# 1 Introduction

The Hardy-Littlewood circle method was originally developed by G.H Hardy and J.E Littlewood based on an initial idea by S. Ramanujan. It was later improved by I.M Vinogradov with the introduction of Weyl Sums. This exposition is based on [3] and [2].

Waring's problem asks if for a given positive integer $k$, there exists a positive integer $s$ such that (1) has a solution, in which all $x_i$ are non negative integers, for all positive integers $N$.

$$N = \sum_{i=1}^{s} x_i^k. \tag{1}$$

Where $g(k)$ denotes the smallest integer $s$ for which (1) is solvable for every possible $N$, we will prove its existence.

However, the number $g(k)$ is restrained by a few integers. For instance, $N = 2^k[(\frac{3}{2})^k]-1$, clearly we have $N < 3^k$, so the most economical way of representing this number is $N = ([(\frac{3}{2})^k]-1)2^k + (2^k-1)1^k$.

So, it is also interesting to consider the smallest $s$, such that for every sufficiently large integer $N$, (1) is solvable, this is denoted $G(k)$.

The Hardy-Littlewood method will yield for some $s$ large enough an asymptotic formula of the number of representations of $N$ as a sum of $k$th powers, denoted $R(N)$.

$$R(N) = \frac{\Gamma(1+\frac{1}{k})^s}{\Gamma(\frac{s}{k})} N^{\frac{s-k}{k}} \mathfrak{S}(N) + \mathcal{O}(N^{\frac{s-k}{k}-\delta'}), \tag{2}$$

where $\Gamma$ denotes the gamma function and $\mathfrak{S}$ denotes the singular series,

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s} (\sum_{c=1}^{q} e(\frac{a}{q}c^k))^s e(-\frac{a}{q}N).$$

Since $\frac{\Gamma(1+\frac{1}{k})^s}{\Gamma(\frac{s}{k})} > 0$, if we bound $\mathfrak{S}(N)$ from below by a bound not depending on $N$, we get that $R(N) \to \infty$ as $N \to \infty$.

The number $\tilde{G}(k)$ denotes the smallest $s$ for which (2) holds.
Before handling the main proof, we will prove some important results, which will be used later.

# 2 Weyl Sums, Weyl's inequality and Hua's lemma

We will first go over some notation. Vinogradov's notation, $f(x) \ll g(x)$ is equivalent to saying $f(x) \in \mathcal{O}(g(x))$. We also define $e(y) \coloneqq e^{i2\pi y}$.

**Definition 2.1.** *Let $P$ be a polynomial with real coefficients, a Weyl sum is a sum of the type:*

$$S = \sum_{n=1}^{N} e(P(n)).$$

We let $S$ be a Weyl sum and define $\Delta_h P(n) \coloneqq P(m+h) - P(m)$, then:

$$\begin{aligned}
|S|^2 &= \Bigg(\sum_{n=1}^{N} e(P(n))\Bigg)\Bigg(\sum_{k=1}^{N} \overline{e(P(k))}\Bigg) = \Bigg(\sum_{n=1}^{N} e(P(n))\Bigg)\Bigg(\sum_{k=1}^{N}(e(-P(k))\Bigg) \\
&= \sum_{n=1}^{N}\sum_{k=1}^{N} e(P(n)-P(k)) = N + \sum_{\substack{h=-(N-1)\\ h\neq 0}}^{N-1} \sum_{n=1}^{N-h} e(\Delta_h P(n)) \\
&= N + \sum_{h=1}^{N-1}\sum_{n=1}^{N-h} e(\Delta_h P(n)) + \overline{e(\Delta_h P(n))} \\
&= N + 2\Re(\sum_{h=1}^{N-1}\sum_{n=1}^{N-h} e(\Delta_h P(n))) \ll N + \sum_{h=1}^{N-1} |\sum_{n=1}^{N-h} e(\Delta_h P(n))|.
\end{aligned}$$

Note that if $P(n)$ is a polynomial of degree $k$ with leading coefficient $\alpha$, then $\Delta_h P(n)$ is a polynomial of degree $k-1$ with leading coefficient $\alpha$.
This process is called Weyl differencing, and will be used in the process of proving the following proposition.

**Proposition 2.2** (Weyl's Inequality)**.** *Let $P$ be a polynomial with real coefficients of degree of $k$ with leading coefficient $\alpha$, and there exist integers $a$, $q$ such that: $|\alpha - \frac{a}{q}| < \frac{c}{q^2}$, $q \geq 1$, $(a,q) = 1$ for some $c > 0$, then:*

$$|\sum_{n=1}^{N} e(P(n))| = \mathcal{O}_{c,\epsilon,k}(N^{1+\epsilon}(N^{-1} + q^{-1} + qN^{-k})^{\frac{1}{K}})$$

*holds $\forall \epsilon > 0$, where $K = 2^{k-1}$.*

*Proof.* The beginning of the proof consists of applying Weyl differencing $k-1$ times. From here on, we let $S_j(f)$ denote a Weyl sum of a polynomial $f$ with degree $j$. Then:

$$\begin{aligned}
|S_k(P)|^4 &\ll |N + \sum_{h=1}^{N-1} |S_{k-1}(\Delta_h P)||^2 \\
&\ll N^2 + N \sum_{h_1=1}^{N-1}\sum_{h_2=1}^{N-1} (N + |S_{k-2}(\Delta_{h_2}\Delta_{h_1} P)|) \\
&\ll N^3 + N \sum_{h_1=1}^{N-1}\sum_{h_2=1}^{N-1} |S_{k-2}(\Delta_{h_2}\Delta_{h_1} P)|.
\end{aligned}$$

Where the first inequality holds by Weyl differencing and the second follows from $|x+y|^2 \leq 2\max\{|x|,|y|\}^2 \leq 2|x|^2+2|y|^2$ along with the Cauchy-Schwarz inequality applied to the the Weyl sums and the identity function. Following this line of thinking we will prove the following formula by induction:

$$|S_k|^{2^l} \ll N^{2^l-1} + N^{2^l-l-1}\sum_{h_1=1}^{N-1}...\sum_{h_l=1}^{N-1}|S_{k-l}(\Delta_{h_l}...\Delta_{h_1}P)|. \tag{3}$$

The computation before this one shows the base case holds. Next we will show the induction step, assuming the statement holds for $1,2,..,l-1$.

$$|S_k|^{2^l} \ll |N^{2^{l-1}-1} + N^{2^{l-1}-(l-1)-1}\sum_{h_1=1}^{N-1}...\sum_{h_{l-1}=1}^{N-1}|S_{k-(l-1)}(\Delta_{h_l}...\Delta_{h_1}P)|^2$$
$$\ll N^{2^l-2} + N^{2^l-2(l-1)-2}N^{l-1}\sum_{h_1=1}^{N-1}...\sum_{h_{l-1}=1}^{N-1}\left(N+\sum_{h_l=1}^{N-1}|S_{k-l}(\Delta_{h_l}...\Delta_{h_1}P)|\right)$$
$$= N^{2^l-2} + N^{2^l-l-1}\left(N^l+\sum_{h_1=1}^{N-1}...\sum_{h_l=1}^{N-1}|S_{k-l}(\Delta_{h_l}...\Delta_{h_1}P)|\right)$$
$$\ll N^{2^l-1} + N^{2^l-l-1}\sum_{h_1=1}^{N-1}...\sum_{h_l=1}^{N-1}|S_{k-l}(\Delta_{h_l}...\Delta_{h_1}P)|.$$

Where the first inequality follows from induction assumption, the second uses a similar strategy as in the base case and the rest is just collecting terms. This shows (3) holds. Next, we will find a bound for a Weyl sum of a polynomial of degree one.

$$\begin{aligned}|\sum_{n=1}^{N}e(\alpha n+\beta)| &= |\sum_{n=1}^{N}e(\alpha n)| = |\frac{e(\alpha)-e(\alpha(N+1))}{1-e(\alpha)}| \\ &\leq \frac{2}{|1-e(\alpha)|} = \frac{2}{|e(\frac{\alpha}{2})-e(-\frac{\alpha}{2})|} = \frac{1}{|\sin(\pi\alpha)|} \leq \frac{1}{2||\alpha||}.\end{aligned} \tag{4}$$

Where $||\alpha||$ denotes the distance from $\alpha$ to the closest integer. Note that the sum also has the trivial bound $N$, since each each term is bounded by 1. Next we apply (3) to $S_k$ $k-1$ times, obtaining:

$$|S_k|^K \ll N^{K-1} + N^{K-k}\sum_{h_1=1}^{N-1}...\sum_{h_k=1}^{N-1}|S_1(\Delta_{h_k}...\Delta_{h_1}P)|). \tag{5}$$

Note that, $\Delta_{h_k}...\Delta_{h_1}P = \alpha h_k\cdot...\cdot h_1 k!n+\beta$ for some $\beta$ implies:

$$\sum_{h_1=1}^{N-1}...\sum_{h_k=1}^{N-1}|S_1(\Delta_{h_k}...\Delta_{h_1}P)| \leq \sum_{h_1=1}^{N-1}...\sum_{h_k=1}^{N-1}\min\{N,\frac{1}{||\alpha h_k\cdot...\cdot h_1 k!||}\}. \tag{6}$$

The expression above can be rewritten in the following way:

$$\begin{aligned} &\sum_{H=1}^{k!(N-1)^k} \min\{N, \frac{1}{||\alpha H||}\} \sum_{\substack{H=k!h_1\cdot\ldots\cdot h_k \\ h_i\in\mathbb{N}, h_i\le (N-1)\forall i}} 1 \\ &= \sum_{||H\alpha||\le\frac{1}{N}} N \sum_{\substack{H=k!h_1\cdot\ldots\cdot h_k \\ h_i\in\mathbb{N}, h_i\le (N-1)\forall i}} 1 + \sum_{||H\alpha||\ge\frac{1}{N}} \frac{1}{||\alpha H||} \sum_{\substack{H=k!h_1\cdot\ldots\cdot h_k \\ h_i\in\mathbb{N}, h_i\le (N-1)\forall i}} 1 \\ &= \sum_{||H\alpha||\le\frac{1}{N}} N \sum_{\substack{H=k!h_1\cdot\ldots\cdot h_k \\ h_i\in\mathbb{N}, h_i\le (N-1)\forall i}} 1 + \sum_{j=1}^{[\frac{N-1}{2}]} \sum_{\frac{j}{N}<||H\alpha||\le\frac{j+1}{N}} \frac{1}{||\alpha H||} \sum_{\substack{H=k!h_1\cdot\ldots\cdot h_k \\ h_i\in\mathbb{N}, h_i\le (N-1)\forall i}} 1 \\ &\le \sum_{||H\alpha||\le\frac{1}{N}} N \sum_{\substack{H=k!h_1\cdot\ldots\cdot h_k \\ h_i\in\mathbb{N}, h_i\le (N-1)\forall i}} 1 + \sum_{j=1}^{[\frac{N-1}{2}]} \sum_{\frac{j}{N}<||H\alpha||\le\frac{j+1}{N}} \frac{N}{j} \sum_{\substack{H=k!h_1\cdot\ldots\cdot h_k \\ h_i\in\mathbb{N}, h_i\le (N-1)\forall i}} 1. \end{aligned} \tag{7}$$

Our next goal is to find a bound on the number of terms inside each box of the type $\frac{j}{N} < ||H\alpha|| \le \frac{j+1}{N}$, we know by assumption, $\alpha = \frac{a}{q} + \frac{d}{q^2}$, where d is a real number, such that $|d| < c$, note also $H = qs + r$, is the standard Euclidean division of $H$.

$$\begin{aligned} ||\alpha H|| &= ||(\frac{a}{q} + \frac{d}{q^2})(qs + r)|| \\ &= ||as + \frac{ar}{q} + \frac{ds}{q} + \frac{dr}{q^2}|| = ||\frac{ar + ds}{q} + \mathcal{O}(\frac{c}{q})||. \end{aligned}$$

We note that $a$ and $d$ are fixed, the maximum amount of values $s$ can take is $1 + \frac{k!(N-1)^{k-1}}{q}$, and r can take $q$ different values.
Realize also that since $(a, q) = 1$, it means $ar$ forms a complete set of residues modulo $q$, which implies that for a fixed $s$, the $q$ terms obtained from changing $r$ are equidistant. This means the upper bound on the number of terms satisfying $\frac{j}{N} < ||\alpha H|| \le \frac{j+1}{N}$ is:

$$\begin{aligned} &\Big(2q\Big(1 + (\frac{j+1}{N} + \frac{c}{q}) - (\frac{j}{N} - \frac{c}{q})\Big)\Big)(1 + \frac{k!(N-1)^k}{q}) \\ &= (2q + \frac{2q}{N} + 2c)(1 + \frac{k!(N-1)^k}{q}) \ll (\frac{N^{k-1}}{q} + \frac{q}{N} + N^{k-2}). \end{aligned} \tag{8}$$

Next we will bound the number of terms satisfying $H = k!h_1 \cdot \ldots \cdot h_k$, clearly the number of possibilities for each is bounded by the number of divisors $(d(H))$.

$$
\begin{aligned}
\frac{d(H)}{H^\epsilon} &= \frac{d(p_1^{\lambda_1}\cdot \ldots \cdot p_m^{\lambda_m})}{(p_1^{\lambda_1}\cdot \ldots \cdot p_m^{\lambda_m})^\epsilon} = \prod_{j=1}^{m} \frac{(\lambda_j+1)}{p_j^{\lambda_j \epsilon}} \\
&\leq \Big( \prod_{p_j \leq 2^{1/\epsilon}} \frac{(\lambda_j+1)}{2^{\lambda_j\epsilon}} \Big) \Big( \prod_{p_j \geq 2^{1/\epsilon}} \frac{(\lambda_j+1)}{2^{\lambda_j}} \Big) \\
&\leq \prod_{p_j \leq 2^{1/\epsilon}} \frac{(\lambda_j+1)}{2^{\lambda_j\epsilon}}.
\end{aligned} \tag{9}
$$

Each of the terms in the product above is bounded from above, since there exist a finite number of terms in the product, it follows that $d(H) \ll H^\epsilon$. Applying this and (8) to (7) yields:

$$
\begin{aligned}
&\sum_{||H\alpha|| \leq \frac{1}{N}} N \sum_{\substack{H=k!h_1\cdot\ldots\cdot h_k \\ h_i\in\mathbb{N}, h_i \leq (N-1)\forall i}} 1 + \sum_{j=1}^{[\frac{N-1}{2}]} \sum_{\frac{j}{N} < ||H\alpha|| \leq \frac{j+1}{N}} \frac{N}{j} \sum_{\substack{H=k!h_1\cdot\ldots\cdot h_k \\ h_i\in\mathbb{N}, h_i \leq (N-1)\forall i}} 1 \\
&\ll N(\frac{N^{k-1}}{q} + \frac{q}{N} + N^{k-2})(N)^\epsilon (1 + \sum_{j=1}^{[\frac{N-1}{2}]} \frac{1}{j}) \\
&\ll N(\frac{N^{k-1}}{q} + \frac{q}{N} + N^{k-2})(N)^\epsilon (1 + \log(N)) \ll (\frac{N^k}{q} + q + N^{k-1})(N^\epsilon).
\end{aligned} \tag{10}
$$

Now we can plug (10) in (5):

$$
\begin{aligned}
|S_k|^k &\ll N^{k-1} + N^{K-k}(\frac{N^k}{q} + q + N^{k-1})(N^\epsilon) \\
&\ll (\frac{N^K}{q} + N^{K-1} + qN^{K-k})N^\epsilon.
\end{aligned} \tag{11}
$$

(11) implies:

$$
|S_k| \ll N^{1+\epsilon}(N^{-1} + q^{-1} + qN^{-k})^{\frac{1}{K}}.
$$

□

Next we will state and proof Hua's Inequality:

**Proposition 2.3** (Hua's Lemma)**.** *The following inequality holds* $\forall \epsilon > 0$*, and* $\forall j \in (\mathbb{Z} \cap [1,k])$*:*

$$
\int_0^1 |\sum_{x=1}^{N} e(x^k y)|^{2^j} dy \ll N^{2^j - j + \epsilon}.
$$

*Proof.* The proof is done by induction, we start by showing the base case, $j = 1$,

$$
\begin{aligned}
\int_0^1 |\sum_{x=1}^{N} e(x^k y)|^2 dy &= \int_0^1 (\sum_{x=1}^{N} e(x^k y))(\sum_{z=1}^{N} e(-x^k y))dy \\
&= \sum_{x=1}^{N}\sum_{z=1}^{N} \int_0^1 e(y(x^k - z^k))dy = N.
\end{aligned} \tag{12}
$$

Now, we move on to the induction step, we assume it holds for $1, 2, ..., j$, and we apply (3) to $|\sum_{x=1}^{N} e(x^k y)|^{2^{j-1}}$

$$|\sum_{x=1}^{N} e(x^k y)|^{2^{j-1}} \ll N^{2^{j-1}-1} +$$
$$N^{2^{j-1}-(j-1)-1} \sum_{h_1=1}^{N-1} ... \sum_{h_{j-1}=1}^{N-1} |S_{k-(j-1)}(\Delta_{h_{j-1}} ... \Delta_{h_1}(x^k y))|.$$

Next, we square both sides, and as in the proof of the previous theorem use $|x+y|^2 \leq 2|x|^2 + 2|y|^2$, the Cauchy-Schwarz inequality and Weyl differencing, obtaining:

$$|\sum_{x=1}^{N} e(x^k y)|^{2^j} \ll N^{2^j-2} + N^{2^j-(2j-2)-2} N^{j-1} \cdot$$
$$\cdot \sum_{h_1=1}^{N-1} ... \sum_{h_{j-1}=1}^{N-1} (N + \sum_{h_j=1}^{N-1} \Re(S_{k-j}(\Delta_{h_j} ... \Delta_{h_1}(x^k y)))$$
$$\ll N^{2^j-2} + N^{2^j-j-1}(N^j + \sum_{h_1=1}^{N-1} ... \sum_{h_{j-1}=1}^{N-1} \Re(\Delta_{h_j} ... \Delta_{h_1}(x^k y)))$$
$$\ll N^{2^j-1} + N^{2^j-j-1} \sum_{h_1=1}^{N-1} ... \sum_{h_{j-1}=1}^{N-1} \Re(\Delta_{h_j} ... \Delta_{h_1}(x^k y)).$$

Next we multiply the first and last terms of the expression above by $|\sum_{x=1}^{N} e(x^k y)|^{2^j}$ and integrate over the interval $[0, 1]$,

$$\int_0^1 |\sum_{x=1}^{N} e(x^k y)|^{2^{j+1}} dy \ll N^{2^j-1} \int_0^1 |\sum_{x=1}^{N} e(x^k y)|^{2^j} dy +$$
$$N^{2^j-j-1} \int_0^1 |\sum_{x=1}^{N} e(x^k y)|^{2^j} \sum_{h_1=1}^{N-1} ... \sum_{h_j=1}^{N-1} \Re(S_{k-j}(\Delta_{h_j} ... \Delta_{h_1}(x^k y)))dy$$
$$= N^{2^j-1} \int_0^1 |\sum_{x=1}^{N} e(x^k y)|^{2^j} dy + N^{2^j-j-1} \cdot \tag{13}$$
$$\cdot \sum_{h_1=1}^{N-1} ... \sum_{h_j=1}^{N-1} \Re(\int_0^1 S_{k-j}(\Delta_{h_j} ... \Delta_{h_1}(x^k y))| \sum_{x=1}^{N} e(x^k y)|^{2^j} dy).$$

Now we are going to analyze the term in the last integral,

$$\int_0^1 S_{k-j}(\Delta_{h_j}...\Delta_{h_1}(x^k y))|\sum_{x=1}^{N} e(x^k y)|^{2^j} dy$$
$$= \sum_{x=1}^{N}\sum_{q_1=1}^{N} ... \sum_{q_{2^j-1}=1}^{N}\sum_{w_1=1}^{N} ... \sum_{w_{2^j-1}=1}^{N}\int_0^1 e((\Delta_{h_j}...\Delta_{h_1}(x^k) + (q_1^k + ...$$
$$... + q_{2^j-1}^k - w_1^k - ... - w_{2^j-1}^k))y)dy.$$

The last expression is equal to the number of integer solutions of the equation, in the interval $[1, N]$:

$$\begin{aligned} \Delta_{h_j}...\Delta_{h_1}(x^k) + q_1^k + ... + q_{2^j-1}^k - w_1^k - ... - w_{2^j-1}^k = 0 \iff \\ h_j \cdot ... \cdot h_1 p_{k-j}(x) = q_1^k + ... + q_{2^j-1}^k - w_1^k - ... - w_{2^j-1}^k. \end{aligned} \tag{14}$$

Where $p_{k-j}$ is a polynomial of degree $k-j$. Note the term on the left hand side is strictly greater than zero, since $x, h_1, ..., h_j > 0$ and it's increasing in $x$ for $x > 0$ (sums of monomials in x with non-negative coefficients). This means that for each fixed $q_1, ..., q_{2^j-1}, w_1, ..., w_{j-1}$ (there exist $N$ possible values for each of them), the number of possible $h_i$ values for each $i$ is bounded by the number of divisors of the right hand side of (14). Since the expression on the right hand side referred previously is bounded by $2^{j-1}N^k$, using the bound for the number of divisors derived in (9) with $\epsilon := \epsilon/jk$ yields that the number of divisors is

$$d(q_1^k + ... + q_{2^j-1}^k - w_1^k - ... - w_{2^j-1}^k) \ll (2^{j-1}N^k)^{\epsilon/jk} \ll N^{\epsilon/j}.$$

Since there exists $j$ such $h$'s, we get that the number of solutions of (14) is bounded by $N^{2^j+\epsilon}$. Now we apply this estimate to (13),

$$N^{2^j-1}\int_0^1 |\sum_{x=1}^{N} e(x^k y)|^{2^j} dy + N^{2^j-j-1}\sum_{h_1=1}^{N-1} ... \sum_{h_j=1}^{N-1} \Re(\int_0^1 S_{k-j}(\Delta_{h_j}...$$
$$...\Delta_{h_1}(x^k y))|\sum_{x=1}^{N} e(x^k y)|^{2^j} dy) \ll N^{2^j-1}N^{2^j-j+\epsilon}+$$
$$N^{2^j-j-1}N^{2^j+\epsilon} \ll N^{2^{j+1}-(j+1)+\epsilon}.$$

□

# 3 Deriving the Asymptotic Formula

We first note the following useful equality,

$$\int_0^1 e(ay)dy = \begin{cases} 1 & \text{if } a = 0. \\ 0 & \text{if } a \in \mathbb{Z} \setminus 0. \end{cases} \tag{15}$$

Now, we will move to the main proof. Note that for each $x_i$ in (1), we have that $x_i \leq N^{\frac{1}{k}}$ must hold for each $i$. Combining this statement with (15) we can deduce:

$$R(N) = \sum_{x_1=0}^{[N^{\frac{1}{k}}]} \dots \sum_{x_s=0}^{[N^{\frac{1}{k}}]} \int_0^1 e\Big(\big((\sum_{i=1}^{s} x_i^k) - N\big)y\Big) dy, \tag{16}$$

where $R(N)$ denotes the number of solutions of (1), and by swapping order of integration theorem and using that $e(a+b) = e(a)e(b)$ we obtain:

$$\begin{aligned} R(N) &= \int_0^1 \Big(\sum_{x_1=1}^{[N^{\frac{1}{k}}]} e(x_1^k y)\Big) \dots \Big(\sum_{x_s=1}^{[N^{\frac{1}{k}}]} e(x_s^k y)\Big) e(-Ny) dy \\ &= \int_0^1 \Big(\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y)\Big)^s e(-Ny) dy. \end{aligned} \tag{17}$$

Now we will divide the interval $[0,1]$, into major and minor arcs. We fix a $\rho > 0$ and a $Q > 0$ such that $\frac{1}{Q^2} > 2\rho$, and define the major arcs:

$$\mathfrak{M}(q,a) = \{x \in \mathbb{R} : |x - \frac{a}{q}| < \rho\}, \quad \text{for each } 1 \leq q \leq Q,\ a \leq q,\ (a,q) = 1. \tag{18}$$

The condition $\frac{1}{Q^2} > 2\rho$, ensures all major arcs are disjoint, since assuming $\frac{a}{q} \neq \frac{b}{r}$:

$$|\frac{a}{q} - \frac{b}{r}| = |\frac{ar - bq}{qr}| \geq \frac{1}{qr} \geq \frac{1}{Q^2} > 2\rho.$$

We define $\mathfrak{M}$, the major arcs, to be the union of all sets of the form (18), and $\mathfrak{m}$, the minor arcs, to be the complement of $\mathfrak{M}$ in the set $[0,1]$. Now we will evaluate the part of (17) over the major arcs.

**Lemma 3.1.** *Let $\rho N \geq 1$ and $2\rho Q^2 < 1$, then the following holds:*

$$\begin{aligned} &\int_{\mathfrak{M}} \Big(\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y)\Big)^s e(-Ny) dy \\ &= \Bigg(\sum_{q=1}^{Q} q^{-s} \Big( \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \big(\sum_{c=1}^{q} e(c^k \frac{a}{q})\big)^s \Big) e(-N\frac{a}{q})\Bigg) \Bigg(\int_{-\rho N}^{\rho N} \Big(\int_0^1 e(u^k v) du\Big)^s \cdot \\ &\cdot N^{\frac{s-k}{k}} e(-v) dv\Bigg) + \mathcal{O}(\rho^{s+1} N^s Q^{s+2} + N\rho^2 Q^3 N^{\frac{s-1}{k}}). \end{aligned} \tag{19}$$

*Proof.* We start by approximating each sum individually, to that end we rewrite $y = \frac{a}{q} + z$ and $x = qb + c$, where $c, b \in \mathbb{Z}$ and $c < q$.

$$\begin{aligned}\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y) = \sum_{x=1}^{[N^{\frac{1}{k}}]} e\big((qb+c)^k(\frac{a}{q}+z)\big) = \sum_{qb+c\leq[N^{\frac{1}{k}}]} e\big((qb+c)^k\frac{a}{q}\big)\cdot \\ \cdot e\big((qb+c)^k z\big) = \sum_{c=1}^{q} e(c^k\frac{a}{q}) \sum_{qb+c\leq[N^{\frac{1}{k}}]} e\big((qb+c)^k z\big).\end{aligned} \tag{20}$$

In the last step of (20), all but one term of the binomial expansion of $(qb+c)^k$ contain $q$, in which case, the term multiplied with $\frac{a}{q}$ will yield an integer. Next, we apply Lemma 8.1 from the appendix (the notation used is also explained in the appendix) to the inner sum from the last expression of (20),

$$\begin{aligned}\sum_{qb+c\leq[N^{\frac{1}{k}}]} e((qb+c)^k z) = \int_0^{[(N^{\frac{1}{k}}-c)/q]} e\big((qb+c)^k z\big)db + \\ \frac{1}{2}\Big(e(c^k z) + e\Big(([(N^{\frac{1}{k}}-c)/q]+c)^k z\Big)\Big) + \\ \int_0^{[(N^{\frac{1}{k}}-c)/q]} e\big((qb+c)^k z\big)2\pi izkq(qb+c)^{k-1}P_1(b)db \\ = \int_c^{q[(N^{\frac{1}{k}}-c)/q]+c} e(t^k z)q^{-1}dt + \mathcal{O}(1) + \mathcal{O}(\rho N) \\ = \int_0^{N^{\frac{1}{k}}} e(t^k z)q^{-1}dt + \mathcal{O}(1+\rho N)\end{aligned} \tag{21}$$

In the second equality, we apply the change of variables $qb + c = t$ to the first integral, use $|e(a)| = 1$ to bound the non integral terms, and use the following computation to bound the last integral:

$$\begin{aligned}|\int_0^{[(N^{\frac{1}{k}}-c)/q]} e\big((qb+c)^k z\big)2\pi izkq(qb+c)^{k-1}P_1(b)db| \\ \leq \int_0^{[(N^{\frac{1}{k}}-c)/q]} 2\pi k|z|q|qb+c|^{k-1}db < \int_0^{[(N^{\frac{1}{k}}-c)/q]} 2\pi\rho kq|qb+c|^{k-1}db \\ = 2\pi\rho[|qb+c|^k]_0^{[(N^{\frac{1}{k}}-c)/q]} = 2\pi\rho([N]-c^k) = \mathcal{O}(\rho N)\end{aligned}$$

We note that, $e(x)$ and $P_1(x)$ are both bounded by 1, and $|z| = |y - \frac{a}{q}| < \rho$. We also note that in the last equality of (21), the changes in the integral endpoints follow from the following two computations:

$$|\int_0^c e(t^k z)q^{-1}dt| \leq \int_0^c |q^{-1}|dt = \frac{c}{q} = \mathcal{O}(1)$$

$$|\int_{N^{\frac{1}{k}}}^{q[(N^{\frac{1}{k}}-c)/q]+c} e(t^k z)q^{-1}dt| \leq \int_{N^{\frac{1}{k}}}^{q[(N^{\frac{1}{k}}-c)/q]+c} |q^{-1}|dt$$
$$= \frac{q[(N^{\frac{1}{k}}-c)/q]+c-N^{\frac{1}{k}}}{q} = [\frac{N^{\frac{1}{k}}-c}{q}] - \frac{N^{\frac{1}{k}}-c}{q} = \mathcal{O}(1).$$

Plugging (21) in (20) yields:

$$\begin{aligned}\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y) &= \sum_{c=1}^{q} e(c^k \frac{a}{q}) \sum_{qb+c\leq[N^{\frac{1}{k}}]} e\big((qb+c)^k z\big)\\ &= \sum_{c=1}^{q} e(c^k \frac{a}{q}) \Big( \int_0^{N^{\frac{1}{k}}} e(t^k z)q^{-1}dt + \mathcal{O}(1+\rho N)\Big)\\ &= q^{-1}\sum_{c=1}^{q} e(c^k \frac{a}{q}) \int_0^{N^{\frac{1}{k}}} e(t^k z)dt + \mathcal{O}(\rho N Q)\end{aligned} \tag{22}$$

Now we will analyze the expression on the left hand side of (19),

$$\int_{\mathfrak{M}} (\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y))^s e(-Ny)dy = \sum_{\substack{a,q\leq Q\\(a,q)=1}} \int_{\mathfrak{M}(a,q)} (\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y))^s e(-Ny)dy$$
$$= \sum_{\substack{a,q\leq Q\\(a,q)=1}} \int_{\mathfrak{M}(a,q)} (q^{-s}(\sum_{c=1}^{q} e(c^k \frac{a}{q}))^s (\int_0^{N^{\frac{1}{k}}} e(t^k z)dt)^s e(-N(\frac{a}{q}+z))+$$
$$+\mathcal{O}(\rho^s N^s Q^s + N\rho Q N^{\frac{s-1}{k}}))dy$$
$$= (\sum_{q=1}^{Q} q^{-s}(\sum_{\substack{a=1\\(a,q)=1}}^{q} e(c^k \frac{a}{q}))^s e(-N\frac{a}{q}))(\int_{-\rho N}^{\rho N} (\int_0^1 e(u^k v)du)^s N^{\frac{s-k}{k}} e(-v)dv)+$$
$$+\mathcal{O}(\rho^{s+1} N^s Q^{s+2} + N\rho^2 Q^3 N^{\frac{s-1}{k}}).$$

The first equality holds due to all the major arcs being disjoint. The second one follows from plugging in (22), and using binomial expansion, the error term consists of both extremes of the binomial expression, note that the sum in (22) is bounded by $q$ and the integral in the same expression is bounded by $N^{\frac{1}{k}}$. In the third step, we rewrite the union as a sum, move the inner sum outside of the integral as it is not dependent on $y$, the error terms change due to integrating over an interval of length $2\rho$, and summing over less than $Q^2$ terms and the integral term change by the following calculation:

$$\int_{\mathfrak{M}(a,q)} (\int_0^{N^{\frac{1}{k}}} e(t^k z)dt)^s e(-Nz)dz = \int_{-\rho}^{\rho} (\int_0^{N^{\frac{1}{k}}} e(t^k z)dt)^s e(-Nz)dz$$
$$= \int_{-\rho}^{\rho} (\int_0^1 e(u^k zN)du)^s N^{\frac{s}{k}} e(-Nz)dz = \int_{-\rho N}^{\rho N} (\int_0^1 e(u^k v)du)^s N^{\frac{s-k}{k}} e(-v)dv$$

□

Next we will bound the portion of (17) over the minor arcs $\mathfrak{m}$.

**Lemma 3.2.** *Let $s \geq 2^k + 1$, then*

$$|\int_{\mathfrak{m}} (\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y))^s e(-Ny)dy| \ll N^{\frac{s}{k}-1-\delta'}$$

*holds for some $\delta' > 0$*

*Proof.*

$$\begin{aligned} &|\int_{\mathfrak{m}} (\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y))^s e(-Ny)dy| \\ &\leq \int_{\mathfrak{m}} |\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y)^{2^k}||\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y)^{s-2^k}|dy \\ &\leq \max_{y\in\mathfrak{m}} |\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y)|^{s-2^k} \int_0^1 |\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y)|^{2^k} dy \\ &\ll \max_{y\in\mathfrak{m}} |\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y)|^{s-2^k} N^{\frac{2^k-k+\epsilon}{k}} = \max_{y\in\mathfrak{m}} |\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y)|^{s-2^k} N^{\frac{2^k}{k}-1+\frac{\epsilon}{k}} \end{aligned} \tag{23}$$

In the first inequality we note $\mathfrak{m} \subseteq [0, 1]$ and in the second inequality we use proposition 2.3 (Hua's Lemma). Next we need to bound the maximum terms. For that we apply theorem 8.3, with $l = 1$, $N = \rho^{-1}$ and $\alpha$ a point in $\mathfrak{m}$, which yields, $|\alpha - \frac{a}{q}| < \frac{\rho}{q}$, for some integer satisfying $1 \leq q \leq \rho^{-1}$, this two statements combined imply $|\alpha - \frac{a}{q}| < q^{-2}$.
Also note $q > Q$ (the upper bound of the divisor in the construction of the major arcs), or else $|\alpha - \frac{a}{q}| \leq \frac{\rho}{q} \leq \rho$, meaning it would belong to the major arc. From $|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}$, we can use proposition 2.2, and obtain:

$$|\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y)| \ll N^{\frac{1}{k}(1+\epsilon)}(N^{-\frac{1}{k}} + q^{-1} + qN^{-1})^{\frac{1}{K}} \tag{24}$$

Now, we choose $\delta > 0$, such that $[N^\delta] = Q < q \leq \rho^{-1} = N^{1-\delta}$ and $\delta \leq \frac{1}{k}$, this two conditions imply the following boundaries for (24),

$$\leq N^{\frac{1+\epsilon}{k}}(N^{-\delta} + [N^{-\delta}] + N^{1-\delta}N^{-1})^{\frac{1}{K}} \ll N^{\frac{1+\epsilon}{k}-\frac{\delta}{K}}$$

Now we can plug in this result in (23), obtaining:

$$\max_{y\in\mathfrak{m}}|\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y))|^{s-2^k} N^{\frac{2^k}{k}-1+\frac{\epsilon}{k}} \ll (N^{\frac{1+\epsilon}{k}-\frac{\delta}{K}})^{s-2^k} N^{\frac{2^k}{k}-1+\frac{\epsilon}{k}}$$
$$= N^{\frac{s}{k}-\frac{2^k}{k}+\frac{2^k}{k}-1+\epsilon(\frac{s-2^k}{k}+\frac{1}{k})-\delta\frac{s-2^k}{K}} = N^{\frac{s}{k}-1+\epsilon'-\delta'}$$

Since we can pick a very small $\epsilon'$, we are done with the proof. □

**Definition 3.3.** *The singular series is denoted by* $\mathfrak{S}$ *and is defined in the following way:*

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s} (\sum_{c=1}^{q} e(\frac{a}{q}c^k))^s e(-\frac{a}{q}N)$$

**Theorem 3.4.** *For* $s \geq 2^k + 1$*, the following asymptotic formula holds,*

$$R(N) = \frac{\Gamma(1+\frac{1}{k})^s}{\Gamma(\frac{s}{k})} N^{\frac{s-k}{k}} \mathfrak{S}(N) + \mathcal{O}(N^{\frac{s-k}{k}-\delta'}). \tag{25}$$

*Proof.* We start by finding a bound for the "tail" of the singular series, for this we apply Proposition 2.2 to the sum $\sum_{c=1}^{q} e(\frac{a}{q}c^k)$, with $N = q$ and $\alpha = \frac{a}{q}$. This yields the bound $q^{1-\frac{1}{K}+\epsilon}$, where $K = 2^{k-1}$. Using this to yield the "tail" of the singular series,

$$|\sum_{q=Q+1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s} (\sum_{c=1}^{q} e(\frac{a}{q}c^k))^s e(-\frac{a}{q}N)|$$
$$\ll \sum_{q=Q+1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s} (q^{1-\frac{1}{K}+\epsilon})^s \leq \sum_{q=Q+1}^{\infty} q^{1-\frac{s}{K}+\epsilon'} \tag{26}$$
$$\leq \int_{q=Q+1}^{\infty} q^{1-\frac{s}{K}+\epsilon'} dq \ll Q^{2-\frac{s}{K}+\epsilon'}.$$

Meaning that the singular series converges. It follows now from lemma 3.1 and (26) that for some $Q$ and $\rho$ satisfying the conditions of the lemma, we have:

$$\int_{\mathfrak{M}} (\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y))^s e(-Ny) dy = (\mathfrak{S} + \mathcal{O}(Q^{2-\frac{s}{K}+\epsilon'})(\int_{-\rho N}^{\rho N} (\int_0^1 e(u^k v) du)^s \cdot \tag{27}$$
$$\cdot N^{\frac{s-k}{k}} e(-v) dv) + \mathcal{O}(\rho^{s+1} N^s Q^{s+2} + N\rho^2 Q^3 N^{\frac{s-1}{k}})$$

Next we will bound the term, $\int_{-\rho N}^{\rho N} (\int_0^1 e(u^k v) du)^s e(-v) dv$.

$$\int_0^1 e(u^k v) du = k^{-1} \int_0^1 e(yv) y^{\frac{1}{k}-1} dy = k^{-1} v^{-\frac{1}{k}} \int_0^v e(c) c^{\frac{1}{k}-1} dc \tag{28}$$

We note $|\int_0^y e(c)|$ is bounded by $y$, and $c^{\frac{1}{k}-1}$ is a positive monotonic decreasing function, so by theorem 8.4 (stated in the Appendix), the integral $\int_0^v e(c)c^{\frac{1}{k}-1}dc$ converges. So we obtain the bound, $|\int_0^1 e(u^k v)du| \ll |v|^{-\frac{1}{k}}$, which further yields,

$$|\int_{\rho N}^{\infty}(\int_0^1 e(u^k v)du)^s e(-v)dv| \ll \int_{\rho N}^{\infty} |v|^{-\frac{s}{k}}dv = (\rho N)^{1-\frac{s}{k}}$$

And from above, we can deduce,

$$\int_{-\rho N}^{\rho N}(\int_0^1 e(u^k v)du)^s e(-v)dv = \int_{-\infty}^{\infty}(\int_0^1 e(u^k v)du)^s e(-v)dv + \mathcal{O}((\rho N)^{1-\frac{s}{k}}). \tag{29}$$

Therefore we analyze the integral over the whole real line:

$$\int_{-\infty}^{\infty}(\int_0^1 e(u^k v)du)^s e(-v)dv = \lim_{\alpha\to\infty}\int_{-\alpha}^{\alpha}(k^{-1}\int_0^1 e(yv)y^{\frac{1}{k}-1}dy)^s e(-v)dv$$

$$= k^{-s}\lim_{\alpha\to\infty}\int_{-\alpha}^{\alpha}\int_0^1 \dots \int_0^1 e((y_1+\dots+y_s-1)v)y_1^{\frac{1}{k}-1}\cdot\dots\cdot y_s^{\frac{1}{k}-1}dy_1\dots dy_s dv$$

$$= k^{-s}\lim_{\alpha\to\infty}\int_0^1 \dots \int_0^1 y_1^{\frac{1}{k}-1}\cdot\dots\cdot y_s^{\frac{1}{k}-1}\int_{-\alpha}^{\alpha} e((y_1+\dots+y_s-1)v)dvdy_1\dots dy_s$$

$$= k^{-s}\lim_{\alpha\to\infty}\int_0^1 \dots \int_0^1 y_1^{\frac{1}{k}-1}\cdot\dots\cdot y_s^{\frac{1}{k}-1}\frac{e((y_1+\dots+y_s-1)\alpha)-e(-(y_1+\dots+y_s-1)\alpha)}{(y_1+\dots+y_s-1)2\pi i}dy_1\dots dy_s$$

$$= k^{-s}\lim_{\alpha\to\infty}\int_0^1 \dots \int_0^1 y_1^{\frac{1}{k}-1}\cdot\dots\cdot y_s^{\frac{1}{k}-1}\frac{\sin((y_1+\dots+y_s-1)\alpha 2\pi)}{(y_1+\dots+y_s-1)\pi}dy_1\dots dy_s \tag{30}$$

Define $f(t) \coloneqq \int_0^1 \dots \int_0^1 (y_1\cdot\dots\cdot y_{s-1}(t-y_1-\dots-y_{s-1}))^{\frac{1}{k}-1}\chi_A(\bar{y})dy_1\dots dy_{s-1}$, where $A$ denotes the set where $\bar{y}=y_1,\dots,y_{s-1}$ satisfy $t-1<y_1+\dots+y_{s-1}<t$. Using this definition, we get that (30) is equal to:

$$k^{-s}\lim_{\alpha\to\infty}\int_0^s f(t)\frac{\sin((t-1)\alpha 2\pi)}{(t-1)\pi}dt \tag{31}$$

Our next goal is to show $f$ is of bounded variation on $[0,s]$, in order to apply theorem 8.5.

$$\int_0^s |f'(t)|dt = \int_0^s |\int_0^1 ... \int_0^1 (y_1 \cdot ... \cdot y_{s-1})^{\frac{1}{k}-1}(t-(y_1+...+y_{s-1}))^{\frac{1}{k}-2}(\frac{1}{k}-1)$$

$$\chi_A(\bar{y})dy_1...dy_{s-1}|dt$$

$$= (1-\frac{1}{k})\int_0^s |\int_0^{\frac{1}{t}} ... \int_0^{\frac{1}{t}} t^{\frac{s-1}{k}-(s-1)}(z_1 \cdot ... \cdot z_{s-1})^{\frac{1}{k}-1} \cdot$$

$$\cdot t^{\frac{1}{k}-2}(1-(z_1+...+z_{s-1}))^{\frac{1}{k}-2}\chi_B(\bar{z})t^{s-1}dz_1...dz_{s-1}|dt$$

$$= (1-\frac{1}{k})\int_0^s |t^{\frac{s}{k}-2}\int_0^{\frac{1}{t}} ... \int_0^{\frac{1}{t}} (z_1 \cdot ... \cdot z_{s-1})^{\frac{1}{k}-1}(1-(z_1+...+z_{s-1}))^{\frac{1}{k}-2} \cdot$$

$$\cdot \chi_B(\bar{z})dz_1...dz_{s-1}|dt. \tag{32}$$

Where $B$ denotes the set where $z_1, ..., z_{s-1}$ satisfy $1-\frac{1}{t} < z_1+...+z_{s-1} < 1$. We know $t^{\frac{s}{k}-2}$ is integrable on the interval $[0,s]$ and $\int_0^{\frac{1}{t}} ... \int_0^{\frac{1}{t}} (z_1 \cdot ... \cdot z_{s-1})^{\frac{1}{k}-1}(1-(z_1+...+z_{s-1}))^{\frac{1}{k}-2} \cdot \chi_{(\{1-\frac{1}{t}<z_1+...+z_{s-1}<1\})}$ is a monotonically decreasing function of $t$, this implies (32) is finite and $f$ is a function of bounded variation, implying we can apply theorem 8.5 to $f$, therefore the expression in (31) is equal to

$$k^{-s}\int_0^1 ... \int_0^1 (y_1 \cdot ... \cdot y_{s-1}(1-(y_1+...+y_{s-1})))^{\frac{1}{k}-1}\chi_{(\{0<y_1+...y_{s-1}<1\})}dy_1...dy_{s-1},$$

The integral term above corresponds to the multivariate Beta function, which means it is equal to:

$$k^{-s}\frac{\Gamma(\frac{1}{k})^s}{\Gamma(\frac{s}{k})} = \frac{\Gamma(1+\frac{1}{k})^s}{\Gamma(\frac{s}{k})}.$$

We can now plug this into (29):

$$\int_{-\rho N}^{\rho N}(\int_0^1 e(u^k v)du)^s e(-v)dv = \frac{\Gamma(1+\frac{1}{k})^s}{\Gamma(\frac{s}{k})} + \mathcal{O}((\rho N)^{1-\frac{s}{k}}).$$

Next we plug the above into (27) yielding:

$$\int_{\mathfrak{M}}(\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y))^s e(-Ny)dy = (\mathfrak{S} + \mathcal{O}(Q^{2-\frac{s}{k}+\epsilon'}))(\frac{\Gamma(1+\frac{1}{k})^s}{\Gamma(\frac{s}{k})} + \\ +\mathcal{O}((\rho N)^{1-\frac{s}{k}}))N^{\frac{s-k}{k}}) + \mathcal{O}(\rho^{s+1}N^s Q^{s+2} + N\rho^2 Q^3 N^{\frac{s-1}{k}}) \tag{33}$$

We let $\delta \in (0, \frac{1}{3})$ and choose $Q = [N^\delta]$ and $\rho = N^{\delta-1}$, this means $N\rho = N^\delta \geq 1$ and $2\rho Q^2 < 2N^{-1-\delta} < 1$, since $N > 2^k+1$ and $k \geq 3$. Applying this means

(33) is equal to

$$
\begin{aligned}
&(\mathfrak{S}(N)+\mathcal{O}(N^{\delta(2-\frac{s}{k}+\epsilon')}))(\frac{\Gamma(1+\frac{1}{k})^s}{\Gamma(\frac{s}{k})}+\\
&+\mathcal{O}((N^\delta)^{1-\frac{s}{k}}))N^{\frac{s-k}{k}})+\mathcal{O}(N^{\delta(s+1)}N^{\delta(s+2)}+N^{2\delta-1}N^{3\delta}N^{\frac{s-1}{k}})\\
&=\mathfrak{S}(N)\frac{\Gamma(1+\frac{1}{k})^s}{\Gamma(\frac{s}{k})}N^{\frac{s-k}{k}}+\mathcal{O}(N^{\delta(s+1)}N^{\delta(s+2)}+N^{2\delta-1}N^{3\delta}N^{\frac{s-1}{k}}+\\
&+(N^\delta)^{1-\frac{s}{k}}+N^{\frac{s-k}{k}}N^{\delta(2-\frac{s}{k}+\epsilon')})=\mathfrak{S}(N)\frac{\Gamma(1+\frac{1}{k})^s}{\Gamma(\frac{s}{k})}N^{\frac{s-k}{k}}+\\
&+\mathcal{O}(N^{2s\delta+3\delta-1}+N^{5\delta-1+\frac{s-1}{k}}+N^{\delta(1-\frac{s}{k})}+N^{\frac{s-k}{k}+\delta(2-\frac{s}{k}+\epsilon')})\\
&=\mathfrak{S}(N)\frac{\Gamma(1+\frac{1}{k})^s}{\Gamma(\frac{s}{k})}N^{\frac{s-k}{k}}+\mathcal{O}(N^{\frac{s-k}{k}-\delta'})
\end{aligned}
$$

Choosing $\delta$ small enough depending on $k, s$ and $\epsilon'$ for some $\delta' > 0$. ☐

Note that, showing that for a certain $s$, $R(N) \to \infty$ as $N \to \infty$ implies $G(k) \leq s$. We also note, that the lower bound for $s$ could be improved, by getting a better estimate for the minor arcs portion of the interval.

# 4 Investigating the singular series

The goal of this section will be to bound the singular series by some constant, this will imply, by the asymptotic formula in (2), that $R(N) \to \infty$ as $N \to \infty$. Our first goal will be to represent the singular series as a product over the primes.

**Proposition 4.1.** *The function,*

$$A_N(q) := \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-s}(\sum_{c=1}^{q} e(\frac{a}{q}c^k))^s e(-\frac{a}{q}N)$$

*is multiplicative.*

*Proof.* Let $q_1$ and $q_2$ be relatively prime positive integers.

$$A_N(q_1q_2) = \sum_{\substack{a=1\\(a,q_1q_2)=1}}^{q_1q_2} q_1^{-s}q_2^{-s}\left(\sum_{c=1}^{q_1q_2} e(\frac{a}{q_1q_2}c^k)\right)^s e(-\frac{a}{q_1q_2}N) \tag{34}$$

From elementary number theory, we know that for any $a \pmod{q_1q_2}$, satisfying $(a, q_1q_2) = 1$, there exist unique $a_1 \pmod{q_1}$ and $a_2 \pmod{q_2}$ relatively prime to $q_1$ and $q_2$ respectively, such that $a_1q_2+q_1a_2 \equiv a (\mathrm{mod}\ q_1q_2)$, this correspondence is bijective (for every pair $a_1$, $a_2$ there exists an unique $a$), now we can deduce:

$$e(-\frac{a}{q_1q_2}N) = e(-\frac{a_1}{q_1}N - \frac{a_2}{q_2}N) = e(-\frac{a_1}{q_1}N) \cdot e(-\frac{a_2}{q_2}N). \tag{35}$$

Next we note $c$ can be decomposed in a similar way (for all $c \pmod{q_1q_2}$, we can find unique $c_1$, $c_2$ such that $c_1q_2 + q_1c_2 = c$), yielding:

$$e(\frac{a}{q_1q_2}c^k) = e\Bigg((\frac{a_1}{q_1} + \frac{a_2}{q_2})(q_1q_2)^k(\frac{c}{q_1q_2})^k\Bigg) = e\Bigg((\frac{a_1}{q_1} + \frac{a_2}{q_2})(q_1q_2)^k \cdot$$
$$\cdot(\frac{c_1}{q_1} + \frac{c_2}{q_2})^k\Bigg) = e\Bigg((\frac{a_1}{q_1} + \frac{a_2}{q_2})(c_1q_2 + c_2q_1)^k\Bigg) = e(\frac{a_1}{q_1}(c_1q_2)^k)e(\frac{a_2}{q_2}(c_2q_1)^k) \tag{36}$$

We note there exists a bijection over the reduced residues mod $q_1$, between $c_1q_2$ and $c_1$, and similarly for mod $q_2$. Using (35) and (36), we can deduce:

$$A_N(q_1q_2) = \sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1} \sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2} q_1^{-s}q_2^{-s}\Bigg(\sum_{c_1=1}^{q_1}\sum_{c_2=1}^{q_2} e(\frac{a_1}{q_1}(c_1)^k)e(\frac{a_2}{q_2}(c_2)^k)\Bigg)^s \cdot$$
$$\cdot e(-\frac{a_1}{q_1}N)\cdot e(-\frac{a_2}{q_2}N) = \Bigg(\sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1} q_1^{-s}(\sum_{c_1=1}^{q_1} e(\frac{a_1}{q_1}(c_1)^k))^s e(-\frac{a_1}{q_1}N)\Bigg)\cdot$$
$$\cdot\Bigg(\sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2} q_2^{-s}(\sum_{c_2=1}^{q_2} e(\frac{a_2}{q_2}(c_2)^k))^s e(-\frac{a_2}{q_2}N)\Bigg) = A_N(q_1)A_N(q_2)$$

□

Applying Proposition 2.2 (Weyl's inequality) to the inner sum of $A_N(q)$ yields:

$$\begin{aligned}|\sum_{c=1}^{q} e(\frac{a}{q}c^k)| &\ll |q^{1+\epsilon}(q^{-1} + q^{-1} + q^{1-k})^{\frac{1}{2^{k-1}}} \\ &\ll |q^{1+\epsilon}q^{-\frac{1}{2k-1}}| = q^{1-\frac{1}{2^{k-1}}+\epsilon}.\end{aligned} \tag{37}$$

We can use (37) to bound:

$$\begin{aligned}|q^{-s}(\sum_{c=1}^{q} e(\frac{a}{q}c^k))^s e(-\frac{a}{q}N)| &\ll q^{-s}q^{s(1-\frac{1}{2^{k-1}}+\epsilon)} \\ &= q^{-\frac{s}{2^{k-1}}+\epsilon'} \ll q^{-2+\frac{1}{2^{k-1}}+\epsilon'}.\end{aligned} \tag{38}$$

In the last step, we assume $s \geq 2^{k-1} + 1$.
It follows from (38) that $\sum_{q=1}^{\infty} A_N(q)$ is absolutely convergent, this statement together with proposition 4.1 fullfill the condition of proposition 8.6 (stated in the Appendix) meaning:

$$\mathfrak{S}(N) = \prod_{\substack{p\\prime}} \sum_{\alpha=0}^{\infty} A_N(p^\alpha)$$

Next, we note that the product being convergent implies, there exists, a prime $P$ large enough, such that:

$$\frac{1}{2} \leq | \prod_{\substack{p>P \\ prime}} \sum_{\alpha=0}^{\infty} A_N(p^\alpha)| \leq \frac{3}{2} \tag{39}$$

**Proposition 4.2.** *The following holds,*

$$\sum_{j=0}^{\infty} A_N(p^j) = \lim_{\beta \to \infty} M_N(p^\beta)/p^{-\beta(s-1)}.$$

*Where $M_N(q)$ denotes the number of solutions of,*

$$x_1^k + ... + x_s^k \equiv N(mod\ q)$$

*where each $x_i \in [1, q] \cap \mathbb{Z}$*

*Proof.* We first note, $\sum_{a=1}^{q} e(\frac{at}{q})$ is equal to $q$ if $q$ divides $t$ and 0 if it does not. This implies:

$$\sum_{a=1}^{q} \sum_{x_1=1}^{q} ... \sum_{x_s=1}^{q} e(\frac{a}{q}(x_1^k + ... + x_s^k - N)) = qM_N(q)$$

Next we collect the terms of the sum over $a$ in the following way;

$$M_N(q) = q^{-1} \sum_{d|q} \sum_{\substack{a=1 \\ gcd(a,q)=d}}^{q} \sum_{x_1=1}^{q} ... \sum_{x_s=1}^{q} e\left(\frac{a'}{q'}(x_1^k + ... + x_s^k - N)\right)$$

where $a' = \frac{a}{d}$ and $q' = \frac{q}{d}$. Now further computation yields:

$$\begin{aligned} M_N(q) &= q^{-1} \sum_{d|q} \sum_{a'=1}^{q'} \left( \sum_{x_1=1}^{q} e(\frac{a'}{q'} x_1^k) \right) ... \left( \sum_{x_s=1}^{q} e(\frac{a'}{q'} x_s^k) \right) e(-\frac{a'}{q'} N) \\ &= q^{-1} \sum_{d|q} \sum_{a'=1}^{q'} \left( \sum_{x_1=1}^{q'} de(\frac{a'}{q'} x_1^k) \right) ... \left( \sum_{x_s=1}^{q'} de(\frac{a'}{q'} x_s^k) \right) e(-\frac{a'}{q'} N) \\ &= q^{-1} \sum_{d|q} d^s \cdot (\frac{q}{d})^s A_N(\frac{q}{d}) = q^{s-1} \sum_{d|q} A_N(\frac{q}{d}) \end{aligned}$$

Plugging in $q = p^\beta$ yields,

$$\sum_{j=0}^{\beta} A_N(p^j) = M_N(p^\beta)/p^{-\beta(s-1)}$$

The proof is completed by letting $\beta$ go to infinity ∎

**Lemma 4.3.** *Let $p$ be a prime and $\tau$ the exponent to which $p$ occurs in the prime factorization of $k$. Let $\gamma$ be equal to $\tau+1$ if $p>2$ and $\tau+2$ if $p=2$. Assume further that for an $m \not\equiv 0(mod\ p)$ and that the equation $y^k \equiv m(mod\ p^\gamma)$ is solvable, then $x^k \equiv m(mod\ p^\delta)$ is solvable for all $\delta > \gamma$.*

Before proving the previous lemma, we remind the function $\phi(n) := \#\{m : (m,n) = 1, 1 \leq m \leq n\}$.

*Proof.* First we treat the case $p=2$, if $\tau = 0$, then $k$ is odd, and $x$ runs through all odd numbers, which is a reduced set of residues modulo $2^\delta$. Since $\delta \geq 2$, $\phi(2^\delta) = 2^{\delta-1}$ and $\phi(2^\delta)$ and $k$ are relatively prime, which implies the set of all $x^k$ is also a reduced set of residues, which means $x^k \equiv m(\text{mod } 2^\delta)$ is solvable.
If $\tau \neq 0$, then $k$ is even, which implies $x^k \equiv 1(\text{mod } 4)$. So for $y^k \equiv m(\text{mod } 2^\gamma)$ to be solvable, $m$ has to be congruent to 1 modulo 4. Those elements form a cyclic group of which 5 is a generator element.

$$x^k \equiv m(\text{mod } 2^\delta) \iff 5^a k \equiv 5^b(\text{mod } 2^\delta) \iff ak \equiv b(\text{mod } 2^{\delta-2})$$

Through a similar process, we obtain $x^k \equiv m(\text{mod } 2^\gamma) \iff ck \equiv b(\text{mod } 2^{\gamma-2})$, where $c$ is chosen such that $5^c \equiv x(\text{mod } 2^\gamma)$.
The statement $ck \equiv b(\text{mod } 2^{\gamma-2})$ implies the greatest common divisor of $k$ and $2^{\gamma-2}$ divides $b$. But $2^{\gamma-2} = 2^\theta$, which is the exponent of 2 in the prime factorization of $k$, therefore it is equal to the greatest common divisor of $k$ and $2^{\delta-2}$, implying $ak \equiv b(\text{mod } 2^{\delta-2})$ has a solution, finishing up the case $p=2$.
For the case $p>2$, choose $g$ to be a primitive root modulo $p^\delta$. Then $x \equiv g^a(\text{mod } p^\delta)$ and $m \equiv g^b(\text{mod } p^\delta)$, since both are coprime to $p$ We note,

$$x^k \equiv m(\text{mod } p^\delta) \iff g^{ka} \equiv g^b(\text{mod } p^\delta) \iff ka \equiv b(\text{mod } \phi(p^\delta))$$

From elementary number theory, we know that if $g$ is a primitive root modulo $p^\delta$, then $g$ is also a primitive root modulo $p^\gamma$.
Since $y^k \equiv m(\text{mod } p^\gamma)$ is solvable, we pick $c$ such that $g^{ck} \equiv y^k \equiv m \equiv g^b(\text{mod } p^\gamma)$, which implies $ck \equiv b(\text{mod } \phi(p^\gamma))$.
From the last equation above, we can deduce the greatest common divisor of $k$ and $p^{\gamma-1}(p-1)$ divides $b$. But $p^{\gamma-1} = p^\theta$, which is the exponent of $p$ in the prime factorization of $k$. This implies $\gcd(k, p^{\gamma-1}(p-1)) = \gcd(k, p^{\delta-1}(p-1))$, which implies $ka \equiv b$ has a solution modulo $\phi(p^\delta)$, proving the statement. □

**Proposition 4.4.** *If the equation:*

$$x_1^k + ... + x_s^k \equiv N(mod\ p^\gamma)$$

*has a solution for all primes $p$ and $\gamma$ chosen as in the previous lemma, such that at least one $x_i \not\equiv 0(mod\ p)$, and $\mathfrak{S}(N)$ is absolutely convergent, then $\mathfrak{S}(N) > 0$*

*Proof.* Fix $p$ and $j$, and let without loss of generality $x_1$ be such that $x_1 \not\equiv 0(\text{mod } p)$. Then consider the equation:

$$y_1^k + ... + y_s^k \equiv N(\text{mod } p^\delta)$$

with $\delta > \gamma$. For each $x_j$, there exist $p^{\delta-\gamma}$ values of $y_j$ such that $y_j \equiv x_j (\text{mod } p^\delta)$. This means, we can apply the previous lemma to:

$$y_1^k \equiv N - (y_2^k + ... + y_s^k)(\text{mod } p^\delta)$$

for each different value of $y_j$ for $j \in \mathbb{Z} \cap [2, s]$ as described above, we can find $(p^{\delta-\gamma})^{s-1}$ solutions to the equation.
Next we use Proposition 4.2, and obtain:

$$\sum_{j=0}^{\infty} A_N(p^j) = \lim_{\delta\to\infty} M_N(p^\delta)p^{-\delta(s-1)} \geq (p^{\delta-\gamma})^{s-1}p^{-\delta(s-1)} = p^{-\gamma(s-1)}. \quad (40)$$

Using (40) and (39), we can obtain the following bound:

$$|\mathfrak{S}(N)| = |\prod_{\substack{p \\ prime}} \sum_{j=0}^{\infty} A_N(p^j)| \geq \frac{1}{2}|\prod_{\substack{p\leq P \\ prime}} p^{-\gamma_p(s-1)}| > 0,$$

where $\gamma_p$ corresponds to the $\gamma$ for each given prime. □

**Lemma 4.5.** *For $s \geq 2k$ (k odd) and $s \geq 4k$ (k even) the equation:*

$$x_1^k + ... + x_s^k \equiv N(mod\ p^\gamma) \quad (41)$$

*has a solution such that at least one $x_i$ is not divisible by $p$*

*Proof.* If $N \not\equiv 0(\text{mod } p)$, we don't need the division condition. On the other hand if $N \equiv 0(\text{mod } p)$, we choose $x_s \equiv 1$ reducing (41) to $x_1^k + ... + x_{s-1}^k \equiv N - 1$. We have reduced the lemma to showing there exists a solution for $s \geq 2k - 1$ for odd k or $s \geq 4k - 1$ for even k and $N \not\equiv 0(\text{mod } p)$.
First we treat the case $p \neq 2$, let $s(N)$ be equal to the least $s$ for which (41) has a solution. We note that for an integer $a \not\equiv 0(\text{mod } p)$, we can multiply both sides of (41) by $a^k$, therefore if (41) is solvable for some $N$, it will be solvable for $a^k N$, the reverse is also seen by multiplying both sides by $(a^k)^{-1}(\text{mod } p^\gamma)$, therefore $s(N) = s(a^k N)$.
Let $g$ be a primitive root modulo $p^\gamma$, then let $a \equiv g^\alpha$ and $b \equiv g^\beta$, then $a^k \equiv b(\text{mod } p^\gamma)$ is solvable $\iff$ $k\alpha \equiv \beta(\text{mod } \phi(p^\gamma))$ is solvable. Which is only solvable if $\gcd(k, \phi(p^\gamma)) = p^\tau \gcd(k, p-1)$ divides $\beta$. Since $\beta$ runs over $p^\tau(p-1)$ different values, the number of different values of $b(\text{mod } p^\gamma)$ for which the equation is solvable is $\frac{p-1}{\gcd(p-1,k)}$.
We now split up the values of $N$ in classes according to the values of $s(N)$, so $N_1$ and $N_2$ belong to the same class if $s(N_1) = s(N_2)$. If a class is nonempty then it has at least $\frac{p-1}{\gcd(p-1,k)}$ elements.
Let $N^i$ denote the set of $N$'s such that $s(N) = i$, in order to show no two consecutive sets are empty (unless all possible $N$'s are already placed in a set), we let $N' \not\equiv 0(\text{mod } p)$ be the least positive integer not belonging to $N^i$ for $i \leq j - 1$. Then either $N' - 1$ or $N' - 2$ is not congruent to zero modulo $p$, since

$N' = N' - 1 + 1^k = N' - 2 + 1^k + 1^k$, and whichever isn't congruent to zero, can be written as a sum of $s$ $k$th powers for some $s \leq j - 1$, implying $s(N') \leq j + 1$. Therefore $s(N') = j$ or $j + 1$.
Let $N^s$ be the last set which is nonempty. Then at least half of the first $s - 1$ sets are nonempty, ergo there exist at least $\frac{s+1}{2}$ nonempty sets, since each set contains at least $\frac{p-1}{\gcd(p-1,k)}$ elements, the sets are disjoint and $N$ runs over $p^\tau(p-1)$ values, we obtain the bound $\frac{s+1}{2} \cdot \frac{p-1}{\gcd(k,p-1)} \leq p^\tau(p-1)$. This implies:

$$s \leq p^\tau(p-1)\frac{2\gcd(k,p-1)}{p-1} - 1 = 2p^\tau \gcd(k,p-1) - 1 \leq 2p^\tau \frac{k}{p^\tau} - 1 \leq 2k - 1.$$

We move to the case $p = 2$, if $\tau = 0$, then $x_1^k = 1 (\bmod\ 2)$ has the trivial solution. If $\tau \geq 1$, then $k$ is even and by choosing all $x_i$ to be one or zero, and $N \leq 2^\gamma - 1$, we can solve (41) for $s \geq 2^\gamma - 1 = 2^{\tau+2} - 1 \leq 4k - 1$, implying for $p = 2$, $\tau \geq 1$, (41) is solvable for $s \geq 4k - 1$. □

From Proposition 4.4 and Lemma 4.5, the following theorem follows:

**Theorem 4.6.** $\forall s \geq 2^k + 1$ *and* $\forall N > 0$

$$\mathfrak{S}(N) > 0.$$

Applying this lower bound to the asymptotic formula in theorem 3.4, we obtain that the number of solutions tends to infinity as $N$ tends to infinity.

# 5 Further Investigating of the singular series

The next goal is obtaining a stronger bound on $s$ for the absolute convergence of the singular series.

**Proposition 5.1.** $\forall N$*, the bound*

$$|A_N(q)| \ll q^{1-\frac{s}{k}}$$

*holds, implying the singular series converges for* $s \geq 2k + 1$*.*

From (35) and (36), it follows that:

$$\sum_{c=1}^{q_1 q_2} e(\frac{a}{q_1 q_2} c^k) = (\sum_{c_1=1}^{q_1} e(\frac{a_1 c_1^k}{q_1}))(\sum_{c_2=1}^{q_2} e(\frac{a_2 c_2^k}{q_2})). \tag{42}$$

We prove the following lemma before proving Proposition 5.1.

**Lemma 5.2.** *If $a$ and $q$ are relatively prime the following estimate holds:*

$$|\sum_{c=1}^{q} e(\frac{a}{q} c^k)| \ll q^{1-\frac{1}{k}}$$

*Proof.* First, we handle the case $q = p^\alpha$ where $\alpha \geq 2$. We let $\tau$ be the maximum integer such that $p^\tau | k$, and if one of the following two conditions is satisfied: $\alpha > k$ or $\gcd(k,p) = 1$, we can make the change of variables, since $\alpha - \tau - 1 \geq 1$, $c = p^{\alpha-\tau-1} f + g$:

$$\sum_{c=1}^{p^\alpha} e(\frac{a}{p^\alpha} c^k) = \sum_{f=1}^{p^{\tau+1}} \sum_{g=1}^{p^{\alpha-\tau-1}} e(\frac{a}{p^\alpha}(p^{\alpha-\tau-1} f + g)^k) \tag{43}$$

We will now analyze the binomial expansion of the last term of (43),

$$(p^{\alpha-\tau-1} f + g)^k = \sum_{j=0}^{k} \binom{k}{j} p^{j(\alpha-\tau-1)} f^j g^{k-j}. \tag{44}$$

We note that $\alpha \geq p^\tau + 1$ and analyze the terms where $j \geq 3$, we have $j(\alpha - \tau - 1) \geq 3(\alpha - \tau - 1) \geq \alpha + 2(2^\tau + 1) - 3(\tau + 1) \geq \alpha$. For the term where $j = 2$ we will split up into cases, if $\tau = 0$, then $2(\alpha - \tau - 1) = 2(\alpha - 1) \geq \alpha$. If $p \neq 2$, then $2(\alpha - \tau - 1) \geq \alpha + 2^\tau + 1 - 2\tau - 2 \geq \alpha$. If $p = 2$, and $\tau = 1$, then $k \geq 6$, since $2^\tau | k$, $2^{\tau+1} \not| k$ and $k \geq 3$, which implies $\alpha > 6 \implies 2(\alpha - \tau - 1) \geq \alpha$. The last case left is $p = 2$ and $\tau \geq 2$, then $2^{\tau-1} | \frac{k}{2}$, implying that $\tau - 1 + 2(\alpha - \tau - 1) = 2\alpha - \tau - 2 \geq \alpha + (2^\tau + 1) - \tau - 2 \geq \alpha$. Implying all terms above except the ones for $j = 0, 1$ are congruent to zero modulo $p^\alpha$, from (43) we obtain:

$$\begin{aligned}\sum_{c=1}^{p^\alpha} e(\frac{a}{p^\alpha} c^k) &= \sum_{f=1}^{p^{\tau+1}} \sum_{g=1}^{p^{\alpha-\tau-1}} e(\frac{a}{p^\alpha}(k p^{\alpha-\tau-1} f g^{k-1} + g^k)) \\ &= \sum_{g=1}^{p^{\alpha-\tau-1}} e(\frac{a}{p^\alpha} g^k) \sum_{f=1}^{p^{\tau+1}} e(\frac{a}{p^\alpha} k p^{\alpha-\tau-1} f g^{k-1}).\end{aligned}$$

As noted in the proof of Proposition 4.2, $\sum_{f=1}^{p^{\tau+1}} e(\frac{a}{p^{\tau+1}} k f g^{k-1})$ is equal to $p^{\tau+1}$ if $p^{\tau+1} | (k g^{k-1})$ and zero if not. Since $p^\tau | k$, the sum is not zero if $p | g$, this implies:

$$\begin{aligned}\sum_{c=1}^{p^\alpha} e(\frac{a}{p^\alpha} c^k) &= \sum_{h=1}^{p^{\alpha-\tau-2}} e(\frac{a}{p^\alpha} p^k h^k) p^{\tau+1} \\ &= \begin{cases} p^{\tau+1} \frac{p^{\alpha-\tau-2}}{p^{\alpha-k}} \sum_{h=1}^{p^{\alpha-k}} e(\frac{a}{p^{\alpha-k}} h^k) & \text{if } \alpha > k \\ p^{\alpha-\tau-2} p^{\tau+1} & \text{if } \alpha \leq k,\ \gcd(k,p) = 1 \end{cases}\end{aligned}$$

$$= \begin{cases} p^{k-1} \sum_{h=1}^{p^{\alpha-k}} e(\frac{a}{p^{\alpha-k}} h^k) & \text{if } \alpha > k \\ p^{\alpha-1} & \text{if } \alpha \leq k,\ \gcd(k,p) = 1 \end{cases} = \begin{cases} p^{k-1} \sum_{h=1}^{p^{\alpha-k}} e(\frac{a}{p^{\alpha-k}} h^k) & \text{if } \alpha > k \\ p^{\alpha-1} & \text{if } \alpha \leq k,\ \gcd(k,p) = 1 \end{cases}. \tag{45}$$

If we now let $r$ be the integer such that $1 \leq \alpha - kr \leq k$, it follows from (45) that if we have $\gcd(k,p) = 1$, we have

$$\sum_{c=1}^{p^\alpha} e(\frac{a}{p^\alpha}c^k) = p^{(k-1)r} \sum_{h=1}^{p^{\alpha-rk}} e(\frac{a}{p^{\alpha-rk}}h^k) = p^{kr-r}p^{\alpha-(kr+1)}$$
$$= p^{\alpha-(r+1)} \leq p^{\alpha-(\frac{\alpha}{k}-1)-1} = (p^\alpha)^{1-\frac{1}{k}} \tag{46}$$

This bounds the sum for $q = p^\alpha$, with $\alpha \geq 2$ and $p$ not in the prime factorisation of $k$.
There exist finitely many primes in the prime factorisation of $k$, so we define

$$U := \max_{\substack{p \\ p|k}} \sum_{h=1}^{p^{\alpha-rk}} e(\frac{a}{p^{\alpha-rk}}h^k). \tag{47}$$

It now follows from (45) and (47) that:

$$\sum_{c=1}^{p^\alpha} e(\frac{a}{p^\alpha}c^k) = p^{(k-1)r} \sum_{h=1}^{p^{\alpha-rk}} e(\frac{a}{p^{\alpha-rk}}h^k) \leq Up^{(k-1)r} \ll p^{(k-1)\frac{\alpha}{k}} = (p^\alpha)^{1-\frac{1}{k}}.$$

Showing the case where $q = p^\alpha$ with $\alpha \geq 2$ and $\gcd(k,p) \neq 1$. Now we handle the case where $\alpha = 1$, then:

$$\sum_{c=1}^{p} e(\frac{a}{p}c^k) = \sum_{s=1}^{p} \nu(s)e(\frac{as}{p}), \tag{48}$$

where $\nu(s) := \#\{c(\text{mod } p) : c^k \equiv s(\text{mod } p)\}$. For $s = p$, the term inside the sum equals one, we are going to use Dirichlet characters (an unfamiliar reader is referenced to chapter 3 of [3]) to simplify the terms where $s \not\equiv p(\text{mod } p)$

$$\nu(s) = \frac{1}{p-1}\sum_{\chi} \hat{\nu}(\chi)\chi(s), \tag{49}$$

where the transform is defined as:

$$\hat{\nu}(\chi) = \sum_{s=1}^{p-1} \nu(s)\overline{\chi(s)} = \sum_{x=1}^{p-1} \overline{\chi(x^k)} = \sum_{x=1}^{p-1} \overline{\chi(x)^k}. \tag{50}$$

The sum in (50) is equal to zero unless $\chi^k$ is equal to the principal character, in which case it is equal to $\phi(p)$, applying this to (49), yields:

$$\nu(s) = \sum_{\chi^k=\chi_0} \chi(s).$$

Going back to (48):

$$\sum_{c=1}^{p} e(\frac{a}{p}c^k) = 1 + \sum_{s=1}^{p-1} \sum_{\chi^k=\chi_0} \chi(s)e(\frac{as}{p}), \tag{51}$$

We pull out the principal character from the sum, since $\chi(s) = 1 \forall s \not\equiv 0 (\text{mod } p)$, adding those terms to one yields $\sum_{s=1}^{p} e(\frac{as}{p}) = 0$, implying (51) is equal to:

$$\sum_{\substack{\chi^k=\chi_0 \\ \chi\neq\chi_0}} \sum_{s=1}^{p-1} \chi(s)e(\frac{as}{p}). \tag{52}$$

The inner sum in (52) is a Gauss sum (defined in Definition 8.7 in the appendix), therefore we can rewrite (52) using the Gauss sum definition and bound it in the following way:

$$|\sum_{\substack{\chi^k=\chi_0 \\ \chi\neq\chi_0}} \tau(\chi, a)| = |\sum_{\substack{\chi^k=\chi_0 \\ \chi\neq\chi_0}} \overline{\chi(s)}\tau(\chi, 1)|$$
$$\leq \sqrt{p} \sum_{\substack{\chi^k=\chi_0 \\ \chi\neq\chi_0}} |\overline{\chi(s)}| \leq \sqrt{p} \sum_{\substack{\chi^k=\chi_0 \\ \chi\neq\chi_0}} 1 \leq p^{1-\frac{1}{k}} \sum_{\substack{\chi^k=\chi_0 \\ \chi\neq\chi_0}} 1.$$

The first step above is justified by Proposition 8.8, and the second one by Proposition 8.9.
To find a bound on the number of primitive characters satisfying $\chi^k = \chi_0$, we note that the value of $\chi$ is determined by the value at a primitive root $g$, implying that $\chi^k = \chi_0 \iff \chi(g)^k = 1$, which is a polynomial equation with at most $k$ solutions, implying:

$$|\sum_{c=1}^{p} e(\frac{a}{p}c^k)| \ll p^{1-\frac{1}{k}}.$$

Let $q = p_1^{\alpha_1}...p_m^{\alpha_m}$ be the usual prime expansion of $q$, let all the $a_i$ for $1 \leq i \leq m$ be defined regressively as in the preliminaries of equation (35). It follows from (42) and that

$$|\sum_{c=1}^{q} e(\frac{a}{q}c^k)| = |\prod_{i=1}^{m} \sum_{c_i=1}^{p_i^{\alpha_i}} e(\frac{a_i}{p_i^{\alpha_i}}c_i^k)| \ll \prod_{i=1}^{m} (p_i^{\alpha_i})^{1-\frac{1}{k}} = q^{1-\frac{1}{k}}$$

□

Now, we prove Proposition 5.1.

*Proof.* Since the definition of $A_N(q)$ includes $a$ and $q$ being relatively prime we are free to apply lemma 5.2 to the inner sum:

$$|A_N(q)| = |\sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s}(\sum_{c=1}^{q} e(\frac{a}{q}c^k))^s e(-\frac{a}{q}N)| \ll \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s}(q^{1-\frac{1}{k}})^s$$
$$\leq qq^{-s}(q^{s-\frac{s}{k}}) = q^{1-\frac{s}{k}}.$$

Which implies:

$$|\mathfrak{S}(N)| \leq \sum_{q=1}^{\infty} |A_N(q)| \ll \sum_{q=1}^{\infty} q^{1-\frac{s}{k}}.$$

The last sum is convergent for $s \geq 2k+1$. $\square$

Combining this with the previous section, we obtain that the singular series is strictly greater than zero for $s \geq 2k+1$ for odd $k$ or $s \geq 4k$ for even $k$.

# 6 A different approach for $G(k)$

In this section, we will show another application of the Hardy-Littlewood circle method, without using the asymptotic formula in (2). Instead our strategy will be to consider $N$ in the following form:

$$N = x_1^k + ... + x_{4k}^k + w_1 + w_2 + \tilde{x}^k \tilde{w}, \tag{53}$$

where $1 \leq x_i \leq \lceil N^{\frac{1}{k}} \rceil$, $1 \leq \tilde{x} \leq \lceil N^{\frac{1}{k}} \rceil^{\frac{1}{2k}}$, $w_1$ and $w_2$ correspond to positive integers smaller than $\frac{1}{4}\lceil N^{\frac{1}{k}} \rceil^k$, which can be represented as a sum of $\tilde{k}$ $k$th powers and $\tilde{w}$ corresponds to a positive integer smaller than $\frac{1}{4}\lceil N^{\frac{1}{k}} \rceil^{k-\frac{1}{2}}$, which can be represented as a sum of $\tilde{k}$ $k$th powers. It is easy to see that $w_1 + w_2 + \tilde{w}\tilde{x}^k \leq \frac{3}{4}\lceil N^{\frac{1}{k}} \rceil^k$, which implies:

$$\frac{1}{5}\lceil N^{\frac{1}{k}} \rceil^k < N - (w_1 + w_2 + \tilde{w}\tilde{x}^k) < \lceil N^{\frac{1}{k}} \rceil^k. \tag{54}$$

The term in (17) under this construction becomes:

$$\int_0^1 \Big(\sum_{x=1}^{\lceil N^{\frac{1}{k}} \rceil} e(x^k y)\Big)^s \Big(\sum_{w} e(wy)\Big)^2 \Big(\sum_{\tilde{w}} \sum_{\tilde{x}} e(\tilde{x}^k \tilde{w} y)\Big) e(-Ny)dy, \tag{55}$$

where the sums are over the terms described above.
We are going to pick our new choice of major arcs now:

$$\begin{aligned} \mathfrak{M}(q,a) &:= \{x \in [0,1] : |x - \frac{a}{q}| < \frac{1}{2qk\lceil N^{\frac{1}{k}} \rceil^{k-1}}\}, \\ &a, q \in \mathbb{N}, (a,q) = 1, q \leq \lceil N^{\frac{1}{k}} \rceil^{\frac{1}{2}} \text{ and } 1 \leq a \leq q. \end{aligned} \tag{56}$$

They are picked this way, so we can apply lemma 8.2 (Stated and proved in the appendix). We let $y$ belong to the major arcs and recall the derivation in (20):

$$\sum_{x=1}^{\lceil N^{\frac{1}{k}} \rceil} e(x^k y) = \sum_{c=1}^{q} e(\frac{a}{q} c^k) \sum_{qb+c \leq \lceil N^{\frac{1}{k}} \rceil} e((qb+c)^k z).$$

We are going to use Lemma 8.2 to approximate the inner sum above.

$$\frac{d}{db}(qb+c)^k z = kzq(qb+c)^{k-1} \leq kzq\lceil N^{\frac{1}{k}}\rceil^{k-1} < k\frac{1}{2qk\lceil N^{\frac{1}{k}}\rceil^{k-1}}q\lceil N^{\frac{1}{k}}\rceil^{k-1} = \frac{1}{2}. \tag{57}$$

Also note,

$$\frac{d}{db}\frac{d}{db}(qb+c)^k z = k(k-1)q^2 z > 0.$$

Therefore, we can apply Lemma 8.2, yielding:

$$\begin{aligned}\sum_{x=1}^{\lceil N^{\frac{1}{k}}\rceil} e(x^k y) &= \sum_{c=1}^{q} e(\frac{a}{q}c^k)(\int_{qb+c\leq\lceil N^{\frac{1}{k}}\rceil} e((qb+c)^k z)db + \mathcal{O}(1)) \\ &= q^{-1}\sum_{c=1}^{q} e(\frac{a}{q}c^k)(\int_0^{\lceil N^{\frac{1}{k}}\rceil} e(t^k z)dt) + \mathcal{O}(q).\end{aligned} \tag{58}$$

Where the last equality follows from the change of variables $qb+c=t$.
By Lemma 5.2 and (28)we have:

$$|q^{-1}\sum_{c=1}^{q} e(\frac{a}{q}c^k)\int_0^{\lceil N^{\frac{1}{k}}\rceil} e(t^k z)dt| \ll q^{-\frac{1}{k}}\min(\lceil N^{\frac{1}{k}}\rceil, |z|^{-\frac{1}{k}}).$$

By (56) we have:

$$q^{-\frac{1}{k}}|z|^{-\frac{1}{k}} = |\alpha q - a|^{-\frac{1}{k}} > |\frac{1}{2k\lceil N^{\frac{1}{k}}\rceil^{k-1}}|^{-\frac{1}{k}} > q,$$

and

$$q^{-\frac{1}{k}}N^{\frac{1}{k}} \geq q^{2-\frac{1}{k}} > q,$$

showing the error term is bounded by the main term in (58). Now we consider (17) over the new set of major arcs but for an $M$ in the interval described in (54):

$$\begin{aligned}&\int_{\mathfrak{M}}\Bigg(\sum_{x=1}^{\lceil N^{\frac{1}{k}}\rceil} e(x^k y)\Bigg)^s e(-My)dy = \int_{\bigcup\mathfrak{M}(q,a)}\Bigg(q^{-1}\sum_{c=1}^{q} e(\frac{a}{q}c^k)\cdot \\ &(\int_0^{\lceil N^{\frac{1}{k}}\rceil} e(t^k z)dt) + \mathcal{O}(q)\Bigg)^s e(-Mz)dz = \int_{\bigcup\mathfrak{M}(q,a)}\Bigg(q^{-s}(\sum_{c=1}^{q} e(\frac{a}{q}c^k))^s\cdot \\ &(\int_0^{\lceil N^{\frac{1}{k}}\rceil} e(t^k z)dt)^s + \mathcal{O}\Big(q^{1-\frac{s-1}{k}}(\min(\lceil N^{\frac{1}{k}}\rceil, |z|^{-\frac{1}{k}}))^{s-1}\Big)\Bigg)e(-Mz)dz.\end{aligned}$$

We now consider the error term,

$$|q^{1-\frac{s-1}{k}}(\min(\lceil N^{\frac{1}{k}}\rceil, |z|^{-\frac{1}{k}}))^{s-1}| \ll |q^{1-\frac{s-1}{k}}N^{1-\frac{s-1}{k}}|.$$

Integrating $|z|^{-\frac{s-1}{k}}$ as in the proof of theorem 3.4 (but over the different major arc bound) yields the inequality above. Next, we consider the error term over all different major arcs, for each fixed $q$ there exist at most $q$ values of $a$, and we have the bound $q \leq \lceil N^{\frac{1}{k}} \rceil^{\frac{1}{2}}$ from the definition of the major arcs, this yields an error term:

$$\sum_{q \leq \lceil N^{\frac{1}{k}} \rceil^{\frac{1}{2}}} q^{2-\frac{s-1}{k}} N^{1-\frac{s-1}{k}} \ll N^{1-\frac{s-1}{k}},$$

for $s > 3k+1$, to ensure the sum is convergent. Now, we consider the main term, since the major arcs are disjoint, we can split up the integral:

$$\sum_{q \leq \lceil N^{\frac{1}{k}} \rceil^{\frac{1}{2}}} \sum_{\substack{a \leq q \\ (a,q)=1}} (q^{-s}(\sum_{c=1}^{q} e(\frac{a}{q}c^k))^s \int_{|z|<(2qk\lceil N^{\frac{1}{k}} \rceil^{k-1})^{-1}} (\int_0^{\lceil N^{\frac{1}{k}} \rceil} e(t^k z) dt)^s e(-Mz) dz. \tag{59}$$

We show now, in the next computation, that extending the integral over $z$ to infinity does not change our error term.

$$\begin{aligned}
&\sum_{q \leq \lceil N^{\frac{1}{k}} \rceil^{\frac{1}{2}}} \sum_{\substack{a \leq q \\ (a,q)=1}} (q^{-s}(\sum_{c=1}^{q} e(\frac{a}{q}c^k))^s \int_{(2qk\lceil N^{\frac{1}{k}} \rceil^{k-1})^{-1}}^{\infty} (\int_0^{\lceil N^{\frac{1}{k}} \rceil} e(t^k z) dt)^s e(-Mz) dz \\
&\ll \sum_{q \leq \lceil N^{\frac{1}{k}} \rceil^{\frac{1}{2}}} q^{1-\frac{s}{k}} \int_{(2qk\lceil N^{\frac{1}{k}} \rceil^{k-1})^{-1}}^{\infty} z^{-\frac{s}{k}} dz \ll \sum_{q \leq \lceil N^{\frac{1}{k}} \rceil^{\frac{1}{2}}} q^{1-\frac{s}{k}} q^{\frac{s}{k}-1} N^{(\frac{s}{k}-1)\frac{k-1}{k}} \\
&\ll \sum_{q \leq \lceil N^{\frac{1}{k}} \rceil^{\frac{1}{2}}} N^{\frac{s}{k}-1-\frac{s}{k^2}+\frac{1}{k}} = N^{\frac{s}{k}-1-\frac{s}{k^2}+\frac{3}{2k}} \ll N^{1-\frac{s-1}{k}}
\end{aligned}$$

Now we consider the outer integral of (59) extended to infinity:

$$\begin{aligned}
&\int_{-\infty}^{\infty} (\int_0^{\lceil N^{\frac{1}{k}} \rceil} e(t^k z) dt)^s e(-Mz) dz \\
&= \int_{-\infty}^{\infty} (\int_0^1 \frac{1}{k} \lceil N^{\frac{1}{k}} \rceil \tilde{t}^{\frac{1}{k}-1} e(\tilde{t} \lceil N^{\frac{1}{k}} \rceil^k z) d\tilde{t})^s e(-Mz) dz \\
&= \int_{-\infty}^{\infty} \lceil N^{\frac{1}{k}} \rceil^s k^{-s} (\int_0^1 \tilde{t}^{\frac{1}{k}-1} e(\tilde{t} \lceil N^{\frac{1}{k}} \rceil^k z) d\tilde{t})^s e(-Mz) dz \\
&= \lceil N^{\frac{1}{k}} \rceil^{s-k} k^{-s} \int_{-\infty}^{\infty} (\int_0^1 \tilde{t}^{\frac{1}{k}-1} e(\tilde{t}\tilde{z}) d\tilde{t})^s e(-\frac{M}{\lceil N^{\frac{1}{k}} \rceil^k} \tilde{z}) d\tilde{z} \\
&= \lceil N^{\frac{1}{k}} \rceil^{s-k} k^{-s} \frac{\Gamma(\frac{1}{k})^s}{\Gamma(\frac{s}{k})} \theta^{\frac{s}{k}-1}.
\end{aligned}$$

Here we do the change of variables $t^k = \tilde{t} \lceil N^{\frac{1}{k}} \rceil^k$ and $z \lceil N^{\frac{1}{k}} \rceil^k = \tilde{z}$. The last equality follows from theorem 8.5 as in the proof of theorem 3.4, where $\theta :=$

$\frac{M}{\lceil N^{\frac{1}{k}}\rceil^k} \in [\frac{1}{5}, 1]$. Since everything but the $\lceil N^{\frac{1}{k}}\rceil^{s-k}$ term is a constant depending on $k$ and $s$, we get that the expression above is bounded by:

$$\gg N^{\frac{s}{k}-1}.$$

Now, we bound the expression before the integral in (59),

$$\sum_{q \leq N^{\frac{1}{2k}}} \sum_{\substack{a \leq q \\ (a,q)=1}} q^{-s}(\sum_{c=1}^{q} e(\frac{a}{q}c^k))^s = \mathfrak{S}(N) + \mathcal{O}(N^{\frac{1}{2k}(-\frac{s}{k}+1)}) = \mathfrak{S}(N) + \mathcal{O}(N^{-\frac{s-k}{2k^2}}).$$

We know by the end of last chapter the singular series is bounded from below, and the error is smaller than our previous error term for $s \geq 4k$, so we obtain the following proposition:

**Proposition 6.1.** *Let $\mathfrak{M}$ be defined as in (56), let $s \geq 4k$ and $M$ be an integer in the interval $[\frac{1}{5}[N^{\frac{1}{k}}]^k, [N^{\frac{1}{k}}]^k]$, then*

$$\int_{\mathfrak{M}} (\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y))^s e(-My) dy \gg \lceil N^{\frac{1}{k}}\rceil^{s-k}$$

Our next goal it to estimate the $w_1$, $w_2$ and $\tilde{w}$ terms of (53), with that in mind, we prove the following lemma:

**Lemma 6.2.** *The number of integers smaller or equal than $P$ that can be written as a sum of $\tilde{k}$ $k$th powers is:*

$$\gg P^{1-(1-\frac{1}{k})^{\tilde{k}}} \tag{60}$$

*Proof.* We prove the statement by induction on $\tilde{k}$. Let $\tilde{k} = 1$, then the number of integers is trivially $[P^{\frac{1}{k}}]$. The term in (60) becomes $P^{1-(1-\frac{1}{k})} = P^{\frac{1}{k}}$, the base case now follows from $[P^{\frac{1}{k}}] \gg P^{\frac{1}{k}}$.
Now, we assume the statement holds for $\tilde{k} \leq n-1$. We let $w$ be an integer representable as a sum of $n-1$ $k$th powers, then the numbers we are searching for are of the form $x^k + w$, we consider the subset where $w \in [0, \frac{1}{2}P^{1-\frac{1}{k}}]$ and $x \in [(\frac{1}{4}P)^{\frac{1}{k}}, (\frac{1}{2}P)^{\frac{1}{k}}]$, we first show these numbers are all distinct. Assume there exist an integer with two different representations $x_1^k + w_1$ and $x_2^k + w_2$ and assume without loss of generality that $x_1 > x_2$. Then we have $x_1^k - x_2^k = w_2 - w_1$.

$$w_2 - w_1 < \frac{1}{2}P^{1-\frac{1}{k}}$$

$$x_1^k - x_2^k \geq (x_2+1)^k - x_2^k > kx_2^{k-1} \geq k(\frac{P}{4})^{1-\frac{1}{k}} > P^{1-\frac{1}{k}}.$$

Therefore there can not be an integer with two different representations of this form. Since the number of different $w$'s is $\gg$ (60) for $\tilde{k} = n-1$ and $P = \frac{1}{2}P^{1-\frac{1}{k}}$, and the number of $x$'s is $\gg P^{\frac{1}{k}}$, we have that the number of integers that can be represented as a sum of $n$ $k$th powers is

$$\gg P^{\frac{1}{k}}(P^{1-\frac{1}{k}})^{1-(1-\frac{1}{k})^{n-1}} = P^{1-\frac{1}{k}-(1-\frac{1}{k})^n+\frac{1}{k}} = P^{1-(1-\frac{1}{k})^n}$$

□

**Lemma 6.3.** *Let $X'$ and $Y'$ be sets of integers contained in intervals of length $X$ and $Y$ respectively. If $z = \frac{a}{q} + \mathcal{O}(q^{-2})$, then the following bound holds:*

$$|\sum_{x\in X'}\sum_{y\in Y'} e(xyz)|^2 \ll (\#X')(\#Y')\frac{\log q}{q}(q+X)(q+Y).$$

*Proof.* We start by using Cauchy-Schwarz, yielding:

$$\begin{aligned}|\sum_{x\in X'}\sum_{y\in Y'} e(xyz)|^2 &\le (\sum_{x\in X'} 1)(\sum_{x\in X'}|\sum_{y\in Y'} e(xyz)|^2) \\ &= (\#X')(\sum_{x\in X'}\sum_{y\in Y'}\sum_{y'\in Y'} e(zx(y-y'))).\end{aligned}$$

Next, we extend the sum from $X'$ to all integers belonging to the interval of length $X$ and note that for each fixed pair $y, y'$ the sum is bounded by the numbers of integers in the interval of length $X$ and by a bound derived in (4) (we recall the notation $||z||$ denotes the distance from $z$ to the closest integer), yielding

$$|\sum_{x\in X'}\sum_{y\in Y'} e(xyz)|^2 \le (\#X')\sum_{y\in Y'}\sum_{y'\in Y'}\min(X, \frac{1}{2||z(y-y')||}). \tag{61}$$

We note that $y - y'$ is at most $Y$, since that's the length of the interval, and that each value can be attained at most $\#Y'$ times, yielding that (61) is less or equal than:

$$(\#X')(\#Y')\sum_{j=-[Y]}^{[Y]}\min(X, \frac{1}{2||zj||}) \ll (\#X')(\#Y')\sum_{j=-[Y]}^{[Y]}\min(X, \frac{1}{||zj||}). \tag{62}$$

We split up the sum into sums of $q$ consecutive terms, since $z = \frac{a}{q} + \mathcal{O}(q^{-2})$. We get that there exist at least $\frac{Y}{q}+1$ of this sums yielding:

$$\begin{aligned}
&(\#X')(\#Y') \sum_{j=-[Y]}^{[Y]} \min(X, \frac{1}{2||zj||}) \\
&\ll (\#X')(\#Y')(\frac{Y}{q}+1) \sum_{j=1}^{q} \min(X, \frac{1}{||zj||}) \\
&= (\#X')(\#Y')(\frac{Y}{q}+1) \sum_{j=1}^{q} \min(X, \frac{1}{||(\frac{a}{q}+\mathcal{O}(q^{-2}))(j+\tilde{q})||}) \\
&= (\#X')(\#Y')(\frac{Y}{q}+1) \sum_{j=1}^{q} \min(X, \frac{1}{||(\frac{aj+\tilde{q}}{q}+\mathcal{O}(q^{-1}))||}).
\end{aligned} \tag{63}$$

where $\tilde{q}$ is some multiple of $q$, since $a$ and $q$ are relatively prime, $aj+\tilde{q}$ runs over a complete set of residues module $q$, implying $||(\frac{aj+\tilde{q}}{q}+\mathcal{O}(q^{-1}))|| \gg \frac{|r|}{q}$, where $r$ denotes the integer belonging to the interval $(-\frac{q}{2}, \frac{q}{2}]$ that is congruent to $aj+\tilde{q}$, using this and taking $X$ in the minimum when $r=0$ we get that (63) is

$$\begin{aligned}
\ll (\#X')(\#Y')(\frac{Y}{q}+1)(N+\sum_{r=1}^{[\frac{q}{2}]} \frac{q}{r}) \ll (\#X')(\#Y')(\frac{Y}{q}+1)(N+q\log(q)) \\
= (\#X')(\#Y')\frac{\log(q)}{q}(Y+q)
\end{aligned}$$

□

**Proposition 6.4.** *Let $\mathfrak{m}$ be the complement of the major arcs defined in (56) and let $U_{\tilde{k}}(X)$ denote the number of integers up to $X$ that can be represented as a sum of $\tilde{k}$ kth powers, then the following holds:*

$$\begin{aligned}
&\int_{\mathfrak{m}} |\sum_{x=1}^{\lceil N^{\frac{1}{k}} \rceil} e(x^k y)||\sum_{w} e(yw)|^2| \sum_{\tilde{w}} \sum_{\tilde{x}} e(y\tilde{w}\tilde{x}^k)|dy \\
&\ll N^3 U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}} \rceil^k)^2 U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}} \rceil^{k-\frac{1}{2}}) \lceil N^{\frac{1}{k}} \rceil^{\frac{1}{2k}} \lceil N^{\frac{1}{k}} \rceil^{-\frac{1}{12k}}
\end{aligned} \tag{64}$$

*for all $\tilde{k} \geq 2k\log(3k)$ and some $\delta > 0$.*

*Proof.* For any fixed $y \in [0,1]$, we can apply theorem 8.3 with $j=1$, $\alpha = z$ and $N = 2k\lceil N^{\frac{1}{k}} \rceil^{k-1}$, obtaining that for any $y$ in the interval $[0,1]$, there exist $q \in [1, 2k\lceil N^{\frac{1}{k}} \rceil^{k-1}]$, and $a$ relatively prime to $q$ such that $|y - \frac{a}{q}| < (2kq\lceil N^{\frac{1}{k}} \rceil^{k-1})^{-1}$. The terms belonging to the minor arcs are the ones for which $q > \lceil N^{\frac{1}{k}} \rceil^{\frac{1}{2}}$. Notice $(2kq\lceil N^{\frac{1}{k}} \rceil^{k-1})^{-1} \ll q^{-2}$, therefore we can apply lemma 6.3 to the innermost double sum of (64).

We let $X'$ be the set of integers up to $\frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^{k-\frac{1}{2}}$ that can be represented as a sum of $\tilde{k}$ $k$th powers, $Y'$ be the integers contained in the interval $[1, \lceil N^{\frac{1}{k}}\rceil^{\frac{1}{2k}}]$, $X = \frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^{k-\frac{1}{2}}$ and $Y = \lceil N^{\frac{1}{k}}\rceil^{\frac{1}{2}}$ (notice $Y < q$), and the lemma yields:

$$\begin{aligned} |\sum_{\tilde{w}}\sum_{\tilde{x}} e(y\tilde{w}\tilde{x}^k)|^2 &\ll (\#X')(\#Y')\frac{\log q}{q}(q+X)(q+Y) \\ &\ll (\#X')\lceil N^{\frac{1}{k}}\rceil^{\frac{1}{2k}}\lceil N^{\frac{1}{k}}\rceil^{k-\frac{1}{2}}\lceil N^{\frac{1}{k}}\rceil^{\epsilon}, \end{aligned}$$

$\forall\epsilon > 0$ notice $\log(q) \leq \log(2k\lceil N^{\frac{1}{k}}\rceil^{k-1}) \ll \lceil N^{\frac{1}{k}}\rceil^{\epsilon}$.

$$\begin{aligned} |\frac{\sum_{\tilde{w}}\sum_{\tilde{x}} e(y\tilde{w}\tilde{x}^k)}{(\#X')\lceil N^{\frac{1}{k}}\rceil^{\frac{1}{2k}}}|^2 &\ll (\#X')^{-1}\lceil N^{\frac{1}{k}}\rceil^{-\frac{1}{2k}}\lceil N^{\frac{1}{k}}\rceil^{k-\frac{1}{2}+\epsilon} \\ &\ll (\lceil N^{\frac{1}{k}}\rceil)^{1-(1-\frac{1}{k})^{\tilde{k}}-1}\lceil N^{\frac{1}{k}}\rceil^{-\frac{1}{2k}(k-\frac{1}{2}}\lceil N^{\frac{1}{k}}\rceil^{k-\frac{1}{2}+\epsilon} \\ &\ll \lceil N^{\frac{1}{k}}\rceil^{(k-\frac{1}{2})(1-(1-\frac{1}{k})^{\tilde{k}}-\frac{1}{2k}+\epsilon}. \end{aligned}$$

Here we use lemma 6.2 to estimate $(\#X')$. It now follows from the equation above that:

$$|\sum_{\tilde{w}}\sum_{\tilde{x}} e(y\tilde{w}\tilde{x}^k)| \ll (\#X')\lceil N^{\frac{1}{k}}\rceil^{\frac{1}{2k}}\lceil N^{\frac{1}{k}}\rceil^{\frac{1}{2}(k-\frac{1}{2})(1-(1-\frac{1}{k})^{\tilde{k}}-\frac{1}{4k}},$$

since $\epsilon$ is arbitrary. Next we bound the term:

$$\int_0^1 |\sum_{w} e(yw)|^2 dy = U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^k) \ll U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^k)^2\lceil N^{\frac{1}{k}}\rceil^{-k(1-\frac{1}{k})^{\tilde{k}}}. \tag{65}$$

The first step holds for any exponential sum (a more detailed derivation is done in (12), note the sum is over $U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^k)$ terms. And the second step follows from lemma 6.2. Now we use (6), (65) to bound (64):

$$\begin{aligned} &\ll (\lceil N^{\frac{1}{k}}\rceil^{4k})(U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^k)^2\lceil N^{\frac{1}{k}}\rceil^{-k(1-\frac{1}{k})^{\tilde{k}}})((\#X')\lceil N^{\frac{1}{k}}\rceil^{\frac{1}{2k}}\lceil N^{\frac{1}{k}}\rceil^{\frac{1}{2}(k-\frac{1}{2})(1-(1-\frac{1}{k})^{\tilde{k}}-\frac{1}{4k}}) \\ &= (\lceil N^{\frac{1}{k}}\rceil^{3k})U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^k)^2 U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^{k-\frac{1}{2}})\lceil N^{\frac{1}{k}}\rceil^{\frac{1}{2k}}\lceil N^{\frac{1}{k}}\rceil^{\frac{3}{2}k(1-\frac{1}{k})^{\tilde{k}}-\frac{1}{4k}}. \end{aligned}$$

Now we just need to bound the term $\lceil N^{\frac{1}{k}}\rceil^{\frac{3}{2}k(1-\frac{1}{k})^{\tilde{k}}-\frac{1}{4k}}$ for $\tilde{k} \geq 2k\log(3k)$.

$$\frac{3}{2}k(1-\frac{1}{k})^{\tilde{k}} - \frac{1}{4k} < -\frac{1}{12k},$$

here we use $(1-\frac{1}{k})^{\tilde{k}} < \frac{1}{9k^2}$, to see this take logarithms on the left side yielding:

$$\tilde{k}\log(1-\frac{1}{k}) < \frac{-\tilde{k}}{k} \leq \frac{-2k\log(3k)}{k} = -2\log(3k).$$

□

**Theorem 6.5.**

$$G(k) \leq 4k + 6\log(3k) + 3$$

*Proof.* We first analyze he major arc portion of (55)

$$\sum_{w_1}\sum_{w_2}\sum_{\tilde{w}}\sum_{\tilde{x}}\int_{\mathfrak{M}}(\sum_{x=1}^{[N^{\frac{1}{k}}]} e(x^k y))^{4k} e(-(N-(w_1+w_2+\tilde{w}\tilde{x}))y)dy \\ \gg \lceil N^{\frac{1}{k}}\rceil^{3k} U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^k)^2 U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^{k-\frac{1}{2}})\lceil N^{\frac{1}{k}}\rceil^{\frac{1}{2k}}. \tag{66}$$

This inequality follows from applying proposition 6.1 with $s = 4k$ for each term of the sum and noting that the number of terms in the sum is $U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^k)^2 U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^{k-\frac{1}{2}})\lceil N^{\frac{1}{k}}\rceil^{\frac{1}{2k}}$.
The bound derived in proposition 6.4 is of lower order than (66) for $\tilde{k} \geq 2k\log(3k)$. It follows that

$$r(N) \gg \lceil N^{\frac{1}{k}}\rceil^{3k} U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^k)^2 U_{\tilde{k}}(\frac{1}{4}\lceil N^{\frac{1}{k}}\rceil^{k-\frac{1}{2}})\lceil N^{\frac{1}{k}}\rceil^{\frac{1}{2k}}.$$

The term above goes to $\infty$ as $N \to \infty$.
It follows that $G(k) \leq 4k + 3\tilde{k} \leq 4k + 6\log(3k) + 3$ □

# 7 Recent developments

The following section is based on [9]
In 1935, Vinogradov introduced a new method of evaluating exponential sums, the Vinogradov Mean Value method.

**Definition 7.1.** *Let $s, k \geq 1$, then $J_{s,k}(X)$ denotes the number of solutions of the system of equations:*

$$\begin{cases} x_1 + ... + x_s = x_{s+1} + ... + x_{2s} \\ x_1^2 + ... + x_s^2 = x_{s+1}^2 + ... + x_{2s}^2 \\ ... \\ x_1^k + ... + x_s^k = x_{s+1}^k + ... + x_{2s}^k, \end{cases} \tag{67}$$

*satisfying $x_i \in [1, X] \cap \mathbb{Z}$, $\forall i = 1, 2, ..., 2s$. $J_{k,s}$ can also be expressed analytically:*

$$J_{k,s} = \int_{(0,1]^k} |f_k(\boldsymbol{\alpha}; X)|^{2s} d\boldsymbol{\alpha},$$

*where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_k)$*

$$f_k(\boldsymbol{\alpha}; X) = \sum_{1 \leq x \leq X} e(\alpha_1 x + ... + \alpha_k x^k).$$

The following conjecture is the central conjecture of the method.

**Conjecture 7.2.** *$\forall s, k \geq 1$, we have the following bound:*

$$J_{s,k}(X) \ll X^{\epsilon}(X^{s} + X^{2s-\frac{1}{2}k(k+1)}) \tag{68}$$

*$\forall X \geq 1$ and $\epsilon > 0$.*

Wooley proved, in 2014, that the conjecture above for the case $k = 3$ and $s \geq 1$, using an efficient congruencing method, the proof can be seen in [4].
In December of 2015 Bourgain, Demeter and Guth proved the conjecture holds for $k \geq 4$ and $s \geq 1$, using $\ell^2$ decoupling methods (a new area in harmonic analysis), the proof can be seen in [5].

Vinogradov first noticed that:

$$\int_0^1 |\sum_{x=1}^{X} e(\alpha x^k)|^{2s} d\alpha$$

corresponds to the number of solutions of:

$$x_1^k + ... + x_s^k = x_{s+1}^k + ... + x_{2s}^k, \tag{69}$$

with all $x_i \leq X$. It follows that:

$$\int_0^1 |\sum_{x=1}^{X} e(\alpha x^k)|^{2s} d\alpha = \sum_{|h_1|<sX} ... \sum_{|h_{s-1}|<sX^{k-1}} \int_{[0,1)^k} |f_k(\boldsymbol{\alpha}; X)|^{2s} \cdot$$
$$\cdot e(-h_1\alpha_1 - ... - h_{k-1}\alpha_{k-1}) d\boldsymbol{\alpha}.$$

To motivate this, note that for each $1 \leq l < k$, $x_1^l + ... x_s^l - x_{s+1}^l - ... - x_{2s}^l$ is equal to an integer with absolute value less or equal than $sX^l$, so by summing over all integers such integers for each $l$, the integral will yield 1, for each solution of (69). It now follows from triangular inequality that:

$$\int_0^1 |\sum_{x=1}^{X} e(\alpha x^k)|^{2s} d\alpha \ll_{s,k} X^{\frac{1}{2}k(k-1)} J_{s,k}(X).$$

If we now apply the recently proved theorem, we obtain:

$$X^{\frac{1}{2}k(k-1)} J_{s,k}(X) \ll X^{\frac{1}{2}k(k-1)} X^{\epsilon}(X^{s} + X^{2s-\frac{1}{2}k(k+1)}) \tag{70}$$

Picking $s = \frac{k(k+1)}{2}$ yields:

$$X^{\epsilon} X^{\frac{1}{2}k(k-1)}(X^{\frac{1}{2}k(k+1)} + X^{\frac{1}{2}k(k+1)}) \ll X^{k^2+\epsilon}.$$

So we improve the bound of Hua's Lemma.

We can now improve lemma 3.2 to $s \geq k(k+1)$, following the proof of this lemma, we get:

$$\int_{\mathfrak{m}} |\sum_{x=1}^{X} e(\alpha x^k)|^s d\alpha \leq \max_{\alpha \in \mathfrak{m}} |\sum_{x=1}^{X} e(\alpha x^k)|^{s-k(k+1)} \int_0^1 |\sum_{x=1}^{X} e(\alpha x^k)|^{k(k+1)} d\alpha$$

$$\ll \max_{\alpha \in \mathfrak{m}} |\sum_{x=1}^{X} e(\alpha x^k)|^{s-k(k+1)} X^{k^2+\epsilon} \ll (X^{1+\epsilon'-\delta})^{s-k(k+1)} X^{k^2+\epsilon} \ll X^{s-k+\epsilon-\delta}. \tag{71}$$

Obtaining a much better bound on $s$ for the minor arcs portion of the asymptotic formula. Combining this bound with the convergence of $\mathfrak{S}$ showed in section 5, we obtain the bound $\tilde{G}(k) \leq k(k+1)$.
Now, we will handle an improvement of Weyl's inequality. The following theorem is theorem 5.2 in [6].

**Theorem 7.3.** *If $j, a$ and $q$ are such that $2 \leq j \leq k$, $|\alpha_j - \frac{a}{q}| \leq q^{-2}$, $a$ and $q$ are relatively prime and $q \leq X^j$, then*

$$|f_k(\boldsymbol{\alpha}; X)| \ll (X^{\frac{1}{2}k(k-1)} J_{s,k-1}(2X)(q^{-1} + X^{-1} + qX^{-j}))^{\frac{1}{2s}} \log(2X) \tag{72}$$

Using the recently proved conjecture, we can further bound (72):

$$\ll (X^{\frac{1}{2}k(k-1)}((2X)^{\epsilon}((2X)^s + (2X)^{2s-\frac{1}{2}k(k-1)})(q^{-1} + X^{-1} + qX^{-j}))^{\frac{1}{2s}} \log(2X).$$

Picking $s = \frac{k(k-1)}{2}$ and using the bound $\log(X) \ll X^{\epsilon}$, yields that the previous expression is

$$\ll X^{\epsilon}(X^{k(k-1)} + X^{k(k-1)})(q^{-1} + X^{-1} + qX^{-j})^{\frac{1}{k(k-1)}} \ll X^{1+\epsilon}(q^{-1} + X^{-1} + qX^{-j})^{\frac{1}{k(k-1)}}.$$

In 2012 Wooley, showed that assuming the conjecture, $\tilde{G}(k) \leq k^2 + 1 - [\frac{\log(k)}{\log(2)}]$, for all $k \geq 3$, this can be seen in [7].
In 2016, Bourgain [8], prove a further improvement to Hua's lemma, using a decoupling method, showing the following inequality holds:

$$\int_0^1 |\sum_{x=1}^{X} e(\alpha x^k)|^{\ell(\ell+1)} d\alpha \ll X^{\ell^2+\epsilon},$$

for any $1 \leq \ell \leq k$. Improving the bound to $\tilde{G}(k) \leq k^2 - k + \mathcal{O}(\sqrt{k})$.

# 8 Appendix

The following is a specific case of Euler summation formula, for a proof of it, the reader is referenced to [3].

**Lemma 8.1.** *Let $f : [n, m] \to \mathbb{R}$ be a continuously differentiable function, then:*

$$\sum_{i=n}^{m} f(i) = \int_n^m f(x)dx + \frac{1}{2}(f(n) + f(m)) + \int_n^m f'(x)P_1(x)dx$$

*where $P_1$ denotes the first Bernoulli periodic function (also known as the saw-tooth function), i.e.*

$$P_1(x) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z}. \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

**Lemma 8.2** (Van der Corput's Lemma)**.** *Let $I$ be a closed interval of the real line and $f$ be a real valued function that is $C^2$ on $I$ satisfying:*

$$0 \leq f'(x) \leq \frac{1}{2} \qquad\qquad f''(x) \geq 0 \quad \forall x \in I$$

*Then*

$$\sum_{n \in I} e(f(n)) = \int_I e(f(x))dx + \mathcal{O}(1) \tag{73}$$

*Proof.* We note that we can multiply both sides of (73) by a constant $e(c)$, and obtain $\sum_{n\in I} e(f(n) + c) = \int_I e(f(x) + c)dx + \mathcal{O}(1)$. Since the first and second derivatives of $f(x) + c$ agree with the first and second derivatives of $f(x)$ we are free to choose $c$ such that $\sum_{n\in I} e(f(n)) - \int_I e(f(x))dx$ is a real number. Therefore, we can replace $e(f(x))$ with $\cos(2\pi f(x))$.
Applying Lemma 8.1 to the sum yields:

$$\begin{aligned}\sum_{n\in I} e(f(n)) &= \int_N^M e(f(x))dx + \frac{1}{2}\Big(e(f(N)) + e(f(M))\Big) + \int_N^M (e(f(x)))'P_1(x)dx \\ &= \int_I e(f(x))dx + \int_N^M (e(f(x)))'P_1(x)dx + \mathcal{O}(1),\end{aligned}$$

where $N$ and $M$, denote the smallest and largest integer contained in $I$. In the second step, we use that $|e(f(x))| \leq 1$ to extend the integral and bound the terms $e(f(N))$ and $e(f(M))$.
Now we analyze the term:

$$\begin{aligned} |\int_N^M (e(f(x)))'P_1(x)dx| &= |\int_N^M \sin(2\pi f(x))2\pi f'(x)P_1(x)dx| \\ &= |\int_N^M \sin(2\pi f(x))2\pi f'(x)\sum_{z=1}^{\infty}\frac{\sin(2\pi zx)}{\pi z}dx| \\ &= 2|\sum_{z=1}^{\infty} z^{-1}\int_N^M f'(x)\sin(2\pi f(x))\sin(2\pi zx)dx| \\ &= |\sum_{z=1}^{\infty} z^{-1}\int_N^M f'(x)(\cos(2\pi(f(x) - zx)) - \cos(2\pi(f(x) + zx)))dx|. \end{aligned} \tag{74}$$

In equation above, we use that the fourier series representation of $x - [x] - \frac{1}{2}$ is $-\sum_{z=1}^{\infty} -\frac{\sin(2\pi zx)}{z\pi}$, the sum is absolutely convergent so we are allowed to switch of order of integration and summation, and the trigonometric identity $2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b)$.
The integral:

$$\begin{aligned} |\int_N^M f'(x)\cos(2\pi(f(x) \pm zx))| &= |\int_N^M f'(x)\frac{\frac{d}{dx}(\sin(2\pi(f(x) \pm zx)))}{2\pi(z \pm f'(x))}dx| \\ &\leq \frac{1}{\pi}|\int_N^M f'(x)(z \pm f'(x))^{-1}dx| \end{aligned} \tag{75}$$

The last inequality follows from,

$$\begin{aligned} \frac{d}{dx}f'(x)(z \pm f'(x))^{-1} &= \frac{f''(x)(z \pm f'(x)) - \pm f''(x)f'(x)}{(z \pm f'(x))^2} \\ &= \frac{f''(x)z}{(z \pm f'(x))^2} \geq 0 \qquad \forall z \in \mathbb{Z}^+, \end{aligned}$$

along with $|\int \frac{d}{dx}(\sin(2\pi(f(x) \pm zx)))| \leq 2$. We follow up the bound on (75),

$$\begin{aligned} \frac{1}{\pi}|\int_N^M f'(x)(z \pm f'(x))^{-1}dx| &\leq \frac{1}{\pi}\int_N^M \frac{1}{2}(z - \frac{1}{2})^{-1}dx \\ &= (\pi(2z-1))^{-1}(M-N) \ll (\pi(2z-1))^{-1} \end{aligned} \tag{76}$$

We can apply (76) to (74) yielding:

$$\begin{aligned} &|\sum_{z=1}^{\infty} z^{-1}\int_N^M f'(x)(\cos(2\pi(f(x) - zx)) - \cos(2\pi(f(x) + zx)))dx| \\ &\ll \sum_{z=1}^{\infty} z^{-1}2(\pi(2z-1))^{-1} = \sum_{z=1}^{\infty} \frac{2}{z(2z-1)\pi} = \mathcal{O}(1) \end{aligned}$$

□

The following theorem is proved in [3]

**Theorem 8.3** (Dirichlet's approximation theorem)**.** *For any real numbers $\alpha_1, ..., \alpha_j, N \geq 2$, $\exists\ a_1, ..., a_j, q \in \mathbb{Z}$, such that $1 \leq q \leq N$, and $|q\alpha_j - a_j| \leq N^{-\frac{1}{j}}$*

The following theorem is stated and proved in [1],

**Theorem 8.4** (Dirichlet's test)**.** *If $f$ is a continuous function on $[a, \infty)$ and $\exists M > 0$ such that $|\int_a^b f(x)dx| \leq M$ holds $\forall b \in [a, \infty)$ and $g$ is a differentiable function on $[a, \infty)$, such that $g(x) \geq 0$ and $g'(x) \leq 0$ hold $\forall x \in [a, \infty)$ and $\lim_{x \to \infty} g(x) = 0$, then $\int_a^{\infty} f(x)g(x)dx$ converges.*

For the following theorem, we refer the reader to [10]

**Theorem 8.5** (Fourier integral theorem)**.** *Let $f$ be a continuous function on the interval $[a, b]$ of bounded variation, i.e:*

$$\sup_{P\in\mathcal{P}} \sum_{j=0}^{n_p-1} |f(x_{j+1}) - f(x_j)| < \infty, \tag{77}$$

*where $\mathcal{P}$ denotes the set of partitions of $[a, b]$, and let $c \in (a, b)$ then:*

$$\lim_{\lambda\to\infty} \int_a^b f(t)\frac{\sin(2\pi\lambda(t-c))}{\pi(t-c)}dt = f(c).$$

**Remark.** *Note that if $f$ is differentiable and its derivative is integrable on $[a, b]$, then (77) is equivalent to:*

$$\int_a^b |f'(x)|dx < \infty.$$

The following proposition, is proved in stated and proved in [3],

**Proposition 8.6.** *Let $f$ be an arithmetic multiplicative function and one of*

$$\sum_{n=1}^{\infty} |f(n)| \qquad\qquad \prod_{\substack{p\\ prime}} \sum_{k=0}^{\infty} |f(p^k)|$$

*be convergent, then both converge and:*

$$\sum_{n=1}^{\infty} f(n) = \prod_{\substack{p\\ prime}} \sum_{k=0}^{\infty} f(p^k)$$

Now, we will state some definitions and results from chapter 3 of [3]:

**Definition 8.7** (Gauss sum)**.** *Let $\chi$ be a Dirichlet character modulo $q$, then a Gauss sum is defined in the following way:*

$$\tau(\chi, n) := \sum_{m=1}^{q} \chi(m)e\Big(\frac{mn}{q}\Big).$$

**Proposition 8.8.** *Let $\chi$ be a Dirichlet character modulo $q$. If $gcd(q, n) = 1$, then*

$$\tau(\chi, n) = \overline{\chi(n)}\tau(\chi, 1).$$

**Proposition 8.9.** *Let $\chi$ be a primitive character modulo $q$, then $|\tau(\chi, 1)| = \sqrt{q}$*

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