

THE PHRAGMÉN-LINDELÖF PRINCIPLE AND ITS APPLICATIONS

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Abstract

This thesis discusses an extension of the classical maximum modulus principle to unbounded domains, called the Phragmén-Lindelöf principle. The principle is illustrated in strips, half planes and sectors, as well as through the related Hadamard theorems. The thesis is concluded by proving applications of the Phragmén-Lindelöf principle within Fourier analysis and operator theory. The applications within Fourier analysis are the Hausdorff-Young inequality and Hardy's uncertainty principle, and the application within operator theory is the Riesz-Thorin interpolation theorem.

Populärvetenskaplig sammanfattning

Komplexa tal kom till som ett verktyg för att lösa andra- och tredjegradsekvationer. Till en början sågs de som värdelösa utanför det specifika användningsområdet, men har sedan dess visat sig vara mycket användbara inom flera matematiska fält och även utanför matematiken. Ett sådant exempel är kvantmekanik inom fysiken.

Den gren av matematiken som tagit de komplexa talen från ekvationslösning till de många tillämpningar vi känner till idag kallas för komplex analys. Detta fält studerar funktioner av komplexa tal. Funktioner av särskild betydelse är så kallade holomorfa funktioner, vilket är funktioner som är deriverbara i komplex mening. Holomorfa funktioner har visat sig ha många egenskaper som generellt inte delas med reella differentierbara funktioner. Exempelvis kan holomorfa funktioner deriveras oändligt många gånger, medan det finns reella differentierbara funktioner som inte kan det.

Detta arbete behandlar en metod för att avgöra om en holomorf funktion är begränsad i en obegränsad domän. En klassisk sats inom komplex analys säger att i en begränsad domän är en holomorf funktion begränsad av sitt maximala absolutbelopp på domänens rand. Phragmén-Lindelöf-principen beskriver en metod som utvidgar denna sats till att gälla i obegränsade domäner. Mer specifikt, om en holomorf funktion är begränsad på randen av en obegränsad domän och funktionen uppfyller en restriktiv tillväxt i domänen så är funktionen begränsad i hela domänen. Tekniken som Phragmén-Lindelöf-principen beskriver används för att bevisa detta för olika domäner.

Phragmén-Lindelöf-principen har flera tillämpningar, både inom komplex analys och andra fält. I detta arbete diskuteras bland annat tillämpningar inom Fourieranalys, som även det är en matematisk gren med många tillämpningar, till exempel inom fysik och kemi.

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Introduction

A classical result within complex analysis is the *maximum modulus principle*. Informally, the principle states that a holomorphic function in a bounded domain is bounded in magnitude by its maximum modulus on the boundary of the domain. See Theorem 1.1.5 for a more precise statement. The assumption that the domain is bounded cannot be removed. The following example shows an instance where the principle fails when applied to an unbounded domain [1, Chapter 12].

Example. Consider the horizontal strip $\Omega = \{z \in \mathbb{C} : -\pi/2 < \text{Im}(z) < \pi/2\}$. Then we have that $\partial\Omega = \{z \in \mathbb{C} : |\text{Im}(z)| = \pi/2\}$. Let $f(z) = \exp(\exp(z))$.

Consider a point on the boundary $x \pm i\pi/2$, where x is a real number. Then we have that

$$f(x \pm i\pi/2) = \exp(\exp(x \pm i\pi/2)) = \exp(\pm ie^x).$$

Since x is real, we get that $|f(z)| = 1$ for $z \in \partial\Omega$. However, if we consider a real point x , which lies in Ω , we have that $f(x) = \exp(\exp(x))$, which is unbounded on the real axis.

Thus, $|f|$ is bounded by 1 on the boundary of Ω , but unbounded in Ω , which shows that the maximum modulus principle does not hold in unbounded domains.

It is then natural to ask if it is possible to formulate a principle similar to that of the maximum modulus principle, that holds for unbounded domains. Indeed, in 1908, Edvard Phragmén and Ernst Lindelöf [2] presented a method to extend the principle to unbounded domains, called the *Phragmén-Lindelöf principle*. In short, the Phragmén-Lindelöf principle describes a technique where the use of an auxiliary function allows us to find a bounded subset where we can apply the maximum modulus principle, while bounding the function outside of this subset. This means that the function is bounded overall. In Chapter 2, we will formally describe the principle, as well as illustrate the technique in a few special domains. In Chapter 3, we discuss applications of the principle within Fourier analysis. More specifically, we prove the Hausdorff-Young inequality and Hardy's uncertainty principle using the Phragmén-Lindelöf principle. Lastly, in Chapter 4, we prove an application within operator theory, namely the Riesz-Thorin interpolation theorem.

CHAPTER I

Preliminaries

In this chapter, we state definitions and results that are either important for understanding the context of the thesis, or will be used directly in later chapters. We omit proofs of the theorems, but the reader can find those in the relevant citations.

I.1 Holomorphic Functions

In this section, we explain the basics of holomorphic functions, as well as a few related results that will be used later. We begin by defining domains.

Definition 1.1.1. A set $\Omega \subset \mathbb{C}$ is a *domain* in $\mathbb{C}$ if it is open and connected in $\mathbb{C}$.

Here, that Ω is open means that for any point $z \in \Omega$, there exists a disc centred at z that is contained in Ω . In this context, Ω being connected means that any two points in Ω can be joined by a continuous curve contained in Ω .

An important definition is the complex derivative [1, Chapter 10].

Definition 1.1.2. Let f be a complex-valued function defined in a domain $\Omega \subset \mathbb{C}$. If $a \in \Omega$ and the limit

$$\lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a}$$

exists, then the limit is the *complex derivative* of f at a , denoted $f'(a)$.

Here we defined the complex derivative at a point in the complex plane. However, it is useful to have a notion of when the complex derivative exists in a set as well [1, Chapter 10].

Definition 1.1.3. Let f be a complex-valued function defined in a domain $\Omega \subset \mathbb{C}$. If $f'(a)$ exists for all a in Ω , then f is *holomorphic* in Ω .

A function that is holomorphic in the whole complex plane is called *entire*. An important property of entire functions in the following theorem [3, Chapter 4].

Theorem 1.1.4 (Liouville's theorem). *If f is a bounded entire function, then f is constant.*

As stated in the Introduction, the Phragmén-Lindelöf principle can be viewed as an extension of the maximum modulus principle, which states that a holomorphic function in a bounded domain attains its maximum modulus on the boundary of the domain [1, Chapter 12].

Theorem 1.1.5 (Maximum modulus principle). *Let $\Omega \subset \mathbb{C}$ be a bounded domain. If f is holomorphic in Ω and continuous in $\overline{\Omega}$, then*

$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in \partial\Omega} |f(z)|.$$

We define the complex line integral [1, Chapter 10].

Definition 1.1.6. Let γ be a C^1 -curve parametrised by $\gamma(t)$, for $a \leq t \leq b$. Suppose that f is a complex function that is continuous on the range of γ . Then we define the *complex line integral* as

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

An important theorem about line integrals of holomorphic functions is Cauchy's theorem [3, Chapter 4].

Theorem 1.1.7 (Cauchy's theorem). *Let Ω be a bounded domain with piecewise smooth boundary. If f is holomorphic in Ω and continuous in $\overline{\Omega}$, then*

$$\int_{\partial\Omega} f(z) dz = 0.$$

As a converse of Cauchy's theorem, Morera's theorem states that line integrals can be used to determine if a continuous function is holomorphic in an open set [1, Theorem 10.17].

Theorem 1.1.8 (Morera's theorem). *Let Ω be an open set and suppose that f is continuous in Ω . If*

$$\int_{\partial T} f(z) dz = 0$$

for every closed triangle $T \subset \Omega$, then f is holomorphic in Ω .

Lastly, we state two theorems that will be used in later chapters. The first of those is a uniqueness theorem [1, Theorem 10.18].

Theorem 1.1.9 (Uniqueness of analytic continuation). *Suppose that Ω is a domain and f is holomorphic in Ω . If the zero set $Z = \{z \in \Omega : f(z) = 0\}$ has a limit point in Ω , then $f(z) = 0$ for all $z \in \Omega$.*

That Z has a limit point in Ω means that there exists a point a in Ω and a sequence $\{a_n\}$ of distinct points in Z such that $a_n \rightarrow a$ when $n \rightarrow \infty$. In other words, the function f has non-isolated zeroes in Ω , which for instance occurs when Z includes a curve [3, Chapter 5].

Lastly, we state the Schwarz reflection principle [1, Theorem 11.17]. Here, we let $\Pi^+ = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ denote the upper half plane. Similarly, we let Π^- denote the lower half plane. We denote the ball with radius r centred at a as $B_r(a)$.

Theorem 1.1.10 (The Schwarz reflection principle). *Let L be a segment of $\mathbb{R}$. Let Ω^+ be a subset of Π^+ with the property that for all $t \in L$, there exists $\varepsilon > 0$ such that $B_\varepsilon(t) \cap \Pi^+$ lies in Ω^+ . Let $\Omega^- = \{\bar{z} : z \in \Omega^+\}$ denote the reflection of Ω^+ . Suppose that $f = u + iv$ is holomorphic in Ω^+ and $\lim_{n \rightarrow \infty} v(z_n) = 0$ for each sequence $\{z_n\}$ in Ω^+ which converges to a point in L . Then there exists a function F that is holomorphic in $\Omega^+ \cup L \cup \Omega^-$, with $F(z) = f(z)$ in Ω^+ and $F(\bar{z}) = \overline{F(z)}$ for all $z \in \Omega^+ \cup L \cup \Omega^-$.*

1.2 Measure Theory

This section concerns the basics of measure theory needed to understand the generalised Hausdorff-Young inequality (Theorem 3.1.6). In particular, we aim to define measurable functions, measure spaces and integrals over a measurable set with respect to a measure. Moreover, we state some important results. This section is based on Rudin's chapter on the topic [1, Chapter 1].

Definition 1.2.1. Let X be a set. A collection of subsets $\mathcal{R}$ of X is a σ -algebra if the following conditions hold:

- (i) $X \in \mathcal{R}$.
- (ii) If $A \in \mathcal{R}$, then the complement of A relative to X is in $\mathcal{R}$.
- (iii) If $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in \mathcal{R}$ for all n , then $A \in \mathcal{R}$.

σ -algebras are used to define measurable spaces and sets.

Definition 1.2.2. A pair $(X, \mathcal{R})$ is a *measurable space* if $\mathcal{R}$ is a σ -algebra in X . Members of $\mathcal{R}$ are called *measurable sets* in $(X, \mathcal{R})$.

As we have defined measurable spaces, it is natural to consider functions on those. In particular, we will consider measurable functions.

Definition 1.2.3. Let $(X, \mathcal{R})$ be a measurable space. A function $f : X \rightarrow \mathbb{C}$ is *complex measurable* if $f^{-1}(V)$ is a measurable set in $(X, \mathcal{R})$ for every open set V in $\mathbb{C}$.

A special function that will be of importance is the positive measure.

Definition 1.2.4. A function μ is a *positive measure* on a σ -algebra $\mathcal{R}$ if the following hold:

- (i) The range of μ is in $[0, \infty]$.
- (ii) $\mu(\emptyset) = 0$.
- (iii) If $\{A_i\}$ is a disjoint countable collection of members in $\mathcal{R}$, then we have that $\mu(\cup A_i) = \sum \mu(A_i)$.

A measurable space $(X, \mathcal{R})$ with a positive measure μ defined on the σ -algebra of its measurable sets is said to be a *measure space* and is denoted $(X, \mathcal{R}, \mu)$. In later chapters, we will for instance often use the classical Lebesgue measure $([0, 1), \mathcal{L}, dx)$, where $\mathcal{L}$ is the σ -algebra of Lebesgue measurable sets. We will also use the counting measure on $\mathbb{Z}$, where the σ -algebra is the collection of all subsets of $\mathbb{Z}$.

For the remainder of this section, we let $(X, \mathcal{R}, \mu)$ denote a measure space with positive measure μ . A function that needs to be defined before we can define the integral is the characteristic function.

Definition 1.2.5. Let E be a measurable set in $(X, \mathcal{R}, \mu)$. We define the *characteristic function* of E as

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}.$$

Defining the general integral requires us to first define the integral of a simple measurable function, which is a function on a measurable space which its range being finitely many points in $[0, \infty)$.

Definition 1.2.6. Let s be a simple measurable function on $(X, \mathcal{R}, \mu)$ that is on the form $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$, where the α_i are the distinct values of s and $A_i = \{x : s(x) = \alpha_i\}$. Suppose that $E \in \mathcal{R}$. Then we define

$$\int_E s d\mu = \sum_{i=1}^n \alpha_i \mu(A_i \cap E).$$

Note that here we use the convention that $0 \cdot \infty = 0$, as we may have that $\alpha_i = 0$ and $\mu(A_i \cap E) = \infty$.

Now we can define the general integral of positive measurable functions.

Definition 1.2.7. Let $f : X \rightarrow [0, \infty]$ be a measurable function and let $E \in \mathcal{R}$. We define

$$\int_E f d\mu = \sup \int_E s d\mu,$$

where the supremum is taken over all simple measurable functions s with $0 \leq s \leq f$.

Defining the integral of a complex function is now a matter of breaking up the function into real and imaginary parts, which are then divided into positive and negative parts. If g is a real function, then the positive part of g is $g^+ = \max \{g, 0\}$ and the negative part is $g^- = -\min \{g, 0\}$.

Definition 1.2.8. Let $f = u + iv$ be a complex function, where u and v are real measurable functions on $(X, \mathcal{R}, \mu)$. If $\int_X |f| d\mu < \infty$ and E is a measurable set in $(X, \mathcal{R}, \mu)$, then we define

$$\int_E f d\mu = \int_E u^+ d\mu - \int_E u^- d\mu + i \int_E v^+ d\mu - i \int_E v^- d\mu.$$

Lastly, we state Fatou's lemma [1, Theorem 1.28].

Lemma 1.2.9 (Fatou's lemma). *If $f_n : X \rightarrow [0, \infty]$ is measurable for each positive integer n , then*

$$\int_X \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu.$$

1.3 Lebesgue spaces

In this section, we let $(X, \mathcal{R}, \mu)$ denote an arbitrary measure space with a positive measure μ .

Definition 1.3.1. Let $0 < p < \infty$ and let f be a complex measurable function on $(X, \mathcal{R}, \mu)$. The L^p -norm of f is defined as

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}.$$

Then we define $L^p(\mu) = \{f \text{ complex measurable} : \|f\|_p < \infty\}$.

If μ is the counting measure on $\mathbb{Z}$, then the corresponding L^p -space is denoted $\ell^p(\mathbb{Z})$. An element of $\ell^p(\mathbb{Z})$ is a complex sequence $x = \{\xi_n\}$. The ℓ^p -norm is then

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |\xi_n|^p \right)^{1/p}.$$

In order to define L^∞ -spaces, we must first define the essential supremum.

Definition 1.3.2. Let $f : X \rightarrow [0, \infty]$ be a measurable function on $(X, \mathcal{R}, \mu)$ and define $S = \{\alpha \in \mathbb{R} : \mu(f^{-1}((\alpha, \infty])) = 0\}$. If $S = \emptyset$, set $\beta = \infty$. Otherwise, set $\beta = \inf S$. Then β is called the *essential supremum* of f .

The essential supremum is then used to define the L^∞ -norm.

Definition 1.3.3. Let f be a complex measurable function on $(X, \mathcal{R}, \mu)$. We define $\|f\|_\infty$ as the essential supremum of $|f|$ and define the corresponding Lebesgue space as $L^\infty(\mu) = \{f \text{ complex measurable} : \|f\|_\infty < \infty\}$.

To conclude this section, we state a few important results. The first of those is Hölder's inequality [1, Theorem 3.5].

Theorem 1.3.4 (Hölder's inequality). *Suppose that $1 < p < \infty$ and $q = p/(p-1)$. If f and g are measurable functions on $(X, \mathcal{R}, \mu)$, then $\|fg\|_1 \leq \|f\|_p \|g\|_q$.*

We state a result about the density of simple functions in $L^p(\mu)$ [1, Theorem 3.13].

Theorem 1.3.5. *Let S be the class of complex measurable simple functions on $(X, \mathcal{R}, \mu)$ with $\mu(\{x : s(x) \neq 0\}) < \infty$. If $1 \leq p < \infty$, then S is dense in $L^p(\mu)$.*

A consequence of this theorem is that any function $f \in L^p(\mu)$ can be approximated arbitrarily well in the L^p -norm by simple functions. In other words, there exists a sequence of simple functions $\{f_j\}$ such that $\lim_{j \rightarrow \infty} \|f_j - f\|_p = 0$.

Lastly, we state a result about the duality of L^p [1].

Lemma 1.3.6. *Let $1 < p \leq \infty$ and $q = p/(p-1)$. Then for any function f on any measure space $(X, \mathcal{R}, \mu)$, it holds that*

$$\|f\|_q = \sup \left| \int_X fg d\mu \right|,$$

where the supremum is taken over all simple functions g with $\|g\|_p = 1$. In other words, the L^q -norm of any function on any measure space equals its norm as a linear functional on L^p .

1.4 Fourier Analysis

As we will discuss applications of the Phragmén-Lindelöf principle within Fourier analysis, we use this section to state a few important related definitions and results. This section is based on Chapters 4 and 7 on Vretblad's book on the topic [4].

A central concept in Fourier analysis is Fourier series. However, as we will only use Fourier coefficients, we only define those.

Definition 1.4.1. Let f be a function with period $2P$ that is Lebesgue integrable over a period. For each integer n , the *Fourier coefficients* of f are defined as

$$\hat{f}(n) = \frac{1}{2P} \int_0^{2P} f(t) e^{-in\frac{\pi}{P}t} dt.$$

Another important concept is the Fourier transform.

Definition 1.4.2. Let f be a function so that $\int_{\mathbb{R}} |f(t)| dt$ is convergent. The *Fourier transform* of f is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i \xi t} dt.$$

The Fourier transform can also refer to the mapping $\mathcal{F} : f \mapsto \hat{f}$. Using this notation, the Fourier transform defined above is written as $\hat{f}(\xi) = \mathcal{F}[f(x)](\xi)$. An important fact is that $\mathcal{F}$ is a linear mapping. Moreover, a result that will be used while proving Hardy's uncertainty principle (Theorem 3.2.14) is the following theorem about the Fourier transform of a Gaussian.

Lemma 1.4.3. Let a be a constant. Then $\mathcal{F}[e^{-ax^2}](\xi) = \sqrt{\pi/a} \cdot e^{-\pi^2 \xi^2 / a}$.

In other words, the Fourier transform of a Gaussian is a Gaussian itself. Consequently, the inverse Fourier transform of a Gaussian is also a Gaussian.

1.5 Operator Theory

Let $(M, \mathcal{M}, \mu)$ and $(N, \mathcal{N}, \nu)$ denote arbitrary measure spaces and let $L^p(\mu)$ and $L^q(\nu)$ be Lebesgue spaces on those respectively. We begin by defining the (p, q) -norm, where $1 \leq p, q \leq \infty$ [5, Chapter 5].

Definition 1.5.1. Let T be a linear operator mapping simple functions in $L^p(\mu)$ into $L^q(\nu)$. If there exists a constant $k > 0$ such that $\|Tf\|_q \leq k\|f\|_p$ for all simple functions $f \in L^p(\mu)$, then T is said to be of *type* (p, q) . The least k for which this holds is called the (p, q) -norm of T .

By Theorem 1.3.5, the subspace of simple functions in L^p is dense in L^p . Moreover, the space is clearly linear. Thus, if the estimate above holds for simple functions, then the following result states that T can be extended to all of $L^p(\mu)$ so that the estimate still holds [6, Chapter 1, Proposition 5.4]. Stein formulates this result for the more general Banach space, but as we are only concerned with L^p -spaces in this context, we state a less general version.

Theorem 1.5.2. *Let $S \subset L^p(\mu)$ be a dense linear subspace of $L^p(\mu)$. Let T_0 be a linear operator mapping S to $L^q(\nu)$ that satisfies $\|T_0 f\|_q \leq C\|f\|_p$ for all $f \in S$, where C is a constant independent of f . Then T_0 has a unique extension T to all of $L^p(\mu)$ so that $\|Tf\|_q \leq C\|f\|_p$ for all $f \in L^p(\mu)$.*

CHAPTER 2

The Phragmén-Lindelöf Principle

The goal of the Phragmén-Lindelöf principle is to extend the maximum modulus principle to unbounded domains. In general, this can be done if the growth of the holomorphic function in question is restricted, however, the exact growth restriction depends on the domain [1, Chapter 12]. In this chapter, we will examine this in more detail. We begin by describing the general Phragmén-Lindelöf principle, and then we illustrate the technique in a few special domains. We conclude with two more refined Phragmén-Lindelöf results, namely the Hadamard theorems.

2.1 Outline of the General Principle

As previously mentioned, the principle was first described by Phragmén and Lindelöf in 1908 [2]. The method that they described is formulated a little differently than the one that we will include here. We choose a formulation that is more aligned with how the technique is performed in the following sections.

Let $\Omega \subset \mathbb{C}$ be an unbounded domain. Suppose that f is a holomorphic function in Ω that is continuous in $\overline{\Omega}$, with $|f| \leq 1$ on $\partial\Omega$. Our goal is to show that $|f| \leq 1$ in Ω . To do so, we introduce an auxiliary function h_ε with the property that $\lim_{\varepsilon \rightarrow 0} h_\varepsilon(z) = 1$ for all $z \in \Omega$. Moreover, for each $\varepsilon > 0$, we require that h_ε satisfies the following conditions:

- (i) fh_ε is holomorphic in Ω and continuous in $\overline{\Omega}$.
- (ii) There exists a bounded subset Ω_{bdd} of Ω such that $|fh_\varepsilon| \leq 1$ on $\partial\Omega_{\text{bdd}}$ and $|fh_\varepsilon| \leq 1$ in $\Omega \setminus \Omega_{\text{bdd}}$.

For a fixed ε , conditions (i) and (ii) allow us to apply the maximum modulus principle to conclude that $|fh_\varepsilon| \leq 1$ in Ω_{bdd} . Since $|fh_\varepsilon| \leq 1$ also in $\Omega \setminus \Omega_{\text{bdd}}$, we conclude that $|fh_\varepsilon| \leq 1$ in Ω . Since the above conditions hold for each $\varepsilon > 0$, the same argument

can be made for each $\varepsilon > 0$, so $|fh_\varepsilon| \leq 1$ in Ω for all $\varepsilon > 0$. By letting $\varepsilon \rightarrow 0$, we finally arrive at the conclusion that $|f| \leq 1$ in Ω .

While the Phragmén-Lindelöf principle might not be that complicated in theory, we will throughout this thesis see that the principle can be used as a quite powerful tool in proving important results.

2.2 The Principle in Special Domains

To illustrate the technique that the Phragmén-Lindelöf principle describes, we will in this section give examples of applying the principle in vertical strips, the half plane and sectors in the complex plane. We begin by considering vertical strips [7, Chapter 12, Theorem 6.1].

Theorem 2.2.1. *Consider the vertical strip $\Omega = \{z \in \mathbb{C} : -\pi/2 < \operatorname{Re}(z) < \pi/2\}$, and a function f that is holomorphic in Ω and continuous in $\overline{\Omega}$, with $|f| \leq 1$ on $\partial\Omega$. Suppose that there exist constants $\alpha \in (0, 1)$ and $C > 0$ such that $|f(z)| \leq \exp(C \exp(\alpha|z|))$ for z in Ω . Then $|f| \leq 1$ in Ω .*

As seen in the Introduction, the condition that $\alpha < 1$ is sharp, since the theorem does not hold if $\alpha = 1$. We turn to proving the theorem.

Proof. Let β be a constant with $\alpha < \beta < 1$, and let $\varepsilon > 0$. Define the auxiliary function $h_\varepsilon(z) = \exp(-2\varepsilon \cos \beta z)$. Note that $h_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$. Set $g_\varepsilon = fh_\varepsilon$.

Notice that by writing $z = x + iy$, where x and y are real numbers, and using the addition formula for cosine, we have that

$$\cos \beta z = \frac{e^{\beta y} + e^{-\beta y}}{2} \cos \beta x + i \frac{e^{\beta y} - e^{-\beta y}}{2} \sin \beta x.$$

Hence, we have that $2\operatorname{Re}(\cos \beta z) = (e^{\beta y} + e^{-\beta y}) \cos \beta x$. Moreover, since $0 < \beta < 1$, there exists $c > 0$ such that

$$c \leq \cos \beta x \tag{2.2.2}$$

for all $x \in [-\pi/2, \pi/2]$.

We now consider $|g_\varepsilon|$ on $\partial\Omega$. With $y \in \mathbb{R}$, we have that

$$|g_\varepsilon(\pm\pi/2 + iy)| = |f(\pm\pi/2 + iy)| |\exp(-2\varepsilon \cos \beta(\pm\pi/2 + iy))|.$$

Since $|f(\pm\pi/2 + iy)| \leq 1$ by assumption, and by using 2.2.2, we get that

$$|g_\varepsilon(\pm\pi/2 + iy)| \leq \left| \exp\left(-c\varepsilon(e^{\beta y} + e^{-\beta y})\right) \right| \leq 1,$$

since $e^{\beta y} + e^{-\beta y} > 0$ for all y , hence $|g_\varepsilon| \leq 1$ on the sides of the strip.

Our goal is now to construct a bounded rectangle R in Ω so that $|g_\varepsilon| \leq 1$ on the boundary of R and outside of R . Then we can apply the maximum modulus principle to conclude that $|g_\varepsilon| \leq 1$ also in R , which implies that $|g_\varepsilon| \leq 1$ in Ω .

To do this, we show that we can draw horizontal lines that form the rectangle R together with the sides of the strip, so that the formed R satisfies the requirements. Note that h_ε can be written as $h_\varepsilon(z) = \exp(-2\varepsilon \operatorname{Re}(\cos \beta z)) \cdot \exp(-2\varepsilon i \operatorname{Im}(\cos \beta z))$. This gives that

$$\begin{aligned} |h_\varepsilon(z)| &= \exp(-2\varepsilon \operatorname{Re}(\cos \beta z)) \\ &= \exp\left(-\varepsilon(e^{\beta y} + e^{-\beta y}) \cos \beta x\right) \\ &\leq \exp\left(-c\varepsilon(e^{\beta y} + e^{-\beta y})\right). \end{aligned}$$

By using this, we get that

$$\begin{aligned} |g_\varepsilon(z)| &\leq \exp\left(Ce^{\alpha|z|}\right) \cdot \exp\left(-c\varepsilon(e^{\beta y} + e^{-\beta y})\right) \\ &= \exp\left(Ce^{\alpha|z|} - c\varepsilon(e^{\beta y} + e^{-\beta y})\right) \\ &\leq \exp\left(Ce^{\alpha|y| + \alpha\pi/2} - c\varepsilon(e^{\beta y} + e^{-\beta y})\right), \end{aligned}$$

where the last step follows from the fact that $|z| \leq |y| + \pi/2$ by the triangle inequality. The exponent in the last expression above is negative for large enough $|y|$ since $\beta > \alpha$, and thus $|g_\varepsilon(z)| \rightarrow 0$ as $|z| \rightarrow \infty$. Therefore, we can choose to draw horizontal lines sufficiently far out to get the desired rectangle R with $|g_\varepsilon| \leq 1$ on the boundary of R , and $|g_\varepsilon| \leq 1$ in $\Omega \setminus R$. By the maximum modulus principle, we get that $|g_\varepsilon| \leq 1$ in R . It follows that $|g_\varepsilon| \leq 1$ in Ω .

Finally, since ε was arbitrary, we have that $|g_\varepsilon| \leq 1$ in Ω for all $\varepsilon > 0$. By letting $\varepsilon \rightarrow 0$, we get that $|f| \leq 1$ in Ω , which concludes the proof. $\square$

It is now straightforward to generalise the statement to any vertical strip by means of applying an appropriate linear transformation. The theorem and proof is provided by the author.

Theorem 2.2.3 (Phragmén-Lindelöf for vertical strips). *Consider the vertical strip $\Omega = \{z \in \mathbb{C} : a < \operatorname{Re}(z) < b\}$, and a function f that is holomorphic in Ω and continuous in $\overline{\Omega}$, with $|f| \leq 1$ on $\partial\Omega$. Suppose that there exist constants $\alpha \in (0, 1)$ and $C > 0$ such that*

$$|f(z)| \leq \exp\left(C \exp\left(\frac{\alpha\pi}{b-a} |z|\right)\right)$$

for z in Ω . Then $|f| \leq 1$ in Ω .

Proof. The linear function

$$\phi(z) = \frac{b-a}{\pi}z + \frac{a+b}{2}$$

maps the strip $U = \{z \in \mathbb{C} : -\pi/2 < \operatorname{Re}(z) < \pi/2\}$ to Ω bijectively. We set $g = f \circ \phi$ and shall apply the previous theorem to g . It is clear that g is holomorphic in U and continuous in $\overline{U}$, since f and ϕ are so in their respective domains. Moreover, we see that $|g| \leq 1$ on ∂U . By using the assumed growth restriction on f , we have that

$$|g(z)| \leq \exp \left(C \exp \left(\frac{\alpha\pi}{b-a} \left(\frac{b-a}{\pi} |z| + \frac{|a+b|}{2} \right) \right) \right) = \exp (D \exp (\alpha |z|)),$$

where $D = C \exp (|a+b|/2)$. Thus, we can apply Theorem 2.2.1 to g , which gives the desired conclusion. $\square$

We now consider the right half plane $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ [7, Exercise 12.1]. The proof is provided by the author.

Theorem 2.2.4 (Phragmén-Lindelöf for half planes). *Let $U = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ be the right half plane, and let f be a function that is holomorphic in U and continuous up to the boundary of U . Suppose that there exist real constants $C > 0$ and $0 < \alpha < 1$ such that $|f(z)| \leq C \exp (|z|^\alpha)$ for all z in U . If $|f| \leq 1$ on the imaginary axis, then $|f| \leq 1$ in U .*

The bound on α is sharp. This is for instance seen if we let $g(z) = e^z$. Then we have that $|g| \leq 1$ on the imaginary axis, but g is not bounded in U . We turn to proving the theorem.

Proof. Let β be a constant such that $\alpha < \beta < 1$, and let $\varepsilon > 0$. Define the function $h_\varepsilon(z) = \exp (-\varepsilon z^\beta)$ and set $g_\varepsilon = fh_\varepsilon$. Note that g_ε is holomorphic in U and continuous in $\overline{U}$, since f and h_ε are.

Firstly, we show that $|g_\varepsilon| \leq 1$ on the imaginary axis. Indeed, if we let y be a real number, then we have that

$$|g_\varepsilon(iy)| \leq \left| \exp \left(-\varepsilon (iy)^\beta \right) \right|,$$

since $|f(iy)| \leq 1$ by assumption. Writing $(iy)^\beta = |y|^\beta (\cos (\pm\pi\beta/2) + i \sin (\pm\pi\beta/2))$ yields that

$$|g_\varepsilon(iy)| \leq \exp \left(-\varepsilon |y|^\beta \cos \left(\pm \frac{\pi}{2} \beta \right) \right) \leq 1,$$

where the last inequality follows from the fact that the exponent is non-positive, since $0 < \beta < 1$.

Now, we want to find a half circle $C_R = \{Re^{it} \in \mathbb{C} : -\pi/2 < t < \pi/2\}$ with radius R so that $|g_\varepsilon| \leq 1$ on C_R and $|g_\varepsilon(z)| \leq 1$ in U for $|z| > R$. We see that

$$\begin{aligned} |g_\varepsilon(Re^{it})| &= |f(Re^{it})| |h_\varepsilon(Re^{it})| \\ &\leq C \exp(R^\alpha) \cdot \exp\left(-\varepsilon R^\beta \cos\left(\frac{\pi}{2}\beta\right)\right), \end{aligned}$$

by using the growth assumption on f in the last step. The last expression can be rewritten as

$$C \exp\left(-R^\beta\left(\varepsilon \cos(\beta\pi/2) - R^{\alpha-\beta}\right)\right).$$

Since $\alpha < \beta$, it follows that $R^{\alpha-\beta} \rightarrow 0$ as $R \rightarrow \infty$, and hence the exponent is non-positive for large enough R . Thus, $|g_\varepsilon(Re^{it})| \leq 1$ for sufficiently large R , so we can choose an R so that $|g_\varepsilon(z)| \leq 1$ on C_R and $|g_\varepsilon(z)| \leq 1$ for $|z| > R$.

Consider the region enclosed by the imaginary axis and C_R . As shown above, $|g_\varepsilon| \leq 1$ on the boundary of this region. The maximum modulus principle gives that $|g_\varepsilon| \leq 1$ in the interior of the region as well. Since we also showed that $|g_\varepsilon| \leq 1$ outside this region, we can conclude that $|g_\varepsilon| \leq 1$ in U .

Lastly, since ε was arbitrary, the above holds for each $\varepsilon > 0$. By letting $\varepsilon \rightarrow 0$ and noting that $|h_\varepsilon| \rightarrow 1$ as $\varepsilon \rightarrow 0$, we conclude that $|f| \leq 1$ in U , as desired. $\square$

The final domain we shall consider is the sector. A sector S in the complex plane that is centred at the positive real axis with angle 2α is defined as

$$S = \{z \in \mathbb{C} : |\arg z| < \alpha\},$$

where $0 < \alpha < \pi$. While it is possible to perform an argument similar to that of the proof of Theorem 2.2.4 to apply the Phragmén-Lindelöf principle in a sector, we instead note that we can regard the right half plane as a sector with $\alpha = \pi/2$. Using this fact, we can quite easily use the above result to get a Phragmén-Lindelöf result for sectors arbitrarily close to the full plane [7, Exercise 12.2]. The proof is provided by the author.

Theorem 2.2.5 (Phragmén-Lindelöf for sectors). *Let $S = \{z \in \mathbb{C} : |\arg z| < \pi/2\beta\}$ be a sector with opening angle π/β , where $\beta > 1/2$. Let f be a holomorphic function in S that is continuous in $\bar{S}$. Assume that there exist $C > 0$ and $0 < \alpha < \beta$ such that $|f(z)| \leq C \exp(|z|^\alpha)$ for all z in U . If $|f| \leq 1$ on the boundary of S , then $|f| \leq 1$ in U .*

An example illustration of the domain can be seen in Figure 2.1.

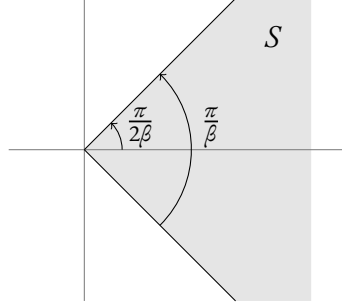


Figure 2.1: The sector considered in Theorem 2.2.5.

Proof. Let U denote the right half plane, as in Theorem 2.2.4. Consider the function $\phi(z) = z^{1/\beta}$, where we consider the principal branch¹ of $z^{1/\beta}$. ϕ is holomorphic in U and maps U to S bijectively.

Define $g = f \circ \phi$, which is holomorphic in U , since f and ϕ are holomorphic in their respective domains. Moreover, g is continuous in $\overline{U}$ by a similar argument.

We verify that g satisfies the conditions of Theorem 2.2.4. Let y be a real number and consider $|g(iy)|$. Since ϕ maps ∂U to ∂S and $|f| \leq 1$ on ∂S by assumption, we have that $|g(iy)| \leq 1$. Now, let $z \in U$. Using the growth assumption on f gives that

$$|g(z)| \leq C \exp(|\phi(z)|^\alpha) = C \exp(|z|^{\alpha/\beta}).$$

Note that $\alpha/\beta \in (0, 1)$, since $\alpha < \beta$. Therefore, we see that g satisfies the conditions of Theorem 2.2.4, so we can apply it. Doing so yields that $|g| \leq 1$ in U , and it then follows that $|f| \leq 1$ in S , which concludes the proof. $\square$

The theorem can be generalised to other rotated sectors. The following theorem and proof is provided by the author.

Theorem 2.2.6. *Let $S = \{z \in \mathbb{C} : \theta_0 < \arg z < \theta_1\}$ where $\theta_1 - \theta_0 = \pi/\beta$ for some $\beta > 1/2$. Let f be a holomorphic function in S that is continuous in $\overline{S}$. Assume that there exist $C > 0$ and $0 < \alpha < \beta$ such that $|f(z)| \leq C \exp(|z|^\alpha)$ for all z in U . If $|f| \leq 1$ on the boundary of S , then $|f| \leq 1$ in U .*

Proof. The function $\phi(z) = \exp(i(\pi/2\beta + \theta_0))z$ maps $\{z \in \mathbb{C} : |\arg z| < \pi/2\beta\}$ to S bijectively. Applying Theorem 2.2.5 to $f \circ \phi$ gives the desired conclusion. $\square$

¹This means that $z^{1/\beta} = \exp\left(\frac{1}{\beta}(\log|z| + i \arg z)\right)$, where $-\pi < \arg z < \pi$.

2.3 Hadamard Theorems

A version of the Phragmén-Lindelöf principle applied to strips is the *Hadamard three lines theorem*. In this case, we assume that a holomorphic function f is bounded in a vertical strip. The Hadamard three lines theorem can be viewed as a more refined version of the Phragmén-Lindelöf theorem (Theorem 2.2.3) with more explicit bounds. In particular, the supremum of $|f|$ taken over a vertical line in the domain is related to the supremums of $|f|$ taken over the sides of the strip, hence the "three lines". The theorem is proven using a Phragmén-Lindelöf technique, and will later be used when proving the Hausdorff-Young inequality (Theorem 3.1.1) and the Riesz-Thorin interpolation theorem (Theorem 4.1.1). We begin by proving a special case of the Hadamard three lines theorem, which we will then use to prove the general case.

Lemma 2.3.1. *Let $U = \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < 1\}$ be a vertical strip. Suppose that f is a holomorphic function in U that is continuous in $\overline{U}$, with $|f| \leq B$ in U , where B is a constant. Define $M(x) = \sup \{|f(x + iy)| : y \in \mathbb{R}\}$ where $0 \leq x \leq 1$. Then $M(x) \leq M(0)^{1-x} M(1)^x$ for all $0 \leq x \leq 1$.*

Proof. We first consider the case when $M(0) = M(1) = 1$. The desired inequality is equivalent to showing that $|f| \leq 1$ in U , and we will use a Phragmén-Lindelöf argument to do so. Let $\varepsilon > 0$ and define

$$h_\varepsilon(z) = \frac{1}{1 + \varepsilon z}$$

for $z \in U$. Note that $|h_\varepsilon| \leq 1$ in $\overline{U}$, since $\operatorname{Re}(1 + \varepsilon z) \geq 1$ for all $z \in \overline{U}$. Thus, $|fh_\varepsilon| \leq 1$ on ∂U . Now, we want to find a rectangle R enclosed by the sides of the strip and two horizontal lines for which $|fh_\varepsilon| \leq 1$ on ∂R and $|fh_\varepsilon| \leq 1$ in $U \setminus R$. By noting that $|1 + \varepsilon z| \geq \varepsilon |y|$, where $z = x + iy \in U$, we see that

$$|f(z)h_\varepsilon(z)| \leq \frac{B}{\varepsilon |y|},$$

for all $z = x + iy \in U$. Therefore, we can let R be the rectangle enclosed by the sides of the strip and the horizontal lines determined by $y = \pm B/\varepsilon$. Then we have that $|fh_\varepsilon| \leq 1$ on ∂R and $|fh_\varepsilon| \leq 1$ in $U \setminus R$. Hence, $|fh_\varepsilon| \leq 1$ in R by the maximum modulus principle, and we conclude that $|fh_\varepsilon| \leq 1$ in U . Since ε was arbitrary, this holds for each $\varepsilon > 0$. By letting $\varepsilon \rightarrow 0$ and noting that $|h_\varepsilon| \rightarrow 1$ as $\varepsilon \rightarrow 0$, we finally conclude that $|f| \leq 1$ in U as desired.

We now consider the case when $M(0) > 0$ and $M(1) > 0$. Define the function $g(z) = M(0)^{1-z} M(1)^z$, which is defined since $M(0) > 0$ and $M(1) > 0$. Note that g is entire and has no zeroes and hence $1/g$ is bounded in $\overline{U}$. By simply using the

definition of g , we see that $|g(iy)| = M(0)$ and $|g(1 + iy)| = M(1)$. This allows us to apply the first case to f/g , and we conclude that $|f/g| \leq 1$ in U , which is equivalent to $|f(z)| \leq |g(z)|$. This gives the desired inequality.

Finally, we consider the case when $M(0) = 0$. In this case, the inequality is equivalent to $f(z) = 0$ for all $z \in U$. Let $\Omega^+ = \{z \in \mathbb{C} : 0 < \text{Im}(z) < 1\}$. The linear function $\phi(z) = -iz$ maps Ω^+ to U bijectively. Let $g = f \circ \phi$. Since f and ϕ are holomorphic and continuous in their respective domains, it follows that g is holomorphic in Ω^+ and continuous up to the boundary of Ω^+ . Since ϕ maps the real axis to the imaginary axis, we have that $g(x) = 0$, where x is a real number, by the assumption that $M(0) = 0$. Hence, we can apply the Schwarz reflection principle (Theorem 1.1.10) to get that there exists a function G that is holomorphic in $\Omega^+ \cup \mathbb{R} \cup \Omega^-$ such that $G(z) = g(z)$ for all $z \in \Omega^+$. Furthermore, $G(x) = 0$ for $x \in \mathbb{R}$ as well. By the uniqueness of analytic continuation (Theorem 1.1.9), we have that $G(z) = 0$ for all $z \in \Omega^+ \cup \mathbb{R} \cup \Omega^-$, and hence $g(z) = 0$ in Ω^+ . Finally, we conclude that this implies that $f(z) = 0$ for $z \in U$.

By symmetry, we can make a similar argument in the case when $M(1) = 0$, and so we conclude the proof. $\square$

The general theorem now follows quite easily from the special case.

Theorem 2.3.2 (The Hadamard three lines theorem). *Consider the strip defined by $U = \{z \in \mathbb{C} : a < \text{Re}(z) < b\}$, where $a < b$. Suppose that f is a holomorphic function in U that is continuous in $\overline{U}$ with $|f| \leq B$ in U , where B is a constant. For $a \leq x \leq b$, define $M(x) = \sup \{|f(x + iy)| : y \in \mathbb{R}\}$. Then $M(x)^{b-a} \leq M(a)^{b-x} M(b)^{x-a}$ for all $a \leq x \leq b$.*

Note that the inequality is equivalent to $\log M(r)$ being convex on (a, b) .

Proof. Let $U_{0,1} = \{z \in \mathbb{C} : 0 < \text{Re}(z) \leq 1\}$. Note that $\phi(z) = (b - a)z + a$ is a linear function that maps $U_{0,1}$ to U bijectively. We set $g = f \circ \phi$, which is holomorphic in $U_{0,1}$ and continuous in $\overline{U_{0,1}}$ since f and ϕ are so in their respective domains. By applying Lemma 2.3.1 to g , we immediately get the general conclusion with f in $U_{a,b}$ by rewriting the inequality using the definition of g . $\square$

A corollary of the Hadamard three lines theorem is the following result [7, Chapter 12, Theorem 6.5].

Corollary 2.3.3 (The Hadamard three circle theorem). *Suppose that f is a holomorphic function on the annulus $A(\alpha, \beta) = \{z \in \mathbb{C} : \alpha \leq |z| \leq \beta\}$. Let $M(r) = \sup_{|z|=r} |f(z)|$, for $\alpha \leq r \leq \beta$. Then*

$$\log M(r) \leq \frac{\log(\beta/r)}{\log(\beta/\alpha)} \log M(\alpha) + \frac{\log(r/\alpha)}{\log(\beta/\alpha)} \log M(\beta). \quad (2.3.4)$$

Similarly to the Hadamard three lines theorem, the inequality 2.3.4 is equivalent to $\log M(r)$ being a convex function of $\log r$.

Proof. Let $g(z) = f(e^z)$. We wish to apply Theorem 2.3.2 to g , as 2.3.4 then follows directly. We verify that the conditions of the theorem hold.

Firstly, let a and b be real numbers so that $e^a = \alpha$ and $e^b = \beta$ and consider the strip $U = \{z \in \mathbb{C} : a \leq \operatorname{Re}(z) \leq b\}$. Note that e^z maps U to $A(\alpha, \beta)$ bijectively. Thus, g is holomorphic in U . Moreover, f is bounded in $A(\alpha, \beta)$ by the maximum modulus principle. It then follows that g is bounded in U . Hence, we can apply the theorem to g and the desired inequality follows. $\square$

CHAPTER 3

Applications to Fourier Analysis

In this chapter, we discuss applications of the Phragmén-Lindelöf principle within Fourier analysis.

3.1 The Hausdorff-Young Inequality

Assume that $f \in L^1((0, 1))$, where $(0, 1)$ denotes the measure space on $[0, 1)$ with the classical Lebesgue measure dx . Consider its Fourier coefficients $\hat{f}(n)$. For any integer n , we have that

$$|\hat{f}(n)| \leq \int_0^1 |f(t)| |e^{-2\pi i n t}| dt \leq \int_0^1 |f(t)| dt,$$

where we used the definition of $\hat{f}(n)$ along with the triangle inequality in the first step. Thus, we see that $\|\hat{f}\|_\infty \leq \|f\|_1$.

Now, assume that $f \in L^2((0, 1))$. In this case, we have that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \sum_{n=-\infty}^{\infty} \left| \int_0^1 f(t) e^{-2\pi i n t} dt \right|^2 \leq \int_0^1 |f(t)|^2 dt,$$

where Bessel's inequality is used in the last step. This inequality is equivalently written as $\|\hat{f}\|_2 \leq \|f\|_2$.

This raises the question of whether it is possible to make a similar estimate when $1 < p < 2$. The Hausdorff-Young inequality shows that it is indeed possible. We shall see that this belongs to a larger family of complex interpolation results in operator theory. Here, "complex" refers to the use of complex methods. The proof is a modification of Rudin's general proof [1, Theorem 12.11] to suit this version of the theorem. Later in this section, we will discuss the general version of the theorem.

Theorem 3.1.1 (The Hausdorff-Young inequality). *Let $1 < p < 2$ and $q = p/(p-1)$. If $f \in L^p((0, 1))$, then $\|\hat{f}\|_q \leq \|f\|_p$.*

To prove the theorem, we first state and prove a lemma.

Lemma 3.1.2. *If Theorem 3.1.1 holds for simple complex functions f with $\|f\|_p = 1$, then it holds for any $f \in L^p((0, 1))$.*

Proof. Let $f \in L^p((0, 1))$. It follows from Theorem 1.3.5 that there exists a sequence of simple functions $\{f_j\}$ such that $\|f_j - f\|_p \rightarrow 0$ as $j \rightarrow \infty$.

Next, we aim to show that if we let $\hat{f}_j(n)$ denote the Fourier coefficients of f_j , then we have that $\hat{f}_j(n) \rightarrow \hat{f}(n)$ as $j \rightarrow \infty$ uniformly in n . By using the definition of the Fourier coefficients and the triangle inequality for integrals, we get that

$$\left| \hat{f}_j(n) - \hat{f}(n) \right| \leq \int_0^1 |(f_j(t) - f(t))e^{-2\pi i n t}| dt$$

for each n . Since $f_j - f \in L^p((0, 1))$ and $e^{-2\pi i n t} \in L^q((0, 1))$, we can apply Hölder's inequality (Theorem 1.3.4). Doing so, and noting that $\|e^{-2\pi i n t}\|_q = 1$, yields that

$$\left| \hat{f}_j(n) - \hat{f}(n) \right| \leq \|f_j - f\|_p.$$

Since the right-hand side is independent of n and $\|f_j - f\|_p \rightarrow 0$ as $j \rightarrow \infty$, we conclude that $\hat{f}_j(n) \rightarrow \hat{f}(n)$ as $j \rightarrow \infty$ uniformly in n .

Lastly, we show that $\|\hat{f}\|_q \leq \|f\|_p$. We can write that

$$\|\hat{f}\|_q^q = \sum_{n=-\infty}^{\infty} \liminf_{j \rightarrow \infty} \left| \hat{f}_j(n) \right|^q \leq \liminf_{j \rightarrow \infty} \sum_{n=-\infty}^{\infty} \left| \hat{f}_j(n) \right|^q,$$

where Fatou's lemma (Lemma 1.2.9) was used in the last step. Hence, we have that $\|\hat{f}\|_q \leq \liminf_{j \rightarrow \infty} \|\hat{f}_j\|_q$. It follows from the assumption that $\|\hat{f}_j\|_q \leq \|f_j\|_p$ holds for each f_j that

$$\|\hat{f}\|_q \leq \limsup_{j \rightarrow \infty} \|f_j\|_p \leq \limsup_{j \rightarrow \infty} \|f_j - f\|_p + \|f\|_p = \|f\|_p,$$

where we used that $\|f_j - f\|_p \rightarrow 0$ as $j \rightarrow \infty$ in the last step, which concludes the proof. $\square$

We now turn to proving Theorem 3.1.1.

Proof. Fix $1 < p < 2$. We consider the case when f is a simple complex function, i.e. that f takes a finite number of values in $\mathbb{C}$, with $\|f\|_p = 1$. Our goal is to show that $\|\hat{f}\|_q \leq 1$, as then we can apply the lemma. We fix an arbitrary positive integer N . Note that

$$\left(\sum_{n=-N}^N |\hat{f}(n)|^q \right)^{1/q} = \sup \left| \sum_{n=-N}^N b_n \hat{f}(n) \right|,$$

where the supremum is taken over the collection of sets $\{b_{-N}, b_{-N+1}, \dots, b_N\} \subset \mathbb{C}$ for which $\sum |b_n|^p = 1$. This follows from Lemma 1.3.6. Thus, we shall show that $\left| \sum b_n \hat{f}(n) \right| \leq 1$ for each set $\{b_{-N}, \dots, b_N\}$ with $\sum |b_n|^p = 1$.

Let $b_{-N}, \dots, b_N$ be complex numbers such that $\sum |b_n|^p = 1$. We define $F = |f|^p$ and $B_n = |b_n|^p$. Note that

$$\int_0^1 F(t) dt = 1, \quad (3.1.3)$$

since $\|f\|_p = 1$, and

$$\sum_{n=-N}^N B_n = 1, \quad (3.1.4)$$

by assumption. Moreover, there exists a function ϕ such that $f = F^{1/p} \phi$ and $|\phi| = 1$, which we see by rewriting $f(t) = |f(t)| e^{i \arg(f(t))}$. By a similar procedure, there exist complex numbers $\beta_{-N}, \dots, \beta_N$ such that $b_n = B_n^{1/p} \beta_n$ and $|\beta_n| = 1$ for each n . By using these equalities to rewrite b_n and $\hat{f}(n)$, we get that

$$\sum_{n=-N}^N b_n \hat{f}(n) = \sum_{n=-N}^N B_n^{1/p} \beta_n \int_0^1 (F(t))^{1/p} \phi(t) e^{-2\pi i n t} dt.$$

Therefore, we define the function

$$g(z) = \sum_{n=-N}^N B_n^{1/p} \beta_n \int_0^1 (F(t))^z \phi(t) e^{-2\pi i n t} dt.$$

The complex powers exist since $F \geq 0$ and $B_n \geq 0$, and thus g is entire as it is a linear combination of those powers. By using the properties that $|\beta_n| = 1$ and $|\phi| = 1$, we can see that g is bounded in any vertical strip $\{z \in \mathbb{C} : a < \operatorname{Re}(z) < b\}$, where a and b are finite real numbers.

Consider the strip $\{z \in \mathbb{C} : 1/2 < \operatorname{Re}(z) < 1\}$. We intend to use the Hadamard three lines theorem (Theorem 2.3.2) to show that $|g| \leq 1$ in the strip, and consequently $|g(1/p)| \leq 1$. To do so, we must verify that $|g| \leq 1$ on the boundary of the strip.

Firstly, we consider $|g(1/2 + iy)|$, where $y \in \mathbb{R}$. Define

$$b_n(y) = \int_0^1 (F(t))^{1/2} (F(t))^{iy} \phi(t) e^{-2\pi i n t} dt.$$

By using Bessel's inequality, and simplifying, we get that

$$\sum_{n=-N}^N |b_n(y)|^2 \leq 1. \quad (3.1.5)$$

Using the triangle inequality followed by the Cauchy-Schwarz inequality yields that

$$\left| g\left(\frac{1}{2} + iy\right) \right| \leq \sum_{n=-N}^N B_n^{1/2} |b_n(y)| \leq \left(\sum_{n=-N}^N B_n \right) \left(\sum_{n=-N}^N |b_n(y)|^2 \right)^{1/2} \leq 1,$$

where we used 3.1.4 and 3.1.5 in the last step.

To show that $|g(1 + iy)| \leq 1$, where $y \in \mathbb{R}$, is quite straightforward, as we simply use the triangle inequality, the properties that $|\beta_n| = 1$ and $|\phi| = 1$, as well as 3.1.3 and 3.1.4. Hence, we can apply the Hadamard three lines theorem to conclude that $|g(1/p)| \leq 1$, which gives the desired result. Since N was arbitrary, we conclude that the theorem holds for f .

Lastly, we can apply Lemma 3.1.2 to conclude the proof. $\square$

The Hausdorff-Young inequality can be generalised to a more general framework, where Theorem 3.1.1 is a special case. Let $(X, \mathcal{R}, \mu)$ be a measure space with a positive measure μ , and $\{\psi_n\}$ an orthonormal set of functions in $L^2(\mu)$, i.e.

$$\int_X \psi_n \overline{\psi_m} d\mu = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases},$$

for $n = 1, 2, 3, \dots$. Assume that $\{\psi_n\}$ is a bounded sequence in $L^\infty(\mu)$, i.e. there exists $M < \infty$ such that $|\psi_n| \leq M$ for all n . Let $f \in L^p(\mu)$, where $1 \leq p \leq 2$. We define $\hat{f}(n) = \int_X f \overline{\psi_n} d\mu$ for $n = 1, 2, 3, \dots$

Assume that $f \in L^1(\mu)$. Then it follows from the boundedness of $\{\psi_n\}$ that $\|\hat{f}\|_\infty \leq M \|f\|_1$.

We now consider $f \in L^2(\mu)$. Then we have that

$$\|\hat{f}\|_2 = \left(\sum_{n=1}^{\infty} |\hat{f}(n)|^2 \right)^{1/2} = \left(\sum_{n=1}^{\infty} \left| \int_X f \overline{\psi_n} d\mu \right|^2 \right)^{1/2}.$$

From this, Bessel's inequality gives that $\|\hat{f}\|_2 \leq \|f\|_2$

Analogous to Theorem 3.1.1, we wish interpolate p between 1 and 2.

Theorem 3.1.6 (General Hausdorff-Young inequality). *Under the above assumptions, suppose that $1 \leq p \leq 2$ and $q = p/(p-1)$. If $f \in L^p(\mu)$, then $\|\hat{f}\|_q \leq M^{(2-p)/p} \|f\|_p$.*

It turns out that the technicality in the generalisation mainly concerns notation, and the proof is identical to the proof of Theorem 3.1.1, in the sense that we only need to change the notation. Since the exact same argument can be made in this case, we omit the proof of the general theorem. We will outline a similar argument in the final chapter.

3.2 Hardy's Uncertainty Principle

Let f be a function such that $f(x) = O(\exp(-ax^2))$, with its Fourier transform $\hat{f}(\xi) = O(\exp(-b\xi^2))$, where a and b are positive real constants. Define the function $f_\delta(x) = f(\delta x)$, where $\delta > 0$. Then we have that $f_\delta(x) = O(\exp(-a\delta^2 x^2))$. Through the definition of the Fourier transform and a change of variables, we see that the Fourier transform of f_δ satisfies $\hat{f}_\delta(\xi) = O(\exp(-b\xi^2/\delta^2))$. Thus, the dilation of f by δ illustrates that if we try to make f decay faster, then its Fourier transform must decay slower. This suggests the existence of some kind of uncertainty principle [8, Chapter 1]. Indeed, Hardy's uncertainty principle deals with this setting. We begin by proving a special case of the theorem, which we will then use to prove the general case.

Theorem 3.2.1. *Let f be a measurable function with the property that $|f(x)| \leq e^{-\pi x^2}$ and $|\hat{f}(\xi)| \leq e^{-\pi \xi^2}$ for all real x and ξ . Then $f(x) = Ce^{-\pi x^2}$, where C is some real constant.*

The proof is based on Terence Tao's proof given in a blog post about the theorem [9], which we aim to formalise here. The proof will be divided into a number of lemmas.

Lemma 3.2.2. *Under the assumptions of the theorem, the Fourier transform $\hat{f}$ can be extended to the complex plane by*

$$\hat{f}(\xi + i\eta) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} e^{2\pi \eta x} dx,$$

where ξ and η are real numbers. Furthermore, this extension of $\hat{f}$ is entire.

Proof. We aim to show that the integral converges. This is done by showing that

$$\int_{\mathbb{R}} |f(x)| e^{2\pi \eta x} dx < \infty.$$

By using the growth bound on f , we have that

$$\int_{\mathbb{R}} |f(x)| e^{2\pi\eta x} dx \leq \int_{\mathbb{R}} e^{-\pi(x^2-2\eta x)} dx = e^{\pi\eta^2} \int_{\mathbb{R}} e^{\pi(x-\eta)^2} dx,$$

where we completed the square in the last step. The integrand is now a Gaussian, so the integral converges.

Secondly, to show that $\hat{f}$ is entire, we will use Morera's theorem (Theorem 1.1.8). Let $T \subset \mathbb{C}$ be a closed triangle. For convenience, we will now set $z = \xi + i\eta$. We shall show that $\int_{\partial T} \hat{f}(z) dz = 0$. Firstly, it is trivial that for each real x , we have that $f(x)e^{-2\pi izx}$ is entire. By using the definition of $\hat{f}$, we get that

$$\begin{aligned} \int_{\partial T} \hat{f}(z) dz &= \int_{\partial T} \left(\int_{\mathbb{R}} f(x) e^{-2\pi izx} dx \right) dz \\ &= \int_{\mathbb{R}} \left(\int_{\partial T} f(x) e^{-2\pi izx} dz \right) dx. \end{aligned}$$

The change of order of integration is allowed by Fubini's theorem (see for instance [1, Theorem 7.8]). The innermost integral is equal to 0 by Cauchy's theorem (Theorem 1.1.7). Hence, we have that $\int_{\partial T} \hat{f}(z) dz = 0$. Since T was arbitrary, we can apply Morera's theorem and conclude that $\hat{f}$ is entire. $\square$

Lemma 3.2.3. *For every real ξ and η , we have that $|\hat{f}(\xi + i\eta)| \leq e^{\pi\eta^2}$.*

Proof. By using the definition of $\hat{f}$ and the triangle inequality for integrals, we get that

$$|\hat{f}(\xi + i\eta)| \leq \int_{\mathbb{R}} |f(x)| e^{2\pi\eta x} dx.$$

Using the assumed bound on f yields that

$$|\hat{f}(\xi + i\eta)| \leq \int_{\mathbb{R}} e^{-\pi x^2} e^{2\pi\eta x} dx,$$

and by completing the square and simplifying, we have that

$$|\hat{f}(\xi + i\eta)| \leq e^{\pi\eta^2} \int_{\mathbb{R}} e^{\pi(x-\eta)^2} dx.$$

Performing a change of variables and using the fact that $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$ evaluates the above integral to 1, which gives the desired inequality. $\square$

Lemma 3.2.4. *Let $F(z) = e^{\pi z^2} \hat{f}(z)$ and define $F_\delta(z) = e^{i\delta z^2} F(z)$, for $\delta > 0$. Consider the sector $S_\theta = \{z \in \mathbb{C} : 0 < \arg z < \theta\}$, where $0 < \theta < \pi/2$. If θ is sufficiently close to $\pi/2$, depending on δ , then $|F_\delta| \leq 1$ in S_θ .*

Remark 3.2.5. If $\delta > 0$, then θ can for instance be chosen so that the inequality $-\delta \sin 2\theta + \pi \cos^2 \theta < 0$ holds. This fact will be motivated in the proof. Note that this means that if δ is small, then θ must be close to $\pi/2$.

Proof. Using Lemma 3.2.3 immediately gives that $|F| \leq 1$ on the real axis. By using the same lemma and simplifying, we get that

$$|F(\xi + i\eta)| \leq e^{\pi(\xi+i\eta)^2} e^{\pi\eta^2} = e^{\pi\xi^2}, \quad (3.2.6)$$

for all real ξ and η . Let $S_\theta = \{z \in \mathbb{C} : 0 < \arg z < \theta\}$, where $0 < \theta < \pi/2$.

Let $\delta > 0$ and consider the function $F_\delta(z)$. We aim to show that if θ is sufficiently close to $\pi/2$, then $|F_\delta(z)| \leq 1$ on the boundary of S_θ . Note that a point z in $\overline{S_\theta}$ can be written as $z = re^{it} = r \cos t + ir \sin t$, where $r \geq 0$ and $0 \leq t \leq \theta$. If $r = 0$, then the bound holds trivially. Let us now consider points on the boundary, where we let $r > 0$ be arbitrary. If $t = 0$, then the bound follows from the fact that $|F| \leq 1$ on the real axis. If $t = \theta$, then by simplifying we have that

$$\left| \exp \left(i\delta r^2 e^{2i\theta} \right) \right| = \left| \exp \left(\delta r^2 (i \cos 2\theta - \sin 2\theta) \right) \right| = \exp \left(-\delta r^2 \sin 2\theta \right). \quad (3.2.7)$$

By using 3.2.6, we have that

$$|F(r \cos \theta + ir \sin \theta)| \leq \exp \left(\pi r^2 \cos^2 \theta \right). \quad (3.2.8)$$

Multiplying 3.2.7 and 3.2.8 yields that

$$|F_\delta(r \cos \theta + ir \sin \theta)| \leq \exp \left(r^2 (-\delta \sin 2\theta + \pi \cos^2 \theta) \right).$$

If θ is sufficiently close to $\pi/2$, then $-\delta \sin 2\theta + \pi \cos^2 \theta < 0$, and therefore

$$|F_\delta| \leq 1 \quad (3.2.9)$$

on the boundary of S_θ .

Now we are in a position to introduce the auxiliary function which we want to use to apply the Phragmén-Lindelöf principle. For $\varepsilon > 0$, define $h_\varepsilon(z) = \exp(i\varepsilon e^{i\varepsilon} z^{2+\varepsilon})$, where we use the principal branch of $z^{2+\varepsilon}$. We show that $|h_\varepsilon F_\delta| \leq 1$ on the boundary of S_θ , for sufficiently small ε . By using 3.2.9, we have that

$$|h_\varepsilon(z) F_\delta(z)| \leq \left| \exp \left(i\varepsilon e^{i\varepsilon} z^{2+\varepsilon} \right) \right|, \quad (3.2.10)$$

for any point z on ∂S_θ . As before, a point in $\overline{S_\theta}$ can be written on the form $z = re^{it}$, where $r \geq 0$ and $0 \leq t \leq \theta$. Again, if $r = 0$, then the bound is trivial. Thus, we let $r > 0$ be arbitrary and first consider the case when $t = 0$. Note that

$$|\exp(i\varepsilon e^{i\varepsilon} z^{2+\varepsilon})| = |\exp(i\varepsilon \cos(\varepsilon) z^{2+\varepsilon})| |\exp(-\varepsilon \sin(\varepsilon) z^{2+\varepsilon})|.$$

Since r is a positive real number, it follows immediately that

$$|\exp(i\varepsilon \cos(\varepsilon) r^{2+\varepsilon})| |\exp(-\varepsilon \sin(\varepsilon) r^{2+\varepsilon})| \leq 1$$

Now, let $t = \theta$. Inserting $z = re^{i\theta}$ in 3.2.10 gives that

$$\begin{aligned} \left| \exp\left(i\varepsilon e^{i\varepsilon} r^{2+\varepsilon} e^{i(2+\varepsilon)\theta}\right) \right| &= \left| \exp\left(i\varepsilon r^{2+\varepsilon} e^{i(2\theta+\varepsilon\theta+\varepsilon)}\right) \right| \\ &= \exp(-\varepsilon r^{2+\varepsilon} \sin(2\theta+\varepsilon\theta+\varepsilon)), \end{aligned}$$

where we wrote $\exp(i(2\theta+\varepsilon\theta+\varepsilon)) = \cos(2\theta+\varepsilon\theta+\varepsilon) + i \sin(2\theta+\varepsilon\theta+\varepsilon)$ in the last step. As $0 < \theta < \pi/2$, we have that $0 < 2\theta+\varepsilon\theta+\varepsilon < \pi$ for sufficiently small ε , and thus $\exp(-\varepsilon r^{2+\varepsilon} \sin(2\theta+\varepsilon\theta+\varepsilon)) \leq 1$ as desired.

Lastly, to conclude a Phragmén-Lindelöf argument, we need to find a bounded subset of S_θ for which $|h_\varepsilon F_\delta| \leq 1$ on the boundary and outside of the region. To do so, we show that $|h_\varepsilon F_\delta| \rightarrow 0$ as $|z| \rightarrow \infty$ in S_θ . Then we can find a circular sector that fulfils the requirements.

Let re^{it} , where $0 < t < \theta$ be a point in S_θ . Using techniques similar to the above, we can write

$$\left| \exp\left(i\varepsilon e^{i\varepsilon} r^{2+\varepsilon} e^{i(2+\varepsilon)t}\right) \right| = \exp(-\varepsilon r^{2+\varepsilon} \sin(\varepsilon+2t+\varepsilon t)). \quad (3.2.11)$$

Similarly, we have that

$$|\exp(i\delta r^2 e^{i2t})| = \exp(-\delta r^2 \sin 2t). \quad (3.2.12)$$

Using 3.2.6, we have that

$$|F(re^{it})| \leq \exp(\pi r^2 \cos^2 t). \quad (3.2.13)$$

By multiplying 3.2.11, 3.2.12 and 3.2.13, we get that

$$|h_\varepsilon F_\delta| \leq \exp(r^2(-\varepsilon r^\varepsilon \sin(\varepsilon+2t+\varepsilon t) - \delta \sin 2t + \pi \cos^2 t)).$$

Again, $0 < 2\theta+\varepsilon\theta+\varepsilon < \pi$ for sufficiently small ε . As r^ε eventually will be the dominant term within the parentheses, we get that $|h_\varepsilon F_\delta| \rightarrow 0$ as $r \rightarrow \infty$. Thus, we can find a bounded region for which $|h_\varepsilon F_\delta| \leq 1$ on the boundary and outside the region.

Finally, we can apply a Phragmén-Lindelöf argument to conclude that $|h_\varepsilon F_\delta| \leq 1$ in S_θ . By letting $\varepsilon \rightarrow 0$, we get that $|F_\delta| \leq 1$ in S_θ . $\square$

We can now prove Theorem 3.2.1.

Proof. By Lemma 3.2.2, we can extend $\hat{f}$ to the complex plane by

$$\hat{f}(\xi + i\eta) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} e^{2\pi \eta x} dx,$$

where ξ and η are real numbers.

Define $F(z) = e^{\pi z^2} \hat{f}(z)$. We aim to show that F is bounded in $\mathbb{C}$. As shown in the proof of Lemma 3.2.4, we have that $|F|$ is bounded by 1 on the real axis. By using Lemma 3.2.3, it is also immediate that $|F|$ is bounded by 1 on the imaginary axis.

Let us now consider the upper right quadrant $\Gamma = \{z \in \mathbb{C} : 0 < \arg z < \pi/2\}$. This is a sector for which $|F| \leq 1$ on the boundary. However, we cannot directly use the previously proven Phragmén-Lindelöf principle applied to a sector (Theorem 2.2.6) as it assumes less than quadratic-exponential growth in this domain, while F has quadratic-exponential growth. Still, we can show that $|F| \leq 1$ in the quadrant by introducing a suitable auxiliary function and employing a Phragmén-Lindelöf argument.

Define $F_{\delta}(z)$ and S_{δ} as in Lemma 3.2.4. The lemma shows that F_{δ} is bounded in magnitude by 1 in S_{δ} , where θ is sufficiently close to $\pi/2$, which depends on δ . We will use this to show that $|F| \leq 1$ in Γ .

Since δ was arbitrary, we have that for each $\delta > 0$, there exists a $\theta_{\delta} \in (0, \pi/2)$ so that $|F_{\delta}| \leq 1$ in $S_{\theta_{\delta}}$. Let $\{\delta_n\}$ be a sequence such that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. Then by Remark 3.2.5, we have that $\{\theta_n\}$ defined as $\theta_n = \theta_{\delta_n}$ for each n is a sequence with $\theta_n \rightarrow \pi/2$ as $n \rightarrow \infty$. By assumption, we have that $|F_{\delta_n}| \leq 1$ on S_{θ_n} .

Let $K \subset \Gamma$ be compact. Since $\theta_n \rightarrow \pi/2$, there exists an N such that $K \subset S_{\theta_n}$ for $n > N$. For such n , we have that

$$|F(z)| = \left| e^{-i\delta_n z^2} F_{\delta_n}(z) \right|.$$

By writing $z = r \cos t + ir \sin t$ and using that $|F_{\delta_n}| \leq 1$ on S_{θ_n} , we get that

$$|F(z)| \leq \exp(\delta_n r^2 \sin^2 t)$$

for all $n > N$. Note that $\exp(\delta_n r^2 \sin^2 t) \rightarrow 1$ uniformly on K as $n \rightarrow \infty$. Since K was arbitrary, we can conclude that $\exp(\delta_n r^2 \sin^2 t)$ converges uniformly to 1 as $n \rightarrow \infty$. This implies that $|F| \leq 1$ in each compact subset of Γ .

Note that Γ can be covered by an increasing sequence of compact subsets defined $K_1 \subset K_2 \subset \dots$ with $\cup_{j=1}^{\infty} K_j = \Gamma$. An example of such sequence is the sequence defined by the subsets $K_j = \{z \in \mathbb{C} : 1/j \leq \arg z \leq \pi/2 - 1/j\}$. On each of these subsets, we have that $|F| \leq 1$. Since the K_j exhaust Γ , this means that $|F| \leq 1$ in Γ , as desired.

Since $|F| \leq 1$ on the real and imaginary axes, a similar argument can be made in the other quadrants, so $|F| \leq 1$ in $\mathbb{C}$. Since F is bounded and entire, it is constant by Liouville's theorem (Theorem 1.1.4). Thus, we have that $\hat{f}(\xi) = Ce^{-\pi\xi^2}$ for all $\xi \in \mathbb{R}$, where C is some constant. Finally, applying Lemma 1.4.3 to calculate the inverse Fourier transform of $\hat{f}$ gives the desired conclusion. $\square$

The general theorem now follows directly.

Theorem 3.2.14 (Hardy's uncertainty principle). *Let f be a measurable function such that $|f(x)| \leq C_1 e^{-\pi\alpha x^2}$ and $|\hat{f}(\xi)| \leq C_2 e^{-\pi\xi^2/\alpha}$ for all real x and ξ , where C_1, C_2 and α are positive constants. Then $f(x) = De^{-\pi\alpha x^2}$, where D is some real constant.*

Proof. Without loss of generality, the constants α, C_1 and C_2 can be normalised to 1. By contracting f by $1/\sqrt{\alpha}$, and thus dilating $\hat{f}$ by $\sqrt{\alpha}$, we see that we can normalise α to 1. By multiplying f by a small constant, we may also normalise C_1 and C_2 to 1. Thus, it is sufficient to prove the theorem for these normalised constants, which is exactly Theorem 3.2.1. $\square$

CHAPTER 4

An Application to Operator Theory

In this chapter, we shall prove an application of the Phragmén-Lindelöf principle within operator theory.

4.1 The Riesz-Thorin Interpolation Theorem

In this section, we will prove the Riesz-Thorin interpolation theorem, which is a result about interpolation of operators. The result is also known as the Riesz-Thorin convexity theorem. This section is based on Stein's chapter about the theorem [5, Chapter 5].

Theorem 4.1.1 (The Riesz-Thorin interpolation theorem). *Let $(M, \mathcal{M}, \mu)$ be a measure space and let T be a linear operator mapping the linear space of all simple measurable functions on $(M, \mathcal{M}, \mu)$ into measurable functions on another measure space $(N, \mathcal{N}, \nu)$. Suppose that $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Assume that T is of type (p_i, q_i) with (p_i, q_i) -norm k_i , where $i = 0, 1$. For $0 \leq t \leq 1$, define*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \text{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Then T is of type (p_t, q_t) with (p_t, q_t) -norm $k_t \leq k_0^{1-t} k_1^t$.

Note that if p_0, p_1, q_0 or q_1 is ∞ , then we use the convention that $1/\infty = 0$. It follows from Theorem 1.5.2 that any linear operator T for which the theorem holds can be uniquely extended to all of $L^{p_t}(\mu)$ so that the estimate of the (p_t, q_t) -norm still holds.

The proof of this theorem is very similar to the proof of the Hausdorff-Young inequality (Theorem 3.1.1).

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Proof. Fix $0 < t < 1$. Let $\alpha_j = 1/p_j$ and $\beta_j = 1/q_j$ for $j = 0, 1$, and let $\alpha = 1/p_t$ and $\beta = 1/q_t$. Define $\alpha(z) = (1-z)\alpha_0 + z\alpha_1$ and $\beta(z) = (1-z)\beta_0 + z\beta_1$. Note that $\alpha(j) = \alpha_j$ and $\beta(j) = \beta_j$ for $j = 0, 1$, and $\alpha(t) = \alpha$ and $\beta(t) = \beta$.

We will first consider the case when $\alpha > 0$ and $\beta < 1$. We estimate $\|Tf\|_{q_t}$, where f is a simple function. By Lemma 1.3.6, we have that

$$\|Tf\|_{q_t} = \sup \left| \int_N (Tf)g d\nu \right|,$$

where the supremum is taken over all simple functions g with $\|g\|_{q'_t} = 1$, where we define $q'_t = q_t/(q_t - 1)$. If we let I_g denote the above integral for a fixed such g , then we want to show that $|I_g| \leq k_0^{1-t} k_1^t \|f\|_{p_t}$ independent of g as above. By dividing by $\|f\|_{p_t}$ if it is non-zero, we may assume, without loss of generality, that $\|f\|_{p_t} = 1$.

Assume that $f = \sum_{j=1}^m a_j \chi_{E_j}$ and $g = \sum_{k=1}^n b_k \chi_{F_k}$ are simple functions satisfying the above. Write $a_j = |a_j| e^{i\theta_j}$ and $b_k = |b_k| e^{i\phi_k}$. From this, define

$$f_z = \sum_{j=1}^m |a_j|^{\frac{\alpha(z)}{\alpha}} e^{i\theta_j} \chi_{E_j} \quad \text{and} \quad g_z = \sum_{k=1}^n |b_k|^{\frac{1-\beta(z)}{1-\beta}} e^{i\phi_k} \chi_{F_k}$$

for $z \in \mathbb{C}$. Division by zero in the exponents is avoided since $\alpha > 0$ and $\beta < 1$. Let

$$F(z) = \int_N (Tf_z)g_z d\nu,$$

which is entire since each complex power is defined. Note that it follows from the definitions of $\alpha(z)$ and $\beta(z)$ that $F(t) = I_g$.

Consider the strip $\Omega = \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < 1\}$. By using the definitions of f_z and g_z as well as the linearity of T , we can rewrite

$$F(z) = \sum_{j=1}^m \sum_{k=1}^n |a_j|^{\frac{\alpha(z)}{\alpha}} |b_k|^{\frac{1-\beta(z)}{1-\beta}} e^{i(\theta_j + \phi_k)} \int_N (T\chi_{E_j}) \chi_{F_k} d\nu.$$

Since each $|a_j|$ and $|b_k|$ is positive, we have that each summand is bounded in Ω and thus F is bounded in Ω . Our goal is to apply the Hadamard three lines theorem (Theorem 2.3.2). We shall show that $|F(iy)| \leq k_0$ and $|F(1+iy)| \leq k_1$, for each real y , as the desired inequality then follows directly.

Let $y \in \mathbb{R}$. We show that $|F(iy)| \leq k_0$. Note that $\alpha(iy) = \alpha_0 + iy(\alpha_1 - \alpha_0)$ and $\beta(iy) = 1 - \beta_0 - iy(\beta_1 - \beta_0)$. Moreover, we can write

$$f_{iy} = |f|^{\frac{\alpha(iy)}{\alpha}} e^{i \arg f_{iy}} \quad \text{and} \quad g_{iy} = |g|^{\frac{1-\beta(iy)}{1-\beta}} e^{i \arg g_{iy}}.$$

By using this and simplifying, we get that

$$|f_{iy}|^{p_0} = |f|^{p_t} \quad \text{and} \quad |g_{iy}|^{q'_0} = |g|^{q'_t}. \quad (4.1.2)$$

By using Hölder's inequality (Theorem 1.3.4), we get that

$$|F(iy)| \leq \int_n |(Tf_{iy})g_{iy}| d\nu \leq \|Tf_{iy}\|_{q_0} \|g_{iy}\|_{q'_0}.$$

Since T is of type (p_0, q_0) , we have that

$$|F(iy)| \leq k_0 \|f_{iy}\|_{p_0} \|g_{iy}\|_{q'_0}.$$

These norms can then be rewritten to p_t - and q_t -norms respectively, using 4.1.2. Then as desired, we have that

$$|F(iy)| \leq k_0 \|f\|_{p_t}^{p_t/p_0} \|g\|_{q'_t}^{q'_t/q'_0} = k_0,$$

where the last step follows from the fact that $\|f\|_{p_t} = 1$ and $\|g\|_{q'_t} = 1$.

A similar procedure can be used to show that $|F(1+iy)| \leq k_1$. Applying the Hadamard three lines theorem immediately gives the desired inequality, which proves the theorem for $\alpha > 0$ and $\beta < 1$.

To conclude the proof, we must consider the other possible values of α and β . If $\alpha = 0$ and $\beta = 1$, then at least one of (p_0, q_0) and (p_1, q_1) is $(\infty, 1)$, and the inequality is trivial. If $\alpha = 0$ and $\beta < 1$, then the above proof is still valid by instead defining $f_z = f$ for all $z \in \mathbb{C}$. Similarly, if $\alpha > 0$ and $\beta = 1$, then the above still holds if we define $g_z = g$ for all $z \in \mathbb{C}$. $\square$

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