

EXISTENCE AND APPROXIMATIONS OF OPTIMAL WAVELETS

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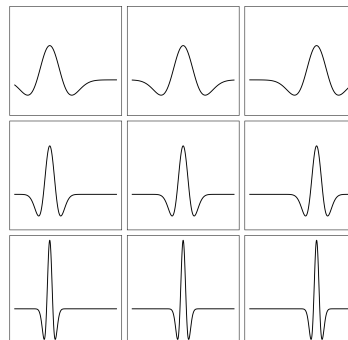
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Abstract

In this thesis we consider two notions of mother wavelet uncertainty for the one-dimensional wavelet transform based on a new localization framework recently introduced by Levie and Sochen. It is shown that minimizers of these uncertainties in the space of linear splines with compact support approximate the minimizers in L^2 in an appropriate sense, validating numerical schemes for searching for minimizers.

Populärvetenskaplig sammanfattning

Wavelettransformen är ett verktyg inom dataanalys för att analysera signaler såsom EKG:er, ljudinspelningar, temperaturavläsningar eller aktiekurser. Transformen bygger på att man låter en funktion som kallas en *wavelet* bygga upp ett bibliotek av förflyttningar och utdragningar av sig själv vilket är illustrerat i bilden till höger. Sedan beskrivs signalen som summan av element ur detta bibliotek. Själva wavelettransformen är informationen om exakt vilka element som utgör signalen.



Figur: Bibliotek av förflyttningar och utsträckningar av en wavelet.

Med hjälp av wavelettransformen kan man komprimera information genom att endast spara de viktigaste termerna i summan. För maskininlärning kan man låta ett neuralt nätverk analysera resultatet av wavelettransformen av signalen istället för signalen. Detta har gjorts med både MR-undersökningar av hjärnan och röntgenbilder av lungor.

Det finns många olika wavelets och vilken vi använder beror på sammanhanget. Gemensamt är att man vill att waveleten ska vara maximalt olik förflyttningar och omskalningar av sig själv för att inte spara överflödigt information. Dock har det historiskt saknats ett matematiskt ramverk för att beskriva detta. Denna uppsats går igenom ett sådant ramverk som nyligen har introducerats och behandlar wavelets som är optimala sett till två olika mått av lämplighet.

Algoritmer för att konstruera optimala wavelets sett till dessa mått har skrivits i år. I datorer kan vi dock bara arbeta med en begränsad samling funktioner som inte kan vara lika välbetorda som de funktionerna som faktiskt är optimala. Uppsatsens stora bidrag är ett bevis för att vi i datorn kan konstruera wavelets som är godtyckligt nära att vara optimala. Förhoppningen är att waveletserna som konstrueras med hjälp av dessa algoritmer därmed ska kunna användas i tillämpningar i framtiden.

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Introduction

Wavelet analysis is a technique introduced in the 1970's which has applications in ECG analysis, seismic wave analysis, preprocessing for machine learning applications, image compression and noise reduction among many others. The fundamental idea is to express a signal as the sum of time shifts and dilates of a *mother wavelet*. This is similar to how a signal can be decomposed in a Fourier series but with each term being localized in time. A fundamental problem is to choose a mother wavelet such that each term conveys as much information about time and scale as possible. This requires the mother wavelet to be well localized in both time and scale.

The first wavelet was constructed in 1910 by Haar [1]. Since the proper mathematical foundation was laid in place in the 70's and 80's, a wealth of new wavelets have been designed. Many wavelets, such as the Daubechies wavelets, are constructed based on qualitative features including vanishing moments and compact support. This is partly due to quantitative effort based around defining uncertainties have been largely unsuccessful. However, a new framework for localization has recently been introduced by Levie and Sochen [2] that allows more meaningful notions of uncertainty to be formulated. This thesis presents the basics of this framework and uses it to define and a notion of uncertainty for wavelets and bound it.

We also discuss another notion of uncertainty proposed by Levie, Avraham and Sochen [23]. Motivated by the problem of designing minimizers for these, the bulk of the thesis is dedicated to proving that by looking for minimizers in the finite dimensional space of linear splines, we can approximate the proper L^2 minimizers. This validates numerical minimizing schemes which can only be carried out in finite dimensional spaces.

The exposition in chapters 1 and 2 are primarily based on [3]–[5]. The definition of the uncertainty presented in Section 2.4 and the proof for the existence of a minimizer are from the PhD thesis of Levie [6]. The main ideas for the convergence of spline minimizers in Chapter 3 and the appendix are from the unpublished notes [7]. The uncertainty bounds in Section 2.4 are novel and so are the results in Chapter 4. The latter should be considered as the main contribution of this thesis.

Chapter 1

Windowed transforms

In this chapter we briefly introduce the short-time Fourier transform and the wavelet transform and discuss some of their elementary properties and a generalization. The exposition is primarily based on the material in the books [3], [4] by Gröchenig and Mallat respectively.

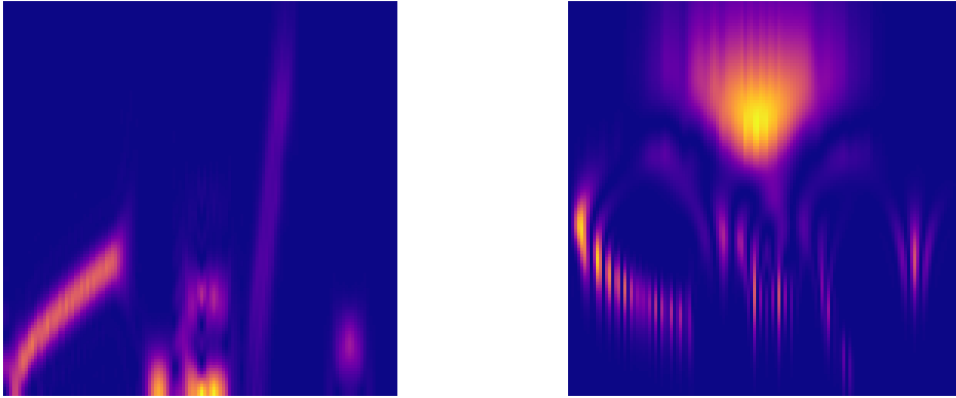


Figure 1.1: The output of the short-time Fourier transform and the wavelet transform applied to the same signal from Example 1.7 below.

Notational conventions

Throughout the thesis we take norms and inner products without subscripts to be in $L^2(\mathbb{R})$ so that $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R})}$ and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2(\mathbb{R})}$. We use the normalization of the Fourier transform where

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} dt$$

and the characteristic function of a set E is denoted by χ_E . For two operators A, B , their commutator is denoted by $[A, B] = AB - BA$ and the adjoint of A is denoted by A^* . A normalized function f in a Hilbert space $\mathcal{H}$ is always a function such that $\|f\| = 1$.

In chapters 3 and 4, all functions and operators are taken to be in the frequency domain unless specified otherwise and the hat notation is dropped.

1.1 Short-time Fourier transform

The most classical windowed transform is the *short-time Fourier transform* or *windowed Fourier transform*. The underlying idea is that with the Fourier transform we can determine which frequency make up a signal, but if those frequencies change over time, the Fourier transform won't be able to tell us at which times they were dominant. Therefore we take the Fourier transform of different time slices of the original signal and end up with a 2 dimensional plot of which frequencies made up the signal at different times.

Definition 1.1 (Short-time Fourier transform). Fix a *window function* $g \neq 0$. The *short-time Fourier transform* of a function f with respect to g is defined as

$$V_g(x, \omega) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i t \omega} dt = \langle f, M_\omega T_x g \rangle$$

where $T_x : g(t) \mapsto g(t-x)$ and $M_\omega : g(t) \mapsto e^{2\pi i t \omega} g(t)$ are the *translation* and *modulation* operators.

The spread in time and frequency of the window function g fundamentally limits the resolution of $V_g f$. Therefore, g is usually taken to be a Gaussian which is optimal in a sense discussed in sections 2.2.1 and 2.3.1.

Somewhat analogously to Parseval's formula for Fourier series, we for the short-time Fourier transform have *Moyal's formula*

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^2)} = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}$$

which has the following direct corollary.

Corollary 1.2. If $f, g \in L^2(\mathbb{R})$, then

$$f = \frac{1}{\|g\|^2} \iint_{\mathbb{R}^2} V_g f(x, \omega) M_\omega T_x g d\omega dx$$

and

$$\|V_g f\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}.$$

In particular, if $\|g\| = 1$, then V_g is an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^2)$.

In view of the above corollary, $|V_g f|^2$ is a good real valued proxy for the short-time Fourier transform which is easier to visualize but still carries information about how the energy of the signal is spread out. It is referred to as the *spectrogram* of f with respect to g .

Example 1.3. To illustrate the short-time Fourier transform we choose the signal

$$f(t) = \begin{cases} \frac{1}{2} \sin(t), & 0 \leq t < 5 \\ \sin(2t), & 5 \leq t < 15, \\ \frac{1}{2} \sin(t), & 15 < t < 20, \\ \sin(t), & 20 \leq t \leq 32. \end{cases}$$

Applying the short-time Fourier transform with the Gaussian window function $g(t) = \pi^{-1/4} e^{-t^2}$ we get the following spectrogram.

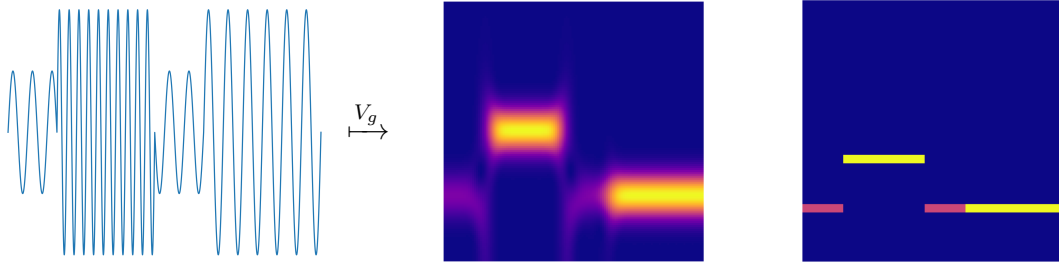


Figure 1.2: The function $f(t)$, its spectrogram and an "ideal" version of its spectrogram.

By introducing a parameter b which determines if g or $\hat{g}$ is well localized as

$$g_b(t) = \frac{1}{b\sqrt{\pi}} e^{-t^2/b^2},$$

we can choose if we want our short-time Fourier transform to have good resolution in time or frequency. This is illustrated in the series of figures below.

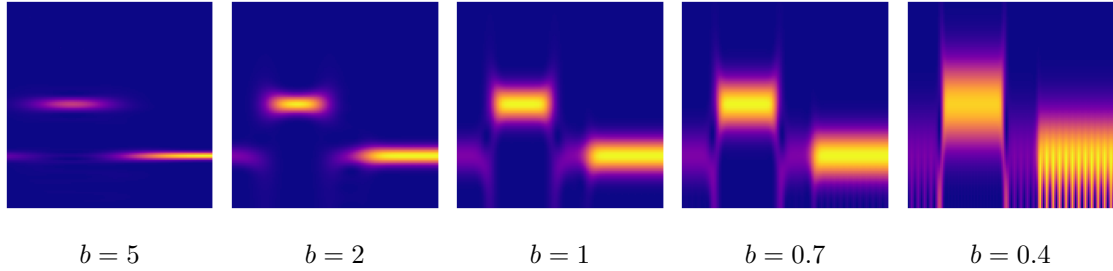


Figure 1.3: The spectrogram of f with respect to g_b for different values of the parameter b . Note that as b increases the time localization increases while the frequency localization decreases.

1.2 Wavelet transform

For the wavelet transform, instead of modulating the window function we dilate it, replacing the frequency parameter by a *scale parameter*. This makes it easier to investigate the signal at different scales. We also place some additional restrictions on which functions we allow to be windows.

Definition 1.4. A function $f \in L^2(\mathbb{R})$ is said to be *admissible* if

$$C_f = \int_{\mathbb{R}} \frac{|\hat{f}(\omega)|^2}{|\omega|} d\omega < \infty.$$

Moreover, if f is also integrable, normalized and satisfies

$$\int_{\mathbb{R}} f(t) dt = \hat{f}(0) = 0,$$

it is said to be a *wavelet*.

Remark. The $C_f < \infty$ condition is commonly referred to as the *admissibility condition* or *Caldéron's condition*.

Definition 1.5 (Continuous wavelet transform). For a wavelet f and a signal $s \in L^2(\mathbb{R})$, the *continuous wavelet transform* of s with respect to f is given by

$$W_f(\alpha, b) = e^{-\alpha/2} \int_{\mathbb{R}} s(t) \overline{f\left(\frac{t-b}{e^\alpha}\right)} dt = \langle s, D_\alpha T_b f \rangle$$

where $D_\alpha : f(t) \mapsto e^{-\alpha/2} f(te^{-\alpha})$ is the dilation operator.

Remark. An alternative (and equivalent) definition is

$$W_f(a, b) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} s(t) \overline{f\left(\frac{t-b}{a}\right)} dt \quad (1.1)$$

which is used in e.g. [4], [8], [9].

Like the short-time Fourier transform, the wavelet transform preserves energy and is invertible in the sense of the following theorem, originally proved in a more theoretical context by Caldéron in [10].

Proposition 1.6 ([4]). *Let f be a real valued wavelet and W_f the wavelet transform of the signal $s \in L^2(\mathbb{R})$ with respect to f . Then*

$$s(t) = \frac{1}{C_f} \int_{\mathbb{R}^+} \int_{\mathbb{R}} W_f(a, b) \frac{1}{\sqrt{a}} f\left(\frac{t-b}{a}\right) db \frac{da}{a^2}$$

and

$$\|s\|^2 = \frac{1}{C_f} \int_{\mathbb{R}^+} \int_{\mathbb{R}} |W_f(a, b)|^2 db \frac{da}{a^2}.$$

The function $|W_f(a, b)|^2$ is known as the *scalogram* and is a natural counterpart to the spectrogram of the short-time Fourier transform. The above formulas also provides some intuition on why we require $C_f < \infty$ as this plays a similar role to $\|g\| < \infty$ in Corollary 1.2.

Example 1.7. Analyzing the signal

$$s(t) = \begin{cases} \sin(|t|^{1.7}/3), & 0 \leq t < 20, \\ e^{\frac{(t-25)^2}{2}} + 2e^{\frac{(t-30)^2}{0.2}} + e^{\frac{(t-32)^2}{0.2}} + 2e^{\frac{(t-34)^2}{0.2}} + e^{\frac{(t-36)^2}{0.2}}, & 20 \leq t < 40, \\ \sin((t-40)^2), & 40 \leq t < 46, \\ (1 - (t-56)^2)e^{-\frac{(t-56)^2}{2}}, & \text{otherwise} \end{cases}$$

with the classical *Mexican hat wavelet* given by $f(t) = \frac{2}{\sqrt{3\sqrt{\pi}}}(1-t^2)e^{-t^2/2}$, we get the following scalogram.

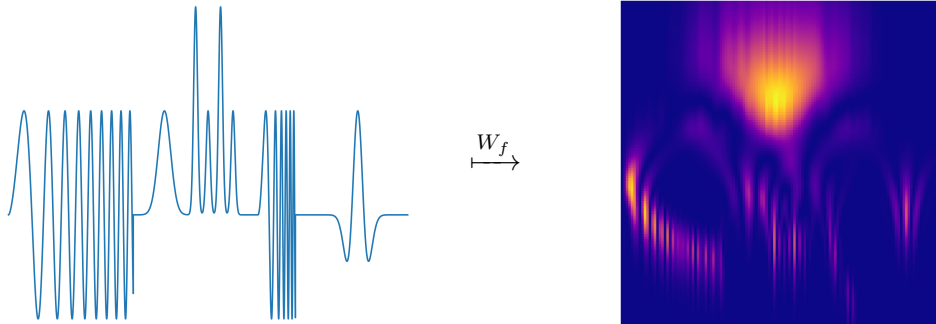


Figure 1.4: The signal $s(t)$ and its scalogram.

1.3 Generalized wavelet transforms

Instead of viewing the above transforms as given, they can be taken to be induced by the translations, modulations and dilations via *representations*, defined as follows.

Definition 1.8. A (unitary) *representation* $(\pi, \mathcal{H})$ of a locally compact group G is a strongly continuous homomorphism from G into the group $\mathcal{U}(\mathcal{H})$ of unitary operators on the Hilbert space $\mathcal{H}$. This means that π satisfies

- (1) $\pi(g_1 g_2) = \pi(g_1) \pi(g_2)$,
- (2) $\pi(g^{-1}) = \pi(g)^{-1} = \pi(g)^*$,
- (3) $g_n \rightarrow g \implies \pi(g_n) f \rightarrow \pi(g) f$.

In the case of the short-time Fourier and wavelet transforms, the group is $\mathbb{R}^2$ and the Hilbert space is $L^2(\mathbb{R})$. The short-time Fourier transform is induced by a representation π via $V_g(x, \omega) = \langle f, \pi(x, \omega) g \rangle$ which can be decomposed as

$$\begin{aligned} M_{g_2} T_{g_1} f(t) &= \pi(g_1, g_2) f(t) = \pi_2(g_2) \pi_1(g_1) f(t), \\ \pi_1(g_1) f(t) &= f(t - g_1), \\ \pi_2(g_2) f(t) &= e^{2\pi i g_2 t} f(t). \end{aligned}$$

While in the case of the wavelet transform

$$\begin{aligned} D_{g_2} T_{g_1} f(t) &= \pi(g_1, g_2) f(t) = \pi_2(g_2) \pi_1(g_1) f(t), \\ \pi_1(g_1) f(t) &= f(t - g_1), \\ \pi_2(g_2) f(t) &= e^{-g_2/2} f(t e^{-g_2}). \end{aligned}$$

These representations induce groups with the set G being the domain and the group law determined by the $\pi(g_1) \pi(g_2) = \pi(g_1 g_2)$ relation. The group induced to the short-time Fourier transform representation is known as the *Heisenberg group* and for the wavelet transform we have the *affine group* provided we use the alternative representation of equation (1.1). We could have viewed the representations and in turn the transforms as being induced by these groups, a view frequently taken in research and introduced for the wavelet transform by Grossman, Morlet and Paul in [11].

From the representations, we can obtain the self-adjoint operators

$$X_k : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad f(t) \mapsto \left. \frac{\partial}{\partial g_k} \pi(g_1, g_2) f(t) \right|_{g=0}$$

for $k = 1, 2$. These are known as (*infinitesimal*) *group generators* and are canonical for the groups in a sense which we do not pursue further. We take up this direction again in Section 2.2.

We want our Hilbert space to be as small as possible to enable certain constructions in Section 2.3 in the sense of the following definition:

Definition 1.9. A representation $(\pi, \mathcal{H})$ is said to be *irreducible* if $\{0\}$ and $\mathcal{H}$ are the only closed subspaces that are invariant under all operators $\pi(h)$.

We can generalize the concept of the short-time Fourier and wavelet transforms to include a larger class of transforms as in [2].

Definition 1.10. Let $\mathcal{H}$ be the Hilbert space of signals to be transformed, G a manifold and $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ a representation of G . For a Radon measure on G , take $L^2(G)$ to be the output space of the *generalized wavelet transform* defined as

$$\begin{aligned} V_f : \mathcal{H} &\rightarrow L^2(G), \\ V_f[s](g) &= \langle s, \pi(g)f \rangle. \end{aligned}$$

The manifold G is called the *phase space* and the vector $f \in \mathcal{H}$ is called the *window function*.

The short-time Fourier transform and continuous wavelet transform are examples of generalized wavelet transforms with the phase space $\mathbb{R}^2$ being the cartesian product of time and frequency and time and scale respectively. Below we list the representations of some other common transforms who operate on signals in $L^2(\mathbb{R}^2)$ to illustrate different generalizations.

- **Ridgelet transform [12]:** The ridgelet transform is perhaps the most obvious two-dimensional generalization of the wavelet transform. It allows for rotations in the following sense

$$\pi(a, b, \theta)f(x_1, x_2) = \frac{1}{\sqrt{a}}f\left(\frac{\cos \theta x_1 + \sin \theta x_2 - b}{a}\right).$$

- **Shearlet transform [13]:** For the shearlet transform, rotations and dilations are replace by *parabolic scalings* and *shears* via

$$\begin{aligned} \pi(a, s, b)f(x) &= a^{-\frac{3}{4}}f(A_a^{-1}S_s^{-1}(x - b)), \\ A_a &= \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}, \quad S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

making the transform better suited for extracting anisotropic features.

- **Curvelet transform [14]:** For the curvelet transform, the window function is decomposed to a polar pair $(W(r), V(t))$ which is affected by the representation as

$$\begin{aligned} \pi(a, b, \theta)(W, V) &= \gamma_{a00}(R_\theta(x - b)), \\ \hat{\gamma}_{a00}(r, \omega) &= W(ar)V(\omega/\sqrt{a})a^{3/4} \end{aligned}$$

where $\hat{\gamma}$ is in polar-Fourier coordinates.

Chapter 2

Localization and uncertainty principles

Having defined the wavelet transform, we now turn to the problem of designing optimal wavelets. One way of doing this is to define an *uncertainty* and let that act as a loss function for numerical purposes. The ultimate goal is to design a wavelet f which makes the wavelet transform W_f as sparse as possible for any arbitrary signal. This requires that changing the input parameter (α, b) slightly should result in a large change in $W_f[s]$. We therefore want an uncertainty which punishes wavelets which are not *well localized* in time or scale.

2.1 Uncertainty principles

The concept of *spread* of a function is most often expressed using expected values and variances. This is analogous to the situation in quantum mechanics where we consider the variance of the modulus square of the wavefunction instead of a wavelet.

Definition 2.1. Given a self-adjoint operator T and a normalized vector $f \in \mathcal{H}$, the *expected value* and *variance* of T with respect to f are the quantities

$$e_f(T) = \langle Tf, f \rangle,$$
$$v_f(T) = \langle (T - e_f(T))^2 f, f \rangle$$

respectively.

We will in this chapter introduce the concepts of *scale* and *scale space* which are analogous to frequency and frequency space, allowing us to define a notion of localization in scale.

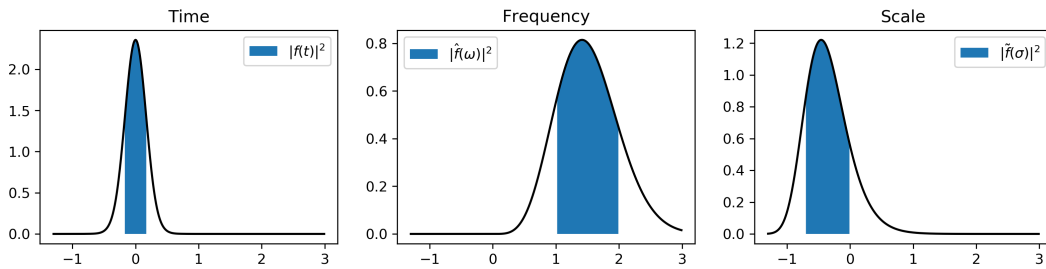


Figure 2.1: The modulus squared of the complex Mexican hat wavelet in time, frequency and scale space together with its standard deviation in each space.

What remains now is to detail what operators are appropriate to consider and what we can say about their variances. The most classical tool for bounding the variances is the *general uncertainty principle*, originally proved by Robertson in [15].

Theorem 2.2. *Let A, B be self-adjoint operators on a Hilbert space $\mathcal{H}$ and a, b constants, then*

$$\|(A - a)u\| \|(B - b)u\| \geq \frac{1}{2} |\langle [A, B]u, u \rangle| \quad \text{for all } u \in \mathcal{H}$$

and equality can hold only if

$$(A - a)u = i\lambda(B - b)u$$

for some $\lambda \in \mathbb{R}$.

The proof is elementary and follows by expanding the right-hand side and using the triangle inequality followed by Cauchy-Schwarz. Since the constants a, b are arbitrary in the above, we can immediately deduce the following corollary bounding the product of the variances by setting $a = e_f(A), b = e_f(B)$.

Corollary 2.3. *Let A, B be self-adjoint operators on a Hilbert space $\mathcal{H}$, then*

$$v_f(A)v_f(B) \geq \frac{1}{4} |\langle [A, B]f, f \rangle|^2$$

and equality can hold only if

$$(A - e_f(A))f = i\lambda(B - e_f(B))f$$

for some $\lambda \in \mathbb{R}$.

The most famous example of the uncertainty principle is *Heisenberg's uncertainty principle* which is most often stated with time and momentum measuring operators. In that case the commutator is a constant multiple of f and since f is normalized, this means that we get a lower limit on the uncertainty.

2.2 Group generators as localization operators

In this section we compute group generators for the short-time Fourier transform and wavelet transform and apply Corollary 2.3 to them.

2.2.1 Short-time Fourier transform

For the short-time Fourier transform from Definition 1.1, the group representation is of the form $\pi(x, \omega)f(t) = f(t - x)e^{2\pi i t \omega}$. The group generators associated to this representation are

$$\begin{aligned} X_x f(t) &= \frac{\partial}{\partial x} \pi(x, \omega) f(t) \Big|_{x=0, \omega=0} \\ &= \frac{\partial}{\partial x} f(t - x) e^{2\pi i t \omega} \Big|_{x=0, \omega=0} = -f'(t) \end{aligned}$$

and

$$\begin{aligned} X_\omega f(t) &= \frac{\partial}{\partial \omega} \pi(x, \omega) f(t) \Big|_{x=0, \omega=0} \\ &= \frac{\partial}{\partial \omega} f(t-x) e^{2\pi i t \omega} \Big|_{x=0, \omega=0} = 2\pi i t f(t). \end{aligned}$$

With this we can compute the commutator as

$$[X_x, X_\omega] f(t) = (X_x X_\omega - X_\omega X_x) f(t) = -2\pi i (t f'(t) + f(t) - t f'(t)) = -2\pi i f(t).$$

Hence, applying Corollary 2.3 to these operators we get a result equivalent to Heisenberg's uncertainty principle with the lower limit of the uncertainty being some constant independent of f and Gaussians being equalizers.

2.2.2 Continuous wavelet transform

An early attempt of extending the concept of using group generators as localization operators to the wavelet transform was made by Dahlke and Maas in [16]. They used the representation $\pi(a, b) f(t) = |a|^{-1/2} f\left(\frac{t-b}{a}\right)$ from equation (1.1) which yields the generators

$$\begin{aligned} \tilde{X}_a f(t) &= \frac{\partial}{\partial a} \pi(a, b) f(t) \Big|_{a=1, b=0} \\ &= \frac{\partial}{\partial a} |a|^{-1/2} f\left(\frac{t-b}{a}\right) \Big|_{a=1, b=0} = -\frac{1}{2} f(t) - t f'(t), \\ \tilde{X}_b f(t) &= \frac{\partial}{\partial b} \pi(a, b) f(t) \Big|_{a=1, b=0} \\ &= \frac{\partial}{\partial b} |a|^{-1/2} f\left(\frac{t-b}{a}\right) \Big|_{a=1, b=0} = -f'(t). \end{aligned}$$

These operators are skew-adjoint so we multiply them by i to make them self-adjoint which we denote by removing the tildes. From here we can apply Corollary 2.3 to get

$$v_f(X_a) v_f(X_b) \geq \frac{1}{4} |\langle (X_a X_b - X_b X_a) f, f \rangle|^2 = \frac{1}{4} |\langle f', f \rangle|^2.$$

Note that this lower limit is zero for real valued f . Compare this to the short-time Fourier transform case where we obtained an absolute constant independent of f as the lower limit. In further investigations by Maass, Sagiv, Sochen and Stark in [8], it was shown that by disregarding the admissibility condition $C_f < \infty$ it is possible to construct a sequence of normalized f_n such that $v_{f_n}(X_a) v_{f_n}(X_b) \rightarrow 0$ as $n \rightarrow \infty$. This is illustrated in the figure below.

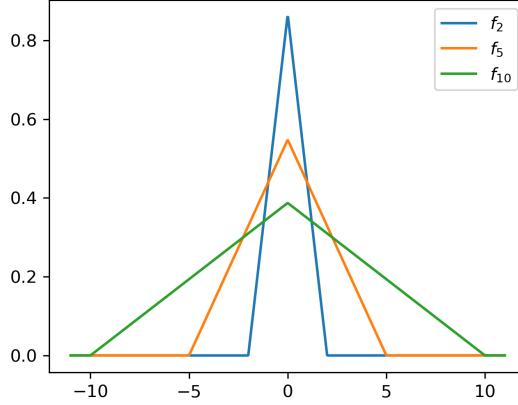


Figure 2.2: The sequence of non-admissible f_n with $v_{f_n}(X_a)v_{f_n}(X_b) \rightarrow 0$ as $n \rightarrow \infty$ from [8].

2.3 Observables as localization operators

Introduced by Levie, Stark, Lieb and Sochen in [5] and further developed by Levie and Sochen in [2], the *local* and *global localization frameworks* is based on the concept of *observables* borrowed from quantum mechanics which are simply self-adjoint or unitary operators. It turns out that if we want our observables to be able to measure the parameters of representations, they need to satisfy the *one parameter canonical commutation relation*

$$\pi(g)^*T\pi(g) = g + T \quad (2.1)$$

where T is an observable. Intuitively, we want observables which satisfy (2.1) since it says that translating our function by g , measuring it, and then translating is back by g has the same effect as adding g to the measurement. The one parameter canonical commutation relation also guarantees that expected values and variances are well behaved in the sense of the following proposition.

Proposition 2.4. *Let T be an observable fulfilling the one parameter canonical commutation relation (2.1), then*

$$\begin{aligned} e_{\pi(g)f}(T) &= e_f(T) + g, \\ v_{\pi(g)f}(T) &= v_f(T). \end{aligned}$$

Proof. We use the relation as $T = \pi(g)T\pi(g)^* + g$ to find

$$\begin{aligned} e_{\pi(g)f}(T) &= \langle \pi(g)f, T\pi(g)f \rangle \\ &= \langle f, \pi(g)^*(\pi(g)T\pi(g)^* + g)\pi(g)f \rangle \\ &= \langle f, Tf \rangle + \langle f, gf \rangle \\ &= e_f(T) + g. \end{aligned}$$

Similarly for $v_{\pi(g)}(f)$ we have

$$\begin{aligned}
v_{\pi(g)f}(T) &= \left\langle \pi(g)f, (T - e_f(T) - g)^2 \pi(g)f \right\rangle \\
&= \left\langle \pi(g)f, (\pi(g)T\pi(g)^* + g - e_f(T) - g)^2 \pi(g)f \right\rangle \\
&= \left\langle \pi(g)f, \pi(g)TTf \right\rangle + e_f(T)^2 \left\langle \pi(g)f, \pi(g)f \right\rangle - 2e_f(T) \left\langle \pi(g)f, \pi(g)Tf \right\rangle \\
&= e_f(T^2) - e_f(T)^2 = v_f(T).
\end{aligned}$$

□

Remark. We can recover (2.1) from $e_{\pi(g)f}(T) = e_f(T) + g$ by noting that

$$\begin{aligned}
e_{\pi(g)f}(T) &= \left\langle \pi(g)f, T\pi(g)f \right\rangle = \left\langle f, Tf \right\rangle + g \left\langle f, f \right\rangle \\
\implies \left\langle f, \pi(g)^* T \pi(g)f \right\rangle &= \left\langle f, (T + g)f \right\rangle
\end{aligned}$$

for all normalized f .

As a consequence of the Stone von-Neumann theorem, it is in [5] derived that any observables which satisfies the one parameter canonical commutation relation is unitarily equivalent to $Tf(x) = xf(x)$. In the two upcoming subsections we will make these unitary operators explicit.

2.3.1 Short-time Fourier transform

For the short-time Fourier transform, the representation is of the form

$$\begin{aligned}
\pi(g_1, g_2)f(t) &= \pi_2(g_2)\pi_1(g_1)f(t), \\
\pi_1(g_1)f(t) &= f(t - g_1), \\
\pi_2(g_2)f(t) &= e^{2\pi i g_2 t} f(t).
\end{aligned}$$

As is the case in quantum mechanics, we use the observables defined in the time and frequency domain respectively as

$$\begin{aligned}
T_x : f(t) &\mapsto tf(t), \\
\hat{T}_\omega : \hat{f}(\omega) &\mapsto \omega \hat{f}(\omega).
\end{aligned}$$

where $\hat{T}_\omega = \mathcal{F}T_\omega\mathcal{F}^*$ acts in the frequency domain. In this case the unitary operator bringing the observable to its canonical form is the Fourier transform. To see that these operators satisfy the one parameter canonical commutation relations (2.1), we first verify

$$\begin{aligned}
\pi_1(g_1)^* T_x \pi_1(g_1)f(t) &= \pi_1(g_1)^* T_x f(t - g_1) \\
&= \pi_1(g_1)^* t f(t - g_1) = (t + g_1)f(t) = (g_1 + T_x)f(t).
\end{aligned}$$

Next we need to compute T_ω which we do as

$$\begin{aligned}
T_\omega f(t) &= \mathcal{F}^* \hat{T}_\omega \mathcal{F}\{f\}(t) \\
&= \mathcal{F}^* \{\omega \hat{f}(\omega)\}(t) = \frac{1}{2\pi i} f'(t)
\end{aligned}$$

allowing us to verify

$$\begin{aligned}
\pi_2(g_2)^* T_\omega \pi_2(g_2) f(t) &= \pi_2(g_2)^* T_\omega e^{2\pi i t g_2} f(t) \\
&= \frac{e^{-2\pi i t g_2}}{2\pi i} \frac{d}{dt} [e^{2\pi i t g_2} f(t)] \\
&= \frac{e^{-2\pi i t g_2}}{2\pi i} \frac{d}{dt} [2\pi i g_2 e^{2\pi i t g_2} f(t) + e^{2\pi i t g_2} f'(t)] \\
&= (g_2 + T_\omega) f(t).
\end{aligned}$$

We can also compute the commutator $[T_x, T_\omega]$ as

$$\begin{aligned}
[T_x, T_\omega] f(t) &= (T_x T_\omega - T_\omega T_x) f(t) \\
&= \frac{1}{2\pi i} (t f'(t) - f(t) - t f'(t)) = \frac{-1}{2\pi i} f(t).
\end{aligned}$$

Note that this commutator is a constant multiple of the commutator based on the group generators. The time and frequency measuring operators are also switched around compared to Section 2.2.1 which doesn't affect the commutator. This means that we still recover a version of Heisenberg's uncertainty principle with Gaussians as equalizers.

2.3.2 Continuous wavelet transform

For the continuous wavelet transform the representation is

$$\begin{aligned}
\pi(g_1, g_2) f(t) &= \pi_2(g_2) \pi_1(g_1) f(t), \\
\pi_1(g_1) f(t) &= f(t - g_1), \\
\pi_2(g_2) f(t) &= e^{-g_2^2/2} f(t e^{-g_2}).
\end{aligned}$$

This representation is only irreducible if we restrict the domain to the *Hardy space*

$$H^2(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \text{supp}(\hat{f}) \subset \mathbb{R}^+ \right\}$$

of *analytic signals*. This since π_1 corresponds to modulation and π_2 corresponds to dilations in the Fourier domain which both keep the support of $\hat{f}$ in $\mathbb{R}^+$. The Hardy space $H^2(\mathbb{R})$ is in a natural correspondence with square integrable real valued functions on $\mathbb{R}$. Intuitively this can be seen as a consequence of any real valued signal being completely determined by only its positive frequencies as there is redundancy in

$$\hat{s}(-\omega) = \int_{\mathbb{R}} s(t) e^{2\pi i \omega t} dt = \overline{\int_{\mathbb{R}} s(t) e^{-2\pi i \omega t} dt} = \overline{\hat{s}(\omega)}.$$

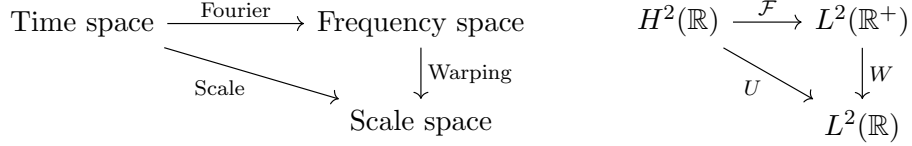
The simplification of only considering wavelets with positive frequency support is made in other works such as [2], [5], [9]. It should be noted that while analytical signals might seem like a purely technical restriction, tools such as the MATLAB Wavelet Toolbox uses analytic wavelets by default [17]. From now on, the ambient Hilbert space should be taken to be $H^2(\mathbb{R})$ unless stated otherwise.

Before we can verify the one parameter canonical commutation relation (2.1), we need to introduce a transform analogous to the Fourier transform, allowing us to measure the scale parameter directly.

Definition 2.5. The *warping transform* $W : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R})$ and *scale transform* $U : H^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ are defined as

$$\begin{aligned} W\{\hat{f}\}(\sigma) &= e^{-\sigma/2} \hat{f}(e^{-\sigma}) = \tilde{f}(\sigma), \\ U &= W\mathcal{F}. \end{aligned}$$

We call the space which the warping transform maps into the *scale space* as an analogue to frequency space. The relations between spaces and transforms are summarized in the diagrams below.



Proposition 2.6. The warping transform $W : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R})$ is a unitary transformation with inverse

$$W^{-1}\{\tilde{f}\}(\omega) = \frac{1}{\sqrt{\omega}} \tilde{f}(-\ln(\omega)).$$

Proof. By direct computation,

$$\begin{aligned} \langle W\hat{f}, W\hat{g} \rangle_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} e^{-\sigma} \hat{f}(e^{-\sigma}) \overline{\hat{g}(e^{-\sigma})} d\sigma \\ &= - \int_{\infty}^0 e^{\sigma} e^{-\sigma} \hat{f}(\omega) \overline{\hat{g}(\omega)} d\omega = \langle f, g \rangle_{L^2(\mathbb{R}^+)}. \end{aligned}$$

For the inverse we verify

$$\begin{aligned} W^{-1}W\{\hat{f}\}(\omega) &= W^{-1}\{\tilde{f}\}(\omega) \\ &= \frac{1}{\sqrt{\omega}} \tilde{f}(-\ln(\omega)) \\ &= \frac{1}{\sqrt{\omega}} e^{-(-\ln(\omega))/2} \hat{f}(e^{-(-\ln(\omega))}) = \hat{f}(\omega) \end{aligned}$$

and

$$\begin{aligned} WW^{-1}\{\tilde{f}\}(\sigma) &= W\left\{\frac{1}{\sqrt{\omega}} \tilde{f}(-\ln(\omega))\right\}(\sigma) \\ &= e^{-\sigma/2} \frac{1}{\sqrt{e^{-\sigma}}} \tilde{f}(-\ln(e^{-\sigma})) = \tilde{f}(\sigma). \end{aligned}$$

□

Associated to an observable $T : H^2(\mathbb{R}) \rightarrow H^2(\mathbb{R})$ and a representation $\pi : G \rightarrow \mathcal{U}(H^2(\mathbb{R}))$ we have

$$\begin{aligned} \hat{T} &= \mathcal{F}T\mathcal{F}^* : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+), & \tilde{T} &= UTU^* : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \\ \hat{\pi} &= \mathcal{F}\pi\mathcal{F}^* : \mathbb{R}^2 \rightarrow \mathcal{U}(L^2(\mathbb{R}^+)), & \tilde{\pi} &= U\pi U^* : \mathbb{R}^2 \rightarrow \mathcal{U}(L^2(\mathbb{R})). \end{aligned}$$

We are now ready to define the observables for the wavelet transform.

Definition 2.7. The *time* and *scale observables* T_x, T_σ are given in time and scale space by

$$\begin{aligned} T_x : f(t) &\mapsto tf(t), \\ \tilde{T}_\sigma : \tilde{f}(\sigma) &\mapsto \sigma \tilde{f}(\sigma) \end{aligned}$$

and their respective domains $\mathcal{D}_t, \mathcal{D}_\sigma$ are given by $\mathcal{D}_t = \{f \in H^2(\mathbb{R}) : \|T_x f\|^2 < \infty\}$ and $\mathcal{D}_\sigma = \{f \in H^2(\mathbb{R}) : \|T_\sigma f\|^2 < \infty\}$.

Note that both of these observables are of the canonical form guaranteed to exist by the Stone von-Neumann theorem as discussed in the beginning of Section 2.3. Therefore the scale transform is in some sense canonical for the wavelet transform. However when doing calculations it is often advantageous to move to the frequency domain as a form of compromise between time and scale.

Proposition 2.8. *In the frequency domain,*

$$\begin{aligned} \hat{T}_t \hat{f}(\omega) &= -i \hat{f}'(\omega), \\ \hat{T}_\sigma \hat{f}(\omega) &= -\ln(\omega) \hat{f}(\omega). \end{aligned}$$

Proof. The first statement follows by elementary properties of the Fourier transform while for the second statement we use the results of Proposition 2.6 to get

$$\begin{aligned} \hat{T}_\sigma \hat{f}(\omega) &= W^* \tilde{T}_\sigma W \{\hat{f}\}(\omega) \\ &= W^* \{\tilde{T}_\sigma e^{-\sigma/2} \hat{f}(e^{-\sigma})\}(\omega) \\ &= W^* \{\sigma e^{-\sigma/2} \hat{f}(e^{-\sigma})\}(\omega) \\ &= \frac{1}{\sqrt{\omega}} (-\ln(\omega)) e^{\ln(\omega)/2} \hat{f}(e^{\ln(\omega)}) = -\ln(\omega) \hat{f}(\omega). \end{aligned}$$

□

In the frequency domain the representation acts as

$$\begin{aligned} \hat{\pi}_1(g_1) \hat{f}(\omega) &= e^{-2\pi i \omega g_1} \hat{f}(\omega), \\ \hat{\pi}_2(g_2) \hat{f}(\omega) &= e^{g_2/2} \hat{f}(\omega e^{g_2}) \end{aligned}$$

while in scale space,

$$\begin{aligned} \tilde{\pi}_2(g_2) \tilde{f}(\sigma) &= W_+ \{\hat{\pi}_2(g_2) \hat{f}\}(\sigma) \\ &= W_+ \{e^{g_2/2} \hat{f}(\omega e^{g_2})\}(\sigma) \\ &= e^{g_2/2} e^{-\sigma/2} \hat{f}(e^{-\sigma} e^{g_2}) = \tilde{f}(\sigma - g_2). \end{aligned}$$

Proposition 2.9. *The observables T_x, T_σ satisfy the one parameter canonical commutation relations*

$$\begin{aligned} \pi_1(g_1)^* T_x \pi_1(g_1) &= g_1 + T_x, \\ \pi_2(g_2)^* T_\sigma \pi_2(g_2) &= g_2 + T_\sigma. \end{aligned} \tag{2.2}$$

Proof. For the first relation, note that

$$\begin{aligned}\langle \pi_1(g_1)f, h \rangle &= \int f(t - g_1) \overline{h(t)} dt \\ &= \int f(s) \overline{h(s + g_1)} ds = \langle f, \pi_1(-g_1)h \rangle \\ \implies \pi_1(g_1)^* &= \pi_1(-g_1)\end{aligned}$$

yielding

$$\begin{aligned}\pi_1(g_1)^* T_x \pi_1(g_1) f(t) &= \pi_1(g_1)^* T_x f(t - g_1) \\ &= \pi_1(g_1)^* t f(t - g_1) \\ &= (t + g_1) f(t) = g_1 f(t) + T_x f(t)\end{aligned}$$

as desired.

For the second relation, note that it suffices to show that

$$U \pi_2(g_2)^* U^* U T_\sigma U^* U \pi_2(g_2) U^* = g_2 + U T_\sigma U^*$$

which is immediate from the above upon using the relations $\tilde{\pi} = U \pi U^*$, $\tilde{T} = U T U^*$ and recalling that $\tilde{\pi}_2(g_2) \tilde{f}(\sigma) = \tilde{f}(\sigma - e_2)$. $\square$

2.4 Wavelet uncertainty

In this subsection we develop and bound an uncertainty quantity for a normalized wavelet f based on the observables T_x, T_σ from Definition 2.7. In [5], this is done by using the same type of product uncertainty as in the short-time Fourier transform case. This can be bounded using Corollary 2.3, the general uncertainty principle.

Corollary 2.10 ([5, Proposition 23]). *Let T_x, T_σ be as in Definition 2.7 and $f \in H^2(\mathbb{R})$ be normalized, then*

$$v_f(T_x) v_f(T_\sigma) \geq \int_0^\infty \frac{|\hat{f}(\omega)|^2}{\omega} d\omega$$

with equality if and only if

$$\hat{f}(\omega) = C e^{\omega(i e_f(T_x) + \lambda(e_f(T_\sigma) + \ln(\omega) - 1))}$$

where C is determined by normalization and λ is a constant.

Proof. By Corollary 2.3, it holds that

$$v_f(T_x) v_f(T_\sigma) \geq \frac{1}{4} |\langle [T_x, T_\sigma] f, f \rangle|^2.$$

We choose to compute the right-hand side in the Fourier domain since we have explicit expressions for both $\hat{T}_t$ and $\hat{T}_\sigma$ there

$$[\hat{T}_t \hat{T}_\sigma - \hat{T}_\sigma \hat{T}_t] \hat{f}(\omega) = -i \frac{d}{d\omega} \left(-\ln(\omega) \hat{f}(\omega) \right) + \ln(\omega) (-i \hat{f}'(\omega)) = \frac{i \hat{f}(\omega)}{\omega}.$$

Plugging this result into the general uncertainty relation yields the desired inequality.

Equalizers must satisfy

$$\begin{aligned}
(-i\frac{d}{d\omega} - e_f(T_x))\hat{f} &= i\lambda(-\ln(\omega) - e_f(T_\sigma))\hat{f} \\
\implies \hat{f}'(\omega) &= \hat{f}(\omega)\left(\lambda(\ln(\omega) + e_f(T_\sigma)) + ie_f(T_x)\right) \\
\implies \hat{f}(\omega) &= C \exp\left(\int_1^\omega \lambda e_f(T_\sigma) + ie_f(T_x) + \lambda \ln(\xi) d\xi\right) \\
&= C \exp\left(\omega[\lambda e_f(T_\sigma) + ie_f(T_x)] + \lambda\omega \ln(\omega) - \lambda\omega + D\right) \\
&= Ce^{\omega(i e_f(T_x) + \lambda(e_f(T_\sigma) + \ln(\omega) - 1))}.
\end{aligned}$$

where we allow C to change value between equality signs. $\square$

Note that in the above, $\hat{f}(0) = C \neq 0$ and hence the equalizer cannot be a wavelet. We resolve this below by adding an additional term to the uncertainty.

In [5], it is noted that this notion of uncertainty is not meaningful because it is not invariant under π , leading to the possibility of arbitrary small uncertainties by scaling the wavelet. Such a construction is detailed below.

Lemma 2.11. *The observables T_x, T_σ satisfy*

$$\begin{aligned}
e_{\pi_1(g_1)f}(T_\sigma) &= e_f(T_\sigma), \\
v_{\pi_1(g_1)f}(T_\sigma) &= v_f(T_\sigma), \\
e_{\pi_2(g_2)f}(T_x) &= e^{g_2}e_f(T_x), \\
v_{\pi_2(g_2)f}(T_x) &= e^{2g_2}v_f(T_x).
\end{aligned} \tag{2.3}$$

Inspired by these relations, the following sequence of wavelets with arbitrary small uncertainty can be constructed.

Example 2.12 ([5, Proposition 25]). Choose a real valued twice continuously differentiable bump function $\hat{f}$ supported in $(0, 1)$ with $\|\hat{f}\| = 1$. Then

$$\hat{f}_n(\omega) = \frac{1}{\sqrt{n}}\hat{f}\left(\frac{\omega - n^2}{n}\right) \implies \begin{cases} e_{f_n}(T_x) &= 0, \\ v_{f_n}(T_x) &\rightarrow 0, \\ e_{f_n}(T_\sigma) &\rightarrow -\infty, \\ v_{f_n}(T_\sigma) &\rightarrow 0. \end{cases}$$

This shortcoming is remedied in [2] where it is noted that it is more meaningful to consider the uncertainty of the *orbit* $\{\pi(g)f\}_{g \in G}$ of the wavelet. This is because it is only when the time and scale of the wavelet is comparable to the signal that we record the coefficients. The direct solution is to add a compensating factor of the form

$$v_f(T_x) \rightarrow e^{-2e_f(T_\sigma)}v_f(T_x)$$

making the variance orbit-invariant by the result in equation (2.3). With this we can define an uncertainty of the form

$$S(f) := \exp\left(-2e_{\frac{f}{\|f\|}}(T_\sigma)\right)v_{\frac{f}{\|f\|}}(T_x) + v_{\frac{f}{\|f\|}}(T_\sigma)$$

which is orbit invariant.

Lastly, we also want to impose the admissibility condition $C_f < \infty$ on wavelets to avoid situations like the one in Corollary 2.10. To do so we introduce the operator $\mathcal{W}$ in the Fourier domain which acts as $\hat{\mathcal{W}}\hat{f}(\omega) = \frac{\hat{f}(\omega)}{\sqrt{\omega}}$ for which it holds that

$$C_f = \|\hat{\mathcal{W}}\hat{f}\|^2, \quad \mathcal{D}_{\mathcal{W}} = \left\{ f \in H^2(\mathbb{R}) : \|\hat{\mathcal{W}}\hat{f}\| < \infty \right\} \quad (2.4)$$

We add this term to the uncertainty and refer to it as a regularization term. To make it orbit invariant we need to add a factor $e^{-e_f(T_\sigma)}$, resulting in the *regularized* wavelet uncertainty

Definition 2.13. For $f \in H^2(\mathbb{R})$, the *regularized wavelet uncertainty* is defined as

$$S(f) := \exp\left(-2e_{\frac{f}{\|f\|}}(T_\sigma)\right) v_{\frac{f}{\|f\|}}(T_x) + v_{\frac{f}{\|f\|}}(T_\sigma) + \exp\left(-e_{\frac{f}{\|f\|}}(T_\sigma)\right) \frac{\|\hat{\mathcal{W}}\hat{f}\|^2}{\|f\|^2}. \quad (2.5)$$

The following proposition is immediate from the construction above.

Proposition 2.14. *The regularized wavelet uncertainty is invariant under translations, dilation and linear scalings, i.e.,*

$$S(f) = S(c\pi(g)f), \quad \text{for } c \in \mathbb{C}, g \in \mathbb{R}^2.$$

In view of the above lemma, if we want to look at minimizers S it by Lemma 2.11 suffices to consider the special case where $e_f(T_x) = e_f(T_\sigma) = 0$ and $\|f\| = 1$ for which S simplifies to

$$S(f) = S_0(f) = \|T_x f\|^2 + \|T_\sigma f\|^2 + \|\hat{\mathcal{W}}\hat{f}\|^2. \quad (2.6)$$

We return to the problems of existence of minimizers of S and the relation between minimizers in different spaces in Chapter 3.

Uncertainty bounds

When defining an uncertainty, it is good to check that it can be bounded from below as a sort of safeguard against counterexamples such as Example 2.12. In the case of the Heisenberg uncertainty principle, this is very straight-forward as the commutator is a constant. For the uncertainty above we have added a term on top of the two variances so we will not be able to use any general results directly.

In this subsection we give a rather broad condition for when we can bound the wavelet uncertainty (2.5). The bounds are not sharp in general so we should not expect equalizers to be optimal in any meaningful sense which can be contrasted with the Gaussians for the short-time Fourier transform from Section 2.3.1 or the waveforms from Corollary 2.10. The only requirements we need to put on our wavelet is that the induced *wavelet system* is a *frame* for the underlying Hilbert space. Wavelet systems are families $\{f_{j,k}\}_{j,k}$, usually chosen with

$$f_{j,k}(t) = a^{j/2} f(a^j t - kb)$$

where the parameters a, b satisfy $a > 1$ and $b > 0$. The classical choices for these are $a = 2$ and $b = 1$. Frames are a generalization of orthogonal bases frequently used in signal processing which we define formally below.

Definition 2.15. A sequence $\{e_j, j \in J\}$ in a separable Hilbert space $\mathcal{H}$ is called a *frame* if there exist positive constants $A, B > 0$ such that for all $g \in \mathcal{H}$

$$A\|g\|_{\mathcal{H}}^2 \leq \sum_{j \in J} |\langle g, e_j \rangle|^2 \leq B\|g\|_{\mathcal{H}}^2.$$

Any two constants A, B satisfying the above are called *frame bounds*. If we can choose $A = B$, then $\{e_j, j \in J\}$ is said to be a *tight frame* [3].

Frames were first considered by Duffin and Schaeffer [18] in 1952 and is still an active area of study in the wavelet context. For an example of an orthonormal wavelet system, consider the classical *Haar wavelet* $f(t) = \chi_{[0,1/2]}(t) - \chi_{[-1/2,0)}(t)$ or see [4, Chapter 7].

A classical theorem first proved by Chui and Shi in [19] gives a necessary condition for being a frame. It was originally proved for $\mathcal{H} = L^2(\mathbb{R})$ case but we note that the proof also works for $H^2(\mathbb{R})$ with minimal modifications.

Theorem 2.16 ([19, Theorem 1]). *Let $\{f_{j,k}\}$ be a frame of $H^2(\mathbb{R})$ with frame bounds A, B . Then $\hat{f}$ satisfies*

$$A \leq \frac{1}{b} \sum_{j \in \mathbb{Z}} |\hat{f}(a^j \omega)|^2 \leq B \quad \text{a.e. on } \mathbb{R}^+. \quad (2.7)$$

The utility of this theorem lies in the following corollary.

Corollary 2.17 ([19]). *Let $f \in H^2(\mathbb{R})$ be as above, then*

$$Ab \ln(a) \leq \int_{\mathbb{R}^+} \frac{|\hat{f}(\omega)|^2}{\omega} d\omega \leq Bb \ln(a).$$

Proof. Multiplying both sides of (2.7) by $\frac{b}{\omega}$ and integrating from 1 to a yields

$$\begin{aligned} Ab \int_1^a \frac{1}{\omega} d\omega &= Ab \ln(a) \leq \int_1^a \sum_{j \in \mathbb{Z}} \frac{|\hat{f}(a^j \omega)|^2}{\omega} d\omega \\ &= \int_0^\infty \frac{|\hat{f}(\omega)|^2}{\omega} d\omega \leq Bb \int_1^a \frac{1}{\omega} d\omega = Bb \ln(a) \end{aligned}$$

as desired. $\square$

With this corollary established, we can prove the following novel uncertainty bound.

Proposition 2.18. *Let $\{f_{j,k}\}$ be a frame of $H^2(\mathbb{R})$ with frame bounds A, B , then*

$$S(f) \geq 2Aba^{-Bb} \ln(a).$$

Proof. We first bound $e^{-2e_f(T_\sigma)} v_f(T_x) + v_f(T_\sigma)$ from below by using

$$\begin{aligned} 0 \leq (a - b)^2 &\implies a + b \geq 2\sqrt{ab} \\ &\implies e^{-2e_f(T_\sigma)} v_f(T_x) + v_f(T_\sigma) \geq 2\sqrt{e^{-2e_f(T_\sigma)} v_f(T_x) v_f(T_\sigma)}. \end{aligned}$$

The quantity in the square root can be bounded using Corollary 2.10 as

$$\begin{aligned} 2\sqrt{e^{-2e_f(T_\sigma)} v_f(T_x) v_f(T_\sigma)} &\geq 2\sqrt{e^{-2e_f(T_\sigma)} \frac{1}{4} \left(\int_{\mathbb{R}^+} \frac{|\hat{f}(\omega)|^2}{\omega} d\omega \right)^2} \\ &= e^{-e_f(T_\sigma)} \int_{\mathbb{R}^+} \frac{|\hat{f}(\omega)|^2}{\omega} d\omega. \end{aligned}$$

Adding the last term of equation (2.5) to this, we get

$$S(f) \geq 2e^{-e_f(T_\sigma)} \int_{\mathbb{R}^+} \frac{|\hat{f}(\omega)|^2}{\omega} d\omega.$$

The first factor can be crudely estimated using the inequality $\ln(\omega) > \frac{-1}{\omega}$ for $\omega \in (0, 1)$ as

$$\begin{aligned} \exp\left(-\int_0^\infty -\ln(\omega)|\hat{f}(\omega)|^2 d\omega\right) &= \exp\left(\int_0^1 \ln(\omega)|\hat{f}(\omega)|^2 d\omega + \int_1^\infty \underbrace{\ln(\omega)|\hat{f}(\omega)|^2}_{\geq 0} d\omega\right) \\ &\geq \exp\left(-\int_0^1 \frac{|\hat{f}(\omega)|^2}{\omega} d\omega\right) \geq \exp\left(-\int_{\mathbb{R}^+} \frac{|\hat{f}(\omega)|^2}{\omega} d\omega\right). \end{aligned}$$

The result of Corollary 2.17 now applies to both factors yielding

$$S(f) \geq 2 \exp(-Bb \ln(a)) Ab \ln(a) = 2Aba^{-Bb} \ln(a).$$

□

We remark that these results can be extended to $f \in L^2(\mathbb{R})$ with

$$S(f) = w_+^2 S_+\left(\frac{f_+}{w_+}\right) + w_-^2 S_-\left(\frac{f_-}{w_-}\right), \quad w_\pm = \|f\|_{H_\pm^2(\mathbb{R})} \quad (2.8)$$

where $H_\pm^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \text{supp}(\hat{f}) \subset \mathbb{R}^\pm\}$, $f_\pm$ is the orthogonal projection of f onto $H_\pm^2(\mathbb{R})$ and $S_\pm$ is the uncertainty on $H_\pm^2(\mathbb{R})$. This results in the following proposition.

Proposition 2.19. *Let $\{f_{j,k}\}$ be a frame of $L^2(\mathbb{R})$ with frame bounds A, B , then*

$$S(f) > 2Ab \ln(a) a^{-2Bb}.$$

The proof is omitted for brevity but is elementary and based on Proposition 2.18.

2.5 Ambiguity function localization

In this section we introduce the ambiguity function and discuss how its localization is related to the sparsifying properties of the wavelet transform.

Definition 2.20. Given a wavelet f , the *ambiguity function* $V_f[f] \in L^2(\mathbb{R})$ is defined as

$$V_f[f](g) = \langle f, \pi(g)f \rangle.$$

One interpretation of the ambiguity function is as the self-correlation of the window. If f would be orthogonal to any time-scale shift of itself, the ambiguity function would just be a Dirac delta peak at $(0, 0)$. Such a construction is not possible and there indeed exists uncertainty principles for the ambiguity function itself such as [20] - albeit in the short-time Fourier transform context. Apart from being fundamentally limited by itself, the ambiguity function localization is also related to the localization of the wavelet transform. The following proposition can be found in [21] and is based on a result from [22].

Proposition 2.21. *Let $f \in \mathcal{H}$ with $C_f = 1$. Then the image space of the wavelet transform $V_f[\mathcal{H}]$ is a reproducing kernel Hilbert space with kernel $V_f[f]$. Precisely, for any $Q \in V_f[\mathcal{H}]$, $Q = V_f[f] * Q$.*

The interpretation is that any element of the image of the wavelet transform is at least as blurry as the ambiguity function.

In [23], the *wavelet-Plancherel theory*, first introduced by Levie and Sochen in [21], is applied to the problem of ambiguity function localization. We recall some of those results here in preparation for the discussion in Chapter 4.

In the wavelet-Plancherel theory applied to the continuous wavelet transform, windows are taken from the *window space* $\mathcal{W}$ equipped with the norm

$$\|f\|_{\mathcal{W}}^2 = C_f = \int_{\mathbb{R}^+} \frac{|\hat{f}(\omega)|^2}{\omega} d\omega = \|\hat{\mathcal{W}}f\|^2$$

while signals are taken from the signal space $\mathcal{S} = L^2(\mathbb{R})$. The *wavelet-Plancherel transform* $V : \mathcal{W} \otimes \mathcal{S} \rightarrow L^2(\mathbb{R}^2)$ is defined for these spaces and is an extension of the wavelet transform which is also an isometric isomorphism. Using the wavelet-Plancherel transform, operators on $L^2(\mathbb{R}^2)$ can be pulled back to the window and signal spaces, making them essentially one-dimensional. Two such operators are the *phase space scale observable* A and the *phase space time observable* B defined as

$$\begin{aligned} AF(a, b) &= aF(a, b), & \text{Dom}(A) &= \left\{ F \in L^2(\mathbb{R}^2, e^{-a}) : aF(a, b) \in L^2(\mathbb{R}^2, e^{-a}) \right\}, \\ BF(a, b) &= bF(a, b), & \text{Dom}(B) &= \left\{ F \in L^2(\mathbb{R}^2, e^{-a}) : bF(a, b) \in L^2(\mathbb{R}^2, e^{-a}) \right\}. \end{aligned}$$

Applying these to a normalized version of the ambiguity function we obtain the following uncertainty definition.

Definition 2.22 ([23, Definition 4]). The *phase space uncertainty* associated to the window f is defined to be

$$\mathcal{L}(f) = v_{K_f}(A) + v_{K_f}(B)$$

where $K_f = V_{\frac{f}{\|f\|_{\mathcal{W}}}} \left[\frac{f}{\|f\|} \right]$ is the *normalized ambiguity function*. The domain of $\mathcal{L}$ is defined to be

$$\text{Dom}(\mathcal{L}) = \left\{ f \in \mathcal{W} \cap \mathcal{S} : f \neq 0, K_f \in \text{Dom}(A) \cap \text{Dom}(B) \right\}.$$

Unlike the regularized wavelet uncertainty S , the phase space uncertainty is not invariant under dilates as can be seen by writing $K_f(g) = \frac{1}{\|f\|\|f\|_{\mathcal{W}}} \langle f, \pi(g)f \rangle$ and recalling that $\|f\|_{\mathcal{W}}$ is not invariant under dilations. Instead of going through the trouble of defining an orbit invariant uncertainty, we restrict our search of minimizers to the set of f where $e_f(T_x) = e_f(T_\sigma) = 0$.

By pulling back the calculation of the variances to the window and signals spaces, the following proposition can be established.

Proposition 2.23 ([23, Proposition 15]). *Let f be a wavelet such that $\hat{f}$ is absolutely continuous, $e_f(T_x) = e_f(T_\sigma) = 0$ and*

$$\begin{aligned} \omega \mapsto \hat{f}(\omega) \ln(\omega) &\in L^2(\mathbb{R}^+), & \omega \mapsto \hat{f}(\omega) \ln(\omega) &\in \mathcal{W}, \\ \omega \mapsto \frac{\hat{f}(\omega)}{\omega} &\in L^2(\mathbb{R}^+), & \omega \mapsto \hat{f}'(\omega) &\in L^2(\mathbb{R}^+), & \omega \mapsto \omega \hat{f}'(\omega) &\in \mathcal{W}. \end{aligned}$$

Then $f \in \text{Dom}(\mathcal{L})$ and the phase space uncertainty $\mathcal{L}(f)$ is given by

$$\mathcal{L}(f) = \|T_x f\|^2 + \|T_\sigma f\|^2 + v \frac{\mathcal{W}_{\hat{f}}}{\|\hat{f}\|_{\mathcal{W}}} \left(i\omega \frac{\partial}{\partial \omega} \right) \left\| \frac{\hat{f}}{\omega} \right\|^2 + v \frac{\mathcal{W}_{\hat{f}}}{\|\hat{f}\|_{\mathcal{W}}} (-\ln(\omega)) \quad (2.9)$$

Note specifically the similarities between the above and S from equation (2.6). In Chapter 4 we prove existence of minimizers and investigate their behavior in different spaces.

Chapter 3

Regularized wavelet uncertainty minimizers

In this chapter we are concerned with the existence of minimizers of the regularized wavelet uncertainty S from definition 2.13 and how they can be approximated by minimizers of S in simpler spaces. Those simpler spaces are more specifically $L^2(0, b)$ and the space of linear splines $P_h(0, b)$. Linear splines in $P_h(0, b)$ are functions $p : [0, b] \rightarrow \mathbb{R}$ which are piecewise linear on the intervals $[0, h], [h, 2h], \dots, [b - h, b]$.

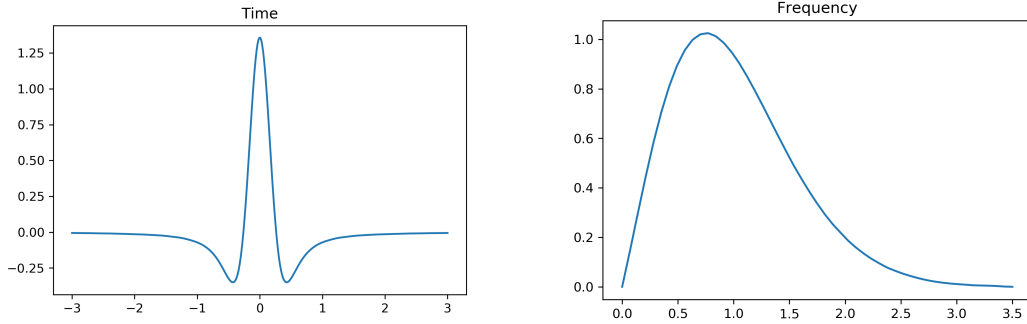


Figure 3.1: A spline minimizer of S computed by minimizing $S(p)$ numerically.

The material in the chapter is based on the ideas laid out in the unpublished notes [7] by Levie.

In the rest of the thesis, all functions and operators are taken to be in the Fourier domain unless stated otherwise. That means that f, T and π now denotes what was previously referred to as $\hat{f}, \hat{T}$ and $\hat{\pi}$ respectively.

The main theorems which we prove in this chapter are the following:

Theorem 3.1. *There exists a minimizer of S in*

$$\mathcal{D}_S := \mathcal{D}_x \cap \mathcal{D}_\sigma \cap \mathcal{D}_W = \left\{ f \in L^2(\mathbb{R}^+) : \|T_x f\| < \infty, \|T_\sigma f\| < \infty, \|Wf\| < \infty \right\}.$$

Theorem 3.2. *There exists a minimizer of S in $\mathcal{D}_S^\sigma \cap L^2(0, b)$ where*

$$\mathcal{D}_S^\sigma = \left\{ f \in \mathcal{D}_S : e_f(T_\sigma) = 0 \right\}.$$

Theorem 3.3. *There exist a minimizer of S in $\mathcal{D}_S^\sigma \cap P_h(0, b)$.*

Remark. Existence of a minimizer of S in $\mathcal{D}_S$ means that there exists a $f \in \mathcal{D}_S$ such that

$$S(f) = \inf_{y \in \mathcal{D}_S} S(y).$$

In the process of proving the above theorems, we will make use of the invariance of S under π discussed in Section 2.4 to restrict our attention to the subset

$$\mathcal{D}_S^0 = \left\{ f \in \mathcal{D}_S : \|f\| = 1, e_f(T_x) = e_f(T_\sigma) = 0 \right\}.$$

Later in Section 3.2.3 we also show that minimizers can be taken to be real valued without loss of generality, leading us to define the subset

$$\mathcal{D}_S^{0,\mathbb{R}} = \left\{ f \in \mathcal{D}_S^0 : f \text{ is real valued} \right\}.$$

With these subsets established, we are ready to state the two main approximation theorems of this chapter.

Theorem 3.4. *Let f_0 be a minimizer of S in $\mathcal{D}_S^{0,\mathbb{R}}$. Then for any $\varepsilon > 0$, there exist $b > 0, h > 0$ and $y \in \mathcal{D}_S^{0,\mathbb{R}} \cap P_h(0, b)$ such that*

$$\|f_0 - y\| < \varepsilon, \quad |S(f_0) - S(y)| < \varepsilon.$$

Theorem 3.5. *Let $\mathcal{M}_h^b$ be the set of minimizers of S in $\mathcal{D}_S^{0,\mathbb{R}} \cap P_h(0, b)$. Then any element of*

$$\mathcal{M} = \left\{ f \in L^2(\mathbb{R}^+) : f = \lim_{n \rightarrow \infty} p_{h_n}^{b_n}, p_{h_n} \in \mathcal{M}_{h_n}^b \text{ for two sequences } b_n \rightarrow \infty, h_n \rightarrow 0 \right\}$$

is a minimizer of S in $\mathcal{D}_S^{0,\mathbb{R}}$.

The interpretation of the above theorem is that if we numerically search for minimizers in $P_h(0, b)$, we should expect the a minimizers to resemble a $L^2(\mathbb{R}^+)$ minimizer f_0 as long as h is small and b is large.

Section 3.1 is dedicated to proving theorems 3.1, 3.2 and 3.3 and in Section 3.2 we show that minimizers are smooth and real valued without loss of generality. In sections 3.3 and 3.4 we prove that minimizers can be approximated by functions in $L^2(0, b)$ and $P_h(0, b)$ respectively. This allows us to prove theorems 3.4 and 3.5.

3.1 Existence of minimizers

We establish existence of minimizers of S in $\mathcal{D}_S$ and $\mathcal{D}_S^\sigma \cap L^2(0, b)$ using a compactness argument that requires quite a bit of work. For minimizers in $\mathcal{D}_S^\sigma \cap P_h(0, b)$ we use of the fact that the space is finite dimensional to establish the same result.

We begin by proving Theorem 3.1 and then outline the changes required to adapt the proof to Theorem 3.2. As mentioned above, it suffices to consider the subset

$$\begin{aligned} \mathcal{D}_S^0 &= \left\{ f \in \mathcal{D}_S : \|f\| = 1, e_f(T_x) = e_f(T_\sigma) = 0 \right\}, \\ e_f(T_x) &= \langle -if', f \rangle, \quad e_f(T_\sigma) = \langle -\ln(\omega)f, f \rangle \end{aligned}$$

where the uncertainty simplifies to equation (2.6), i.e.,

$$S(f) = \|T_x f\|^2 + \|T_\sigma f\|^2 + \|\mathcal{W}f\|^2.$$

The following proposition is the key to proving existence.

Proposition 3.6. Let $\{f_n\}_n \subset \mathcal{D}_S^0$ be a minimizing sequence of $S(f)$ in the sense that

$$\lim_{n \rightarrow \infty} S(f_n) = \inf_{y \in \mathcal{D}_S^0} S(y).$$

Then, there exist $N \in \mathbb{N}$ and a subset $\mathcal{K} \subset \mathcal{D}_S^0$ such that $f_n \in \mathcal{K}$ for $n > N$. Moreover, $\mathcal{K}$ is compact.

The first step is specifying the subset $\mathcal{K} \subset \mathcal{D}_S^0$. To do so, we first bound the uncertainty from above by considering a normalized real valued tent function g with support in $\mathbb{R}^+$. It then holds that $\pi_2(e_g(T_\sigma))^{-1}g \in \mathcal{D}_S^0$ and

$$S_0(\pi_2(e_g(T_\sigma))^{-1}g) = S(g) = 0.5K > 0$$

for some $K > 0$. Now, since $\inf_{y \in \mathcal{D}_S^0} S(y) < K$, for any minimizing sequence $(f_n)_n \subset \mathcal{D}_S^0$, there exists a N such that $S(f_n) \leq K$ for $n > N$. In particular, this means that

$$\|T_x f_n\|^2 \leq K, \quad \|T_\sigma f_n\|^2 \leq K, \quad \|\mathcal{W} f_n\|^2 \leq K.$$

The subset $\mathcal{K}$ can now be defined as the set of $f \in \mathcal{D}_S^0$ such that the above inequalities hold. For reference, we write it here as

$$\mathcal{K} = \left\{ f \in \mathcal{D}_S : \|f\| = 1, e_f(T_x) = 0, e_f(T_\sigma) = 0, \right. \\ \left. \|T_x f\|^2 \leq K, \|T_\sigma f\|^2 \leq K, \|\mathcal{W} f\|^2 \leq K \right\} \quad (3.1)$$

We prove that $\mathcal{K}$ is compact by showing that it is both closed and pre-compact. For the closedness, we will need a couple of lemmas.

Lemma 3.7. The subset $\{f \in L^2(\mathbb{R}) : \|xf\|^2 \leq M\}$ is closed in $L^2(\mathbb{R})$.

Proof. Let $(f_n)_n$ be a sequence such that $\|xf_n\|^2 \leq M$ and write $f = f_n + \delta_n$. We show that $\int_{-N}^N x^2 |f|^2 dx \leq M$ for all $N \in \mathbb{N}$ by noting that for all n ,

$$\begin{aligned} \int_{-N}^N x^2 |f|^2 dx &= \int_{-N}^N x^2 |f_n|^2 dx + 2 \operatorname{Re} \int_{-N}^N x^2 f_n \bar{\delta}_n dx + \int_{-N}^N x^2 |\delta_n|^2 dx \\ &\leq M + 2 \underbrace{\sqrt{\int_{-N}^N x^2 |f_n|^2 dx}}_{< \sqrt{M}} \underbrace{\sqrt{N^2 \int_{-N}^N |\delta_n|^2 dx}}_{\xrightarrow{n \rightarrow \infty} 0} + N^2 \underbrace{\int_{-N}^N |\delta_n|^2 dx}_{\xrightarrow{n \rightarrow \infty} 0} \xrightarrow{n \rightarrow \infty} M. \end{aligned}$$

Letting $N \rightarrow \infty$, we obtain the desired inequality. $\square$

Lemma 3.8. The subset

$$\left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} x |f(x)|^2 dx = 0, \int_{\mathbb{R}} x^2 |f(x)|^2 dx \leq M \right\}$$

is closed in $L^2(\mathbb{R})$.

Proof. We remark that since $f \in L^2(\mathbb{R})$ and $\int_{\mathbb{R}} x^2 |f(x)|^2 dx < \infty$, $x|f(x)|^2$ is integrable. Now, let f_n be a sequence in this subset that converges to f and write $f = f_n + \delta_n$. That the second condition will hold for f follows by Lemma 3.7. For the first condition, note first that by the triangle inequality, $|a - b|^2 \leq 2|a|^2 + 2|b|^2$ and Lemma 3.7,

$$\int_{\mathbb{R}} x^2 |\delta_n|^2 dx = 2 \int_{\mathbb{R}} x^2 |f|^2 dx + 2 \int_{\mathbb{R}} x^2 |f_n|^2 dx \leq 4M.$$

Also, for all $a > 0$,

$$\begin{aligned} \int_{\mathbb{R}} |x| |\delta_n|^2 dx &= \int_{|x| < a} |x| |\delta_n|^2 dx + \int_{|x| > a} |x| |\delta_n|^2 dx \\ &\leq \underbrace{a \int_{|x| < a} |\delta_n|^2 dx}_{\rightarrow 0} + \underbrace{\frac{1}{a} \int_{|x| > a} |x|^2 |\delta_n|^2 dx}_{\leq 4M} \xrightarrow{n \rightarrow \infty} L \leq \frac{4M}{a} \end{aligned}$$

and since this holds for arbitrarily large a , we have that

$$\int_{\mathbb{R}} |x| |\delta_n|^2 dx \rightarrow 0.$$

Next note that

$$\int_{\mathbb{R}} x |f|^2 dx = \int_{\mathbb{R}} x |f_n + \delta_n|^2 dx = \underbrace{\int_{\mathbb{R}} x |f_n|^2 dx}_{=0} + 2 \operatorname{Re} \int_{\mathbb{R}} x f_n \bar{\delta}_n dx + \int_{\mathbb{R}} x |\delta_n|^2 dx.$$

Therefore, we can estimate

$$\left| \int_{\mathbb{R}} x |f|^2 dx \right| \leq 2 \sqrt{\int_{\mathbb{R}} |x| |f_n|^2 dx} \underbrace{\sqrt{\int_{\mathbb{R}} |x| |\delta_n|^2 dx}}_{\rightarrow 0} + \underbrace{\int_{\mathbb{R}} |x| |\delta_n|^2 dx}_{\rightarrow 0}$$

which will yield the desired conclusion once we show that $\int |x| |f_n|^2 dx$ is bounded in n . Indeed,

$$\begin{aligned} \int_{\mathbb{R}} |x| |f_n|^2 dx &= \int_{|x| \leq 1} |x| |f_n|^2 dx + \int_{|x| > 1} |x| |f_n|^2 dx \\ &\leq \int_{|x| \leq 1} |f_n|^2 dx + \int_{|x| > 1} |x|^2 |f_n|^2 dx \leq 1 + M. \end{aligned}$$

□

Lemma 3.9. *The following subsets are closed:*

- (A) $\{\|T_x f\|^2 \leq K, e_f(T_x) = 0\},$
- (B) $\{\|T_\sigma f\|^2 \leq K, e_f(T_\sigma) = 0\},$
- (C) $\{\|\mathcal{W}f\|^2 \leq K\}.$

Proof. For (A) and (B), we can write these restrictions in time (which we denote by $\check{f}$) and scale space as

$$\begin{aligned} \int t^2 |\check{f}(t)|^2 dt &\leq K, & \int t |\check{f}(t)|^2 dt &= 0, \\ \int \sigma^2 |\tilde{f}(\sigma)|^2 d\sigma &\leq K, & \int \sigma |\tilde{f}(\sigma)|^2 d\sigma &= 0 \end{aligned}$$

respectively. That the set of f satisfying each of these conditions are closed follows by Lemma 3.8.

Meanwhile for (C), we introduce the mapping

$$\mathcal{U} : L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+), \quad \mathcal{U}\{f\}(s) = \frac{1}{s} f\left(\frac{1}{s}\right).$$

which can be seen to be an isometry by making the change of variables $\omega = \frac{1}{s}$. It furthermore holds that if $\|\mathcal{W}f\|^2 \leq K$, then

$$K \geq \|\mathcal{W}f\|^2 = \int_0^\infty \frac{1}{\omega} |f(\omega)|^2 d\omega = \int_0^\infty s |\mathcal{U}\{f\}(s)|^2 ds.$$

Since s is always positive, the above can be written as $\|\sqrt{s}\mathcal{U}\{f\}\|^2 \leq K$ which can be showed to correspond to a closed set by making a minor modification to the proof of Lemma 3.7. $\square$

Proposition 3.10. *The subset $\mathcal{K}$ is closed in $L^2(\mathbb{R})$.*

Proof. By writing $\mathcal{K}$ as an intersection as in (3.1) and noting that all subsets are closed by Lemma 3.9, it follows that $\mathcal{K}$ is closed too. $\square$

For pre-compactness, we show that $\mathcal{K}$ can be approximated by compact sets with arbitrary small error.

Lemma 3.11. *For any $\varepsilon > 0$, there exists a compact subset $C_{a,b}$ of $L^2(\mathbb{R}^+)$ such that for any $f \in \mathcal{K}$, there is a $y \in C_{a,b}$ such that*

$$\|f - y\| < \varepsilon.$$

Proof. For $f \in \mathcal{K}$, consider the scale space representation $|\tilde{f}(\sigma)|^2$ as a probability distribution with mean value 0 and variance $\|T_\sigma f\|^2 \leq K$. Applying Chebyshev's inequality to the associated random variable, we have for any $\alpha > 0$ that

$$\int_{[e^{-\alpha}, e^\alpha]^c} |f(\omega)|^2 d\omega = \int_{[-\alpha, \alpha]^c} |\tilde{f}(\sigma)|^2 d\sigma \leq \frac{K}{\alpha^2}.$$

Now, fix $\varepsilon > 0$ and choose α so large that $\frac{K}{\alpha^2} < \varepsilon$. Then, with $y = f|_{[e^{-\alpha}, e^\alpha]} \in L^2(e^{-\alpha}, e^\alpha)$, it holds that $\|f - y\| < \varepsilon$ by the above inequality.

Now write $a = e^{-\alpha}$, $b = e^\alpha$ and consider the closed subset

$$\mathcal{K}_{[a,b]} = \left\{ f|_{[a,b]} : f \in \mathcal{K} \right\},$$

where $f|_{[a,b]}$ is the restriction of f to $[a, b]$ padded with zeros. We show that $\mathcal{K}_{[a,b]}$ is contained in a compact subset.

For all $f \in \mathcal{K}$, we have

$$\int_a^b |f'|^2 d\omega \leq \|f'\|^2 = \|T_x f\|^2 \leq K$$

We want to expand f in a uniformly convergent Fourier series which is possible if f is absolutely continuous with periodic extensions. Since $f(a) \neq f(b)$ is general, we mirror f around $\omega = b$ and denote the even extension on $[a, 2b - a]$ by f_e . This function has the desired properties together with

$$\begin{aligned} \|f_e'\|^2 \leq 2K &\implies \|f_e'\|^2 = \sum_n \frac{\pi}{(b-a)^2} n^2 |c_n|^2 \leq 2K \\ &\implies |c_n| \leq \frac{\sqrt{2K}(b-a)}{\pi} \frac{1}{|n|} \end{aligned}$$

where $(c_n)_n$ are the Fourier coefficients of f_e . The above inequality defines a compact subset by Parseval's formula known as a *Hilbert cube* which f . Letting $H_{a,b}$ denote the closed subset of square integrable even functions around b whose Fourier coefficients are in the Hilbert cube specified above, we have that this set is compact too. Lastly, we define $C_{a,b}$ as the set of $y \in H_{a,b}$ restricted to $L^2(a,b)$ and note that $f \in C_{a,b}$. This set is compact since we can mirror a sequence to be even and use the compactness of $H_{a,b}$ to extract a convergent subsequence. Then the unmirrored subsequence converges to the restriction of that limit to $[a, b]$. $\square$

Proposition 3.12. *The subset $\mathcal{K}$ is pre-compact.*

Proof. Fix a sequence $(f_n)_n \subset \mathcal{K}$. We wish to prove that $(f_n)_n$ has a convergent subsequence which we do by constructing a Cauchy subsequence. By the above lemma, we can for each m choose a_m, b_m such that any function in $\mathcal{K}$ can be approximated by a function in C_{a_m, b_m} with error less than $\frac{1}{m}$. For each f_n we let f_n^m denote these approximations so that

$$f_n^m \in C_{a_m, b_m}, \quad \|f_n - f_n^m\| < \frac{1}{m}. \quad (3.2)$$

Fixing m , we view $(f_n^m)_n \subset C_{a_m, b_m}$ as a sequence in the compact set and extract a convergent subsequence

$$(f_{n_j^m}^m)_j \subset C_{a_m, b_m}, \quad f_{n_j^m}^m \xrightarrow{j \rightarrow \infty} f^m \text{ in } C_{a_m, b_m}.$$

We also filter the subsequences so that $(n_j^m)_j$ is a subsequence of $(n_j^{m'})_j$ for $m > m'$.

Keeping m fixed, by (3.2) we have that

$$\|f_{n_j^m}^m - f_{n_j^{m'}}^{m'}\| < \frac{1}{m}$$

Now since $f_{n_j^m}^m \xrightarrow{j \rightarrow \infty} f^m$ in C_{a_m, b_m} , we can choose j_m so large that

$$\|f_{n_j^m}^m - f^m\| < \frac{1}{m} \implies \|f_{n_j^m}^m - f^m\| < \frac{2}{m}$$

for all $j \geq j_m$. This implies that $(f_{n_{j_m}^m})_m$ is Cauchy as can be seen by noting that for all $m > m'$ it holds that

$$\|f_{n_{j_m}^m}^m - f^{m'}\| < \frac{2}{m'} \implies \|f_{n_{j_{m_1}}^{m_1}}^{m_1} - f_{n_{j_{m_2}}^{m_2}}^{m_2}\| < \frac{4}{m'}$$

for $m_1, m_2 > m'$. This finishes the proof. $\square$

With this we have what we need to prove Proposition 3.6.

Proof of Proposition 3.6. That the tail of any minimizing sequence is in $\mathcal{K}$ was shown above and so only compactness remains. This follows by closedness via Proposition 3.10 and pre-compactness via Proposition 3.12. $\square$

Proof of Theorem 3.1. As established during the discussion on the invariance of S under π in Section 2.4, the existence of a minimizer follows if we can find a minimizer of S_0 in $\mathcal{D}_S^0$. Since a minimizing sequence can be taken to be in the compact subset $\mathcal{K} \subset \mathcal{D}_S^0$ without loss of generality by the above proposition, we can pass to a convergent subsequence converging to some point $f_0 \in \mathcal{K}$. Moreover, by passing to a subsequence repeatedly we have that

$$\|T_x f_n\|^2 \xrightarrow{n \rightarrow \infty} \sigma_x, \quad \|T_\sigma f_n\|^2 \xrightarrow{n \rightarrow \infty} \sigma_\sigma, \quad \|\mathcal{W} f_n\|^2 \xrightarrow{n \rightarrow \infty} \sigma_{\mathcal{W}}$$

for some $\sigma_x, \sigma_\sigma, \sigma_{\mathcal{W}} \leq K \in \mathbb{R}$ such that $\sigma_x + \sigma_\sigma + \sigma_{\mathcal{W}} = \inf_{y \in \mathcal{D}_S^0} S(y)$.

To show that f_0 really is a minimizer, define

$$\mathcal{K}_{\alpha, \beta, \gamma} = \left\{ y \in \mathcal{K} : \|T_x y\|^2 \leq \alpha, \|T_\sigma y\|^2 \leq \beta, \|\mathcal{W} y\|^2 \leq \gamma \right\}$$

and note that this subset is closed by Lemma 3.9 and thus compact. Now for each $\varepsilon > 0$, it holds that the tail of the minimizing sequence belongs to $\mathcal{K}_{\sigma_x + \varepsilon, \sigma_\sigma + \varepsilon, \sigma_{\mathcal{W}} + \varepsilon}$. Thus

$$S(f_0) \leq \sigma_x + \sigma_\sigma + \sigma_{\mathcal{W}} + 3\varepsilon \quad \text{for all } \varepsilon > 0 \implies S(f_0) = \sigma_x + \sigma_\sigma + \sigma_{\mathcal{W}}$$

meaning that f_0 indeed is a minimizer. $\square$

Proof of Theorem 3.2. The same argument of restricting the domain to a subset of $\mathcal{D}_S^0$ works for Theorem 3.2 as for 3.1. The only difference is that we intersect with $L^2(0, b)$ which preserves compactness, yielding existence again. $\square$

Proof of Theorem 3.3. We write $S(p)$ explicitly as

$$S(p) = \int_0^b \left(|p'(\omega) - e_p(T_x)|^2 + |\ln(\omega)p(\omega)|^2 + \frac{|p(\omega)|^2}{\omega} \right) d\omega.$$

This can be decomposed into a sum where each term will depend continuously on the values $p(0), p(h), p(2h), \dots, p(b)$. Since it suffices to search for normalized minimizers on a subset of the compact unit sphere of $P_h(0, b)$ and the mapping $p \mapsto S(p)$ is continuous by the above, we get existence of a minimizer. $\square$

3.2 Properties of minimizers

The goal of this section is to establish ODE's which are solved by minimizers in $\mathcal{D}_S^0$ and $\mathcal{D}_S^0 \cap L^2(0, b)$. To do so, we establish an analogue of Fermat's theorem for stationary points formulated in the language of calculus of variations.

Definition 3.13. Let $\mathcal{H}$ be a separable Hilbert space and $F : \mathcal{D}_F \rightarrow \mathbb{R}$ a functional such that $\mathcal{D}_F \cup \{0\}$ is dense in $\mathcal{H}$, where $\mathcal{D}_F$ is the domain of F . The *Gateaux differential* of F at $f \in \mathcal{D}_F$ in the direction $h \in \mathcal{D}_F$ is defined as

$$dF(f, h) = \lim_{t \rightarrow 0} \frac{F(f + th) - F(f)}{t}$$

whenever the limit exists. In the case where there exists a vector $\delta F = \delta F(f) \in \mathcal{H}$ such that

$$dF(f, h) = \langle \delta F(f), h \rangle$$

for all $h \in \mathcal{D}_F$, we call $\delta F(f)$ the *variation* of F at f .

It turns out that choosing $L^2(\mathbb{R}^+)$ as the ambient Hilbert space means that not all the variations we want to compute exist. Specifically, not even the variation of $F(f) = e_f(T)$ for a self adjoint T exists. However, by moving to the *realification* $\mathcal{H}^R$ of $\mathcal{H}$ we can get around this.

Definition 3.14. The *realification* of a complex Hilbert space $\mathcal{H}$ is the real Hilbert space $\mathcal{H}^R$ defined with the same set but with the underlying field for multiplication by scalars being $\mathbb{R}$ and with the inner product

$$\langle \cdot, \cdot \rangle_{\mathcal{H}^R} = \operatorname{Re} \langle \cdot, \cdot \rangle_{\mathcal{H}}.$$

For the computation of the variation we will drop the R superscript and take all inner products to be in the realification of $L^2(\mathbb{R}^+)$.

In the sequel, we will always implicitly assume that all domains are appropriately dense in $\mathcal{H}$ and that the variation is defined for any f .

3.2.1 Establishing the Euler-Lagrange equations

The goal of this section is to establish the Euler-Lagrange equations satisfied by stationary points of the uncertainty functional. We state the two main theorems here.

Theorem 3.15 (The $L^2(\mathbb{R}^+)$ Euler-Lagrange equation). *A minimizer f_0 of S in $\mathcal{D}_S^0$ satisfies*

$$f_0''(\omega) = H(\omega)f_0(\omega), \quad H(\omega) = \ln(\omega)^2 + \frac{1}{\omega} + \kappa \ln(\omega) - \sigma$$

where $\kappa = 2\|T_x f_0\|^2 + \|\mathcal{W}f_0\|^2$ and $\sigma = \|T_x f_0\|^2 + \|T_\sigma f_0\|^2 + \|\mathcal{W}f_0\|^2$.

Theorem 3.16 (The $L^2(0, b)$ Euler-Lagrange equation). *A minimizer f_0 of S in $\mathcal{D}_S^0 \cap L^2(0, b)$ satisfies*

$$f_0''(\omega) = H_0(\omega)f_0(\omega), \quad H_0(\omega) = \ln(\omega)^2 + \frac{1}{\omega} + \kappa \ln(\omega) + \beta \ln(\omega) - \sigma$$

where $\kappa = 2\|T_x f_0\|^2 + \|\mathcal{W}f_0\|^2$, $\beta \in \mathbb{R}$ and $\sigma = \|T_x f_0\|^2 + \|T_\sigma f_0\|^2 + \|\mathcal{W}f_0\|^2$.

Before attempting a proof, we need to compute the variation of the uncertainty. To that end, we write the uncertainty from (2.5) using only expected values and variances as

$$S(f) = \exp\left(-2e_{\frac{f}{\|f\|}}(T_\sigma)\right) v_{\frac{f}{\|f\|}}(T_x) + v_{\frac{f}{\|f\|}}(T_\sigma) + \exp\left(-e_{\frac{f}{\|f\|}}(T_\sigma)\right) e_{\frac{f}{\|f\|}}(\mathcal{W}^2). \quad (3.3)$$

Proposition 3.17. *The variation of the uncertainty S at $f \in \mathcal{D}_S^0$ is given by*

$$\delta S(f)(\omega) = -2f''(\omega) + \left(4\sigma_x \ln(\omega) - 2\sigma_x + 2\sigma_{\mathcal{W}} \ln(\omega) + \frac{2}{\omega} - 2\sigma_{\mathcal{W}} + 2\ln(\omega)^2 - 2\sigma_\sigma\right) f(\omega)$$

where $\sigma_x = \|T_x f\|^2$, $\sigma_\sigma = \|T_\sigma f\|^2$ and $\sigma_{\mathcal{W}} = \|\mathcal{W}f\|^2$.

To prove this proposition we need to compute the variation of expected values and variances which is the contents of the lemmas below. Throughout, we work with an arbitrary self-adjoint operator T to be able to apply the results to T_x, T_σ and $\mathcal{W}^2$.

Lemma 3.18. *When $\|f\| = 1$, the variations of the expected values and variances are given by*

$$\begin{aligned}\delta e_f(T) &= 2Tf, \\ \delta v_f(T) &= 2(T - e_f(T))^2 f.\end{aligned}$$

Proof. To compute the variation, we use the method outlined in Definition 3.13. For the expected value $e_f(T) = \langle Tf, f \rangle$ we have

$$\begin{aligned}e_{f+th}(T) &= \langle T(f+th), f+th \rangle \\ &= \langle Tf, f \rangle + 2t \langle Tf, h \rangle + O(t^2) \\ \implies \delta e_f(T) &= 2Tf.\end{aligned}$$

Meanwhile for the variance $v_f(T) = e_f((T - e_f(T))^2)$,

$$\begin{aligned}v_{f+th}(T) &= \langle (T - e_{f+th}(T))^2 (f+th), f+th \rangle \\ &= \langle (T - e_f(T) - 2t \langle Tf, h \rangle)^2 (f+th), f+th \rangle.\end{aligned}$$

Expanding the square as $(a - b - c)^2 = (a - b)^2 + c^2 + 2c(a - b)$ and noting that $c^2 = o(t)$, we find

$$\begin{aligned}v_{f+th}(T) &= \langle (T - e_f(T))^2 (f+th), f+th \rangle \\ &\quad + 2t \langle Tf, h \rangle \langle (T - e_f(T)) (f+th), f+th \rangle + o(t) \\ &= \langle (T - e_f(T))^2 f, f \rangle + 2t \langle (T - e_f(T))^2 f, h \rangle \\ &\quad + 2t \langle Tf, h \rangle \langle (T - e_f(T)) f, f \rangle + o(t) \\ &= v_f(T) + 2t \left\langle \left[(T - e_f(T))^2 + \langle (T - e_f(T)) f, f \rangle T \right] f, h \right\rangle + o(t).\end{aligned}$$

Now note that

$$\langle (T - e_f(T)) f, f \rangle = e_f(T)(1 - \|f\|^2) = 0$$

since we assumed $\|f\| = 1$. Thus the last term of the above vanishes and we get that the variation simplifies to

$$\delta v_f(T) = 2(T - e_f(T))^2 f$$

which is what we wanted to show. $\square$

Next we want to establish a form of the chain rule for compositions of the form $F(f) = H(Tf)$ where $T : \mathcal{H} \rightarrow \mathcal{H}$ is a possibly nonlinear operator and F, H are functionals on $\mathcal{H}$. To that end, we say that if there exists a bounded linear operator $\delta T : \mathcal{H} \rightarrow \mathcal{H}$ which satisfies

$$T(f+h) - T(f) = \delta T(h) + o(\|h\|),$$

then δT called the *variation* of T at f . The $o(\|h\|)$ term is to be interpreted as a vector with norm comparable to $o(\|h\|)$.

Lemma 3.19. *If F and H are functionals on $\mathcal{H}$ and $T : \mathcal{H} \rightarrow \mathcal{H}$ is an operator such that $F(f) = H(Tf)$, then*

$$\delta F(f) = \delta T^*(\delta H(Tf)).$$

Proof. Using the rules

$$\begin{aligned} H(y + th) - H(y) &= \langle \delta H(y), th \rangle + o(t), \\ T(f + th) - T(f) &= \delta T(th) + o(t) \end{aligned}$$

we compute

$$\begin{aligned} F(f + th) - F(f) &= H(T(f + th)) - H(Tf) \\ &= H(Tf + \delta T(th) + o(t)) - H(Tf) \\ &= \langle \delta H(Tf), \delta T(th) + o(t) \rangle + o(t) \\ &= \langle \delta T^*(\delta H(Tf)), th \rangle + o(t) \end{aligned}$$

where we in the last step used that $\langle \delta H(Tf), o(t) \rangle = o(t)$ by Cauchy-Schwarz. $\square$

Lemma 3.20. *If F and H are functionals such that $F(f) = H\left(\frac{f}{\|f\|}\right)$, then*

$$\delta F(f) = \frac{1}{\|f\|} \delta H\left(\frac{f}{\|f\|}\right) + \left\langle \delta H\left(\frac{f}{\|f\|}\right), f \right\rangle \frac{f}{\|f\|^3}.$$

Proof. Let T be the nonlinear operator $T : f \mapsto \frac{f}{\|f\|}$. To use it in Lemma 3.19, we need to calculate its variation, i.e., how $\frac{f+h}{\|f+h\|} - \frac{f}{\|f\|}$ depends on h . To that end, take $\approx$ to mean equality modulo $o(\|h\|)$ and note that

$$\begin{aligned} \frac{f+h}{\|f+h\|} &\approx \frac{f+h}{\sqrt{\|f\|^2 + 2\langle f, h \rangle}} = \frac{f+h}{\|f\| \sqrt{1 + \frac{2\langle f, h \rangle}{\|f\|^2}}} \approx \frac{f+h}{\|f\| \left(1 + \frac{\langle f, h \rangle}{\|f\|^2}\right)} \\ &\approx \frac{f+h}{\|f\|} \left(1 - \frac{\langle f, h \rangle}{\|f\|^2}\right) \approx \frac{f}{\|f\|} + \left(\frac{h}{\|f\|} - \frac{\langle f, h \rangle}{\|f\|^2} \frac{f}{\|f\|}\right) \end{aligned}$$

since $\sqrt{1+x} \approx 1+x/2$ and $\frac{1}{1+x} \approx (1-x)$. This yields that $\delta T(h) = \frac{h}{\|f\|} - \frac{\langle f, h \rangle}{\|f\|^2} \frac{f}{\|f\|}$ which can be seen to be self-adjoint by computing

$$\begin{aligned} \langle \delta T(a), b \rangle &= \left\langle \frac{a}{\|f\|} - \frac{\langle f, a \rangle}{\|f\|^2} \frac{f}{\|f\|}, b \right\rangle \\ &= \frac{\langle a, b \rangle}{\|f\|} - \frac{\langle f, a \rangle \langle f, b \rangle}{\|f\|^3} = \langle a, \delta T(b) \rangle. \end{aligned}$$

Plugging this into the chain rule we find

$$\delta F(f) = \frac{1}{\|f\|} \delta H\left(\frac{f}{\|f\|}\right) + \left\langle \delta H\left(\frac{f}{\|f\|}\right), f \right\rangle \frac{f}{\|f\|^3}.$$

$\square$

Lemma 3.21. For all f , the variation of the expected value and variance are given by

$$\delta e_{\frac{f}{\|f\|}}(T) = 2T \frac{f}{\|f\|^2} - 2e_{\frac{f}{\|f\|}}(T) \frac{f}{\|f\|^2}$$

and

$$\delta v_{\frac{f}{\|f\|}}(T) = 2(T - e_{\frac{f}{\|f\|}}(T))^2 \frac{f}{\|f\|^2} - 2v_{\frac{f}{\|f\|}}(T) \frac{f}{\|f\|^2}.$$

Proof. For both variations, we apply the chain rule to the results of Lemma 3.18. For $H(f) = e_f(T)$ we have

$$\begin{aligned} \delta e_{\frac{f}{\|f\|}}(T) &= \frac{1}{\|f\|} \delta H\left(\frac{f}{\|f\|}\right) + \left\langle \delta H\left(\frac{f}{\|f\|}\right), f \right\rangle \frac{f}{\|f\|^3} \\ &= \frac{1}{\|f\|} 2T \frac{f}{\|f\|} + \left\langle 2T \frac{f}{\|f\|}, f \right\rangle \frac{f}{\|f\|^3} \\ &= 2T \frac{f}{\|f\|^2} - 2e_{\frac{f}{\|f\|}}(T) \frac{f}{\|f\|^2}. \end{aligned}$$

Similarly with $H(f) = v_f(T)$ we have

$$\begin{aligned} \delta v_{\frac{f}{\|f\|}}(T) &= \frac{1}{\|f\|} \delta H\left(\frac{f}{\|f\|}\right) + \left\langle \delta H\left(\frac{f}{\|f\|}\right), f \right\rangle \frac{f}{\|f\|^3} \\ &= \frac{1}{\|f\|} 2(T - e_{\frac{f}{\|f\|}}(T))^2 \frac{f}{\|f\|} + \left\langle 2(T - e_{\frac{f}{\|f\|}}(T))^2 \frac{f}{\|f\|}, f \right\rangle \frac{f}{\|f\|^3} \\ &= 2(T - e_{\frac{f}{\|f\|}}(T))^2 \frac{f}{\|f\|^2} - 2v_{\frac{f}{\|f\|}}(T) \frac{f}{\|f\|^2} \end{aligned}$$

as desired. $\square$

A product rule also holds for variations with the proof being straight-forward.

Lemma 3.22. If A and B are functionals on $\mathcal{H}$ with variations δA and δB , then the variation of $C(f) = A(f)B(f)$ is given by

$$\delta C(f) = \delta A(f)B(f) + A(f)\delta B(f).$$

With all the lemmas established, we are ready to prove the proposition.

Proof of Proposition 3.17. For reference we here recall the expression for the uncertainty in expected values and variances from equation (3.3):

$$S(f) = \exp\left(-2e_{\frac{f}{\|f\|}}(T_\sigma)\right) v_{\frac{f}{\|f\|}}(T_x) + v_{\frac{f}{\|f\|}}(T_\sigma) + \exp\left(-e_{\frac{f}{\|f\|}}(T_\sigma)\right) e_{\frac{f}{\|f\|}}(\mathcal{W}^2).$$

By the product rule applied to the above using Lemma 3.21, we have that

$$\begin{aligned}
\delta S(f) &= \exp\left(-2e_{\frac{f}{\|f\|}}(T_\sigma)\right) \left[-2\delta e_{\frac{f}{\|f\|}}(T_\sigma) v_{\frac{f}{\|f\|}}(T_x) + \delta v_{\frac{f}{\|f\|}}(T_x)\right] \\
&\quad + \exp\left(-e_{\frac{f}{\|f\|}}(T_\sigma)\right) \left[-\delta e_{\frac{f}{\|f\|}}(T_\sigma) e_{\frac{f}{\|f\|}}(\mathcal{W}^2) + \delta e_{\frac{f}{\|f\|}}(\mathcal{W}^2)\right] \\
&\quad + \delta v_{\frac{f}{\|f\|}}(T_\sigma) \\
&= \exp\left(-2e_{\frac{f}{\|f\|}}(T_\sigma)\right) \left[-2\left(2T_\sigma \frac{f}{\|f\|^2} - 2e_{\frac{f}{\|f\|}}(T_\sigma) \frac{f}{\|f\|^2}\right) v_{\frac{f}{\|f\|}}(T_x)\right] \\
&\quad + \exp\left(-2e_{\frac{f}{\|f\|}}(T_\sigma)\right) \left[2(T_x - e_{\frac{f}{\|f\|}}(T_x))^2 \frac{f}{\|f\|^2} - 2v_{\frac{f}{\|f\|}}(T_x) \frac{f}{\|f\|^2}\right] \\
&\quad + \exp\left(-e_{\frac{f}{\|f\|}}(T_\sigma)\right) \left[-\left(2T_\sigma \frac{f}{\|f\|^2} - 2e_{\frac{f}{\|f\|}}(T_\sigma) \frac{f}{\|f\|^2}\right) e_{\frac{f}{\|f\|}}(\mathcal{W}^2)\right] \\
&\quad + \exp\left(-e_{\frac{f}{\|f\|}}(T_\sigma)\right) \left[2\mathcal{W}^2 \frac{f}{\|f\|^2} - 2e_{\frac{f}{\|f\|}}(\mathcal{W}^2) \frac{f}{\|f\|^2}\right] \\
&\quad + 2(T_\sigma - e_{\frac{f}{\|f\|}}(T_\sigma))^2 \frac{f}{\|f\|^2} - 2v_{\frac{f}{\|f\|}}(T_\sigma) \frac{f}{\|f\|^2}.
\end{aligned} \tag{3.4}$$

In the case where $f \in \mathcal{D}_S^0$, i.e., $e_f(T_x) = e_f(T_\sigma) = 0$ and $\|f\| = 1$, this simplifies to

$$\begin{aligned}
\delta S(f) &= -4v_f(T_x)T_\sigma f \\
&\quad + 2T_x^2 f - 2v_f(T_x)f \\
&\quad - 2e_f(\mathcal{W}^2)T_\sigma f \\
&\quad + 2\mathcal{W}^2 f - 2e_f(\mathcal{W}^2)f \\
&\quad + 2T_\sigma^2 f - 2v_f(T_\sigma)f.
\end{aligned} \tag{3.5}$$

By writing out $T_x = i\frac{\partial}{\partial\omega}$, $T_\sigma = -\ln(\omega)$, $\mathcal{W}^2 = \frac{1}{\omega}$ and putting

$$\sigma_x = v_f(T_x), \quad \sigma_\sigma = v_f(T_\sigma), \quad \sigma_{\mathcal{W}} = e_f(\mathcal{W}^2) = \|\mathcal{W}f\|^2$$

this reads as

$$\delta S(f)(\omega) = -2f''(\omega) + \left(4\sigma_x \ln(\omega) - 2\sigma_x + 2\sigma_{\mathcal{W}} \ln(\omega) + \frac{2}{\omega} - 2\sigma_{\mathcal{W}} + 2\ln(\omega)^2 - 2\sigma_\sigma\right) f(\omega).$$

□

With the variation established, we move to proving the first theorem.

Proof of Theorem 3.15. Let f_0 be a minimizer of S in $\mathcal{D}_S^0$ and assume towards a contradiction that $\delta S(f_0) \neq 0$. Then we can find a $h \in \mathcal{D}_S$ such that

$$\langle \delta S(f_0), h \rangle = e < 0.$$

By multiplying h by $t > 0$ we have

$$S(f_0 + th) = S(f_0) + te + o(t)$$

since $o(\|th\|) = o(t)$ so we can choose t so small that $S(f_0 + th) - S(f_0) < 0$ in the above which yields that $S(f_0 + th) < S(f_0)$, a contradiction to f_0 being a minimizer.

Plugging $\delta S(f_0) = 0$ into the result of Proposition 3.17 we obtain

$$\begin{aligned} & -2f_0''(\omega) + \left(4\sigma_x \ln(\omega) - 2\sigma_x + 4\sigma_{\mathcal{W}} \ln(\omega) + \frac{2}{\omega} - 2\sigma_{\mathcal{W}} + 2\ln(\omega)^2 - 2\sigma_{\sigma}\right) f_0(\omega) = 0 \\ \implies & f_0''(\omega) = \left(\ln(\omega)^2 + \frac{1}{\omega} + [2\sigma_x + 2\sigma_{\mathcal{W}}] \ln(\omega) - \sigma_x - \sigma_{\sigma} - \sigma_{\mathcal{W}}\right) f_0(\omega) \end{aligned}$$

which is the desired conclusion. $\square$

The same argument cannot be applied the problem of finding a minimizer in $L^2(0, b)$ since in that case the variation needs to take into consideration that $[0, b]$ is not invariant under dilations. To incorporate this restrictions we can use a Lagrangian multipliers approach where we only search for minimizers among functions with zero scale. This is unfortunately technically difficult due to the underlying space being infinite dimensional. For this reason, the proof of Theorem 3.16 can be found in Appendix A but we mention here that the Lagrangian multipliers approach works in our case for the infinite dimensional space and we essentially recover the differential equation $\delta S(f_0) - \beta \delta e_{f_0}(\check{T}_2) = 0$.

Remark. The above theorems does not give us uniqueness of minimizers although it serves as a strong indicator that we should expect minimizers to be unique.

3.2.2 Smoothness of the minimizers

Having established the variational properties of minimizers, we move to showing smoothness properties and bounds. We will need the following simple auxiliary lemma.

Lemma 3.23. *Let $f \in L^2(\mathbb{R}^+)$ be absolutely continuous with $f(0) = 0$ and $\|f'\|^2 \leq K$, then*

$$|f(\omega)| \leq \sqrt{K} \sqrt{\omega}.$$

Proof.

$$|f(\omega)| = |f(\omega) - f(0)| = \left| \int_0^\omega f'(t) dt \right| \leq \int_0^\omega |f'(t)| dt \leq \|f'\| \|\chi_{[0, \omega]}\| \leq \sqrt{K} \sqrt{\omega}.$$

$\square$

Proposition 3.24. *Let f_0 be a minimizer of S in $\mathcal{D}_S^0$. Then f_0 is smooth and $f_0''|_{(0, b)} \in L^2(0, b)$ for any $b > 0$.*

Proof. By Theorem 3.15, f_0 satisfies the ODE

$$f_0'' = H f_0, \quad H(\omega) = \ln(\omega)^2 + \frac{1}{\omega} - \kappa \ln(\omega) - \sigma$$

where $\kappa, \sigma \in \mathbb{R}$. We claim first that $H f_0|_{(0, b)} \in L^1(0, b)$. Indeed, using that $|f_0(\omega)| \leq \sqrt{K} \sqrt{\omega}$ by the lemma above, we have that

$$\int_0^b |H(\omega) f_0(\omega)| d\omega \leq \int_0^b \left| \ln(\omega)^2 \sqrt{K} \sqrt{\omega} + \frac{\sqrt{K}}{\sqrt{\omega}} + \kappa |\ln(\omega)| \sqrt{K} \sqrt{\omega} + \sigma \sqrt{K} \sqrt{\omega} \right| d\omega < \infty$$

Since $f_0'' = H f_0$, we conclude that $f_0''|_{(0, b)} \in L^1(0, b)$ implying that f_0' can be uniformly bounded by some constant M by the compactness of $[0, b]$. It thus holds that

$$|f_0(\omega)| \leq M\omega$$

which we can use to show that $Hf_0|_{(0,b)} \in L^2(0,b)$. Using the inequality $(a+b+c+d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)$ and $|f_0(\omega)| < M\omega$ we get

$$\|Hf_0\|_{L^2(0,b)}^2 \leq \int_0^b 4 \left(M^2 \ln(\omega)^4 \omega^2 + \frac{M^2 \omega^2}{\omega^2} + \kappa^2 \ln(\omega)^2 M^2 \omega^2 + M^2 \sigma^2 \omega^2 \right) d\omega < \infty.$$

We can establish smoothness inductively by noting that f_0 and H being continuous and $Hf_0 = f_0''$ implies that f_0 is twice continuously differentiable. Since H is smooth we can repeat this argument that $f_0'' = Hf_0 \in C^k \implies f_0 \in C^{k+2}$ for all even k . Inductively, we get that f_0 is C^∞ as desired. $\square$

Proposition 3.25. *Let f_0 be a minimizer of S in $\mathcal{D}_S^0 \cap L^2(0,b)$. Then*

- (1) f_0 is smooth in $(0,b)$,
- (2) There exist bounds $M_k > 0$ such that

$$\|f_0^{(k)}\|_{L^\infty(0,b)} \leq M_k, \quad k = 0, 1, 2,$$

- (3) There exists a constant M_3 such that

$$|\ln(\omega)f_0(\omega)| \leq M_3.$$

Proof. To conclude that f_0 is bounded, note that f_0 is absolutely continuous and so in particular uniformly continuous and bounded since $[0,b]$ is compact. This gives the M_0 bound.

For smoothness and the M_1 bound of f_0' , we use the same argument as in Proposition 3.24 but with

$$H_0(\omega) = H(\omega) + \beta \ln(\omega) = \ln(\omega)^2 + \frac{1}{\omega} - \kappa \ln(\omega) + \beta \ln(\omega)$$

corresponding to Euler-Lagrange equation for the minimizers in $\mathcal{D}_S^0 \cap L^2(0,b)$ from Theorem 3.16. To bound $f_0'' = H_0 f_0$ we need to bound H_0 since f_0 is bounded already. Note that for $\omega < \frac{1}{2}$, $\ln(\omega)^2 < |\ln(\omega)| < \frac{1}{\omega}$ and so

$$|f_0''(\omega)| = |H_0(\omega)f_0(\omega)| \leq \frac{C}{\omega} M_1 \omega = CM_1.$$

For $\omega \in [1/2, b]$, H_0 is bounded trivially so we get another bound there yielding M_2 as the maximum.

Lastly, note that we only need to show that the last estimate holds near 0 since f_0 is bounded and $[0,b]$ is compact. It holds since f_0 is right differentiable at 0 and $f_0(0) = 0$. $\square$

3.2.3 Existence of real valued minimizers

Let f be a minimizer of S in $L^2(\mathbb{R}^+)$. Then by Proposition 3.24 and Theorem 3.15, f is smooth and satisfies the Euler-Lagrange equation

$$f'' + Hf = 0.$$

By writing $f = f_R + if_I$ where f_R and f_I are real valued, we see that f_R and f_I both are solutions to the above ODE since H is real valued. Since $e_f(T_x) = 0$ is real, we have that

$$\begin{aligned} 0 &= e_f(T_x) = e_f\left(i\frac{\partial}{\partial\omega}\right) = \operatorname{Re}\left(e_f\left(i\frac{\partial}{\partial\omega}\right)\right) = \operatorname{Re}\int_0^\infty if'\bar{f}d\omega \\ &= \int_0^\infty \operatorname{Re}(if'\bar{f})d\omega = \int_0^\infty (-f'_If_R + f'_Rf_I)d\omega = \int_0^\infty W(f_I, f_R)d\omega \end{aligned}$$

where

$$W(f, g)(\omega) = f'(\omega)g(\omega) - g'(\omega)f(\omega)$$

is the *Wronskian*. Note that we implicitly used that $f' \in L^2(\mathbb{R})$ which follows by $\|T_x f\|^2 < \infty$. Now, if f_R and f_I are linearly dependent, then $f_I = cf_R$ and $f = f_R + icf_R = (1+ic)f_R$ implies that $\frac{f}{1+ic}$ is real valued. Then, since S is invariant under linear scalings, $S(f) = S\left(\frac{f}{1+ic}\right)$ and we have a real valued minimizer.

If instead f_R and f_I are linearly independent, then the Wronskian cannot be identically zero. However, since the ODE is of the form $f'' = Hf$, the Wronskian is constant by

$$\begin{aligned} W'(f_R, f_I)(\omega) &= f_R(\omega)f_I''(\omega) - f_I(\omega)f_R''(\omega) \\ &= f_R(\omega)H(\omega)f_I(\omega) - f_I(\omega)H(\omega)f_R(\omega) = 0 \end{aligned}$$

meaning it must be constantly nonzero. This contradicts $0 = \int_0^\infty W(f_I, f_R)d\omega$ however, so we conclude that f_R and f_I must be linearly dependant.

In the case where f is a minimizer of S in $\mathcal{D}_S^g \cap L^2(0, b)$, similarly to in the proof of Proposition 3.25, we can replace H by H_0 for which the above argument also works.

3.3 Convergence of compactly supported minimizers

The following two propositions show in which sense the minimizers in $L^2(0, b)$ approximate the minimizers in $L^2(\mathbb{R}^+)$.

Proposition 3.26. *Let f_0 be a minimizer of S in $\mathcal{D}_S^{0, \mathbb{R}}$. Then for any $\varepsilon > 0$, there exist $b > 0$ and $y \in \mathcal{D}_S^{0, \mathbb{R}} \cap L^2(0, b)$ such that*

$$\|f_0 - y\| < \varepsilon, \quad |S(f_0) - S(y)| < \varepsilon.$$

Proposition 3.27. *Let $\mathcal{M}_b$ be the set of minimizers of S in $\mathcal{D}_S^{0, \mathbb{R}} \cap L^2(0, b)$. Then any element of*

$$\mathcal{M} = \left\{ f \in L^2(\mathbb{R}^+) : f = \lim_{n \rightarrow \infty} f_{b_n}, f_{b_n} \in \mathcal{M}_{b_n} \text{ for some sequence } b_n \rightarrow \infty \right\}$$

is a minimizer of S in $\mathcal{D}_S^{0, \mathbb{R}}$.

We now devote the rest of this section to proving the two above propositions. To do so we will need the following auxiliary lemma.

Lemma 3.28. *If $f, f' \in L^2(\mathbb{R})$, then*

$$\lim_{\omega \rightarrow \infty} f(\omega) = 0.$$

Proof. Since f is an L^2 function, we note that the $f' \in L^2(\mathbb{R})$ condition implies that it is meaningful to discuss pointwise values. This since f is Hölder continuous with exponent $1/2$ by

$$|f(\beta) - f(\alpha)| = \left| \int_{\alpha}^{\beta} f'(\omega) d\omega \right| \leq \sqrt{\int_{\alpha}^{\beta} |f'(\omega)|^2 d\omega} \sqrt{\beta - \alpha} = C\sqrt{\beta - \alpha}.$$

for some constant C . Now, to get a contradiction, suppose that there exists an increasing sequence $\omega_n \rightarrow \infty$ such that $|f(\omega_n)|^2 > \varepsilon$ for all n . By moving to a subsequence, we can require

$$|\omega_{n+1} - \omega_n| > \eta, \quad \text{for all } n$$

for some arbitrary η . We now want derive the contradiction $f \notin L^2(\mathbb{R})$ by bounding $\|f\|^2$ from below as

$$\|f\|^2 \geq \sum_n \int_{\omega_n}^{\omega_n + \eta} |f(\omega)|^2 d\omega.$$

Since f is Hölder, it holds for all $\omega \in [\omega_n, \omega_n + \eta]$ that

$$|f(\omega) - f(\omega_n)| \leq C\sqrt{\eta}$$

and we can choose η so small that

$$\left| |f(\omega)|^2 - |f(\omega_n)|^2 \right| < \frac{\varepsilon}{2} \implies |f(\omega)|^2 > \frac{\varepsilon}{2}.$$

We now get our contradiction as

$$\|f\|^2 \geq \sum_n \int_{\omega_n}^{\omega_n + \eta} |f(\omega)|^2 d\omega > \sum_n \int_{\omega_n}^{\omega_n + \eta} \frac{\varepsilon}{2} d\omega = \infty.$$

□

Note that this lemma applies to all $f \in \mathcal{D}_S$ since $\|T_x f\| < \infty \implies f' \in L^2(\mathbb{R}^+)$. We now move to proving the two propositions.

Proof of Proposition 3.26. With f_0 a minimizer in $\mathcal{D}_S^{0,\mathbb{R}}$, the inequality $\|T_{\sigma} f_0\| < \infty$ implies that we can find an increasing sequence $b_n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} |\ln(b_n)| |f_0(b_n)| = 0.$$

With this sequence we construct the following sequences of approximations leading up to f_n :

Write $c_n = |f_0(b_n)| = |c_n|e^{i\theta_n}$, $\theta_n \in \{0, \pi\}$ and define

$$\begin{aligned} f_n^1(\omega) &= \begin{cases} f_0(\omega), & \omega \in (0, b_n), \\ c_n - e^{i\theta_n}(\omega - b_n), & \omega \in [b_n, b_n + |c_n|], \\ 0, & \text{otherwise,} \end{cases} \\ f_n^2(\omega) &= \frac{f_n^1(\omega)}{\|f_n^1\|}, \quad \begin{cases} e_1^n = e_{f_n^2}(T_x), \\ e_2^n = e_{f_n^2}(T_{\sigma}), \end{cases} \\ f_n(\omega) &= \pi(e_1^n, e_2^n)^{-1} f_n^2(\omega). \end{aligned}$$

Note that the last step yields that $e_{f_n}(T_x) = e_{f_n}(T_{\sigma}) = 0$. From here we show that

$$\|f_0 - f_n\| \rightarrow 0, \quad |S(f_0) - S(f_n)| \rightarrow 0$$

by showing that the following series of approximations hold:

$$(1) \quad \|f_0 - f_n^1\| \xrightarrow{n \rightarrow \infty} 0:$$

$$\begin{aligned} \|f_0 - f_n^1\|^2 &= \int_{b_n}^{\infty} |f_0(\omega) - f_n^1(\omega)|^2 d\omega \\ &\leq \underbrace{\int_{b_n}^{\infty} |f_0(\omega)|^2 d\omega}_{\rightarrow 0} + \int_{b_n}^{b_n+|c_n|} \underbrace{|c_n - e^{i\theta_n}(\omega - b_n)|^2}_{\leq 2|c_n| \rightarrow 0} d\omega \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

$$(2) \quad \|f'_0 - f_n^{1'}\| \xrightarrow{n \rightarrow \infty} 0:$$

$$\|f'_0 - f_n^{1'}\|^2 \leq \int_{b_n}^{\infty} |f'_0(\omega)|^2 d\omega + \int_{b_n}^{b_n+|c_n|} \underbrace{|f_n^{1'}(\omega)|^2}_{=1} d\omega \xrightarrow{n \rightarrow \infty} 0.$$

$$(3) \quad \|f_n^1 - f_n^2\| \xrightarrow{n \rightarrow \infty} 0:$$

$$\|f_n^1 - f_n^2\| = \left| 1 - \frac{1}{\|f_n^1\|} \right| \|f_n^1\| \xrightarrow{n \rightarrow \infty} \|f_0\| - 1 = 0.$$

$$(4) \quad \|f_n^{1'} - f_n^{2'}\| \xrightarrow{n \rightarrow \infty} 0:$$

$$\|f_n^{1'} - f_n^{2'}\| = \underbrace{\left| 1 - \frac{1}{\|f_n^1\|} \right|}_{\rightarrow 0} \underbrace{\|f_n^{1'}\|}_{< \infty} \xrightarrow{n \rightarrow \infty} 0.$$

$$(5) \quad e_{f_n^1}(T_x) \xrightarrow{n \rightarrow \infty} 0:$$

$$\begin{aligned} |e_{f_n^1}(T_x)| &= \left| \int_0^{\infty} f_n^{1'}(\omega) \overline{f_n^1(\omega)} d\omega \right| \\ &\leq \underbrace{\left| \int_0^{b_n} f'_0(\omega) \overline{f_0(\omega)} d\omega \right|}_{\rightarrow e_{f_0}(T_x)=0} + \int_{b_n}^{b_n+|c_n|} \underbrace{|f_n^{1'}(\omega)|}_{=1} \underbrace{|f_n^1(\omega)|}_{\leq |c_n| \rightarrow 0} d\omega \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

$$(6) \quad e_{f_n^2}(T_x) \xrightarrow{n \rightarrow \infty} 0: \text{ By (5), it suffices to show that}$$

$$|e_{f_n^1}(T_x) - e_{f_n^2}(T_x)| = |\langle T_x f_n^1, f_n^1 \rangle| \left(1 - \frac{1}{\|f_n^1\|^2} \right) \xrightarrow{n \rightarrow \infty} 0.$$

$$(7) \quad e_{f_n^1}(T_\sigma) \xrightarrow{n \rightarrow \infty} 0:$$

$$e_{f_n^1}(T_\sigma) = \underbrace{\int_0^{b_n} \ln(\omega) |f_0(\omega)|^2 d\omega}_{\rightarrow e_{f_0}(T_\sigma)=0} + \int_{b_n}^{b_n+|c_n|} \underbrace{\ln(\omega) |f_n^1(\omega)|^2}_{\leq \ln(b_n+|c_n|)|c_n|^2} d\omega \xrightarrow{n \rightarrow \infty} 0$$

where the last limit follows by $|c_n| \rightarrow 0$ and $|\ln(b_n)| |f_0(b_n)| \rightarrow 0$.

(8) $e_{f_n^2}(T_\sigma) \xrightarrow{n \rightarrow \infty} 0$: By (7) it suffices to note that

$$|e_{f_n^1}(T_\sigma) - e_{f_n^2}(T_\sigma)| = |\langle T_\sigma f_n^1, f_n^1 \rangle| \left| 1 - \frac{1}{\|f_n^1\|^2} \right| \xrightarrow{n \rightarrow \infty} 0$$

(9) $\|f_n^2 - f_n\| \xrightarrow{n \rightarrow \infty} 0$: This follows from the continuity of π since $e_1^n \rightarrow 0$ by (6) and $e_2^n \rightarrow 0$ by (8).

(10) $\|f_0 - f_n\| \xrightarrow{n \rightarrow \infty} 0$: This follows from the triangle inequality applied to (1), (3) and (9).

(11) $\|T_x f_n^2\| \xrightarrow{n \rightarrow \infty} \|T_x f_0\|$: This follows immediately from (2) and (4)

(12) $\left| \|T_x f_n\| - \|T_x f_n^2\| \right| \xrightarrow{n \rightarrow \infty} 0$:

$$\left| \|T_x f_n^2\| - \|T_x f_n\| \right| \leq \int_0^\infty \left| |f_n^{2'}(\omega)|^2 - |f_n'(\omega)|^2 \right| d\omega \xrightarrow{n \rightarrow \infty} 0$$

by the dominated convergence theorem motivated by $f_0 \in \mathcal{D}_S$ and the continuity of π together with (6) and (8).

(13) $\|T_x f_n\| \xrightarrow{n \rightarrow \infty} \|T_x f_0\|$: This follows from the triangle inequality applied to (11) and (12).

(14) $\|T_\sigma f_n^1\| \xrightarrow{n \rightarrow \infty} \|T_\sigma f_0\|$:

$$\left| \|T_\sigma f_n^1\| - \|T_\sigma f_0\| \right| \leq \|T_\sigma f_n^1 - T_\sigma f_0\| + \underbrace{\left| \|T_\sigma f_0\| - \|T_\sigma f_0\| \right|}_{=0}.$$

The first term converges to 0 by

$$\begin{aligned} \|T_\sigma f_n^1 - T_\sigma f_0\|^2 &= \|\ln(\omega)[f_0 - f_n^1]\|^2 \\ &\leq \underbrace{\int_{b_n}^\infty |\ln(\omega)|^2 |f_0(\omega)|^2 d\omega}_{\rightarrow 0} + \int_{b_n}^{b_n + |c_n|} |\ln(\omega)|^2 \underbrace{|c_n - e^{i\theta_n}(\omega - b_n)|^2}_{\leq |c_n|} d\omega \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

since $|\ln(b_n)| |f_0(b_n)| \rightarrow 0$.

(15) $\left| \|T_\sigma f_n^2\| - \|T_\sigma f_n^1\| \right| \xrightarrow{n \rightarrow \infty} 0$:

$$\left| \|T_\sigma f_n^2\| - \|T_\sigma f_n^1\| \right| = \underbrace{\|T_\sigma f_n^1\|}_{< \infty} \left| \frac{1}{\|f_n^1\|} - 1 \right| \xrightarrow{n \rightarrow \infty} 0.$$

(16) $\left| \|T_\sigma f_n^2\| - \|T_\sigma f_n\| \right| \xrightarrow{n \rightarrow \infty} 0$:

$$\left| \|T_\sigma f_n^2\| - \|T_\sigma f_n\| \right| \leq \int_0^\infty |\ln(\omega)|^2 \left| |f_n^2(\omega)|^2 - |f_n(\omega)|^2 \right| d\omega \xrightarrow{n \rightarrow \infty} 0$$

by the dominated convergence theorem motivated by $f_0 \in \mathcal{D}_S$ and the continuity of π together with (6) and (8).

(17) $\|T_\sigma f_n\| \xrightarrow{n \rightarrow \infty} \|T_\sigma f_0\|$: This follows from the triangle inequality applied to (14), (15) and (16).

(18) $\|\mathcal{W}f_n^1\| \xrightarrow{n \rightarrow \infty} \|\mathcal{W}f_0\|$:

$$\|\mathcal{W}f_n^1\|^2 = \underbrace{\int_0^{b_n} \frac{|f_0(\omega)|^2}{\omega} d\omega}_{\rightarrow \|\mathcal{W}f_0\|^2} + \int_{b_n}^{b_n+|c_n|} \frac{1}{\omega} \underbrace{\left| \frac{f_n^1(\omega)}{\omega} \right|^2}_{\leq 2|c_n| \rightarrow 0} d\omega \xrightarrow{n \rightarrow \infty} \|\mathcal{W}f_0\|^2.$$

(19) $\left| \|\mathcal{W}f_n^2\| - \|\mathcal{W}f_n^1\| \right| \xrightarrow{n \rightarrow \infty} 0$:

$$\left| \|\mathcal{W}f_n^2\| - \|\mathcal{W}f_n^1\| \right| = \underbrace{\|\mathcal{W}f_n^1\|}_{< \infty} \left(\frac{1}{\|\mathcal{W}f_n^1\|} - 1 \right) \xrightarrow{n \rightarrow \infty} 0.$$

(20) $\left| \|\mathcal{W}f_n^2\| - \|\mathcal{W}f_n\| \right| \xrightarrow{n \rightarrow \infty} 0$:

$$\left| \|\mathcal{W}f_n^2\| - \|\mathcal{W}f_n\| \right| \leq \int_0^\infty \frac{||f_n^2(\omega)|^2 - |f_n(\omega)|^2|}{\omega} d\omega \xrightarrow{n \rightarrow \infty} 0$$

by the dominated convergence theorem motivated by $f_0 \in \mathcal{D}_S$ and the continuity of π together with (6) and (8).

(21) $\|\mathcal{W}f_n\| \xrightarrow{n \rightarrow \infty} \|\mathcal{W}f_0\|$: This follows from the triangle inequality applied to (18), (19) and (20).

(22) $|S(f_0) - S(f_n)| \xrightarrow{n \rightarrow \infty} 0$: This follows from the triangle inequality applied to (13), (17) and (21).

To finish the proof, let $\varepsilon > 0$ and choose n so large that $\|f_0 - f_n\| < \varepsilon$ and $|S(f_0) - S(f_n)| < \varepsilon$. Then f_n is compactly supported since it is the dilation of the compactly supported f_n^1 . That $f_n \in \mathcal{D}_S^{0,\mathbb{R}}$ is immediate from the construction. $\square$

Proof of Proposition 3.27. Fix $f \in \mathcal{M}$ so that $f = \lim_{n \rightarrow \infty} f_{b_n}$ for a sequence $b_n \rightarrow \infty$ where f_{b_n} is a minimizer of S in $\mathcal{D}_S^{0,\mathbb{R}} \cap L^2(0, b_n)$. By Proposition 3.26,

$$S(f_{b_n}) \xrightarrow{n \rightarrow \infty} S(f_0)$$

where f_0 is a minimizer of S in $\mathcal{D}_S^{0,\mathbb{R}}$. We can therefore conclude that $f_{b_n} \in \mathcal{K}$ for large enough n by the definition of $\mathcal{K}$. Now in the same way as in the proof of Theorem 3.1 on page 28 with $\mathcal{K}_{\alpha,\beta,\gamma}$, we can show that $S(f) = S(f_0)$ since $f_{b_n} \rightarrow f$ and so f is indeed a minimizer. $\square$

3.4 Convergence of compactly supported spline minimizers

The goal of this section is to establish the following two propositions which will allow us to prove theorems 3.4 and 3.5.

Proposition 3.29. *Let f_0 be a minimizer of S in $\mathcal{D}_S^{0,\mathbb{R}} \cap L^2(0, b)$. Then for any $\varepsilon > 0$, there exist $h > 0$ and $y \in \mathcal{D}_S^{0,\mathbb{R}} \cap P_h(0, b)$ such that*

$$\|f_0 - y\| < \varepsilon, \quad |S(f_0) - S(y)| < \varepsilon.$$

Proposition 3.30. *Let $\mathcal{M}_h$ be the set of minimizers of S in $\mathcal{D}_S^{0,\mathbb{R}} \cap P_h(0, b)$. Then any element of*

$$\mathcal{M} = \left\{ f \in L^2(0, b) : f = \lim_{n \rightarrow \infty} p_{h_n}, p_{h_n} \in \mathcal{M}_{h_n} \text{ for some sequence } h_n \rightarrow 0 \right\}$$

is a minimizer of S in $\mathcal{D}_S^{0,\mathbb{R}} \cap L^2(0, b)$.

The following diagram illustrates how we obtain splines in $\mathcal{D}_S^{0,\mathbb{R}} \cap P_h(0, b)$ which approximate minimizers $f_0 \in \mathcal{D}_S^{0,\mathbb{R}} \cap L^2(0, b)$:

$$f_0 \xrightarrow[L^\infty + L^2 \text{ bounds}]{\text{Interpolation}} p \xrightarrow[L^2 \text{ bounds}]{\text{Nudge scale to 0}} k \xrightarrow[L^2 \text{ bounds}]{\text{Normalize}} \frac{k}{\|k\|}.$$

Interpolating f_0 means that we construct a linear spline p for which $p(\omega) = f_0(\omega)$ for $\omega = 0, h, 2h, \dots, b$. The following lemma gives quantitative estimates on the L^2 error between a continuous function and its interpolation. Note that unless we state otherwise, all interpolations are to be taken to be with step size h .

Lemma 3.31 ([24, Theorem 3.4]). *Let $[a, b] \subset \mathbb{R}$ be a compact interval. Then for any $f \in L^2(a, b)$ and its interpolation p , the following holds.*

(1) *If f is absolutely continuous in $[a, b]$, with $f' \in L^2(a, b)$, then*

$$\|f - p\| \leq \frac{1}{\pi} \|f'\| h.$$

(2) *If f' is absolutely continuous in $[a, b]$, with $f'' \in L^2(a, b)$, then*

$$\|f - p\| \leq \frac{1}{\pi^2} \|f''\| h^2.$$

In particular, in the case where f_0 is a minimizer of S in $\mathcal{D}_S^{0,\mathbb{R}} \cap L^2(0, b)$ and p its interpolation, the following approximation lemma holds.

Lemma 3.32. *With f_0 and p as above, it holds that $\|f_0 - p\| = O(h^2)$ and*

$$|f_0(\omega) - p(\omega)| \leq M_2 h^2$$

for all $\omega \in [0, b]$.

Proof. For the L^2 error, since p is the interpolation of f_0 , it holds that

$$\|f_0 - p\| \leq \underbrace{\frac{1}{\pi^2} \|f_0''\| h^2}_{\text{by Lemma 3.31}} \leq \underbrace{\frac{1}{\pi^2} M_2 h^2}_{\text{by Proposition 3.25}} = O(h^2).$$

For the pointwise error, let $g(\omega) = f_0(\omega) - p(\omega)$ and note that $g(kh) = 0$ for all integer k and $g'' = f_0''$ a.e. For a fixed $\omega \in [0, b]$ we can find an integer n such that

$$nh \leq \omega < (n+1)h$$

and so $|\omega - nh| < h$, yielding

$$|g(\omega)| = \left| \int_{nh}^{\omega} g'(t) dt \right| = \left| \int_{nh}^{\omega} \int_{nh}^t g''(s) ds dt \right| \leq \int_{nh}^{\omega} \int_{nh}^t M_2 ds dt \leq M_2 h^2$$

using the M_2 bound from Proposition 3.25 since $g'' = f_0''$ a.e. □

We now show how T_σ affects p compared to f_0 .

Lemma 3.33. *With f_0 and p as above, it holds that*

$$\begin{aligned} |e_{f_0}(T_\sigma) - e_p(T_\sigma)| &= O(h^2), \\ \left| \|T_\sigma f_0\|^2 - \|T_\sigma p\|^2 \right| &= O(h^2). \end{aligned}$$

Proof. For the expected scale, we estimate directly as

$$\begin{aligned} |e_{f_0}(T_\sigma) - e_p(T_\sigma)| &= |\langle -\ln(\omega)f_0, f_0 \rangle - \langle -\ln(\omega)p, p \rangle| \\ &= |\langle -\ln(\omega)f_0, f_0 - p \rangle + \langle -\ln(\omega)(f_0 - p), p \rangle| \\ &\leq \|\ln(\omega)f_0\| \|f_0 - p\| + \|f_0 - p\| \|\ln(\omega)p\| \\ &= \|f_0 - p\| (\|\ln(\omega)f_0\| + \|\ln(\omega)p\|). \end{aligned}$$

By Lemma 3.32, the first factor is $O(h^2)$ so the desired conclusion will follow if we can bound the two terms. For the first term it suffices to note that $f_0 \in \mathcal{K}$ while the second term can be crudely estimated as

$$\|\ln(\omega)p\| \leq \max_{\omega \in [0, b]} |p(\omega)| \sqrt{\int_0^b |\ln(\omega)|^2 d\omega} \leq \max_{\omega \in [0, b]} |f_0(\omega)| \cdot O(\ln(b)\sqrt{b})$$

which, for b fixed, is bounded since it is independent of h .

For the differences in norm, since both f_0 and p are real valued, we have

$$\begin{aligned} \left| \|T_\sigma f_0\|^2 - \|T_\sigma p\|^2 \right| &= \left| \int_0^b |\ln(\omega)|^2 (f_0(\omega)^2 - p(\omega)^2) d\omega \right| \\ &\leq \int_0^b |\ln(\omega)|^2 |f_0(\omega) + p(\omega)| |f_0(\omega) - p(\omega)| d\omega \\ &\leq \underbrace{\max_{\omega \in [0, b]} |f_0(\omega) + p(\omega)|}_{\leq 2M_0} \underbrace{\max_{\omega \in [0, b]} |f_0(\omega) - p(\omega)|}_{\leq M_2 h^2} \underbrace{\left| \int_0^b |\ln(\omega)|^2 d\omega \right|}_{< \infty} = O(h^2) \end{aligned}$$

where we used the estimates from Proposition 3.25 and Lemma 3.32. $\square$

Since we want to approximate f_0 with a spline $k \in P_h(0, b)$ with $e_k(T_\sigma) = 0$, we need to modify p . To that end, let

$$L : C(0, b) \rightarrow P_h(0, b)$$

be the operator that, applied to a function f , interpolates the product $\ln(\omega)f$. With L we define the spline

$$k = p + \alpha Lp$$

where α is a real parameter. This next lemma states that we can choose α so that k is close to p and has expected scale zero.

Lemma 3.34. *There exists $\alpha = O(h^2)$ such that $k = p + O(h^2)$ and $e_k(T_\sigma) = 0$.*

Proof. In order for $e_k(T_\sigma) = 0$ to hold we need to have that

$$\begin{aligned} 0 = e_k(T_\sigma) &= \langle -\ln(\omega)(p + \alpha Lp), (p + \alpha Lp) \rangle \\ &= \underbrace{\langle -\ln(\omega)Lp, Lp \rangle}_A \alpha^2 + 2\underbrace{\langle -\ln(\omega)p, Lp \rangle}_B \alpha + \underbrace{\langle -\ln(\omega)p, p \rangle}_C \\ &= A\alpha^2 + B\alpha + C. \end{aligned}$$

If we can find a solution $\alpha = \alpha(h) = O(h^2)$ to this equation, we will be able to choose k as in the lemma since Lp is bounded in h . Recall that the solutions are of the form

$$\alpha = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Since p is the interpolation of $f_0 \in \mathcal{D}_S^{0,\mathbb{R}} \cap L^2(0, b)$, $B = \langle -\ln(\omega)p, Lp \rangle$ is real valued. Therefore, if we can show that $4AC \ll B^2 \implies \sqrt{B^2 - 4AC} \approx |B|$, we can choose the sign in front of the square root to get cancellation. More specifically, we show that A is bounded from above, $|B|$ is bounded from below and note that $e_p(T_\sigma) = C = O(h^2)$ by the first statement of Lemma 3.33. By Taylor expanding the expression for α , this yields that $\alpha = O(h^2)$ when the sign before the square root is chosen appropriately.

To bound A from above, note that since p is the interpolation of f_0 , $Lp = Lf_0$. Furthermore, since f_0 is differentiable at 0 and $f_0(0) = 0$, $\ln(\omega)f_0(\omega) = 0$ at $\omega = 0$. Moreover, by Proposition 3.25, $|\ln(\omega)f_0(\omega)| \leq M_3 \implies |Lp| \leq M_3$ and so

$$|\langle -\ln(\omega)Lp, Lp \rangle| = \int_0^b |\ln(\omega)||Lp|^2 d\omega \leq M_3^2 \int_0^b |\ln(\omega)| d\omega < \infty.$$

To bound $|B| = |\langle \ln(\omega)p, Lp \rangle|$ from below, we show that it behaves as $\|T_\sigma f_0\|$ for small h . The first step in that direction is to recall that by Lemma 3.33, $|\|T_\sigma f_0\|^2 - \|T_\sigma p\|^2| = O(h^2)$ so we only need to bound

$$\begin{aligned} |B - \|T_\sigma p\|^2| &= |\langle \ln(\omega)p, Lp \rangle - \langle \ln(\omega)p, \ln(\omega)p \rangle| \\ &= |\langle \ln(\omega)p, Lp - \ln(\omega)p \rangle| \\ &\leq \|\ln(\omega)p\| \|Lp - \ln(\omega)p\|. \end{aligned}$$

The first factor is bounded by Lemma 3.33 again while the second factor is the norm of the difference between the smooth function $\ln(\omega)p$ and its interpolation Lp . By Lemma 3.31 this difference is $O(h)$ if we can bound the norm of the derivative. For this we have

$$\|[\ln(\omega)p(\omega)]'\| \leq \|\omega^{-1}p\| + \|\ln(\omega)p'\|.$$

The first term can be bounded using Proposition 3.25 which states that $|f_0'(\omega)| \leq M_1$ which combined with $f_0(0) = 0$ yields $|f_0(\omega)| \leq M_1\omega$. The same argument also works for p by the mean value theorem. We get

$$\|\omega^{-1}p\| \leq \int_0^1 \omega^{-2}p(\omega)^2 d\omega + \int_1^\infty \omega^{-2}p(\omega) d\omega \leq M_1^2 + 1 + o(1)$$

since the norm of p is close to the norm of f_0 by the triangle inequality and Lemma 3.32. For the second term, we simply use that the support of p is compact combined with the same mean value theorem argument to get that

$$\|\ln(\omega)p'\| \leq M_1 \int_0^b |\ln(\omega)|^2 d\omega < \infty.$$

Having established this, we can make the approximation argument explicit as

$$\begin{aligned} \left| B - \|T_\sigma f_0\|^2 \right| &= \left| \langle \ln(\omega)p, Lp \rangle - \|T_\sigma f_0\|^2 \right| \\ &\leq \underbrace{\left| \langle \ln(\omega)p, Lp \rangle - \|T_\sigma p\|^2 \right|}_{\rightarrow 0 \text{ by above}} + \underbrace{\left| \|T_\sigma p\|^2 - \|T_\sigma f_0\|^2 \right|}_{\rightarrow 0 \text{ by Lemma 3.33}}. \end{aligned}$$

Taking a step back, this means that B indeed is bounded from below and we can find an α satisfying the desired properties.

All that is left to show is that $k = p + O(h^2)$, i.e., $\|k - p\| = \alpha \|Lp\| = O(h^2)$. Since we have already showed that $\alpha = O(h^2)$, it suffices to bound $\|Lp\|$. In the above, we showed that $\|Lp - \ln(\omega)p\| = O(h)$ but that same argument works to show that $\|Lp - \ln(\omega)f_0\| = O(h)$ and so

$$\|Lp\| = \|\ln(\omega)f_0\| + O(h) < \infty$$

as desired. $\square$

With k defined we show that as $h \rightarrow 0$, $S(k) \rightarrow S(f_0)$.

Lemma 3.35. *The following three estimates hold:*

- (A) $\left| \|T_x f_0\|^2 - \frac{1}{\|k\|^2} \|T_x k\|^2 \right| = O(h),$
- (B) $\left| \|T_\sigma f_0\|^2 - \frac{1}{\|k\|^2} \|T_\sigma k\|^2 \right| = O(h^2),$
- (C) $\left| \|\mathcal{W} f_0\|^2 - \frac{1}{\|k\|^2} \|\mathcal{W} k\|^2 \right| = O(h^2).$

Note first that $\|k\| = 1 + O(h^2)$ by lemmas 3.34 and 3.32 meaning we can dismiss the normalizations. We now treat each estimate separately.

Proof of Estimate (A). For T_x , we first note that

$$\begin{aligned} \left| \|T_x f_0\|^2 - \|T_x k\|^2 \right| &= \left| \|f'_0\|^2 - \|k'\|^2 \right| \\ &= \left| \|p' + \alpha[Lp]'\|^2 - \|f'_0\|^2 \right| \\ &\leq \left| \left(\|p'\| + \alpha\|[Lp]'\| \right)^2 - \|f'_0\|^2 \right| \\ &= \left| \|p'\|^2 - \|f'_0\|^2 \right| + \alpha \left(2\|p'\|\|[Lp]'\| + \alpha\|[Lp]'\|^2 \right). \end{aligned}$$

For the first term we have that

$$\left| \|p'\|^2 - \|f'_0\|^2 \right| \leq \|p' - f'_0\| \left(\|p'\| + \|f'_0\| \right) = O(h)$$

the reverse triangle inequality and Lemma 3.31. Meanwhile the second term is $O(h^2)$ since $\alpha = O(h^2)$ and both p' and $[Lp]'$ are bounded.

Proof of Estimate (B). For T_σ , we estimate in steps via the triangle inequality as

$$\left| \|T_\sigma f_0\|^2 - \|T_\sigma k\|^2 \right| \leq \left| \|T_\sigma f_0\|^2 - \|T_\sigma p\|^2 \right| + \left| \|T_\sigma p\|^2 - \|T_\sigma k\|^2 \right|.$$

The first term is $O(h^2)$ by Lemma 3.33 while for the second term we have

$$\begin{aligned} \left| \|T_\sigma p\|^2 - \|T_\sigma k\|^2 \right| &\leq \int_0^b |\ln(\omega)|^2 |(p+k)(p-k)| d\omega \\ &= \int_0^b |\ln(\omega)|^2 |(p+k)(\alpha Lp)| d\omega. \end{aligned}$$

Since Lp is the interpolation of $\ln(\omega)f_0(\omega)$, we can bound it by M_3 by Proposition 3.25. Similarly we can bound the sum $p+k$ as

$$\begin{aligned} p+k &= 2p + \alpha Lp \leq 2M_0 + \alpha M_3 \\ \implies \left| \|T_\sigma p\|^2 - \|T_\sigma k\|^2 \right| &\leq M_3 \alpha \int_0^b |\ln(\omega)|^2 (2M_0 + \alpha M_3) d\omega = O(h^2) \end{aligned}$$

which is the desired conclusion.

Proof of Estimate (C). For the $\mathcal{W}$ estimate, we first show that for the interpolation p ,

$$\begin{aligned} \left| \|\mathcal{W}f_0\|^2 - \|\mathcal{W}p\|^2 \right| &= |\langle \omega^{-1}f_0, f_0 \rangle - \langle \omega^{-1}p, p \rangle| \\ &= |\langle \omega^{-1}f_0, f_0 - p \rangle + \langle f_0 - p, \omega^{-1}p \rangle| \\ &\leq \|\omega^{-1}f_0\| \|f_0 - p\| + \|f_0 - p\| \|\omega^{-1}p\| \\ &= \|f_0 - p\| (\|\omega^{-1}f_0\| + \|\omega^{-1}p\|). \end{aligned}$$

That this quantity is $O(h^2)$ will follow if we can show that $\|\omega^{-1}f_0\|$ and $\|\omega^{-1}p\|$ are bounded since $\|f_0 - p\| = O(h)^2$ by Lemma 3.32. This is done similarly to the proof of Lemma 3.34: By Proposition 3.25, we have that $|f'_0(\omega)| \leq M_1 \implies |f_0(\omega)| \leq M_1\omega$ since $f_0(0) = 0$ and that same bound holds for p by the mean value theorem. Therefore we can estimate $\|\omega^{-1}f_0\|$ and $\|\omega^{-1}p\|$ as

$$\int_0^1 \omega^{-2} p(\omega)^2 d\omega \leq M_1^2, \quad \int_0^1 \omega^{-2} f_0(\omega)^2 d\omega \leq M_1^2.$$

For the second parts of the integrals we have

$$\begin{aligned} \int_1^b \omega^{-2} p(\omega)^2 d\omega &\leq \int_1^b p(\omega)^2 d\omega \leq \|p\|^2 \leq (\|p - f_0\| + \|f_0\|)^2 = 1 + o(1) \\ \int_1^b \omega^{-2} f_0(\omega)^2 d\omega &\leq \int_1^b f_0(\omega)^2 d\omega \leq \|f_0\|^2 = 1 \end{aligned}$$

by Lemma 3.32. In conclusion,

$$\left| \|\mathcal{W}f_0\|^2 - \|\mathcal{W}p\|^2 \right| = O(h^2).$$

By the triangle inequality, it suffices to show that $\left| \|\mathcal{W}p\|^2 - \|\mathcal{W}k\|^2 \right| = O(h^2)$ to get the desired conclusion. We can show this by estimating $\|\mathcal{W}(p-k)\|$ since

$$\|\mathcal{W}p\| - \|\mathcal{W}k\| \leq \|\mathcal{W}(p-k)\| + \|\mathcal{W}k\| - \|\mathcal{W}k\| = \|\mathcal{W}(p-k)\|$$

and a similar inequality holds for the $\|\mathcal{W}k\| - \|\mathcal{W}p\|$ case. Since $k-p = \alpha Lp$ we have reduced to problem to bounding $\|\mathcal{W}Lp\|$ since $\alpha = O(h^2)$.

Note first that $|[Lp](\omega)| \leq M_3$ by Proposition 3.25 and we therefore pointwise have that

$$\begin{aligned} |([Lp]^2)(\omega)| &\leq \int_0^\omega |([Lp]^2)'(t)| dt \leq \int_0^\omega 2M_3 |[Lp]'(t)| dt \\ &\leq 2M_3 \sqrt{\int_0^\omega |[Lp]'(t)|^2 dt} \sqrt{\omega} \leq M_4 \sqrt{\omega} \end{aligned}$$

where we bounded the integral by recalling that $\|[Lp]'\| < \infty$ as shown in the proof of Lemma 3.34. This inequality yields that $\frac{|[Lp](\omega)|}{\omega} \leq \frac{M_4}{\sqrt{\omega}}$ which is integrable providing the desired bound. $\square$

Remark. Estimate (A) can with some more work be improved to $O(h^2)$ using an interpolation result of the form $\|p'\| \leq \|f'_0\|$.

With the above lemma established, the proof of Proposition 3.29 follows without much extra work.

Proof of Proposition 3.29. The spline by which we approximate f_0 is $y = \frac{k}{\|k\|}$. By lemmas 3.31 and 3.34 and the fact that $\|k\| = 1 + O(h^2)$ we can choose h so small that

$$\|f_0 - p\| < \frac{\varepsilon}{3}, \quad \|p - k\| < \frac{\varepsilon}{3}, \quad \left\| k - \frac{k}{\|k\|} \right\| < \frac{\varepsilon}{3}.$$

By Lemma 3.35, we can also choose h so small that

$$\left| \|T_x f_0\|^2 - \frac{1}{\|k\|^2} \|T_x k\|^2 \right|, \quad \left| \|T_\sigma f_0\|^2 - \frac{1}{\|k\|^2} \|T_\sigma k\|^2 \right|, \quad \left| \|\mathcal{W} f_0\|^2 - \frac{1}{\|k\|^2} \|\mathcal{W} k\|^2 \right|$$

are all smaller than $\frac{\varepsilon}{3}$. It thus follows by the triangle inequality that

$$\|f_0 - y\| < \varepsilon, \quad |S(f_0) - S(y)| < \varepsilon$$

as desired. $\square$

Proof of Proposition 3.30. This proof works the same way as that of Proposition 3.27. Fix $f \in \mathcal{M}$ so that $f = \lim_{n \rightarrow \infty} p_{h_n}$ for a sequence $h_n \rightarrow 0$ where p_{h_n} is a minimizer of S in $\mathcal{D}_S^{0, \mathbb{R}} \cap P_h(0, b)$. By Proposition 3.29,

$$S(p_{h_n}) \xrightarrow{n \rightarrow \infty} S(f_0)$$

where f_0 is a minimizer of S in $\mathcal{D}_S^{0, \mathbb{R}} \cap L^2(0, b)$. The tail of the sequence $(p_{h_n})_n$ is therefore in a closed subset of $\mathcal{K} \cap L^2(0, b)$ which is compact, hence $S(f) = S(f_0)$ and f is a minimizer as desired. $\square$

We are now ready to prove the original theorems of this chapter.

Proof of Theorem 3.4. Letting f_0 be a minimizer of S in $\mathcal{D}_S^{0, \mathbb{R}}$, by Proposition 3.26 there exist a $b > 0$ and $f \in \mathcal{D}_S^{0, \mathbb{R}} \cap L^2(0, b)$ such that

$$\|f_0 - f\| < \frac{\varepsilon}{2}, \quad |S(f_0) - S(f)| < \frac{\varepsilon}{2}.$$

We now want to repeat this step by approximating f by a spline but Proposition 3.29 requires the $L^2(0, b)$ element to be a minimizer. We can do away with this condition if we

can show that f satisfies the estimates of Proposition 3.25. This is however immediate from the construction of f in the proof of Proposition 3.26. Therefore we can find a $h > 0$ and $y \in \mathcal{D}_S^{0,\mathbb{R}} \cap P_h(0, b)$ that satisfies

$$\|f - y\| < \frac{\varepsilon}{2}, \quad |S(f) - S(y)| < \frac{\varepsilon}{2}.$$

Using the triangle inequality, we can obtain the two inequalities of the theorem, finishing the proof. $\square$

Proof of Theorem 3.5. We again use the same strategy as for propositions 3.27 and 3.30. Fix $f \in \mathcal{M}$ so that $f = \lim_{n \rightarrow \infty} p_{h_n}^{b_n}$ where $p_{h_n}^{b_n}$ is a minimizer of S in $\mathcal{D}_S^{0,\mathbb{R}} \cap P_{h_n}(0, b_n)$. By Theorem 3.4,

$$S(p_{h_n}^{b_n}) \xrightarrow{n \rightarrow \infty} S(f_0)$$

where f_0 is a minimizer of S in $\mathcal{D}_S^{0,\mathbb{R}}$. We can therefore conclude that $p_{h_n}^{b_n} \in \mathcal{K}$ for large enough n by the definition of $\mathcal{K}$. Using the same 3ε argument as in the proof of Theorem 3.1 on page 28 we can conclude that $S(f) = S(f_0)$ which yields the desired conclusion. $\square$

Chapter 4

Phase space uncertainty minimizers

Inspired by numerical constructions of minimizers and a calculation of the variation of the phase space uncertainty $\mathcal{L}$ from Section 2.5, introduced by Levie, Avraham and Sochen in [23], we in this chapter investigate existence and properties of minimizer in an analogous way to Chapter 3. Later we also show that spline minimizers approximate proper minimizers in an appropriate way as shown for the regularized uncertainty in Chapter 3. These results are novel and validates the numerical scheme used to construct minimizers in [23].

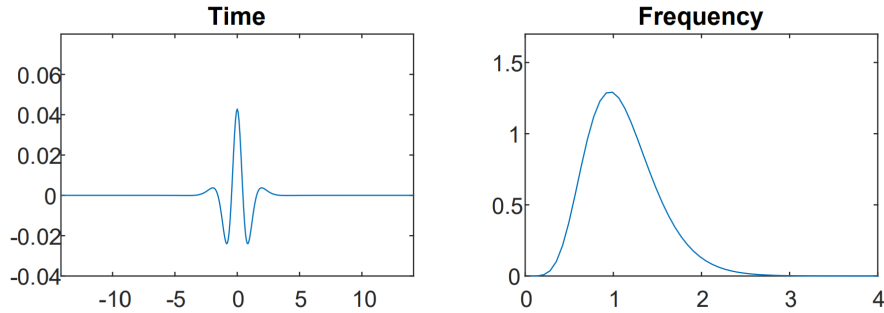


Figure 4.1: Numerical spline minimizer of $\mathcal{L}$, reproduced with permission from [23]. Compared with Figure 3.1, a noticeable difference is that f is smaller near zero for the phase space uncertainty minimizer.

For reference, we repeat here the expression for the phase space uncertainty in the special case where $e_f(T_x) = e_f(T_\sigma) = 0$ and $\|f\| = 1$ from Section 2.5

$$\mathcal{L}(f) = \|T_x f\|^2 + \|T_\sigma f\|^2 + v \frac{\mathcal{W}_f}{\|f\|_{\mathcal{W}}} \left(i\omega \frac{\partial}{\partial \omega} \right) \left\| \frac{f}{\omega} \right\|^2 + v \frac{\mathcal{W}_f}{\|f\|_{\mathcal{W}}} (-\ln(\omega)). \quad (4.1)$$

where $T_x f(\omega) = -if'(t)$, $T_\sigma f(\omega) = -\ln(\omega)$ and $\mathcal{W}$ is the window space from Section 2.5 with norm $\|f\|_{\mathcal{W}} = \left(\int_{\mathbb{R}} \frac{|f(\omega)|^2}{\omega} d\omega \right)^{1/2}$.

The theorems that we set out to prove are the following.

Theorem 4.1. *There exists a minimizer f_0 of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^0$ where*

$$\mathcal{D}_{\mathcal{L}}^0 = \left\{ f \in \text{Dom}(\mathcal{L}) : \|f\| = 1, e_f(T_x) = 0, e_f(T_\sigma) = 0 \right. \\ \left. \|f''\|^2 \leq M, \|\omega f'\|_{\mathcal{W}}^2 \leq M, \left\| \frac{f}{\omega} \right\|^2 \leq M, \omega f \in L^2(\mathbb{R}^+) \right\} \quad (4.2)$$

for any $M > 50$.

Remark. The $\omega f \in L^2(\mathbb{R}^+)$ condition is not strictly required for existence but only for the next two theorems. The $\leq M$ inequalities are a purely technical detail needed for the proof. In practice, the fact that M can be arbitrary large means that any minimizer for which all norms are finite can be attained which is what we want. The reason for the lower limit of 50 is that it is hard to prove that the set is nonempty for $M < 50$.

We expect that the above quantities are finite for any minimizer and so present the following conjecture.

Conjecture 4.2. *There exists a minimizer of $\mathcal{L}$ in*

$$\left\{ f \in \text{Dom}(\mathcal{L}) : \|f\| = 1, e_f(T_x) = 0, e_f(T_\sigma) = 0 \right\}.$$

Theorem 4.3. *There exists a minimizer f_0 of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^0 \cap L^2(0, b)$.*

Theorem 4.4. *There exists a minimizer p_0 of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^0 \cap P_h(0, b)$.*

For the convergence theorems, we once again define the spaces of real valued functions with expected time and scale zero as

$$\mathcal{D}_{\mathcal{L}}^{0, \mathbb{R}} = \left\{ f \in \mathcal{D}_{\mathcal{L}}^0 : f \text{ is real valued} \right\}.$$

Analogously to the situation in Chapter 3, it suffices to consider real valued functions. We prove this in Section 4.2.

Theorem 4.5. *Let f_0 be a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^{0, \mathbb{R}}$. Then for any $\varepsilon > 0$, there exist $b > 0, h > 0$ and $y \in \mathcal{D}_{\mathcal{L}}^{0, \mathbb{R}} \cap P_h(0, b)$ such that*

$$\|f_0 - y\| < \varepsilon, \quad |\mathcal{L}(f_0) - \mathcal{L}(y)| < \varepsilon.$$

Theorem 4.6. *Let $\mathcal{M}_h^b$ be the set of minimizers of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^{0, \mathbb{R}} \cap P_h(0, b)$. Then any element of*

$$\mathcal{M} = \left\{ f \in L^2(\mathbb{R}^+) : f = \lim_{n \rightarrow \infty} p_{h_n}^{b_n}, p_{h_n}^{b_n} \in \mathcal{M}_{h_n}^{b_n} \text{ for two sequences } b_n \rightarrow \infty, h_n \rightarrow 0 \right\}$$

is a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^{0, \mathbb{R}}$.

Remark. When proving the above theorems and all results leading up to them below, we will assume that M is chosen so large that none of the $\leq M$ inequalities defining $\mathcal{D}_{\mathcal{L}}^0$ are sharp.

4.1 Existence of minimizers

The goal of this section is to prove the three existence theorems. We treat Theorem 4.1 first and conclude Theorem 4.3 as a consequence. Theorem 4.4 is proved separately at the end of this section.

As mentioned in Section 2.5, we limit our search to

$$\mathcal{D}_{\mathcal{L}}^0 = \left\{ f \in \mathcal{D}_{\mathcal{L}} : \|f\| = 1, e_f(T_x) = e_f(T_\sigma) = 0 \right\}$$

Note that when the expected signal time $e_f(T_x)$ is zero, so is the expected window time $e_f^{\mathcal{W}}(T_x)$ since

$$e_{\frac{f}{\|f\|_{\mathcal{W}}}}^{\mathcal{W}} \left(i\omega \frac{\partial}{\partial \omega} \right) = \frac{1}{\|f\|_{\mathcal{W}}^2} e_f \left(i\omega \frac{\partial}{\partial \omega} \right) = 0.$$

We can therefore further simplify the uncertainty (4.1) for $f \in \mathcal{D}_{\mathcal{L}}^0$ to

$$\mathcal{L}(f) = \|T_x f\|^2 + \|T_\sigma f\|^2 + \frac{1}{\|f\|_{\mathcal{W}}^2} \|i\omega f'\|_{\mathcal{W}}^2 \left\| \frac{f}{\omega} \right\|^2 + \frac{1}{\|f\|_{\mathcal{W}}^2} v_f^{\mathcal{W}}(-\ln(\omega)) \quad (4.3)$$

We will prove the theorems in the same way as in Section 3.1. The first step in this direction is to produce a function $g \in \mathcal{D}_{\mathcal{L}}^0$ with non-zero uncertainty. Due to the $\leq M$ limits in the definition of $\mathcal{D}_{\mathcal{L}}^0$, this requires a bit more finesse than in the $\mathcal{D}_S^0$ case. For brevity, we mention that it is possible to construct a bump function g with expected time and scale 0 such that

$$\|f''\|^2 \leq 50, \quad \|\omega f'\|_{\mathcal{W}}^2 \leq 50, \quad \left\| \frac{f}{\omega} \right\|^2 \leq 50.$$

We write $\mathcal{L}(g) = 0.5K$ for this g and some nonzero K and note that for a minimizing sequence $(f_n)_n$ in $\mathcal{D}_{\mathcal{L}}^0$,

$$\mathcal{L}(f_n) \leq K$$

for large enough n . Since each term of $\mathcal{L}(f_n)$ is positive, we can conclude that

$$\begin{aligned} \|T_x f_n\|^2 &\leq K, & \frac{1}{\|f_n\|_{\mathcal{W}}^2} \|i\omega f_n'\|_{\mathcal{W}}^2 \left\| \frac{f_n}{\omega} \right\|^2 &\leq K, \\ \|T_\sigma f_n\|^2 &\leq K, & \frac{1}{\|f_n\|_{\mathcal{W}}^2} v_{f_n}^{\mathcal{W}}(-\ln(\omega)) &\leq K \end{aligned} \quad (4.4)$$

for large enough n . Now define $\mathcal{K} \subset \mathcal{D}_{\mathcal{L}}^0$ as the space of all $f \in \mathcal{D}_{\mathcal{L}}^0$ such that the above inequalities hold. For reference, we write it here as

$$\begin{aligned} \mathcal{K} = \left\{ f \in \text{Dom}(\mathcal{L}) : \|f\| = 1, e_f(T_x) = 0, e_f(T_\sigma) = 0, \omega f \in L^2(\mathbb{R}^+), \right. \\ \left. \|f''\|^2 \leq M, \|\omega f'\|_{\mathcal{W}}^2 \leq M, \left\| \frac{f}{\omega} \right\|^2 \leq M, \right. \\ \left. \|T_x f\|^2 \leq K, \frac{1}{\|f\|_{\mathcal{W}}^2} \|i\omega f'\|_{\mathcal{W}}^2 \left\| \frac{f}{\omega} \right\|^2 \leq K, \right. \\ \left. \|T_\sigma f\|^2 \leq K, \frac{1}{\|f\|_{\mathcal{W}}^2} v_f^{\mathcal{W}}(-\ln(\omega)) \leq K \right\}. \end{aligned} \quad (4.5)$$

If we show that $\mathcal{K}$ is compact, we will be able to prove Theorem 4.1 using the same method as for Theorem 3.1.

Before we can prove that $\mathcal{K}$ is compact however, we need to specify a topology. We can choose this rather freely but the tougher we choose the norm, the harder it will be to prove pre-compactness but easier to prove closedness. We make the choice

$$\|f\|_{\mathcal{WS}} = \frac{1}{4} \left(\|f\| + \|f\|_{\mathcal{W}} + \|f'\| + \|\sqrt{|\ln(\omega)|}f\|_{\mathcal{W}} \right).$$

For pre-compactness we will need to do some extra work to show convergence in each term compared to the Lemma 3.11.

Lemma 4.7. *For any $\varepsilon > 0$, there exists a compact subset $C_{a,b} \subset L^2(\mathbb{R}^+)$ such that for any $f \in \mathcal{K}$, there is a $y \in C_{a,b}$ such that*

$$\|f - y\|_{\mathcal{WS}} < \varepsilon.$$

Proof. The proof comes in two parts; first we show that there exist a, b such that any f can be well approximated by a function in $L^2(a, b)$ and then we show that those functions can, without loss of generality, be taken from a compact subset of $L^2(a, b)$.

We parameterize a, b by $\alpha \in \mathbb{R}$ so that $[a, b] = [e^{-\alpha} - e^{-\alpha}/2, e^{\alpha} + e^{-\alpha}/2]$ and show that $y = f\phi$ works as an approximation where ϕ is the C^0 smoothened indicator function

$$\phi(\omega) = \begin{cases} \frac{\omega}{e^{-\alpha}/2} - 1, & \omega \in [e^{-\alpha}/2, e^{-\alpha}), \\ 1, & \omega \in [e^{-\alpha}, e^{\alpha}], \\ 1 - \frac{\omega - e^{\alpha}}{e^{-\alpha}/2}, & \omega \in (e^{\alpha}, e^{\alpha} + e^{-\alpha}/2]. \end{cases}$$

The reason we do not use $f\chi_{[e^{-\alpha}, e^{\alpha}]}$ is that we want $f - y$ to be differentiable. This choice yields that $f - y = 0$ on $[e^{-\alpha}, e^{\alpha}]$ and $|f - y| < |f|$ otherwise. Therefore it suffices to show that all four terms of $\|f\chi_{[e^{-\alpha}, e^{\alpha}]^c}\|_{\mathcal{WS}}$ can be made small uniformly in α :

- (1) For $\|f\chi_{[e^{-\alpha}, e^{\alpha}]^c}\|$, we can move to the scale domain to get

$$\int_{[e^{-\alpha}, e^{\alpha}]^c} |f(\omega)|^2 d\omega = \int_{[-\alpha, \alpha]^c} |\tilde{f}(\sigma)|^2 d\sigma \leq \frac{K}{\alpha^2}$$

using Chebyshev's inequality since $v_{\tilde{f}}(\sigma) = \|T_{\sigma}f\|^2 \leq K$ by $f \in \mathcal{K}$.

- (2) For $\|f\|_{\mathcal{W}}$, we can estimate as

$$\begin{aligned} \int_{[e^{-\alpha}, e^{\alpha}]^c} \frac{|f(\omega)|^2}{\omega} d\omega &= \int_0^{e^{-\alpha}} \frac{|f(\omega)|^2}{\omega} d\omega + \int_{e^{\alpha}}^{\infty} \frac{|f(\omega)|^2}{\omega} d\omega \\ &\leq \int_0^{e^{-\alpha}} \omega \frac{|f(\omega)|^2}{\omega^2} d\omega + \int_{e^{\alpha}}^{\infty} |f(\omega)|^2 d\omega \\ &\leq e^{-\alpha} \int_0^{e^{-\alpha}} \frac{|f(\omega)|^2}{\omega^2} d\omega + \frac{K}{\alpha^2} \\ &\leq Me^{-\alpha} + \frac{K}{\alpha^2} \end{aligned}$$

where we used the result from (1) and that $\left\|\frac{f}{\omega}\right\|^2 \leq M$.

(3) For $\|f' \chi_{[e^{-\alpha}, e^{\alpha}]^c}\|$, we can estimate the tail as

$$\int_{e^{\alpha}}^{\infty} |f'(\omega)|^2 d\omega = \int_{e^{\alpha}}^{\infty} \frac{\omega}{\omega} |f'(\omega)|^2 d\omega \leq M e^{-\alpha}$$

by $\|\omega f'\|_{\mathcal{W}}^2 \leq M$. Meanwhile for the $[0, e^{-\alpha}]$ interval, we need to use the $\|f''\|^2 \leq M$ inequality. For $\omega \in [0, e^{-\alpha}]$ it holds that

$$\begin{aligned} |f'(\omega)| &= \left| \int_0^{\omega} f''(\xi) d\xi \right| \leq \|\chi_{[0, \omega]} f''\|_{L^1} \leq \sqrt{\omega} \sqrt{M} \leq e^{-\alpha/2} \sqrt{M} \\ \implies \int_0^{e^{-\alpha}} |f'(\omega)|^2 d\omega &\leq e^{-\alpha} e^{-\alpha} M = e^{-2\alpha} M. \end{aligned}$$

(4) Lastly, for $\left\| \sqrt{|\ln(\omega)|} f \chi_{[e^{-\alpha}, e^{\alpha}]^c} \right\|_{\mathcal{W}}$, the tail can trivially be estimated as

$$\int_{e^{\alpha}}^{\infty} \frac{|\ln(\omega)|}{\omega} |f(\omega)|^2 d\omega \leq \int_{e^{\alpha}}^{\infty} |f(\omega)|^2 d\omega \leq \frac{K}{\alpha^2}$$

by (1). Meanwhile, the $[0, e^{-\alpha}]$ interval can be estimated as

$$\int_0^{e^{-\alpha}} \frac{|\ln(\omega)|}{\omega} |f(\omega)|^2 d\omega = \int_0^{e^{-\alpha}} \omega |\ln(\omega)| \frac{|f(\omega)|^2}{\omega^2} d\omega \leq \alpha e^{-\alpha} M$$

by $\left\| \frac{f}{\omega} \right\|^2 \leq M$.

For the second part, we construct the subset is $C_{a,b}$ similarly to how it was done in Lemma 3.11 with the change that we also require the Fourier coefficients $(c_n)_n$ to satisfy $|c_n| \leq \frac{C(M)}{n^2}$ for a constant C depending on M . The reason that we can require this is that $\|f''\|^2 \leq M$. With this change, $C_{a,b}$ is compact with respect to the $\|f'\|$ norm. To get compactness of $C_{a,b}$ in the $\mathcal{WS}$ norm it only remains to show that any sequence in $C_{a,b}$ converges in the norms $\|f\|_{\mathcal{W}}$ and $\left\| \sqrt{|\ln(\omega)|} f \right\|_{\mathcal{W}}$. This follows from the compactness of $[a, b]$ and the continuity of $\frac{1}{\omega}$ and $\frac{|\ln(\omega)|}{\omega}$ in $[a, b]$. $\square$

Proposition 4.8. *The space $\mathcal{K}$ is pre-compact.*

Proof. The only part which does not carry over directly from the proof Proposition 3.12 is completeness of the underlying space. To see that this holds, let $(f_n)_n$ be a Cauchy sequence in $\mathcal{WS}$. Since this sequence is also Cauchy in L^2 , it converges to some function f . By the completeness of Sobolev spaces, $\|f_n - f_0\| + \|f'_n - f'_0\| \rightarrow 0$ and so it only remains to show that the last two terms of $\|f_n - f\|_{\mathcal{WS}}$ converges. The proof for both terms are similar so we illustrate the procedure only for the $\mathcal{W}$ norm.

Since $(f_n)_n$ is Cauchy in $\mathcal{WS}$, it is convergent in $\mathcal{W}$, i.e., $f_n/\sqrt{\omega}$ is Cauchy in L^2 . Since L^2 is complete, $f_n/\sqrt{\omega}$ converges to some element $g \in L^2$. It thus holds that

$$\left\| \frac{f_n}{\sqrt{\omega}} - g \right\| = \left\| \frac{f_n}{\sqrt{\omega}} - \frac{\sqrt{\omega}g}{\sqrt{\omega}} \right\| \rightarrow 0 \implies \|f_n - \sqrt{\omega}g\|_{\mathcal{W}} \rightarrow 0$$

and so f_n converges also in $\mathcal{W}$. $\square$

We now move to showing that $\mathcal{K}$ is closed. The first step in this direction is to show that a sequence in $\mathcal{K}$ cannot leave $\mathcal{D}_{\mathcal{L}}$.

Lemma 4.9. *The window space $\mathcal{W}$ is closed in $(L^2(\mathbb{R}^+), \|\cdot\|_{\mathcal{WS}})$.*

Proof. Since we have established that $\mathcal{W}$ is complete in the above proposition, let $(f_n)_n$ be a sequence in $\mathcal{W}$ that converges in $\mathcal{WS}$. Then $(f_n)_n$ is Cauchy in $\mathcal{WS}$ and hence also in $\mathcal{W}$. From here, completeness of $\mathcal{W}$ yields the desired result. $\square$

Meanwhile for the three last three conditions in equation (4.2) we employ another method.

Lemma 4.10. *The subset $\{f \in L^2(\mathbb{R}^+) : \|f'\|^2 \leq M\}$ is closed in $(L^2(\mathbb{R}^+), \|\cdot\|_{\mathcal{WS}})$.*

The proof is completely analogous to that of Lemma 3.7 after moving to the time domain so we omit it for brevity.

Lemma 4.11. *The subset $\{f : \|f'\|_{\mathcal{W}}^2 \leq M\}$ is closed in $(L^2(\mathbb{R}^+), \|\cdot\|_{\mathcal{WS}})$.*

Proof. We again use an argument similar to that used in the proof of Lemma 3.7. Let $f_n \rightarrow f$ in $\mathcal{WS}$ and write $\delta_n = f - f_n$ to get for each N that

$$\begin{aligned} \int_0^N \omega |f'(\omega)|^2 d\omega &= \int_0^N \omega |f'_n(\omega)|^2 d\omega + 2 \operatorname{Re} \int_0^N \omega f'_n(\omega) \overline{\delta'_n(\omega)} d\omega + \int_0^N \omega |\delta'_n(\omega)|^2 d\omega \\ &\leq M + 2 \sqrt{N \int_0^N |f'_n(\omega)|^2 d\omega} \sqrt{N \int_0^N |\delta'_n(\omega)|^2 d\omega} + N \int_0^N |\delta'_n(\omega)|^2 d\omega \rightarrow M \end{aligned}$$

by $\|f' - f'_n\| \rightarrow 0$. By letting $N \rightarrow \infty$ we obtain our desired inequality. $\square$

Lemma 4.12. *The subset $\{f : \left\| \frac{f}{\omega} \right\|^2 \leq M\}$ is closed in $(L^2(\mathbb{R}^+), \|\cdot\|_{\mathcal{WS}})$.*

Proof. We use a similar argument to the above lemmas; let $f_n \rightarrow f$ in $\mathcal{WS}$ and write $\delta_n = f - f_n$ to get for each N that

$$\begin{aligned} \int_{1/N}^N \frac{|f(\omega)|^2}{\omega^2} d\omega &= \int_{1/N}^N \frac{|f_n(\omega)|^2}{\omega^2} d\omega + 2 \operatorname{Re} \int_{1/N}^N \frac{1}{\omega^2} f'_n(\omega) \overline{\delta'_n(\omega)} d\omega + \int_{1/N}^N \frac{1}{\omega^2} |\delta'_n(\omega)|^2 d\omega \\ &\leq M + 2 \sqrt{N^2 \int_{1/N}^N |f_n(\omega)|^2 d\omega} \sqrt{N^2 \int_{1/N}^N |\delta_n(\omega)|^2 d\omega} + N^2 \int_{1/N}^N |\delta_n(\omega)|^2 d\omega \rightarrow M \end{aligned}$$

by $\|f' - f'_n\| \rightarrow 0$. By once again letting $N \rightarrow \infty$ we obtain our desired inequality. $\square$

Lemma 4.13. *The subset $\left\{f : \frac{1}{\|f\|_{\mathcal{W}}^2} \|i\omega f'\|_{\mathcal{W}}^2 \left\| \frac{f}{\omega} \right\|^2 \leq K\right\}$ is closed in $(L^2(\mathbb{R}^+), \|\cdot\|_{\mathcal{WS}})$.*

Proof. Write

$$A(f) = \|i\omega f'\|_{\mathcal{W}}^2, \quad B(f) = \left\| \frac{f}{\omega} \right\|^2$$

so that the condition can be written as

$$A(f)B(f) \leq K \|f\|_{\mathcal{W}}^2.$$

Now let $(f_n)_n$ be a sequence in the set such that $f_n \rightarrow f$ in $\mathcal{WS}$. Then for each N it holds that

$$\begin{aligned} & \left(\int_{1/N}^N \omega |f'(\omega)|^2 d\omega \right) \left(\int_{1/N}^N \frac{1}{\omega^2} |f(\omega)|^2 d\omega \right) \\ & \leq \left(\underbrace{\int_{1/N}^N \omega |f'_n(\omega)|^2 d\omega}_{\leq A(f_n)} + 2 \sqrt{N \underbrace{\int_{1/N}^N |f'_n(\omega)|^2 d\omega}_{\leq \|f'_n\|^2 \leq K}} \sqrt{N \int_{1/N}^N |\delta'_n(\omega)|^2 d\omega} + N \int_{1/N}^N |\delta'_n(\omega)|^2 d\omega \right) \\ & \times \left(\underbrace{\int_{1/N}^N \frac{1}{\omega^2} |f_n(\omega)|^2 d\omega}_{\leq B(f_n)} + 2 \sqrt{N^2 \underbrace{\int_{1/N}^N |f_n(\omega)|^2 d\omega}_{\leq \|f_n\|^2 \leq 1}} \sqrt{N^2 \int_{1/N}^N |\delta_n(\omega)|^2 d\omega} + N^2 \int_{1/N}^N |\delta_n(\omega)|^2 d\omega \right). \end{aligned}$$

Now since

$$\begin{aligned} \|f' - f'_n\| &\leq 4\|f - f_n\|_{\mathcal{WS}} \rightarrow 0, & \|f - f_n\| &\leq 4\|f - f_n\|_{\mathcal{WS}}, \\ \|f - f_n\|_{\mathcal{WS}} &\rightarrow 0 \implies \|f - f_n\|_{\mathcal{W}} \rightarrow 0 \implies \|f_n\|_{\mathcal{W}} \rightarrow \|f\|_{\mathcal{W}}, \end{aligned}$$

we can, by letting $n \rightarrow \infty$, conclude that

$$\left(\int_{1/N}^N \omega |f'(\omega)|^2 d\omega \right) \left(\int_{1/N}^N \frac{1}{\omega^2} |f(\omega)|^2 d\omega \right) \leq K \|f\|_{\mathcal{W}}^2$$

for all N . The left-hand side converges to $A(f)B(f)$ as $N \rightarrow \infty$, allowing us to conclude that

$$A(f)B(f) = \|i\omega f'\|_{\mathcal{W}}^2 \left\| \frac{f}{\omega} \right\|_{\mathcal{W}}^2 \leq K \|f\|_{\mathcal{W}}^2$$

as desired. $\square$

Lemma 4.14. *The subset $\left\{ f : \frac{1}{\|f\|_{\mathcal{W}}^2} v_f^{\mathcal{W}}(-\ln(\omega)) \leq K \right\}$ is closed in $(L^2(\mathbb{R}^+), \|\cdot\|_{\mathcal{WS}})$.*

Proof. We begin by writing out the variance as

$$v_f^{\mathcal{W}}(-\ln(\omega)) = e_f^{\mathcal{W}}(\ln(\omega)^2) - e_f^{\mathcal{W}}(-\ln(\omega))^2$$

which implies that $\frac{1}{\|f\|_{\mathcal{W}}^2} v_f^{\mathcal{W}}(-\ln(\omega)) \leq K$ can be written as

$$\int_0^\infty \frac{\ln(\omega)^2}{\omega} |f(\omega)|^2 d\omega \leq K \|f\|_{\mathcal{W}}^2 + \left(\int_0^\infty \frac{-\ln(\omega)}{\omega} |f(\omega)|^2 d\omega \right)^2 = K \|f\|_{\mathcal{W}}^2 + e_f^{\mathcal{W}}(-\ln(\omega))^2.$$

Before showing that this corresponds to a closed subset, note that

$$\|f - f_n\|_{\mathcal{WS}} \rightarrow 0 \implies \|f - f_n\|_{\mathcal{W}} \rightarrow 0 \implies \|f_n\|_{\mathcal{W}} \rightarrow \|f\|_{\mathcal{W}} \quad (4.6)$$

by the reverse triangle inequality. Similarly, we have that

$$e_{f_n}^{\mathcal{W}}(-\ln(\omega)) \rightarrow e_f^{\mathcal{W}}(-\ln(\omega)) \quad (4.7)$$

as $n \rightarrow \infty$. Indeed, this follows if we can show that

$$\int_0^1 \frac{1 - \ln(\omega)}{\omega} |f_n(\omega)|^2 d\omega \rightarrow \int_0^1 \frac{1 - \ln(\omega)}{\omega} |f(\omega)|^2 d\omega$$

and

$$\int_1^\infty \frac{\ln(\omega)}{\omega} |f_n(\omega)|^2 d\omega \rightarrow \int_1^\infty \frac{\ln(\omega)}{\omega} |f(\omega)|^2 d\omega.$$

To see this, consider the two weighted spaces $L^2(\chi_{[0,1]} \frac{-\ln(\omega)}{\omega})$ and $L^2(\chi_{(1,\infty)} \frac{\ln(\omega)}{\omega})$. We can then use the reverse triangle inequality as

$$\begin{aligned} \left| \|f_n\|_{L^2(\chi_{[0,1]} \frac{-\ln(\omega)}{\omega})} - \|f\|_{L^2(\chi_{[0,1]} \frac{-\ln(\omega)}{\omega})} \right| &\leq \|f_n - f\|_{L^2(\chi_{[0,1]} \frac{-\ln(\omega)}{\omega})} \\ &\leq \left\| \sqrt{|\ln(\omega)|} (f_n - f) \right\|_{\mathcal{W}} \rightarrow 0. \end{aligned}$$

The same argument works for $(1, \infty)$, yielding the desired conclusion.

We are now ready to prove that the condition corresponds to a closed subset. Writing $\delta_n = f - f_n$, we have that for each N ,

$$\begin{aligned} \int_{1/N}^N \frac{\ln(\omega)^2}{\omega} |f(\omega)|^2 d\omega &= \int_{1/N}^N \frac{\ln(\omega)^2}{\omega} |f_n(\omega)|^2 d\omega + 2 \operatorname{Re} \int_{1/N}^N \frac{\ln(\omega)^2}{\omega} f_n(\omega) \overline{\delta_n(\omega)} d\omega \\ &\quad + \int_{1/N}^N \frac{\ln(\omega)^2}{\omega} |\delta_n(\omega)|^2 d\omega \\ &\leq K \|f_n\|_{\mathcal{W}}^2 + e_f^{\mathcal{W}} (-\ln(\omega))^2 \\ &\quad + 2 \sqrt{\frac{\ln(1/N)^2}{N} \int_{1/N}^N |f_n(\omega)|^2 d\omega} \sqrt{\frac{\ln(1/N)^2}{N} \int_{1/N}^N |\delta_n(\omega)|^2 d\omega} \\ &\quad + \frac{\ln(1/N)^2}{N} \int_{1/N}^N |\delta_n(\omega)|^2 d\omega. \end{aligned}$$

By equations (4.6) and (4.7), the first two terms converge to $K \|f\|_{\mathcal{W}}^2 + e_f^{\mathcal{W}} (-\ln(\omega))^2$. Meanwhile, the remaining terms all vanish as $n \rightarrow \infty$ since $\|\delta_n\| \rightarrow 0$. We can now conclude that

$$\int_0^\infty \frac{\ln(\omega)^2}{\omega} |f(\omega)|^2 d\omega = \lim_{N \rightarrow \infty} \int_{1/N}^N \frac{\ln(\omega)^2}{\omega} |f(\omega)|^2 d\omega \leq K \|f\|_{\mathcal{W}}^2 + e_f^{\mathcal{W}} (-\ln(\omega))^2$$

as desired. $\square$

Proposition 4.15. *The subset $\mathcal{K}$ is closed in $(L^2(\mathbb{R}^+), \|\cdot\|_{\mathcal{WS}})$.*

Proof. To prove this we need to show that each condition defining $\mathcal{K}$ from equation 4.5 corresponds to a closed subset. Functions which satisfy the $\leq K$ inequalities from equation (4.4) are in $\operatorname{Dom}(\mathcal{L})$. The remaining conditions corresponding to closed subsets was shown in lemmas 3.9, 4.9, 4.10, 4.11, 4.12, 4.13 and 4.14. $\square$

Having established compactness of $\mathcal{K}$ from propositions 4.8 and 4.15, we can prove theorems 4.1 and 4.3 in the exactly the same way as the analogous theorems 3.1 and 3.2.

Proof of Theorem 4.4. As in the proof of Theorem 3.3, we can write out $\mathcal{L}(p)$ explicitly and note that each term depends continuously on the spline parameters to conclude that the minimum is attained since the unit sphere in $P_h(0, b)$ is compact. $\square$

4.2 Properties of minimizers

The variation and Euler-Lagrange equations for the phase space uncertainty were recently established in [23]. We recall those results here and use them to conclude that minimizers are smooth and can be taken to be real valued without loss of generality, similarly to the results in Sections 3.2.2 and 3.2.3.

Theorem 4.16 ([23, Theorem 32]). *The variation of $\mathcal{L}$ for f satisfying $e_f(T_x) = e_f(T_\sigma) = 0$ is given by*

$$\begin{aligned}\delta\mathcal{L}(f) &= \frac{\delta v_{K_f}(A)}{\delta f} + \frac{\delta v_{K_f}(B)}{\delta f}, \\ \frac{\delta v_{K_f}(A)}{\delta f} &= 2\omega \left(\ln(\omega)^2 - v_{\frac{f}{\|f\|}}(\ln(\omega)^2) \right) \frac{f}{\|f\|^2} \\ &\quad + 2 \left(\left(\ln(\omega) - e_{\frac{f}{\|f\|_{\mathcal{W}}}}^{\mathcal{W}}(\ln(\omega)) \right)^2 - v_{\frac{f}{\|f\|_{\mathcal{W}}}}^{\mathcal{W}}(\ln(\omega)) \right) \frac{f}{\|f\|_{\mathcal{W}}^2}, \\ \frac{\delta v_{K_f}(B)}{\delta f} &= 2\omega \left(-\frac{\partial^2}{\partial \omega^2} - v_{\frac{f}{\|f\|}} \left(i \frac{\partial}{\partial \omega} \right) \right) \frac{f}{\|f\|^2} \\ &\quad + 2 \left(\left(-\omega^2 \frac{\partial^2}{\partial \omega^2} - \omega \frac{\partial}{\partial \omega} - v_{\frac{f}{\|f\|_{\mathcal{W}}}}^{\mathcal{W}} \left(i\omega \frac{\partial}{\partial \omega} \right) \right) \right) \frac{\left\| \frac{f}{\omega} \right\|^2}{\|f\|^2} \frac{f}{\|f\|_{\mathcal{W}}^2} \\ &\quad + 2v_{\frac{f}{\|f\|_{\mathcal{W}}}}^{\mathcal{W}} \left(i\omega \frac{\partial}{\partial \omega} \right) \left(\frac{1}{\omega} \frac{f}{\|f\|^2} - \omega \frac{\left\| \frac{f}{\omega} \right\|^2}{\|f\|^4} f \right).\end{aligned}$$

When we are looking for minimizers in $\mathcal{D}_{\mathcal{L}}^0$ as in theorems 4.3 and 4.4, we say that the variation is *constrained*. The following proposition gives the Euler-Lagrange equation in this case.

Proposition 4.17 ([23, Proposition 35]). *The constrained variation of $\mathcal{L}$ for f satisfying $e_f(T_x) = e_f(T_\sigma) = 0$ is given by*

$$\delta\mathcal{L}^{\text{cons}}(f) = \delta\mathcal{L}(f) - \lambda \ln(\omega)f - \mu i \frac{\partial}{\partial \omega} f$$

for some Lagrange multipliers $\mu, \lambda \in \mathbb{C}$. Moreover, if f is real valued, then $\mu = 0$.

From here we can establish Euler-Lagrange equations. This is formulated in Theorems 19 and 34 in [23] as $\delta\mathcal{L}(f) = 0$ and $\delta\mathcal{L}^{\text{cons}}(f) = 0$ respectively. We make this explicit in the following propositions.

Proposition 4.18 (The $L^2(\mathbb{R}^+)$ Euler-Lagrange equation). *A minimizer f_0 of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^0$ satisfies*

$$\begin{aligned}f_0''(\omega)(c_0\omega + c_1\omega^2) + f_0'(\omega)(c_2\omega) \\ + f_0(\omega) \left(c_3\omega \ln(\omega)^2 + c_4\omega + c_5 \ln(\omega)^2 + c_6 \ln(\omega) + \frac{c_7}{\omega} + c_8 \right) = 0\end{aligned}$$

for constants $c_0, \dots, c_8$ depending on f_0 .

Proposition 4.19 (The $L^2(0, b)$ Euler-Lagrange equation). *A minimizer f_0 of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^0 \cap L^2(0, b)$ satisfies*

$$f_0''(\omega)(c_0\omega + c_1\omega^2) + f_0'(\omega)(c_2\omega - \mu i) + f_0(\omega) \left(c_3\omega \ln(\omega)^2 + c_4\omega + c_5 \ln(\omega)^2 + c_6 \ln(\omega) + \frac{c_7}{\omega} + c_8 \right) = 0$$

for constants $c_0, \dots, c_8, \mu$ depending on f_0 .

We are now ready to establish smoothness results analogous to propositions 3.24 and 3.25.

Proposition 4.20. *Let f_0 be a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^0$. Then f_0 is smooth and $f_0''|_{(0,b)} \in L^2(0, b)$ for any $b > 0$.*

Proof. For smoothness, note that we can write

$$f_0''(\omega) = a(\omega)f_0'(\omega) + b(\omega)f_0(\omega)$$

which allows us to inductively show that f_0 is smooth. The $f_0''|_{(0,b)} \in L^2(0, b)$ property follows directly from $\|f_0\| \leq M$ since $f_0 \in \mathcal{D}_{\mathcal{L}}^0$. $\square$

Proposition 4.21. *Let f_0 be a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^0 \cap L^2(0, b)$. Then*

- (1) f_0 is smooth in $(0, b)$,
- (2) There exist bounds $M_k > 0$ such that

$$\|f_0^{(k)}\|_{L^\infty(0,b)} \leq M_k, \quad k = 0, 1, 2,$$

- (3) There exists a constant M_3 such that

$$|\ln(\omega)f_0(\omega)| \leq M_3.$$

Proof. Smoothness follows by the same argument as in Proposition 4.20. For the L^∞ bounds, the $k = 0$ case follows by the smoothness of f_0 and compactness of $[0, b]$. Both the $k = 1$ and $k = 2$ bounds follow by $f_0 \in \mathcal{D}_{\mathcal{L}}^0$. Lastly, the M_3 estimate can be shown in the same way as for Proposition 3.25. $\square$

We now fulfill the promise of showing that minimizers can be taken to be real valued without loss of generality.

Proposition 4.22. *Let f be a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^0$, then there exists a real valued minimizer $f^{\text{real}} = cf \in \mathcal{D}_{\mathcal{L}}^{0, \mathbb{R}}$ for a $c \in \mathbb{C}$ with unit norm.*

Proposition 4.23. *Let f be a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^0 \cap L^2(0, b)$, then there exists a real valued minimizer $f^{\text{real}} = cf \in \mathcal{D}_{\mathcal{L}}^{0, \mathbb{R}} \cap L^2(0, b)$ for a $c \in \mathbb{C}$ with unit norm.*

The proof of both of these propositions are analogous to that in Section 3.2.3 with the main difference being that it is not immediate that Wronskian is not constant. This can however be proved by standard methods and we omit the proof for brevity.

4.3 Convergence of compactly supported minimizers

In this section we aim to establish results analogous to those in Section 3.3.

Proposition 4.24. *Let f_0 be a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^{0,\mathbb{R}}$. Then for any $\varepsilon > 0$, there exist $b > 0$ and $y \in \mathcal{D}_{\mathcal{L}}^{0,\mathbb{R}} \cap L^2(0, b)$ such that*

$$\|f_0 - y\| < \varepsilon, \quad |\mathcal{L}(f_0) - \mathcal{L}(y)| < \varepsilon.$$

Proposition 4.25. *Let $\mathcal{M}_b$ be the set of minimizers of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^{0,\mathbb{R}} \cap L^2(0, b)$. Then any element of*

$$\mathcal{M} = \left\{ f \in L^2(\mathbb{R}^+) : f = \lim_{n \rightarrow \infty} f_{b_n}, f_{b_n} \in \mathcal{M}_{b_n} \text{ for some sequence } b_n \rightarrow \infty \right\}$$

is a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^{0,\mathbb{R}}$.

Proof of Proposition 4.24. We build on the construction from Proposition 3.26 with the change that we are using the condition $\omega f \in L^2(\mathbb{R}^+)$ to get an increasing sequence $b_n \rightarrow \infty$ such that

$$b_n |f_0(b_n)| \rightarrow 0.$$

From here the approximation is constructed in the same way which we recall here is by writing $c_n = f_0(b_n) = |c_n|e^{i\theta_n}$, $\theta_n \in \{0, \pi\}$ and defining

$$\begin{aligned} f_n^1(\omega) &= \begin{cases} f_0(\omega), & \omega \in (0, b_n), \\ c_n - e^{i\theta_n}(\omega - b_n), & \omega \in [b_n, b_n + |c_n|], \\ 0, & \text{otherwise,} \end{cases} \\ f_n^2(\omega) &= \frac{f_n^1(\omega)}{\|f_n^1\|}, \quad \begin{cases} e_1^n = e_{f_n^2}(T_x), \\ e_2^n = e_{f_n^2}(T_\sigma), \end{cases} \\ f_n(\omega) &= \pi(e_1^n, e_2^n)^{-1} f_n^2(\omega). \end{aligned}$$

Recall also that since $f_0 \in \mathcal{D}_{\mathcal{L}}^0$, $\mathcal{L}(f_0)$ can be written as in equation (4.3), i.e.,

$$\mathcal{L}(f) = \|T_x f\|^2 + \|T_\sigma f\|^2 + \frac{1}{\|f\|_{\mathcal{W}}^2} \|i\omega f'\|_{\mathcal{W}}^2 \left\| \frac{f}{\omega} \right\|^2 + \frac{1}{\|f\|_{\mathcal{W}}^2} v_f^{\mathcal{W}}(-\ln(\omega))$$

and convergence of the first two terms and the $\|f\|_{\mathcal{W}}$ factor was already established in Proposition 3.26. Therefore to finish the proof we need to show that the last two terms of $\mathcal{L}(f_n)$ converges, i.e.,

$$\frac{1}{\|f_n\|_{\mathcal{W}}^2} \|i\omega f_n'\|_{\mathcal{W}}^2 \left\| \frac{f_n}{\omega} \right\|^2 + v_{\frac{f_n}{\|f_n\|_{\mathcal{W}}}}^{\mathcal{W}}(-\ln(\omega)) \xrightarrow{n \rightarrow \infty} \frac{1}{\|f_0\|_{\mathcal{W}}^2} \|i\omega f_0'\|_{\mathcal{W}}^2 \left\| \frac{f_0}{\omega} \right\|^2 + v_{\frac{f_0}{\|f_0\|_{\mathcal{W}}}}^{\mathcal{W}}(-\ln(\omega)).$$

We do this in the following sequence of approximations where some easier proofs are omitted.

$$(1) \quad \|\omega f_n^{1'}\|_{\mathcal{W}} \xrightarrow{n \rightarrow \infty} \|\omega f_0'\|_{\mathcal{W}}:$$

$$\|\omega f_n^{1'}\|_{\mathcal{W}} = \underbrace{\int_0^{b_n} \omega |f_0'(\omega)|^2 d\omega}_{\rightarrow \|\omega f_0'\|_{\mathcal{W}}} + \underbrace{\int_{b_n}^{b_n + |c_n|} \omega |f_n^{1'}(\omega)|^2 d\omega}_{=O(b_n |c_n|) \rightarrow 0}.$$

$$(2) \quad \left| \|\omega f_n^{2'}\|_{\mathcal{W}} - \|\omega f_n^{1'}\|_{\mathcal{W}} \right| \xrightarrow{n \rightarrow \infty} 0.$$

$$(3) \quad \left| \|\omega f_n'\|_{\mathcal{W}} - \|\omega f_n^{2'}\|_{\mathcal{W}} \right| \xrightarrow{n \rightarrow \infty} 0:$$

$$\left| \|\omega f_n'\|_{\mathcal{W}} - \|\omega f_n^{2'}\|_{\mathcal{W}} \right| \leq \int_0^\infty \omega \left| |f_n'(\omega)|^2 - |f_n^{2'}(\omega)|^2 \right| d\omega \xrightarrow{n \rightarrow \infty} 0$$

by the dominated convergence theorem motivated by $f_0 \in \mathcal{D}_{\mathcal{L}}$ and the continuity of π .

$$(4) \quad \|\omega f_n'\|_{\mathcal{W}} \xrightarrow{n \rightarrow \infty} \|\omega f_0'\|_{\mathcal{W}}: \text{ This follows from the triangle inequality applied to (1), (2) and (3).}$$

$$(5) \quad \frac{1}{\|f_n\|_{\mathcal{W}}^2} \xrightarrow{n \rightarrow \infty} \frac{1}{\|f_0\|_{\mathcal{W}}^2}.$$

$$(6) \quad \left\| \frac{f_n^1}{\omega} \right\|^2 \xrightarrow{n \rightarrow \infty} \left\| \frac{f_0}{\omega} \right\|^2:$$

$$\left\| \frac{f_n^1}{\omega} \right\|^2 = \int_0^{b_n} \frac{|f_0(\omega)|^2}{\omega^2} d\omega + \int_{b_n}^{b_n+|c_n|} \frac{1}{\omega^2} \underbrace{|f_n^1(\omega)|^2}_{\leq |c_n| \rightarrow 0} d\omega \xrightarrow{n \rightarrow \infty} \left\| \frac{f_0}{\omega} \right\|^2.$$

$$(7) \quad \left| \left\| \frac{f_n^2}{\omega} \right\|^2 - \left\| \frac{f_n^1}{\omega} \right\|^2 \right| \xrightarrow{n \rightarrow \infty} 0.$$

$$(8) \quad \left| \left\| \frac{f_n^2}{\omega} \right\|^2 - \left\| \frac{f_n}{\omega} \right\|^2 \right| \xrightarrow{n \rightarrow \infty} 0:$$

$$\left| \left\| \frac{f_n^2}{\omega} \right\|^2 - \left\| \frac{f_n}{\omega} \right\|^2 \right| \leq \int_0^\infty \frac{||f_n^2(\omega)|^2 - |f_n(\omega)|^2|}{\omega^2} d\omega \xrightarrow{n \rightarrow \infty} 0$$

by the dominated convergence theorem motivated by $f_0 \in \mathcal{D}_{\mathcal{L}}$ and the continuity of π .

$$(9) \quad e_{f_n^1}^{\mathcal{W}}(\ln(\omega)^2) \xrightarrow{n \rightarrow \infty} e_{f_0}^{\mathcal{W}}(\ln(\omega)^2):$$

$$e_{f_n^1}^{\mathcal{W}}(\ln(\omega)^2) = \int_0^{b_n} \frac{\ln(\omega)^2}{\omega} |f_0(\omega)|^2 d\omega + \int_{b_n}^{b_n+|c_n|} \frac{\ln(\omega)^2}{\omega} |f_n^1(\omega)|^2 d\omega \xrightarrow{n \rightarrow \infty} e_{f_0}^{\mathcal{W}}(\ln(\omega)^2).$$

$$(10) \quad e_{f_n^1}^{\mathcal{W}}(\ln(\omega)) \xrightarrow{n \rightarrow \infty} e_{f_0}^{\mathcal{W}}(\ln(\omega)).$$

$$(11) \quad \left| e_{f_n^2}^{\mathcal{W}}(\ln(\omega)^2) - e_{f_n^1}^{\mathcal{W}}(\ln(\omega)^2) \right| \xrightarrow{n \rightarrow \infty} 0:$$

$$\left| e_{f_n^2}^{\mathcal{W}}(\ln(\omega)^2) - e_{f_n^1}^{\mathcal{W}}(\ln(\omega)^2) \right| = e_{f_n^1}^{\mathcal{W}}(\ln(\omega)^2) \left| \frac{1}{\|f_n^1\|^2} - 1 \right| \xrightarrow{n \rightarrow \infty} 0.$$

$$(12) \quad \left| e_{f_n^2}^{\mathcal{W}}(\ln(\omega)) - e_{f_n^1}^{\mathcal{W}}(\ln(\omega)) \right| \xrightarrow{n \rightarrow \infty} 0.$$

$$(13) \quad \left| e_{f_n}^{\mathcal{W}}(\ln(\omega)^2) - e_{f_n^2}^{\mathcal{W}}(\ln(\omega)^2) \right| \xrightarrow{n \rightarrow \infty} 0:$$

$$\left| e_{f_n}^{\mathcal{W}}(\ln(\omega)^2) - e_{f_n^2}^{\mathcal{W}}(\ln(\omega)^2) \right| \leq \int_0^\infty \frac{\ln(\omega)^2}{\omega} \left| |f_n^2(\omega)|^2 - |f_n(\omega)|^2 \right| d\omega \xrightarrow{n \rightarrow \infty} 0$$

by the dominated convergence theorem motivated by $f_0 \in \mathcal{D}_{\mathcal{L}}$ and the continuity of π .

$$(14) \quad \left| e_{f_n}^{\mathcal{W}}(\ln(\omega)) - e_{f_n^2}^{\mathcal{W}}(\ln(\omega)) \right| \xrightarrow{n \rightarrow \infty} 0:$$

$$\left| e_{f_n}^{\mathcal{W}}(\ln(\omega)) - e_{f_n^2}^{\mathcal{W}}(\ln(\omega)) \right| \leq \int_0^\infty \frac{|\ln(\omega)|}{\omega} \left| |f_n^2(\omega)|^2 - |f_n(\omega)|^2 \right| d\omega \xrightarrow{n \rightarrow \infty} 0$$

by the dominated convergence theorem motivated by $f_0 \in \mathcal{D}_{\mathcal{L}}$ and the continuity of π .

$$(15) \quad e_{f_n}^{\mathcal{W}}(\ln(\omega)^2) \xrightarrow{n \rightarrow \infty} e_{f_0}^{\mathcal{W}}(\ln(\omega)^2): \text{ This follow by the triangle inequality applied to (9), (11) and (13).}$$

$$(16) \quad e_{f_n}^{\mathcal{W}}(\ln(\omega))^2 \xrightarrow{n \rightarrow \infty} e_{f_0}^{\mathcal{W}}(\ln(\omega))^2: \text{ This follow by the triangle inequality applied to (10), (12) and (14).}$$

That $\|f_n - f_0\| \rightarrow 0$ was established in the proof of Proposition 3.26 and having established that $\mathcal{L}(f_n) \rightarrow \mathcal{L}(f_0)$, we can for any ε find a f_n which satisfies the conditions of the proposition. $\square$

The proof of Proposition 4.25 is completely analogous to that of Proposition 3.27.

Proof of Proposition 4.25. Fix $f \in \mathcal{M}$ so that $f = \lim_{n \rightarrow \infty} f_{b_n}$ for a sequence $b_n \rightarrow \infty$ where f_{b_n} is a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^{0, \mathbb{R}} \cap L^2(0, b_n)$. By Proposition 4.24,

$$\mathcal{L}(f_{b_n}) \xrightarrow{n \rightarrow \infty} \mathcal{L}(f_0)$$

where f_0 is a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^{0, \mathbb{R}}$. We can therefore conclude that $f_{b_n} \in \mathcal{K}$ for large enough n by the definition of $\mathcal{K}$. Now in the same way as in the proof of theorems 3.1 and 4.1 with $\mathcal{K}_{\alpha, \beta, \gamma}$, we can show that $\mathcal{L}(f) = \mathcal{L}(f_0)$ since $f_{b_n} \rightarrow f$ and so f is indeed a minimizer. $\square$

4.4 Convergence of compactly supported spline minimizers

In this section we extend the results of Section 3.4 to the remaining terms of $\mathcal{L}(f)$. To do so, we reuse the construction of the splines k from Lemma 3.34 for which it only remains to show that the last two term of $\mathcal{L}(k)$ converges to the corresponding terms in $\mathcal{L}(f_0)$, i.e.,

$$\begin{aligned} \frac{1}{\|k\|_{\mathcal{W}}^2} \|i\omega k'\|_{\mathcal{W}}^2 \left\| \frac{1}{\omega} \frac{k}{\|k\|} \right\|^2 &\xrightarrow{n \rightarrow \infty} \frac{1}{\|f_0\|_{\mathcal{W}}^2} \|i\omega f_0'\|_{\mathcal{W}}^2 \left\| \frac{f_0}{\omega} \right\|^2, \\ v_{\frac{k}{\|k\|_{\mathcal{W}}}}^{\mathcal{W}}(-\ln(\omega)) &\xrightarrow{n \rightarrow \infty} v_{\frac{f_0}{\|f_0\|_{\mathcal{W}}}}^{\mathcal{W}}(-\ln(\omega)). \end{aligned}$$

Proposition 4.26. *Let f_0 be a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^{0,\mathbb{R}} \cap L^2(0, b)$. Then for any $\varepsilon > 0$, there exist $h > 0$ and $y \in \mathcal{D}_{\mathcal{L}}^{0,\mathbb{R}} \cap P_h(0, b)$ such that*

$$\|f_0 - y\| < \varepsilon, \quad |\mathcal{L}(f_0) - \mathcal{L}(p_0)| < \varepsilon.$$

Before attempting a proof, we need to establish the following lemma which should be seen as an extension of Lemma 3.35.

Lemma 4.27. *The following three estimates hold:*

$$(A) \quad \left\| i\omega \frac{k'}{\|k\|} \right\|_{\mathcal{W}}^2 - \|i\omega f'_0\|_{\mathcal{W}}^2 = O(h),$$

$$(B) \quad \left\| \frac{k}{\|k\|\omega} \right\|^2 - \left\| \frac{f_0}{\omega} \right\|^2 = O(h^2),$$

$$(C) \quad v_{\frac{k}{\|k\|}}^{\mathcal{W}}(-\ln(\omega)) - v_{f_0}^{\mathcal{W}}(-\ln(\omega)) = O(h).$$

We again use that $\|k\| = 1 + O(h^2)$ by lemmas 3.34 and 3.32, meaning we can dismiss the normalizations. From here we treat each estimate separately.

Proof of Estimate (A). This can be seen as a consequence of the T_x convergence established in Lemma 3.35. Indeed,

$$\left| \|\omega k'\|_{\mathcal{W}}^2 - \|\omega f'_0\|_{\mathcal{W}}^2 \right| \leq \int_0^b \omega |k'(\omega)|^2 - |f'_0(\omega)|^2 d\omega \leq b \left| \|k'\|^2 - \|f'_0\|^2 \right| = O(h)$$

as desired.

Proof of Estimate (B). Here we first approximate by p similarly to the proof of Lemma 3.35:

$$\begin{aligned} \left| \left\| \frac{f_0}{\omega} \right\|^2 - \left\| \frac{p}{\omega} \right\|^2 \right| &= |\langle \omega^{-1} f_0, \omega^{-1} f_0 \rangle - \langle \omega^{-1} p, \omega^{-1} p \rangle| \\ &= |\langle \omega^{-1} f_0, \omega^{-1} (f_0 - p) \rangle - \langle \omega^{-1} (f_0 - p), \omega^{-1} p \rangle| \\ &\leq \|\omega^{-1} f_0\| \|\omega^{-1} (f_0 - p)\| + \|\omega^{-1} (f_0 - p)\| \|\omega^{-1} p\| \\ &= \|\omega^{-1} (f_0 - p)\| (\|\omega^{-1} f_0\| + \|\omega^{-1} p\|). \end{aligned}$$

The two terms in the second factor are both bounded in the proof of Lemma 3.35 so it only remains to show that the first factor is $O(h^2)$. Note that the pointwise bound established in Lemma 3.32 on $|f_0(\omega) - p(\omega)|$ can easily be improved for $\omega \in [0, h]$ to

$$|f_0(\omega) - p(\omega)| \leq M_2 \omega^2.$$

Hence

$$\begin{aligned} \int_0^b \frac{|f_0(\omega) - p(\omega)|^2}{\omega^2} d\omega &= \int_0^h \frac{|f_0(\omega) - p(\omega)|^2}{\omega^2} d\omega + \int_h^b \frac{|f_0(\omega) - p(\omega)|^2}{\omega^2} d\omega \\ &\leq \int_0^h M_2^2 \omega^2 d\omega + \frac{1}{h^2} \int_h^b |f_0(\omega) - p(\omega)|^2 d\omega \\ &= O(h^3) + \frac{1}{h^2} \|f_0 - p\|^2 = O(h^2) \end{aligned}$$

where we used that $\|f_0 - p\| = O(h^2)$.

Now we need to show that approximating k by p works as well. Indeed, we can repeat the above argument with f_0 replaced by k yielding that we need to show that $\int_0^b \frac{|p(\omega) - k(\omega)|^2}{\omega^2} d\omega$ is small. To see this, note that

$$\int_0^b \frac{|p(\omega) - k(\omega)|^2}{\omega^2} d\omega = \alpha^2 \int_0^b \frac{|Lp(\omega)|^2}{\omega^2} d\omega \leq \alpha^2 \int_0^h \frac{|Lp(\omega)|^2}{\omega^2} d\omega + \frac{\alpha^2}{h^2} \int_0^b |Lp(\omega)|^2 d\omega.$$

From here, the second term can be estimated as

$$\frac{\alpha^2}{h^2} \int_0^b |Lp(\omega)|^2 d\omega \leq \frac{\alpha^2}{h^2} M_3^2 b = O(h^2)$$

while for the first term, Lp is linear and has the value $\ln(h)p(h)$ at $\omega = h$. Hence

$$\alpha^2 \int_0^h \frac{1}{\omega^2} \left| \ln(h)p(h) \frac{\omega}{h} \right|^2 d\omega \leq \alpha^2 h \ln(h)^2 M_0^2 \frac{1}{h^2} = O(h^2)$$

since $h \ln(h)^2 \leq 1$ for small h .

Proof of Estimate (C). As done previously, we split the variance as

$$v_k^{\mathcal{W}}(-\ln(\omega)) = e_k^{\mathcal{W}}(\ln(\omega)^2) - e_k^{\mathcal{W}}(-\ln(\omega))^2$$

and show that we have convergence in each term. This is a weaker statement than the convergence in the second estimate since $\frac{1}{\omega^2}$ is less well-behaved near 0 than $\frac{\ln(\omega)}{\omega}$ and so follows by the same argument. We illustrate it for $e_k^{\mathcal{W}}(\ln(\omega)^2)$ and note that the same procedure works for $e_k^{\mathcal{W}}(-\ln(\omega))$.

$$\begin{aligned} \left| e_{f_0}^{\mathcal{W}}(\ln(\omega)^2) - e_k^{\mathcal{W}}(\ln(\omega)^2) \right| &= \left| \langle \ln(\omega)^2 \omega^{-1} f_0, f_0 \rangle - \langle \ln(\omega)^2 \omega^{-1} p, p \rangle \right| \\ &= \left| \langle \ln(\omega)^2 \omega^{-1} f_0, f_0 - p \rangle - \langle \ln(\omega)^2 \omega^{-1} (f_0 - p), p \rangle \right| \\ &= \left| \langle \omega^{-1} f_0, \ln(\omega)^2 (f_0 - p) \rangle - \langle \ln(\omega)^2 (f_0 - p), \omega^{-1} p \rangle \right| \\ &\leq \|\omega^{-1} f_0\| \|\ln(\omega)^2 (f_0 - p)\| + \|\ln(\omega)^2 (f_0 - p)\| \|\omega^{-1} p\| \\ &= \|\ln(\omega)^2 (f_0 - p)\| (\|\omega^{-1} f_0\| + \|\omega^{-1} p\|) \end{aligned}$$

The second factor is bounded in the proof of Lemma 3.35, so we move to bounding the first factor. On $[0, 1]$, $\ln(\omega)^4 \leq \frac{1}{\omega^2}$ while on $[1, b]$, $\ln(\omega)^4 \leq \ln(b)^4 < \infty$ and we conclude that $\|\ln(\omega)^2 (f_0 - p)\| = O(h)$.

To approximate this by $e_k^{\mathcal{W}}(\ln(\omega)^2)$, we can repeat the same argument used for estimate (B). $\square$

Proof of Proposition 4.26. With the above lemma established we are ready to finish the proof. As in Proposition 4.24, we have reused the constructions from Chapter 3 and so can recover the corresponding results. In particular, from Lemma 3.35 we have that

$$\begin{aligned} \left| \|T_x f_0\|^2 - \frac{1}{\|k\|^2} \|T_x k\|^2 \right| &= O(h), \\ \left| \|T_\sigma f_0\|^2 - \frac{1}{\|k\|} \|T_\sigma k\|^2 \right| &= O(h^2), \\ \left| \|\mathcal{W} f_0\|^2 - \frac{1}{\|k\|^2} \|\mathcal{W} k\|^2 \right| &= O(h^2). \end{aligned}$$

Therefore by combining these estimates with those established in Lemma 4.27 above, we obtain the desired result that

$$\left| \mathcal{L}(f_0) - \mathcal{L}\left(\frac{k}{\|k\|}\right) \right| < \varepsilon$$

for small enough h . □

We again also have the following limit set result.

Proposition 4.28. *Let $\mathcal{M}_h$ be the set of minimizers of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^{0,\mathbb{R}} \cap P_h(0, b)$. Then any element of*

$$\mathcal{M} = \left\{ f \in L^2(0, b) : f = \lim_{n \rightarrow \infty} p_{h_n}, p_{h_n} \in \mathcal{M}_{h_n} \text{ for some sequence } h_n \rightarrow 0 \right\}$$

is a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^{0,\mathbb{R}} \cap L^2(0, b)$.

The proof is analogous to that of Proposition 4.25.

We are now in a position to prove the two main convergence theorems of this chapter. This works analogously to theorems 3.4 and 3.5

Proof of Theorem 4.5. Let f_0 be a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^{0,\mathbb{R}}$. By Proposition 4.24 there exist $b > 0$ and $f \in \mathcal{D}_{\mathcal{L}}^{0,\mathbb{R}} \cap L^2(0, b)$ such that

$$\|f_0 - f\| < \frac{\varepsilon}{2}, \quad |\mathcal{L}(f_0) - \mathcal{L}(f)| < \frac{\varepsilon}{2}.$$

We now want to repeat this step by approximating f by a spline. However, Proposition 4.26 requires the $L^2(0, b)$ element to be a minimizer. We can do away with this condition if we can show that f satisfies the estimates of Proposition 4.21. This is however immediate from the construction of f in the proof of Proposition 4.24. Therefore we can find a $h > 0$ and $y \in \mathcal{D}_{\mathcal{L}}^{0,\mathbb{R}} \cap P_h(0, b)$ that satisfies

$$\|f - y\| < \frac{\varepsilon}{2}, \quad |\mathcal{L}(f) - \mathcal{L}(y)| < \frac{\varepsilon}{2}.$$

Using the triangle inequality we can obtain the two inequalities of the theorem, finishing the proof. □

Proof of Theorem 4.6. Fix $f \in \mathcal{M}$ so that $f = \lim_{n \rightarrow \infty} p_{h_n}^{b_n}$ where $p_{h_n}^{b_n}$ is a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^0 \cap P_{h_n}(0, b_n)$. By Theorem 4.5,

$$\mathcal{L}(p_{h_n}^{b_n}) \xrightarrow{n \rightarrow \infty} \mathcal{L}(f_0)$$

where f_0 is a minimizer of $\mathcal{L}$ in $\mathcal{D}_{\mathcal{L}}^{0,\mathbb{R}}$. We can therefore conclude that $p_{h_n}^{b_n} \in \mathcal{K}$ for large enough n by the definition of $\mathcal{K}$. Using the same argument as in the proof of Theorem 3.1 on page 28 we can conclude that $\mathcal{L}(f) = \mathcal{L}(f_0)$ which yields the desired conclusion. □

Appendix A

A proof of Theorem 3.16

The purpose of this appendix is to prove Theorem 3.16 which we restate here for convenience.

Theorem A.1 (The $L^2(0, b)$ Euler-Lagrange equation). *A minimizer f_0 of S in $\mathcal{D}_S^0 \cap L^2(0, b)$ satisfies*

$$f_0''(\omega) = H_0(\omega)f_0(\omega), \quad H_0(\omega) = \ln(\omega)^2 + \frac{1}{\omega} + \kappa \ln(\omega) + \beta \ln(\omega) - \sigma$$

where $\kappa = 2\|T_x f_0\|^2 + \|\mathcal{W}f_0\|^2$, $\beta \in \mathbb{R}$ and $\sigma = \|T_x f_0\|^2 + \|T_\sigma f_0\|^2 + \|\mathcal{W}f_0\|^2$.

By the discussion in Section 3.2.1, this will follow if we can show that

$$\delta S(f_0) = \beta e_{f_0}(T_\sigma)$$

for a minimizer $f_0 \in \mathcal{D}_S^0 \cap L^2(0, b)$.

As mentioned in the end of Section 3.2.1, when we are searching for minimizers in $L^2(0, b)$, we want to only consider wavelets with $e_f(T_\sigma) = 0$ so as not to find a minimizer which would leave $L^2(0, b)$ when having π applied to it. Had this been a finite dimensional problem we would have used the following Lagrangian function to express this

$$L(f, \beta) = S(f) - \beta e_f(T_\sigma)$$

and with the gradient corresponding to the variation, we would have obtained $\delta S(f_0) - \beta \delta e_{f_0}(T_\sigma) = 0$ for $f_0 \in \mathcal{D}_S^0$, which is the ODE in Theorem 3.16. The remainder of this appendix is spent on formalizing this argument for $L^2(0, b)$. The arguments are based on a collection of unpublished notes from [7].

A.1 Sobolev boundedness of the uncertainty differential

The first part of showing that our ODE holds is to show that it holds in terms of Gateaux differentials, i.e.,

$$dS(f_0) = \beta de_{\frac{f_0}{\|f_0\|}}(T_\sigma). \tag{A.1}$$

To do so, we first show that Gateaux differentials can be viewed as bounded linear functionals on an appropriate space and that those functionals are uniquely determined by their values at $\{\sin_m\}_m$ where $\sin_m : [0, b] \rightarrow \mathbb{R}$ is defined as

$$\sin_m(\omega) = \sqrt{\frac{2}{b}} \sin\left(\frac{\pi}{b} m \omega\right).$$

Note also that the collection $\{\sin_m\}_{m \in \mathbb{N}}$ is an orthonormal basis of $L^2(0, b)$. A more general version of the space we will be working in is defined for $s \in \mathbb{N}$ as

$$W^s(0, b) = \left\{ f \in L^2(0, b) : \sum_{m \in \mathbb{N}} |\langle f, \sin_m \rangle|^2 (1 + m^2)^s < \infty \right\}$$

with the inner product and induced norm

$$\begin{aligned} \langle f, g \rangle_s &= \sum_{m \in \mathbb{N}} \langle f, \sin_m \rangle \overline{\langle g, \sin_m \rangle} (1 + m^2)^s, \\ \|f\|_{W^2(0, b)}^2 &= \sum_{m \in \mathbb{N}} |\langle f, \sin_m \rangle|^2 (1 + m^2)^s. \end{aligned}$$

We wish to prove the following proposition.

Proposition A.2. *Let $f \in \mathcal{D}_S \cap L^2(0, b)$, then $dS(f, \cdot) \in W^2(0, b)^*$.*

To do so we will need the following three lemmas, the first two of which we omit for brevity.

Lemma A.3. *Let k be even and satisfy $k \leq s$. Functions $f \in W^s(0, b)$ can be term-by-term differentiated according the formula*

$$f^{(k)} = (-1)^{k/2} \sum_{m \in \mathbb{N}} \frac{\pi^k}{b^k} m^k \langle f, \sin_m \rangle \sin_m$$

where the convergence of the sine series is uniform for $k < s$ in $W^s(0, b)$, and in $W^{s-k}(0, b)$ for $k \leq s$.

Lemma A.4. *Let $k < s$ and $W^s(0, b) \ni \eta_n \rightarrow \eta \in W^s(0, b)$ in the topology of $W^s(0, b)$. Then $\eta_n^{(k)} \rightarrow \eta^{(k)}$ uniformly.*

Lemma A.5. *The space $W^2(0, b)$ is a dense subspace of $\mathcal{D}_S \cap L^2(0, b)$.*

Proof. We need to show that all $f \in W^2(0, b)$ satisfy

$$\|T_x f\| < \infty, \quad \|T_\sigma f\| < \infty, \quad \|\mathcal{W}f\| < \infty.$$

For T_x , by Lemma A.3

$$\|f'\| = \sum_{m \in \mathbb{N}} |\langle f', \sin_m \rangle|^2 \leq C \sum_{m \in \mathbb{N}} |\langle f, \sin_m \rangle|^2 m^2 \leq C \|f\|_{W^2} < \infty.$$

It also holds that $f(0) = 0$ since f can be written as a sine series by Lemma A.3. Therefore by Lemma 3.23 we get the desired conclusion for T_σ and $\mathcal{W}$.

Denseness follows by denseness of trigonometric polynomials in $L^2(0, b)$. $\square$

With the above lemmas established, we are ready to prove the proposition.

Proof of Proposition A.2. In order for $dS(f, h)$ to be defined, we need $S(f + h)$ to be defined which it is since $W^2(0, b) \subset \mathcal{D}_S$ by Lemma A.5. Linearity is clear and so it only remains to show continuity.

By collecting terms in equation (3.5) and using the fact that T_x, T_σ and $\mathcal{W}$ are self-adjoint, we can expand the Gateaux differential $dS(f, h)$ as

$$dS(f, h) = \langle d_x(f), T_x h \rangle + \langle d_\sigma(f), T_\sigma h \rangle + \langle d_{\mathcal{W}}(f), \mathcal{W}h \rangle + \langle d_I(f), h \rangle$$

for functions $d_x, d_\sigma, d_{\mathcal{W}}, d_I$. Now let $h_n \rightarrow h$ in $W^2(0, b)$ and conclude, using the Cauchy-Schwarz inequality, that

$$\begin{aligned} |dS(f, h) - dS(f, h_n)| &\leq \|d_x(f)\|_{L^2} \|T_x(h - h_n)\|_{L^2} + \|d_\sigma(f)\|_{L^2} \|T_\sigma(h - h_n)\|_{L^2} \\ &\quad + \|d_{\mathcal{W}}(f)\|_{L^2} \|\mathcal{W}(h - h_n)\|_{L^2} + \|d_I(f)\|_{L^2} \|h - h_n\|_{L^2}. \end{aligned}$$

Each of the d functions are in L^2 and so continuity follows if we can show that each of $T_x, T_\sigma, \mathcal{W}$ and the identity operator is continuous from $W^2(0, b)$ to $L^2(0, b)$. For the identity mapping we need to show that

$$\|g\|_{L^2} \leq C\|g\|_{W^2}$$

for $g \in W^2(0, b)$. Using that $\{\sin_m\}_m$ is an orthonormal basis, we get

$$\|g\|_{L^2}^2 = \sum_{m \in \mathbb{N}} |\langle g, \sin_m \rangle|^2 \leq \sum_{m \in \mathbb{N}} |\langle g, \sin_m \rangle|^2 (1 + m^2)^2 = \|g\|_{W^2}^2.$$

Similarly for T_x ,

$$\begin{aligned} \|g'\|_{L^2}^2 &= \sum_{m \in \mathbb{N}} |\langle g', \sin_m \rangle|^2 = C \sum_{m \in \mathbb{N}} |m \langle g, \sin_m \rangle|^2 \\ &\leq C \sum_{m \in \mathbb{N}} |\langle g, \sin_m \rangle|^2 (1 + m^2)^2 = C\|g\|_{W^2}^2. \end{aligned}$$

For T_σ and $\mathcal{W}$ we will need to use a more intricate method which implicitly uses that $\|g\|_{L^2} \leq C\|g\|_{W^2}$. By Lemma A.3, $h(0) = h_n(0) = 0$ and by Lemma A.4, $h'_n \rightarrow h'$ uniformly. Combining this with the uniform continuity of h', h'_n , we can bound both h and h_n simultaneously as

$$|h(\omega)| + |h_n(\omega)| \leq 3|h'(0)|\omega$$

for $\omega < R$ and $n > N$ for some R and N . Therefore for any $r < R$ we can split up $\|\mathcal{W}(h - h_n)\|^2$ as

$$\begin{aligned} \|\mathcal{W}(h - h_n)\|^2 &= \int_0^\infty \frac{|h - h_n|^2}{\omega} d\omega = \int_0^b \frac{|h - h_n|^2}{\omega} d\omega + \int_b^\infty \frac{|h - h_n|^2}{\omega} d\omega \\ &\leq \int_0^r \frac{|3h'(0)\omega|^2}{\omega} d\omega + \frac{1}{r} \|h - h_n\|^2. \end{aligned}$$

By choosing $r = r_n = \|h - h_n\| \rightarrow 0$, the above converges to zero as $n \rightarrow \infty$. In the special case where $r_n = 0$, $\|\mathcal{W}(h - h_n)\|^2 = 0$ too. This same argument carries over directly to the T_σ case and so we are done. $\square$

By writing the variations of $e_f(T)$ and $v_f(T)$ in the same form as S above, the above argument works to show that they too can be viewed as linear functionals on $W^2(0, b)$.

A.2 The uncertainty differential on a finite dimensional subspace

In view of the following lemma, we are only interested in the values $dS(f_0)(\sin_m)$.

Lemma A.6. *The sine coefficient mapping $\mathcal{S} : y \mapsto \{y(\sin_m)\}_{m \in \mathbb{N}}$ is a one-to-one linear mapping from $W^2(0, b)^*$ to $\mathbb{C}^{\mathbb{N}}$.*

Proof. Assume that $\mathcal{S}(y_1) = \mathcal{S}(y_2)$, i.e., $y_1(\sin_m) = y_2(\sin_m)$ for all m . Any test function $h \in W^2(0, b)$ can be written as the infinite sum

$$h = \sum_{m \in \mathbb{N}} \langle h, \sin_m \rangle \sin_m$$

since $\{\sin_m\}$ is an orthogonal basis of $L^2(0, b)$. Now since both y_1, y_2 are linear and continuous we have

$$\begin{aligned} y_1(h) &= y_1 \left(\lim_{N \rightarrow \infty} \sum_{m=1}^N \langle \eta, \sin_m \rangle \sin_m \right) \\ &= \lim_{N \rightarrow \infty} \sum_{m=1}^N y_1(\sin_m) \langle \eta, \sin_m \rangle \\ &= \lim_{N \rightarrow \infty} \sum_{m=1}^N y_2(\sin_m) \langle \eta, \sin_m \rangle \\ &= y_2 \left(\lim_{N \rightarrow \infty} \sum_{m=1}^N \langle \eta, \sin_m \rangle \sin_m \right) = y_2(h). \end{aligned}$$

□

We now restrict ourselves to the subspace of $W^2(0, b)$ consisting of sinusoidal polynomials

$$\Pi_N = \text{span} \{ \sin_m \}_{m=1}^N.$$

For $K_1, K_2, K_3, \varepsilon > 0$, we define $\mathcal{K}_{K_1, K_2, K_3}^\varepsilon$ as the space of all $f \in \mathcal{D}_S \cap L^2(0, b)$ such that

$$\frac{1}{\|f\|^2} \|T_x f\|^2 \leq K_1 e^{2\varepsilon}, \quad \frac{1}{\|f\|^2} \|T_\sigma f\|^2 \leq K_2, \quad \frac{1}{\|f\|^2} \|\mathcal{W}f\|^2 \leq K_3 e^\varepsilon, \quad \left| e_{\frac{f}{\|f\|}}(T_\sigma) \right| \leq \varepsilon.$$

Lemma A.7. *The space $\mathcal{K}_{K_1, K_2, K_3}^\varepsilon$ is compact.*

Proof. Closedness follows from writing $\mathcal{K}_{K_1, K_2, K_3}^\varepsilon$ as the intersection of closed sets corresponding to the above conditions while pre-compactness can be proved as with Lemma 3.11 and Proposition 3.12. □

Proposition A.8. *Consider the domain*

$$R_N = \left\{ p \in \Pi_N[0, b] : \frac{-1}{\sqrt{N}} \leq e_{\frac{p}{\|p\|}}(T_\sigma) \leq \frac{1}{\sqrt{N}} \right\}.$$

Let p_N^0 be a minimizer of S in R_N . Then there exists a subsequence $(p_{N_m}^0)_m$ of $(p_N^0)_N$ converging to a minimizer f_0 of S in $\mathcal{D}_S^{0, \mathbb{R}} \cap L^2(0, b)$ such that

$$S(p_{N_m}^0) \rightarrow S(f_0).$$

Proof. Let y_0 be a minimizer of S in $\mathcal{D}_S^{0,\mathbb{R}} \cap L^2(0, b)$ and p_N the truncation of its sine series, i.e.,

$$p_N = \sum_{n=1}^N \langle y_0, \sin_n \rangle \sin_n.$$

We now claim that $S(p_N) \xrightarrow{N \rightarrow \infty} S(y_0)$ which we prove by showing convergence in each term below.

- (1) $\|p_N\| \xrightarrow{N \rightarrow \infty} \|y_0\| = 1$: Since $y_0 \in \mathcal{D}_S$, $y'_0 \in L^2(0, b)$ and so by Parseval with $c_n = \langle y_0, \sin_n \rangle$ we have that

$$\sum_{n=1}^{\infty} n^2 c_n^2 \leq K$$

for some $K > 0$. Therefore,

$$\|y_0 - p_N\|^2 = \sum_{n=N+1}^{\infty} c_n^2 \leq \frac{1}{N^2} \sum_{n=N+1}^{\infty} n^2 c_n^2 = o(N^{-2})$$

and using that $\{\sin_n\}_n$ is a basis we also have that

$$\|y_0\|^2 - \|p_N\|^2 = \|y_0 - p_N\|^2 = o(N^{-2}).$$

- (2) $e_{\frac{p_N}{\|p_N\|}}(T_\sigma) \xrightarrow{N \rightarrow \infty} 0$: By (1) we can dismiss the normalization factor and just compute

$$\begin{aligned} e_{p_N}(T_\sigma) &= e_{p_N}(T_\sigma) - e_{y_0}(T_\sigma) = \int_0^b \ln(\omega) (p_N^2 - y_0^2) d\omega \\ &\leq \sqrt{\int_0^b \ln(\omega)^2 |p_N + y_0|^2 d\omega} \|p_N - y_0\| \\ &\leq \underbrace{\sqrt{\int_0^b \ln(\omega)^2 |2\|y'_0\|\sqrt{\omega}|^2 d\omega}}_{<\infty} \underbrace{\|p_N - y_0\|}_{\rightarrow 0} \xrightarrow{N \rightarrow \infty} 0 \end{aligned}$$

where we used Lemma 3.23 and the fact that $\|p'_N\| \leq \|y_0\|$ by Parseval. Note also that this convergence is $o(N^{-2})$ by (1).

- (3) $v_{\frac{p_N}{\|p_N\|}}(T_x) \xrightarrow{N \rightarrow \infty} \|T_x y_0\|^2$: Note first that since y_0 is real valued, so is p_N and hence $e_{p_N}(T_x) = 0$ yielding

$$v_{\frac{p_N}{\|p_N\|}}(T_x) = \frac{1}{\|p_N\|^2} \|p'_N\|^2.$$

By (1), we can ignore the normalization factor and just show that

$$\|p'_N\| \rightarrow \|y'_0\| \iff \|p'_N - y'_0\| \xrightarrow{N \rightarrow \infty} 0.$$

This holds since y'_0 is an L^2 function and Fourier series converge in L^2 .

- (4) $e^{-2e_{\frac{p_N}{\|p_N\|}}(T_\sigma)} v_{\frac{p_N}{\|p_N\|}}(T_x) \xrightarrow{N \rightarrow \infty} \|T_x y_0\|^2$: This follows by (2) and (3).

- (5) $v_{\frac{p_N}{\|p_N\|}}(T_\sigma) \xrightarrow{N \rightarrow \infty} \|T_\sigma y_0\|^2$: By (1), we can disregard the normalization. The first step is to show that $v_{p_N}(T_\sigma)$ behaves as $\|T_\sigma p_N\|^2$. Indeed,

$$v_{p_N}(T_\sigma) - \|T_\sigma p_N\|^2 = -2e_{p_N}(T_\sigma)^2 + e_{p_N}(T_\sigma)\|p_N\|^2 \xrightarrow{N \rightarrow \infty} 0$$

by (2). We can now proceed similarly to (2) to find

$$\begin{aligned} \left| \|T_\sigma p_N\|^2 - \|T_\sigma y_0\|^2 \right| &= \left| \int_0^b \ln(\omega)^2 (p_N^2 - y_0^2) d\omega \right| \\ &= \int_0^b \ln(\omega)^2 (p_N + y_0)(p_N - y_0) d\omega \\ &\leq \underbrace{\sqrt{\int_0^b \ln(\omega)^4 |p_N + y_0|^2 d\omega}}_{< \infty} \underbrace{\|p_N - y_0\|}_{\rightarrow 0} \xrightarrow{N \rightarrow \infty} 0 \end{aligned}$$

as desired.

- (6) $\left| \frac{e^{-\frac{p_N}{\|p_N\|}}(T_\sigma)}{\frac{e}{\|p_N\|^2}} \|W p_N\|^2 - \|W y_0\|^2 \right| \xrightarrow{N \rightarrow \infty} 0$: By (1) and (2) we can dismiss the first factor and write the difference as

$$\begin{aligned} \left| \|W p_N\|^2 - \|W y_0\|^2 \right| &= \left| \int_{\mathbb{R}^+} \frac{1}{\omega} (p_N(\omega)^2 - y_0(\omega)^2) d\omega \right| \\ &\leq \int_{\mathbb{R}^+} \frac{1}{\omega} |p_N + y_0| |p_N - y_0| d\omega. \end{aligned}$$

Both y_0 and p_n can be bounded using Lemma 3.23 since $\|p'_N\| \leq \|y'_0\|$ by Parseval to yield $|p_N(\omega) + y_0(\omega)| \leq 2\|y'_0\|\sqrt{\omega}$. If we apply the same estimate to $|p_N - y_0|$ to bound the integrand we can move the limit inside the integral by the compactness of $[0, b]$ and the dominated convergence theorem. The a.e. pointwise convergence of Fourier series now yields the desired conclusion.

Now since each of the terms of $S(p_N)$ converges to those of $S(y_0)$, $S(p_N) \xrightarrow{N \rightarrow \infty} S(y_0)$ as desired.

To get existence of a minimizer p_N^0 , note that since S is invariant under linear scalings and R_N is finite dimensional, we can search for minimizers in the compact unit sphere of R_N and since S is continuous in each parameter, we get existence of the minimizers $p_N^0 \in R_N$. Since $e_{\frac{p_N}{\|p_N\|}}(T_\sigma) = o(N^{-2})$ by (2), for large enough N we have that $p_N \in R_N$ and thus

$$S(p_N^0) \leq S(p_N) \xrightarrow{N \rightarrow \infty} S(y_0) \implies S(p_N^0) \xrightarrow{N \rightarrow \infty} S(y_0).$$

In particular, the above convergence holds for any subsequence $(p_{N_m}^0)_m$ and so it only remains to show that p_N^0 has a subsequence which converges to a $L^2(0, b)$ minimizer.

Since $S(p_N^0) \leq S(p_N)$ and all terms of $S(p_N)$ converge to the corresponding terms of $S(y_0)$, we have that $p_N^0 \in \mathcal{K}_{K,K,K}^1$ for $K = 1 + S(y_0)$ for large enough N . This space is compact by Lemma A.7 and so we get a convergent subsequence $p_{N_m}^0 \rightarrow f_0$. To show that f_0 indeed is a $L^2(0, b)$ minimizer, note that we can pass to an additional three subsequences if necessary to get that

$$v_{q_m^0}(T_x) \rightarrow \sigma_1^0, \quad v_{q_m^0}(T_\sigma) \rightarrow \sigma_2^0, \quad \|W q_m^0\|^2 \rightarrow \sigma_3^0, \quad S(y_0) = \sigma_1^0 + \sigma_2^0 + \sigma_3^0.$$

Hence for each $\delta, \varepsilon > 0$ there exists an integer M such that $q_m^0 \in \mathcal{K}_{\sigma_1^0 + \delta, \sigma_2^0 + \delta, \sigma_3^0 + \delta}^\varepsilon$ for $m > M$. By virtue of this holding for all ε, δ , we conclude that

$$e_{f_0}(T_\sigma) \leq 0, \quad \|T_x f_0\|^2 \leq \sigma_1^0, \quad \|T_\sigma f_0\|^2 \leq \sigma_2^0, \quad \|\mathcal{W} f_0\|^2 \leq \sigma_3^0$$

and since y_0 is a minimizer of S , $S(f_0) = S(y_0)$ and so f_0 is a minimizer as desired. $\square$

A.3 An Euler-Lagrange equation for the uncertainty differential

We now claim that each q_m^0 satisfies an Euler-Lagrange type equation. Indeed, q_m^0 is a solution to the Lagrangian multipliers problem of minimizing $S(q)$ subject to $e_{\frac{q}{\|q\|}}(T_\sigma) \leq \frac{1}{\sqrt{N_m}}$ and $e_{\frac{q}{\|q\|}}(T_\sigma) \geq \frac{-1}{\sqrt{N_m}}$. Now recall that solutions of Lagrangian multiplier problems with inequality conditions of the form $g(x) \leq 0$ satisfy

$$dS(q_m^0) - \beta_m de_{\frac{q_m^0}{\|q_m^0\|}}(T_\sigma) = 0$$

with $\beta_m = 0$ for those m for which q_m^0 is in the interior of the allowed region. This equality only holds in the domain in which we found q_m^0 and since we added the bounds on the expected scale as a restriction, this is Π_{N_m} , i.e.,

$$dS(q_m^0)(\sin_k) = \beta_m de_{\frac{q_m^0}{\|q_m^0\|}}(T_\sigma)(\sin_k) \quad (\text{A.2})$$

for $k \leq N_m$. Our goal is now to lift this result to the $L^2(0, b)$ minimizer f_0 with the first step being to move to a subsequence where β_m converges.

Proposition A.9. *The sequence $(\beta_m)_m$ has a convergent subsequence.*

Before proceeding with the proof we need to establish three lemmas.

Lemma A.10. *Let $f \in \mathcal{K}_{K_1, K_2, K_3}^\varepsilon$ for some $K_1, K_2, K_3, \varepsilon > 0$. Then $de_{\frac{f}{\|f\|}}(T_\sigma)$ is given as an L^2 inner product with the L^2 function $-2 \ln(\omega) \frac{f(\omega)}{\|f\|^2} - 2e_{\frac{f}{\|f\|}}(T_\sigma) \frac{f(\omega)}{\|f\|^2}$ which we denote by $\delta e_{\frac{f}{\|f\|}}(T_\sigma)$, i.e.,*

$$de_{\frac{f}{\|f\|}}(T_\sigma)(h) = \left\langle \underbrace{-\frac{2}{\|f\|^2} \left(\ln(\omega) + e_{\frac{f}{\|f\|}}(T_\sigma) \right) f(\omega)}_{=\delta e_{\frac{f}{\|f\|}}(T_\sigma)(\omega)}, h \right\rangle_{L^2}.$$

Proof. This is a direct consequence of the calculation in Lemma 3.21. $\square$

By translating variations to L^2 functions we can show equality in distribution by showing equality in L^2 . In particular, this holds for the sequence q_m^0 .

Lemma A.11. *With $\delta e_{\frac{q_m^0}{\|q_m^0\|}}(T_\sigma)$ and $\delta e_{\frac{f_0}{\|f_0\|}}(T_\sigma)$ as L^2 functions by Lemma A.10,*

$$\delta e_{\frac{q_m^0}{\|q_m^0\|}}(T_\sigma) \xrightarrow{m \rightarrow \infty} \delta e_{\frac{f_0}{\|f_0\|}}(T_\sigma).$$

Proof. From how q_m^0 is defined in Proposition A.8, it hold that $\|q_m^0\| = 1 = \|f_0\|$ and $e_{q_m^0}(T_\sigma) \rightarrow 0$ and so the only nontrivial part left to verify is that

$$\|T_\sigma(q_m^0 - f_0)\| \xrightarrow{m \rightarrow \infty} 0.$$

We derive this as a consequence of $\|q_m^0 - f_0\| \rightarrow 0$. Note that Lemma 3.23 applies to both q_m^0 and f_0 meaning we can bound them by $\sqrt{K}\sqrt{\omega}$ and so by splitting up the above integral at $r_m = \|q_m^0 - f_0\|$, using the elementary inequality $(a + b)^2 \leq 2(a^2 + b^2)$ and assuming that $r_m < b$ we have

$$\begin{aligned} \|T_\sigma(q_m^0 - f_0)\|^2 &= \int_0^{r_m} |-\ln(\omega)q_m^0 + \ln(\omega)f_0|^2 d\omega + \int_{r_m}^b |-\ln(\omega)q_m^0 + \ln(\omega)f_0|^2 d\omega \\ &\leq 4K \ln(b)^2 \int_0^{r_m} \omega d\omega + \ln(b)^2 \int_{r_m}^b |q_m^0 - f_0|^2 d\omega \leq C\|q_m^0 - f_0\|^2 \rightarrow 0. \end{aligned}$$

When $r_m \geq b$, we simply do not split up the integral. $\square$

From this lemma and Lemma A.10 we conclude that

$$de_{\frac{q_m^0}{\|q_m^0\|}}(T_\sigma) \rightarrow de_{\frac{q_m^0}{\|q_m^0\|}}(T_\sigma). \quad (\text{A.3})$$

Lemma A.12. *For any integer k ,*

$$\lim_{m \rightarrow \infty} dS(q_m^0)(\sin_k) = dS(f_0)(\sin_k).$$

Proof. The convergence we want to show here is similar to that in Proposition A.2. Indeed, using equation (3.4) we can write

$$\begin{aligned} |dS(q_m^0)(\sin_k) - dS(f_0)(\sin_k)| &\leq |\langle d_x(f) - d_x(q_m^0), T_x h \rangle| + |\langle d_\sigma(f) - d_\sigma(q_m^0), T_\sigma h \rangle| \\ &\quad + |\langle d_{\mathcal{W}}(f) - d_{\mathcal{W}}(q_m^0), \mathcal{W}h \rangle| + |\langle d_I(f) - d_I(q_m^0), h \rangle| \end{aligned}$$

where $d_x, d_\sigma, d_{\mathcal{W}}$ can be derived from (3.4). We now verify that each nontrivial term goes to 0. Note that in the following the integer k is fixed.

- (1) $\langle q_m^0 - f_0, \sin_k \rangle \xrightarrow{m \rightarrow \infty} 0$: This follows immediately by Cauchy-Schwarz and $\|q_m^0 - f_0\| \rightarrow 0$.
- (2) $\langle q_m^{0''} - f_0'', \sin_k \rangle \xrightarrow{m \rightarrow \infty} 0$: By moving the derivatives to the sine we can get the desired conclusion using Cauchy-Schwarz combined with $\|q_m^0 - f_0\| \rightarrow 0$.
- (3) $\langle q_m^{0'} - f_0', \sin_k \rangle \xrightarrow{m \rightarrow \infty} 0$: Again we can move the derivative to the sine and use Cauchy-Schwarz since $\|\sin'_k\| < \infty$.
- (4) $\langle \frac{1}{\omega}(q_m^0 - f_0''), \sin_k \rangle \xrightarrow{m \rightarrow \infty} 0$: By moving the $1/\omega$ to $\sin_k$ and applying Cauchy-Schwarz, we see that it suffices to show that

$$\|q_m^0 - f_0\| \|\omega^{-1} \sin_k\| \rightarrow 0.$$

The second factor is bounded since $\sin_k$ has a zero of order 1 at 0 while the first factor converges to 0.

- (5) $\langle \ln(\omega)[q_m^0 - f_0], \sin_k \rangle \xrightarrow{m \rightarrow \infty} 0$: Similarly here we can move the $\ln(\omega)$ to $\sin_k$ and note that the resulting product is bounded near 0 and then apply Cauchy-Schwarz.

□

Proof of Proposition A.9. Existence of a convergent subsequence follows if we can show that $(\beta_m)_m$ is bounded. If $de_{\frac{f_0}{\|f_0\|}}(T_\sigma)(\sin_k) = 0$ for all k we can take $\beta_m = 0$ for all m . If not, then we can write

$$\beta_m = \frac{dS(q_m^0)(\sin_k)}{de_{\frac{q_m^0}{\|q_m^0\|}}(T_\sigma)(\sin_k)}$$

and it will hold that the numerator converges to $de_{\frac{f_0}{\|f_0\|}}(T_\sigma)(\sin_k) \neq 0$ as $m \rightarrow \infty$ by Lemma A.11, so it is bounded from below. Therefore it only remains to bound $dS(q_m^0)(\sin_k)$ from above, but this holds since it converges to $dS(f_0)(\sin_k)$ by Lemma A.12. □

By abuse of notation, we denote this convergent subsequence by β_m and note that the results established above suffice to lift the result in equation (A.2) to f_0 :

$$\underbrace{dS(q_m^0)(\sin_k)}_{\substack{\rightarrow dS(f_0)(\sin_k) \\ \text{by Lemma A.12}}} = \underbrace{\beta_m}_{\substack{\rightarrow \beta \\ \text{by Prop. A.9}}} \cdot \underbrace{de_{\frac{q_m^0}{\|q_m^0\|}}(T_\sigma)(\sin_k)}_{\substack{\rightarrow de_{\frac{f_0}{\|f_0\|}}(T_\sigma)(\sin_k) \\ \text{by eq. (A.3)}}}$$

which implies that

$$dS(f_0)(\sin_k) = \beta de_{\frac{f_0}{\|f_0\|}}(T_\sigma)(\sin_k).$$

By Proposition A.2, this differential can be viewed as a linear functional on $W^2(0, b)$ and by Lemma A.6 we have

$$dS(f_0) = \beta de_{\frac{f_0}{\|f_0\|}}(T_\sigma)$$

which is precisely equation (A.1).

A.4 From the uncertainty differential to the uncertainty variation

Finally we prove a representation theorem for functionals on $W^2(0, b)$ that will allow us to translate the uncertainty differential to a variation.

Lemma A.13. *Let $y \in W^2(0, b)^*$ satisfy $\{y(\sin_m)\}_{m \in \mathbb{N}} \in \ell^2$, then there exists a unique function $q \in L^2(0, b)$ such that for any $h \in W^2(0, b)$,*

$$y(h) = \langle h, q \rangle_{L^2}.$$

Proof. Since $\{y(\sin_m)\}_{m \in \mathbb{N}} \in \ell^2$, we have convergence in the sum

$$q = \sum_{m \in \mathbb{N}} \overline{y(\sin_m)} \sin_m.$$

By writing an arbitrary $h \in W^2(0, b)$ as $h = \sum_{m \in \mathbb{N}} \langle h, \sin_m \rangle_{L^2} \sin_m$ as in the proof of Lemma A.6 and using the linearity and continuity of y , we have that

$$\begin{aligned} y(h) &= y \left(\sum_{m \in \mathbb{N}} \langle h, \sin_m \rangle_{L^2} \sin_m \right) = \sum_{m \in \mathbb{N}} \langle h, \sin_m \rangle y(\sin_m) \\ &= \left\langle \langle h, \sin_m \rangle_{L^2}, \overline{\{y(\sin_m)\}} \right\rangle_{\ell^2} = \langle h, q \rangle_{L^2}. \end{aligned}$$

For uniqueness, if $y(h) = \langle h, q_1 \rangle_{L^2} = \langle h, q_2 \rangle_{L^2}$ for all h , we form $\tilde{q} = q_1 - q_2$ and note that

$$\langle h, \tilde{q} \rangle_{L^2} = 0 \quad \text{for all } h \in W^2(0, b).$$

Now since $W^2(0, b)$ is dense in $L^2(0, b)$ by Lemma A.5 and inner products are continuous, we conclude that $\tilde{q} = 0$. $\square$

Proof of Theorem 3.16. Both $dS(f_0)$ and $de_{\frac{f_0}{\|f_0\|}}(T_\sigma)$ satisfy the ℓ^2 condition of the above lemma by Lemma A.10 and equation (A.1). Thus since $de_{\frac{f_0}{\|f_0\|}}(T_\sigma)$ was given as an inner product in Lemma A.10 and $dS(f_0)$ was defined via the inner product with the variation in the first place, we have equality in the L^2 sense of

$$\delta S(f_0) = \beta(-2 \ln(\omega) f_0).$$

The factor -2 can be absorbed β to yield the desired equality, completing the proof. $\square$

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