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Derivation and stability determination of black hole metrics

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Abstract

The Schwarzschild metric and the Kerr metric describe the gravitational fields around static and rotating black holes, respectively. Here, we derive the Kerr metric in a simpler way than how it was derived originally and determine the stability of the Schwarzschild metric. The Kerr metric was derived by using the Einstein-Hilbert action as well as directly from the Einstein field equations. In order to do this, we first made an anzats with the help of the Weyl method [1, 2]. By using the same method as Chandrasekhar [3], we found that the Schwarzschild metric is stable.

Popular scientific summary: Black holes could be gateways to the future

Black holes are not only graves of older stars in the universe; they are also potential time machines. Because of their incredibly high gravitation, they are able to bend space-time so much that if you were to travel to one and then come back, hundreds of years could have passed of Earth. Many would think that traveling into a black hole would mean the traveler's demise, but if the black hole was rotating, this may not be the case.

Black holes are some of the most fascinating objects that we know about today. With the first announcement of gravitational waves given in 2016 by the LIGO observation, by taking data from colliding black holes, they have opened up a large range of studies regarding the cosmos. There is data pointing towards the fact that there may exist a super-massive black hole at the center of the Milky Way and there have been discussions about the possibility that the universe may have been created out of one. They have also opened up discussions about the possible existence of worm holes and parallel universes.

The idea that an object with an escape velocity larger than the speed of light was proposed as early as in 1783 by John Michell, but at that time they were called "dark stars". It took more than 100 years, when Einstein had published his theory of general relativity, to give further calculations pointing to the existence of such objects. A solution of Einstein's equations pointing towards the existence of black holes was done by Karl Schwarzschild and independently by Johannes Droste, less than a year after Einstein published his theory of general relativity.

A black hole is formed as a result of the decomposition of a massive star. The force of gravity is so strong that the pressure preserving the structure of the star eventually fails, which means that the star collapses into itself. As a result, something with a gravitational force so strong that not even light can escape it is formed. This new object is called a *black hole*, namely because it is "black" in the sense that we can not see it and "hole" since everything that gets too close to it falls into it and can not escape, like a deep hole in the ground.

One of the things that makes black holes so interesting is their incredible density. The corresponding radius of a black hole for a given mass can be simply calculated by using an equation derived by Schwarzschild. This calculation shows that if the Earth were to collapse into a black hole, it would be as small as a grape!

One of the many characteristics of a black hole is that it can rotate. It took almost 50 years after Einstein published his theory of relativity to correctly describe this rotation, due to the calculations being so complicated. Before the solution done by Kerr, there had been discussions regarding the fact that rotation could perhaps slow down the process of a collapsing star. With the picture given by Schwarzschild, traveling to a black hole would mean that you would end up in a so-called singularity, but with Kerr this may not be the case.

My bachelor thesis deals with the rotation and the stability of black holes. The solution obtained by Kerr is first derived in a more pedestrian fashion and then the stability of the Schwarzschild solution is determined. Could the existence of stable, rotating black holes mean a possibility to travel into black holes? Could they be the doors to possible parallel universes? It remains to be seen.

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1 Introduction

Only a few months after Einstein's field equation was published, the so-called Schwarzschild solution was found, which describes the gravitational field that is outside a spherical source. Almost 50 years later, the Kerr metric was found, which describes a rotating black hole. After Kerr derived this metric, a lot of new properties were found, increasing the knowledge about rotational gravitational sources and black holes, which has made an impact on what is known about the cosmos [4].

This thesis consists of two parts, where the aim of the first one is to derive the Kerr metric, while the aim of the second part is to determine the stability of the Schwarzschild metric. In section 2, we go through a short historical background and in section 3 some theoretical background that is used is brought up[1]. The performed derivation of the Kerr metric is presented in section 4 and the determination of the Schwarzschild metric can be seen in section 5. The thesis is using natural coordinates, meaning $c = 1$ and $G = 1$.

2 Historical background

Despite the common belief that a black hole is a modern concept, it was actually brought up long before the contributions Albert Einstein made to modern physics. As early as in 1783, the natural philosopher John Michell used Newtonian physics to describe what was then called dark stars. He proposed that a star has a critical circumference linked to the escape velocity and that this escape velocity could be larger than the speed of light for a small enough star. Pierre-Simone Laplace proposed the same in 1795, independently of Michell. However, it turned out that this description did not follow the wave description of light, which later became favored by the scientists and further confirmed by the laws of electromagnetism of James Clerk Maxwell in 1864. Thus, the theory of dark stars seemed for a time no longer probable. It took general relativity to give a fuller understanding of those objects.

It was in 1915 that Albert Einstein published his final piece of the theory of general relativity. Only a year later, the German astrophysicist Karl Schwarzschild made the first steps towards a new possible description of the dark stars, when using the theory of general relativity to describe the field outside a star [5]. What Schwarzschild did was to look at the Einstein field equation and make an exact solution for a field that is spherically symmetric and in vacuum. He used what is known as the geodesic principle for a body falling freely when being close to the sun. His solution showed a more precise solution for the orbits of the planets than that made by Newton, since the work by Newton implied problems when looking at the orbit of Mercury. This solution has further been shown to give the possibility of black holes [6]. This solution did, however, only consider non-spinning, perfectly spherically symmetric objects [5].

[1]It should be noted that the reader is assumed to be familiar with some basic concepts of general relativity.

After the solution made by Schwarzschild, what remained was to find a similar solution that described a rotating body. One of the reasons why such a solution was of interest was because many scientists at the time had discussed if there could be an angular momentum halting down the process of a collapsing star. This was, however, not as easily obtained as for the non-rotational case, which explains why this solution wasn't obtained until almost 50 years later. The reason for this being much more difficult is that if the same procedure as the one of Schwarzschild is done for a rotating body, the equations are very complicated.

In 1932, there were some attempts by Lewis, who used the work done by Weyl in 1917 [4]. What Weyl had done was to introduce new coordinates in order to simplify a non-rotating case. Later, in 1968, Ernst formulated the equations differently and thus found a solution by using this course of action. However, Kerr managed to solve this problem five years before this, using a different method [7]. His exact way of doing this will not be covered here. For a gravitational field of a rotating body, the Kerr solution is thought to be the only possible solution that is asymptotically flat and axisymmetric [8].

3 Theoretical background

3.1 Underlying definitions and notations

This section is meant to give the reader some background regarding the definitions and notations used in this report, as they can differ in different texts.

The metric tensor is a geometrical description of space-time and has the general form

$$\mathrm{d}s^2 = \sum_{\mu\nu} g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu, \tag{3.1}$$

where $g_{\mu\nu} = \eta_{\mu\nu}$ for the Minkowski metric.

In spherical coordinates, the metric in flat space is written as

$$\mathrm{d}s^2 = \sum_{\mu\nu} g_{\mu\nu}\mathrm{d}x^\mu\, \mathrm{d}x^\nu = -\,\mathrm{d}t^2 + \,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \tag{3.2}$$

When differentiating with respect to α (say) one may use the notation

$$f_{,\alpha} \equiv \frac{\partial}{\partial\alpha} f \tag{3.3}$$

The Ricci tensor, in terms of Christoffel symbols, can be written as

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\alpha}, \tag{3.4}$$

with each Christoffel symbol being

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right) \tag{3.5}$$

and by using this we define the Ricci scalar:

$$R = g^{\mu\nu} R_{\mu\nu} \tag{3.6}$$

The Einstein tensor can then be defined in terms of the Ricci tensor and Ricci scalar:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{3.7}$$

Finally, it may be appropriate to use the so-called tortoise coordinate

$$r_* = r + 2\,M\,\ln\left(\frac{r}{2\,M} - 1\right), \tag{3.8}$$

For a Schwarzschild black hole, the event horizon is given by the Schwarzschild radius $r = 2\,M$, so this coordinate has the property to approach the event horizon as $r_* \to -\infty$.

3.2 Simplified metrics for symmetric cases

3.2.1 Stationary axisymmetric space-times

Stationary axisymmetric space-times means that for a rotation around a given space-like line the geometry does not change and is time independent. Let's consider a general metric, as was shown in equation (3.1), and let the axis of symmetry be the time $x^0 = t$ and the angle $x^1 = \phi$, such that the space-time is independent of those variables and only depends on x^2 and x^3. This implies $g_{ij} = g_{ij}(x^2, x^3)$. The symmetry also implies invariance for $t \to -t$ and $\phi \to -\phi$ simultaneously and this in turn gives $g_{02} = g_{03} = g_{12} = g_{13} = 0$. This in the general form (3.1) gives

$$\begin{aligned} \mathrm{d}s^2 &= g_{00}\,\mathrm{d}t^2 + 2\,g_{01}\,\mathrm{d}t\,\mathrm{d}\phi + g_{11}\,\mathrm{d}\phi^2 \\ &\quad + g_{22}\,(\mathrm{d}x^2)^2 + 2\,g_{23}\,\mathrm{d}x^2\,\mathrm{d}x^3 + g_{33}\left(\mathrm{d}x^3\right)^2, \end{aligned} \tag{3.9}$$

Letting $x^2 = r$ and $x^3 = \theta$ implies that $g_{23} = \vec{e}_r \cdot \vec{e}_\theta = 0$ because of orthogonality[2]. It follows that the appropriate form for the stationary axisymmetric space-times is [3, 9]

$$ds^2 = -e^{2\,\nu}\,dt^2 + e^{2\,\psi}\,(d\phi - \omega\,dt)^2 + e^{2\,\mu_2}\,dr^2 + e^{2\,\mu_3}\,d\theta^2, \tag{3.10}$$

where ν, ψ, ϕ, μ_2 and μ_3 are functions of r and θ.

3.2.2 Non-stationary axisymmetric space-times

Here, we want to make (3.10) more general by allowing for time dependence, meaning $g_{ij} = g_{ij}(t, x^2, x^3)$. Here, one can use the Cotton-Darboux theorem [3], which states that a metric with three coordinates in a three-dimensional space can always be diagonalized. In this case, this implies that $g_{02} = g_{03} = g_{23} = 0$. An appropriate form of this metric has turned out to be [3]

$$ds^2 = -e^{2\,\nu}\,dt^2 + e^{2\,\psi}\left(\mathrm{d}\phi - \omega\,\mathrm{d}t + q_2\,\mathrm{d}x^2 + q_3\,\mathrm{d}x^3\right)^2 + e^{2\,\mu_2}\,(\mathrm{d}\,x^2)^2 + e^{2\,\mu_3}\,(\mathrm{d}\,x^3)^2, \tag{3.11}$$

where ν, ψ, ϕ, ω, q_2, q_3, μ_2 and μ_3 are functions of t, x^2 and x^3.

[2] This can also be shown to be true for general x^2 and x^3 [3].

3.3 The Schwarzschild metric

In spherical coordinates, the Schwarzschild metric has the form

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\,\sin^2\theta\,\mathrm{d}\phi^2 \tag{3.12}$$

and describes the exterior geometry of a static, spherically symmetric object of mass M. This object is typically a star or a black hole.

3.4 The Kerr metric

The Kerr metric also describes the exterior geometry of a massive object (typically a star or a black hole). Unlike the Schwarzschild metric, however, this metric is not static and spherically symmetric, but axisymmetric and allows for rotation around the axis of symmetry.

In oblate spherical coordinates, the Kerr metric has the form [2]

$$ds^2 = -dt^2 + \frac{\Sigma\,dr^2}{a^2 + r^2 - 2\,M\,r} + \Sigma\,d\theta^2 + (a^2 + r^2)\,\sin^2\theta\,d\phi^2 + \frac{2\,M\,r}{\Sigma}(dt - a\,\sin^2\theta\,d\phi)^2, \tag{3.13}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $a \equiv J/M$, where J is the angular momentum and M is the total mass of the object [10].

3.5 Perturbation theory

Perturbation theory is often used for mathematical problems that either have no exact solution or whose solutions are very complicated.

The general approach of finding an approximate solution with perturbation theory is by starting from the exact solution from either a special case of the problem at hand or a related simpler problem. One then forms the solution of the problem in terms of parameters related to the known solution. These parameters are kept small and thus introduce a small change, a perturbation, allowing the perturbed solution propagate through the answer. One may add a small parameter ϵ (say) such that one has a unperturbed problem at $\epsilon = 0$ and then look at the change of the solution for small non-zero ϵ. This can often be done with the help of a Taylor expansion around ϵ.

The application of such a general prescription is more of an art than a simple formula. The easiest way of depicting the concept of perturbation theory is via practice, see section 5.1.

4 Derivation of the Kerr metric

This section covers the derivation of the Kerr metric. The Schwarzschild metric - which is a non-rotating case of the Kerr metric - can be solved relatively easy using pen and paper. When deriving the Kerr metric, however, things start to get more complicated. Thankfully, deriving the Kerr metric is much easier nowadays than it was when Kerr first derived it. One reason for it being easier is because it has already been solved once, giving the person deriving it the advantage of knowing the expected result. Another reason is because computers have been evolving and can give a quicker solution to differential equations than a human would. Here, we choose to derive the Kerr metric by first making an ansatz and then solving the unknown functions by using the Einstein-Hilbert action as well as deriving it directly from the Einstein field equations. For this we applied the Weyl method [1, 2], making it possible to only solve for two unknown functions. All of the calculations were performed in Maple.

4.1 Kerr metric ansatz

Here, we will make an ansatz for the Kerr metric, by considering what is known from section 3. It will turn out that the ansatz will be similar to that of Deser and Franklin [2] as well as Teukolsky [4].

The Kerr metric gives a description of the space-time for a rotating body. For this, we first assume stationary, axisymmetric space-times [11]. As mentioned before, this can be represented by the metric in equation (3.10). Thus, what's left is a metric with components dependent of x^2 and x^3 and unlike the Minkowski metric there is also a $\mathrm{d}t\,\mathrm{d}\phi$ term present.

Now, what remains is to find a suitable coordinate system for what has been given so far. In Newtonian physics, it is generally known that due to the fictitious centrifugal force, a massive rotating body, like a planet, that would have otherwise been nearly spherical, gets a shape similar to that of an oblate spheroid. One can, as a starting point, therefore assume that using spheroidal coordinates is a good choice for the Kerr metric. For this, we add a parameter a for the equatorial radius, so that the coordinate system approaches spherical coordinates as $a \to 0$. Since spherical coordinates are a good choice in the non-rotating case, we can conclude that $a \to 0$ should imply that the metric approaches the Schwarzschild metric.

The ellipsoidal coordinates are given by [4]:

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin\theta\cos\phi \\ y &= \sqrt{r^2 + a^2}\sin\theta\sin\phi \\ z &= r\,\cos\theta \end{aligned} \tag{4.1}$$

Now, the flat metric in those coordinates, after doing a coordinate transformation from the Cartesian coordinates, becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left(\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\right)\mathrm{d}r^2 + \left(r^2 + a^2\cos^2\theta\right)\mathrm{d}\theta^2 + \left(r^2 + a^2\right)\sin^2\theta\,\mathrm{d}\phi^2 \tag{4.2}$$

Looking back at the expression obtained in equation (3.10), one can see that there is a $\mathrm{d}t\,\mathrm{d}\phi$ missing in equation (4.2). However, when making a starting point for the anzats, one can get around this by adding 0. We choose to follow the same argument as Teukolsky did in his paper [4]. By first multiplying the $\mathrm{d}t$ and $\mathrm{d}\phi$ terms with Σ/Σ, where $\Sigma = r^2 + a^2\,\cos^2\theta$ and them adding them, one could rewrite their sum as

$$\begin{aligned} -\mathrm{d}t^2 + \left(r^2 + a^2\right)\sin^2\theta\,\mathrm{d}\phi^2 = &-\frac{r^2+a^2}{\Sigma}\left(\mathrm{d}t - a\,\sin^2\theta\,\mathrm{d}\phi\right)^2 \\ &+ \frac{\sin^2\theta}{\Sigma}\left(a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi\right)^2 \end{aligned} \tag{4.3}$$

Using this in equation (4.2) gives

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{r^2+a^2}{\Sigma}\left(\mathrm{d}t - a\,\sin^2\theta\,\mathrm{d}\phi\right)^2 + \left(\frac{\Sigma}{r^2+a^2}\right)\mathrm{d}r^2 \\ &+ \left(r^2 + a^2\cos^2\theta\right)\mathrm{d}\theta + \frac{\sin^2\theta}{\Sigma}\left(a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi\right)^2 \end{aligned} \tag{4.4}$$

Now, let $r^2 + a^2 \to r^2 + a^2 - Z(r,\theta)$ in the first term and $r^2 + a^2 \to D(r,\theta)$ in the second term. This gives the following metric:

$$ds^2 = -dt^2 + \frac{\Sigma}{D(r,\theta)}\,dr^2 + \Sigma\,d\theta^2 + (a^2+r^2)\,\sin^2\theta\,d\phi^2 + \frac{Z(r,\theta)}{\Sigma}(dt - a\,\sin^2\theta\,d\phi)^2, \tag{4.5}$$

where $D(r,\theta)$ and $Z(r,\theta)$ are two unknown functions. This ansatz, (4.5), is the one that was mentioned in the beginning of the section as similar[3] to that of Deser and Franklin [2].

The solutions of the two unknown functions were derived in two different ways. The first way was to vary the Einstein-Hilbert action for the unknown functions and the other was to use the Einstein field equations.

4.2 Derivation using the Einstein-Hilbert action

The Einstein-Hilbert action is

$$S = \pm\frac{2}{K}\int R\sqrt{-g}\,d^4x \tag{4.6}$$

where R is the Ricci scalar and g is the determinant of the metric tensor. $R\sqrt{-g}$ forms the Lagrangian $\mathcal{L}$ and the constant before the integral can be regarded as 1 without changing the relating equations.

By using the metric in equation (4.5), one gets a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}\left(Z, Z_{,r}, Z_{,r,r}, Z_{,\theta}, Z_{,\theta,\theta}, D, D_{,r}, D_{,r,r}, D_{,\theta}, D_{,\theta,\theta}, r, \theta\right) \tag{4.7}$$

[3]The difference is that the two unknown functions are dependent of r as well as θ in this report, but were dependent of only r in that of Deser and Franklin.

and the action is

$$S = \int \mathcal{L}\,\mathrm{d}^4x \tag{4.8}$$

Since there are second derivatives in the Lagrangian, one needs to derive a corresponding Euler-Lagrange equation that includes second derivatives. Let's consider a function $f(r,\theta)$ and then use the derived relation for $D(r,\theta)$ and $Z(r,\theta)$. When varying the action, one gets

$$\delta S = \iint \delta\mathcal{L}\,\mathrm{d}r\,\mathrm{d}\theta = \iint \frac{\partial\mathcal{L}}{\partial f}\delta f + \frac{\partial\mathcal{L}}{\partial f_{,r}}\delta f_{,r} + \frac{\partial\mathcal{L}}{\partial f_{,r,r}}\delta f_{,r,r} + \frac{\partial\mathcal{L}}{\partial f_{,\theta}}\delta f_{,\theta} + \frac{\partial\mathcal{L}}{\partial f_{,\theta,\theta}}\delta f_{,\theta,\theta}\,\mathrm{d}r\,\mathrm{d}\theta \tag{4.9}$$

By integrating by parts and neglecting the surface terms, one gets

$$\begin{aligned}\delta S &= \iint \frac{\partial\mathcal{L}}{\partial f}\delta f - \frac{\partial}{\partial r}\frac{\partial\mathcal{L}}{\partial f_{,r}}\delta f + \frac{\partial^2}{\partial r^2}\frac{\partial\mathcal{L}}{\partial f_{,r,r}}\delta f - \frac{\partial}{\partial\theta}\frac{\partial\mathcal{L}}{\partial f_{,\theta}}\delta f + \frac{\partial^2}{\partial\theta^2}\frac{\partial\mathcal{L}}{\partial f_{,\theta,\theta}}\delta f\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \iint \delta f\left[\frac{\partial\mathcal{L}}{\partial f} - \frac{\partial}{\partial r}\frac{\partial\mathcal{L}}{\partial f_{,r}} + \frac{\partial^2}{\partial r^2}\frac{\partial\mathcal{L}}{\partial f_{,r,r}} - \frac{\partial}{\partial\theta}\frac{\partial\mathcal{L}}{\partial f_{,\theta}} + \frac{\partial^2}{\partial\theta^2}\frac{\partial\mathcal{L}}{\partial f_{,\theta,\theta}}\right]\mathrm{d}r\,\mathrm{d}\theta = 0\end{aligned} \tag{4.10}$$

For this to be equal to zero in all cases, the expression inside the brackets should be zero. Using the same calculations for $D(r,\theta)$ and $Z(r,\theta)$, this gives the relations

$$\frac{\partial\mathcal{L}}{\partial D} - \frac{\partial}{\partial r}\frac{\partial\mathcal{L}}{\partial D_{,r}} + \frac{\partial^2}{\partial r^2}\frac{\partial\mathcal{L}}{\partial D_{,r,r}} - \frac{\partial}{\partial\theta}\frac{\partial\mathcal{L}}{\partial D_{,\theta}} + \frac{\partial^2}{\partial\theta^2}\frac{\partial\mathcal{L}}{\partial D_{,\theta,\theta}} = 0 \tag{4.11}$$

and

$$\frac{\partial\mathcal{L}}{\partial Z} - \frac{\partial}{\partial r}\frac{\partial\mathcal{L}}{\partial Z_{,r}} + \frac{\partial^2}{\partial r^2}\frac{\partial\mathcal{L}}{\partial Z_{,r,r}} - \frac{\partial}{\partial\theta}\frac{\partial\mathcal{L}}{\partial Z_{,\theta}} + \frac{\partial^2}{\partial\theta^2}\frac{\partial\mathcal{L}}{\partial Z_{,\theta,\theta}} = 0 \tag{4.12}$$

We now want to find the solutions for $D(r,\theta)$ and $Z(r,\theta)$. From previous publications, we know that $D(r,\theta) \to D(r)$ and $Z(r,\theta) \to Z(r)$ provides a solution. Proving this in a more general case is no simple matter. However, it can be argued that this should be the case using the following.

First, let's assume that $Z(r,\theta)$ and $D(r,\theta)$ can be written as Fourier series,

$$Z(r,\theta) = \sum_{n=0}^{l} f_n(r)\cos n\,\theta + g_n(r)\sin n\,\theta \tag{4.13}$$

and

$$D(r,\theta) = \sum_{n=0}^{s} h_n(r)\cos n\,\theta + k_n(r)\sin n\,\theta \tag{4.14}$$

where l and s are to be determined.

Now, one may look at small variations of θ, here denoted $\delta\theta$. Around $\theta = 0$ this implies $\sin\theta \to \delta\theta$ and $\cos\theta \to 1$. For $Z(r,\theta)$, this implies

$$Z(r,\theta) = \sum_{n=0}^{l} f_n(r) + \delta\theta \sum_{n=0}^{l} n\, g_n(r) \tag{4.15}$$

and similarly for $D(r,\theta)$ and their respective partial derivatives. Since we can break out $\delta\theta$, one could (with the assumption that all sums give a finite answer) regard the sums as functions of r. Because of a recurrent pattern, we found it appropriate to define a function $X_\beta^{(\alpha)}(r)$ as

$$X_\beta^{(\alpha)}(r) \equiv \sum_{n=0}^{\beta} n^\alpha\, x_n(r) \tag{4.16}$$

where the functions $x_n(r)$ for $n \in [0,\alpha]$ are assumed to be linearly independent.

For $Z(r,\theta)$, this further gives

$$\left.\begin{aligned} Z(r,\theta) &= F_l^{(0)}(r) + \delta\theta\, G_l^{(1)}(r) \\ Z_{,\theta} &= G_l^{(1)}(r) - \delta\theta\, F_l^{(2)}(r) \\ Z_{,\theta,\theta} &= -F_l^{(2)}(r) - \delta\theta\, G_l^{(3)}(r) \end{aligned}\right\}$$

One can simply obtain the corresponding for $D(r,\theta)$ by replacing F by H, G by K and l by s.

The obtained expressions for $Z(r,\theta)$ and $D(r,\theta)$ with $\theta \to \delta\theta$ were used in equation (4.11) and (4.12). Let's first consider function (4.11). This resulted in an expression with five separable equations inside $\delta\theta^4$, $\delta\theta^3$, $\delta\theta^2$, $\delta\theta$ and 1. The equation inside 1 was

$$2\,(a^2+r^2)^2\,(a^2+r^2-F_l^{(0)})\,G_l^{(1)} = 0 \tag{4.18}$$

This is true for $G_l^{(1)} = 0$ or $F_l^{(0)} = a^2+r^2$. The equation inside $\delta\theta^4$ was

$$\left(4\,r\,K_s^{(1)}(r)\,(a^2+r^2)\,\left(\frac{\mathrm{d}}{\mathrm{d}r}G_l^{(1)}(r)\right) + 4\,a^2\,G_l^{(1)}(r)^2 - 4\,K_s^{(1)}(r)\,r^2\,G_l^{(1)}(r)\right)\,G_l^{(1)}(r) = 0 \tag{4.19}$$

Here, $G_l^{(1)}(r) = 0$ clearly gives a solution. With the assumption of linear independence, this could imply one of two things. It could either implies that $l = 0$ or it could imply that $g_n(r) = 0$, where $n \in [1,l]$. In both cases, this gives $G_l^{(2)}(r) = G_l^{(3)}(r) = 0$. After letting $G_l^{(1)} = 0$, the equation inside $\delta\theta^3$ became

$$(a^2+r^2)^2\,F_l^{(2)}(r) = 0, \tag{4.20}$$

which clearly has the solution $F_l^{(2)}(r) = 0$. Again, this solution could either imply that $l = 0$ or $f_n(r) = 0$, where $n \in [1,l]$, which in either case implies $F_l^{(1)}(r) = F_l^{(3)}(r) = 0$.

The solutions $G_l^{(1)}(r) = G_l^{(2)}(r) = G_l^{(3)}(r) = 0$ and $F_l^{(1)}(r) = F_l^{(2)}(r) = F_l^{(3)}(r) = 0$ imply that we have $Z(r,\theta) = f_0(r)$, which means that the function Z has no θ dependence.

The equations inside $\delta\theta^2$ and $\delta\theta$ are still not vanishing when using this, but we will come back to that later.

When using equation (4.12) and then letting $G_l^{(1)}(r) = G_l^{(2)}(r) = G_l^{(3)}(r) = 0$ and $F_l^{(1)}(r) = F_l^{(2)}(r) = F_l^{(3)}(r) = 0$, one does, again, get separable equations. This time there are four of them, inside $\delta\theta^3$, $\delta\theta^2$, $\delta\theta$ and 1. These equations (as well as the two equations that were left when using equation (4.11)) turned out to be vanishing for $F_l^{(0)} = a^2 + r^2$, providing a solution. However, this implies that we get no information regarding $D(r,\theta)$. Because of this, we consider a case when $F_l^{(0)}(r)$ does not have this solution[4].

When looking back at the equation inside $\delta\theta^2$, that was obtained using equation (4.11), we find that

$$K_s^{(3)}(r)\, 2\,(a^2+r^2)^2 + 4\, r\, K_s^{(1)}(r)\, \left[(a^2+r^2)\left(r - \frac{\mathrm{d}}{\mathrm{d}r}F_l^{(0)}(r)\right) + r\, F_l^{(0)}(r)\right] = 0 \tag{4.21}$$

By applying the definition in equation (4.16) for $K_s^{(3)}(r)$ and $K_s^{(1)}(r)$, one can break out $n\, k_n(r)$ inside the sum. It is then clear that $s = 0$ and $k_n(r) = 0$ for $n \in [1, s]$ provide solutions, which clearly also imply that $K_s^{(1)}(r) = K_s^{(2)}(r) = K_s^{(3)}(r) = 0$. If this would not be the case, then $F_l^{(0)}(r)$ would provide no real solution.

From equation (4.12), the equation inside $\delta\theta^3$ now becomes

$$3\,(a^2+r^2)^2\, H_s^{(2)}(r)^2 = 0 \tag{4.22}$$

This is true for $H_s^{(2)}(r) = 0$, which, using the same argument as before, implies $H_s^{(1)}(r) = H_s^{(2)}(r) = H_s^{(3)}(r) = 0$.

With the assumptions at hand, it has now been argued that the equations should satisfy $G_l^{(1)}(r) = G_l^{(2)}(r) = G_l^{(3)}(r) = 0$, $F_l^{(1)}(r) = F_l^{(2)}(r) = F_l^{(3)}(r) = 0$, $K_s^{(1)}(r) = K_s^{(2)}(r) = K_s^{(3)}(r) = 0$ and $H_s^{(1)} = H_s^{(2)} = H_s^{(3)} = 0$. This in equation (4.13) and (4.14) implies $Z(r,\theta) = f_0(r)$ and $D(r,\theta) = h_0(r)$, meaning that they have no θ dependence.

By extremizing the action integral with respect to $D(r,\theta)$ and letting $D(r,\theta) \to D(r)$ and $Z(r,\theta) \to Z(r)$, one gets:

$$\begin{aligned} &r^2\left[r\, D(r)\left(\frac{d}{d\,r}Z(r)\right) - (r^2 + Z(r))D(r) + r^2(a^2 + r^2 - Z(r))\right] + \\ &+a^2\left[r\, D(r)\left(\frac{d}{dr}Z(r)\right) - D(r)\, r^2 + \left(r^2 - Z(r)\right)\left(a^2 + r^2 - Z(r)\right)\right]\cos^2\theta = 0 \end{aligned} \tag{4.23}$$

[4] This also allows us to cancel out $F_l^{(0)}(r) - a^2 + r^2$ in the equations.

Similarly, when extremizing the action integral with respect to $Z(r,\theta)$, one gets:

$$\begin{aligned}
& a^4\,\cos^2\theta\left[2a^2 - r\left(\frac{d}{dr}D(r)\right) - 5\,Z(r) - 2D(r)\right] \\
& + r^2 a^2\,\cos^2\theta\left[5a^2 + 3r^2 - r\left(\frac{d}{dr}D(r)\right) - 6\,Z(r) - D(r)\right] \\
& + Z(r)\,a^2\,\cos^2\theta\left[r\left(\frac{d}{dr}D(r)\right) + 3\,Z(r) + 2\,D(r)\right] \\
& + r^2\left[r^2\left(a^2 + r^2 + D(r)\right) - r\left(a^2 + r^2 - Z(r)\right)\left(\frac{d}{dr}D(r)\right) - \left(r^2 + D(r)\right)Z(r)\right] = 0
\end{aligned} \tag{4.24}$$

It should be noted that equation (4.23) and (4.24) contain two separate equations each, since $a^2\cos^2\theta$ can be taken out of parts of the equations. For $\cos\theta = 0$ we are left with two equations that are only dependent of r, that can be solved. Since (4.23) and (4.24) should hold for any θ this should also hold for any other value of θ.

The equations (4.23) an (4.24) had the solutions:

- $Z(r) = 0,\ D(r) = a^2 + r^2$
- $Z(r) = a^2 + r^2,\ D(r) = 0$
- $Z(r) = C\,r,\ D(r) = a^2 + r^2 - C\,r$

The second solution gives $g = 0$, which implies it gives the null metric. This leaves us with the third solution, with the first one just being a special case of the third one. By adding these solutions in equation (4.5) and letting the metric approach the Schwarzschild metric (defined in equation (3.12)) as $a \to 0$, one ends up with $C = 2\,M$, which gives:

$$ds^2 = -dt^2 + \frac{\Sigma\,dr^2}{a^2 + r^2 - 2\,M\,r} + \Sigma\,d\theta^2 + (a^2 + r^2)\,\sin^2\theta\,d\phi^2 + \frac{2\,M\,r}{\Sigma}(dt - a\,\sin^2\theta\,d\phi)^2 \tag{4.25}$$

This metric is the same as the Kerr metric defined in equation (3.13).

4.3 Derivation using the Einstein field equations

The second method was to look at the Ricci tensors. Here, it is already assumed that the two unknown functions only depend on r. In vacuum, $R_{\mu\nu} = 0$. The simplest Ricci tensor to simplify was the R_{22} tensor. After letting $R_{22} = 0$ and simplifying, one got the following:

$$
\begin{aligned}
&a^2\cos^2\theta\left[-r\,\frac{\mathrm{d}D(r)}{\mathrm{d}r}\left(a^2+r^2-Z(r)\right)+2\,a^4+8\,a^2\,r^2+6\,r^4\right]\\
&+a^2\cos^2\theta\left[D(r)\left(r\,\frac{\mathrm{d}Z(r)}{\mathrm{d}r}-2\,a^2-4\,r^2\right)+Z(r)\left(4\,Z(r)+2\,D(r)-6\,a^2-10\,r^2\right)\right]\\
=&r^2\left[r\,\frac{\mathrm{d}D(r)}{\mathrm{d}r}\left(a^2+r^2-Z(r)\right)+2(a^4-r^4)\right]\\
+&r^2\left[-D(r)\left(r\,\frac{\mathrm{d}Z(r)}{\mathrm{d}r}+2\,a^2\right)+2\,Z(r)\left(r^2-a^2+D(r)\right)\right]
\end{aligned}
\tag{4.26}
$$

For this to be true for all r and θ, the left hand side and the right hand side should equal 0. This results in two ordinary differential equations (ODEs):

$$
\begin{aligned}
ODE1=&-r\,\frac{\mathrm{d}D(r)}{\mathrm{d}r}\left(a^2+r^2-Z(r)\right)+2\,a^4+8\,a^2\,r^2+6\,r^4\\
&+D(r)\left(r\,\frac{\mathrm{d}Z(r)}{\mathrm{d}r}-2\,a^2-4\,r^2\right)+Z(r)\left(4\,Z(r)+2\,D(r)-6\,a^2-10\,r^2\right)=0
\end{aligned}
\tag{4.27}
$$

and

$$
ODE2=r\,\frac{\mathrm{d}D(r)}{\mathrm{d}r}\left(a^2+r^2-Z(r)\right)+2(a^4-r^4)-D(r)\left(r\,\frac{\mathrm{d}Z(r)}{\mathrm{d}r}+2\,a^2\right) \tag{4.28}
$$

$$
+2\,Z(r)\left(r^2-a^2+D(r)\right)=0 \tag{4.29}
$$

Now, if one adds these ODEs the derivatives cancel, giving

$$
\frac{ODE1+ODE2}{4}=(a^2+r^2-Z(r))(a^2+r^2-Z(r)-D(r))=0 \tag{4.30}
$$

This gives $D(r)=a^2+r^2-Z(r)$. After adding this in equation (4.27), one could see that $Z(r)=C\,r$, which in turn gave $D(r)=a^2+r^2-C\,r$. Note that we so far only looked at the R_{22} tensor. The solutions did, however, also turn out to give $R_{\mu\nu}=0$ for all $\mu,\nu\in[0,3]$ and consequently a vanishing Ricci scalar. One could therefore conclude that the solutions give a metric in vacuum [2].

Again, by inserting this in equation (4.5) and using the fact that $a\to 0$ should give the Schwarzschild metric, one can see that the constant C should equal $2\,M$, which is the same result as in the previous method (Section 4.2).

5 The stability of the Schwarzschild metric

Here, the stability of the Schwarzschild metric is determined using perturbation theory (as previously mentioned in section 3.5). This is done following the approach of Chandrasekhar [3].

5.1 Creating a base

First, we restrict ourselves to time-dependent axisymmetric modes, that can be described by the metric given in (3.11). It can be argued that this is allowed due to the fact that there is no preferred rotational axis for a spherically symmetric system.

When determining the stability of the Schwarzschild metric, we want to make small variations around the Schwarzschild values, meaning that we get $\nu+\delta\nu$, $\mu_2+\delta\mu_2$, $\mu_3+\delta\mu_3$, $\psi+\delta\psi$, $\omega+\delta\omega$, $q_2+\delta q_2$ and $q_3+\delta q_3$. For Schwarzschild, we have $e^{2\,\nu}=e^{-2\,\mu_2}=1-\frac{2\,M}{r}$, $e^{\mu_3}=r$, $e^{\psi}=r\,\sin\theta$ and $\omega=q_2=q_3=0$. We also have $x^2=r$ and $x^3=\theta$. Since the variations are small, we can make a first order Taylor expansion around those values in the metric. This resulted in the following metric:

$$\begin{aligned} ds^2 = &-\left(1-\frac{2\,M}{r}\right)(1+2\,\delta\nu)\;dt^2+r^2\,\sin^2\theta\,(1+2\,\delta\psi)\,(d\varphi-\delta\omega\,dt-\delta q_2\,dr-\delta q_3\,d\theta)^2 \\ &+\frac{1}{\left(1-\frac{2\,M}{r}\right)}\,(1+2\,\delta\mu_2)\;dr^2+r^2\,(1+2\,\delta\mu_3)\;d\theta^2 \end{aligned} \tag{5.1}$$

This is our base for the following perturbation calculations. One can see that $\delta\omega$, δq_2 and δq_3 will cause a dragging of the inertial frames and hence a rotation, as they give the metric off-diagonal terms. However, $\delta\nu$, $\delta\psi$, $\delta\mu_2$ and $\delta\mu_3$ will not. Chandrasekhar called these perturbations the axial and polar perturbations, respectively. This choice of names can be further justified when looking at the effect of the metric when reversing the sign of ϕ. Axial and polar vectors are linearly independent, and it can therefore be argued that these perturbations are independent of each other. An intuitive picture for a two-dimensional space can be seen in Figure 1.

5.2 Axial perturbations

We let the perturbations be of the first order and then calculated for $\delta R_{12}=0$ and $\delta R_{13}=0$ in Maple. This gave the same result as Chandrasekhar originally got in his book, i.e.

$$\frac{\partial}{\partial\theta}\,Q(t,r,\theta)=\frac{\partial}{\partial t}\left[r^4\sin^3\theta\left(\frac{\partial}{\partial r}\delta\omega-\frac{\partial}{\partial t}\delta q_2\right)\right] \tag{5.2}$$

and

$$\frac{\partial}{\partial r}\,Q(t,r,\theta)=-\frac{\partial}{\partial t}\left[\frac{r^3\sin^3\theta}{r-2\,M}\left(\frac{\partial}{\partial\theta}\delta\omega-\frac{\partial}{\partial t}\delta q_3\right)\right], \tag{5.3}$$

where

$$Q(t,r,\theta)=r^2\sin^3\theta\left(1-\frac{2\,M}{r}\right)\left(\frac{\partial}{\partial r}\delta q_3-\frac{\partial}{\partial\theta}\delta q_2\right) \tag{5.4}$$

Since the equations are linear in the perturbations, the time derivatives can be replaced by a factor $i\sigma$, where σ is a frequency. This comes from the general definitions of the Fourier transform.

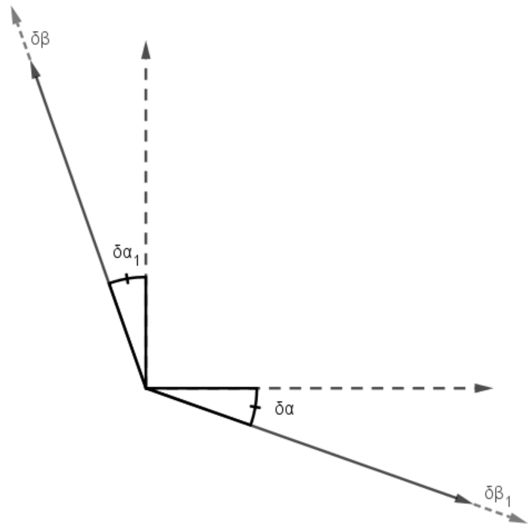

Figure 1: The figure shows a two-dimensional space, for which $\delta\alpha$ denotes the angular difference from the dragging of the inertial frames, while $\delta\beta$ denotes the change of the length of the vectors. These are clearly independent, since the angular growth direction is perpendicular to $\delta\beta$.

Now, our equations can be written as

$$\frac{1}{r^4\,\sin^3\theta}\frac{\partial}{\partial\theta}\,Q(t,r,\theta) = i\,\sigma\,\frac{\partial}{\partial r}\delta\omega + \sigma^2\,\delta q_2 \tag{5.5}$$

and

$$\frac{r-2\,M}{r^3\,\sin^3\theta}\frac{\partial}{\partial r}\,Q(t,r,\theta) = -i\,\sigma\,\frac{\partial}{\partial\theta}\delta\omega - \sigma^2\,\delta q_3 \tag{5.6}$$

By differentiating (5.5) with respect to r and (5.6) with respect to θ and then adding them, the $\delta\omega$ disappeared. After rewriting and using our definition of $Q(t,r,\theta)$, we ended up with

$$\sin^3\theta\frac{\partial}{\partial\theta}\left[\frac{1}{\sin^3\theta}\frac{\partial\,Q}{\partial\theta}\right] + r^4\frac{\partial}{\partial r}\left[\frac{r-2\,M}{r^3}\frac{\partial\,Q}{\partial r}\right] + \frac{\sigma^2 r^2}{\left(1-\frac{2\,M}{r}\right)}Q = 0 \tag{5.7}$$

One may notice that equation (5.7) has some similarities with the Gegenbauer functions, $C_n^{\nu}(\theta)$, when $\nu = -3/2$. These have the property

$$\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\sin^{2\nu}\theta + n\,(n+2\,\nu)\right]\,C_n^{\nu}(\theta) = 0 \tag{5.8}$$

After using separation of variables for equation (5.7) and letting the angular part be equal to $C_{\ell+2}^{-3/2}(\theta)$ and the radial part be equal to $g(r)$, we found some cancellations, making it possible for (5.7) to be written as a radial equation:

$$r\,(r-2\,M)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r\,(r-2\,M)}{r^4}\frac{\mathrm{d}}{\mathrm{d}r}g(r)\right)-2\,n\frac{r\,(r-2\,M)}{r^4}\,g(r)+\sigma^2 g(r)=0, \tag{5.9}$$

where

$$n\equiv\frac{(\ell+2)(\ell-1)}{2} \tag{5.10}$$

Chandrasekhar has shown that by letting $g(r) = r\,Z^{(-)}$ and introducing the tortoise coordinate defined in (3.8), one would get that (5.9) could be rewritten as

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}+V^{(-)}\right)\,Z^{(-)}=\sigma^2\,Z^{(-)}, \tag{5.11}$$

where

$$V^{(-)}=\frac{r\,(r-2\,M)}{r^5}\,[2\,(n+1)\,r-6\,M] \tag{5.12}$$

By using Maple it could be verified that this was, indeed, the case. Thus, the axial perturbation gives a solution that can be expressed similar to that of the one-dimensional Schrödinger equation. The solution shows that the axial perturbations can be described by a wave equation with a potential barrier $V^{(-)}$. This potential approaches 0 as $r \to 2\,M$ and as $r \to \infty$, or as $r_* \to -\infty$ and $r_* \to \infty$, meaning that it is vanishing at the event horizon and at infinity. Figure 2 shows a plot of this potential as a function of r_* for $\ell = 2$, $\ell = 3$ and $\ell = 4$.

5.3 Polar perturbations

For the polar perturbations, we looked at vanishing δR_{02}, δR_{03}, δR_{23} and δG_{22} as they only were found to be dependent of $\delta\nu$, $\delta\mu_2$, $\delta\mu_3$ and $\delta\psi$. The following expressions were shown in Chandrasekhar's book, and are treated as an ansatz in this report:

$$\begin{cases}\delta\nu = N(r)\,P_\ell(\cos\theta)\\ \delta\mu_2 = L(r)\,P_\ell(\cos\theta)\\ \delta\mu_3 = [T(r)\,P_\ell(\cos\theta)+V(r)\,\frac{\partial^2}{\partial\theta^2}P_\ell(\cos\theta)]\\ \delta\psi = [T(r)\,P_\ell(\cos\theta)+V(r)\cot\theta\,\frac{\partial}{\partial\theta}P_\ell(\cos\theta)]\end{cases} \tag{5.13}$$

After letting $\delta\nu$, $\delta\mu_2$, $\delta\mu_3$ and $\delta\psi$ be of the first order and then using (5.13) we found that, after simplifying, $\delta R_{03} = 0$ gave $T(r) = V(r) - L(r)$, which was used for the following calculations.

After simplifying $\delta R_{02} = 0$, we got the following:

$$\frac{\mathrm{d}}{\mathrm{d}r}(n\,V(r)+L(r))=-\frac{1}{r\,(2\,M-r)}\,[(3\,M-r)\,n\,V(r)+(5\,M-2\,r)\,L(r)] \tag{5.14}$$

By letting $\delta R_{23} = 0$ and then simplify, we found that:

$$\frac{\mathrm{d}}{\mathrm{d}r}L(r)=\frac{\mathrm{d}}{\mathrm{d}r}N(r)-\frac{1}{r(2\,M-r)}\,[(3\,M-r)\,N(r)+(M-r)\,L(r)] \tag{5.15}$$

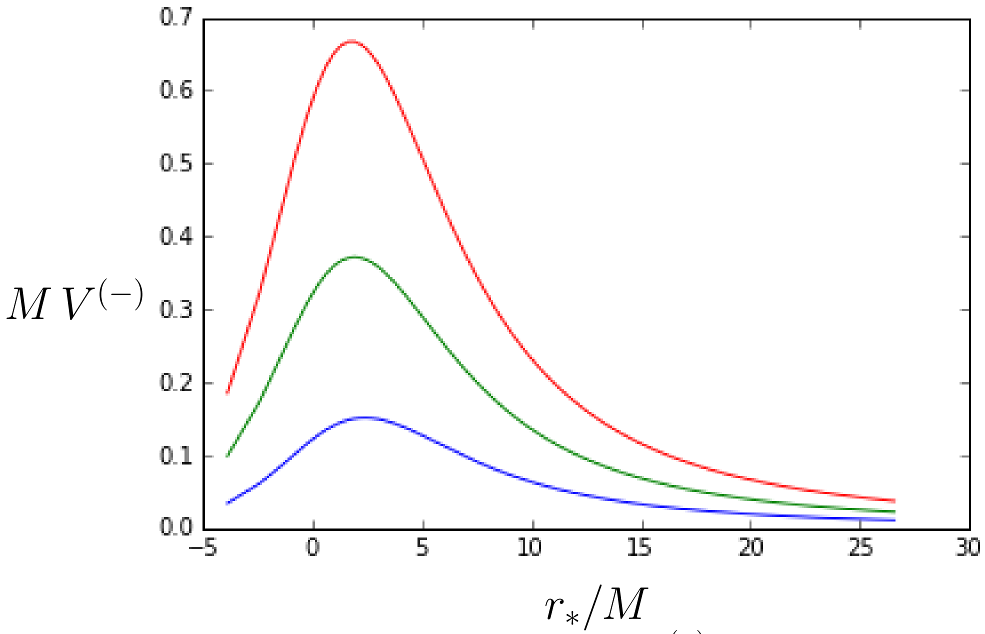

Figure 2: The plot shows the potential barrier, $V^{(-)}$, for the axial perturbations. The lowest curve has $\ell = 2$, while the other two have $\ell = 3$ and $\ell = 4$. At first glance, the potentials seem to have a maximum for the same value of r_*, but they do vary slightly depending on ℓ.

Finally, $\delta G_{22} = 0$ implied:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}N(r) &= \frac{1}{2\,M-r}\left[(M-r)\,\frac{\mathrm{d}}{\mathrm{d}r}(n\,V(r)+L(r))\right] \\ &+\frac{1}{2\,M-r}\left[\left(\frac{\sigma^2\,r^3}{(2\,M-r)}-1\right)(n\,V(r)+L(r))-\frac{l\,(l+1)}{2}N(r)+n\,L(r)\right] \end{aligned} \tag{5.16}$$

Regarding the derivatives, this is a system with three unknown functions and three derivatives, meaning that the system can be solved using elementary algebra. By inserting equation (5.14) in equation (5.16) one gets

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}N(r) &= \left[\frac{(r-M)\,(3\,M-r)}{r\,(2\,M-r)^2}+\frac{\sigma^2 r^3}{(2\,M-r)^2}-\frac{1}{2\,M-r}\right]n\,V(r) \\ &+\left[\frac{(r-M)\,(5\,M-2\,r)}{r\,(2\,M-r)^2}+\frac{\sigma^2 r^3}{(2\,M-r)^2}+\frac{n-1}{2\,M-r}\right]L(r) \\ &+\left[-\frac{l\,(l+1)}{2\,(2\,M-r)}\right]N(r) \end{aligned} \tag{5.17}$$

This in (5.15) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}L(r) = &\left[\frac{(r-M)\,(3\,M-r)}{r\,(2\,M-r)^2} + \frac{\sigma^2 r^3}{(2\,M-r)^2} - \frac{1}{2\,M-r}\right] n\,V(r) \\ &+ \left[\frac{(r-M)\,(5\,M-2\,r)}{r\,(2\,M-r)^2} + \frac{\sigma^2 r^3}{(2\,M-r)^2} + \frac{n-1}{2\,M-r} - \frac{M-r}{r\,(2\,M-r)}\right] L(r) \\ &+ \left[-\frac{l\,(l+1)}{2\,(2\,M-r)} - \frac{3\,M-r}{r\,(2\,M-r)}\right] N(r)\end{aligned} \tag{5.18}$$

Finally, this in (5.14) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}n\,V(r) = &-\left[\frac{3\,M-r}{r\,(2\,M-r)} + \frac{(r-M)\,(3\,M-r)}{r\,(2\,M-r)^2} + \frac{\sigma^2 r^3}{(2\,M-r)^2} - \frac{1}{2\,M-r}\right] n\,V(r) \\ &- \left[\frac{4\,M-r}{r\,(2\,M-r)} + \frac{(r-M)\,(5\,M-2\,r)}{r\,(2\,M-r)^2} + \frac{\sigma^2 r^3}{(2\,M-r)^2} + \frac{n-1}{2\,M-r}\right] L(r) \\ &- \left[-\frac{l\,(l+1)}{2\,(2\,M-r)} + \frac{r-3\,M}{r\,(2\,M-r)}\right] N(r)\end{aligned} \tag{5.19}$$

This gave us a third-order system, that can be reduced to a second-order equation. A solution, that allows for an expression similar to the one-dimensional Schrödinger equation, is, according to Chandrasekhar [3]

$$Z^{(+)} = r\,V(r) - \frac{r^2}{n\,r+3\,M}\,(L(r) + n\,V(r)) \tag{5.20}$$

It will now be determined if this is really the case. It can be noted that the expression for $Z^{(+)}$ has an empirical origin.

By taking the derivative of (5.20) with respect to r_*, while remembering that $\frac{\mathrm{d}}{\mathrm{d}r_*} = \left(1 - \frac{2\,M}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}$, we ended up having

$$\frac{\mathrm{d}}{\mathrm{d}r_*}Z^{(+)} = (r-2\,M)\frac{\mathrm{d}}{\mathrm{d}r}V(r) + \frac{3\,M\,(r-2\,M)}{r\,(n\,r+3\,M)}\,V(r) + \frac{n\,r^2 - 3\,M\,n\,r - 3\,M^2}{(n\,r+3\,M)^2}\,(L(r) + n\,V(r)) \tag{5.21}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}Z^{(+)} = &-\frac{(2\,M-r)^2}{r\,(n\,r+3\,M)}\left(\frac{\mathrm{d}}{\mathrm{d}r}n\,V(r)\right) + \frac{2\,M-r}{r^2}\,N(r) \\ &+ \left[\frac{-n^2\,(n+2)\,r^4 + 2\,n^2\,M\,\left(n+\frac{5}{2}\right) r^3 - 3\,M^2\,n\,(n+1)\,r^2 - 9\,M^3\,r + M^4}{r^2\,(n\,r+3\,M)^3}\right] L(r) \\ &+ \left[-r\,\sigma^2 + \frac{-n^3\,r^2 + 9\,M\,n^3\,r - 15\,M^2\,n\,\left(n-\frac{3}{5}\right)\,(n-1)}{(n\,r+3\,M)^3}\right] V(r) \\ &-27\,M^3\left[\frac{2\,\left(n-\frac{3}{2}\right)\,n\,r^2 + 5\,M\left(n-\frac{2}{5}\right)\,r - 4\,M^2}{r^3\,(n\,r+3\,M)^3}\right] V(r)\end{aligned} \tag{5.22}$$

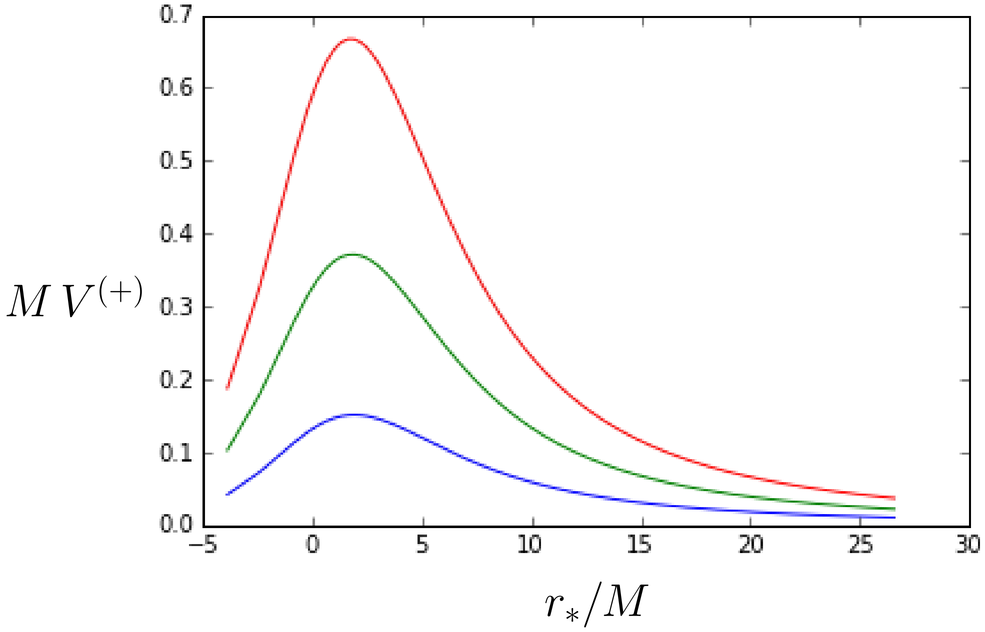

Figure 3: The plot shows the potential barrier, $V^{(+)}$, for the polar perturbations. The lowest curve has $\ell = 2$, while the other two have $\ell = 3$ and $\ell = 4$.

Chandrasekhar has proved that, with the $Z^{(+)}$ previously written, the following would hold:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r_*^2} + V^{(+)}\right)\, Z^{(+)} = \sigma^2\, Z^{(+)} \tag{5.23}$$

where

$$V^{(+)} = \frac{2\,(r-2\,M)}{r^4\,(n\,r+3\,M)^2}[n^2\,(n+1)\,r^3 + 3\,M\,n^2\,r^2 + 9\,M^2\,n\,r + 9\,M^3] \tag{5.24}$$

We could verify, using Maple, that this was the case. Again, the potential is vanishing at the event horizon and at infinity. Figure 3 shows a plot of this as a function of r_*/M for $\ell = 2$, $\ell = 3$ and $\ell = 4$. Compared to Figure 2, this plot looks very similar and one can easily show that $V^{(+)} \to V^{(-)}$ as $n \to \infty$.

Looking back at $\delta R_{02} = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}r}L(r) + \frac{5\,M - 2\,r}{r\,(2\,M-r)}\,L(r) = -\left(\frac{\mathrm{d}}{\mathrm{d}r}n\,V(r) + \frac{3\,M - r}{r\,(2\,M-r)}\,n\,V(r)\right) \tag{5.25}$$

If $\frac{\mathrm{d}}{\mathrm{d}r}f(r) = \frac{5\,M-2\,r}{r\,(2\,M-r)}$, then $e^{f(r)} = \frac{r^2}{\sqrt{1-\frac{2\,M}{r}}}$. Similarly, if $\frac{\mathrm{d}}{\mathrm{d}r}g(r) = \frac{3\,M-r}{r\,(2\,M-r)}$, then $e^{g(r)} = \frac{r}{\sqrt{1-\frac{2\,M}{r}}} = \frac{1}{r}\,e^{f(r)}$. Thus, by multiplying both sides by $e^{f(r)}$, we end up having

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\sqrt{1-\frac{2\,M}{r}}}\,L(r)\right) = -r\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r}{\sqrt{1-\frac{2\,M}{r}}}\,n\,V(r)\right) \tag{5.26}$$

By rewriting (5.20), one gets

$$V(r) = \frac{r\,(n\,r + 3\,M)}{3\,M\,r^2}\,Z^{(+)} + \frac{r}{3\,M}\,L(r) \tag{5.27}$$

and by inserting this in equation (5.26), one gets

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{r^2}{\sqrt{1-\frac{2\,M}{r}}}\,L(r) = \frac{-n\,r}{3\,M + n\,r}\,\frac{\mathrm{d}}{\mathrm{d}r}\frac{n\,r + 3\,M}{\sqrt{1-\frac{2\,M}{r}}}\,Z^{(+)} \tag{5.28}$$

After integrating and using integration by parts, one ends up having

$$L(r) = \frac{3\,M}{r^2}\,\Phi - \frac{n}{r}\,Z^{(+)}, \tag{5.29}$$

where

$$\Phi = n\sqrt{1-\frac{2\,M}{r}}\int\frac{Z^{(+)}}{\sqrt{1-\frac{2\,M}{r}}\,(3\,M + n\,r)}\,\mathrm{d}r. \tag{5.30}$$

Looking back at (5.27), we can now calculate $V(r)$ as:

$$V(r) = \frac{1}{r}\,\left(Z^{(+)} + \Phi\right). \tag{5.31}$$

Finally, we want to calculate $N(r)$. Before, we calculated the differentiation of $V(r)$, $N(r)$ and $L(r)$ with respect to r. In (5.21), there is a $\frac{\mathrm{d}}{\mathrm{d}r}V(r)$ present, suggesting we can replace it by what we got in (5.17). Then we can use our obtained values of $L(r)$ and $V(r)$ in (5.29) and (5.31). This gave the following:

$$\begin{aligned} N(r) =& \frac{(r^4\,\sigma^2 + 3\,M^2 - M\,r)\,\Phi}{r^2\,(2\,M - r)} - Z^{(+)}\,\frac{n}{r\,(n\,r + 3\,M)^2}\,\left(n^2\,r^2 + r\,(3\,M + r)\,n + 6\,M^2\right) \\ &- \frac{n\,r}{n\,r + 3\,M}\left(\frac{\mathrm{d}}{\mathrm{d}r_*}Z^{(+)}\right) \end{aligned} \tag{5.32}$$

5.4 The relation between the axial and polar perturbations

Previously, we found two expressions for $V^{(+)}$ and $V^{(-)}$. Now, we want to create a general potential, $V^{(\pm)}$. We find that:

$$\begin{cases} V^{(+)} - V^{(-)} = \dfrac{6\,M\,(2\,M - r)\,(12\,M^2 + 3\,(2\,n - 1)\,r\,M - 2\,n\,r^2)}{r^4(n\,r + 3\,M)^2} = 2\,\alpha \\ \\ V^{(+)} + V^{(-)} = \dfrac{6\,M\,(2\,M - r)\,(6\,M^3 - 3\,M^2 r - 2\,n^2(n+1)\,r^3)}{r^4(n\,r + 3\,M)^2} = 2\,\gamma \end{cases} \tag{5.33}$$

so $V^{(\pm)} = \gamma \pm \alpha$. With the help of Maple, we could verify that $\alpha = \beta \frac{\mathrm{d}}{\mathrm{d}r_*} f$ and $\gamma = \beta^2 f^2 + \kappa\, f$, where $\beta = 6\,M$, $\kappa = 4\,n\,(n+1)$ and $f = \frac{r-2\,M}{2\,r^2 (n\,r+3\,M)}$. Thus, when adding this into the wave equations, one gets:

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}r_*^2} Z^{(+)} + \sigma^2 Z^{(+)} = V^{(+)}\, Z^{(+)} = \left(\beta \frac{\mathrm{d}}{\mathrm{d}r_*} f + \beta^2\, f^2 + \kappa\, f\right) Z^{(+)} \\ \frac{\mathrm{d}^2}{\mathrm{d}r_*^2} Z^{(-)} + \sigma^2 Z^{(-)} = V^{(-)}\, Z^{(-)} = \left(-\beta \frac{\mathrm{d}}{\mathrm{d}r_*} f + \beta^2\, f^2 + \kappa\, f\right) Z^{(-)} \end{cases} \tag{5.34}$$

Now, we want a solution $Z^{(+)} = p\, Z^{(-)} + q\, Z^{(-)}_{,r_*}$, where p and q are two unknown functions. By differentiating this twice with respect to r_*, while using $Z^{(-)}_{,r_*,r_*} = (V^{(-)} - \sigma^2) Z^{(-)}$ and $Z^{(+)}_{,r_*,r_*} = (V^{(+)} - \sigma^2) Z^{(+)}$, we got the following:

$$\begin{aligned} Z^{(+)}_{,r_*,r_*} = &\left[p_{,r_*,r_*} + (2\,q_{,r_*} + p)(V^{(-)} - \sigma^2) + q\, V^{(-)}\right]\, Z^{(-)} \\ &+ \left[2\, p_{r_*} + q\,(V^{(-)} - \sigma^2) + q_{,r_*,r_*}\right] Z^{(-)}_{,r_*} \\ =&p\,(V^{(+)} - \sigma^2) Z^{(-)} + q\,(V^{(+)} - \sigma^2) Z^{(-)}_{,r_*} \end{aligned} \tag{5.35}$$

Thus, $p_{,r_*,r_*} + (2\,q_{,r_*} + p)(V^{(-)} - \sigma^2) + q\, V^{(-)} = p\,(V^{(+)} - \sigma^2)$ and $2\, p_{r_*} + q\,(V^{(-)} - \sigma^2) + q_{,r_*,r_*} = q\,(V^{(+)} - \sigma^2)$, which gave

$$\frac{\mathrm{d}}{\mathrm{d}r_*} \left[p_{,r_*}\, q - p\, q_{,r_*} + q^2 (V^{(-)} - \sigma^2) - p^2\right] = 0, \tag{5.36}$$

which implies that the integral of this with respect to r_* is equal to a constant. By inserting $V^{(-)} = -\beta \frac{\mathrm{d}}{\mathrm{d}r_*} f + \beta^2\, f^2 + \kappa\, f$, one can see that we should have a solution of p of the form $p = a + b\, f$, where a and b are constants. Inserting this after integrating gave

$$(b\, q - q^2\, \beta)\, f_{,r_*} + (q^2\, \beta^2 - b^2)\, f^2 + (q^2 \kappa - 2\, a\, b)\, f - q^2\, \sigma^2 - (a + b\, f)\, q_{,r_*} - a^2 = \text{Constant} \tag{5.37}$$

For this to be a constant, the function f must be vanishing. This clearly implies that q is a constant and $q_{,r_*} = 0$. The resulting a and b turned out to be $a = \frac{q\,\kappa}{2\,\beta}$ and $b = q\, \beta$, which in turn gave $p = q\, \left(\frac{\kappa}{2\,\beta} + \beta\, f\right)$ and $q = C_1$, where C_1 is a constant. Hence,

$$Z^{(+)} = C_1 \left[\left(\frac{\kappa}{2\,\beta} + \beta\, f\right) Z^{(-)} + Z^{(-)}_{,r_*}\right] \tag{5.38}$$

Similarly, by using the same method for $Z^{(-)} = p_2\, Z^{(+)} + q_2\, Z^{(+)}_{,r_*}$, we found that

$$Z^{(-)} = C_2 \left[-\left(\frac{\kappa}{2\,\beta} + \beta\, f\right) Z^{(+)} + Z^{(+)}_{,r_*}\right] \tag{5.39}$$

What remains now is to find the constants C_1 and C_2. By using our obtained $Z^{(-)}$ in our obtained expression of $Z^{(+)}$, we find that

$$Z^{(+)} = -C_1\, C_2 \left(\sigma^2 + \frac{\kappa^2}{4\,\beta^2}\right) Z^{(+)} \Rightarrow \frac{1}{C_1\, C_2} = -\left(\frac{\kappa}{2\,\beta} + i\,\sigma\right)\left(\frac{\kappa}{2\,\beta} - i\sigma\right) \tag{5.40}$$

By letting $\frac{1}{C_1} = \frac{\kappa}{2\,\beta} + i\,\sigma$ and $\frac{1}{C_2} = -\left(\frac{\kappa}{2\,\beta} - i\,\sigma\right)$ we ended up having

$$\begin{cases}(\kappa + 2\beta i\sigma)Z^{(+)} = (\kappa + 2\beta^2 f)Z^{(-)} + 2\,\beta Z^{(-)}_{,r_*} \\ (\kappa - 2\beta i\sigma)Z^{(-)} = (\kappa + 2\beta^2 f)Z^{(+)} - 2\,\beta Z^{(+)}_{,r_*}\end{cases} \tag{5.41}$$

These are the same relations that Chandrasekhar ended up with.

5.5 Reflection and transmission

Now that we have established some expressions for the potentials, let's take a look at how the potentials behave when letting $r_* \to \pm\infty$. One can simply verify that

$$V^{(\pm)} \to \frac{2\,(n+1)}{r^2} \text{ as } r_* \to +\infty \tag{5.42}$$

For $r_* \to -\infty$, one needs to keep in mind that $r_* \to -\infty$ as $r \to 2\,M$. By using the logarithm of $V^{(+)}$ and $V^{(-)}$ and using the fact that $r_* = r + 2\,M\,\ln\left(\frac{2\,M}{r} - 1\right)$, one can see that the potentials will approach 0 as $e^{\pm r_*/2\,M}$. Now, one can compare those limits to the p-series, which is generally known to be convergent for $p > 1$. In this case we choose to look at when $p = 2$. One gets

$$\lim_{s\to\infty} \sum_{r_*=k}^{s} \frac{2\,(n+1)}{r_*^2} \geq \lim_{s\to\infty} \sum_{r_*=k}^{s} \frac{2\,(n+1)}{r^2} \geq \lim_{s\to\infty} \int_{k+1}^{s} \frac{2\,(n+1)}{r^2}\,\mathrm{d}r_* \tag{5.43}$$

and

$$\lim_{s\to\infty} \sum_{r_*=k}^{s} \frac{1}{r_*^2} \geq \lim_{s\to\infty} \sum_{r_*=k}^{s} e^{-r_*/2\,M} \geq \lim_{s\to\infty} \int_{k+1}^{s} e^{-r_*/2\,M}\,\mathrm{d}r_* \tag{5.44}$$

so one can conclude that $V^{(\pm)}$ has a finite integral. In the wave equations in (5.34), $f \to 0$ as $r_* \to \pm\infty$. For vanishing $V^{(+)}$, $V^{(-)}$ and f, one ends up having $\frac{\mathrm{d}^2}{\mathrm{d}r_*^2} Z^{(+)} = -\sigma^2 Z^{(+)}$ and $\frac{\mathrm{d}^2}{\mathrm{d}r_*^2} Z^{(-)} = -\sigma^2 Z^{(-)}$, meaning that $Z^{(\pm)} \to e^{\pm i\,\sigma\,r_*}$ as $r_* \to \pm\infty$. Thus $Z^{(\pm)}$ approaches a wave equation, with $e^{+i\,\sigma\,r_*}$ being waves moving to the left while $e^{-i\,\sigma\,r_*}$ is waves moving to the right. For waves moving from $+\infty$, once they meet the potential $V^{(\pm)}$, a part will be reflected and the rest will be transmitted. Thus, for $r_* \to +\infty$, $Z^{(\pm)}$ will approach the incident as well as the reflected waves and for $r_* \to -\infty$, $Z^{(\pm)}$ will approach the transmitted waves. In more mathematical terms,

$$\begin{aligned} Z^{(\pm)} &\to e^{+i\,\sigma\,r_*} + R^{(\pm)}(\sigma)\, e^{-i\,\sigma\,r_*} \text{ for } r_* \to +\infty \\ &\to T^{(\pm)}(\sigma)\, e^{+i\,\sigma\,r_*} \text{ for } r_* \to -\infty \end{aligned} \tag{5.45}$$

where $R^{(\pm)}(\sigma)$ and $T^{(\pm)}(\sigma)$ denote the amplitudes of the reflected and transmitted wave, respectively. The amplitude of the incident wave is assumed to be 1.

The relation between $R^{(\pm)}(\sigma)$ and $T^{(\pm)}(\sigma)$ was obtained by using the Wronskian in the book by Chandrasekhar [3]. This relation can, however, be obtained in a much more simple manner. Since the reflected and transmitted waves are assumed to be fractions of the incident wave, the probability density of the reflected waves plus the probability density of the transmitted waves should correspond to the probability density of the incident wave. As has been observed before, the functions $Z^{(\pm)}$ are behaving like Schrödinger-like wave equations and thus $\left|Z^{(\pm)}\right|^2$ gives a probability distribution, giving us the relation

$$|R^{(\pm)}(\sigma)|^2 + |T^{(\pm)}(\sigma)|^2 = 1 \tag{5.46}$$

Now, let's take a look at what was found in (5.34) and (5.41). As stated before,

$$\frac{\mathrm{d}^2}{\mathrm{d}r_*^2} Z^{(\pm)} = -\sigma^2 \, Z^{(\pm)} \text{ if } V^{(\pm)} = f = 0 \tag{5.47}$$

and we also obtain the relations

$$\begin{cases} (\kappa + 2\beta i\sigma) Z^{(+)} = \kappa \, Z^{(-)} + 2\,\beta Z^{(-)}_{,r_*} \text{ if } f = 0 \\ (\kappa - 2\beta i\sigma) Z^{(-)} = \kappa \, Z^{(+)} - 2\,\beta Z^{(+)}_{,r_*} \text{ if } f = 0 \end{cases} \tag{5.48}$$

Now, if $Z^{(+)} = e^{+i\,\sigma\,r_*}$, then $Z^{(-)} = \frac{\kappa - 2\,i\,\sigma\,\beta}{\kappa - 2\,i\,\sigma\,\beta} \, e^{i\,\sigma\,r_*} = e^{i\,\sigma\,r_*}$ and if $Z^{(+)} = e^{-i\,\sigma\,r_*}$, then $Z^{(-)} = \frac{\kappa + 2\,i\,\sigma\,\beta}{\kappa - 2\,i\,\sigma\,\beta} \, e^{-i\,\sigma\,r_*}$. From (5.45), $r_* \to -\infty$ implies $Z^{(+)} \to T^{(+)}(\sigma) \, e^{+i\,\sigma\,r_*}$ and $Z^{(-)} \to T^{(-)}(\sigma) \, e^{+i\,\sigma\,r_*}$. Hence

$$T^{(+)}(\sigma) = T^{(-)}(\sigma) \tag{5.49}$$

Since the $e^{-i\,\sigma\,r_*}$ is a wave moving to the right, the reflected waves have the relation

$$R^{(-)}(\sigma) = \frac{\kappa + 2\,i\,\sigma\,\beta}{\kappa - 2\,i\,\sigma\,\beta} \, R^{(+)}(\sigma) \tag{5.50}$$

which implies $|R^{(+)}(\sigma)|^2 = |R^{(-)}(\sigma)|^2$. What this says is thus that these amplitudes only differ with a phase difference.

5.6 The stability of the Schwarzschild metric

It has now been established that the axial and polar perturbations satisfy a wave equation similar to that of the one-dimensional Schrödinger equation. For Z being the wave function and V being the potential, these equations had the form

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r_*^2} + V \right) Z = \sigma^2 \, Z \tag{5.51}$$

This looks like the familiar one-dimensional, time-independent Schrödinger equation, with the Hamiltonian to the left and the energy, σ^2, to the right. This expression can now be multiplied by Z and after an integration by parts, one gets

$$\int \left(\left(\frac{\partial Z}{\partial r_*} \right)^2 + V \, Z^2 \right) \mathrm{d}r_* = \sigma^2 \int Z^2 \, \mathrm{d}r_*$$

With the previous comments about $V^{(+)}$ and $V^{(-)}$ being positive, this makes the expression inside the integral on the left hand side positive, which further implies that $\sigma^2 > 0$, meaning $\sigma \in \mathbb{R}$. For the earlier mentioned time dependence $e^{i\sigma t}$, this implies that there is no unbounded growth in t.

6 Conclusions

The metric obtained section 4 did align with the previously obtained Kerr metric, which has also been defined in equation (3.13). It may therefore be concluded that the ansatz together with the Weyl method is a suitable method for more labor-saving way of deriving the Kerr metric.

In section 5, we found that an axial as well as a polar perturbation around the Schwarzschild metric satisfied a wave equation similar to the Schrödinger equation. From this, we found that the corresponding energy had no exponential growth as $t \to \pm\infty$, which further shows that it is bounded. Hence, the Schwarzschild metric is stable.

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