An introduction to some ordinary differential equations governing stellar structures

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Popular scientific summary

The Lane-Emden equation is a nonlinear second order ordinary differential equation which models many phenomena in mathematical physics and astrophysics. This equation describes the equilibrium density distribution in a self-gravitating sphere of polytropic or isothermal gas and has crucial relevance in the field of radiative cooling and modeling of clusters of galaxies. Moreover, this equation has been considered quite versatile when examining aspects such as the analysis of isothermal cores, convective stellar interiors, and fully degenerate stellar configurations. In addition, recent observation lead to the conclusion that the density profiles of dark matter halos too are often modeled by the isothermal Lane-Emden equation with suitable boundary conditions at the origin. The Lane-Emden equation was first introduced in 1869 by the American astrophysicist Jonathan Homer Lane (1819-1880), who was interested in computing the temperature and the density of the solar surface. It is to be noted the contribution to this equation by the Swiss mathematician Robert Emden (1862-1940), who explained the expansion and compression of gas spheres through a mathematical model. Since Stefan's law was published a decade later, Lane used some other experimental results concerning the rate of emission of radiant energy by a heated surface, and the value that he obtained for the solar temperature was 30,000 degrees Kelvin, roughly five times larger than the actual number. Despite getting wrong results concerning the surface, Lane's temperature and density results for the stellar interior, turned out to be very reasonable, and his equation is still used nowadays for computing the inner structure of polytropic stars. The Emden-Chandrasekhar equation is the isothermal case of the Lane-Emden equation and was first introduced by Robert Emden in 1907; in this case, the polytropic index has an infinite value. Later on, Subrahmanyan Chandrasekhar (1910-1995) made important contributions by analyzing the system in the phase plane. In this dissertation, we will go through explicit, numerical and qualitative aspects of the polytropic and the isothermal equation.

Abstract

The Lane-Emden equation is a non-linear differential equation governing the equilibrium of polytropic stationary self-gravitating, spherically symmetric star models;

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n=0.$$

In the isothermal cases we have the Chandrasekhar equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right)-e^{-\psi}=0$$

After having derived these models, we will go through all cases for which analytic solutions are achievable. Moreover, we will discuss the existence and uniqueness of positive solutions under specific boundary conditions by transforming the equations to autonomous ones. The analysis depends upon the value of the polytropic index n. We also compute some solutions numerically.

Contents

1 Introduction: Lane-Emden Equation

In this dissertation we will consider spherically symmetric mass distribution of stellar structures. We start from the concept that pressure and gravity are the main forces determining such a structure and they must be balanced in order for the star not to collapse under its own weight. The equations describing the structure of such models are the *hydrostatic balance equation* [6] :

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho \tag{1}$$

and the *mass conservation equation*:

$$\frac{dm}{dr} = 4\pi\rho r^2 \tag{2}$$

with mass m, pressure P, density ρ and radius r. Here G is the gravitational constant. We can combine the equations above into a single second order ODE. Multiplying (1) by r^2/ρ and differentiating gives us

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr}. \tag{3}$$

By substituting eq.(2) in eq.(3) we get the hydrostatic equilibrium equation in the form:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi\rho G. \tag{4}$$

1.1 Polytropes

In order to describe stellar models, we will go over the so called *polytropes* as they turn out to be very practical when it comes to dealing with the internal structure of a star. Their usefulness is explained by the simplicity of the provided solutions which can be utilized for the estimation of numerous physical quantities (see section 2). In astrophysics, a polytrope describes a relationship in which the pressure depends on the density and the equation describing such relationship is the following:

$$P = K\rho^\gamma \tag{5}$$

where K is a constant, and

$$\gamma = 1 + 1/n \tag{6}$$

is called the adiabatic index (a parameter characterizing the behavior of the specific heat of a gas) and n is the polytropic index. Eq.(6) is called the Polytropic equation and describes how the pressure changes in relation to the density throughout a star. Different values of n describe how a given substance (e.g. gas, fluid) reacts to compression, for instance:

- $n = 0$, represents an incompressible fluid [16, p. 20]. In this case, the pressure can change but the density remains constant [14, p.47], [16, p.51].
- Neutron stars are well modeled for $0.5 < n < 1$ [16, p.28].
- $n = 1.5$ models quite accurately, fully convective star cores like red giants and brown dwarfs [7, p. 331]
- $n = 3$ (Eddington's approximation) is an interesting case since it corresponds to a useful approximation of the Sun [16, pp. 50,51] .
- $n = 5$ applies to "stars" with an infinite radius [16, p. 51]; moreover, it is implied that only polytropes with $n < 5$ have a surface [9].

N.B. generally speaking, stars and gaseous/rocky planets, are composed by different layers, each being modeled by a different value of n, therefore the polytropic indices used in the above examples merely represent a rough average.

By substituting (5) and (6) into (4) we obtain

$$\frac{(n+1)K}{4\pi G n r^2}\left[\frac{d}{dr}\left(r^2\rho^{\frac{1}{n}-1}\frac{d\rho}{dr}\right)\right] = -\rho \tag{7}$$

and after introducing the following dimensionless variables:

$$r = a\xi, \quad a = \left(\frac{(n+1)K}{4\pi G}\lambda^{(1/n)-1}\right)^{\frac{1}{2}} \tag{8}$$

with $\rho = \lambda\theta^n$, and λ being an arbitrary constant, we get the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0. \tag{9}$$

Choosing $\lambda = \rho_c = \rho(0)$ where ρ_c is the central density, we obtain in addition the condition $\theta(0) = 1$. We shall see below that we also need to take $\theta'(0) = 0$ in order to get a bounded solution. Moreover, we shall find explicit solutions for $n = 0, 1$ and 5 (section 1.3). For other values, qualitative and numerical methods are necessary to learn about the behavior of $\theta(\xi)$. What we have used above to derive the standard form of the Lane-Emden Equation, is called non-dimensionalization procedure. The main idea is to rewrite all the physical quantities present in equations like the hydrostatic balance equation used above as dimensionless numbers times their characteristic values. For example, the radius has been changed as $r = a\xi$. This approach assures that all the dimensional quantities are factored out and leave behind only a pure mathematical equation, as the polytropic structure behavior depends only on n. Note that we are only interested in positive values of θ, therefore θ^n is well defined, even if n is not an integer. However, we will sometimes extend θ^n to negative values of θ for convenience, see e.g. the remark after proposition 1.2. Before moving on, some time should be spent in discussing the initial conditions above, specifically, we will prove that $\frac{d\theta}{d\xi}$ vanishes at the origin for bounded solutions. We start by writing the Lane-Emden Equation in the form

$$\frac{1}{\xi^2}(\xi^2\theta'(\xi))' = f(\theta(\xi)),$$

where $f(\theta) = -\theta^n$.

Proposition 1.1. *Let $f \in C(\mathbb{R})$. Assuming that θ is $C^2(0,T]$, for some $T > 0$, and bounded and that $\frac{1}{\xi^2}(\xi^2\theta'(\xi))' = f(\theta)$ on $(0,T)$, we have that $\theta'(\xi) \to 0$ as $\xi \to 0$.*

Proof. Let $\xi_0 \in (0,T)$, then

$$\xi^2\theta'(\xi) = \int_{\xi_0}^{\xi} t^2 f(\theta(t))dt + A$$

for $A = {\xi_0}^2\theta'(\xi_0)$. The right-hand side has a limit as $\xi \to 0$:

$$B = \int_{\xi_0}^{0} t^2 f(\theta(t))dt + A = -\int_{0}^{\xi_0} t^2 f(\theta(t))dt + A.$$

Case 1- $B \neq 0$ In this case:

$$\xi^2\theta'(\xi) \to B \neq 0 \quad as \quad \xi \to 0.$$

If $B > 0$ we get

$$\theta'(\xi) \geq \frac{B}{2\xi^2}$$

if $0 < \xi \leq \varepsilon$ for some $\varepsilon > 0$. Now we integrate

$$\theta(\varepsilon) - \theta(\xi) \geq \int_{\xi}^{\varepsilon} \frac{B}{2t^2} dt = -\frac{B}{2\varepsilon} + \frac{B}{2\xi}$$

hence,

$$\theta(\xi) \leq \theta(\varepsilon) + \frac{B}{2\varepsilon} - \frac{B}{2\xi} \to -\infty$$

as $\xi \to 0^+$. If $B < 0$, a similar argument gives $\theta(\xi) \to +\infty$ as $\xi \to 0^+$. This contradicts the assumption that θ is bounded.

Case 2- $B = 0$. We now write

$$\begin{aligned}
\xi^2\theta'(\xi) &= \int_{\xi_0}^{\xi} t^2 f(\theta(t))dt + A \\
&= \int_0^{\xi} t^2 f(\theta(t)dt + \underbrace{A - \int_0^{\xi_0} t^2 f(\theta(t))dt}_{\text{B=0}} \\
&= \int_0^{\xi} t^2 f(\theta(t))dt.
\end{aligned}$$

Hence

$$|\theta'(\xi)| \leq \frac{1}{\xi^2} \int_0^{\xi} t^2 C = \frac{C\xi}{3} \to 0 \quad \text{as} \quad \xi \to 0$$

where $C = \max |f(\theta(t))|, t \in (0, T]$. □

N.B. the function f in the proposition is supposed to be defined and continuous on $\mathbb{R}$, while $f(\theta) = -\theta^n$ is in general only defined for $\theta \geq 0$. We can solve this by setting $f(\theta) = -|\theta|^n$ when n is not an integer. The theorem then applies if $n \geq 0$. Since we are only interested in positive solutions, this does not affect anything.

1.2 Existence and uniqueness

Note that equation (9) has a singularity at the origin. Therefore it is important to discuss the existence and uniqueness for the initial value problem:

$$\begin{cases} (\xi^2\theta')' + \xi^2\theta^n = 0 \\ \theta\,(0) = 1, \theta'\,(0) = 0. \end{cases} \tag{10}$$

The proof that will be given here is based on Ni and Nussbaum's paper ([12], Proposition 2.35) and concerns the more general problem

$$\begin{cases} (\xi^{m-1}u')' - \xi^{m-1}f(u(\xi),\xi) = 0 \\ u(0) = d, u'(0) = 0 \end{cases} \tag{11}$$

Eq. (10) is a special case of (11) where $m = 3, d = 1$ and $f(u, \xi) = -u^n$.

Proposition 1.2. *Let $m \geq 1$ and $f : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be of class C^1. For every $d \in \mathbb{R}$, (11) has a unique C^2 solution $u(\xi) = u(\xi, d)$*

Proof. We introduce the auxiliary variable $v = u'$ and rewrite (11) as

$$\begin{cases} \quad u' = v \\ \quad (\xi^{m-1}v)' - \xi^{m-1}f(u(\xi),\xi) = 0 \\ \quad u(0) = d, v(0) = 0 \end{cases}$$

which can be rewritten as a system of two integral equations:

$$\begin{cases} \quad u(\xi) = d + \int_0^\xi v(t)dt \\ \quad v(\xi) = \frac{1}{\xi^{m-1}} \int_0^\xi t^{m-1} f(u(t),t)dt. \end{cases}.$$

Note that the initial conditions $u(0) = d$ and $v(0) = 0$ are automatically satisfied by continuous solutions to these integral equations. We let K be the right hand side:

$$K(u, v) = (K_1(u, v), K_2(u, v)),$$

$$K(u, v) = \left(d + \int_0^\xi v(t)dt; \frac{1}{\xi^{m-1}} \int_0^\xi t^{m-1} f(u(t), t)dt \right).$$

We define a Banach space X for (u, v)

$$X = \{(u, v)|u, v \in C[0, \epsilon]\}$$

with norm

$$||(u, v)|| = \max(\max_{0 \leq t \leq \epsilon} |u(t)|, \max_{0 \leq t \leq \epsilon} |v(t)|).$$

Here ϵ is a positive number which will be chosen sufficiently small later on. Our goal is to find a closed subset $T \subseteq X$ such that $K : T \to T$ and show that K is a contraction, so that we can apply the Banach fixed point theorem [15]. We let $x_0 = (d, 0) \in X$ and T be the closed ball of radius δ around x_0 in X for some $\delta > 0, T = \{x \in X : ||x - x_0|| \leq \delta\}$. Then we have

$$\begin{aligned}||K(u,v) - x_0|| &= \max\left(\max_{0\leq\xi\leq\epsilon} |\int_0^\xi v(t)dt|, \max_{0\leq\xi\leq\epsilon} \frac{1}{\xi^{m-1}} |\int_0^\xi t^{m-1} f(u(t),t)dt|\right)\\ &\leq \max\left(\max_{0\leq\xi\leq\epsilon} \int_0^\xi |v(t)|dt, \max_{0\leq\xi\leq\epsilon} \frac{1}{\xi^{m-1}} \int_0^\xi t^{m-1}|f(u(t),t)|dt\right)\\ &\leq \max(\epsilon\delta, \frac{M\epsilon}{m}),\end{aligned}$$

from which it follows that $K : T \to T$ if $\epsilon \leq 1$ and $\epsilon \leq \frac{\delta m}{M}$ where

$$M = \max\{\max\{|f(u,t)|, |f_u(u,t)|\} : |u - d| \leq \delta, 0 \leq t \leq 1\}.$$

We now have

$$\begin{gathered}K(u_1, v_1) - K(u_2, v_2)\\ = \int_0^\xi (v_1(t) - v_2(t))dt; \frac{1}{\xi^{m-1}} \int_0^\xi t^{m-1}(f(u_1(t),t)) - f(u_2(t),t)dt\end{gathered}$$

therefore

$$\begin{aligned}||K(u_1, v_1) - K(u_2, v_2)|| = \max(&\max_{0\leq\xi\leq\epsilon} |\int_0^\xi v_1(t) - v_2(t)dt|,\\ &\max_{0\leq\xi\leq\epsilon} \frac{1}{\xi^{m-1}} |\int_0^\xi t^{m-1}(f(u_1(t),t)) - f(u_2(t),t)|dt\\ \leq \max(&\max_{0\leq\xi\leq\epsilon} \int_0^\xi |v_1(t) - v_2(t)|dt,\\ &\max_{0\leq\xi\leq\epsilon} \frac{1}{\xi^{m-1}} \int_0^\xi t^{m-1}|(f(u_1(t),t)) - f(u_2(t),t)|dt)\end{aligned}$$

so we can either have

$$||K(u_1, v_1) - K(u_2, v_2)|| \leq \max_{0\leq\xi\leq\epsilon} \int_0^\xi |v_1(t) - v_2(t)|dt$$

or

$$||K(u_1, v_1) - K(u_2, v_2)|| \leq \max_{0\leq\xi\leq\epsilon} \frac{1}{\xi^{m-1}} \int_0^\xi t^{m-1}|(f(u_1(t),t)) - f(u_2(t),t)|dt).$$

In the first case we have

$$\max_{0\leq\xi\leq\epsilon}\int_0^{\xi}|v_1(t)-v_2(t)|dt \leq \epsilon \max_{0\leq\xi\leq\epsilon}|v_1(t)-v_2(t)| \leq \epsilon||(u_1,v_1)-(u_2,v_2)||$$

In the second case we have

$$\max_{0\leq\xi\leq\epsilon}\frac{1}{\xi^{m-1}}\int_0^{\xi} t^{m-1}\underbrace{|f(u_1(t),t)-f(u_2(t),t)|}_{\leq M|u_1(t)-u_2(t)|\leq M||(u_1,v_1)-(u_2,v_2)||}dt$$
$$\leq \frac{M\epsilon}{m}||(u_1,v_1)-(u_2,v_2)||.$$

Finally we have

$$||K(u_1,v_1)-K(u_2,v_2)|| \leq \max(\epsilon,\frac{M\epsilon}{m})||(u_1,v_1)-(u_2,v_2)||$$

therefore K will be a contraction if $\epsilon, \frac{M\epsilon}{m} < 1$.

In summary, the Banach fixed point theorem applies if $\delta \leq 1$ and $\epsilon < \min(1, \frac{m}{M})$.

□

Again, when n is not an integer, we set $f(u,\xi) = -|u|^n$ when applying the result to (10). The proposition applies if $n \geq 1$.

1.3 Explicit solutions

For certain values of n, it is in fact possible to find explicit solutions to the Lane-Emden equation. The following transformations are needed:

a)for

$$\theta = \frac{\chi}{\xi} \tag{12}$$

equation (9) reduces to

$$\frac{d^2\chi}{d\xi^2} = -\frac{\chi^n}{\xi^{n-1}} \tag{13}$$

b)for

$$x = \frac{1}{\xi}; \tag{14}$$

(9) reduces to

$$x^4\frac{d^2\theta}{dx^2} = -\theta^n \tag{15}$$

c) Observe that (15) has a solution of the form

$$\theta = a x^{\omega} \tag{16}$$

with

$$\omega = \frac{2}{n-1}; \qquad a = [\frac{2(n-3)}{(n-1)^2}]^{\frac{1}{n-1}} \tag{17}$$

when $n > 3$. Therefore, we consider the substitution

$$\theta = A x^{\omega} z; \qquad \omega = \frac{2}{n-1} \tag{18}$$

with A being an arbitrary constant. From (18) we obtain

$$\frac{d^2\theta}{dx^2} = A\left[x^{\omega}\frac{d^2 z}{dx^2} + 2\omega x^{\omega-1}\frac{dz}{dx} + \omega(\omega-1)x^{\omega-2}z\right] \tag{19}$$

and by substituting (19) in (15) while using the relation in (18) we obtain

$$x^2\frac{d^2 z}{dx^2} + 2\omega x\frac{dz}{dx} + \omega(\omega-1)z + A^{n-1}z^n = 0. \tag{20}$$

One more substitution is needed to eliminate x, namely

$$x = \frac{1}{\xi} = e^t; \quad t = \log x = -\log\xi. \tag{21}$$

this transforms (20) to

$$\frac{d^2 z}{dt^2} + (2\omega-1)\frac{dz}{dt} + \omega(\omega-1)z + A^{n-1}z^n = 0. \tag{22}$$

If we assume that $n > 3$ and choose A=a we will have

$$A^{n-1} = \omega(1-\omega). \tag{23}$$

Therefore (22) becomes

$$\frac{d^2 z}{dt^2} + (2\omega-1)\frac{dz}{dt} + \omega(\omega-1)z(1-z^{n-1}) = 0 \tag{24}$$

and by substituting $\omega = \frac{2}{n-1}$ into (24), we get

$$\frac{d^2 z}{dt^2} + (\frac{5-n}{n-1})\frac{dz}{dt} - \frac{2(n-3)}{(n-1)^2}z(1-z^{n-1}) = 0. \tag{25}$$

If we instead set A = 1, our equation becomes

$$\frac{d^2z}{dt^2} + (2\omega - 1)\frac{dz}{dt} + \omega(\omega - 1)z + z^n = 0 \tag{26}$$

or equivalently

$$\frac{d^2z}{dt^2} + (\frac{5-n}{n-1})\frac{dz}{dt} + \frac{2(3-n)}{(n-1)^2}z + z^n = 0. \tag{27}$$

Note that the transformation used in (18) is called *homology*. When studying differential equations, if a whole family of solutions can be obtained by 'scaling', we can say that a '*homology transformation*' is admitted [6]; in our case, the constant A is our scaling factor. We are now ready to discuss the various explicit solutions.

For n = 0, (9) becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -1. \tag{28}$$

After integrating twice we obtain

$$\theta = D - \frac{C}{\xi} - \frac{1}{6}\xi^2 \tag{29}$$

where C and D are our constants of integration. Using the initial conditions, we get

$$\theta = 1 - \frac{1}{6}\xi^2, \tag{30}$$

so the first zero of this function happens when $\xi = \sqrt{6}$.

For n=1, equation (13) becomes

$$\frac{d^2\chi}{d\xi^2} = -\chi, \tag{31}$$

which has the general solution

$$G\sin(\xi - I) \tag{32}$$

with G and I being constants of integration. Taking into account the initial conditions and equation (12) we get

$$\theta = \frac{\sin(\xi)}{\xi} \tag{33}$$

which has its first zero at $\xi = \pi$.

For n = 5 equation (25) reduces to

$$\frac{d^2z}{dt^2} = \frac{1}{4}z(1 - z^4) \tag{34}$$

which multiplied by $\frac{dz}{dt}$ on both sides, gives us

$$\frac{1}{2}\frac{d}{dt}\left[\left(\frac{dz}{dt}\right)^2\right] = \frac{1}{4}z(1 - z^4)\frac{dz}{dt}. \tag{35}$$

After integrating and rearranging we get

$$\frac{dz}{\pm(2D + \frac{1}{4}z^2 - \frac{1}{12}z^6)^{\frac{1}{2}}} = dt \tag{36}$$

and by setting D=0, while choosing the sign so that $t \to \infty$ we get

$$\frac{dz}{z(1 - \frac{1}{3}z^4)^{\frac{1}{2}}} = -\frac{1}{2}dt. \tag{37}$$

At this point we make the substitution

$$\frac{1}{3}z^4 = \sin^2\zeta. \tag{38}$$

Differentiating both sides of (38) we get

$$\frac{4}{3}z^3 dz = 2\sin\zeta\cos\zeta d\zeta, \tag{39}$$

which leads to

$$\frac{dz}{z} = \frac{1}{2}\frac{\cos\zeta}{\sin\zeta}d\zeta. \tag{40}$$

By substituting (38) and (40) in (37) we obtain

$$\csc\zeta d\zeta = -dt, \tag{41}$$

which once integrated gives

$$\tan\frac{1}{2}\zeta = Ce^{-t} \tag{42}$$

where C is an integration constant. From (38) we obtain

$$\frac{1}{3}z^4 = \frac{4\tan^2\frac{1}{2}\zeta}{(1 + \tan^2\frac{1}{2}\zeta)^2}, \tag{43}$$

and after solving for z and substituting (42) in (43) we get

$$z = \pm \left[\frac{12C^2 e^{-2t}}{(1 + C^2 e^{-2t})^2} \right]^{\frac{1}{4}} \tag{44}$$

which leads to

$$\theta = \pm [\frac{3C^2}{(1 + C^2 \xi^2)^2}]^{\frac{1}{4}}. \tag{45}$$

Taking into account the initial conditions, we get

$$\theta = \frac{1}{(1 + \frac{1}{3}\xi^2)^{\frac{1}{2}}}. \tag{46}$$

In this case we notice that θ has no finite zero, but that $\theta(\xi) \to \infty$ as $\xi \to 0$. The graph below has been obtained with python and shows all the solutions of the Lane-Emden Equation for $0 \leq n \leq 6$. The solutions for $n = 0, 1, 5$ were analyti-

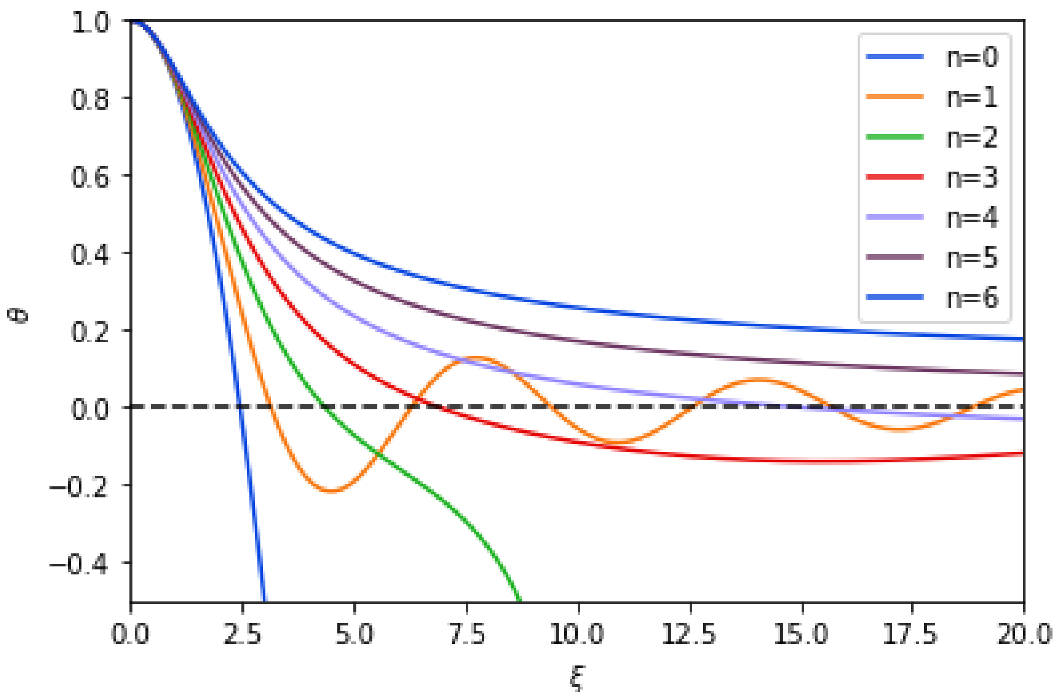


Figure 1: Lane-Emden Equation n=0,1,2,3,4,5,6

cally obtained above as well, but for $n = 2, 3, 4, 6$ the numerical approach has been necessary.

2 The (y,z) plane

In order to study the solutions qualitatively, we use the change of variables in (18) and (21) with $A = 1$. We restrict our attention to the case $\omega > 0$, implying $n > 1$, which means that Propositions 1.1 and 1.2 apply. By setting $y = \frac{dz}{dt}$ then we can rewrite (26) as the first order system

$$\begin{cases} z' = y \\ y' = -(2\omega - 1)y - \omega(\omega - 1)z - z^n. \end{cases} \tag{47}$$

We will study the phase portrait for this system, and the analysis will be restricted to $z \geq 0$ since we are interested in positive θ. Whenever $z' = y \neq 0$, we can also express y as a function of z and obtain the first order equation

$$y\frac{dy}{dz} + (2\omega - 1)y + \omega(\omega - 1)z + z^n = 0. \tag{48}$$

since

$$\frac{d^2z}{dt^2} = \frac{dy}{dt} = \frac{dy}{dz}\frac{dz}{dt} = y\frac{dy}{dz}. \tag{49}$$

In terms of the original variables ξ and θ, we have

$$z = \xi^\omega \theta \tag{50}$$

and

$$y = \frac{dz}{d\xi}\frac{d\xi}{dt} = -\xi\frac{dz}{d\xi} = -\xi^{\omega+1}\frac{d\theta}{d\xi} - \omega z. \tag{51}$$

Therefore

$$\frac{d\theta}{d\xi} = -\xi^{-\omega-1}(y + \omega z). \tag{52}$$

Moreover, from the substitution in (21) we can infer that the limit $t \to \infty$ corresponds to $\xi \to 0$, and $t \to -\infty$ corresponds to $\xi \to \infty$.

2.1 Fixed points and nullclines

We are now ready to calculate our fixed points and go over the solution-curves in the (y,z) plane.

From (47) we can easily check that there are two fixed points, namely O_1 and O_2 :

$$y = z = 0; y = 0, z = [\omega(1-\omega)]^{\frac{1}{n-1}}. \tag{53}$$

The second fixed point only exists when $\omega(1-\omega) \geq 0$, which implies the restriction $0 < \omega < 1$. In order to get some insight into the behavior of the solutions near the origin, we linearize (47) at O_1 and compute the eigenvalues and eigenvectors. The linearization is

$$\begin{bmatrix} z' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega(\omega-1) & -(2\omega-1) \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix}$$

with eigenvalues

$$\lambda_1 = -\omega \quad \lambda_2 = -(\omega - 1) \tag{54}$$

and eigenvectors

$$(X, -\omega X); \quad (X, -(\omega-1)X), X \neq 0. \tag{55}$$

Hence, O_1 is a stable node if $\omega > 1$ and a saddle when $0 < \omega < 1$. From (55) we respectively obtain the following directions:

$$X : y = -(\omega-1)z; \qquad Y : y = -\omega z.$$

If we solve (48) for $\frac{dy}{dz}$ and use the substitutions for y and z in (50) and (51), we get:

$$\frac{dy}{dz} = -\frac{(2\omega-1)y + \omega(\omega-1)z + z^n}{y} = -\frac{(2\omega-1)\xi\frac{d\theta}{d\xi} + \omega^2\theta - \xi^2\theta^n}{\xi\frac{d\theta}{d\xi} + \omega\theta}. \tag{56}$$

Note that since our solution curve is characterized by $\theta' = 0$ at $\xi = 0$, when observing (50) and (51) it is shown that it must converge to the origin as $t \to \infty$ and from (56) we obtain

$$\frac{dy}{dz} \to -\omega = -\frac{2}{n-1}, \tag{57}$$

therefore our curve touches the Y-direction at the origin.

Proposition 2.1. *There is at most one solution which converges to the origin tangentially to the Y-direction as $t \to \infty$ [6, pp. 109-110].*

For the case $\omega < 1, (n > 3)$ we obtain the following linearization at O_2:

$$\begin{bmatrix} z' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 + 2\omega & -(2\omega - 1) \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix}$$

which leads to the eigenvalues

$$\lambda = \frac{-(2\omega - 1) \pm \sqrt{4\omega^2 + 4\omega - 7}}{2}. \tag{58}$$

Note that inside the square root of (58), we obtain zero when $\omega = \frac{2\sqrt{2}-1}{2} \approxeq 0.91$ since $\omega > 0$, and we will call this value ω^*. When $\omega < \omega^*$, we obtain complex eigenvalues, therefore O_2 is a stable focus. On the other hand, when $\omega > \omega^*$, both eigenvalues turn out to be negative, which implies that our fixed point is a stable node. When $n = 5$, just like our previous case where $\frac{1}{2} < \omega < 1, 3 < n < 5$, our linearization at O_2 will then be

$$\begin{bmatrix} z' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix}$$

which leads to

$$\lambda = \pm i. \tag{59}$$

Note that our λ in (59), lacks the real part,therefore we have a solution around the z-axis, shown by the graph on figure 8. Before analyzing the phase portraits, it comes useful to know the slopes values of y' and z' for the different values of n, since the solution trajectories are affected. When $1 < n < 5$, from (56) we can clearly see that $\frac{dy}{dz} = 0$ when $(2\omega - 1)y = -\omega(\omega - 1)z - z^n$. This curve is called the y'-*nullcline* and it represents the change of direction of y' (and hence $\frac{dy}{dz}$.) On the other hand, when $y = 0$, we obtain the z'-*nullcline* which will always overlap the z-axis. Here below, we obtained with python graphs illustrating these three different cases.

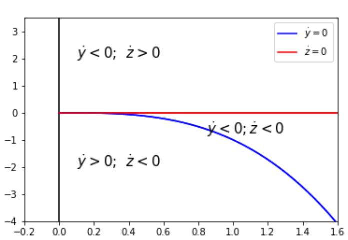


Figure 2: $n = 3$

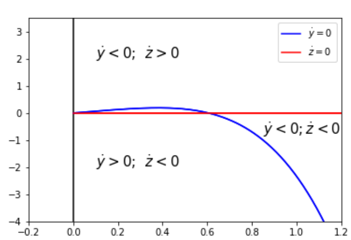


Figure 3: $n = 4$

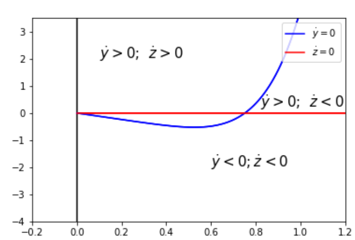


Figure 4: $n = 6$

We are now ready to break down our solutions into different cases depending upon the value of n, however, only the first case ($n \leq 3$) will be investigated in detail.

2.2 Case $\omega \geq 1, n \leq 3$

We will now rewrite (48) in the form

$$\frac{dy}{dz} = -(2\omega - 1) - \frac{\omega(\omega - 1)z + z^n}{y}. \tag{60}$$

We investigate the solution-curves in the z-positive half-plane, since we are exclusively considering θ and therefore z being greater or equal to zero. Moreover, given $\omega \geq 1$, O_1 is the only fixed point for this case.

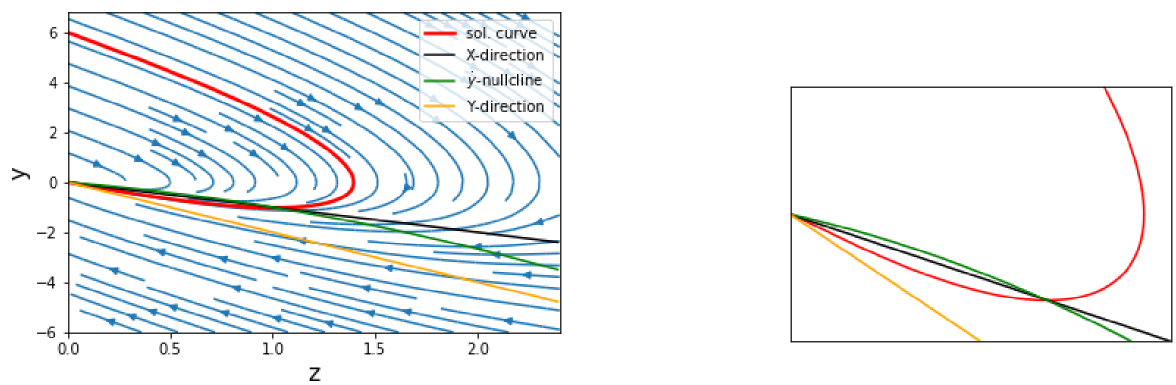


Figure 5: $n = 2, \omega = 2$

Both graphs (1) and (2) were obtained with python, and we can see that the solution curves obtained, match the information given by our eigenvalues and eigenvectors. N.B. in order for the reader to be able to clearly see near the origin the nullcline trajectory together with the solution curve and both X and Y directions, our interested area had to be zoomed in (see graph on the right.) At this point, we state and prove the following theorem:

Proposition 2.2. *All solutions starting from the positive y-axis, fall towards the z-axis as y decreases and z increases. Once the solution curve crosses the z-axis at* $y = 0$, *it continues to fall as both y and z decrease, until the intersection with the curve* $y' = 0$, *after which as y increases, z continues to decrease, therefore either a point is reached on the negative y-axis in finite time or the origin is reached as* $t \to \infty$ *[6].*

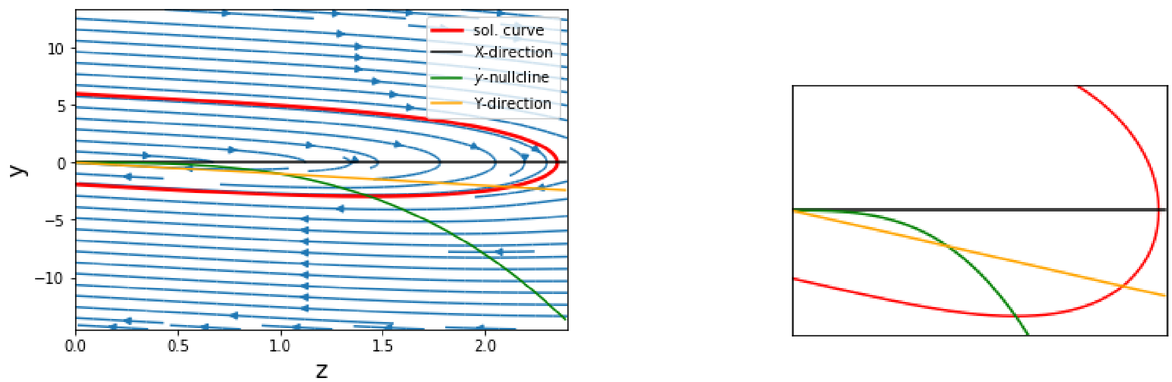


Figure 6: $n = 3, \omega = 1$

Proof. In order to prove the behavior of this solution, consider the point where the solution curve starts at the positive y-axis; $y(t_0) = y_0 > 0$ and $z(t_0) = z_0 = 0$. It follows that as long as we are in the upper quadrant, $z(t)$ increases and $y(t)$ decreases. From (47) and the assumption that $\omega \geq 1$, we also see that $y' < -z^n$ or $y' < -{z_1}^n$ for $t > t_1 > 0$, where $z_1 = z(t_1)$ in the upper quadrant. Thus $y \leq y(t_1) - {z_1}^n(t - t_1) < y_0 - {z_1}^n(t - t_1)$ for $t > t_1$. It follows that the curve will cross the z-axis for some $t < t_1 + y_0 {z_1}^{-n}$. Once the solution curve crosses the z-axis, y keeps decreasing and the slope of z turns negative too as when $z' = 0$ the $z'-nullcline$ is crossed, until the $y'-nullcline$ is intersected. After that, as y increases and z continues to decrease, the solution curve either approaches the fixed point O_1 as $t \to \infty$ or for some $t < \infty$ another point with $y < 0$ and $z = 0$ will be reached. □

Proposition 2.3. *For $1 < n \leq 3$, all solution curves are of the form described in Proposition 3.2 [6].*

Proof. By following the solution backward in time, if our starting point is the negative $y - axis$ for z sufficiently small, we are in a region where $\frac{dy}{dz}$ is positive, therefore

$$y < -\frac{\omega(\omega - 1)z + z^n}{(2\omega - 1)} \tag{61}$$

and when $y < 0$ from (60) we have

$$\frac{dy}{dz} > -(2\omega - 1) \tag{62}$$

or

$$y \geq y_1 - (2\omega - 1)(z - z_1) \tag{63}$$

for y_1 and z_1 being our initial points. If z is large, the equations above are contradictory, therefore we can safely infer that our solution curve must cross the $y'-nullcline$ for a finite z. Once the nullcline is crossed, both y and z increase. We will call y_2 the point where the solution curve enters the new region and this is the smallest value of y, therefore as long as we are in the lower quadrant, $|y_2| > |y|$. From (60) it follows that

$$\frac{dy}{dz} > -(2\omega - 1) + \frac{\omega(\omega - 1)z + z^n}{|y_2|} \tag{64}$$

therefore

$$y > -(2\omega - 1)z + \frac{\frac{1}{2}\omega(\omega - 1)z^2 + \frac{1}{n+1}z^{n+1}}{|y_2|} + C \tag{65}$$

where C is an integration constant. From this integration, we see that y becomes zero from some finite z therefore the $z - axis$ must be crossed vertically. Once our solution curve is in the first quadrant, y keeps increasing and z decreases, therefore, since $\frac{dy}{dz}$ is bounded, a point on the positive $y - axis$ is reached. □

Proposition 2.4. *There is a solution starting on the positive $y - axis$ which converges to the origin tangentially to the Y-direction as $t \to \infty$ [15].*

Proof. Using a version of the stable manifold theorem, [15, Theorem 9.3] we know that there is a solution curve which converges to the origin as $t \to \infty$ tangentially to the Y-direction. By Proposition 2.3 this curve, for t decreasing, after crossing both nullclines, intercepts the positive $y - axis$. □

The solution described above is called the E-curve and gives us a solution to the original problem (10). One can actually prove a bit more. Consider the solution $y(z; y_0)$ which passes through the point $y_0 < 0$ on the negative $y - axis$, then it can be shown that the E-curve is the limit of this curve as $y_0 \to 0$ [6, pp. 117-120]. It follows that the E-curve divides the phase portrait into regions by different solution 'families'. Let $y_0(E) > 0$ be the starting point of the E-curve on the positive $y-axis$. The solutions belonging to other families can easily be classified depending upon their starting point on the positive y-axis. If we consider a starting point $0 < y_0(M) < y_0(E)$, the M-solution will be bounded by the E-curve while eventually reaching the origin as $t \to \infty$. On the other hand if we consider a starting point $y_0(F) > y_0(E)$, the F-solution curve that will be generated, will lie outside of the area bounded by the E-curve and will reach a point $z = 0, y < 0$ at some finite t.

2.3 Case $\frac{1}{2} < \omega < 1, 3 < n < 5$

This case turns out to be slightly more difficult to deal with since there is the fixed point O_2, which as we have seen, can either be a stable focus or a stable node, depending on the value of ω; moreover, it is to be noted that O_1 is a saddle.

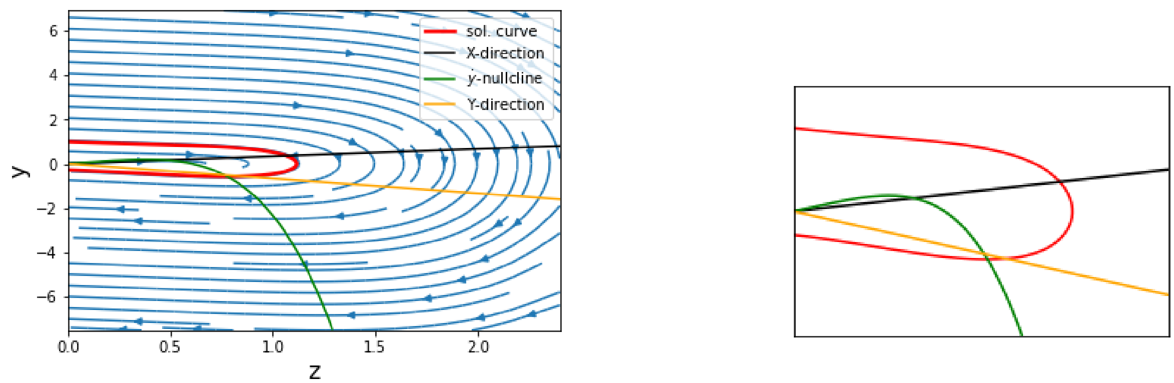


Figure 7: $n = 4, \omega = \frac{2}{3}$

The analysis of the solution curves for this case is somehow similar but presents a few more characterisics than case 2.2, given the presence of both fixed points and the new value of n giving the $y'-nullcline$ a different trajectory (see fig. 2). In this case, we still have the E-curve starting on the positive $y - axis$ and approaching O_1 tangentially to the Y-direction; this is the stable manifold at the origin [15, theorem 9.4]. When $\omega < \omega^*$, the curve tends to spiral towards O_2 given the imaginary root, and when $\omega > \omega^*$, our curve behaves very similarly to when $\omega \geq 1$. [6].

2.4 Case $\omega = \frac{1}{2}, n = 5$

by substituting these values in (47), it follows that

$$\begin{cases} z' = y \\ y' = \frac{1}{4}z - z^5. \end{cases} \tag{66}$$

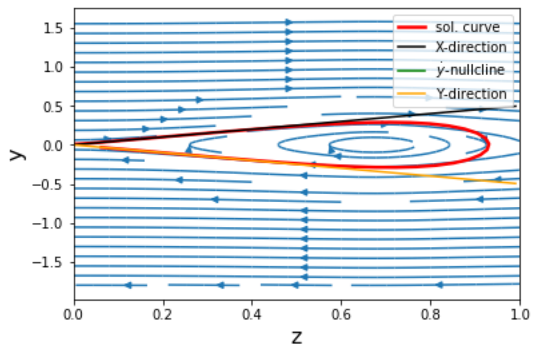


Figure 8: $n = 5, \omega = \frac{1}{2}$

In this case, our E-curve is tangential at the origin to the X and Y directions which are given by $y \pm \frac{1}{2}z = 0$. As was briefly mentioned in Sec 2, when the origin is approached along the Y-direction for $t \to \infty$, it corresponds to $\xi \to 0$, while the origin approached along the X-direction for $t \to -\infty$ corresponds to $\xi \to \infty$. In order for the solution curves to be analyzed in detail in this case, we integrate (47) for $n = 5, \omega = \frac{1}{2}$ which will give us

$$y^2 = \frac{1}{4}z^2 - \frac{1}{3}z^6 + D \tag{67}$$

where the sign of the integration constant D determines the family of the solution curve. For D=0 we obtain the E-curve, when D is positive, we obtain the F-solutions lying outside of the E-region, and finally, for negative D, the solutions we obtain will be of class M which are bounded by the E-curve. Note that in this particular case, the stable and unstable manifolds of the origin coincide.

2.5 Case $0 < \omega < \frac{1}{2}$, $n > 5$.

In this case, as $\omega < \frac{1}{2}$, we still have our fixed point O_2 being an unstable focus. Just like our previous case, after linearizing, we obtain an eigenvalue with an imaginary part indicating a rotation as well and a real part being positive, indicative of a growing solution curve. Given the new value of n, the y'-nullcline will initially go through the lower quadrant, and after intersecting the $z-axis$, it will tend to infinity in the upper quadrant (see fig. 3), therefore the solution curve will be influenced accordingly. In this case, none of the solutions starting on the positive $y-axis$ reach any of the fixed points. It is therefore easier to classify the solutions according to the point on the negative $y-axis$ which they pass through. We then describe their behavior backward in time. The only type of solution that approaches the origin when $t \to -\infty$ is what we call the D-curve, with some starting point $y_0(D)$ on the negative $y-axis$. As $t \to -\infty$, such a curve reaches O_1 tangential to the X-direction, this is the unstable manifold of the origin. For a starting point $y_0 < y_0(D) < 0$ the family of solutions that we obtain are entirely outside the region marked by the D-curve, and such solutions intersect the positive $y - axis$ as well backward in time. When our starting point is some $0 > y_0 > y_0(D)$, our solutions remain bounded by the D-curve and as $t \to -\infty$ and O_2 is approached spirally. Within this region, if we let the starting point $y_0 \to 0$, the E-curve tangential to the Y-direction is obtained. This curve spirals towards O_2 as well. This is the stable manifold of the origin.

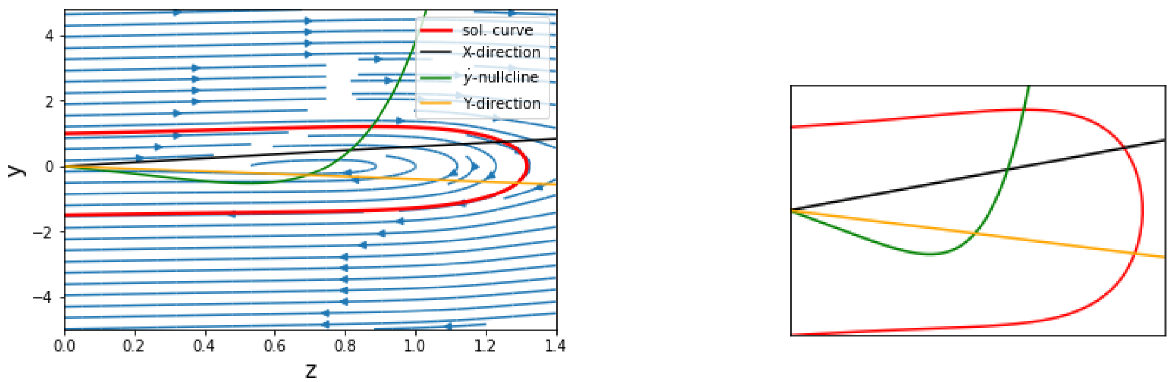


Figure 9: $n = 6, w = \frac{2}{5}$

It is clear from (21) and (50) that $z(t) = 0$ for some $t \in (-\infty, \infty)$ implies that θ becomes 0 at a finite $\xi = e^{-t}$ Throughout the analysis concerning $\theta(\xi)$ we can conclude that when $1 < n < 5$, the solution becomes 0 at a finite ξ and when $n \geq 5$, the solution tends to 0 as $\xi \to \infty$. Indeed, from (21) and (50) it follows that if $z(t) = 0$ for some $t \in \mathbb{R}$, then $\theta(\xi) = 0$ where $\xi = e^{-t}$. On the other hand, if $z(t)$ remains positive and bounded as $t \to -\infty$, then $\theta(\xi) = \xi^{-\omega} z(t) \to 0$ as $\xi = e^{-t} \to \infty$.

3 Physical quantities

From the Lane-Emden equation, important formulae are derived to describe polytropic stars' physical characteristics, and knowing the results for all the different values of the polytropic index n turns out to be very useful. Once the solution of the Lane-Emden equation for the different values of n are known, we may then express physical quantities like the radius of the star as well as its mass. For $n < 5$, the Lane-Emden function θ goes to zero at a finite value of ξ which leads to the density ρ being equal to zero according to the relationship $\rho = \rho_c \theta^n$.

3.1 Stellar radius

At this point we are ready to derive the stellar radius:

$$R = a\xi_1 = \left[\frac{(n+1)K}{4\pi G} \right]^{\frac{1}{2}} \rho_c^{\frac{1-n}{2n}} \xi_1 \tag{68}$$

where ξ_1 is the first zero and it represents the surface of the star. It is clear that for n=1 the equation above becomes independent of the central density ρ_c ; moreover, since $\xi_1 = \pi$, we obtain:

$$R = \left[\frac{K}{2\pi G} \right]^{\frac{1}{2}} \pi \tag{69}$$

therefore we see that the radius of a polytrope of index 1 shows exclusively a dependence on K.

3.2 Total mass

By substituting the values $r = a\xi$ and $\rho = \theta^n \rho_c$, the mass interior to ξ_0 will be

$$M(\xi_0) = \int_0^{a\xi_0} 4\pi \rho r^2 dr = 4\pi a^3 \rho_c \int_0^{\xi_0} \xi^2 \theta^n d\xi \tag{70}$$

but according to (9), $-\theta^n\xi^2 = \frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi})$, therefore we will get (70) in this form:

$$M(\xi_0) = -4\pi a^3\rho_c \int_0^{\xi_0} \frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -4\pi a^3\rho_c\xi_0^2\frac{d\theta}{d\xi}. \tag{71}$$

Now we will use the value of a from (8) so (71) becomes

$$M(\xi_0) = -4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{\frac{3}{2}} \rho_c^{\frac{(3-n)}{2n}}(\xi_0^2\frac{d\theta}{d\xi}); \tag{72}$$

the total mass M is given by (72) with $\xi_0 = \xi_1$. For n=3 we get another case where ρ_c disappears and our mass is

$$M = -4\pi\left[\frac{K}{\pi G}\right]^{\frac{3}{2}}(\xi_1^2\frac{d\theta}{d\xi}), \tag{73}$$

therefore, in this case the mass will not depend on the central density.

4 Chandrasekhar's equation

In this section we will go over spherically symmetric, self-gravitating gas spheres consisting of an isothermal fluid. These are described by Chandrasekhar's equation. In a system subjected to isothermal change, the temperature tends to remain constant. In order to derive this model, the introduction of the following physical quantities is necessary:

- W = mean molecular weight
- H = proton mass
- T = star temperature
- k_B= Boltzmann constant.
- σ= Stefan-Boltzmann constant.
- c= speed of light.

For isothermal gas spheres, the pressure is given by

$$P = \rho T\frac{k_B}{WH} + \frac{4\sigma}{3c}T^4 \tag{74}$$

we can now rewrite eq.(4) as:

$$K\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\log\rho}{dr}) = -4\pi\rho G. \tag{75}$$

where $K = \frac{k_B}{WH}T$ and log stands for the natural logarithm. By making the substitutions

$$\rho = \lambda e^{-\psi}; \quad r = a\xi = \left[\frac{K}{4\pi G\lambda}\right]^{\frac{1}{2}}\xi, \tag{76}$$

where λ is an arbitrary constant, we obtain the Chandrasekhar's equation for isothermal spheres:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) - e^{-\psi} = 0. \tag{77}$$

N.B. Since the relationship between P and ρ in (74) is linear, we can formally think of Chandrasekhar's equation as the Lane-Emden Equation with $n = \infty$ (see (5) and (6)). For complete isothermal gas spheres, we choose $\lambda = \rho_c$, therefore the relationship in (76) gives us the boundary condition $\psi(0) = 0$. Moreover, it follows from Proposition 1.1 that $\psi'(\xi) \to 0$ if ψ is a bounded solution. Therefore we have the following boundary conditions:

$$\psi = 0, \quad \frac{d\psi}{d\xi} = 0, \quad \xi = 0. \tag{78}$$

Moreover, for existence and uniqueness, the analysis of (77) is similar to the analysis of (10). In this case our initial value problem

$$\begin{cases} (\xi^2\psi')' - \xi^2 e^{-\psi} = 0 \\ \psi\,(0) = 0, \psi'\,(0) = 0, \end{cases}$$

with $f(u,\xi) = e^{-u}$ is another special case of (11) where $m = 3$ and $d = 0$. Therefore Proposition 1.2 applies here as well.

4.1 Transformations of Chandrasekhar's equation and singular solutions

By setting

$$x = \frac{1}{\xi} \tag{79}$$

we obtain

$$x^4 \frac{d^2\psi}{dx^2} = e^{-\psi} \tag{80}$$

which can be satisfied by the singular solution:

$$\psi = -\log 2 - 2\log x. \tag{81}$$

Because of the existence of such solutions, we can now introduce a new variable z:

$$z = -\psi - 2\log x. \tag{82}$$

After differentiating, we obtain:

$$\frac{d\psi}{dx} = -\frac{2}{x} - \frac{dz}{dx}; \quad \frac{d^2\psi}{dx^2} = \frac{2}{x^2} - \frac{d^2z}{dx^2} \tag{83}$$

and (80) becomes:

$$x^2 \frac{d^2z}{dx^2} + e^z - 2 = 0. \tag{84}$$

In order to eliminate x from (84) we can set

$$x = \frac{1}{\xi} = e^t \tag{85}$$

then

$$\frac{dz}{dx} = e^{-t}\frac{dz}{dt}; \quad \frac{d^2z}{dx^2} = e^{-2t}\left(\frac{d^2z}{dt^2} - \frac{dz}{dt}\right) \tag{86}$$

and we finally obtain

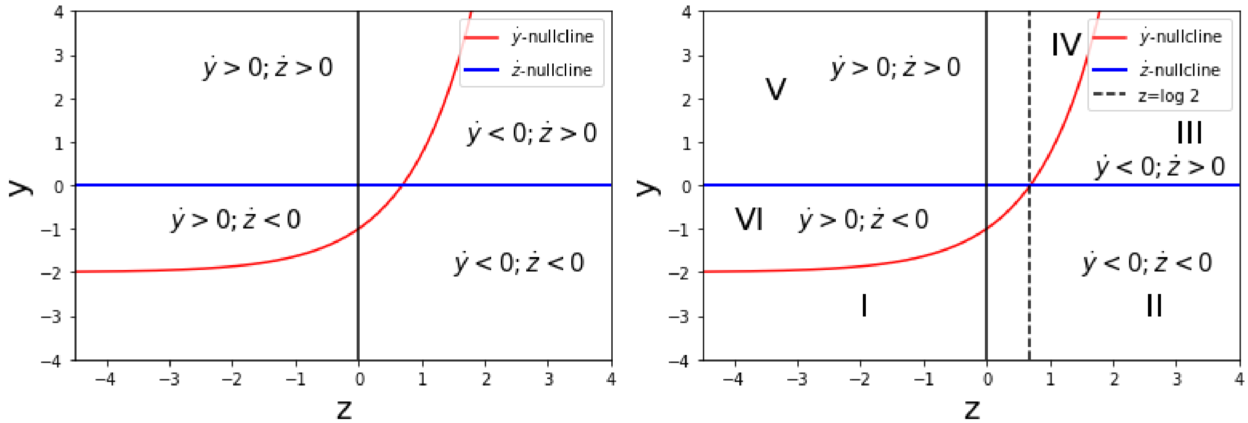
$$\frac{d^2z}{dt^2} - \frac{dz}{dt} + e^z - 2 = 0. \tag{87}$$

N.B. Unlike the polytropic case, we are no longer considering the restriction $z \geq 0$.

4.2 Chandrasekhar's equation in the (y,z) plane and fixed points.

Eq. (87) can be rewritten as the first order system

$$\begin{cases} z' = y \\ y' = y - e^z + 2, \end{cases} \tag{88}$$



(a) Nullclines for Chandrasekhar's Equation

(b) Different regions of the phase plane for Chandrasekhar's equation

Figure 10

where $y = \frac{dz}{dt}$. Whenever $z' = y \neq 0$ we can write y as a function of z and obtain the equation

$$y\frac{dy}{dz} - y + e^z - 2 = 0. \tag{89}$$

The nullclines of (88) are respectively

$$y = e^z - 2 \quad and \quad y = 0.$$

The only fixed point of (88) is

$$(z, y) = (\log 2, 0). \tag{90}$$

We will now recall (82)

$$z = -\psi + 2\log\xi, \tag{91}$$

and since $y = \frac{dz}{dt}$ it follows that

$$y = \frac{dz}{dt} = -\xi\frac{dz}{d\xi} = \xi\frac{d\psi}{d\xi} - 2. \tag{92}$$

Solving (89) for $\frac{dy}{dz}$ gives us:

$$\frac{dy}{dz} = \frac{2 + y - e^z}{y} \tag{93}$$

and according to the values of y and z in (91) and (92), (93) becomes

$$\frac{dy}{dz} = \frac{\xi\psi' - \xi^2 e^{-\psi}}{\xi\psi' - 2}. \tag{94}$$

From (91), (92) and (94) we observe that

$$z \to -\infty, \quad t \to \infty, \quad y \to -2, \quad \frac{dy}{dz} \to 0 \tag{95}$$

as $\xi \to 0$, therefore we see that our solution curve is asymptotic to the line $y = -2$, as $z \to -\infty$.

Proposition 4.1. *There is at most one solution asymptotic to the line $y = -2$ as $z \to -\infty$ [6, pp.162-163].*

In order to study the behavior of the solution curve near our fixed point $(\log 2, 0)$ we will linearize (88):

$$\begin{bmatrix} z' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix}.$$

Hence the eigenvalues are

$$\lambda_1, \lambda_2 = \frac{1}{2} \pm i\frac{1}{2}\sqrt{7} \tag{96}$$

which means that our fixed point is an unstable focus, implying that solutions starting sufficiently close converge to it as $t \to -\infty$. In fact, a stronger result can be proven.

Proposition 4.2. *All solutions of (88) converge to the fixed point $(\log 2, 0)$ as $t \to -\infty$.*

Proof. Let's consider our solution curve starting below the y-asymptote in region I at some point C with coordinates (z_0, y_0) and let's follow the solution backward in time (see figure 10 and 11). In this starting region, both $z', y' < 0$ which means that such a curve either enters region II, reaching some point E with coordinates $(z_1, y_1) > (z_0, y_0)$ or converges to the fixed point. In region II, y is still negative and we have $e^z \geq 2$ as $z = \log 2$ is crossed, therefore from (93) we obtain

$$\frac{dy}{dz} = \frac{e^z - 2 - y}{-y} \geq \frac{-y}{-y} = 1$$

which implies that region III must be entered. In this region $z' > 0, y' < 0$ so the solution curve must enter region IV where z' stays positive and y' turns positive.

Hence, region V must be entered or the fixed point must be reached. If region V is approached, we have $e^z \leq 2$ and from (93) we derive

$$\frac{dy}{dz} = \frac{2 - e^z + y}{y} \geq \frac{y}{y} = 1$$

which leads the solution curve to region VI. In region VI we have $z' < 0, y' > 0$ which leads to the solution curve entering region I staying above the $y-$asymptote after crossing the $y'-$nullcline at some point D parallel to E. Since the y component increases from now on, the horizontal line between D and E together with the curve between these two points form a backward invariant region, this implies that as $t \to -\infty$ all solutions remain bounded. Above we mentioned the possibility of the solution converging to the fixed point from region I and IV, but in fact this cannot happen since the fixed point is an unstable focus [10, pp.188-195].

Let's now consider a solution starting at some point A in any other region. If A lies within the backward invariant region discussed above, there is no way the curve can "escape", given that it will be bounded by the outer solution. However, if A lies outside of what we just labeled as the outer solution, it must reach a point B on the y'-nullcline and enter region I while staying above the $y-$asymptote. Since the B coordinates are lower than the D coordinates, the curve is not in the backward invariant set, and while this solution is continued backward in time, it must remain above the solution starting at C. Since in region I y increases, the solution must eventually hit the horizontal line between D and E, which implies that the backward invariant region is entered. According to the generalized Poincare-Bendixon theorem [15, Theorem 7.16] the limit set of the solution is either a fixed point, a periodic orbit or a heteroclinic cycle. The last alternative can be excluded as there is only one fixed point which is an unstable focus (hence there is no homoclinic orbit connected to it). The second one can be ruled out as well using Bendixson's criterion [15, p. 227]: $\text{div}(y, e^z - y - z) = \frac{\partial}{\partial z}(-y) + \frac{\partial}{\partial y}(e^z - y - z) = -1 \neq 0$. Hence, the limit set is just the fixed point $(\log 2, 0)$ and it then follows that the solution converges to the fixed point [15, Lemma 6.7].

□

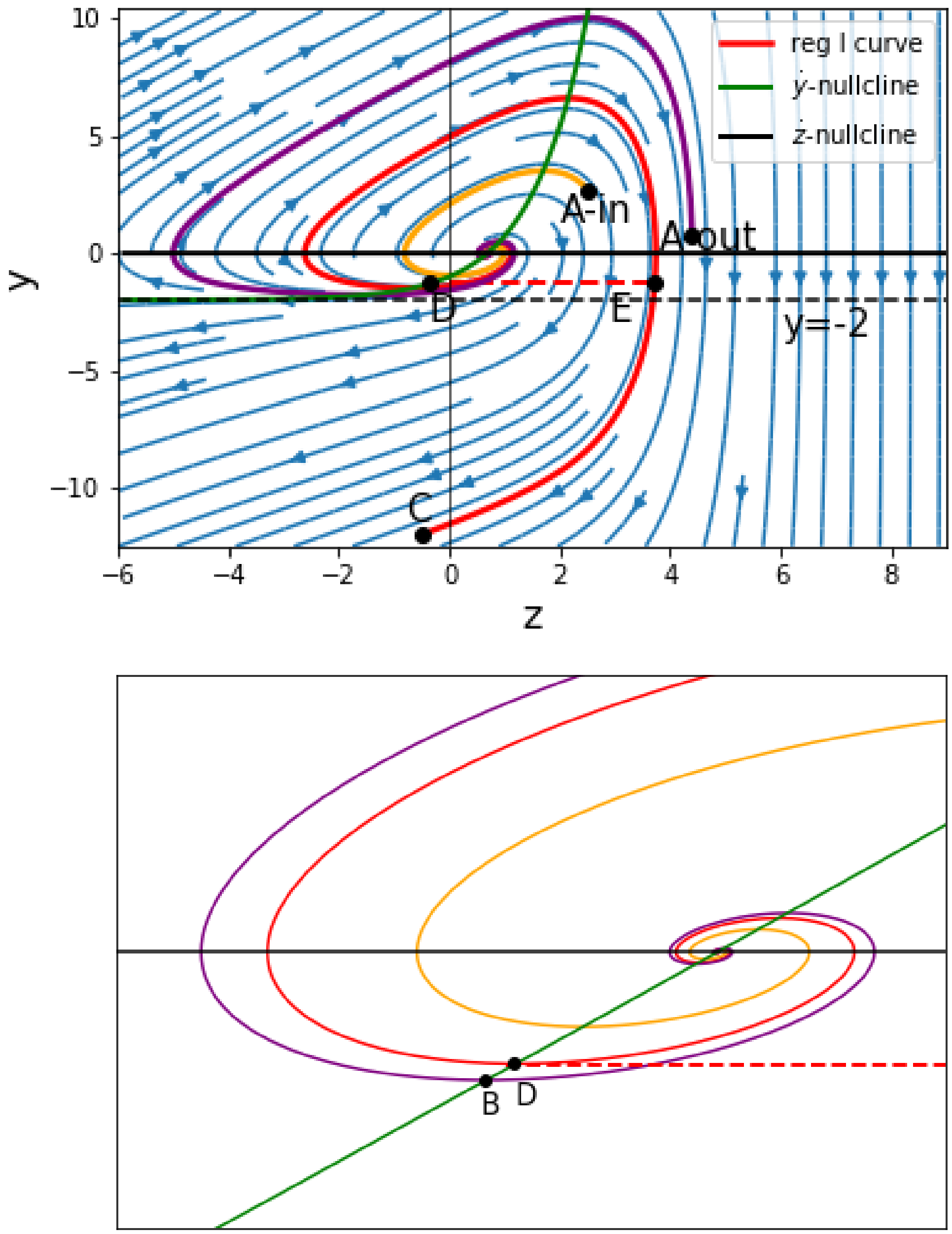


Figure 11: Phase plane for the Chandrasekhar's equation with zoomed in backward invariant region

Similarly to the analysis in Sec.2 of $\theta(\xi)$ when $n \geq 5$, here the solution converges to the fixed point when $\xi \to \infty$ for t decreasing.

5 Conclusion: stellar structure from the 1800's to modern research

The study of stellar structures has long been a fascinating topic to the scientific community. Even though one of the first attempts for a mathematical model governing gas spheres did not happen until Newton's time (see his equation of hydrostatic equilibrium in ch.1), the story, at least conceptually, started with the Greek philosopher Anaxagoras (510 B.C.-428 B.C.) who described stars as flaming stones, and continued with another Greek philosopher, namely Aristotle (384 B.C.-322 B.C.) who appealed to the analogy of observing a lead projectile melting as a result of its passage through the air, in order to explain that stars are heated the same way [1, p.211].

Moving to more modern times, the Lane-Emden Equation was the first one describing gaseous stellar structure models considering spherically symmetric stars in hydrostatic equilibrium and such a representation did not come until the 1800's. The equation was investigated thoroughly in the late 19th century and the early 20th century. However, other forms than the equations we worked with are a current research topic. The Lane-Emden polytropic and isothermal equations are another form of the Euler-Poisson's equation, which is a partial differential equation that has been lately used to model rotating gaseous stars. The first problems aimed to find a family of rotating star solutions were solved around 1971 by Auchmuty and Beals [3], and many more results followed ever since. As an example, extrasolar planet structures are a contemporary interest where a solid rocky core together with its gravitational potential need to be included. The Euler-Poisson equation system is needed to describe the dynamical evolution of a star without any symmetry restriction. As we analyzed the spherically symmetric case, we could see that the solutions and their behaviors are sometimes explicit, as opposed to the rotating star solutions which are provided by abstract existence theorems from the calculus of variations [17] and nonlinear analysis [2] [8].

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