mathematics **103**, Academic Press, New York (1983).

- [21] F. Palomo, A. Romera, *Certain actual topics on modern Lorentzian geometry*, Elsevier (2006), 513-546.
- [22] F. Warner, *Foundations of differentiable manifolds and Lie groups*, Corrected reprint of the 1971 edition, Graduate Texts in Mathematics **94** Springer (1983).

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