

A Note on Chvátal's Conjecture

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Abstract

The main purpose of this thesis is to treat and clarify two results found by Hunter Snevily on the open problem of Chvátal's conjecture. Chvátal's conjecture states that a certain class of sets, called ideals, always have a maximum intersecting family of subsets, where all members contain a common element. Such a family of subsets is called a star. Snevily presents his results in an article, the structure of which will be followed when presenting and proving both his and earlier acquired results, constituting the basis of his conclusions.

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1 Introduction

Extremal set theory is a branch within extremal combinatorics, the latter applicable in computer science and in many aspects coinciding with the field of extremal graph theory, wherein Václav Chvátal has contributed extensively. Extremal set theory studies the minimal and maximal cardinalities of classes of sets, if these have to satisfy certain restrictions. The still open conjecture of Chvátal, treating such a class called ideals or sets with the hereditary property and the associated results presented by the late Hunter Snevily in [6], are the main objects of this thesis. We are to examine the results presented in [6] thoroughly, together with the results acquired by Claude Berge, Peter Stein and Jochanan Schönheim, among others, leading up to Snevily's conclusions. We begin by stating some definitions and the conjecture itself.

Definition 1.1 Let X be a finite set. A collection I of subsets of X is called an ideal if $A \in I, B \subseteq A \Rightarrow B \in I$. We call the elements in any collection of subsets members.

Definition 1.2 An intersecting subfamily G of I is a collection of subsets of I such that no two members of G are disjoint.

Definition 1.3 An intersecting subfamily S of I is called a star if there is an element $x \in X$ such that $x \in A$ for all $A \in S$.

Conjecture 1.4 (Chvátal [3]). If I is an ideal of the finite set X , then among the maximum intersecting subfamilies of I , there is a star.

By definition, the cardinality of a star is less than or equal to the cardinality of a maximum intersecting subfamily of I . Chvátal conjectured that if we search for a largest possible intersecting subfamily of an ideal, we can limit ourselves to the search for a star. The conjecture has not been proved in general yet, but partial results have been obtained for ideals satisfying certain restrictions. The problem of coming up with a general proof is still open and a cash prize has been offered by Chvátal for the solution.

Two interesting results are the ones presented by Snevily in [6], the article which can be considered to constitute the basis of this thesis. Snevily proved that every ideal containing a so-called dominant element satisfies the conjecture. He also proved that every ideal for which the maximum members can be partitioned into two stars, in which the pairwise intersections between the bases are all equal, also satisfies the conjecture. Before presenting these results in greater detail, we observe some other results and definitions that one may find interesting.

1.1 Chvátal's Result

Chvátal [3] proved himself that I satisfies the conjecture if the elements of the finite set X are given a linear order and I has the property that that $\{a_1, a_2, \dots, a_n\} \in I$ and $b_i \leq a_i$ for all i , implies that $\{b_1, b_2, \dots, b_n\} \in I$. This statement can also be formulated in the following way. I satisfies the conjecture if I has the property that if $A \in I$ and there is a bijection f from A to B such that

$$f(a) \leq a, \quad a \in A$$

then $B \in I$.

1.2 Wang's and Wang's Result

Wang and Wang [9] generalized the above result in the following way. Consider a tree T with root r and leaves $a_1, \dots, a_n$ and let v be any node in T . Denote the set of nodes in the unique path from r to v by $[r, v]$ and define $x \leq y$ if $[r, x] \subseteq [r, y]$. An ideal I is said to have hereditary tree structure if

- (a) every set of I is contained in $[r, a_i]$ for some a_i ;
- (b) if $\{x_1, \dots, x_n\} \in I$, and $y_i \leq x_i$, $i = 1, \dots, n$, then $\{y_1, \dots, y_n\} \in I$.

Wang and Wang proved that if I has hereditary tree structure T , then Chvátal's conjecture is true. We omit the proofs of Chvátal's and Wang and Wang's results here and refer the interested reader to the references made.

2 Ideals and Dominant Elements

As mentioned above, Snevily proved the conjecture when I has a so-called dominant element. We present that result in this section.

2.1 Berge's Theorem

To present and prove the first main result of Snevily on the conjecture, we are in need of the following theorem of Berge and some definitions.

Theorem 2.1 (Berge [2]). If I is an ideal of a finite set X , then I is the disjoint union of pairs of disjoint subsets of X , together with $\emptyset$ if $|I|$ is odd.

Presenting and proving this result, Berge uses a graph-theoretical approach. We shall therefore state some necessary graph-theoretical definitions. In this, we assume that the reader is somewhat familiar with basic concepts of graph theory.

Definition 2.2 Let I be an ideal of the finite set X . Denote by $G(I)$ a simple graph whose vertices represent the members of I , two vertices of $G(I)$ being joined by an edge if and only if they represent two disjoint subsets of X .

Definition 2.3 Let G be a simple graph. A matching of G is a set of pairwise independent edges. A vertex which constitutes an endpoint of any one of these edges is said to be matched. A matching that matches all the vertices except, possibly, the vertex $\emptyset$, is said to be perfect.

A perfect matching of $G(I)$ as above is in other words the disjoint union of pairs of disjoint subsets of X , since two vertices of $G(I)$ are being joined by an edge if and only if they represent two disjoint subsets of X . Every member of I that is matched is included in one of these disjoint pairs, that together with $\emptyset$ construct I . Thus, the graph-theoretical definition coincides with the formulation in Theorem 2.1.

Proof [of Theorem 2.1] Assume that the theorem holds for all ideals of X with less than m members. Let I be an ideal of X with m members. We want to show that $G(I)$ has a perfect matching $\mathcal{F}$, by induction on m . Let a be an element of X and let $A_1 = \{a\}, A_2, A_3, \dots, A_s$ denote the members of I which contain a . If we put $B_i = A_i - \{a\}$, we can write $I = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, with

$$\mathcal{A} = (A_1, A_2, \dots, A_s), \quad \mathcal{B} = (B_1, B_2, \dots, B_s), \quad \mathcal{C} = (C_1, C_2, \dots, C_t).$$

$\mathcal{A}$ consists of all members of I containing a , $\mathcal{B}$ consists of all members in $\mathcal{A}$ modified to not contain a , and $\mathcal{C}$ consists of the members of I not containing a . Note that $I = \mathcal{A} \cup \mathcal{C}$ is a partition of I , since every member of I either contains the element a or not.

$I_1 = \mathcal{B} \cup \mathcal{C}$ is an ideal of X with less than m members and by the induction hypothesis, $G(I_1)$ has a perfect matching $\mathcal{F}_1$. $I_2 = \mathcal{B}$ is also an ideal of X with less than m members and by the induction hypothesis, $G(I_2)$ has a perfect matching $\mathcal{F}_2$.

Note that $B_1 = A_1 - \{a\} = \emptyset \in \mathcal{B}$ and may by the theorem be unmatched in $\mathcal{F}_1$ or $\mathcal{F}_2$. $\mathcal{F}_1 \cup \mathcal{F}_2$ generates a subgraph of $G(I_1)$, that consists of all members of $\mathcal{B}$ and $\mathcal{C}$, with two members connected if and only if they are connected in $\mathcal{F}_1$, $\mathcal{F}_2$, or in both. The maximum degree of this subgraph of $G(I_1)$ is 2, i.e. the vertices in the subgraph are incident to at most two edges each.

We use this partial graph to associate some edges to be included in the required matching of $G(I)$. This by looking at each connected component G_0 of $\mathcal{F}_1 \cup \mathcal{F}_2$. The following cases can be observed:

1. G_0 consists of one single vertex. In this case, this vertex is $\emptyset = B_1$, since all other components are connected in the partial graph. We take the edge $[\emptyset, a]$ for the required matching $\mathcal{F}$.
2. G_0 consists of one isolated edge $[B_i, B_j]$, which must belong to both $\mathcal{F}_1$ and $\mathcal{F}_2$. This is since one of B_i, B_j must be different from $\emptyset$, say B_i , and if $[B_i, B_j]$ is in only one of $\mathcal{F}_1$ and $\mathcal{F}_2$, B_i has to be connected to at least one other member in $\mathcal{F}_1 \cup \mathcal{F}_2$, which contradicts the fact that it is an isolated edge.

By definition, the members B_i and B_j are disjoint, which implies that $B_i \cap A_j = B_i \cap (B_j \cup \{a\}) = \emptyset$ and $B_j \cap A_i = B_j \cap (B_i \cup \{a\}) = \emptyset$. Thus, $[B_i, A_j]$ and $[B_j, A_i]$ are two edges of $G(I)$, which we use in the required matching $\mathcal{F}$.

3. G_0 consists of a cycle $[B_{i_1}, B_{i_2}, \dots, B_{i_k}, B_{i_1}]$, the edges of which must belong alternately to $\mathcal{F}_1$ and $\mathcal{F}_2$. This is since $\mathcal{F}_1$ and $\mathcal{F}_2$ are perfect matchings, which implies that two adjacent edges cannot belong to the same $\mathcal{F}_i$. This also excludes the possibility for a member of $\mathcal{C}$ to belong to the cycle, since such a member can only be connected with an edge in $\mathcal{F}_1$. We include the following edges in $\mathcal{F}$: $[B_{i_1}, A_{i_2}], [B_{i_2}, A_{i_3}], \dots, [B_{i_k}, A_{i_1}]$.
4. G_0 consists of an edge $[C_i, C_j]$, which must belong to $\mathcal{F}_1$. We include $[C_i, C_j]$ in $\mathcal{F}$.
5. The only remaining case is that G_0 consists of an elementary chain μ consisting of at least 3 members. The members of $\mathcal{C}$ cannot be included in the chain, except as for endpoints, for earlier mentioned reasons. In fact, the endpoints must consist of members of $\mathcal{C}$ or be equal to $\emptyset$. If we let the left endpoint be a member of $\mathcal{B}$, the chain is of the form $[B_{i_1}, B_{i_2}, \dots]$ and we cannot have that $[B_{i_1}, B_{i_2}]$ is in both $\mathcal{F}_1$ and $\mathcal{F}_2$. Thus we must have $B_{i_1} = \emptyset$, since we otherwise would be able to connect B_{i_1} further to the left. The first element in μ must therefore be $\emptyset$ or a member of $\mathcal{C}$. Since the order lacks importance, we can say the same about the last element in μ . Since μ is not a cycle, the first and last elements must differ. This means that at least one of them must be a member of $\mathcal{C}$ and the other must be $\emptyset$ or another member of $\mathcal{C}$. We can thus write either

$$\mu = [C_i, B_{j_1}, B_{j_2}, \dots, B_{j_k}, C_j]$$

or

$$\mu = [C_i, B_{j_1}, B_{j_2}, \dots, B_{j_k} = \emptyset].$$

In the first case, the required matching $\mathcal{F}$ will include the edges:

$$[C_i, A_{j_1}], [B_{j_1}, A_{j_2}], \dots, [B_{j_{k-1}}, A_{j_k}], [B_{j_k}, C_j]$$

In the second case, the required matching $\mathcal{F}$ will not include the last edge (and the vertex $B_{j_k} = \emptyset$ will remain unmatched in $\mathcal{F}$).

Since we have covered all possible forms the connected components can take above and found suitable matchings, the obtained set of edges $\mathcal{F}$ will be a matching of $G(I)$ which matches all vertices of $G(I)$, except possibly the vertex $\emptyset$. ■

Example 2.4 (See [1], p. 103). Let I be an ideal of the set $\{1, 2, 3, 4, 5\}$, consisting of the members: $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}, \{1, 5\}, \{3, 5\}, \{4, 5\}, \{1, 2, 3\}, \{1, 2, 4\}$ and $\{1, 3, 5\}$. Then, by Theorem 2.1, I is the union of the following pairs of disjoint sets together with $\emptyset$: $\{1\}$ and $\{4, 5\}$; $\{2\}$ and $\{1, 3, 5\}$; $\{3\}$ and $\{1, 2, 4\}$; $\{4\}$ and $\{1, 2, 3\}$; $\{5\}$ and $\{1, 4\}$; $\{1, 3\}$ and $\{2, 4\}$; $\{1, 2\}$ and $\{3, 5\}$; $\{1, 5\}$ and $\{2, 3\}$.

2.2 Snevily's First Result

To formulate the first theorem stated by Snevily [6], we need to define the notion of a dominant element.

Definition 2.5 Let I be an ideal of the finite set X . If $A \in I$ and $a, b \in X$, we write

$$A_{ab} = \begin{cases} (A - \{a\}) \cup \{b\} & \text{if } a \in A, b \notin A \\ A & \text{otherwise} \end{cases}$$

$b \in X$ is a dominant element in I if $A_{ab} \in I$ for all $A \in I$ and for all $a \in X$.

We clarify the meaning of this in an example.

Example 2.6 Let $X = \{1, 2, 3, 4\}$ be a set. $I = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{1\}, \{2\}, \{3\}, \emptyset\}$ is an ideal of X . 3 is a dominant element of X since if we replace one element in a member of I with 3, we obtain another member in I . By letting $A = \{1, 2\}$, we have $A_{13} = (\{1, 2\} - \{1\}) \cup \{3\} = \{2\} \cup \{3\} = \{2, 3\} \in I$. By inspection, this holds for all members $A \in I$.

Definition 2.7 If I is an ideal, $A \in I$ is a base of I if $A \subseteq B, B \in I \Rightarrow A = B$.

The interpretation of the definition is that A is a member of I not contained in any other member of I . Thus, A need not to be of the largest cardinality of the members of I , but merely has to be a member of I that is not a proper subset of any other member of I , therefore denoted as a base of I . We are now ready to state and prove the first main result in [6].

Theorem 2.8 (Snevily [6]). Let I be an ideal of the set $X = \{1, 2, \dots, n\}$ and suppose that I has a dominant element. Then I has a maximum intersecting subfamily F with the dominant element contained in each member of F .

Thus, for an ideal I with a dominant element, Chvátal's conjecture is true, i.e. among the maximum intersecting subfamilies of I , there is a star. This result could be considered a major breakthrough in the work with proving the conjecture as it greatly generalizes for

which types of ideals the conjecture is true. It also strengthens earlier acquired results such as one of Schönheim [5], which follows as a corollary.

Using Berge's theorem, the definition of a base and induction on $|I|$, the following proof can be constructed.

Proof We first consider the special cases $I = \emptyset$ or $I = \{\emptyset\}$, i.e. the smallest possible ideals of X . $A \in \emptyset \Rightarrow A_{ab} \in \emptyset$ is true, since $A \in \emptyset$ is false. $A \in \{\emptyset\} \Rightarrow A = \emptyset \Rightarrow A_{ab} = \emptyset \in \{\emptyset\}$ is also true. Thus both $I = \emptyset$ and $I = \{\emptyset\}$ have a dominant element. In both these cases, $F = \emptyset$ is the only intersecting subfamily. Since F has no members, the dominant element lies in every member of F . The theorem thus holds in these special cases.

Suppose I is a minimal ideal violating the claim and that the theorem holds for ideals of less cardinality than I . Then $I \neq \emptyset$ and $I \neq \{\emptyset\}$ by above. Hence, $|I| > 1$. Suppose without loss of generality that 1 is the dominant element of I . Note that $\emptyset$ is not a base of I . Let $\{B_1, \dots, B_k\}$ be the bases of I not containing 1. Suppose that there is a base C containing 1, such that $C \neq (B_i)_{a1}$ for $i = 1, \dots, k$ and $a \in X$. Then $I' = I - \{C\}$ is an ideal and further $|I'| < |I|$. 1 is a dominant element of I' as well. Suppose that $C = A_{a1}$, where $A \neq C$ if $A \in I'$ and $1 \notin A$, $a \in A$, $a \notin C$ (recall Definition 2.5). By the assumptions, A is not a base. This implies that for a base D , $A \subset D$. If $1 \in D$, then $C = A_{a1} \subset D$, which contradicts the fact that C is a base. If $1 \notin D$, then $D = B_i$ and $C = A_{a1} \subseteq D_{a1} = (B_i)_{a1}$, which implies that C is either not a base, or a base of the form $(B_i)_{a1}$, contradicting the assumption that $C \neq (B_i)_{a1}$. Thus $C \neq A_{a1}$ and therefore 1 is a dominant element of I' . By the induction hypothesis, the theorem thus holds for I' .

Let F be a maximum intersecting subfamily in I . If $C \notin F$, then F is a maximum intersecting subfamily in I' as well. By the induction hypothesis, there is then an equal sized intersecting subfamily G in I' , such that the dominant element 1 is contained in each member of G . But then G is a maximum intersecting subfamily in I as well and the theorem then holds for I . We can therefore assume that $C \in F$ for every maximum intersecting subfamily F in I .

I' has a maximum intersecting subfamily F' , where each member contains 1 by the induction hypothesis. $F' \cup \{C\}$ is then an intersecting subfamily for I , where each member contains 1. If I has an intersecting subfamily F with $F = G \cup \{C\}$, that is larger than $F' \cup \{C\}$, we must have $|G| > |F'|$. But G is an intersecting subfamily in I' as well, and since we have assumed that F' is a maximum intersecting subfamily in I' , this cannot hold. Thus, $F' \cup \{C\}$ is an intersecting subfamily of maximum size for I and the theorem holds.

By the above and the assumption that I is a minimal ideal violating the claim of the theorem, we can assume that every base of I containing 1 must be of the form $(B_i)_{a1}$. Let I'' be the ideal generated by $\{B_1, \dots, B_k\}$, i.e. the bases of I that do not contain 1. Let $F'' = F \cap I''$, where F is a maximum intersecting subfamily for I . The members of F'' do not contain 1 since the members of I'' do not contain 1. Every member $A \in I - I''$ does contain 1 by the following. $A \subseteq C$ for some base $C \neq B_i$ containing 1. If $1 \notin A$, then $A \subseteq B_j$, which contradicts that $A \in I - I''$. Thus, 1 must be contained in every member of $I - I''$.

We now want to replace F'' in F with $|F''|$ members that contain 1 to obtain a star. Using Berge's result (Theorem 2.1), we can find disjoint members A_i and C_i of I'' , $i = 1, \dots, k$, such that $I'' = \{A_1, C_1\} \cup \{A_2, C_2\} \cup \dots \cup \{A_k, C_k\}$, including $\{\emptyset\}$ if $|I''|$ is odd. Since F'' is intersecting, F'' can contain at most one of A_i and C_i for every i . Suppose that $A_1, \dots, A_m \in F''$. Let $H' = \{C_1, \dots, C_m\}$. Then $H' \cap F'' = \emptyset$ and further, $C_i \cup \{1\}$ and A_i are disjoint since $1 \notin A_i$. Since F is intersecting and $F'' \subset F$, we have $C_i \cup \{1\} \notin F$. Let $H = \{C_1 \cup \{1\}, \dots, C_m \cup \{1\}\}$. Then $(F - F'') \cup H$ is a star of I with the same size as F if we can show that $H \subseteq I$. This is true by the following. For the bases B_i of I not containing 1, we have $B_i \in F$ for every i , since if $B_i \notin F$, $I''' = I - \{B_i\}$ is a smaller ideal with 1 as a

dominant element and F as a maximum intersecting subfamily. By the induction hypothesis, I''' and therefore I contains a star of the same size as F . If $1 \notin B_i$, we have $B_i \in F''$ and since $F'' \cap H' = \emptyset$, $B_i \notin H'$. Thus the members C_i of I are not bases of I . This implies that $C_i = A - \{a\}$ for some $A \in I, a \in X$. $C_i \cup \{1\} = A_{a1} \in I$ for every i , since 1 is assumed to be a dominant element of I . Thus $H \subseteq I$, which implies that $(F - F'') \cup H$ is a star in I , with the same size as the maximum intersecting family F of I , and we are done. ■

Corollary 2.9 (Schönheim [5]). Let $A_1, A_2, \dots, A_n$ be the bases of an ideal I such that $A_1 \cap A_2 \cap \dots \cap A_n \neq \emptyset$. Then Chvátal's conjecture is true.

Proof Any element in the intersection of the bases of I is a dominant element, by the following. Let $a \in X$ be an element in the intersection of the bases A_i . Let $B \in I$. By the definition of a base, $B \subseteq A_i$ for some i . Let $b \in X$. If $b \notin B$ or $a \in B$, then by the definition of a dominant element, $B_{ba} = B$ and thus $B_{ba} \in I$. If $b \in B$ and $a \notin B$, we can write $B - \{b\} \subseteq B \subseteq A_i$. Since $a \in A_i$, we also have $B_{ba} = (B - \{b\}) \cup \{a\} \subseteq A_i$. Since all possible subsets of A_i belong to I , it follows that $B_{ba} \in I$ in this case as well. The element a is therefore dominant for I and the result follows. ■

Theorem 2.8 can be used to obtain a proof of the following theorem.

Theorem 2.10 (Snevily [6]). Let $\mathcal{A}$ be a family of subsets of $X = \{1, 2, \dots, n\}$ such that if $A, B \in \mathcal{A}$, then $A \cap B \neq \emptyset$ and $A \cup B \neq X$. Then $|\mathcal{A}| \leq 2^{n-2}$.

Proof Apply repeatedly the following shift operations S_j to $\mathcal{A}$. For $1 < j \leq n$, define

$$S_j(A) = \begin{cases} A_{j1} & \text{if } j \in A, 1 \notin A, A_{j1} \notin \mathcal{A} \\ A & \text{otherwise} \end{cases}$$

After the application of one shift operator to each member in $\mathcal{A}$, we obtain a new set $\mathcal{A}$. When the next shift operator is applied, $\mathcal{A}$ in the definition of S_j is replaced with that new set. By representing each member $A \in \mathcal{A}$ with a number in the binary system, we show that we obtain a new collection of subsets that are invariant under S_j in a finite number of steps. Call that invariant set $\mathcal{A}^*$. If the element $j \in A$, the corresponding binary number attains a one in position j (from the right). Otherwise it attains a zero in position j . When applying S_j to A , the size of the corresponding binary number will not increase, since we either exchange places of the one in place j with the zero in place 1, or do nothing. Neither can it decrease endlessly. The process must thus end in a finite number of steps.

We show that the function $A \mapsto S_j(A)$ is injective. If $j \notin A, B$ or $1 \in A, B$ or $A_{j1}, B_{j1} \in \mathcal{A}$, then $S_j(A) = S_j(B) \Rightarrow A = B$. If $j \in A, B$ and $1 \notin A, B$ and $A_{j1}, B_{j1} \notin \mathcal{A}$ and we have an element k , $k \neq 1$ and $k \neq j$, that is in A but not in B , $k \in A_{j1}$ as well and then we cannot have $S_j(A) = S_j(B)$. Thus $A \neq B \Rightarrow S_j(A) \neq S_j(B)$ in that case. The only remaining case is that A and B are such that $S_j(A) = A_{j1}$ and $S_j(B) = B$, or the other way around. In the first case, $A_{j1} \notin \mathcal{A}$ and $B \in \mathcal{A}$ and we cannot have $S_j(A) = S_j(B)$. By symmetry, the same type of argument holds for $S_j(A) = A$ and $S_j(B) = B_{j1}$. Thus the function is injective and therefore $|\mathcal{A}^*| = |\mathcal{A}|$.

Suppose that $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{A}$. We show that this implies $S_j(A) \cap S_j(B) \neq \emptyset$ for every $A, B \in \mathcal{A}$. If A and B are of the same form, i.e. either $S_j(A) = A_{j1}$ and $S_j(B) = B_{j1}$, or $S_j(A) = A$ and $S_j(B) = B$, the implication follows since either $1 \in S_j(A) \cap S_j(B)$ or $S_j(A) \cap S_j(B) = A \cap B \neq \emptyset$. Suppose instead that A and B are such that $S_j(A) = A_{j1}$ and $S_j(B) = B$. If $1 \in B$, then $1 \in S_j(A) \cap S_j(B)$. If $k \in A \cap B$ for some k , $k \neq 1$, $k \neq j$,

then k is not affected by the shift operator, and thus $k \in S_j(A) \cap S_j(B)$. By symmetry, the same type of argument holds when $S_j(A) = A$ and $S_j(B) = B_{j1}$. The remaining case is that $1 \notin B, j \in B$ and $A \cap B = \{j\}$, or the other way around. In the first case, B is such that $S_j(B) = B$. We must thus have $B_{j1} \in \mathcal{A}$. By assumption $A \cap B_{j1} \neq \emptyset$. Neither 1 nor j can belong to $A \cap B_{j1}$. There must therefore exist an element $k, k \neq 1$ and $k \neq j$, such that $k \in A \cap B_{j1}$. But then $k \in A \cap B$ as well and by the above, $k \in S_j(A) \cap S_j(B)$. By symmetry, the same type of argument holds when $1 \notin A, j \in A$ and $A_{j1} \in \mathcal{A}$.

Suppose that $A \cup B \neq X$ for every $A, B \in \mathcal{A}$. We show that this implies $S_j(A) \cup S_j(B) \neq X$ for every $A, B \in \mathcal{A}$. If A and B are of the same form, i.e. $S_j(A) = A_{j1}$ and $S_j(B) = B_{j1}$ or $S_j(A) = A$ and $S_j(B) = B$, the implication follows since either $j \notin S_j(A) \cup S_j(B)$ or $S_j(A) \cup S_j(B) = A \cup B \neq X$. Suppose instead that A and B are such that $S_j(A) = A_{j1}$ and $S_j(B) = B$. If $j \notin B$, then $j \notin S_j(A) \cup S_j(B)$ and thus $S_j(A) \cup S_j(B) \neq X$. Let $j \in B$. If $1 \notin B$, we must have that $B_{j1} \in \mathcal{A}$ since $S_j(B) = B$. But then $A \cup B_{j1} \neq X$ by assumption. Since $j \in A$ and $1 \in B_{j1}$, there must be another element k such that $k \neq 1, j$ and $k \notin A, B_{j1}$. Then $k \notin A, B$ and therefore $k \notin A_{j1}, B$. Thus $k \notin S_j(A) \cup S_j(B)$. Suppose that $1 \in B$. Then 1 and $j \in A \cup B$. There must then be another element $k \neq 1, j$, such that $k \notin A, B$. Then $k \notin A_{j1}, B$ and thus $S_j(A) \cup S_j(B) \neq X$. By symmetry, the conclusion holds when $S_j(A) = A$ and $S_j(B) = B_{j1}$ as well.

$A_{j1} \in \mathcal{A}^*$ for every $A \in \mathcal{A}^*$ for which $j \in A$ and $1 \notin A$. The members of $\mathcal{A}^*$ are invariant under the shift operator, which implies that $S_j(A) \in \mathcal{A}^*$. 1 is thereby a dominant element in at least $\mathcal{A}^*$. Let I be the ideal generated by the members of $\mathcal{A}^*$. We want to show that 1 is a dominant element for I as well. Let $B \in I$. Then $B \subseteq A$ for some $A \in \mathcal{A}^*$. Suppose that $j \in B$ and $1 \notin B$. If $1 \in A$, we have $B_{j1} \subseteq A$. If $1 \notin A$, we have $B_{j1} \subseteq A_{j1}$. In either case, $B_{j1} \subseteq A_{j1} \in I \Rightarrow B_{j1} \in I$. 1 is thereby a dominant element for I . Thus, by Theorem 2.8, I has a maximum intersecting subfamily F with the element 1 contained in each of its members.

If $S \in F$ and $(X - S) \cup \{1\} \in F$, then $S \in I$ and $(X - S) \cup \{1\} \in I$. Thus $S \subseteq A$ and $(X - S) \cup \{1\} \subseteq B$ for some $A, B \in \mathcal{A}^*$. But then $X = S \cup (X - S) \cup \{1\} \subseteq A \cup B$, which contradicts that $A \cup B \neq X$. Thus for $S \in F$, we must have $(X - S) \cup \{1\} \notin F$. Let $Y = X - \{1\}$. The number of subsets of Y is 2^{n-1} . If $S \in F$, we can write $S = T \cup \{1\}$, where $T \subseteq Y$. But then $(Y - T) \cup \{1\} \notin F$. At most half of the subsets of Y can therefore cause subsets of the form $S = T \cup \{1\} \in F$. Since $\mathcal{A}^*$ is intersecting and F is intersecting and of maximum size, $|\mathcal{A}| = |\mathcal{A}^*| \leq |F| \leq 2^{n-2}$. ■

3 Strong Stars

The second result obtained by Snevily and presented in [6] concerns ideals that can be partitioned in a certain way. We start with a definition.

Definition 3.1 Let $C = \{B_1, \dots, B_n\}$ be a collection of subsets of a finite set X such that all pairwise intersections of the B_i are identical and non-empty. Then C is called a strong star.

3.1 Stein's Result

Stein [8] has proved Chvátal's conjecture for ideals I with $n > 2$ bases, where $n - 1$ bases form a strong star. This result constitutes a crucial part in Snevily's proof of the second main result in [6]. To prove Stein's result, we are in need of some lemmas. The lemmas are mostly

technical and makes use of a wide range of notation. We thus encourage the reader to study this subsection with patience.

Let $B_1, B_2, \dots, B_n$ be the bases of an ideal I . Without loss of generality, we suppose that $B_1, B_2, \dots, B_{n-1}$ form a strong star, i.e. $B_i \cap B_j = B_k \cap B_l \neq \emptyset$ for $1 \leq i < j \leq n-1$ and $1 \leq k < l \leq n-1$. Let G be an intersecting subfamily of I . Suppose that $G \cap \mathcal{P}(B_i) \neq \emptyset$ for $1 \leq i \leq n$, where $\mathcal{P}(B_i)$ denotes the power set of B_i . Put

$$\begin{aligned} G_j &= G \cap \mathcal{P}(B_j) \\ P_{ij} &= \{X : X = g \cap B_j, g \in G_i\}, \quad |P_{ij}| = p_{ij} \\ X_{ij} &= X_{ji} = B_j \cap B_i, \quad |X_{ij}| = x_{ij} \end{aligned}$$

Note that $X_{ij} = X_{kl} = X_0$ for all $1 \leq i < j \leq n-1$ and $1 \leq k < l \leq n-1$. Also $P_{ji} = P_{jk} = P_{j0}$ and thus $p_{ji} = p_{j0}$ for all $1 \leq i \leq n-1$, $1 \leq j \leq n-1$, $1 \leq k < n-1$ and $j \neq i, j \neq k$. We further have that $P_{ij} \neq \emptyset$ and $\emptyset \notin P_{ij}$. The first statement follows from the assumption $G_i \neq \emptyset$. The second statement follows from that $G_j \neq \emptyset \Rightarrow \exists h \in G_j, h \cap g \neq \emptyset$ for every $g \in G_i$ and $h \cap g \subseteq g \cap B_j \Rightarrow g \cap B_j \neq \emptyset$.

Lemma 3.2 Let $A, B \subseteq \mathcal{P}(S)$, $|S| = m$. If every member of A has a non-empty intersection with every member of B , then

$$|A| + |B| \leq 2^m$$

Proof The complements of the members of A cannot be included in B . It follows that $|B| \leq 2^m - |A|$. ■

Lemma 3.3 Let G be an intersecting subfamily of the ideal I . Let $B_1, B_2, \dots, B_n$ be the bases of I and suppose that $B_1, B_2, \dots, B_{n-1}$ form a strong star. If $n-1 \geq 3$, then for each $i \leq n-1$, there are at least $n-3$ different j , such that $p_{ji} \leq 2^{x_{ji}-1}$, $j \leq n-1$, $j \neq i$.

Proof $g \in G_i$ and $h \in G_j$ intersect in $B_i \cap B_j = X_{ij}$. This implies that $P_{ij}, P_{ji} \subseteq \mathcal{P}(X_{ij})$. Thus for $1 \leq i < j \leq n-1$, we have $p_{ij} + p_{ji} \leq 2^{x_{ij}}$. Suppose that $p_{ji} > 2^{x_{ji}-1} = 2^{x_0-1}$ and $p_{ki} > 2^{x_{ki}-1} = 2^{x_0-1}$ where $i \neq j$, $i \neq k$, $j \neq k$ and $j, k \leq n-1$. We get the following contradiction: $2^{x_0} = 2^{x_0-1} + 2^{x_0-1} < p_{ji} + p_{ki} = p_{jk} + p_{kj} \leq 2^{x_0}$. Since we further must have $j \neq i$ and $j \neq n$ to obtain $p_{ji} \leq 2^{x_{ji}-1} = 2^{x_0-1}$, we can choose j in at least $n-3$ ways. ■

Lemma 3.4 Let $p_1 + p_2 \leq 2^x$. Let p_1, p_2, x and a be positive integers and $p_1 \leq 2^{x-1}$. Then

$$p_1 2^a \leq 2^{x+a-1} - (2^x - 2p_1) \leq 2^{x+a-1} - (p_2 - p_1).$$

Proof We have

$$p_1(2^a - 2) \leq 2^{x-1}(2^a - 2) = 2^{x+a-1} - 2^x.$$

Thus

$$\begin{aligned} p_1 2^a &\leq 2^{x-1}(2^a - 2) + 2p_1 = 2^{x+a-1} - 2^x + 2p_1 \\ &\leq 2^{x+a-1} - (p_1 + p_2) + 2p_1 = 2^{x+a-1} - (p_2 - p_1). \quad \blacksquare \end{aligned}$$

Lemma 3.5 Let $X_1 \cup X_2 \cup A$ be a partition of the finite set B , with $X_1, X_2 \neq \emptyset$. Let H be an intersecting subfamily of $\mathcal{P}(B)$, such that $h \cap X_i \neq \emptyset$, $i = 1, 2$, for all $h \in H$. Moreover, let $S_i = \{h \cap X_i, h \in H\}$, $i = 1, 2$. Then $|H| \leq 2^{|B|-1} - 2^{|A|}(2^{|X_1|} - |S_1|)$ for $i = 1, 2$.

Proof Let H' be the collection of subsets of the form $X_1 \cup M_2 \cup M_3$, where $M_2 \in \mathcal{P}(X_2) - S_2$ and $M_3 \subseteq A$. Then the following facts hold:

- (i) H' and H are disjoint.
- (ii) Any two members of H' have a non-empty intersection since $X_1 \subseteq h$ for every $h \in H'$.
- (iii) Every member of H' has non-empty intersection with every member of H since every member in H intersects X_1 and $X_1 \subseteq h$ for every $h \in H'$.
- (iv) Any two members of $H' \cup H$ have a non-empty intersection.
- (v) $|H'| = 1 \cdot 2^{|A|}(2^{|X_2|} - |S_2|)$ by the multiplication principle.

Consequently, $2^{|B|-1} \geq |H' \cup H| = |H'| + |H| = 1 \cdot 2^{|A|}(2^{|X_2|} - |S_2|) + |H|$. This proves the lemma for $i = 2$. The proof for $i = 1$ is analogous. ■

It immediately follows that $|H| \leq 2^{|B|-1} - 2^{|X_i|} + |S_i|$ for $i = 1, 2$, since $2^{|A|} \geq 1$, $2^{|X_i|} - |S_i| \geq 1$ and $\emptyset \notin S_i$ for $i = 1, 2$.

Lemma 3.6 Let $B = X_1 \cup X_2 \cup X_3 \cup A$ be a partition of the finite set B with $X_i \neq \emptyset$, $i = 1, 2, 3$. Let H be an intersecting subfamily of $\mathcal{P}(B)$, such that for each $h \in H$, $h \cap X_i \neq \emptyset$, $i = 1, 2, 3$ holds. Then $|H| \leq 2^{|B|-1} - 2^{|X_i|}$ for $i = 1, 2, 3$.

Proof Let $A' = X_3 \cup A$. $B = X_1 \cup X_2 \cup A'$ then fulfils the conditions in Lemma 3.5. With S_2 as in the proof of Lemma 3.5 we thus have $|H| \leq 2^{|B|-1} - 2^{|A'|}(2^{|X_2|} - |S_2|) = 2^{|B|-1} - 2^{|A|}2^{|X_3|}(2^{|X_2|} - |S_2|) \leq 2^{|B|-1} - 2^{|X_3|}$ since $2^{|A|} \geq 1$ and $2^{|X_2|} - |S_2| \geq 1$. The latter inequality holds since $\emptyset \notin S_2$. This proves the lemma when $i = 2$. The proof when $i = 1$ and $i = 3$ follows by symmetry. ■

To continue, we are in need of further assumptions and notations. We also repeat some notations used earlier in this section, to assist the reader. Let, as before, $B_1, B_2, \dots, B_{n-1}, B_n$ be the bases of I and suppose that $B_1, B_2, \dots, B_{n-1}$ form a strong star. Let G be an intersecting subfamily of I . Let $n \geq 4$. Put

$$\begin{aligned}
&X_0 = X_{ij} = X_{ji} \text{ for all } 1 \leq i < j \leq n-1; \quad |X_0| = x_0; \\
&P_{j0} = P_{ji}, \text{ for all } 1 \leq i < j \leq n-1; \quad |P_{j0}| = p_{j0} = p_{ji}; \\
&|B_j| = b_j \text{ for every } j \leq n; \\
&A_j = B_j - (X_0 \cup X_{jn}) \text{ for every } j \leq n-1; \\
&|A_j| = a_j \text{ for every } j \leq n-1; \\
&A_n = B_n - (X_{n1} \cup X_{n2} \cup \dots \cup X_{n,n-1}); \\
&|A_n| = a_n; \\
&G_j = G \cap \mathcal{P}(B_j) \text{ for every } j \leq n; \\
&\bar{G}_j = p_{j0}p_{jn} |\mathcal{P}(B_j - (X_0 \cup X_{jn}))| = p_{j0}p_{jn}2^{a_j} \text{ for every } j \leq n-1; \\
&\bar{G}_n = p_{n1}p_{n2} \dots p_{n,n-1} |\mathcal{P}(B_n - (X_{n1} \cup X_{n2} \cup \dots \cup X_{n,n-1}))| = p_{n1}p_{n2} \dots p_{n,n-1}2^{a_n}.
\end{aligned}$$

Note that $|G_j| \leq \bar{G}_j$.

Lemma 3.7 If $(\bigcap_{i=1}^{n-1} B_i) \cap B_n = \emptyset$ and $G_h = G \cap \mathcal{P}(B_h) \neq \emptyset$ for every $h \leq n$, then for every pair i, j , with $1 \leq i \leq n-1$, $1 \leq j \leq n-1$, $i \neq j$, either

$$|G| \leq \sum_{h \neq j}^{n-1} |G_h| + p_{ji}2^{x_{nj}}2^{a_j}$$

or

$$|G| \leq \sum_{h \neq j}^{n-1} |G_h| + 2^{x_{nj}} \bar{G}_n p_{nj}^{-1}$$

hold.

Proof We have

$$\begin{aligned} |G| &= \left| \bigcup_{h=1}^n G_h \right| \leq \sum_{h=1}^n |G_h| \\ &\leq \sum_{h \neq j}^{n-1} |G_h| + \bar{G}_n + \bar{G}_j. \end{aligned}$$

Either $\bar{G}_j/p_{jn} \geq \bar{G}_n/p_{nj}$ holds, or the opposite. The first case gives

$$\begin{aligned} \bar{G}_j + \bar{G}_n &= p_{ji} 2^{a_j} p_{jn} + p_{nj} \frac{\bar{G}_n}{p_{nj}} \leq p_{ji} 2^{a_j} p_{jn} + p_{nj} \frac{\bar{G}_j}{p_{jn}} \\ &= p_{ji} 2^{a_j} p_{jn} + p_{nj} p_{ji} 2^{a_j} = p_{ji} 2^{a_j} (p_{jn} + p_{nj}) \leq p_{ji} 2^{a_j} 2^{x_{nj}}. \end{aligned}$$

The second case gives

$$\begin{aligned} \bar{G}_j + \bar{G}_n &= p_{jn} \frac{\bar{G}_j}{p_{jn}} + \bar{G}_n \leq p_{jn} \frac{\bar{G}_n}{p_{nj}} + \bar{G}_n \\ &= p_{nj}^{-1} \bar{G}_n (p_{jn} + p_{nj}) \leq p_{nj}^{-1} \bar{G}_n 2^{x_{nj}}, \end{aligned}$$

where the last inequality in both cases follows from Lemma 3.2. ■

Lemma 3.8 Lemma 3.7 holds even when replacing G_n in $G = G_1 \cup G_2 \cup \dots \cup G_n$ with

$$G'_n = \left\{ \bigcup_{i=1}^{n-1} M_i \cup M'_n : M_i \in \{g \cap B_i : g \in G_n\}, \text{ for } i = 1, 2, \dots, n-1; M'_n \subseteq \left(B_n - \bigcup_{i=1}^{n-1} B_i \right) \right\}.$$

We let $G' = G_1 \cup G_2 \cup \dots \cup G'_n$.

Note that since every $g \in G_n$ is in G'_n as well, $G_n \subseteq G'_n$. Thus $G \subseteq G'$ and $|G| \leq |G'|$, for we are able to construct the set $(g \cap B_1) \cup (g \cap B_2) \cup \dots \cup (g \cap B_{n-1}) \cup (g \cap (B_n - \bigcup_{i=1}^{n-1} B_i))$, with this g in every M_i . This set is in turn a partition of G , since M_i and M_j are disjoint for $1 \leq i < j \leq n-1$. Otherwise, there would exist an element x lying in B_i and B_j and further in a $g \subseteq B_n$, which contradicts the assumption $(B_i \cap B_j) \cap B_n = \emptyset$.

The sets M_i are sets in P_{ni} when $g \in G_n$. Thus, M_i can be chosen in p_{ni} ways. Further, $B_n - \bigcup_{i=1}^{n-1} B_i = B_n - \bigcup_{i=1}^{n-1} X_{in}$, since we are only removing the parts of B_i that are parts of B_n as well. The sets X_{in} and X_{jn} are disjoint if $1 \leq i < j \leq n-1$. Otherwise we would have $(B_i \cap B_j) \cap B_n \neq \emptyset$. The number of elements in $B_n - \bigcup_{i=1}^{n-1} X_{in}$ is therefore equal to $b_n - (x_{1n} + x_{2n} + \dots + x_{n,n-1})$ and the number of possible subsets of $B_n - \bigcup_{i=1}^{n-1} X_{in}$ that can be chosen as M'_n is $2^{b_n - (x_{1n} + x_{2n} + \dots + x_{n,n-1})} = 2^{a_n}$. Thus $|G'_n| = \bar{G}_n$.

P_{ij} consists of the sets of the form $\{g \cap B_j, g \in G_i\}$. If $i \leq n-1$ and $g \in G_i$, g cannot be a subset of B_n since then X_0 would contain an element that lies in B_n . G_n and G'_n are subsets of B_n . Thus, G_i and G_j do not depend on the choice between G_n and G'_n . This means that P_{ij} , for $1 \leq i \leq n-1, 1 \leq j \leq n-1, i \neq j$, is also independent of that choice. The

intersections between the sets in G_n and B_j on the one hand, and the intersections between the sets in G'_n and B_j on the other, are the same. The conclusion is that also the sets P_{nj} , $j \leq n-1$, remain the same when G_n is replaced with G'_n . The numbers x_{ji} and a_j do not change either, since these only depend on the bases. We shall use these facts in the proof of Stein's theorem, that is soon to be stated.

Lemma 3.9 If $(\bigcap_{i=1}^{n-1} B_i) \cap B_n = \emptyset$, $G_i = G \cap \mathcal{P}(B_i) \neq \emptyset$ for every $i \leq n$, and $p_{j0} \leq 2^{x_0-1}$ for some $j \leq n-1$, then

$$\sum_{h \neq j}^{n-1} |G_h| + p_{j0} 2^{a_j} 2^{x_{jn}} \leq \sum_{h=1}^{n-1} 2^{b_h-1} - (n-2) 2^{x_0-1}.$$

Proof By assumption, $p_{j0} \leq 2^{x_0-1}$, and thus $2^{x_0} - 2p_{j0} \geq 0$. Lemma 3.4 with $a_j > 0$, $p_{j0} > 0$ and $x_0 > 0$ gives

$$p_{j0} 2^{a_j} \leq 2^{x_0+a_j-1} - (2^{x_0} - 2p_{j0}).$$

Thus,

$$p_{j0} 2^{a_j} 2^{x_{jn}} \leq 2^{x_0+a_j-1+x_{jn}} - (2^{x_0} - 2p_{j0}) 2^{x_{jn}} \leq 2^{b_j-1} - (2^{x_0} - 2p_{j0}),$$

since $x_0 + a_j + x_{jn} = b_j$ and $2^{x_{jn}} \geq 1$.

Using Lemma 3.5 with the partition $B_h = X_0 \cup X_{hn} \cup A_h$, we get

$$|G_h| \leq 2^{b_h-1} - 2^{a_h} (2^{x_0} - p_{h0}) \leq 2^{b_h-1} - (2^{x_0} - p_{h0}),$$

since $a_h \geq 1$.

By Lemma 3.2, $p_{h0} + p_{j0} \leq 2^{x_0}$, which gives

$$|G_h| \leq 2^{b_h-1} - p_{j0}.$$

By Lemma 3.3, there is at most one i with $i \leq n-1$, $i \neq j$, such that $p_{ij} > 2^{x_0-1}$. Hence

$$\begin{aligned} \sum_{h \neq j}^{n-1} |G_h| + p_{j0} 2^{a_j} 2^{x_{jn}} &= \sum_{h \neq i, h \neq j}^{n-1} |G_h| + p_{j0} 2^{a_j} 2^{x_{jn}} + |G_i| \\ &\leq \sum_{h \neq i, h \neq j}^{n-1} (2^{b_h-1} - (2^{x_0} - p_{h0})) + 2^{b_j-1} - (2^{x_0} - 2p_{j0}) + 2^{b_i-1} - p_{j0} \\ &= \sum_{h=1}^{n-1} 2^{b_h-1} - (n-2) 2^{x_0} + \sum_{h \neq i}^{n-1} p_{h0} \\ &\leq \sum_{h=1}^{n-1} 2^{b_h-1} - (n-2) 2^{x_0} + \sum_{h \neq i}^{n-1} 2^{x_0-1} \\ &= \sum_{h=1}^{n-1} 2^{b_h-1} - 2(n-2) 2^{x_0-1} + (n-2) 2^{x_0-1} \\ &= \sum_{h=1}^{n-1} 2^{b_h-1} - (n-2) 2^{x_0-1}. \blacksquare \end{aligned}$$

Lemma 3.10 Let a , b , m , r_1 and r_2 be non-negative numbers. If $a + b \leq 2m$, $a \leq m$ and $r_1 \geq r_2$, then

$$ar_1 + br_2 \leq m(r_1 + r_2).$$

Proof

$$\begin{aligned} ar_1 + br_2 &\leq ar_1 + (2m - a)r_2 = a(r_1 - r_2) + 2mr_2 \\ &\leq m(r_1 - r_2) + 2mr_2 = m(r_1 + r_2). \blacksquare \end{aligned}$$

Lemma 3.11 Let $G_i = G \cap \mathcal{P}(B_i) \neq \emptyset$ for every $i \leq n$, and $p_{in}/2^{x_{in}} \leq \frac{1}{2}$ for every $i \leq n-1$. Then either

$$\left| \bigcup_{i=1}^{n-1} G_i \right| \leq \sum_{i=1}^{n-1} \bar{G}_i \leq \sum_{i=1}^{n-1} 2^{b_i-1} p_{in} 2^{-x_{in}}$$

holds or

$$\left| \bigcup_{i=1}^{n-1} G_i \right| \leq \sum_{i=1}^{n-1} \bar{G}_i \leq 2^{b_j-1}$$

for some $j \leq n-1$.

Proof By Lemma 3.3 there is at most one j such that $p_{j0} > 2^{x_0-1}$ with $j \leq n-1$. Without loss of generality, assume that $j = n-1$ and $p_{i0} \leq p_{10} \leq 2^{x_0-1}$ for $i \leq n-2$. Put $\bar{G}'_i = p_{in} p_{10} 2^{a_i}$ for $i \leq n-2$. By assumption, we then have $\bar{G}_i \leq \bar{G}'_i$ for $i \leq n-2$, and therefore

$$\begin{aligned} \sum_{i=1}^{n-1} \bar{G}_i &\leq \sum_{i=1}^{n-2} \bar{G}'_i + \bar{G}_{n-1} \\ &= T = p_{10} \sum_{i=1}^{n-2} \bar{G}'_i p_{10}^{-1} + p_{n-1,0} (\bar{G}_{n-1} p_{n-1,0}^{-1}). \end{aligned}$$

Put $r_1 = \sum_{i=1}^{n-2} \bar{G}'_i p_{10}^{-1}$ and $r_2 = \bar{G}_{n-1} p_{n-1,0}^{-1}$.

Case 1: $r_1 \geq r_2$. Put $a = p_{10} \leq 2^{x_0-1}$ and $b = p_{n-1,0}$. Then $a + b \leq 2^{x_0} = 2 \cdot 2^{x_0-1} = 2m$ by Lemma 3.2, with $m = 2^{x_0-1}$. By Lemma 3.10,

$$\begin{aligned} T = ar_1 + br_2 &\leq 2^{x_0-1} (r_1 + r_2) = 2^{x_0-1} \left(\sum_{i=1}^{n-2} \bar{G}'_i p_{10}^{-1} + \bar{G}_{n-1} p_{n-1,0}^{-1} \right) \\ &= 2^{x_0-1} \sum_{i=1}^{n-2} p_{in} 2^{a_i} + 2^{x_0-1} p_{n-1,n} 2^{a_{n-1}}. \end{aligned}$$

Since $2^{x_0+a_i+x_{in}} = 2^{b_i}$, this gives

$$T \leq \sum_{i=1}^{n-2} 2^{b_i-1} p_{in} / 2^{x_{in}} + 2^{b_{n-1}-1} p_{n-1,n} / 2^{x_{n-1,n}} = \sum_{i=1}^{n-1} 2^{b_i-1} p_{in} 2^{-x_{in}}.$$

Case 2: $r_1 < r_2$. We have

$$\begin{aligned} T &\leq p_{10} \bar{G}_{n-1} p_{n-1,0}^{-1} + p_{n-1,0} \bar{G}_{n-1} p_{n-1,0}^{-1} = p_{n-1,n} 2^{a_{n-1}} (p_{10} + p_{n-1,0}) \\ &\leq p_{n-1,n} 2^{a_{n-1}} 2^{x_0} = p_{n-1,n} 2^{-x_{n-1,n}} 2^{a_{n-1}+x_0+x_{n-1,n}} \leq \frac{1}{2} \cdot 2^{b_{n-1}} = 2^{b_j-1}. \blacksquare \end{aligned}$$

Lemma 3.12 If $y_1, y_2, \dots, y_m$ are positive numbers with $y_1 \geq y_i$ for $i = 1, 2, \dots, m$ and $c \geq 0$, $c_j \geq 0$ for $j = 1, 2, \dots, m$, and if further $cy_1 \leq \sum_{i=1}^m c_i y_i$ holds, then $\sum_{i=1}^m c_i \geq c$.

Proof $0 \leq cy_1 \leq \sum_{i=1}^m c_i y_i \leq \sum_{i=1}^m c_i y_1 \Rightarrow \sum_{i=1}^m c_i \geq c$. ■

Lemma 3.13 Let $B_1 \cap B_2 \cap B_3 = \emptyset$ and $G_i = G \cap \mathcal{P}(B_i) \neq \emptyset$ for $i = 1, 2, 3$. Put

$$G_3'' = \{y \cup M_{31} \cup M_{32}; M_{31} \in \{g \cap B_1; g \in G_3\}, M_{32} \in \{g \cap B_2; g \in G_3\}\}$$

where $y = B_3 - (B_1 \cup B_2)$, and suppose that $p_{31} \leq 2^{x_{31}-1}$. Then

$$|G_1| + |G_2| + p_{31}p_{32}2^{b_3+x_{31}-x_{32}} \leq 2^{b_h-1} + 2^{b_k-1} - 2^{x_{hk}-1}$$

for some h and k with $h \neq k$ and $h, k \in \{1, 2, 3\}$.

Proof The proof of this lemma is analogous with the proof of the “main lemma” within [7]. If $p_{21} \leq 2^{x_{21}-1}$, then the proof is like case 1 of [7]. If $p_{21} > 2^{x_{21}-1}$, then $p_{12} < 2^{x_{21}-1}$ and either $p_{32} < 2^{x_{32}-1}$ or $p_{23} < 2^{x_{32}-1}$, like in case 1 respectively case 2 of the proof within [7]. ■

We are now ready to state and prove Stein's result.

Theorem 3.14 (Stein [8]). If I is an ideal having $n \geq 3$ bases such that $n - 1$ bases form a strong star, then Chvátal's conjecture is true.

Proof If $n = 3$, the theorem is true by Corollary 3.7 in [9]. We proceed with induction over n . Suppose that $n \geq 4$ and that the theorem is true for fewer than n bases. If $\bigcup_{i=1}^n B_i$ is a star, we are done by Corollary 2.9. From now on, we therefore suppose that $(\bigcup_{i=1}^{n-1} B_i) \cap B_n = \emptyset$. Suppose that no intersecting subfamily G of I fulfils the condition $G \cap \mathcal{P}(B_i) \neq \emptyset$ for all $i \leq n$. If there is a maximum intersecting family G with $G \cap \mathcal{P}(B_n) = \emptyset$, then G is a maximum intersecting subfamily of $\mathcal{P}(\bigcup_{i=1}^{n-1} B_i)$, and we are done since $\bigcup_{i=1}^{n-1} B_i$ is a star. Otherwise, the intersecting subfamily G of I is an intersecting subfamily of $\mathcal{P}((\bigcup_{i \neq j}^{n-1} B_i) \cup B_n)$, for some $j \leq n - 1$. The statement then follows from the induction hypothesis.

Only the case when $(\bigcup_{i=1}^{n-1} B_i) \cap B_n = \emptyset$ and there is an intersecting subfamily G of I such that $G \cap \mathcal{P}(B_i) \neq \emptyset$ for all $i \leq n$ remains.

One of the following two cases holds:

- (1) There is an $i \leq n - 1$ such that $p_{ni} \leq 2^{x_{ni}-1}$.
- (2) $p_{ni} > 2^{x_{ni}-1}$ for every $i \leq n - 1$.

Case 1. Choose an $i \leq n - 1$ with $p_{ni} \leq 2^{x_{ni}-1}$. Let $j \leq n - 1$, $j \neq i$ and put

$$e_{n-1,j} = \sum_{h \neq j}^{n-1} |G_h| + p_{j0}2^{a_j}2^{x_{jn}}$$

$$e_{n,j} = \sum_{h \neq j}^{n-1} |G_h| + 2^{x_{jn}}\bar{G}_n p_{nj}^{-1}.$$

By Lemma 3.7, $|G| \leq e_{n-1,j}$ or $|G| \leq e_{n,j}$. By Lemma 3.3, there is a j with $j \leq n - 1$, $j \neq i$ and $p_{ji} \leq 2^{x_{ji}-1}$. First suppose that $|G| \leq e_{n-1,j}$ for such a j . By Lemma 3.9, we then have

$$|G| \leq \sum_{h=1}^{n-1} 2^{b_h-1} - (n-2)2^{x_0-1}.$$

We have $X_0 = \bigcap_{h=1}^{n-1} B_h \neq \emptyset$. Let $x \in X_0$. The collection of all subsets of $\bigcup_{h=1}^{n-1} B_h$ that contain the element x is a star with $\sum_{h=1}^{n-1} 2^{b_n-1} - (n-2)2^{x_0-1}$ elements, which shows the theorem in this case.

Now suppose that $e_{n-1,j} < e_{n,j}$ for all $j \leq n-1$ with $j \neq i$, $p_{ji} \leq 2^{x_{ji}-1}$. Then

$$|G| \leq \sum_{h \neq j}^{n-1} |G_h| + \bar{G}_n 2^{x_{nj}} p_{nj}^{-1} \quad (1)$$

for some such j . The following holds:

$$\begin{aligned} \bar{G}_n 2^{x_{nj}} p_{nj}^{-1} &= \left(\prod_{h=1}^{n-1} p_{nh} \right) 2^{a_n} 2^{x_{nj}} p_{nj}^{-1} = \left(\prod_{h=1}^{n-1} p_{nh} \right) 2^{b_n - \sum_{h=1}^{n-1} x_{nh}} 2^{x_{nj}} p_{nj}^{-1} \\ &= \left(\prod_{h \neq j}^{n-1} p_{nh} \right) 2^{b_n - \sum_{h \neq j}^{n-1} x_{nh}} = |G'_n|, \end{aligned}$$

where G'_n is as in Lemma 3.8, defined from the bases B_h with $h \neq j$. We now have

$$|G| \leq |G'| \leq \sum_{h \neq j}^{n-1} |G_h| + |G'_n|,$$

where $G' = \bigcup_{h \neq j}^n B_h$. We use Lemma 3.8 to obtain

$$|G| \leq |G'| \leq e'_{n-1,k} = \sum_{h \neq j, h \neq k}^{n-1} |G_h| + p_{ki} 2^{x_{nk}} 2^{a_k}$$

or

$$|G| \leq |G'| \leq e'_{n,k} = \sum_{h \neq j, h \neq k}^{n-1} |G_h| + \bar{G}_n 2^{x_{nk}} p_{nk}^{-1}$$

for every $k \leq n-1$, such that $k \neq i$, $k \neq j$. $\bar{G}_n$ is here defined from G' . Since $e_{n-1,j} < e_{n,j}$ for all $j \leq n-1$, $j \neq i$, $p_{ji} \leq 2^{x_{ji}-1}$, we have $p_{ji} 2^{a_j} 2^{x_{jn}} \leq \bar{G}_n 2^{x_{nj}} p_{nj}^{-1}$, with $\bar{G}_n$ as in (1) for all such j . The new numbers $p_{ki} 2^{a_k} 2^{x_{nk}}$ are the same as the old, and the new $\bar{G}_n 2^{x_{nk}} p_{nk}^{-1}$ cannot be smaller than the old since $2^{x_{nk}} p_{nk}^{-1} \geq 1$. Thus $e'_{n-1,k} < e'_{n,k}$ for all $k \leq n-1$ with $k \neq i$, $k \neq j$ and $p_{ki} \leq 2^{x_{ki}-1}$. By this, $|G| \leq |G'| \leq e'_{n,k}$ and we can repeat the procedure as long as we have four bases left. At last, we therefore obtain

$$|G| \leq e_n = |G_i| + |G_q| + \bar{G}'_n,$$

where $\bar{G}'_n = p_{ni} p_{nq} 2^{b_n - x_{ni} - x_{nq}}$. By Lemma 3.13, $|G| \leq 2^{b_h-1} + 2^{b_k-1} - 2^{x_{hk}-1}$ for some different indices h and k belonging to the set $\{i, q, n\}$. By assumption, $X_{hk} \neq \emptyset$ and $2^{b_h-1} + 2^{b_k-1} - 2^{x_{hk}-1}$ is the size of the star containing all subsets of $B_h \cup B_k$ that contain a fixed element $x \in X_{hk}$. This concludes case 1 of the proof.

Case 2. Here, $p_{ni} > 2^{x_{ni}-1}$ for $i \leq n-1$. Without loss of generality, we may assume that $|B_1| \geq |B_i|$ for $i \leq n-1$. If we take a fixed element x in $B_1 \cap B_n$, which by assumption is non-empty, and construct the set of all subsets of $B_1 \cup B_n$ that contain x , we obtain a star with $2^{b_1-1} + 2^{b_n-1} - 2^{x_{1n}-1}$ elements. Thus $|G| \geq 2^{b_1-1} + 2^{b_n-1} - 2^{x_{1n}-1}$. By Lemma 3.6, $|G_n| \leq 2^{b_n-1} - 2^{x_{1n}}$. Since G is maximal, we have

$$\begin{aligned}
\sum_{h=1}^{n-1} \bar{G}_h &\geq \sum_{h=1}^{n-1} |G_h| = |G| - |G_n| \\
&\geq 2^{b_1-1} + 2^{b_n-1} - 2^{x_{1n}-1} - (2^{b_n-1} - 2^{x_{1n}}) \\
&= 2^{b_1-1} + 2^{x_{1n}-1} > 2^{b_1-1}.
\end{aligned} \tag{2}$$

Put $\sum_{h=1}^{n-1} \bar{G}_h = (1+w)2^{b_1-1}$ with $w > 0$. The assumption $p_{ni} > 2^{x_{ni}-1}$, $i \leq n-1$, and Lemma 3.2 gives $p_{in} < 2^{x_{in}-1}$. By Lemma 3.11,

$$\sum_{h=1}^{n-1} \bar{G}_h \leq 2^{b_j-1} \leq 2^{b_1-1}$$

or

$$\sum_{h=1}^{n-1} \bar{G}_h = (1+w)2^{b_1-1} \leq \sum_{i=1}^{n-1} 2^{b_i-1} p_{in} 2^{-x_{in}}. \tag{3}$$

The first inequality contradicts (2). Thus, we suppose that the second inequality holds. Lemma 3.12 with $c = 1+w$, $y_i = 2^{b_i-1}$ and $c_i = p_{in} 2^{-x_{in}}$ gives

$$\sum_{i=1}^{n-1} p_{in} 2^{-x_{ni}} \geq 1+w.$$

Thus

$$\sum_{i=1}^{n-1} p_{ni} 2^{-x_{ni}} \leq \sum_{i=1}^{n-1} (1 - p_{in} 2^{-x_{in}}) = n-1 - \sum_{i=1}^{n-1} p_{in} 2^{-x_{in}} \leq n-1 - (1+w),$$

where the first inequality follows from Lemma 3.2.

Choose a fixed element $x \in X_0$ and let H be the set of all subsets of X_0 containing x . Let

$$H_i = \{M_1 \cup M_2 \cup M_3; M_1 \in H, M_2 \subseteq A_i, M_3 \in P_{in}\}, \quad i \leq n-1$$

and finally $G' = (\bigcup_{i=1}^{n-1} H_i) \cup G_n$. G' is then an intersecting subfamily with

$$|G'| = \sum_{i=1}^{n-1} 2^{x_0-1} 2^{a_i} p_{in} + |G_n|$$

elements. Since

$$\sum_{i=1}^{n-1} \bar{G}_i \leq \sum_{i=1}^{n-1} 2^{x_0-1} 2^{a_i} p_{in},$$

we have $|G| \leq |G'|$ and since G is a maximum intersecting subfamily of I , $|G| = |G'|$. It is therefore possible to replace G with G' , which we from now on call G . In this family G , we have $P_{10} = P_{20} = \dots = P_{n-1,0}$ and $p_{i0} = 2^{x_0-1}$ for $i \leq n-1$. Put

$$K_i = \{M_1 \cup M_2 \cup M_3; M_1 \in H, M_2 \subseteq A_i, M_3 \subseteq X_{in}\}, \quad i \leq n-1$$

and $K = \bigcup_{i=1}^{n-1} K_i$. K is an intersecting subfamily of $\bigcup_{i=1}^{n-1} B_i$ and therefore also of I . The number of elements in K is

$$\sum_{i=1}^{n-1} 2^{x_0-1} 2^{a_i} 2^{x_{in}} - (n-2)2^{x_0-1},$$

where the negative term follows since H has been counted $n-1$ times in the sum. The family K consists of

$$\sum_{i=1}^{n-1} 2^{x_0-1} 2^{a_i} (2^{x_{in}} - p_{in}) - (n-2) 2^{x_0-1}$$

more sets than $\bigcup_{i=1}^{n-1} H_i$. By Lemma 3.2, $2^{x_{in}} - p_{in} \geq p_{ni}$ and by the assumption $p_{ni} > 2^{x_{in}-1}$, $2^{x_{in}} - p_{in} \geq 2^{x_{in}-1} + 1$ (note that p_{ni} is the same for the new and old G). This gives

$$\begin{aligned} \sum_{i=1}^{n-1} 2^{x_0-1} 2^{a_i} (2^{x_{in}} - p_{in}) - (n-2) 2^{x_0-1} &\geq \sum_{i=1}^{n-1} 2^{x_0-1} 2^{a_i} (2^{x_{in}-1} + 1) - (n-2) 2^{x_0-1} \\ &= \sum_{i=1}^{n-1} 2^{b_i-2} + \sum_{i=1}^{n-1} 2^{x_0-1} 2^{a_i} - (n-2) 2^{x_0-1} > \sum_{i=1}^{n-1} 2^{b_i-2}. \end{aligned}$$

We use the new G in the following, and for this G , $p_{i0} = 2^{x_0-1}$. By this

$$\bar{G}_i = 2^{x_0-1} p_{in} 2^{a_i} = 2^{x_0-1} p_{in} 2^{b_i-x_0-x_{in}} = 2^{b_i-1} p_{in} 2^{-x_{in}}, \quad i \leq n-1,$$

from which we obtain

$$(1+w) 2^{b_1-1} = \sum_{i=1}^{n-1} \bar{G}_i = \sum_{i=1}^{n-1} 2^{b_i-1} p_{in} 2^{-x_{in}} < \sum_{i=1}^{n-1} 2^{b_i-2},$$

since $p_{in} < 2^{x_{in}-1}$ by assumption. The number of sets we have added to the sets in $\bigcup_{i=1}^{n-1} H_i$ is therefore strictly greater than $\sum_{i=1}^{n-1} 2^{b_i-2} > (1+w) 2^{b_1-1}$. Since (the new) G is maximal, we therefore have

$$\bar{G}_n = 2^{a_n} \prod_{i=1}^{n-1} 2^{x_{in}} p_{ni} 2^{-x_{in}} \geq |G_n| > \sum_{i=1}^{n-1} 2^{b_i-2} > (1+w) 2^{b_1-1}. \quad (4)$$

By the inequality between arithmetic and geometric mean,

$$\begin{aligned} \bar{G}_n &= 2^{b_n} \prod_{i=1}^{n-1} p_{ni} 2^{-x_{ni}} \leq 2^{b_n} \left(\frac{\sum_{i=1}^{n-1} p_{ni} 2^{-x_{ni}}}{n-1} \right)^{n-1} \\ &\leq 2^{b_n} \left(\frac{n-1-(1+w)}{n-1} \right)^{n-1}. \end{aligned}$$

Therefore,

$$2^{b_n} \left(1 - \frac{1+w}{n-1} \right)^{n-1} \geq \bar{G}_n > (1+w) 2^{b_1-1}$$

and

$$(1+w) 2^{b_1-1} = \sum_{i=1}^{n-1} \bar{G}_i$$

with $w \geq 0$. By this

$$\left| \bigcup_{i=1}^{n-1} G_i \right| \leq \sum_{i=1}^{n-1} \bar{G}_i + \bar{G}_n \leq (1+w) 2^{b_1-1} + 2^{b_n} \left(1 - \frac{1+w}{n-1} \right)^{n-1} \leq 2^{b_1-1} + f(w) 2^{b_n}$$

where

$$\begin{aligned} f(w) &= w \left(1 - \frac{1+w}{n-1}\right)^{n-1} / (w+1) + \left(1 - \frac{1+w}{n-1}\right)^{n-1} \\ &= \frac{2w+1}{w+1} \left(1 - \frac{1+w}{n-1}\right)^{n-1}. \end{aligned}$$

The derivative is

$$f'(w) = - \left(\frac{n-2-w}{n-1} \right)^{n-1} \frac{(2n-2)w^2 + (3n-2)w + 1}{(w+1)^2(n-2-w)}.$$

The derivative is negative when $0 \leq w < n-2$. Since $n > 1$, we have $\frac{1}{2}(n-3) < n-2$. The derivative is therefore negative when $0 \leq w \leq \frac{1}{2}(n-3)$, and by this

$$f(w) < f(0) = \left(1 - \frac{1}{n-1}\right)^{n-1} = \frac{1}{\left(1 + \frac{1}{n-2}\right)^{n-1}} < \frac{1}{\left(1 + \frac{1}{n-1}\right)^{n-1}}$$

when $0 \leq w \leq \frac{1}{2}(n-3)$.

The last inequality in (4), together with Lemma 3.12, where $c = 1 + w$, $c_i = \frac{1}{2}$ and $y_i = 2^{b_i-1}$, gives $1 + w \leq \frac{1}{2}(n-1)$, whence $0 \leq w \leq \frac{1}{2}(n-3)$. Therefore,

$$|G| < 2^{b_1-1} + \frac{2^{b_n}}{\left(1 + \frac{1}{n-1}\right)^{n-1}}.$$

The denominator is a strictly increasing function of $n \geq 2$. Since $n \geq 4$,

$$\begin{aligned} |G| &< 2^{b_1-1} + \frac{2^{b_n}}{\left(1 + \frac{1}{3}\right)^3} = 2^{b_1-1} + \frac{27}{64} 2^{b_n} = 2^{b_1-1} + \frac{27}{32} 2^{b_n-1} \\ &= 2^{b_1-1} + 2^{b_n-1} - \frac{5}{32} 2^{b_n-1} = 2^{b_1-1} + 2^{b_n-1} - \frac{5}{4} 2^{b_n-4} = \bar{G}^+. \end{aligned}$$

We now show that there is a star in $\mathcal{P}(B_1) \cup \mathcal{P}(B_n)$ with at least $\bar{G}^+$ members. Since $2^{b_n-1} > |G_n|$ by Lemma 3.6 and $|G_n| \geq 2^{b_1-1}$ by (4), we have $b_n > b_1$.

A star of $\mathcal{P}(B_1) \cup \mathcal{P}(B_n)$ can have

$$2^{b_1-1} + 2^{b_n-1} - 2^{x_{1n}-1} \tag{5}$$

members. It is therefore sufficient to show that $2^{x_{1n}-1} \leq \frac{5}{4} \cdot 2^{b_n-4}$.

If $b_n = b_1 + 1$ and $0 \leq x_{1n} - 1 \leq b_1 - 3 = b_n - 4$, then by (5), there is a star with more than $\bar{G}^+$ members.

If $b_n = b_1 + 1$ and $x_{1n} - 1 > b_1 - 3 = b_n - 4$, it follows from definition that $x_{1n} = b_1 - 1 = b_n - 2$, since $x_{1n} = |B_1 \cap B_n| \leq |B_1| - 1 = b_n - 2$ and $x_{1n} \geq b_1 - 1 = b_n - 2$. Since $n \geq 4$ by assumption, we must have $n = 4$ and $p_{24} = p_{34} = p_{42} = p_{43} = 1$, which satisfies $p_{ni} \leq 2^{x_{ni}-1}$. The latter is as known the condition of case 1 and we are done.

If $b_n > b_1 + 1$, we have $x_{1n} - 1 = |B_1 \cap B_n| - 1 \leq b_1 - 2 \leq b_n - 4$, and it follows from (5) that I has a star with more than $\bar{G}^+$ members.

This concludes the proof. ■

3.2 Kleitman's and Magnanti's Results

Kleitman and Magnanti [4] have obtained two results regarding intersecting collections of finite sets, the latter used by Snevily in his second result in [6], concerning Chvátal's conjecture. We shall state both theorems here since the first is necessary in the proof of the second. We omit the proof in the first case and refer the reader to [4]. We begin by a definition.

Definition 3.15 Let F be a subfamily of the finite set $X = \{1, 2, \dots, n\}$. We define the latent subsets F^L of F to be those subsets of X which are a subset of members in F , but which are not themselves in F , i.e.

$$F^L = \{A \subseteq X : A \subseteq B \in F, A \notin F\}.$$

Lemma 3.16 (Kleitman and Magnanti [4]). Let F_1 and F_2 be two subfamilies of the finite set $X = \{1, 2, \dots, n\}$ where each member of F_1 intersects each member of F_2 . Then

$$|F_1^L| |F_2^L| \geq |F_1| |F_2|$$

Proof The interested reader is referred to [4], where the proof is stated. ■

Theorem 3.17 (Kleitman and Magnanti [4]). Let F be an intersecting subfamily of the finite set X with the property that each member of F contains at least one of the elements $x_1, x_2 \in X$. Then there is an intersecting subfamily F' of X subordinate to F (i.e., $A \in F'$ implies $A \subseteq A_1 \in F$) that satisfies:

- (i) $|F'| \geq |F|$
- (ii) either x_1 is contained in each member of F' or x_2 is.

Proof If $\{x_1\}$ or $\{x_2\}$ are members of F , we are done, since F is an intersecting family. Suppose therefore that $\{x_j\} \notin F$, $j = 1, 2$. Let

$$\begin{aligned} G_1 &= \{A \in F : x_1 \in A, x_2 \notin A\} \\ G_2 &= \{A \in F : x_2 \in A, x_1 \notin A\} \\ G_3 &= F - G_1 - G_2. \end{aligned}$$

Since every member of F contains at least one of $x_1, x_2 \in X$, we can partition F in the following way:

$$F = G_1 \cup G_2 \cup G_3.$$

Let further

$$\bar{G}_j = \{A - \{x_j\} : A \in G_j\}$$

for $j = 1, 2$. By the assumptions on F and the fact that a member of $\bar{G}_1$ and a member of $\bar{G}_2$ must have a common element other than the x_j 's, every member of $\bar{G}_1$ intersects every member of $\bar{G}_2$. By Lemma 3.16, $|\bar{G}_1^L| |\bar{G}_2^L| \geq |\bar{G}_1| |\bar{G}_2|$. Suppose that $|\bar{G}_1^L| \geq |\bar{G}_2|$. The family

$$F' = G_1 \cup G_3 \cup \{A \cup \{x_1\} : A \in \bar{G}_1^L\}$$

then satisfies the conditions above. $|\bar{G}_1^L| \geq |\bar{G}_2| = |G_2|$ gives $|F'| \geq |F|$. It is clear that every member of F' contains x_1 . Assume that $B \in F'$. If $B \in G_1$ or $B \in G_3$, then $B \subseteq A \in F$. If $B = A \cup \{x_1\} \in \{A \cup \{x_1\} : A \in \bar{G}_1^L\}$, then $A \subseteq C \in \bar{G}_1$, and hence $B \subseteq C \cup \{x_1\} \in G_1 \subseteq F$. ■

F' as above is a star, and thus if we can find an intersecting subfamily F of I fulfilling Theorem 3.17, then I satisfies Chvátal's conjecture.

3.3 Snevily's Second Result

Proceeding with the previous definitions and results of the section as basis, together with the definition below, we consider the second main result in [6].

Definition 3.18 If $C = \{A_1, A_2, \dots, A_n\}$ is a strong star with $\bigcap_{i=1}^n A_i = D$ and $|D| = k$, then we call C a strong k -star and D the k -set of C . When $k = 1$ we will just call C a 1-star.

Example 3.19 Let $X = \{1, 2, 3, 4, 5\}$. Then $C = \{\{1, 2, 3\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 3\}\}$ is a strong 2-star of X with the 2-set $\{2, 3\}$.

Theorem 3.20 (Snevily [6]). Let I be an ideal of the finite set $X = \{1, 2, \dots, n\}$ and let $B_1, B_2, \dots, B_q$ denote the bases of I . If the set of bases $B_1, B_2, \dots, B_q$ can be partitioned into two 1-stars, then I satisfies Chvátal's conjecture.

Proof Let C_1 and C_2 be any partition of the bases $B_1, B_2, \dots, B_q$ into two 1-stars. We can assume that their 1-sets are disjoint. Otherwise, $B_1 \cap B_2 \cap \dots \cap B_q \neq \emptyset$ and by Corollary 2.9, the conjecture is true for I . Suppose, without loss of generality, that $\{1\}$ is the 1-set of C_1 and $\{2\}$ is the 1-set of C_2 . We may assume that $n > 2$. If $n = 1$ or $n = 2$, the conjecture follows directly, since for an ideal of such a set, the only possible intersecting subfamilies are stars. Since $n > 2$, $q > 2$. If either $|C_1| = 1$ or $|C_2| = 1$, then $q - 1$ of the bases of I form a strong star and we are done by Theorem 3.14. We can therefore assume that $|C_i| \geq 2$, $i = 1, 2$.

Let F be a maximum intersecting subfamily for I . If F contains a subset of a base, F must also contain the base. Otherwise we can obtain a larger intersecting family by adding the base to F since a base has non-empty intersection with all of its subsets. Assume that $|F \cap C_1| \leq 1$. Then either $F \cap C_1 = \emptyset$ or $F \cap C_1 = B_1$ where $B_1 \in C_1$. In the first case, F does not contain any of the bases in C_1 and thus cannot contain any subsets of these bases. Thus, each member in F must be a subset of one of the bases in C_2 . F is then a star, since the intersection between the bases in C_2 only contains 2, and the bases must be contained in F . In the second case, each member of F is a subset of B_1 or of one of the bases in C_2 . Thus, F is a maximum intersecting subfamily for the ideal generated by the bases in C_2 and the base $B_1 \in C_1$. We know from earlier that $|C_i| \geq 2$, $i = 1, 2$. This ideal therefore has $m > 2$ bases, $m - 1$ of which are contained in C_2 and thus form a strong star. By Theorem 3.14, Chvátal's conjecture is then true for the ideal. By symmetry, the same type of arguments holds when $|F \cap C_2| \leq 1$. We can therefore assume that $|F \cap C_i| \geq 2$ for $i = 1, 2$.

Let I_1 be the ideal generated by the bases in C_1 and let I_2 be the ideal generated by the bases in C_2 . F is intersecting and if F contains subsets of the bases in C_1 , it must also contain the bases. The only element in the intersection of the bases in C_1 is 1, by assumption. Thus, if $D, E \in F \cap I_1$, then we must have $\{1\} \subseteq D \cap E$. Similarly, if $G, H \in F \cap I_2$, then $\{2\} \subseteq G \cap H$. Thus, every member of F contains either 1 or 2. By Theorem 3.17, we are done. ■

Example 3.21 (See [6], p. 140). Let I be the ideal of the set $X = \{1, 2, \dots, 20\}$, generated by the following bases: $\{1, 2, 10, 14, 15\}$, $\{1, 2, 6, 11, 14, 19\}$, $\{4, 10, 11, 19\}$, $\{3, 5, 6\}$, $\{3, 5, 10\}$ and $\{6, 7, 8, 9, 10, 12, 13, 16, 17, 18, 20\}$. Then the bases of I can be partitioned into the following 1-stars:

$$\begin{aligned} C_1 &= \{\{1, 2, 10, 14, 15\}, \{4, 10, 11, 19\}, \{3, 5, 10\}, \{6, 7, 8, 9, 10, 12, 13, 16, 17, 18, 20\}\} \\ C_2 &= \{\{1, 2, 6, 11, 14, 19\}, \{3, 5, 6\}\} \end{aligned}$$

$\{10\}$ is the 1-set of C_1 and $\{6\}$ is the 1-set of C_2 . By the theorem above, we see that I satisfies Chvátal's conjecture.

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