

AN INTRODUCTION TO QUASI-CONFORMAL MAPPINGS

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Abstract

This thesis introduces the theory of quasiconformal mappings on the plane. We give three different definitions of quasiconformal mappings: Differentiable quasiconformal map and two geometric definitions, one based on boundedness of the modulus of quadrilaterals and the other on ring domains. We also showcase when these definitions are equivalent. Then we prove that the geometric definition for quasiconformality is a local behaviour. Additionally we include some extension theorems for quasiconformal mappings, and end by proving Mori's Theorem.

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List of symbols

Symbol	Meaning	Page
Ω	Jordan domain	7
$\phi_z, \phi_{\bar{z}}$	Complex derivatives of ϕ	8
J_ϕ	Jacobian determinant of ϕ	9
$\mu(z)$	Complex dilatation	10
D_ϕ	Dilatation of ϕ	11
ϕ	Quasiconformal mapping (local)	11,22,(36)
Γ	Family of curves	15
$l(\gamma)$	Length of γ	16
$m^n(E)$	n -dimensional Lebesgue measure of E	
$L(\Gamma, \rho)$	Infimum of the ρ -weighted lengths for the curve family Γ	17
$\lambda(\Gamma)$	Extremal length of the curve family Γ	17
Γ^+, Γ^-	Upper and lower families of curves	19
Q	Quadrilateral	21
R_Q	Canonical rectangle associated to the quadrilateral Q	22
π_Q	Canonical Conformal map from Q to R_Q	22
$\text{mod}(Q)$ or $M(Q)$	Modulus of quadrilateral Q	22
$R(a, b, c, d)$	Rectangle with vertices $(a, b, c, d) \in \mathbb{C}^4$	22
$k(z_1, z_2)$	Spherical distance	25
R	Ring domain or Doubly connected region	28
R'	Canonical annulus associated with Ring domain R	28
π_R	Canonical conformal map from R to R'	28
$\Phi(r), \Psi(p), \Xi(\lambda)$	Modules for the extremal problems, Grötzsch, Teichmüller and Mori	44
$K(\Omega)$	Maximal dilation Ω under ϕ	45

1 Introduction

The purpose of this thesis, is to introduce the reader to some fundamental concepts in the theory of planar quasiconformal mappings. Three key books serve as primary references for this subject: [Ahlfors (1966)], [Lehto & Virtanen (1973)], and [Fletcher & Markovic (2007)]. While these works focus on different aspects of the theory in their later chapters, they share a somewhat common starting point.

Conformal mapping theory has been of great importance in both mathematics and in its applications in physics and engineering. However, its limitations were recognized early on. For instance, when humans attempted to map the Earth onto a plane, which famously cannot preserve both country areas and geographical angles simultaneously. The first introduction to quasiconformal mappings was by H. Grötzsch in 1928 when he posed the problem of finding a function that maps one rectangle onto another while preserving vertex correspondences. Since a conformal map cannot always achieve this, he sought a mapping that is as close to being conformal as possible, in the sense of minimizing the supremum of the dilation at any point in the rectangle. Later, both Lars Ahlfors and Oswald Teichmüller made significant contributions to the development of the theory.

By the Riemann mapping theorem, any open simply connected plane domain can be mapped conformally onto the open unit disk. However, Liouville's theorem implies that the only conformal mappings in dimensions $n \geq 3$ are Möbius transformations, so there is no analogue of the Riemann mapping theorem in higher dimensions. This limitation, along with the ability to control the dilation imposed by a conformal mapping, led to the need for a more general class of functions, quasiconformal mappings. There are also generalizations of quasiconformal mappings, such as quasiregular and quasisymmetric mappings, but these will not be studied in this thesis.

Ahlfors' style of writing leaves many details for the reader to work out independently. During the academic year 2022, my advisor, Magnus Aspenberg, conducted a class for two PhD students, Olof Rubin and Mats Bylund. They decided to compile notes based on Ahlfors' book to fill in some of these gaps. These notes existed only in paper form. Initially, the goal of this thesis was to transcribe those notes into $\text{\LaTeX}$, but the project grew into a more extensive task—providing additional context, proving all necessary statements, and structuring the material cohesively. This thesis does not claim originality beyond its presentation.

The thesis begins with the study of differentiable quasiconformal mappings in Chapter 2, focusing on the dilation of space under the mapping. This requires functions to be continuously differentiable. Using this framework, we introduce Grötzsch's problem and provide a solution.

In the next chapter, we aim to remove the previous differentiability condition and instead study the maps which bounds the modulus of quadrilaterals or ring domains. There are two ways to define the modulus of a quadrilateral: one can either express it as the ratio of the side lengths (radii) of its conformally equivalent rectangle (annuli) or define it purely in terms of extremal length. In the planar case, these two definitions are equivalent. This chapter also explores the connections between the initial geometric objects studied, as established in Lemma 3.24. We prove that quasiconformality is a local property and conclude the chapter by

proving several extension theorems analogous to those for conformal maps and Mori's Theorem.

This project would not have been possible if not for the multiple insights and perspectives I gained from the conversations I had with my advisor Magnus Aspenberg. At the same time I would like to extend my gratitude to close friends and family for being there when I needed any support.

2 Differentiable quasiconformal mappings

This chapter comprises three sections. First, we define quasiconformality using a geometric approach, focusing on the idea of distortion. Geometrically, a map is quasiconformal if it transforms infinitesimal circles into infinitesimal ellipses. The distortion is measured by the ratio of the ellipse's major axis to its minor axis. A limitation of this definition is that it requires that functions be continuously differentiable, $\phi \in \mathcal{C}^1$, since derivatives are required to calculate the distortion. However, we will show that this condition can be removed in a later chapter.

In the next section we introduce the classical Grötzsch problem, which initially motivated the definition of quasiconformality. Given two rectangles R_1, R_2 , one tries to find some map between them which is "closest" to a conformal map. The last section proves results concerning the behaviour of quasiconformal mappings under composition.

2.1 Local behaviour for differentiable quasiconformal mappings

Since Jordan domains play an important role in this thesis, we begin by recalling the following definitions.

Definition 2.1 (Jordan curve). *A Jordan curve (or simple closed curve) in the complex plane is the image of a continuous injective map*

$$\gamma: S^1 \rightarrow \mathbb{C}.$$

With S^1 begin the unit circle in $\mathbb{C}$.

By the Jordan curve theorem, the complement of a Jordan curve in $\mathbb{C}$ consists of exactly two connected components.

Definition 2.2 (Jordan domain). *Let γ be a Jordan curve. The Jordan domain (or interior) determined by γ is the bounded connected component of the complement $\mathbb{C} \setminus \gamma(S^1)$.*

Unless stated otherwise, we let $\Omega \subset \mathbb{C}$ denote a Jordan domain. Let $\phi: \Omega \rightarrow \phi(\Omega)$ be a $\mathcal{C}^1$ -diffeomorphism, with $z = x + iy$ coordinate on Ω at z_0 and $w = u + iv$ coordinate on $\phi(\Omega)$ at $\phi(z_0)$. From complex analysis we can write ϕ as,

$$\phi(z) = u(z) + iv(z) \quad u, v \in \mathcal{C}^1$$

Fix $z = z_0$, then locally

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \underbrace{\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}}_A \begin{pmatrix} dx \\ dy \end{pmatrix} \quad (2.1)$$

We can view this as a local linear transformation $T: x \mapsto Ax$, which maps circles onto ellipses. We have

$$\begin{aligned} du^2 + dv^2 &= (u_x dx + u_y dy)^2 + (v_x dx + v_y dy)^2 \\ &= u_x^2 dx^2 + 2u_x u_y dx dy + u_y^2 dy^2 + v_x^2 dx^2 + 2v_x v_y dx dy + v_y^2 dy^2 \\ &= E dx^2 + 2F dx dy + G dy^2, \end{aligned}$$

with

$$E = u_x^2 + v_x^2 \quad F = u_x u_y + v_x v_y \quad G = v_y^2 + u_y^2.$$

Writing this in quadratic form,

$$du^2 + dv^2 = (dx \ dy) \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_S \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Since S is symmetric, it is orthogonally diagonalizable over $\mathbb{R}$. Let λ_1, λ_2 be the eigenvalues corresponding to the eigenvectors e_1, e_2 , assume without loss of generality that, $|\lambda_1| > |\lambda_2|$, since $|\lambda_1| = |\lambda_2|$ is the conformal case.

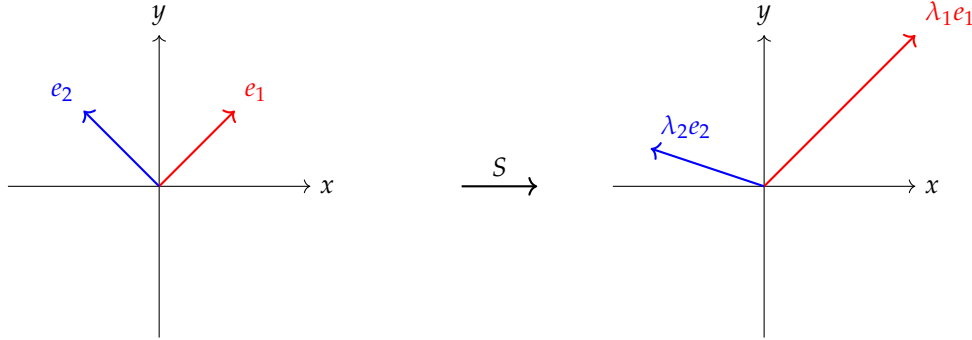


Figure 1: Linear transformation S locally around z_0

Computing the eigenvalues for S ,

$$\begin{vmatrix} E - \lambda & F \\ F & G - \lambda \end{vmatrix} = (E - \lambda)(G - \lambda) - F^2 = 0 \implies \lambda^2 - (E + G)\lambda + EG - F^2 = 0$$

$$\lambda_{1,2} = \frac{1}{2} \left(E + G \pm \sqrt{(E - G)^2 + 4F^2} \right).$$

After multiplying with the conjugate, the ratio is

$$\sqrt{\frac{\lambda_1}{\lambda_2}} = \left(\frac{E + G + \sqrt{(E - G)^2 + 4F^2}}{E + G - \sqrt{(E - G)^2 + 4F^2}} \right)^{1/2} = \frac{E + G + \sqrt{(E - G)^2 + 4F^2}}{2(EG - F^2)^{1/2}}$$

Now we use the condition that ϕ is continuously differentiable to get a general formula for the above ratio. We use derivatives in the complex form to get,

$$\begin{aligned} \phi_z &= \frac{1}{2}(\phi_x - i\phi_y) = \frac{1}{2}(u_x + v_y + i(v_x - u_y)) \\ \phi_{\bar{z}} &= \frac{1}{2}(\phi_x + i\phi_y) = \frac{1}{2}(u_x - v_y + i(v_x + u_y)). \end{aligned} \tag{2.2}$$

By applying Cauchy Riemann equation we relate derivatives in the complex form to conformality.

Recall that a map is conformal, if it preserves local angles between intersecting smooth curves. More precisely, suppose that $\gamma : [a, b] \rightarrow \Omega$ is a smooth curve and that $\gamma'(t_0) \neq 0$,

then we have a tangent line at the point $z_0 = \gamma(t_0)$ pointing at the direction of $\gamma'(t_0)$, with the slope $\tan(\arg \gamma'(t_0))$. If γ_1 and γ_2 are such smooth curves intersecting at some point $z_0 = \gamma_1(t_1) = \gamma_2(t_2)$, with non zero derivatives, then define the angle between the curves γ_1 and γ_2 at z_0 to be

$$\arg \gamma_2'(t_2) - \arg \gamma_1'(t_1).$$

Definition 2.3. Let $\Omega \subset \mathbb{C}$ be a Jordan domain. A function $\phi : \Omega \rightarrow \mathbb{C}$ with existing derivative

$$\lim_{z \rightarrow a} \frac{|\phi(z) - \phi(a)|}{|z - a|},$$

is conformal if for every two smooth curves γ_1, γ_2 , with $\gamma_1(t_1) = \gamma_2(t_2)$ and non-zero derivatives, $\gamma_1'(t_1) \neq 0, \gamma_2'(t_2) \neq 0$. The angle between them is preserved under the map ϕ . That is

$$\arg \gamma_2'(t_2) - \arg \gamma_1'(t_1) = \arg \phi'(\gamma_2(t_2)) - \arg \phi'(\gamma_1(t_1)).$$

For the following lemma we need to relate conformality with analytical functions to use the Cauchy Riemann equation. This is obtained in the following lemma.

Lemma 2.4. Let $\phi : \Omega \rightarrow \mathbb{C}$ and $\phi \in C^1$. Then ϕ is conformal if and only if ϕ is analytical with non-vanishing derivatives, $\phi'(z) \neq 0$ for every $z \in \Omega$.

Lemma 2.5. If $\phi \in C^1$ is conformal then $\phi_{\bar{z}} = 0$ and $\phi_z = \phi'$.

Proof. From the previous lemma we get that ϕ is analytical with non-vanishing derivative everywhere, therefore satisfying Cauchy Riemann equation

$$u_x = v_y, \quad u_y = -v_x.$$

hence

$$\phi_{\bar{z}} = \frac{1}{2}(u_x - v_y + i(v_x + u_y)) = 0.$$

The second statement follows form,

$$\phi_z = \frac{1}{2}(u_x + v_y + i(v_x - u_y)) = u_x + iv_x = \phi'.$$

□

The complex form summarizes compactly equation (2.1) according to the following,

$$\begin{aligned} \phi_z dz + \phi_{\bar{z}} d\bar{z} &= \frac{1}{2}(u_x + v_y + i(v_x - u_y))(dx + idy) + \frac{1}{2}(u_x - v_y + i(v_x + u_y))(dx - idy) \\ &= \frac{1}{2}dx(u_x + v_y + i(v_x - u_y) + u_x - v_y + i(v_x + u_y)) + \\ &\quad + \frac{1}{2}dy(iu_x + iv_y - v_x + u_y - iu_x + iv_y + v_x + u_y) \\ &= dx(u_x + iv_x) + dy(u_y + iv_y) = u_x dx + u_y dy + i(v_x dx + v_y dy) = du + idv = dw. \end{aligned}$$

Further we get the Jacobian as

$$\begin{aligned} |\phi_z|^2 - |\phi_{\bar{z}}|^2 &= \frac{1}{4}((u_x + v_y)^2 + (v_x - u_y)^2) - \frac{1}{4}((u_x - v_y)^2 + (v_x + u_y)^2) \\ &= \frac{1}{4}(u_x^2 + 2u_x v_y + v_y^2 + v_x^2 - 2u_y v_x + u_y^2 - u_x^2 + 2u_x v_y - v_y^2 - v_x^2 - 2u_y v_x - u_y^2) \\ &= u_x v_y - u_y v_x = J_\phi > 0. \end{aligned}$$

A map is called **orientation preserving** if the Jacobian is positive and **orientation reversing** if it is negative. We will onward only consider the orientation preserving case, which is equivalent with $|\phi_z| > |\phi_{\bar{z}}|$. Applying the triangle inequality to dw we get.

$$(|\phi_z| - |\phi_{\bar{z}}|)|dz| \leq |dw| \leq (|\phi_z| + |\phi_{\bar{z}}|)|dz|.$$

With $|dw|$ being maximal if $\arg(\phi_z dz) = \arg(\phi_{\bar{z}} d\bar{z})$, that is maximal when

$$\frac{\phi_z dz}{\phi_{\bar{z}} d\bar{z}} > 0,$$

and minimal when

$$\frac{\phi_z dz}{\phi_{\bar{z}} d\bar{z}} < 0.$$

This follows from either writing out the fraction in polar form or that the triangle inequality is maximal/minimal when vectors are parallel hence the fraction is positive or negative real numbers. Now define the **complex dilation** to be

$$\mu(z) = \frac{\phi_{\bar{z}}}{\phi_z}.$$

Let $\alpha = \arg(dz)$, now $|dw|$ is maximal when $\arg(\phi_z dz) = \arg(\phi_{\bar{z}} d\bar{z})$. Because $\arg$ is linear over multiplication, we get that $|dw|$ is maximal if

$$\arg\left(\frac{\phi_{\bar{z}}}{\phi_z}\right) = \arg(\phi_{\bar{z}}) - \arg(\phi_z) = \arg(dz) - \arg(d\bar{z}) = \arg\left(\frac{dz}{d\bar{z}}\right).$$

Hence the following holds,

$$\arg(\mu(z)) = \arg\left(\frac{dz}{d\bar{z}}\right) = 2\alpha \implies \alpha = \frac{1}{2} \arg(\mu(z))$$

If $\beta = \arg(dw) = \arg(\phi_z dz) = \arg(\phi_{\bar{z}} d\bar{z})$ then

$$\begin{aligned} 2\beta &= \arg(\phi_z dz) + \arg(\phi_{\bar{z}} d\bar{z}) \\ &= \arg(\phi_z) + \arg(dz) + \arg(\phi_{\bar{z}}) + \arg(d\bar{z}) \\ &= \arg(\phi_z) + \alpha + \arg(\phi_{\bar{z}}) - \alpha \\ &= \arg\left(\frac{\phi_{\bar{z}}}{\phi_z}\right) \end{aligned}$$

with the **second complex dilatation**, $\nu(z) = \frac{\phi_{\bar{z}}}{\phi_z}$ we get $\beta = \frac{1}{2} \arg(\nu(z))$.

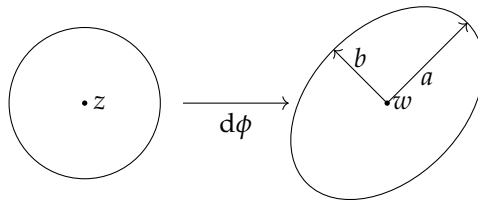


Figure 2: Distortion on infinitesimal circles

The linear map T sends infinitesimal circles onto ellipses with major axis of length a and minor axis of length b . The ratio between the major axis to the minor at a given point is called **dilatation**, and is given by

$$D_\phi(z) := \frac{a}{b} = \frac{|\phi_z(z)| + |\phi_{\bar{z}}(z)|}{|\phi_z(z)| - |\phi_{\bar{z}}(z)|} \geq 1.$$

Note that when ϕ is conformal this means $\phi_{\bar{z}} = 0$ and $D_\phi = 1$. Let $d_\phi(z) = |\mu(z)| = \frac{|\phi_{\bar{z}}(z)|}{|\phi_z(z)|} < 1$, then

$$D_\phi(z) = \frac{1 + d_\phi(z)}{1 - d_\phi(z)}.$$

Definition 2.6. A C^1 - diffeomorphism $\phi : \Omega \rightarrow \phi(\Omega)$ is said to be a differentiable **quasiconformal map** (DQC) if $D_\phi < \infty$ for all $z \in \Omega$. It's a K -quasiconformal (K -q.c) if $D_\phi \leq K$.

2.2 Grötzsch's problem

Let R and R' be two closed rectangles, then Grötzsch's problem seeks to determine the minimum K for which there exists a K -quasiconformal mapping $\phi : R \rightarrow R'$ mapping the vertex c_i to c'_i , that is $\phi(c_i) = c'_i$ for $i = 1, 2, 3, 4$.

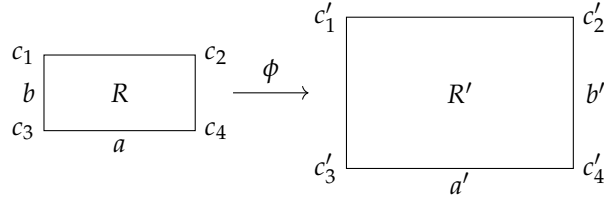


Figure 3: Rectangles in Grötzsch's problem

Without loss of generality, one can assume that $\frac{a}{b} \leq \frac{a'}{b'}$ (otherwise swap a and b). Let γ be a horizontal line and set $dw = \phi_z dz + \phi_{\bar{z}} d\bar{z}$. Then by what is usually called Grötzsch's argument,

$$\begin{aligned} a' &\leq \text{length of } \phi(\gamma) = \int_0^a |dw| = \int_0^a |\phi_z dz + \phi_{\bar{z}} d\bar{z}| \leq \int_0^a |\phi_z dz| + |\phi_{\bar{z}} d\bar{z}| = \int_0^a |\phi_z| + |\phi_{\bar{z}}| |dz| \\ &\leq \int_0^a |\phi_z| + |\phi_{\bar{z}}| dx + \underbrace{\int_0^a |\phi_z| + |\phi_{\bar{z}}| dy}_{=0} = \int_0^a (|\phi_z| + |\phi_{\bar{z}}|) dx. \end{aligned}$$

Both inequalities follow from the triangle inequality, while the last equality follows from $dy = 0$, (since γ is a horizontal line). Now integrate over R and then apply the Cauchy-Schwarz inequality gives,

$$\begin{aligned} \int_0^b a' dy &= a'b \leq \int_0^b \int_0^a (|\phi_z| + |\phi_{\bar{z}}|) dx dy = \int_0^b \int_0^a \frac{|\phi_z| + |\phi_{\bar{z}}|}{\sqrt{|\phi_z|^2 - |\phi_{\bar{z}}|^2}} \sqrt{|\phi_z|^2 - |\phi_{\bar{z}}|^2} dx dy \\ &= \int_0^b \int_0^a \sqrt{\frac{|\phi_z| + |\phi_{\bar{z}}|}{|\phi_z| - |\phi_{\bar{z}}|}} J_\phi^{1/2} dx dy \leq \int_0^b \int_0^a D_\phi^{1/2} J_\phi^{1/2} dx dy \end{aligned}$$

$$\underbrace{\leq}_{\text{CauchySchwarz}} \left(\int_0^b \int_0^a D_\phi dx dy \right)^{1/2} \left(\int_0^b \int_0^a J_\phi dx dy \right)^{1/2} \implies a'^2 b^2 \leq \int_0^b \int_0^a D_\phi dx dy \cdot a' b'$$

After dividing by $a' b'$ we get the following inequality

$$\frac{a' b^2}{b'} \leq \int_0^b \int_0^a D_\phi dx dy \implies \frac{a'}{b'} \bigg/ \frac{a}{b} \leq \frac{1}{ab} \int_0^b \int_0^a D_\phi dx dy \leq \sup_{z \in R} D_\phi(z).$$

The minimum is attained for the affine mapping given by

$$A(z) = \frac{1}{2} \left(\frac{a'}{a} + \frac{b'}{b} \right) z + \frac{1}{2} \left(\frac{a'}{a} - \frac{b'}{b} \right) \bar{z}$$

$$\mu_A(z) = \frac{\frac{1}{2} \left(\frac{a'}{a} - \frac{b'}{b} \right)}{\frac{1}{2} \left(\frac{a'}{a} + \frac{b'}{b} \right)} = \frac{a'b - ab'}{a'b + ab'} = \frac{\frac{a'}{b'} - \frac{a}{b}}{\frac{a'}{b'} + \frac{a}{b}} \geq 0$$

The inequality follows from the initial assumption, $\frac{a}{b} \leq \frac{a'}{b'}$.

$$D_A = \frac{1 + |\mu_A|}{1 - |\mu_A|} = \frac{a'b + ab' + a'b - ab'}{a'b + ab' - (a'b - ab')} = \frac{a'b}{ab'}$$

So the affine map A gives the smallest K , i.e. for every $f : R \rightarrow R'$ satisfying the conditions for the Grötzsch's problem, we have $K_f \geq K_A$.

2.3 Composed mappings

We begin by proving the following lemma, which will be used throughout this section.

Lemma 2.7. *Let $\phi, \psi \in \mathcal{C}^1$, then we have*

$$\overline{(\phi_z)} = (\bar{\phi})_{\bar{z}} \tag{2.3}$$

$$\overline{(\phi_{\bar{z}})} = (\bar{\phi})_z \tag{2.4}$$

$$(\psi \circ \phi)_z = (\psi_{\xi} \circ \phi) \phi_z + (\psi_{\bar{\xi}} \circ \phi) \overline{(\phi_{\bar{z}})} \tag{2.5}$$

$$(\psi \circ \phi)_{\bar{z}} = (\psi_{\xi} \circ \phi) \phi_{\bar{z}} + (\psi_{\bar{\xi}} \circ \phi) \overline{(\phi_z)} \tag{2.6}$$

Proof. The first statement follows from equation (2.2)

$$\overline{(\phi_z)} = \frac{1}{2} \overline{(u_x + v_y + i(v_x - u_y))} = \frac{1}{2} (u_x + v_y + i(u_y - v_x)).$$

$$(\bar{\phi})_{\bar{z}} = \frac{1}{2} (\bar{\phi}_x + i \bar{\phi}_y) = \frac{1}{2} (u_x - i v_x + i u_y + v_y) = \frac{1}{2} (u_x + v_y + i(u_y - v_x)).$$

Hence proving the equality. The second statement follows in the same way

$$\overline{(\phi_{\bar{z}})} = \frac{1}{2} \overline{(\phi_x + i \phi_y)} = \frac{1}{2} \overline{(u_x - v_y + i(u_y + v_x))} = \frac{1}{2} (u_x - v_y - i(u_y + v_x)),$$

$$(\bar{\phi})_z = \frac{1}{2}(\bar{\phi}_x - i\bar{\phi}_y) = \frac{1}{2}(u_x - iv_x - i(u_y - iv_y)) = \frac{1}{2}(u_x - v_y - i(u_y + v_x)).$$

For the third statement, let $\zeta = \phi(z)$ and $\psi = \psi(\zeta)$, giving

$$\begin{aligned} d\psi &= \psi_\zeta d\zeta + \psi_{\bar{\zeta}} d\bar{\zeta} \\ &\stackrel{\text{Chain rule}}{=} (\psi_\zeta \circ \phi)(\phi_z dz + \phi_{\bar{z}} d\bar{z}) + (\psi_{\bar{\zeta}} \circ \phi)(\bar{\phi}_z dz + \bar{\phi}_{\bar{z}} d\bar{z}) \\ &\stackrel{\text{Expanding}}{=} [(\psi_\zeta \circ \phi)\phi_z + (\psi_{\bar{\zeta}} \circ \phi)\bar{\phi}_z] dz + [(\psi_\zeta \circ \phi)\phi_{\bar{z}} + (\psi_{\bar{\zeta}} \circ \phi)\bar{\phi}_{\bar{z}}] d\bar{z} \\ &= (\psi \circ \phi)_z dz + (\psi \circ \phi)_{\bar{z}} d\bar{z}. \end{aligned}$$

Last equality follows from applying the chain rule on the composed map $(\psi \circ \phi)$. $\square$

Now we wish to solve for $\psi_\zeta \circ \phi$ and $\psi_{\bar{\zeta}} \circ \phi$ from (2.6) and (??). To that extent multiply by $\frac{\phi_{\bar{z}}}{\phi_z}$ in first equation

$$\frac{\phi_{\bar{z}}}{\phi_z}(\psi \circ \phi)_z = \phi_{\bar{z}}(\psi_\zeta \circ \phi) + \frac{\overline{(\phi_z)} \cdot \phi_{\bar{z}}}{\phi_z}(\psi_{\bar{\zeta}} \circ \phi).$$

Subtracting this from the second equation gives,

$$(\psi \circ \phi)_{\bar{z}} - \frac{\phi_{\bar{z}}}{\phi_z}(\psi \circ \phi)_z = \left[\overline{(\phi_z)} - \frac{\overline{(\phi_z)}\phi_{\bar{z}}}{\phi_z} \right] (\psi_{\bar{\zeta}} \circ \phi).$$

Using $J_\phi = |\phi_z|^2 - |\phi_{\bar{z}}|^2$, simplifies the expression to

$$(\psi \circ \phi)_{\bar{z}} - \frac{\phi_{\bar{z}}}{\phi_z}(\psi \circ \phi)_z = \frac{J_\phi}{\phi_z}(\psi_{\bar{\zeta}} \circ \phi) \implies (\psi_{\bar{\zeta}} \circ \phi) = \frac{1}{J_\phi}((\psi \circ \phi)_{\bar{z}}\phi_z - (\psi \circ \phi)_z\phi_{\bar{z}}).$$

Substituting this into the first equation and solving for $\psi_\zeta \circ \phi$ gives

$$\psi_\zeta \circ \phi = \frac{1}{\phi_z}((\psi \circ \phi)_z - (\psi_{\bar{\zeta}} \circ \phi)\overline{(\phi_z)}) = \frac{1}{\phi_z}((\psi \circ \phi)_z - \frac{\overline{(\phi_z)}}{J_\phi}((\psi \circ \phi)_{\bar{z}}\phi_z - (\psi \circ \phi)_z\phi_{\bar{z}})).$$

Collecting terms

$$\psi_\zeta \circ \phi = \left(\frac{1}{\phi_z} + \frac{\overline{(\phi_z)}\phi_{\bar{z}}}{J_\phi\phi_z} \right) (\psi \circ \phi)_z - \frac{\overline{(\phi_z)}\phi_z}{J_\phi\phi_z} (\psi \circ \phi)_{\bar{z}}.$$

Using $\overline{(\phi_z)} = \phi_{\bar{z}}$ and $\overline{(\phi_{\bar{z}})} = \phi_z$ one can further simplify to get,

$$\psi_\zeta \circ \phi = \frac{1}{J_\phi}((\psi \circ \phi)_z\phi_{\bar{z}} - (\psi \circ \phi)_{\bar{z}}\phi_z).$$

Finally we have proved the following

$$\begin{cases} \psi_\zeta \circ \phi = \frac{1}{J_\phi}((\psi \circ \phi)_z\phi_{\bar{z}} - (\psi \circ \phi)_{\bar{z}}\phi_z) \\ \psi_{\bar{\zeta}} \circ \phi = \frac{1}{J_\phi}((\psi \circ \phi)_{\bar{z}}\phi_z - (\psi \circ \phi)_z\phi_{\bar{z}}). \end{cases}$$

With this we can compute the complex dilation $\mu_{\psi \circ \phi}$

$$\mu_{\psi \circ \phi} = \frac{\psi_{\bar{\zeta}} \circ \phi}{\psi_\zeta \circ \phi} = \frac{(\psi \circ \phi)_{\bar{z}}\phi_z - (\psi \circ \phi)_z\phi_{\bar{z}}}{(\psi \circ \phi)_z\phi_{\bar{z}} - (\psi \circ \phi)_{\bar{z}}\phi_z}$$

$$= \frac{\phi_z}{\phi_{\bar{z}}} \cdot \frac{\frac{(\psi \circ \phi)_{\bar{z}}}{(\psi \circ \phi)_z} - \frac{\phi_{\bar{z}}}{\phi_z}}{1 - \frac{(\psi \circ \phi)_{\bar{z}}}{(\psi \circ \phi)_z} \cdot \frac{\phi_{\bar{z}}}{\phi_z}} = \frac{\phi_z}{\phi_{\bar{z}}} \cdot \frac{\mu_{\psi \circ \phi} - \mu_{\phi}}{1 - \mu_{\psi \circ \phi} \overline{\mu_{\phi}}}.$$

Let $r_{\phi} = \overline{(\phi_z)}/\phi_{\bar{z}}$, solving for the composed dilation $\mu_{\psi \circ \phi}$ gives

$$\mu_{\psi \circ \phi} = \frac{\mu_{\phi} + r_{\phi}(\mu_{\psi \circ \phi})}{1 + r_{\phi} \overline{\mu_{\phi}}(\mu_{\psi \circ \phi})} \quad (2.7)$$

Remark 2.8. The special cases with $\psi = \phi^{-1}$ gives,

$$\mu_{\phi^{-1} \circ \phi} = \frac{\phi_z}{\phi_{\bar{z}}} \cdot \frac{0 - \mu_{\phi}}{1 - 0} = -\nu_{\phi}.$$

When ψ is conformal, $\mu_{\psi} = 0$, plugging this into equation (2.7) results in

$$\mu_{\psi \circ \phi} = \mu_{\phi}.$$

That is the complex dilation is invariant under post-compositions with conformal mappings. If ϕ is conformal $\mu_{\phi} = 0$, then

$$\mu_{\psi \circ \phi} = r_{\phi}(\mu_{\psi \circ \phi}).$$

Which can also be written as

$$\nu_{\psi \circ \phi} = \nu_{\psi \circ \phi}.$$

hence also invariant under compositions with conformal maps. Finally, let $\omega = \psi \circ \phi$ we find that

$$\mu_{\omega \circ \phi^{-1} \circ \phi} = \frac{\phi_z}{\phi_{\bar{z}}} \cdot \frac{\mu_{\omega} - \mu_{\phi}}{1 - \overline{\mu_{\phi}} \mu_{\omega}}.$$

Finalizing this section we prove how the dilation is compounded under compositions of quasiconformal mapping.

Theorem 2.9. If ϕ and ψ are K_{ϕ} -qc, K_{ψ} -qc respectively then both of $\phi \circ \psi$ and $\psi \circ \phi$ are $K_{\phi} \cdot K_{\psi}$ -qc.

Proof. From a geometric perspective, this follows since the greatest dilatation occurs when the major axes of expansion for both ϕ and ψ are aligned, which yields $K_{\phi \circ \psi} \leq K_{\phi} K_{\psi}$. An algebraic proof follows from, equation (2.1) and equation (2.7). Since the proof is similar we only prove the $\phi \circ \psi$ case.

$$\begin{aligned} |K_{\phi \circ \psi}| &= \left| \frac{1 + d_{\phi \circ \psi}}{1 - d_{\phi \circ \psi}} \right| = \left| \left(1 + \frac{\mu_{\phi} + r_{\phi}(\mu_{\psi \circ \phi})}{1 + r_{\phi} \overline{\mu_{\phi}}(\mu_{\psi \circ \phi})} \right) / \left(1 - \frac{\mu_{\phi} + r_{\phi}(\mu_{\psi \circ \phi})}{1 + r_{\phi} \overline{\mu_{\phi}}(\mu_{\psi \circ \phi})} \right) \right| \\ &\leq \frac{|1 + r_{\phi} \overline{\mu_{\phi}} \mu_{\psi} + \mu_{\phi} + r_{\phi} \mu_{\psi}|}{|1 + r_{\phi} \overline{\mu_{\phi}} \mu_{\psi} - \mu_{\phi} - r_{\phi} \mu_{\psi}|} \leq \frac{1 + d_{\phi} d_{\psi} + d_{\phi} + d_{\psi}}{|1 - \mu_{\phi} + r_{\phi} \mu_{\psi}(\overline{\mu_{\phi}} - 1)|} \\ &\leq \frac{1 + d_{\phi} d_{\psi} + d_{\phi} + d_{\psi}}{1 - d_{\phi} - d_{\psi} + d_{\phi} d_{\psi}} = \left(\frac{1 + d_{\phi}}{1 - d_{\phi}} \right) \cdot \left(\frac{1 + d_{\psi}}{1 - d_{\psi}} \right) = K_{\phi} \cdot K_{\psi}. \end{aligned}$$

With $|r_{\phi}| = 1$, and $|\mu_{\phi}| \geq 1$. The last inequality follows from

$$|1 - \mu_{\phi} + r_{\phi} \mu_{\psi}(\overline{\mu_{\phi}} - 1)| \geq |1 - \mu_{\phi}| - |r_{\phi} \mu_{\psi}(\overline{\mu_{\phi}} - 1)|.$$

□

By induction, we obtain the following corollary.

Corollary 2.10. *Let $\{\phi_i\}_{i=1}^N$ be a finite sequence of K_i -quasiconformal $\phi_i : \Omega \rightarrow \Omega$ mappings. The dilation of any compositions will be bounded above by the product of the corresponding dilations.*

3 Geometric definition of quasiconformal maps

This chapter presents a second definition of quasiconformal mappings based on the boundedness of the modules taken over different underlying geometric sets, beginning with quadrilaterals. In Section 3.3, we establish several properties of the module, including its continuity. Importantly, we also prove the equivalence between the geometric quasiconformal maps and the differentiable quasiconformal maps. Next section 3.4 extends this approach to ring domains as the underlying geometry, establishing analogous results and proving a connection between the two different geometric starting points.

We then demonstrate that the geometric definition of quasiconformal mappings is entirely determined by the local behaviour of the function. Following this, we investigate the problem of finding a ring domain that maximizes the module while separating two distinct sets.

Finally, the last section proves the well-known Mori's theorem, an important step toward solving the dilation problem—specifically, establishing upper bounds on the distance between the images of points on the open unit disc under a quasiconformal mapping.

3.1 Extremal length

Let Ω be a Jordan domain, Γ family of curves in Ω . Every curve $\gamma \in \Gamma$ is assumed to be locally rectifiable.

Definition 3.1. [*Length of curve & (locally) rectifiable curves*]

Let $\gamma : I \rightarrow \mathbb{R}^n$ be a curve and let $\{t_k\}_{k=0}^n$ be an ordered partition of I . The **length of curve** γ is defined by

$$l(\gamma) = \sup \sum_{k=1}^n |\gamma(t_k) - \gamma(t_{k-1})|.$$

With supremum taken over all possible ordered partitions. Note that $0 \leq l(\gamma) \leq \infty$. A curve γ is called **rectifiable** if $l(\gamma) < \infty$. Further if for each compact subset $J \subset I$, the curve restricted to J is rectifiable, then the curve is called **locally rectifiable**.

An example of a locally rectifiable curve that is not fully rectifiable is the curve parametrized with $\gamma(t) = \frac{1}{t}$ for $t \in I = (0, 1]$. This additional condition is crucial for later proving properties of extremal length. For instance, the superadditivity of extremal length requires that any restricted sub-curve must be rectifiable. If not stated otherwise, the family of curves will be locally rectifiable in the remaining of this thesis.

Let Γ be a family of locally rectifiable curves in the plane. We shall now introduce a geometric quantity called the extremal length $\lambda(\Gamma)$, which intuitively is the average of the minimum length for a given curve family measured with some admissible metric. This quantity will later prove to be of great importance for conformal and quasiconformal mappings. We will show that it is invariant under conformal mappings and remains bounded under quasiconformal mappings.

Definition 3.2. A function $\rho(z) : \mathbb{C} \rightarrow \mathbb{R}$, defined on the whole plane, is an **admissible metric** if

- $\rho \geq 0$,
- ρ is measurable,

- $A(\rho) := \int \int_{\mathbb{C}} \rho^2 dx dy, \quad 0 < A(\rho) < \infty.$

Denote by $\mathcal{P}$ the class of all **admissible metrics**.

The length of a curve $\gamma \in \Gamma$ with respect to ρ is

$$L_\gamma(\rho) = \int_\gamma \rho |dz|.$$

Further we define $L(\Gamma, \rho)$ to be the shortest length in the family of curves with respect to ρ , that is

$$L(\Gamma, \rho) := \inf_{\gamma \in \Gamma} L_\gamma(\rho).$$

Definition 3.3. Let Γ be a family of locally rectifiable curves in Ω . The **extremal length** of Γ is

$$\lambda(\Gamma) = \sup_{\rho \in \mathcal{P}} \frac{L(\Gamma, \rho)^2}{A(\rho)}$$

Example 3.4. Let R be a rectangle, and let Γ be the family of locally rectifiable curves inside R that joins any two opposite vertical edges of R . Our goal is to compute the extremal length of Γ . For any $\rho \in \mathcal{P}$ and any horizontal line segment in Γ we have $|dz| = dx$,

$$L_\gamma(\rho) = \int_\gamma \rho(x + iy) |dx + idy| = \int_0^a \rho(x + iy) dx.$$

Taking the infimum we get

$$L(\Gamma, \rho) \leq \int_0^a \rho(x + iy) dx$$

Moreover, since this inequality holds for every y we integrate both sides giving us,

$$\int_0^b L(\Gamma, \rho) dy \leq \int_0^b \int_0^a \rho(x + iy) dx dy.$$

The left hand side equals $bL(\Gamma, \rho)$ and applying Fubini's theorem to the right and side we get

$$bL(\Gamma, \rho) \leq \int \int_R \rho(x + iy) dx dy.$$

Squaring both sides and applying Cauchy Schwarz inequality to the right hand side gives,

$$(bL(\Gamma, \rho))^2 \leq \left(\int \int_R \rho(x + iy) dx dy \right)^2 \leq \int \int_R 1 dx dy \int \int_R \rho(x + iy)^2 dx dy \leq abA(\rho).$$

Dividing by $b^2A(\rho)$,

$$\frac{L(\Gamma, \rho)^2}{A(\rho)} \leq \frac{a}{b}.$$

Since this holds for every $\rho \in \mathcal{P}$ taking the supremum over $\mathcal{P}$ will gives us the following inequality

$$\lambda(\Gamma) = \sup_{\rho \in \mathcal{P}} \frac{L(\Gamma, \rho)^2}{A(\rho)} \leq \frac{a}{b}.$$

Note that if $\rho_0(z) = \chi_R(z)$ (where $\chi_R(z)$ is the characteristic function on the set R), then

$$\frac{L(\Gamma, \rho_0)^2}{A(\rho_0)} = \frac{a}{b}.$$

Hence the upper bound for $\lambda(\Gamma)$ is attained and $\lambda(\Gamma) = a/b$.

We will now collect some properties of extremal lengths. We begin by studying their action on the set theoretic operations. We say $\Gamma_1 < \Gamma_2$ if for any $\gamma_2 \in \Gamma_2$ there exists some $\gamma_1 \in \Gamma_1$ such that $\gamma_1 \subset \gamma_2$. Further let

$$\Gamma_1 + \Gamma_2 := \{\gamma_1 + \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2 \text{ whenever this gives a new curve}\}$$

With $\gamma_1 + \gamma_2$ meaning, γ_1 followed by γ_2 . We have the following lemma.

Theorem 3.5. *Let Γ_1 and Γ_2 be two families of curves. Then,*

- a) *If $\Gamma_1 < \Gamma_2$ then $\lambda(\Gamma_1) \leq \lambda(\Gamma_2)$*
- b) *If $\Gamma_1 \subset \Gamma_2$ then $\lambda(\Gamma_1) \geq \lambda(\Gamma_2)$*
- c) *$\lambda(\Gamma_1 + \Gamma_2) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2)$*
- d) *$\lambda(\Gamma_1 \cup \Gamma_2)^{-1} \geq \lambda(\Gamma_1)^{-1} + \lambda(\Gamma_2)^{-1}$, if Γ_1 and Γ_2 are contained in two disjoint measurable sets.*

Proof. Fix some $\rho \in \mathcal{P}$ then for any $\gamma_2 \in \Gamma_2$ there exist a $\gamma_1 \in \Gamma_1$ such that $\gamma_1 \subset \gamma_2$, then

$$\int_{\gamma_1} \rho(z) |dz| \leq \int_{\gamma_2} \rho(z) |dz|.$$

Taking infimum on both sides gives

$$L(\Gamma_1, \rho) \leq L(\Gamma_2, \rho),$$

hence

$$\lambda(\Gamma_1) = \sup_{\rho \in \mathcal{P}} \frac{L(\Gamma_1, \rho)^2}{A(\rho)} \leq \sup_{\rho \in \mathcal{P}} \frac{L(\Gamma_2, \rho)^2}{A(\rho)} = \lambda(\Gamma_2).$$

For the second part, note that if $\Gamma_1 \subset \Gamma_2$ implies $\Gamma_2 < \Gamma_1$ and applying the first statement of the lemma gives the desired result.

For the third statement, we can begin by assuming that $\lambda(\Gamma_i) \in (0, \infty)$ otherwise there is nothing to prove. By the assumption, we can normalize the family of admissible metrics to $\hat{\mathcal{P}}$. So that $L(\Gamma_1, \rho_1) = A(\rho_1)$ and $L(\Gamma_2, \rho_2) = A(\rho_2)$ for any metrics $\rho_1, \rho_2 \in \hat{\mathcal{P}}$. Let $\rho = \max(\rho_1, \rho_2)$, giving

$$L(\Gamma_1 + \Gamma_2, \rho) \geq L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2) = A(\rho_1) + A(\rho_2) \geq A(\rho).$$

Hence we get

$$\lambda(\Gamma_1 + \Gamma_2) = \sup_{\rho \in \hat{\mathcal{P}}} \frac{L(\Gamma_1 + \Gamma_2, \rho)^2}{A(\rho)} \geq \frac{(L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2))A(\rho)}{A(\rho)} = \frac{L(\Gamma_1, \rho_1)^2}{A(\rho_1)} + \frac{L(\Gamma_2, \rho_2)^2}{A(\rho_2)}.$$

Giving the desired inequality.

For the last statement, we can assume that $\lambda(\Gamma_1 \cup \Gamma_2) \neq 0$. Consider an admissible metric ρ with $L(\Gamma_1 \cup \Gamma_2, \rho) > 0$ and set $\rho_1 = \rho$ on $E_1 \supset \Gamma_1$ and $\rho_2 = \rho$ on $E_2 \supset \Gamma_2$ and 0 otherwise, by the assumption of the statement this can be done with E_1 and E_2 complementary disjoint measurable set. Thus we have

$$L(\Gamma_i, \rho_i) \geq L(\Gamma_1 \cup \Gamma_2, \rho)$$

And $A(\rho) = A(\rho_1) + A(\rho_2)$. Thus

$$\frac{A(\rho)}{L(\Gamma_1 \cup \Gamma_2)^2} \geq \frac{A(\rho_1)}{L(\Gamma_1, \rho_1)^2} + \frac{A(\rho_2)}{L(\Gamma_2, \rho_2)^2}.$$

□

Now we prove the boundedness of extremal lengths under quasiconformal mappings.

Theorem 3.6. *Let $\phi : \Omega \rightarrow \phi(\Omega)$ be a C^1 K -qc map, and let Γ be a family of curves contained in Ω and $\hat{\Gamma} = \phi(\Gamma)$, then*

$$K^{-1}\lambda(\Gamma) \leq \lambda(\hat{\Gamma}) \leq K\lambda(\Gamma).$$

Proof. For a given $\rho(z)$ define $\hat{\rho}(\xi) = 0$ outside of $\phi(\Omega)$ and

$$\hat{\rho}(\xi) = \begin{cases} 0, & \text{if } \xi \notin \phi(\Omega), \\ \left(\frac{\rho}{|\phi_z| - |\phi_{\bar{z}}|} \circ \phi^{-1} \right)(\xi), & \text{if } \xi \in \phi(\Omega). \end{cases}$$

Where $\phi(z) = \xi$. The new metric $\hat{\rho}$ is admissible for $\hat{\Gamma}$. Then

$$\int_{\mathbb{C}} \hat{\rho}^2 d\xi d\eta = \int_{\phi(\Omega)} \hat{\rho}^2 d\xi d\eta = \int_{\Omega} \rho^2 \frac{|\phi_z| + |\phi_{\bar{z}}|}{|\phi_z| - |\phi_{\bar{z}}|} dx dy \leq KA(\rho).$$

If $\gamma \in \Gamma$ let $\hat{\gamma} = \phi(\gamma)$ be a curve in $\hat{\Gamma}$, then

$$\int_{\hat{\gamma}} \hat{\rho} |d\xi| \geq \int_{\gamma} \rho |dz|,$$

which implies

$$L(\hat{\rho}, \hat{\Gamma}) \geq L(\rho, \Gamma).$$

Combining this we get $\lambda(\Gamma') \geq K^{-1}\lambda(\Gamma)$. The upper inequality follows from considering ϕ^{-1} . □

It follows from this theorem that extremal lengths are conformally invariant.

Corollary 3.7. *The extremal length $\lambda(\Gamma)$ is invariant under conformal mappings.*

3.2 Symmetry Principle

For any curve γ let γ^- be its reflection on the real line, while γ^+ be the curve attained by reflecting the part of γ below the real axis and retaining the part above the curve, note that $\gamma \cup \gamma^- = \gamma^+ \cup (\gamma^+)^-$. In the case Γ is a curve family then Γ^- consists all the curves after applying $(\cdot)^-$ on each curve. In the same way Γ^+ is the set of all curves after applying $(\cdot)^+$. We aim to calculate the extremal length of a family of symmetric curve families, that is $\Gamma^- = \Gamma$.

Theorem 3.8. *Let Γ be a curve family such that $\Gamma^- = \Gamma$, then $\lambda(\Gamma) = \frac{1}{2}\lambda(\Gamma^+)$.*

Proof. We prove two inequalities. For a given ρ , set $\hat{\rho}(z) = \sup(\rho(z), \rho(\bar{z}))$. First we prove that $L_\gamma(\hat{\rho}) = L_{\gamma^+}(\hat{\rho})$. For the case $\gamma^+ = \gamma$ it follows immediately, otherwise note that the metric $\hat{\rho}$ is symmetric with respect to conjugation, that is $\hat{\rho}(\bar{z}) = \hat{\rho}(z)$. Hence when calculating the length of a curve γ with respect to $\hat{\rho}$ we can consider only γ^+ , giving

$$L_\gamma(\hat{\rho}) = L_{\gamma^+}(\hat{\rho}) = \int_{\gamma^+} \sup(\rho(z), \rho(\bar{z})) |dz| \geq \int_{\gamma^+} \rho(z) |dz| \geq \inf_{\gamma \in \Gamma^+} \int_\gamma \rho(z) |dz| = L(\Gamma^+, \rho)$$

this holds for every $\gamma \in \Gamma$ hence we get $L(\Gamma, \hat{\rho}) \geq L(\Gamma^+, \rho)$. Denote $U = \{z \in \mathbb{C} : \rho(z) \geq \rho(\bar{z})\}$. The area is then,

$$\begin{aligned} A(\hat{\rho}) &= \int \int_{\mathbb{C}} (\sup(\rho(z), \rho(\bar{z})))^2 dx dy = \int \int_{\mathbb{C}} (\mathbb{1}_U \rho(z) + \mathbb{1}_{U^c} \rho(\bar{z}))^2 dx dy \\ &= \int \int_{\mathbb{C}} \mathbb{1}_U \rho(z)^2 + \mathbb{1}_{U^c} \rho(\bar{z})^2 dx dy + \int \int_{\mathbb{C}} \underbrace{\mathbb{1}_U \mathbb{1}_{U^c}}_{=0} \rho(z) \rho(\bar{z}) dx dy \\ &= \int \int_A \rho(z)^2 dx dy + \int \int_{A^c} \rho(\bar{z})^2 dx dy \leq \int \int_{\mathbb{C}} \rho(z)^2 dx dy + \int \int_{\mathbb{C}} \rho(\bar{z})^2 dx dy \\ &= A(\rho(z)) + A(\rho(\bar{z})) = 2A(\rho). \end{aligned}$$

The last equality follows because we are integrating over the complex plane and the functions are symmetric with respect to complex conjugation. Together we have

$$\begin{aligned} L(\Gamma^+, \rho) &\leq L(\Gamma, \hat{\rho}) \\ A(\rho) &\geq \frac{1}{2} A(\hat{\rho}). \end{aligned}$$

Combined gives

$$\frac{L(\Gamma^+, \rho)^2}{A(\rho)} \leq 2 \frac{L(\Gamma, \hat{\rho})^2}{A(\hat{\rho})} \leq 2 \sup_{\rho_1 \in \mathcal{P}} \frac{L(\Gamma, \rho_1)^2}{A(\rho_1)} = 2\lambda(\Gamma).$$

This holds for every ρ hence $\lambda(\Gamma) \geq \frac{1}{2} \lambda(\Gamma^+)$.

For the second inequality we instead set

$$\tilde{\rho}(z) = \begin{cases} \rho(z) + \rho(\bar{z}), & \text{if } \text{Im}(z) \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Then we get

$$\begin{aligned} L_{\gamma^+}(\tilde{\rho}) &= \int_{\gamma^+} \rho(z) + \rho(\bar{z}) |dz| = \int_{\gamma^+} \rho(z) |dz| + \int_{\gamma^+} \rho(\bar{z}) |dz| \\ &= \int_{\gamma^+} \rho(z) |dz| + \int_{(\gamma^+)^-} \rho(z) |dz| = \int_{\gamma^+ + (\gamma^+)^-} \rho(z) |dz| \\ &= \int_{\gamma + \gamma^-} \rho(z) |dz| = L_{\gamma + \gamma^-}(\rho) = L_\gamma(\rho) + L_{\gamma^-}(\rho) \geq 2 \inf_{\gamma \in \Gamma} L_\gamma(\rho) = 2L(\Gamma, \rho). \end{aligned}$$

Again this holds for all $\gamma \in \Gamma^+$ giving $L(\Gamma, \rho) \leq \frac{1}{2} L(\Gamma^+, \tilde{\rho})$. Denote $\mathbb{H} = \{z \in \mathbb{C} : \text{im}(z) \geq 0\}$. The area is

$$A(\tilde{\rho}) \leq \int \int_{\mathbb{C}} (\rho(z)^2 + \rho(\bar{z})^2 + 2\rho(z)\rho(\bar{z})) \mathbb{1}_{\mathbb{H}} dx dy$$

$$\begin{aligned}
&= \int \int_{\mathbb{C}} (2\rho(z)^2 + 2\rho(\bar{z})^2 - (\rho(z) - \rho(\bar{z}))^2) \mathbb{1}_{\mathbb{H}} dx dy \\
&\leq \int \int_{\mathbb{C}} (\rho(z)^2 + \rho(\bar{z})^2) \mathbb{1}_{\mathbb{H}} dx dy \\
&\leq \int \int_{\mathbb{C}} \rho(z)^2 + \rho(\bar{z})^2 dx dy = 2A(\rho)
\end{aligned}$$

Last equality follows again because we are integrating over the whole complex plane, hence both $\rho(z)$ and $\rho(\bar{z})$ will contribute to the integral by the same amount. Combined we have

$$\begin{aligned}
L(\Gamma, \rho) &\leq \frac{1}{2} L(\Gamma, \tilde{\rho}) \\
A(\rho) &\geq \frac{1}{2} A(\tilde{\rho})
\end{aligned}$$

Using this we can estimate the extremal length by,

$$\frac{L(\Gamma, \rho)^2}{A(\rho)} \leq \frac{1}{2} \frac{L(\Gamma^+, \tilde{\rho})^2}{A(\tilde{\rho})} \leq \frac{1}{2} \lambda(\Gamma^+)$$

resulting in the other inequality $\lambda(\Gamma) \leq \frac{1}{2} \lambda(\Gamma^+)$. Which combined with previous inequality proves the desired equality $\lambda(\Gamma) = \frac{1}{2} \lambda(\Gamma^+)$. $\square$

3.3 Quasiconformality by means of quadrilaterals

Since functions in this chapter wont be differentiable we need a new definition for orientation preserving maps, especially concerned with homomorphisms. For a given Jordan curve C , consider all homeomorphisms $f : \mathbb{D} \rightarrow C$ with $\mathbb{D} = \{re^{i\theta} : 0 \leq \theta \leq 2\pi\}$. We divide these maps into two equivalence classes. Two maps f_1 and f_2 are called equivalent if $f_2^{-1}(f_1(\theta))$ is increasing in θ . The pairs $C^+ = (C, \alpha)$ and $C^- = (C, \beta)$ are called oriented curves. With α as the class of positively-oriented parametrizations, and β as the negatively-oriented parametrizations. Homomorphisms that preserve the orientations are called orientation preserving.

Definition 3.9. Let $f : A \rightarrow A'$ where A and A' are two subsets of the complex plane. The mapping f is called orientation preserving if f preserves the orientation of the boundary of every Jordan domain D such that $\bar{D} \subset A$.

Let $\phi : \Omega \rightarrow \phi(\Omega)$ be an orientation preserving homeomorphism. A quadrilateral Q is the closure of a Jordan domain D with 4 distinguished boundary points z_1, z_2, z_3, z_4 , denoted by $Q = \{D, z_1, z_2, z_3, z_4\}$. The points z_i are called vertices of the quadrilateral, further we will label the vertices in the positive orientation with regard to D . The vertices divides ∂D into four continuous non-intersecting arcs, otherwise known as Jordan arcs. The Jordan arcs from $\widehat{z_1 z_2}$ and $\widehat{z_3 z_4}$ are called the a -sides and remaining two arcs the b -sides.

When discussing homeomorphisms $\phi : \bar{Q} \rightarrow \bar{Q}'$, between the two quadrilaterals $Q = \{D, z_1, z_2, z_3, z_4\}$ and $Q' = \{D', z'_1, z'_2, z'_3, z'_4\}$, we will mean homeomorphisms that sends the vertices accordingly, i.e. $\phi(z_i) = z'_i$. If the restriction ϕ onto D is conformal then ϕ is called a conformal mapping. In general it is not possible to map conformally any two quadrilaterals. Instead by Riemann mapping theorem any quadrilateral can be mapped via some conformal map π_Q to a rectangle R with side lengths a and b , which are unique up to multiplication of

some constant. The map and the corresponding rectangle will be called **canonical mapping** π_Q and **canonical rectangle** R_Q . Note that two quadrilaterals are conformal equivalent if and only if the respective canonical rectangles are similar.

For each canonical rectangle, we fix one vertex at 0 and another at 1, so the rectangle $R(0, 1, 1 + Mi, Mi)$ is a representative of an equivalence class of quadrilaterals up to similarities for the rectangles. The rotation $R(0, M, M + i, i)$ belongs to the same class. The modulus of a quadrilateral is defined as

$$M = \text{mod}(Q) = \frac{a}{b},$$

where a and b are the lengths of the corresponding a and b -sides of the canonical rectangle R_Q . If Γ is the family of curves connecting the b -sides of R , then by Example 3.4

$$\lambda(\Gamma) = \frac{a}{b}.$$

The quadrilateral Q_s is obtained from Q by swapping the roles of the a - and b -sides. In this case, if Γ_s denotes the family of curves connecting the a -sides, then,

$$\text{mod}(Q_s) = \lambda(\Gamma_s) = \frac{a}{b} = \frac{1}{\lambda(\Gamma)} = \frac{1}{\text{mod}(Q)}.$$

Instead for defining the module with application of conformal mappings, one can define it completely via the extremal length. This definition is more easy to carry over to higher dimensions. The definition via conformal mappings carries poorly over to higher dimension since there is no equivalence to Riemann mapping theorem for higher dimensional spaces. For our concern these definitions are equivalent on the complex plane and on the Riemann sphere.

Definition 3.10. Let $\phi : \Omega \rightarrow \phi(\Omega)$ be a homeomorphism. We say that ϕ is K -quasiconformal in Ω if for all quadrilaterals $Q \subset \Omega$,

$$\frac{\text{mod}(Q)}{K} \leq \text{mod}(\phi(Q)) \leq K \text{mod}(Q).$$

Since a quasiconformal map is a homeomorphism, in particular injective the image of a quadrilateral will also be a quadrilateral. We will use Rengel's inequality multiple times in this thesis. For the case of completeness we state it here. We will use m^d to denote the Lebesgue measure in dimension d . Later on we will use the notion m_ρ^d to mean the ρ weighted Lebesgue measure in d dimension.

Theorem 3.11. [Rengel's inequality]

For a Jordan domain let $Q = \{D, z_1, z_2, z_3, z_4\}$ be a quadrilateral. Now let s_a be the euclidean length of the shortest path in D from the side $\widehat{z_1 z_2}$ to the side $\widehat{z_3 z_4}$ (a -sides), and s_b the euclidean length of the shortest path in D from the side $\widehat{z_2 z_3}$ to the side $\widehat{z_1 z_4}$. We have that

$$\frac{s_b^2}{m^2(Q)} \leq \text{mod}(Q) \leq \frac{m^2(Q)}{s_a^2}$$

Equality holds if and only if Q is a rectangle.

Now follows some initial properties of quasiconformal mappings.

Proposition 3.12. *Let $\phi : \Omega \rightarrow \phi(\Omega)$ be a K_ϕ -quasiconformal mapping then,*

- 1) ϕ^{-1} is K_ϕ -quasiconformal
- 2) If f is conformal, then both $\phi \circ f$ and $f \circ \phi$ are K_ϕ -quasiconformal.
- 3) If ψ is K_ψ -quasiconformal, then both $\psi \circ \phi$ and $\phi \circ \psi$ are $K_\phi K_\psi$ -quasiconformal.
- 4) If $\phi \in C^1$ then definition 3.10 agrees with the previous definition 2.6.

Proof. The first proposition follows from the fact that ϕ is a homeomorphism then it has a well defined inverse and since $\phi^{-1}(Q)$ is a quadrilateral then by substitution into the requirement for ϕ being a K_ϕ quasiconformal map we get

$$\frac{\text{mod}(\phi^{-1}(Q))}{K_\phi} \leq \text{mod}(Q) \leq K_\phi \text{mod}(\phi^{-1}(Q))$$

Which gives the desired inequalities.

Second statement follows because the module of any quadrilateral is conformally invariant, this since if it were not the underlying canonical rectangle will change module under conformal mappings which it cannot from the previous remark 2.8. Hence,

$$\frac{\text{mod}(Q)}{K_\phi} = \frac{\text{mod}(f(Q))}{K_\phi} \leq \text{mod}(\phi \circ f(Q)) \leq K_\phi \text{mod}(f(Q)) = K_\phi \text{mod}(Q)$$

and

$$\frac{\text{mod}(Q)}{K_\phi} \leq \text{mod}(\phi(Q)) = \text{mod}(f \circ \phi(Q)) = \text{mod}(\phi(Q)) \leq K_\phi \text{mod}(Q)$$

The third statement follows from similar consideration

$$\frac{\text{mod}(Q)}{K_\phi K_\psi} \leq \frac{\text{mod}(\psi(Q))}{K_\phi} \leq \text{mod}(\phi \circ \psi(Q)) \leq K_\phi \text{mod}(\psi(Q)) \leq K_\phi K_\psi \text{mod}(Q)$$

For the last statement, assume that $D_\phi \leq K_\phi$ for all $z \in \Omega$. Let $Q \subset \Omega$ be a quadrilateral, and $f : R_Q \rightarrow R_{\phi(Q)}$ be the map induced from the canonical rectangle $R_Q = R(0, M, M + i, i)$ of Q onto the canonical rectangle $R_{\phi(Q)} = R(0, M', M' + i, i)$, with $M' = \text{mod}(\phi(Q))$ and $M = \text{mod}(Q)$. Because of conformal invariance of the dilation we get $D_f \leq K_\phi$. Hence after multiplying with J_f , we get

$$\frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \leq K_\phi \implies (|f_z| + |f_{\bar{z}}|)^2 \leq K_\phi J_f$$

Since $|f_z|^2 + |f_{\bar{z}}|^2 \leq |f_z|^2 + |f_{\bar{z}}|^2 + 2|f_z||f_{\bar{z}}|$ we get, $|f_z|^2 + |f_{\bar{z}}|^2 \leq K_\phi J_f$. Further note that

$$|f_z|^2 + |f_{\bar{z}}|^2 = |f_x - if_y|^2 + |f_x + if_y|^2 = 2(|f_x|^2 + |f_y|^2) \geq |f_x|^2,$$

hence giving

$$|f_x|^2 \leq K_\phi J_f.$$

Since the module $\text{mod}(R')$ equals the area of the rectangle, we get

$$M' = \text{mod}(R') = \int_R J_f(z) dx dy \geq \frac{1}{K_\phi} \int_R |f_x(z)|^2 dx dy.$$

By applying Cauchy-Schwarz inequality we get

$$\left(\int_0^M |f_x(z)| dx \right)^2 \leq M \int_0^M |f_x(z)|^2 dx.$$

Integrating over $y \in [0, 1]$ we get

$$\int_0^1 \left(\int_0^M |f_x(z)| dx \right)^2 dy \leq M \int_0^1 \int_0^M |f_x(z)|^2 dx dy.$$

We conclude that

$$\frac{1}{K} \int_0^1 \int_0^M |f_x|^2 dx dy \geq \frac{1}{MK} \int_0^1 dy \underbrace{\left(\int_0^M |f_x(z)| dx \right)^2}_{\geq M'} \geq \frac{M'^2}{MK},$$

rearranging gives the first inequality,

$$\text{mod}(\phi(Q)) = M' \leq KM.$$

For the other inequality we compose with suitable conformal mappings, without loss of generality we can look at the point $z = 0$ and assume ϕ is a K -quasiconformal, such that $\phi(0) = 0$, $\phi_z(0) = |\phi_z(0)|$ and $\phi_{\bar{z}}(0) = |\phi_{\bar{z}}(0)|$. Recall

$$D_\phi(0) = \frac{|\phi_z(0)| + |\phi_{\bar{z}}(0)|}{|\phi_z(0)| - |\phi_{\bar{z}}(0)|}.$$

Then

$$\phi(z) = u(z) + iv(z) = |\phi_z(0)|z + |\phi_{\bar{z}}(0)|\bar{z} + o(z).$$

Chose a $\delta > 0$ such that the closure of the rectangle

$$R_\delta = R((-1-i)\delta, (1-i)\delta, (1+i)\delta, (-1+i)\delta)$$

is contained in Ω . Then the b -sides of $\phi(R_\delta)$ are at a distance

$$s_b = 2\delta(|\phi_z(0)| + |\phi_{\bar{z}}(0)|) + o(\delta),$$

with the area

$$m^2(\phi(R_\delta)) = 4\delta^2(|\phi_z(0)|^2 - |\phi_{\bar{z}}(0)|^2 + o(\delta^2))$$

By Rengel's inequality we get

$$\text{mod}(\phi(R_\delta)) \geq \frac{4\delta^2(|\phi_z(0)| + |\phi_{\bar{z}}(0)|)^2 + o(\delta^2)}{4\delta^2(|\phi_z(0)|^2 - |\phi_{\bar{z}}(0)|^2) + o(\delta^2)}$$

Since $\text{mod}(R_\delta) = 1$, by the first inequality in this proof we have,

$$\text{mod}(\phi(R_\delta)) = \frac{\text{mod}(\phi(R_\delta))}{\text{mod}(R_\delta)} \leq K.$$

Rearranging we get,

$$(|\phi_z(0)| + |\phi_{\bar{z}}(0)|)^2 + \frac{o(\delta)}{\delta} \leq K(|\phi_z(0)|^2 - |\phi_{\bar{z}}(0)|^2 + \frac{o(\delta)}{\delta})$$

If we let $\delta \rightarrow 0$ we obtain the required result. $\square$

As an application of Rengel's theorem, we shall prove the superadditivity relation of the module.

Lemma 3.13. *Let $Q_n, n = 1, 2, \dots$, be quadrilaterals with $\overline{Q_n} \subset \overline{Q}$ for each n and no two quadrilaterals contain common interior points. If the a -sides of every Q_n are contained in respective a -side of Q , then*

$$\sum_{n=1}^{\infty} \text{mod}(Q_n) \leq \text{mod}(Q).$$

Further we get equality if and only if all Q_n are rectangles, and $\sum_{n=1}^{\infty} m^2(Q_n) = m^2(Q)$

Proof. Due to conformal invariance we may suppose that Q is a rectangle $R(0, \text{mod}(Q), \text{mod}(Q) + i, i)$. Let $\Gamma_{Q_n}^a$ denote the family of curves contained in Q_n connecting the a -sides of each of Q_n . Then the shortest path is at least 1. That is

$$s_a(Q_n) = \inf_{\gamma \in \Gamma_{Q_n}^a} l(\gamma) \geq 1$$

with the a -sides being the horizontal lines and the shortest path is a straight line. By Rengel's inequality Theorem (3.11) we obtain

$$\sum_n \text{mod}(Q_n) \leq \sum_n \frac{m^2(Q_n)}{s_a(Q_n)} \leq \sum_n m^2(Q_n) \leq m^2(Q) = \text{mod}(Q)$$

By Rengel's inequality we obtain equality if and only if each Q_n is a rectangle and $\sum_n m^2(Q_n) = m^2(Q)$. $\square$

We will now build to prove sequential continuity of the module over the quadrilaterals, and since the topological space containing all the quadrilaterals and the topology from $\mathbb{C}$ is first countable it follows that the module is continuous. For two finite points z_1, z_2 denote by $k(z_1, z_2)$ the **spherical distance**, that is

$$k(z_1, z_2) = \arctan \left| \frac{z_1 - z_2}{1 + \bar{z}_1 z_2} \right|$$

For $z_2 = \infty$ we have $k(z_1, \infty) = \arctan |1/z_1|$. This distance is defined on the Riemann sphere. Now we will define two notions of convergence for quadrilaterals. Let us consider a sequence of quadrilaterals Q_n with sides a_1^n, a_2^n, b_1^n and b_2^n for each n . We say that the sequence of quadrilaterals converge to Q from inside if $\overline{Q_n} \subset \overline{Q}$ for every n , and that for any $\epsilon > 0$ there exists some n_ϵ such that for any $n \geq n_\epsilon$ and every point on a_i^n, b_i^n ($i = 1, 2$) has a spherical distance less than ϵ from the corresponding sides a_i and b_i . With this definition we get the following lemma.

Lemma 3.14. *If a sequence of quadrilaterals Q_n converge to Q from inside, then*

$$\lim_{n \rightarrow \infty} \text{mod}(Q_n) = \text{mod}(Q).$$

Proof. Note that the canonical map $\pi_Q : \bar{Q} \rightarrow R_Q$ is uniformly continuous, since it is continuous and defined on a compact set $\bar{Q}$, with $R_Q = R(0, M(Q), M(Q) + i, i)$. That is for any $z_1, z_2 \in \bar{Q}$, we have

$$\lim_{n \rightarrow \infty} \sup_{k(z_1, z_2) < \delta} k(\pi_Q(z_1), \pi_Q(z_2)) = 0$$

It follows that $\pi_Q(Q_n) = Q'_n$ converges to R_Q from inside in both the spherical and euclidean distance, since the two convergence coincide on the Complex plane. Now let $0 < \epsilon < \min(1/2, \text{mod}(Q)/2)$, because of convergence of the quadrilaterals there exists some $n_\epsilon > 0$ with the property that for any $n > n_\epsilon$, every side of Q'_n will be contained in a closed slit of thickness ϵ around the corresponding side of R_Q . Hence from Rengel's inequality, Theorem (3.11), $s_a(Q'_n) \geq 1 - 2\epsilon$ and $s_b(Q'_n) \geq \text{mod}(Q) - 2\epsilon$. Furthermore the canonical map is conformal hence $\text{mod}(Q'_n) = \text{mod}(Q_n)$ and $m^2(Q'_n) \leq \text{mod}(Q)$, we get

$$\frac{(\text{mod}(Q) - 2\epsilon)^2}{\text{mod}(Q)} \leq \text{mod}(Q_n) \leq \frac{\text{mod}(Q)}{(1 - 2\epsilon)^2}.$$

Letting $\epsilon \rightarrow 0$ will give the desired result. $\square$

For any quadrilateral we can construct a sequence of quadrilaterals converging to Q from inside. This is done by first mapping the interior of Q which is a simply connected open set, conformally by a function f onto the quadrilateral that consisting of the open unit disc and the vertices $z_i, i = 1, 2, 3, 4$. After this we consider the quadrilaterals Q'_n consisting of the disc $|z| < 1 - \frac{1}{n}$ with the scaled down vertices $(1 - \frac{1}{n})z_i, i = 1, 2, 3, 4$. The inverse image of the quadrilaterals $Q_n = f^{-1}(Q'_n)$ converge to Q from inside, since the inverse function is uniformly continuous on the closed disc.

More generally we say that the sequence of quadrilaterals Q_n converges to Q if for any $\epsilon > 0$ there corresponds an n_ϵ such that for $n \geq n_\epsilon$ every point on a_i^n and b_i^n and interior points of Q_n have distance at most ϵ from a_i, b_i and Q respectively. The more general statement of Lemma (3.14), is the following.

Theorem 3.15. *If a sequence of quadrilaterals Q_n converges to a quadrilateral Q , then*

$$\lim_{n \rightarrow \infty} \text{mod}(Q_n) = \text{mod}(Q).$$

Proof. We construct a sequence of nested Jordan domains $G_1 \supset G_2 \supset \dots \supset \bar{Q}$, which converges monotonically to Q , where additionally the boundary converges in a stronger sense due to Fréchet.

Definition 3.16. [Convergence in the Fréchet sense] *A sequence of Jordan domains G_n is said to converge in the Fréchet sense to a Jordan domain Q if, for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that for all $n > n_\epsilon$ there exists a homeomorphism*

$$g_n : \partial G_n \rightarrow \partial Q$$

satisfying

$$\sup_{z \in \partial G_n} k(g_n(z), z) < \epsilon,$$

where $k(\cdot, \cdot)$ denotes the spherical metric on $\hat{\mathbb{C}}$.

This can be done by first conformally mapping the complement of $\bar{Q}$ onto the open unit disc $\mathbb{D}$, then taking the complement of the preimage of a sequence of disc $\mathbb{D}_n$ that converge to $\mathbb{D}$ from inside. This will form the sequence G_n as desired. We will use the following lemma

Lemma 3.17. [Radó] *Let $G_1 \supset G_2 \supset \dots \supset \bar{Q}$ be a sequence of Jordan domains that converge in the sense of Fréchet. Further let $f_n : R \rightarrow G_n$ and $f : R \rightarrow Q$ be conformal mapping of a Jordan domain R , which are normalized at a point $z_0 \in R$, that is;*

$$f_n(z_0) = f(z_0) \quad \arg f'_n(z_0) = \arg f'(z_0).$$

Then $f_n(z) \rightarrow f(z)$ uniformly on $\bar{R}$.

We apply Radó's lemma in the following way. Let R_Q denote the canonical rectangle of Q , $f = \pi_Q^{-1}$ and f_n be the normalized conformal mapping of the domain R onto G_n at the point $z_0 \in R_Q$. Because of Radó we know f_n converges uniformly to f . Note that $f^{-1} = \pi_Q$ which is continuous and defined on a compact set R_Q , hence uniformly continuous. It follows that $f_n^{-1}(Q)$ converges to R_Q from inside. Thus for every $\epsilon > 0$ there exists some $N > 0$ such that for every every point on a side of $f_N^{-1}(Q) = Q'$ is at most $\epsilon/2$ away from the respective side in R .

Now consider the original sequence Q_n of quadrilaterals converging to Q . For some n_1 onward, every Q_n is contained in $\bar{G}_N$, thus $f_n^{-1}(Q_n) = Q'_n$ is contained in $\bar{R}_Q$. From the uniform continuity of f_N^{-1} it follows that Q'_n , $n \geq n_1$ converges to the quadrilateral Q' . Hence we can find some $n_\epsilon > 0$ such that $n \geq n_\epsilon$, the points of each side of Q'_n lies in strip of width ϵ from the corresponding sides of R_Q . Hence $\text{mod}(Q'_n) = \text{mod}(Q_n)$. Hence we can apply the inequality in the proof of Lemma (3.14), proving the statement. $\square$

The following is a further justification for the geometric definition of quasiconformality.

Proposition 3.18.

- 1) A 1-Quasiconformal map is conformal
- 2) Let ϕ_n be a sequence of K -quasiconformal mappings that converge uniformly on every compact subset of Ω to a homeomorphism ϕ , then ϕ is also a K -quasiconformal map.

Proof. Since every point of Ω is contained in the interior for some quadrilateral Q with $\bar{Q} \subset \Omega$. It is sufficient to prove conformality of the 1-quasiconformal map ϕ on the quadrilaterals. First we map both Q and the quadrilateral $Q' = \phi(Q)$ conformally down to the canonical rectangles $R_Q = R(0, M, M + i, i)$ and $R'_Q = R(0, M', M' + i, i)$, respectively. For a 1-quasiconformal mapping we have by the definition that $\text{mod}(Q) = \text{mod}(Q')$. Following the diagram below we have $f = \pi_2 \circ \phi \circ \pi_1^{-1}$.

$$\begin{array}{ccc} Q & \xrightarrow{\phi} & \phi(Q) \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ R & \xrightarrow{f} & R' \end{array}$$

We now aim to prove, f reduces to the identity map. Which implies that the quadrilaterals Q and Q' are in the same conformal equivalence class, implying that ϕ is conformal.

Chose an interior point of R_Q , $z_0 = x_0 + iy_0$. We divide the rectangle R_Q into two rectangles $R_{Q_1} = R(0, x_0, x_0 + i, i)$ and $R_{Q_2} = R(x_0, M, M + i, x_0 + i)$, let $R'_i = f(R_{Q_i})$. Then by 1-quasiconformality we have

$$\text{mod}(R'_1) + \text{mod}(R'_2) = \text{mod}(R_1) + \text{mod}(R_2) = x_0 + (\text{mod}(R) - x_0) = \text{mod}(R')$$

However since R' is a rectangle and we have equality $\text{mod}(R'_1) + \text{mod}(R'_2) = \text{mod}(R')$ which is equivalent by Lemma 3.13 with R'_1 and R'_2 are rectangles. With this we get that the real part of $f(z_0)$ equals x_0 hence x_0 is invariant under f , that same can be done for y_0 , Since we picked z_0 arbitrary we have demonstrated that f is the identity map.

For the second proposition it suffices to prove for any quadrilateral, $Q, \overline{Q} \subset \Omega$ the bound $\text{mod}(\phi(Q)) \leq K\text{mod}(Q)$ holds. To do so, we construct a sequence of quadrilaterals converging to Q from inside. This can be done following the construction discussed before Theorem 3.15. By selecting a subtle subsequence starting at some $N > 0$, we get $\phi_n(Q_n) \subset \phi(Q)$. Since ϕ_n converges uniformly to ϕ we obtain that $\phi_n(Q_n)$ converge to $\phi(Q)$ from inside. Thus by Lemma 3.14 we get $\lim_{n \rightarrow \infty} \text{mod}(\phi_n(Q_n)) = \text{mod}(\phi(Q))$ and $\lim_{n \rightarrow \infty} \text{mod}(Q_n) = \text{mod}(Q)$. Further since for each n we have $\text{mod}(\phi_n(Q_n)) \leq K\text{mod}(Q_n)$, we have the desired result, $\text{mod}(\phi(Q)) \leq K\text{mod}(Q)$ $\square$

Another extension of the second proposition is one can remove the condition that the sequence of ϕ_n converges to a homeomorphism. This will not be proven here, consult [Lehto & Virtanen (1973) chapter 2.5].

3.4 Module of a Ring domain

We will show in this section that the theory can be developed starting from doubly connected regions, also called ring domains. Conformal equivalence classes of such domains are completely determined by a single numerical invariant, the modulus of the ring domain. The definition of this modulus is based on the following theorem.

Theorem 3.19. [Ahlfors (2021), Chapter 6 Theorem 10]

Let Ω be a 2-multiply connected region (that is the complement of Ω contains exactly 2 connected components). There exists a conformal mapping of Ω onto some annulus $\{z : 0 \leq r_1 \leq |z| \leq r_2 \leq \infty\}$.

This theorem, ensures that we can calculate the moduli of a ring domain region by first conformally mapping it onto two concentric circles. The moduli will then be determined by the radii of these concentric circles. The conformal equivalent annulus of a ring domain will be named the **canonical annulus** and the corresponding map the **canonical map**, which we will denote by R' and π_R respectively. Let R be a ring domain and R' be the canonical annulus, then if $r_1 > 0$ and $r_2 < \infty$, the ratio r_2/r_1 is the same for each canonical images of R . We define the module of a ring domain to be the following

$$\text{mod}(R') = \log \frac{r_2}{r_1}. \quad (3.1)$$

If $r_1 = 0$ or $r_2 = \infty$ then we define $\text{mod}(R') = \infty$, which occurs if and only if the boundary components contain precisely a single point. In the case for $\text{mod}(R') = \infty$ the ring domains fall into two conformal equivalence cases. The first case is classified by $r_1 = 0, r_2 < \infty, r_1 > 0, r_2 = \infty$ which is equivalent by inversion with the second case $r_1 = 0, r_2 = \infty$.

Next we will show that the module of a ring module can be determined without the use of conformal mappings. Let R be some ring domain, with R_1 and R_2 the bounded and the unbounded components of the complement of R , let π_R be the canonical map of R onto $R' = \{w : r_1 \leq |w| \leq r_2\}$.

Denote by $\mathcal{C}_1$ preimages of the circles $\{w : |w| = r\}$ for all $r_1 < r < r_2$. Now every curve $\gamma \in \mathcal{C}_1$ separate both of the components R_1 and R_2 . Hence $\mathcal{C}_1 \subset \mathcal{C}$, with $\mathcal{C}$ being the family of locally rectifiable curves separating R_1 from R_2 . Denote by $\mathcal{P}$ a family of non-negative Borel-measurable with finite weighted area on R .

$$\int_R \rho^2 dm^2(z) < \infty.$$

The family $\mathcal{P}$ are the admissible metrics on R . Associate for every function $\rho \in \mathcal{P}$ the function $\rho_1(\pi_R(z)) = \rho(z)|\pi_R(z)/\pi'_R(z)|$, further we will assume that $|\pi'_R/\pi_R| \in \mathcal{P}$. We will estimate the ρ weighted plane Leagues measure of the ring domain, $m_\rho^2(R)$. From the first chapter notation we have $A(\rho\chi_R) = m_\rho^2(R)$. First by a change of variables via the conformal map π_R to the concentric annulus R' , with the Jacobian $dxdy = \frac{dudv}{|\pi'_R|^2}$, followed by a change to polar coordinates and applying Cauchy-Schwartz's inequality we obtain,

$$\begin{aligned} m_\rho^2(R) &= \int \int_R \rho^2(z) dxdy = \int \int_A \rho^2(\pi_R^{-1}(w)) \frac{dudv}{|\pi'_R(\pi_R^{-1}(w))|^2} \\ &= \int_{r_1}^{r_2} \int_0^{2\pi} \frac{\rho^2(\pi_R^{-1}(re^{i\alpha}))}{|\pi'_R(\pi_R^{-1}(re^{i\alpha}))|^2} r dr d\alpha = \int_{r_1}^{r_2} \frac{1}{r} \int_0^{2\pi} \rho_1^2(re^{i\alpha}) d\alpha dr \\ &\geq \frac{1}{2\pi} \int_{r_1}^{r_2} \frac{1}{r} \left(\int_0^{2\pi} \rho_1(re^{i\alpha}) d\alpha \right)^2 dr. \end{aligned}$$

The forth equality follows from the following,

$$\rho_1^2(\pi_R(z)) = \rho^2(z) \frac{|\pi_R(z)|^2}{|\pi'_R(z)|^2}.$$

Evaluating for $z = \pi_R^{-1}(w)$ with $w = re^{i\alpha}$

$$\rho_1^2(re^{i\alpha}) = \frac{\rho^2(\pi_R^{-1}(re^{i\alpha}))}{|\pi'_R(\pi_R^{-1}(re^{i\alpha}))|^2} |\pi_R(\pi_R^{-1}(re^{i\alpha}))|^2 = \frac{\rho^2(\pi_R^{-1}(re^{i\alpha}))}{|\pi'_R(\pi_R^{-1}(re^{i\alpha}))|^2} r^2.$$

Rearranging will give,

$$\frac{\rho^2(\pi_R^{-1}(re^{i\alpha}))}{|\pi'_R(\pi_R^{-1}(re^{i\alpha}))|^2} = \frac{\rho_1^2(re^{i\alpha})}{r^2}.$$

If $C_1 \in \mathcal{C}_1$, that is $\pi_R(C_1)$ is a circle, and we can parametrize the curve C_1 by $z(\alpha) = \pi_R^{-1}(re^{i\alpha})$ so, $|dz| = |(\pi_R^{-1})'(re^{i\alpha})re^{i\alpha}|d\alpha = \frac{r}{|\pi'_R(\pi_R^{-1}(re^{i\alpha}))|}d\alpha$.

$$\int_0^{2\pi} \rho_1(re^{i\alpha}) d\alpha = \int_0^{2\pi} \rho(\pi_R^{-1}(re^{i\alpha})) \frac{r}{|\pi'_R(\pi_R^{-1}(re^{i\alpha}))|} \cdot \frac{\pi'_R(\pi_R^{-1}(re^{i\alpha}))|dz|}{r}$$

$$= \int_0^{2\pi} \rho(\pi_R^{-1}(re^{i\alpha}))|dz| = \int_{C_1} \rho(z)|dz| \geq \inf_{C \in \mathcal{C}_1} l_\rho(C) \geq \inf_{C \in \mathcal{C}} l_\rho(C).$$

It follows from above that, if $\text{mod}(R) < \infty$,

$$\begin{aligned} m_\rho^2(R) &\geq \frac{1}{2\pi} \int_{r_1}^{r_2} \left(\inf_{C \in \mathcal{C}} l_\rho(C) \right)^2 \frac{dr}{r} \\ &= \frac{1}{2\pi} \log \frac{r_2}{r_1} \left(\inf_{C \in \mathcal{C}} l_\rho(C) \right)^2 = \frac{1}{2\pi} \text{mod}(R) \left(\inf_{C \in \mathcal{C}} l_\rho(C) \right)^2. \end{aligned}$$

Solving for $\text{mod}(R)$ gives

$$\text{mod}(R) \leq \frac{2\pi m_\rho^2(R)}{\left(\inf_{C \in \mathcal{C}} l_\rho(C) \right)^2}.$$

Since this inequality holds for each $\rho \in \mathcal{P}$, we can take the infimum to obtain

$$\text{mod}(R) \leq \inf_{\rho \in \mathcal{P}} \frac{2\pi m_\rho^2(R)}{\left(\inf_{C \in \mathcal{C}} l_\rho(C) \right)^2}.$$

Following the inequalities and plugging in $\rho = |\pi'_R / \pi_R|$ gives equality the whole way. Hence we attain the equality

$$\text{mod}(R) = \inf_{\rho \in \mathcal{P}} \frac{2\pi m_\rho^2(R)}{\left(\inf_{C \in \mathcal{C}} l_\rho(C) \right)^2} \quad (3.2)$$

In the case $\text{mod}(R) = \infty$ we have $m_\rho^2(R) = \infty$, that is if, $\inf_{C \in \mathcal{C}} l_\rho(C) > 0$.

One can characterize the module of a ring domain in a similar way, but by taking the supremum instead over all the admissible metrics on R . Denote by $\mathcal{C}_2$ the family of preimages $f^{-1}(C)$ of the segments $C = \{w : r_1 < |w| < r_2, \arg w = \alpha\}$. Let $\mathcal{C}^*$ be the family of locally rectifiable curves in R which are connecting the components of the complement of R , having a limit point on each of the components. Note $\mathcal{C}_2 \subset \mathcal{C}^*$. Again let $\mathcal{P}$ be family of admissible metrics on R . Following the same type of calculations gives,

$$m_\rho^2(R) = \int \int_R \rho^2 dm^2(z) = \int_0^{2\pi} \int_{r_1}^{r_2} \rho_1^2(re^{i\alpha}) \frac{dr}{r} d\alpha \geq \frac{1}{\log \frac{r_2}{r_1}} \int_0^{2\pi} \left(\int_{r_1}^{r_2} \rho_1(re^{i\alpha}) \frac{dr}{r} \right)^2 d\alpha,$$

The inequality follows from Cauchy-Schwarz inequality,

$$\left(\int_{\log(r_1)}^{\log(r_2)} \rho_1(e^t e^{i\alpha}) dt \right)^2 \leq \left(\int_{\log(r_1)}^{\log(r_2)} 1 dt \right) \cdot \left(\int_{\log(r_1)}^{\log(r_2)} \rho_1^2(e^t e^{i\alpha}) dt \right) = \log \left(\frac{r_2}{r_1} \right) \int_{r_1}^{r_2} \frac{\rho_1^2(re^{i\alpha})}{r} dr.$$

The length of the curve is bounded below,

$$l_\rho(C) = \int_{r_1}^{r_2} \rho_1(re^{i\alpha}) \frac{dr}{r} \geq \inf_{C \in \mathcal{C}^*} l_\rho(C).$$

Hence,

$$m_\rho^2(R) \geq \frac{1}{\text{mod}(R)} \int_0^{2\pi} \left(\int_{r_1}^{r_2} \rho_1(re^{i\alpha}) \frac{dr}{r} \right)^2 d\alpha \geq \frac{1}{\text{mod}(R)} \int_0^{2\pi} \left(\inf_{C \in \mathcal{C}^*} l_\rho(C) \right)^2 d\alpha$$

$$= \frac{2\pi}{\text{mod}(R)} \left(\inf_{C \in \mathcal{C}^*} l_\rho(C) \right)^2.$$

This inequality holds for each $\rho \in \mathcal{P}$ hence

$$\text{mod}(R) \geq \sup_{\rho \in \mathcal{P}} \frac{2\pi \left(\inf_{C \in \mathcal{C}^*} l_\rho(C) \right)^2}{m_\rho^2(R)}$$

As before we attain equality for $\rho = |\pi'_R / \pi_R|$. The above calculations gives us the following theorem.

Theorem 3.20. *Let R be a ringdomain and $\mathcal{C}$ and $\mathcal{C}^*$ be locally rectifiable separating or joining the two components of R respectively. Further let $\mathcal{P}$ be the family of admissible metrics on R containing $\rho = |\pi'_R / \pi_R|$, then the module of the ring domain is,*

$$\text{mod}(R) = 2\pi \inf_{\rho \in \mathcal{P}} \frac{m_\rho^2(R)}{\left(\inf_{C \in \mathcal{C}} l_\rho(C) \right)^2} \quad (3.3)$$

$$\text{mod}(R) = 2\pi \sup_{\rho \in \mathcal{P}} \frac{\left(\inf_{C \in \mathcal{C}^*} l_\rho(C) \right)^2}{m_\rho^2(R)}. \quad (3.4)$$

With the infimum/supremum is attained when $\rho = |\pi'_R / \pi_R|$.

This theory can be extended into more general frameworks, considering both more general families of functions $\mathcal{P}$ and curve families $\mathcal{C}$. We will not do this here, but rather refer to chapter 3.4 in [Lehto & Virtanen (1973)].

With the definition established, the rest of this section will focus on proving results for ring domains that are analogous to those for quadrilaterals. Specifically, we will present an analogue of Rengel's inequality for ring domains, prove the superadditivity and continuity of the module, and relate the two notions of modules: those of ring domains and those of quadrilaterals.

Let R be a ring domain with $\mathcal{C}$ be the locally rectifiable curve family separating the two components, if R separate 0 and ∞ then each curve $\gamma \in \mathcal{C}$ will also separate these two points, further we have

$$2\pi = \left| \int_\gamma d \arg z \right| \leq \int_\gamma \frac{|dz|}{|z|}. \quad (3.5)$$

Setting $\rho = 1/|z|$ in equation (3.3), we get the analogue of Rengel's inequality,

$$\text{mod}(R) \leq \frac{1}{2\pi} \int \int_R \frac{dm^2(z)}{|z|^2}. \quad (3.6)$$

If R is some annulus centered at 0, then with $\pi_R = z$ we get from Theorem (3.20) that infimum is attained at when $\rho = |\pi'_R / \pi_R| = 1/|z|$, hence we get equality in equation (3.6). To get a necessary condition for equality, consider the inverse of the canonical map $g = \pi_R^{-1}$ onto the annulus $\{w : r_1 < |w| < r_2\}$, hence we get $z = g(w)$ and $dm^2(z) = |g'(w)|^2 dm^2(w)$, combined with equality in equation (3.6), then

$$\log \frac{r_2}{r_1} = \text{mod}(R) = \frac{1}{2\pi} \int \int_R \frac{dm^2(z)}{|z|^2}$$

$$= \frac{1}{2\pi} \int \int_{g(R)} \frac{|g'(w)|^2 dm^2(w)}{|g(w)|^2} = \frac{1}{2\pi} \int_{r_1}^{r_2} \int_0^{2\pi} \left| \frac{g'(w)}{g(w)} \right|^2 r dr d\alpha. \quad (3.7)$$

With last equality follows by polar coordinate change, where $w = re^{i\alpha}$. If $|w| = r$ is constant we have since the winding number is 1 around the origin

$$\int_0^{2\pi} \frac{g'(w)}{g(w)} w d\alpha = \int_{|w|=r} d \arg g(w) = \pm 2\pi. \quad (3.8)$$

If $\text{mod}(R) < \infty$ the inequality

$$\frac{1}{2\pi r} \left| \int_0^{2\pi} \frac{g'(w)}{g(w)} w d\alpha \right|^2 \leq \frac{1}{2\pi r} \int_0^{2\pi} \left| \frac{g'(w)}{g(w)} \right|^2 d\alpha \cdot \int_0^{2\pi} |w|^2 d\alpha = \int_0^{2\pi} \left| \frac{g'(w)}{g(w)} \right|^2 r d\alpha.$$

Which simplifies to

$$\frac{1}{2\pi r} \left| \int_0^{2\pi} \frac{g'(w)}{g(w)} w d\alpha \right|^2 \leq \int_0^{2\pi} \left| \frac{g'(w)}{g(w)} \right|^2 r d\alpha.$$

This becomes an equality for every r , $r_1 < r < r_2$, since if we assume there exists some $\hat{r} \in (r_1, r_2)$ such that we have strict inequality, then after integrating over r the strict inequality is preserved giving

$$\frac{1}{2\pi} \int_{r_1}^{r_2} \frac{1}{r} \left| \int_0^{2\pi} \frac{g'(w)}{g(w)} w d\alpha \right|^2 dr < \int_{r_1}^{r_2} \int_0^{2\pi} \left| \frac{g'(w)}{g(w)} \right|^2 \hat{r} d\alpha dr.$$

Because of equation (3.8), the left hand side simplifies to $2\pi \log(\frac{r_2}{r_1})$, while the right hand side to $\log(\frac{r_2}{r_1})$ by using the equality assumption in equation (3.7). This is a contradiction.

This is only possible if $wg'(w)/g(w)$ has a constant value on every circle $|w| = r$. Since

$$\int_0^{2\pi} \frac{g'(w)}{g(w)} w d\alpha = \pm 2\pi,$$

implies that either $wg'(w)/g(w) = \pm 1$ hence either $g(w)/w$ or $wg(w)$ is constant and we get equality in equation (3.6) if and only if R is a annulus with center 0 or $\text{mod}(R) = \infty$.

Next we prove super additivity for the module of ring domains.

Theorem 3.21. [Superadditivity of the module for ring domains]

Let R and R_n , $n = 1, 2, \dots$ be ring domains such that $R = \cup_n R_n$ and each pair of R_n are mutually disjoint. If every ring domain R_n separates the two components of the complement of R , then

$$\sum_n \text{mod}(R_n) \leq \text{mod}(R),$$

further if $\text{mod}(R) < \infty$, equality is attained if and only if every R_n is an annulus and $\sum_n m^2(R_n) = m^2(R)$.

Proof. Without loss of generality we can assume that R is an annulus $R = \{z : 0 < r_1 < |z| < r_2 < \infty\}$, other wise we can conformally map it onto one by its canonical map π_R . Each $\text{mod}(R_n)$ is finite now by equation (3.6) we get

$$\text{mod}(R_n) \leq \frac{1}{2\pi} \int \int_{R_n} \frac{dm^2(z)}{|z|^2}$$

where equality holds if and only if R_n is an annulus of center 0. Hence

$$\sum_n \text{mod}(R_n) \leq \sum_n \frac{1}{2\pi} \int \int_{R_n} \frac{dm^2(z)}{|z|^2} = \frac{1}{2\pi} \int \int_{\cup R_n} \frac{dm^2(z)}{|z|^2} \stackrel{\text{Disjoint}}{=} \frac{1}{2\pi} \int \int_R \frac{dm^2(z)}{|z|^2} = \text{mod}(R).$$

With last equality follow since R is an annulus. $\square$

Next we state the continuity of the module for ring domains, which corresponds to Lemma 3.14, for the case of quadrilaterals. First we formulate the definition of convergence of ring domains from inside.

Definition 3.22. A sequence of ring domains R_n converges to the ring domain R from inside $R_n \subset R$, if for every $\epsilon > 0$ there exists some n_ϵ such that for every $n \geq n_\epsilon$ and every point in the two components of the complement of R_n is a spherical distance less than ϵ away from the corresponding component in complement of R .

Further we call a ring domain analytical if its boundary consist of two analytical curves, note that for each ring domain R there exists some sequence of analytical ring domains R_n which converge to R from inside. With this we can state the following continuity theorem.

Lemma 3.23. If R_n is a sequence of ring domains which converge from inside to the ring domain R , then

$$\lim_{n \rightarrow \infty} \text{mod}(R_n) = \text{mod}(R).$$

For the proof note that for some n onward the complement of R_n will lie outside a compact subset of R , hence by the Theorem 3.19, we can map R_n and R conformally onto concentric circles, further the images of R_n will converge from inside to the canonical ring domain R' . From some n onward R'_n will be contained in R' , and the lemma follows from the superadditivity of the module.

The notions of moduli for quadrilaterals and ring domains are closely connected. This connection is made more clear if one cut an annulus $R = \{z : 0 < r_1 < |z| < r_2 < \infty\}$ along the real axis and map the new simply connected domain conformally via the logarithmic function onto a rectangle with the ratios of sides

$$\frac{1}{2\pi} \log \frac{r_2}{r_1} = \frac{1}{2\pi} \text{mod}(R).$$

We will now state a lemma estimating the moduli of quadrilaterals with the moduli of a specific ring domain, and vice versa.

Lemma 3.24. Let Q_n , $n = 1, 2, \dots$ be quadrilaterals having no common interior points. Let R be a ring domain separating the a -sides of each quadrilateral. That is for each quadrilateral Q_n the corresponding a -sides are separated by the ringdomain R . Then

$$\sum \text{mod}(Q_n) \leq \frac{2\pi}{\text{mod}(R)}.$$

Equality is attained if and only if R is an annulus, each of the a -sides lie on the boundary of R , the b -sides are radial segments and $\sum m^2(Q_n) = m^2(R)$.

One can characterize quasiconformality by bounding modules for ring domains. That is the family of orientation-preserving homomorphisms which bound the module for quadrilaterals and ring domains coincide.

Theorem 3.25. *An orientation-preserving homeomorphism ϕ of a domain Ω is an K -quasiconformal map if and only if*

$$\text{mod}(\phi(R)) \leq K \text{mod}(R) \quad (3.9)$$

for every ring domain $R, \bar{R} \subset \Omega$.

Proof. Let ϕ be a orientation-preserving homomorphism, first we prove that if (3.9) holds for every ring domain A , then ϕ is a K -quasiconformal mapping. To do this we need to show that the following inequality holds,

$$\text{mod}(\phi(Q)) \leq K \text{mod}(Q),$$

for any quadrilateral Q . Since the module of a ring domain are conformally invariant, we can replace Q and $\phi(Q)$ by their canonical rectangles $R_Q = R(0, M, M + i, i)$ and $R_{\phi(Q)} = R(0, M', M' + i, i)$ respectively, with $M = \text{mod}(Q)$ and $M' = \text{mod}(\phi(Q))$. Let $R'_n \subset R_{\phi(Q)}$ be a sequence of rectangles whose sides are a constant distance $1/n$ away from the corresponding sides of $R_{\phi(Q)}$, to ensure $R'_n \subset R_{\phi(Q)}$ we take n big enough such that $2/n < \min(1, M)$, hence they converge from inside to $R_{\phi(Q)}$. Let $R_n = \phi^{-1}(R'_n)$, note that we also have convergence from inside of R_n to R , giving the convergence of the module

$$\lim_{n \rightarrow \infty} \text{mod}(R_n) = M.$$

Subdivide R'_n into n^3 vertical rectangles $R'_{n,k}$, $1 \leq k \leq n^3$, with equal width, which is

$$\frac{(M' - 2/n)}{n^3}.$$

Each one of these rectangles will be considered as a quadrilateral with horizontal a -sides. Let $R_{n,k} = \phi^{-1}(R'_{n,k})$. Since $\bigcup_{k=1}^{n^3} R_{n,k} = R_n$ then by superadditivity of modules we get

$$\sum_{k=1}^{n^3} \text{mod}(R_{n,k}) \leq \text{mod}(R_n).$$

Hence by the pigeon hole principle, there exists some p , with

$$\text{mod}(R_{n,p}) \leq \frac{\text{mod}(R_n)}{n^3}. \quad (3.10)$$

Now consider the following figure,

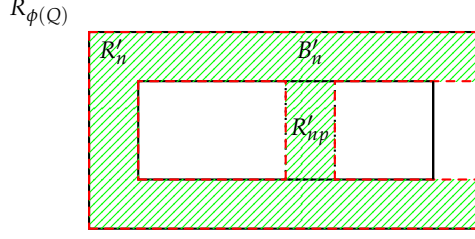


Figure 4: The outer rectangle is $R_{\phi(Q)}$, while R'_n is the inner rectangle. The ringdomain B'_n is defined to be the green shaded region with the dashed red lines as the boundary. Lastly R'_{np} is the rectangle inside R'_n with the b -sides contained in each of the boundary components of B'_n

Let B'_n be the ring domain that has the red dashed lines as boundary and the green shaded region as the interior illustrated in the figure 4. Let B_n be the pre image of B'_n , that is $B_n = \phi^{-1}(B'_n)$. Since the b -sides of R'_{np} lie one on each boundary component of B'_n then the same is true about R_{np} , that is each of the b -sides of R_{np} lie one on each boundary component of B_n . It now follows from Lemma 3.24, that

$$\frac{1}{\text{mod}(R_{n,p})} \leq \frac{2\pi}{\text{mod}(B_n)}.$$

Using the bound in equation (3.10), we get

$$\text{mod}(B_n) \leq 2\pi \text{mod}(R_{n,p}) \leq \frac{2\pi}{n^3} \text{mod}(R_n).$$

We need a lower bound for $\text{mod}(B'_n)$, to this extent, chose $n^2 > M'$, and set

$$\rho(z) = \begin{cases} \frac{n^3}{M' - \frac{2}{n}}, & \text{if } z \in R'_{n,p} \\ n, & \text{otherwise in } B'_n. \end{cases}$$

Then for any arc γ joining the boundary components of B'_n we have $l_\rho(\gamma) \geq 1$, follows since the shortest path from $\partial R'_{n,p}$ to B'_n has euclidean length $1/n$. The curve γ will completely be contained in B'_n ,

$$l_\rho(\gamma) = \int_\gamma n |dz| = n \int_\gamma |dz| \geq n \frac{1}{n} = 1.$$

Lemma 3.20 gives,

$$\text{mod}(B'_n) \geq \frac{2\pi}{m_\rho^2(B'_n)} \geq \frac{2\pi}{n^3 \left(\frac{M'}{n} + \frac{1}{M' - \frac{2}{n}} \right)}.$$

Finally, we deduce that

$$\frac{\text{mod}(B'_n)}{\text{mod}(B_n)} \geq \frac{1}{\left(\frac{M'}{n} + \frac{1}{M' - \frac{2}{n}} \right) \text{mod}(R_n)}$$

The expression on the right tend to M'/M as $n \rightarrow \infty$, since $\lim_{n \rightarrow \infty} \text{mod}(R_n) = M$, it follows that $M' \leq KM$, which proves the first implication.

For the converse let B be any ring domain and $B' = \phi(B)$. We prove that if ϕ is a K -quasiconformal map then it bounds the module of $\phi(B)$ by the module of B , that is

$$\frac{1}{K} \text{mod}(B) \leq \text{mod}(B') \leq K \text{mod}(B).$$

Note we only need to prove one of these inequalities, the other inequality follows from the same proof but applied on the inverse of ϕ , hence we will focus on the upper bound. Since no assumption was made about the boundary behaviour of the mapping ϕ , we first approximate B' by analytic ring domains. If $\epsilon > 0$ is given, we can construct an analytic ringdomain $B'_\epsilon, \overline{B'_\epsilon} \subset B'$, which separates the boundary components of B' such that,

$$\text{mod}(B'_\epsilon) > \text{mod}(B') - \epsilon.$$

Then $B_\epsilon = \phi^{-1}(B'_\epsilon)$ is a ring domain bounded by two Jordan curves, and it follow from the monotonicity of the module that,

$$\text{mod}(B_\epsilon) < \text{mod}(B).$$

The aim now is to prove $\text{mod}(B'_\epsilon) \leq K \text{mod}(B_\epsilon)$ this will immediately give the desired result by letting $\epsilon \rightarrow 0$, that is because we would have,

$$\text{mod}(B') - \epsilon < \text{mod}(B'_\epsilon) \leq K \text{mod}(B_\epsilon) < \text{mod}(B).$$

Since the boundary of both B_ϵ and B'_ϵ consist of Jordan curves, we can extend ϕ topologically to the boundary. Since the canonical mapping of B_ϵ can be topologically extended to $\overline{B_\epsilon}$ we may restrict ourself to the annulus case. Cut the annuli along the real axis giving us two domains which can be viewed of as two quadrilaterals, with the cuts as the b -sides. Name them Q_1 and Q_2 with $Q'_i = \phi(Q_i)$, now it follows from Lemma 3.24 that,

$$\text{mod}(Q'_1) + \text{mod}(Q'_2) \leq \frac{2\pi}{\text{mod}(B'_\epsilon)}, \quad \text{mod}(Q_1) + \text{mod}(Q_2) = \frac{2\pi}{\text{mod}(B_\epsilon)}.$$

Since ϕ is a K -quasiconformal mapping we get,

$$\text{mod}(B'_\epsilon) \leq \frac{2\pi}{\text{mod}(Q'_1) + \text{mod}(Q'_2)} \leq \frac{2\pi}{\frac{1}{K}(\text{mod}(Q_1) + \text{mod}(Q_2))} = K \text{mod}(B_\epsilon).$$

With last equality follows from the equality condition in Lemma 3.24. We have proved that $\text{mod}(B'_\epsilon) \leq K \text{mod}(B_\epsilon)$ and the argument ends by letting $\epsilon \rightarrow 0$. $\square$

3.5 Local characterization of quasiconformality

Next we will prove that K -quasiconformality is a local property.

Definition 3.26. We say $\phi : \Omega \rightarrow \phi(\Omega)$ is **locally K -quasiconformal** if for every $z \in \Omega$ there exists some neighbourhood $V \ni z$ such that the restriction

$$\phi_V = \phi|_V : V \longrightarrow \phi(V)$$

is geometrically K -quasiconformal, i.e. for every quadrilateral $Q \subset V$ one has

$$\frac{1}{K} \text{mod}(Q) \leq \text{mod}(\phi_V(Q)) \leq K \text{mod}(Q).$$

Remark 3.27. Note that this is genuinely a local condition: a large quadrilateral $Q \subset \Omega$ need not be contained in the neighbourhood V_z chosen for every point $z \in \Omega$. Nevertheless, the upcoming theorem shows that one can approximate any “big” quadrilateral by sufficiently small ones, so the local quasiconformal bounds on these pieces imply the global K -quasiconformal property on Ω .

To prove the main result of this section, Theorem 3.29, we need the following Lemma.

Lemma 3.28 (Fletcher & Markovic (2007), Lemma 2.3.1). Let R be a rectangle in the complex plane $R = R(0, M, M + i, i)$, with module $\text{mod}(R) = M$, now let l be a curve connecting the opposite horizontal edges. Separating R into a left and right quadrilaterals $R = Q_1 \cup Q_2$.

$$\text{mod}(Q_1) + \text{mod}(Q_2) \leq \text{mod}(R) \quad (3.11)$$

with equality if and only if the line l is vertical moreover, $l = \text{mod}(Q_1)$,

Theorem 3.29. The map $\phi : \Omega \rightarrow \Omega'$ is locally K -quasiconformal if and only if it is geometrically K -quasiconformal on Ω

Proof. The aim of the proof is to take an arbitrary quadrilateral Q map it to the canonical rectangle R_Q with a rectangular subdivision (Q_{ij}) , this will be denoted as (R_Q, Q_{ij}) , that is the quadrilateral R_Q with the subdivision Q_{ij} . After mapping R_Q with the subdivision via ϕ , $Q'_{ij} = \phi(Q_{ij})$, giving a new subdivision $(\phi(R_Q), Q'_{ij})$ we do not necessary have rectangular subdivision any more. Next we restrict to each vertical segment Q'_k with the corresponding subdivision Q'_{kj} , for $0 \leq j \leq N$, for some large enough N , that is the subdivision (Q'_k, Q'_{kj}) . Now mapping Q'_k via the canonical map $\pi_{R_{Q'_k}}$ to the canonical rectangle while pushing forward the subdivision Q_{kj} , we get a new subdivision $(R_{Q'_k}, Q''_{kj})$, with $R_{Q'_k} = \pi_{R_{Q'_k}}(Q'_k)$ and $Q''_{kj} = \pi_{R_{Q'_k}}(Q'_{kj})$.

We later prove that each one of these quadrilaterals Q''_{kj} contains a horizontal line segment. The horizontal line segments in turn defines a new subdivision which is rectangular, name this new subdivision as $\hat{Q}''_{kj}$. We do this for each vertical strip Q'_k then including them back into $\phi(R_Q)$. Finally we map them back to R_Q to get the subdivision $(R_Q, \hat{Q}_{ij})$. This last subdivision will be a basis to obtain bounds on the modules for the quadrilateral $\phi(R_Q)$.

To begin, let Q be any quadrilateral in Ω , since the module is conformally invariant we study the canonical rectangle $R_Q = R(0, M, M + i, i)$. We subdivide R_Q into a mesh of small rectangles Q_{ij} such that each small rectangle has module less than $1/K$,

$$\text{mod}(Q_{ij}) \leq \frac{1}{K}.$$

We denote the tuple (Q, Q_{ij}) to be the quadrilateral Q with the subdivision given by Q_{ij} , where Q_{ij} is the j 'th quadrilateral in the i 'th vertical segment. Also Q_k will denote the k 'th vertical strip of quadrilaterals, see the following figure that illustrate the subdivision of R_Q .

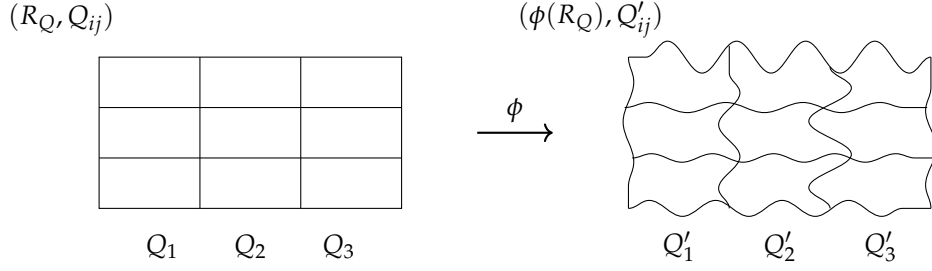


Figure 5: The subdivision of R_Q and the image of this subdivision under ϕ . Each Q_k and Q'_k is the k 'th vertical strip of the corresponding subdivision.

Since ϕ is locally K -quasiconformal, then any pair of vertically adjacent small rectangles have a neighbourhood for which ϕ is K -quasiconformal. By considering each one of the vertical strips Q'_k and mapping to the canonical rectangle,

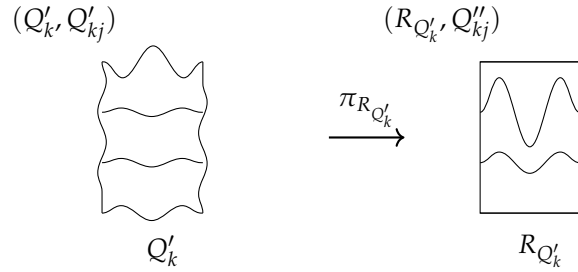


Figure 6: Restricting to a vertical strip with the subdivision from $(\phi(R_Q), Q'_{ij})$, that is (Q'_k, Q'_{kj}) and mapping to the canonical rectangle $R_{Q'_k}$.

By what is the difficult step in this proof, and will be proved later. Each one of the quadrilaterals in the subdivision Q''_{kj} will contain a horizontal line segment, drawn as dotted red line segments in figure 7.

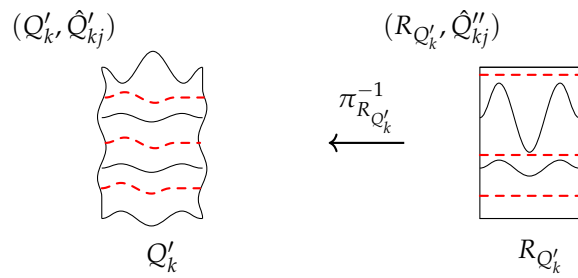


Figure 7: The red horizontal dotted lines constitute a new subdivision of the vertical strip $R_{Q'_k}$ denoted by $\hat{Q}''_{kj}$. Pulling this subdivision back to Q'_k we get the new subdivision $\hat{Q}'_{kj} = \pi_{R_{Q'_k}}^{-1}(\hat{Q}''_{kj})$.

By doing this for each vertical segment Q'_k and including this new subdivision back into $\phi(R_Q)$, and finally including them into R_Q we get the subdivision $(R_Q, \hat{Q}_{ij})$.

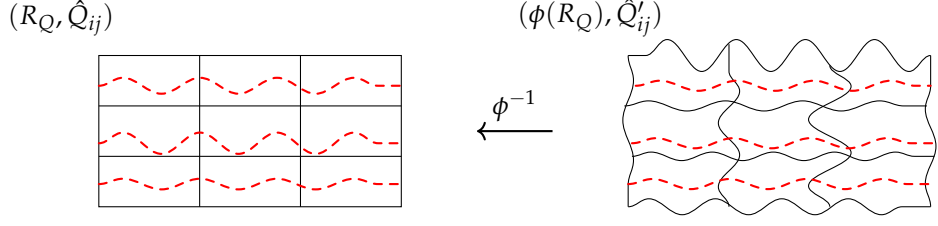


Figure 8: The initial rectangle R_Q with the new subdivision (red-dotted lines), $(R_Q, \hat{Q}_{ij})$, with $\hat{Q}_{ij} = \phi^{-1}(\hat{Q}'_{ij})$.

With the new subdivision we get the following bounds for the modules.

$$\begin{aligned} \text{mod}(Q) &= \sum_i \text{mod}(\hat{Q}_i) & \sum_i \text{mod}(\hat{Q}'_i) &\leq \text{mod}(Q') \\ \sum_j \frac{1}{\text{mod}(\hat{Q}_{i,j})} &\leq \frac{1}{\text{mod}(\hat{Q}_i)} & \sum_j \frac{1}{\text{mod}(\hat{Q}'_{i,j})} &= \frac{1}{\text{mod}(\hat{Q}'_i)}. \end{aligned}$$

The top left equality follows directly by Lemma (3.28). The inequality below it follows from Theorem (3.1,d). First note that the inequality is equivalent to

$$\sum_j \text{mod}((\hat{Q}_{i,j})_s) \leq \text{mod}((\hat{Q}_i)_s).$$

With $(\hat{Q}_{i,j})_s$ denotes the quadrilateral with swapped a -sides and b -sides. Taking $(\Gamma_s)_{ij}$ to denote the family of locally rectifiable curves connecting the horizontal curves of the mesh $\hat{Q}_{ij}$. While $(\Gamma_s)_i$ to be the locally rectifiable curves connecting the horizontal lines of $\hat{Q}_i$. Then the inequality is again equivalent to

$$\sum_j \lambda((\Gamma_s)_{ij}) \leq \lambda((\Gamma_s)_i).$$

This is exactly the inequality in Theorem (3.1,d), since

$$(\Gamma_s)_{i1} + (\Gamma_s)_{i2} + \cdots + (\Gamma_s)_{ik} = (\Gamma_s)_i.$$

The top right inequality follows in the same way as the previous inequality by Theorem (3.1,d). Compared to the top left equality note we lose strict equality because we cant ensure straight vertical lines any more.

The last equality follows since the subdivision is rectangular, and relies on the fact that this new subdivision $\hat{Q}'_{ij}$ exists. Also we have local K -quasiconformality on $\phi : \hat{Q}_{ij} \rightarrow \hat{Q}'_{ij}$.

$$\frac{1}{K} \text{mod}(\hat{Q}_{ij}) \leq \text{mod}(\hat{Q}'_{ij}) \leq K \text{mod}(\hat{Q}_{ij}).$$

With this in mind we get the desired inequality by the following inequities

$$\text{mod}(Q) = \sum_i \text{mod}(\hat{Q}_i) \leq \sum_i \left(\sum_j \frac{1}{\text{mod}(\hat{Q}_{ij})} \right)^{-1} \leq \sum_i \left(\sum_j \frac{1}{K \text{mod}(\hat{Q}'_{ij})} \right)^{-1}$$

$$= K \sum_i \left(\frac{1}{\text{mod}(\hat{Q}'_i)} \right)^{-1} = K \sum_i \text{mod}(\hat{Q}'_i) \leq K \text{mod}(Q')$$

All that is left to be shown, is the existence of a rectangular subdivision $\hat{Q}'_{ij}$ of each vertical segment $R_{Q'_k}$ for each k . That is for every quadrilateral $\tilde{Q} \in \{\pi_{R_{Q'_k}}(Q_{kj}) : 1 \leq j \leq N_k\}$, with N_k equals the number of subdividing quadrilaterals of $R_{Q'_k}$, there exists some horizontal line segment connecting the b -sides of $\tilde{Q}$. We proceed by contradiction, assume there do not exist a horizontal line in $\tilde{Q}$. By rotating the rectangle and then applying the exponential map (see figure 9), the assumption will be equivalent to having no circle inside of the ringdomain created by mapping $\tilde{Q}$ via the exponential map.

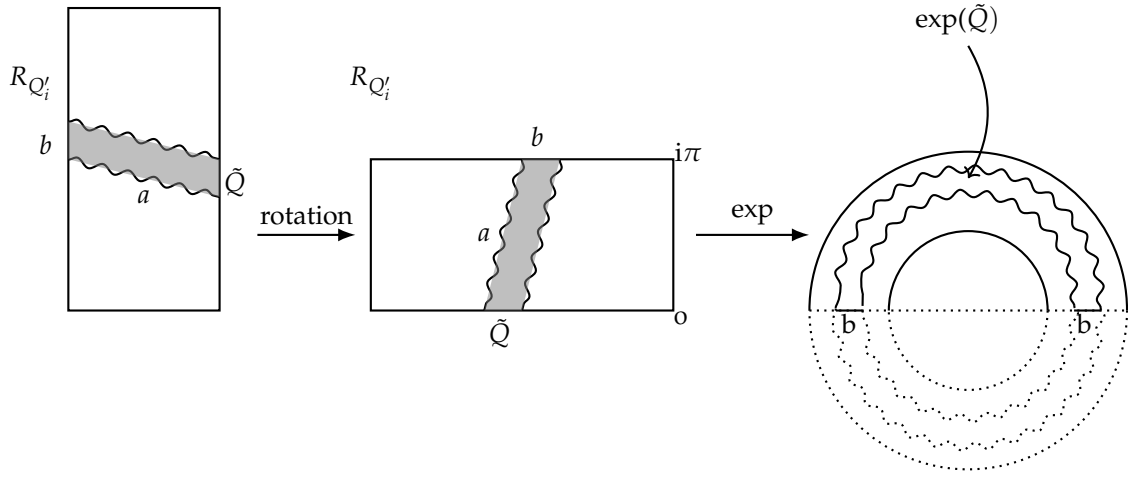


Figure 9: Conformally mapping the quadrilateral Q_i to a ringdomain

Let Γ denote the family of curves connecting the b -sides of $\tilde{Q}$, then

$$\text{mod}(\tilde{Q}) = \lambda(\Gamma)$$

The module of the ringdomain $A = \tilde{Q} \cup -\tilde{Q}$

$$\text{mod}(A) = \frac{1}{2} \frac{1}{\text{mod}(R_{Q'_i})} \geq \frac{1}{2}$$

With first equality follows from the fact Γ is a symmetric curve family relating to Theorem 3.8. Let γ_1 and γ_2 denote the inner Jordan curve respectively the outer Jordan curve for the ringdomain A . The assertion that there is no circle inside of $\exp(\tilde{Q})$ connecting the b -sides is equivalent to

$$\sup_{z \in \gamma_1} |z| > \inf_{z \in \gamma_2} |z|.$$

Let z_1 and z_2 be two points for which the supremum and infimum is attained, hence $|z_1| > |z_2|$. Then the ringdomain A will separate $\{0, z_1\}$ from $\{z_2, \infty\}$.

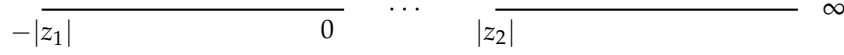


Figure 10: The Ringdomain used for the Teichmüller extremal problem.

Combining this we get that the extremal ring domain as in figure (12),

$$\begin{aligned} \frac{1}{2} \leq \text{mod}(\Omega) &\leq \text{mod}(\hat{\mathbb{C}} \setminus [-|z_1|, 0] \cup [|z_2|, \infty)) \\ &< \text{mod}(\hat{\mathbb{C}} \setminus [-|z_1|, 0] \cup [|z_1|, \infty)) \underbrace{=}_{\text{conformal}} \text{mod}(\hat{\mathbb{C}} \setminus [-1, 0] \cup [1, \infty)) = \frac{1}{2}. \end{aligned}$$

Which is a contradiction. The last equality follows from evaluating the module explicitly, this can be done using Weierstrass $\wp$ -functions [Ahlfors (1966), Chapter 3], or a more geometric approach [Lehto & Virtanen (1973), Chapter 2]. Hence $\tilde{Q}$ contains a straight line segment connecting the b -sides. Which then implies that the subdivision $\tilde{Q}'_{kj}$ exists. We have proved that local quasiconformality implies global quasiconformality. The other way follows since every neighbourhood of a point will be contained in some quadrilateral for which we have bounded moduli by global quasiconformality. $\square$

3.6 Three extremal problems

First let Ω denote a doubly connected region, that is the complement Ω^c contains two connected components. Let C_1 denote the bounded component and C_2 the unbounded component. We will state three extremal geometric problems of the following type: Among all the doubly connected domains that separate C_1 and C_2 find the one with the largest module. We will solve the extremal problem under the first two of the following conditions, while the case of Mori is solved in [Ahlfors (1966)].

I (Grötzsch) $C_1 = \{z : |z| \leq 1\}$ is the unit circle and C_2 contains some real point r with $r > 1$.

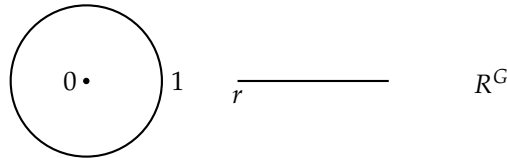


Figure 11: Extremal ring domain for the Grötzsch problem

II (Teichmüller) C_1 contains 0 and -1 while C_2 contains a point of distance p away from the origin.

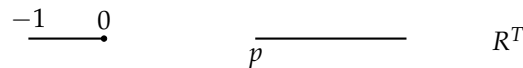


Figure 12: Extremal ring domain for the Teichmüller problem

III (Mori) For some $\lambda > 0$, assume C_1 satisfies

$$\text{diam}(C_1 \cap \{|z| \leq 1\}) \geq \lambda$$

while C_2 contains the origin.

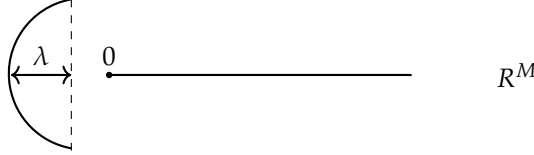


Figure 13: Extremal ring domain for Mori's problem

Each figure from above represent the ring domains which attains the extremal. Denote respective ring domain by R^G , R^T , R^M . Explicitly we have

$$R^G = \mathbb{C} \setminus (\overline{\mathbb{D}} \cup (r, \infty)), \quad R^T = \mathbb{C} \setminus ([-1, 0] \cup (p, \infty)), \quad R^M = \mathbb{C} \setminus (A_\lambda \cup (0, \infty)). \quad (3.12)$$

With A_λ is the semicircle in the figure 13. We will prove the first two cases.

For Grötzsch's problem we, let Γ denote the family of closed curves that separate C_1 and C_2 . We know that $\lambda(\Gamma) = \frac{1}{\text{mod}(\Omega)}$. To use the symmetry principle consider, $\tilde{\Gamma}$ which is the family of closed curves in the complement of $C_1 \cup \{r\}$, with zero winding number about r and a non-zero winding number around the origin. Any curve in Γ will also be a curve in $\tilde{\Gamma}$, hence $\Gamma \subset \tilde{\Gamma}$ giving $\lambda(\Gamma) \geq \lambda(\tilde{\Gamma})$. Since for every curve $\tilde{\gamma}$ in $\tilde{\Gamma}$ the curve $\tilde{\gamma}^-$ will also separate the domains C_1 and C_2 , it will also have zero winding number around r and a non-zero winding number around 0, hence $\tilde{\Gamma}^- = \tilde{\Gamma}$. Now utilizing the symmetric principle, Theorem 3.8, gives $\lambda(\tilde{\Gamma}) = \frac{1}{2}\lambda(\tilde{\Gamma}^+)$. In the same way, if we let Γ_0 be the curve family in the extremal case R^G which can contain curves with segments on the edges of the cut (r, ∞) , then $\lambda(\Gamma_0) = \frac{1}{2}\lambda(\Gamma_0^+)$.

Next we aim to prove $\tilde{\Gamma}^+ = \Gamma_0^+$, because with it we get $\lambda(\Gamma) \geq \lambda(\tilde{\Gamma}) = \lambda(\Gamma_0)$, which implies that $\text{mod}(R) \leq \text{mod}(R^G)$. Let $\tilde{\gamma} \in \tilde{\Gamma}$ since this curve has a non-zero winding number around the origin and a zero winding number around r , there exists two points P_1 and P_2 which are respectively on $(-\infty, -1)$ and $(1, r)$, they divide $\tilde{\gamma}$ in two arcs $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ such that $\tilde{\gamma} = \tilde{\gamma}_1 + \tilde{\gamma}_2$, note that there maybe exists multiple points on the segments such as P_1 and P_2 , for this case pick the first instance of such points, that is $\gamma : I \rightarrow \mathbb{C}$ pick the smallest $t_i \in I$ such that the conditions for the point $P_i = \gamma(t_i)$ is satisfied. Further by the linearity of the operation $(\cdot)^+$ we get,

$$\tilde{\gamma}^+ = \tilde{\gamma}_1^+ + \tilde{\gamma}_2^+ = (\tilde{\gamma}_1^+ + \tilde{\gamma}_2^{+-})^+.$$

The arc $\tilde{\gamma}_1^+ + \tilde{\gamma}_2^{+-}$ belongs to Γ_0 hence $\tilde{\gamma}^+ \in \Gamma_0^+$. We have proved the first inclusion $\tilde{\Gamma}^+ \subset \Gamma_0^+$. Since every curve in Γ_0^+ is the positive reflection of a curve in Γ_0 but $\Gamma_0 \subset \tilde{\Gamma}$ it follows that $\Gamma_0^+ \subset \tilde{\Gamma}^+$. Which proves that R^G is extremal.

For the case of Teichmüller we will need the Koebe's one-quarter distortion theorem.

Theorem 3.30. [Carleson & Gamelin (1993)] Let $\mathcal{S}$ be the collection of function, which are analytical, injective functions f defined on the open disk such that $f(0) = 0$ and $f'(0) = 1$. If $f \in \mathcal{S}$, then the image of f will cover the disk centered at 0 of radius $\frac{1}{4}$. Further we also have

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2},$$

By Riemann mapping theorem we can map the open disk conformally onto the open simply connected region $f : \mathbb{D} \rightarrow C_1 \cup \overset{\circ}{\Omega}$ such that $f(0) = 0$. Let $g : \mathbb{D} \rightarrow \mathbb{C} \setminus (-\infty, -\frac{1}{4}]$ be the Koebe's function, that is

$$g(z) = \frac{z}{(1-z)^2}.$$

this is a conformal map and is usually given in the context of demonstrating the upper bound for the Koebe one quarter theorem. For $p > 1$, denote $\phi_p = -pg(z)$, which maps the open disc conformally onto $\mathbb{C} \setminus [p, \infty)$. Koebe one-quarter Theorem 3.30 gives $|f'(0)| \leq 4p$, further if we let $\Omega = R^T$ for some $p > 1$ then $f(z) = \phi_p(z)$, hence by differentiating we get

$$\phi_p'(z) = \frac{-4p(1-z)^2 + 8pz(1-z)}{(1-z)^4} \implies |\phi_p'(0)| = 4p.$$

Suppose $a_f, a_{\phi_p} \in \mathbb{D}$ are the inverse images of -1 under f and ϕ respectively, that is $f(a_f) = \phi_p(a_{\phi_p}) = -1$. By the second part of Koebe one-quarter Theorem applied to the normalized conformal map $f(z)/f'(0)$, gives

$$1 = |f(a_f)| \leq \frac{|a_f||f'(0)|}{(1-|a_f|)^2} \leq \frac{4p|a_f|}{(1-|a_f|)^2}.$$

With equality if $\Omega = R^T$, because of $|f(z)| = |\phi_p(z)|$. In other words $|a_{\phi_p}| \leq |a_f|$ this follows because the function $x/(1-x)^2$ is strictly increasing for $x \in [0, 1]$,

$$\frac{4p|a_{\phi_p}|}{(1-|a_{\phi_p}|)^2} = |\phi_p(a_{\phi_p})| = 1 = |f(a_f)| \leq \frac{4p|a_f|}{(1-|a_f|)^2}.$$

Together with the monotonicity of the function yields $|a_{\phi_p}| \leq |a_f|$. Note that f is an arbitrary normalized conformal map of the disc onto $C_1 \cup \overset{\circ}{\Omega}$ hence a_{ϕ_p} has smallest absolute value of all inverses of -1 .

Consider the ring domain $\mathbb{D} \setminus f^{-1}(C_1)$. Because f is normalized the inverse image of C_1 will certainly contain the origin. Now by inverting the open disc via $\psi(z) = 1/z$, the boundary of the open disc stays invariant and we get a ring domain $\hat{R}$ with the open disc as the bounded component and $\psi(f^{-1}(C_1))$ as the unbounded component which will contain the point $1/a_f$ obviously $1/|a_f| > 1$. Rotating by $-\arg(1/a_f)$ around the origin, will align $1/a_f$ along the real axis. We satisfy the conditions for the Grötzsch problem. The extremal of the module is attained when the ring domain is $\mathbb{C} \setminus \mathbb{D} \cup [\frac{1}{|a_f|}, \infty)$. Because of f being conformal we get

$$\begin{aligned} \text{mod}(\Omega) &= \text{mod}(\mathbb{D} \setminus f^{-1}(C_1)) = \text{mod}(\mathbb{C} \setminus \mathbb{D} \cup \psi(f^{-1}(C_1))) \\ &\leq \text{mod}(\mathbb{C} \setminus \mathbb{D} \cup [\frac{1}{|a_f|}, \infty)) \leq \text{mod}(\mathbb{C} \setminus \mathbb{D} \cup [\frac{1}{|a_{\phi_p}|}, \infty)). \end{aligned}$$

For the last inequality note the module will increase when $|a_f|$ decreases, this follows from the monotonicity of the module. Now since $|a_{\phi_p}| \leq |a_f|$ which implies this is the largest ringdomain we can get. Applying the conformal map $\phi_p \circ \psi$ to $\mathbb{C} \setminus \mathbb{D} \cup [\frac{1}{|a_{\phi_p}|}, \infty)$ gives the ring domain R^T .

Let $\Phi(r)$, $\Psi(p)$ and $\Xi(\lambda)$ denote the module for respective extremal case. By some calculation done in Ahlfors book [Ahlfors (1966)], connecting the module with Elliptic and Modular functions, one can prove the following inequalities that will be used to prove Mori's theorem,

$$\begin{aligned}\Phi(r) &\leq 4r, \\ \Psi(p) &\leq 16(p+1), \\ \lambda\Xi(\lambda) &\leq 16.\end{aligned}\tag{3.13}$$

3.7 Extension theorems for quasiconformal maps

The aim of this section is to prove extension theorems similar for the conformal case but for quasiconformal mappings. That is for $\phi : \Omega \setminus S \rightarrow \phi(\Omega)$ and $S \subset \overline{\Omega}$ with some different properties, such as being an isolated boundary point or some analytical curve. The aim is to prove that we can extend ϕ to S , in a way that preserves some K -quasiconformality. We then move on to prove Mori's theorem which states that K -quasiconformal mappings are $1/K$ Hölder continuous on the unit disc, which has can be used to state compactness theorems for the space of quasiconformal mappings on the disc.

Definition 3.31 (Isolated boundary point). *Let (X, τ) be a topological space and $U \subset X$. We say $x \in \partial U$ is an isolated boundary point of U if there exists an open neighbourhood V of x with*

$$V \cap \partial U = \{x\}.$$

Example 3.32. *Fix $z_0 \in \Omega$. Let $\overline{\mathbb{D}}$ be the closed disc about z_0 with $\overline{\mathbb{D}} \subset \Omega$, and set $\mathbb{D}_0 = \overline{\mathbb{D}} \setminus \{z_0\}$. Then z_0 is an isolated boundary point of $\mathbb{D}_0$, and since $\mathbb{D}_0$ is conformally a degenerate annulus its modulus is infinite. Hence for any K -quasiconformal ϕ ,*

$$\infty = \text{mod}(\mathbb{D}_0) \leq K \text{mod}(\phi(\mathbb{D}_0)) \implies \text{mod}(\phi(\mathbb{D}_0)) = \infty,$$

so $\phi(\mathbb{D}_0)$ is again topologically a punctured disc.

Definition 3.33 (Removable singularity). *Let $\phi : \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$ be K -quasiconformal. We call z_0 a removable singularity if ϕ extends to a K -quasiconformal map on all of Ω .*

Theorem 3.34 (Removability of isolated boundary points). *Every isolated boundary point of a domain is a removable singularity for any K -quasiconformal map.*

Proof. Let $\phi : \Omega \rightarrow \Omega'$ be a K -quasiconformal mapping, furthermore let z_0 be an isolated boundary point. We will first prove the existence of a homeomorphic extension of ϕ to $\Omega \cup \{z_0\}$ and then show this extension yields a K -quasiconformal mapping. Since z_0 is a isolated boundary point of Ω , we can choose a small $R > 0$ such that $\mathbb{D}(z_0, R) = \{z \in \mathbb{C} : |z - z_0| < R\}$ is contained in Ω , in other words, $\mathbb{D}(z_0, R) \cap \partial\Omega = \{z_0\}$. Then for each $0 < r < R$, the annulus $A(r, R) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$ lies in $\Omega \cup \{z_0\}$. By the previous example 3.32 the module is unbounded when $r \rightarrow 0$,

$$\lim_{r \rightarrow 0} \text{mod}(A(r, R)) = \infty.$$

Since ϕ is K -quasiconformal on $\Omega \setminus \{z_0\}$ then

$$\frac{1}{K} \text{mod}(A(r, R)) \leq \text{mod}(\phi(A(r, R))) \leq K \text{mod}(A(r, R)).$$

As $r \rightarrow 0$ we get that $\text{mod}(\phi(A(r, R))) \rightarrow \infty$. This implies that the annulus $\phi(A(r, R))$ degenerate to a punctured disk, with the inner boundary shrinks to a point, hence also punctured at some point w_0 . We correspond z_0 with w_0 that is the define the extension to be $\bar{\phi}(z) = \phi(z)\chi_{\Omega \setminus \{z_0\}} + w_0\chi_{\{z_0\}}$ (χ_A is the characteristic function on the set A). Every sequence of points converging to $z_n \rightarrow z_0$ will give a convergent sequence $\phi(z_n) \rightarrow w_0$, hence $\bar{\phi}$ is also continuous. Since ϕ is one-to-one it implies that $\bar{\phi}$ is one-to-one. Therefore a homeomorphism.

To prove that the extension $\bar{\phi}$ is K -quasiconformal map. Let Q be some quadrilateral with $\bar{Q} \subset \Omega \cup \{z_0\}$. If $z_0 \notin Q$ we have by the K -quasiconformality of ϕ that,

$$\text{mod}(\bar{\phi}(Q)) = \text{mod}(\phi(Q)) \leq K \text{mod}(Q).$$

If $z_0 \in \partial Q$, consider a sequence of quadrilaterals $Q_n \subset Q$ that converge to Q from inside such that the distance between the boundary of Q_n and Q is at most $1/n$. By continuity of the module we get that $\lim_{n \rightarrow \infty} \text{mod}(Q_n) = \text{mod}(Q)$. Since $\phi(Q_n) \subset \phi(Q)$ is also a sequence of quadrilaterals converging to $\phi(Q)$ from inside, we get,

$$\text{mod}(\bar{\phi}(Q)) = \lim_{n \rightarrow \infty} \text{mod}(\bar{\phi}(Q_n)) \leq K \lim_{n \rightarrow \infty} \text{mod}(Q_n) = K \text{mod}(Q).$$

Lastly, if $z_0 \in Q$ is an interior point of Q . Consider the canonical rectangle of $\bar{\phi}(Q)$, that is $R_{\bar{\phi}(Q)} = R(0, \text{mod}(\bar{\phi}(Q)), \text{mod}(\bar{\phi}(Q)) + i, i)$, with the respective canonical map $\pi_{\bar{\phi}(Q)}$, we divide the canonical rectangle vertically into two rectangles R'_1 and R'_2 , along the point $l_0 = \pi_{\bar{\phi}(Q)}(\bar{\phi}(z_0))$, it follows now from Lemma 3.13 that

$$\text{mod}(R'_1) + \text{mod}(R'_2) = \text{mod}(R_{\bar{\phi}(Q)}) = \text{mod}(\bar{\phi}(Q)).$$

Further because l_0 is a boundary point for each rectangle R_i , $i = 1, 2$, the preimages $R_i = \phi^{-1}(R'_i)$ will contain z_0 as a boundary point. From the previous case in this proof we obtain

$$\text{mod}(R'_i) \leq K \text{mod}(R_i)$$

combining this we get

$$\text{mod}(\bar{\phi}(Q)) = \text{mod}(R_{\bar{\phi}(Q)}) = \text{mod}(R'_1) + \text{mod}(R'_2) \leq K(\text{mod}(R_1) + \text{mod}(R_2)) \leq K \text{mod}(Q),$$

where we have applied the inequality in Lemma 3.13. $\square$

Theorem 3.35. [Removability of analytical arc]

Let $\phi : \Omega \rightarrow \phi(\Omega)$ be a homeomorphism and γ be a closed analytical curve in Ω with the possible exception of its end points, then

$$K(\Omega) := \sup_{Q \subset \Omega} \frac{\text{mod}(\phi(Q))}{\text{mod}(Q)} = \sup_{Q \subset \Omega \setminus \gamma} \frac{\text{mod}(\phi(Q))}{\text{mod}(Q)} := K(\Omega \setminus \gamma).$$

Proof. If $\psi : I \rightarrow \gamma$ is analytical in a neighborhood of I we get that $K(\Omega) \geq K(\Omega \setminus \gamma)$, since

$$\sup_{Q \subset \Omega} \frac{\text{mod}(\phi(Q))}{\text{mod}(Q)} \geq \sup_{Q \subset \Omega \setminus \gamma} \frac{\text{mod}(\phi(Q))}{\text{mod}(Q)},$$

because we are taking supremum over a larger set i.e $\Omega \setminus \gamma \subset \Omega$.

For the other direction assume $K(\Omega \setminus \gamma) < \infty$, and let $Q, \bar{Q} \subset \Omega$ be a quadrilateral whose canonical mapping has a conformal extension to a domain containing $\bar{Q}$. It is sufficient to show that the dilation of Q under ϕ is at most $K(\Omega \setminus \gamma)$. In other words, we need to prove that ϕ is $K(\Omega \setminus \gamma)$ -quasiconformal, since

$$K(\Omega) = \sup_{Q \subset \Omega} \frac{\text{mod}(\phi(Q))}{\text{mod}(Q)} \leq \sup_{Q \subset \Omega} \frac{K(\Omega \setminus \gamma) \text{mod}(Q)}{\text{mod}(Q)} = K(\Omega \setminus \gamma).$$

To this extent, using the canonical conformal maps we pass to the case with Q and $\phi(Q) = Q'$ are rectangles. The problem can be stated as follows. Let ϕ be a homomorphisms of the rectangle $R_Q = R(0, M, M + i, i)$ onto the rectangle $R_{\phi(Q)} = R(0, M', M' + i, i)$, with $\{\gamma_i\}_{i=1}^n$ be closed disjoint analytical arcs in $\bar{R}_Q$ and let the maximal dilation of ϕ on $R \setminus \bigcup_{i=1}^n \gamma_i$ be at most $K(\Omega \setminus \gamma)$, we wish to show $M' \leq K(\Omega \setminus \gamma)M$.

If a curve γ_i is not completely horizontal we can divide it into finite number of sub arcs which intersect any horizontal line only once, that is every sub-arc of C_i is strictly increasing or decreasing. Thus we can divide R_Q into sub-rectangles $R_h(y_{h-1}i, M + y_{h-1}i, M + y_h i, y_h i)$, where $h = 1, \dots, m$ and $0 = y_0 < y_1 < \dots y_m = 1$, so that the following two conditions hold,

- i) For every h and i the set $R_h \cap \gamma_i$ is a finite union of finite analytical arc joining the horizontal sides of R_h .
- ii) For every h and $\epsilon > 0$, then $\sum_{k=0}^{N_h} \text{diam}(\phi(Q_{hk} \cap (\cup \gamma_i))) < \epsilon$.

From the first condition and that each curve is analytic and monotone we get that each rectangle R_h is split up in a finite number N_h of domains. This in turn gives us a finite number, N_h of quadrilaterals $Q_{h,k}$, with the horizontal lines y_h and y_{h-1} as the a -sides and the curves inside of R_h as b -sides (it is possible for the quadrilaterals to be degenerate). Note that, $R_h = \bigcup_{k=1}^{N_h} \bar{Q}_{hk}$. Denote by $s'_{h,k}$ the shortest distance between the b -sides of $Q'_{h,k} = \phi(Q_{hk})$, then it follows from the second condition that,

$$\sum_{k=0}^{N_h} s'_{h,k} \geq M' - \sum_{h=0}^m \text{diam}(\phi(Q_{hk} \cap \cup \gamma_i)) > M' - \epsilon.$$

The first inequality follows since $s'_{h,k}$ will be negligible in the clustered region of the images of the arcs. On the other hand applying Rengel's inequality to $Q'_{h,k}$ we obtain,

$$\text{mod}(Q'_{h,k}) \geq \frac{(s'_{h,k})^2}{m^2(Q'_{h,k})}.$$

By the special case of Schwarz inequality named Sedrakyan's lemma we have

$$\sum_{k=0}^{N_h} \text{mod}(Q'_{h,k}) \geq \frac{(\sum_k s'_{h,k})^2}{\sum_{k=0}^{N_h} m^2(Q'_{h,k})} > \frac{(M' - \epsilon)^2}{m^2(Q'_k)}.$$

Combined with the fact ϕ is a $K(\Omega \setminus \gamma)$ -quasiconformal on the each Q_{hk} and super additivity of the module gives,

$$\sum_{k=0}^{N_h} \text{mod}(Q'_{h,k}) \leq K(\Omega \setminus \gamma) \sum_{k=0}^{N_h} \text{mod}(Q_{h,k}) \leq K(\Omega \setminus \gamma) \text{mod}(R_h) = \frac{MK(\Omega \setminus \gamma)}{y_h - y_{h-1}}.$$

Implies

$$\frac{y_h - y_{h-1}}{MK(\Omega \setminus \gamma)} < \frac{m^2(R'_h)}{(M' - \epsilon)^2}.$$

Summing over h we get

$$\sum_{h=1}^m \frac{y_h - y_{h-1}}{MK(\Omega \setminus \gamma)} = \frac{1}{MK(\Omega \setminus \gamma)} \leq \sum_{h=1}^m \frac{m^2(R'_h)}{(M' - \epsilon)^2} = \frac{M'}{(M' - \epsilon)^2}$$

Letting $\epsilon \rightarrow 0$ gives the desired inequality, $M' \leq K(\Omega \setminus \gamma)M$. $\square$

What follows next is the Mori's theorem, which gives an upper bound for the maximum distortion for a K -quasiconformal map.

Theorem 3.36. [Mori's theorem]

Let $\mathbb{D}$ denote the open unit disc in the complex plane. Let $\phi : \mathbb{D} \rightarrow \mathbb{D}$ be a K -quasiconformal mapping, which is normalized $\phi(0) = 0$. Then for two distinct points z_1 and z_2 ,

$$|\phi(z_1) - \phi(z_2)| \leq 16|z_1 - z_2|^{1/K}.$$

Further 16 cannot be replaced by a smaller constant.

Proof. First, we assume that ϕ has a K -quasiconformal extension to the boundary of the disc, $\phi : \mathbb{D} \rightarrow \mathbb{D}$. We will later remove this assumption. Since ϕ maps into the unit disc, the maximum possible distance is 1. Thus, if $|z_1 - z_2| \geq 1$, there is nothing to prove as the inequality holds.

Now, assume $|z_1 - z_2| \leq 1$. To relate the distortion to the K -quasiconformality of ϕ , we study the behaviour of a specific annulus. Specifically, consider an annulus R formed by two concentric circles: the inner circle has a diameter of $|z_1 - z_2|$, and the outer circle has a radius of $1/2$. Additionally, both points z_1 and z_2 lie on the boundary of the inner circle. The module of the annuli is

$$\text{mod}(R) = \frac{1}{2\pi} \log \frac{1}{|z_1 - z_2|}.$$

Note that we have the distance $|z_1 - z_2|$ in the expression of the module. Also since ϕ is K -quasiconformal we get

$$\text{mod}(R) \leq K \text{mod}(\phi(R)).$$

The aim now is to calculate $\text{mod}(\phi(R))$ by first conformally mapping to the extremal case of Grötzsch extremal problem in Figure 11 and then using the inequities in equation (3.13). The proof will be in two parts when R is completely inside of $\mathbb{D}$ and when it has points outside of $\mathbb{D}$.

For the first case $R \subset \mathbb{D}$. Let $\xi_i = \phi(z_i)$, $i = 1, 2$. Let $\hat{R}$ be the ring-domain with inner

component same as for $\phi(R)$ but outer boundary the circle. By the monotonicity of the module we get $\text{mod}(\phi(R)) \leq \text{mod}(\hat{R})$, now consider the Blaschke factor,

$$T(\xi) = \left(\frac{\xi - \xi_1}{1 - \bar{\xi}_1 \xi} \right)^{-1}.$$

We have that $T(\xi_1) = \infty$ and $|T(\xi_2)| > 1$, hence we are in the case of the Grötzsch extremal problem, the map T is conformal and we get

$$\begin{aligned} \text{mod}(\hat{R}) &= \text{mod}(T(\hat{R})) \leq \frac{1}{2\pi} \log \phi(|T(\xi_2)|) \\ &\leq \frac{1}{2\pi} \log \phi\left(\left| \frac{1 - \bar{\xi}_1 \xi_2}{\xi_2 - \xi_1} \right|\right) \leq \frac{1}{2\pi} \log 4 \cdot \left(\left| \frac{1 - \bar{\xi}_1 \xi_2}{\xi_2 - \xi_1} \right| \right) \\ &\leq \frac{1}{2\pi} \log 8 \cdot \left(\frac{1}{|\xi_2 - \xi_1|} \right). \end{aligned}$$

With last step follows because $|1 - \bar{\xi}_1 \xi_2| \leq 2$. Combining we get

$$\frac{1}{2\pi} \log \frac{1}{|z_1 - z_2|} \leq K \text{mod}(\phi(R)) \leq K \text{mod}(\hat{R}) \leq \frac{K}{2\pi} \log 8 \frac{1}{|\xi_1 - \xi_2|}.$$

Hence for the first case we get,

$$|\xi_1 - \xi_2| \leq 8|z_1 - z_2|^{1/K}. \quad (3.14)$$

The second case $R \cap \mathbb{D} \neq R$, let R_1 denote the bounded component for the ring domain R , R_2 the unbounded component and $\xi_i = \phi(z_i)$, now

$$\text{diam}(\phi(R_1) \cap \mathbb{D}) \geq |\xi_1 - \xi_2|.$$

While $0 \in \phi(R_2)$. Then we get

$$\text{mod}(R) = \frac{1}{2\pi} \log \frac{1}{|z_1 - z_2|} \leq \frac{K}{2\pi} \log \phi(R) \leq \frac{K}{2\pi} \log \Xi(|\xi_1 - \xi_2|) \leq \frac{K}{2\pi} \log \frac{16}{|\xi_1 - \xi_2|}.$$

Obtaining

$$|\xi_1 - \xi_2| \leq 16|z_1 - z_2|^{1/K}.$$

All that is left now is to remove the hypothesis that ϕ is continuous on the boundary. Consider the Riemann map ψ_r that maps the open disk $\{z : |z| < r\}$ conformally onto the unit disk, that satisfies $\psi_r(0) = 0$, $\psi_r'(0) > 0$, for $0 < r < 1$. Note that for $r \rightarrow 1$ we get that $\psi_r(\mathbb{D}) \rightarrow \mathbb{D}$, in the Hausdorff metric. Since K -quasiconformal maps are closed under compositions, this An introduction to quasi-conformal mappingsimplies that $\psi_r(\phi(rz))$ is K -quasiconformal and is continuous on $|z| = 1$, hence we get

$$|\psi_r(\phi(rz_1)) - \psi_r(\phi(rz_2))| \leq 16|z_1 - z_2|^{1/K}.$$

Since $\psi_r(\mathbb{D}) \rightarrow \mathbb{D}$ for $r \rightarrow 1$ then ψ converges to the identity and we get

$$|\psi(z_1) - \psi(z_2)| \leq 16|z_1 - z_2|^{1/K}.$$

We will skip the proof of optimality of 16, consult chapter 2 theorem 3.2 of [Lehto & Virtanen (1973)] for a proof of this fact. $\square$

Remark 3.37. *The Hölder constant cannot be improved in Mori's theorem. This is because the map $\phi(z) = z|z|^{(1-K)/K}$, is K -quasiconformal. While the Hölder condition cannot be fulfilled for any $\alpha > 1/K$, this since for points z close to 0, we have*

$$\frac{|\phi(z) - \phi(0)|}{|z - 0|^\alpha} = |z|^{\frac{1}{K} - \alpha} \rightarrow \infty \quad \alpha > \frac{1}{K}, |z| \rightarrow 0.$$

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