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2014

Document Version:
Publisher's PDF, also known as Version of record

[Link to publication](#)

Citation for published version (APA):
Kristensson, G. (2014). *Multiple scattering by a collection of randomly located obstacles Part I: Theory - coherent fields*. (Technical Report LUTEDX/(TEAT-7235)/1-52/(2014); Vol. TEAT-7235). The Department of Electrical and Information Technology.

Total number of authors:
1

Multiple scattering by a collection of randomly located obstacles
Part I: Theory — coherent fields

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Abstract

Scattering of electromagnetic waves by discrete, randomly distributed objects is addressed. In general, the non-intersecting scattering objects can be of arbitrary form, material and shape. The main aim of this paper is to calculate the coherent reflection and transmission characteristics of a finite or semi-infinite slab containing discrete, randomly distributed scatterers. Typical applications of the results are found at a wide range of frequencies (radar up to optics), such as attenuation of electromagnetic propagation in rain, fog, and clouds *etc*. The integral representation of the solution of the deterministic problem constitutes the underlying framework of the stochastic problem. Conditional averaging and the employment of the Quasi Crystalline Approximation lead to a system of integral equations in the unknown expansion coefficients. Of special interest is the slab geometry, which implies a system of integral equations in the depth variable. Explicit solutions for tenuous media and low frequency approximations can be obtained for spherical obstacles.

1   Introduction

Multiple scattering of electromagnetic waves by a discrete collection of scatterers has a long and well-documented history in the literature. The topic is covered in detail in several textbooks and the interested reader is referred to the excellent treatments in *e.g.*, [16, 17, 25, 29, 44–46] for discussions on the subject.

The journal literature is also extensive. The starting point of multiple scattering can be set to the pioneer work by Foldy [13]. Several important works followed, see *e.g.*, [2, 5, 12, 21, 22, 26, 28, 38, 42, 43, 47–50, 52, 54, 57–59], and further references to the subject are found therein. Some of the theories are tested experimentally [18, 24, 59].

The main stress in the cited literature above, is on finding the effective electromagnetic properties, *i.e.*, the bulk permittivity and the permeability, of the many-scatterer system. This is effectively done by the introduction of an exponential trial solution in the final equations, which leads to a determinant system in the unknowns. The field inside and outside the scattering region are then computed. One purpose of this paper is to develop an alternative, computationally effective method that does not rely on the homogenized properties of the problem, but has a potential to be useful at frequencies outside the frequency region of homogenization.

The analysis presented in this paper shows initial similarities with previous treatments of the topic, see *e.g.*, [42], but then proceeds along a somewhat different route, and the transmission and reflection properties of a slab geometry are in focus. The transmitted and reflected intensities are conveniently represented as a sum of two terms — the coherent and the incoherent contributions. In this paper we focus on the analysis of the coherent term. The remaining part — the incoherent or diffuse part — is postponed to a future paper.

A system of integral equations is identified, and the reflected and transmitted (coherent) fields are identified. Integral equations, especially if the kernel is smooth, have well established properties and are easy to solve. This is a major advantage of the approach presented in this paper.

If desired, the transmitted field may then be used to find the effective bulk material parameters, and, consequently, the field inside the material. This procedure also introduces a means of estimating the accuracy of the bulk property approximation by comparing the accuracy of the transmitted and reflected field with the corresponding results obtained by a homogeneous slab. If the bulk material parameters model the material appropriately, both transmission and reflection fields computed with the two methods agree. This procedure is illustrated numerically in a subsequent paper [15].

The transmitted coherent contribution is frequently modeled by the use of the Radiative Transfer Equation (RTE) — more precisely the Bouguer-Beer law[1] [7, 16, 29]. The RTE is derived under certain assumptions, *e.g.*, the scatterers are far enough apart, so that they are located in the far field of all other scatterers, and plane wave excitation of the scatterers (single scatterer extinction cross section is used). These assumptions are not made in this paper. In this respect, the present analysis generalizes this law.

The paper is organized as follows. In Section 2, the deterministic analysis of the multiple electromagnetic scattering is given and solved with the use of spherical vector wave and their translation properties. This analysis follows to a large extent the pioneer work made by Peterson and Ström [38] in the early 70's. The final expressions, however, differ, due to different aim of the analysis. The stochastic description is made in Section 3. Conditional averaging and the employment of the Quasi Crystalline Approximation lead to a system of integral equations in the unknown expansion coefficients. The details of this analysis is given in Section 4. The pertinent system of integral equations for the slab (and the half space) is developed in Section 5, and two natural and important approximations — tenuous media and low-frequency approximations — are developed in Section 6. The paper ends with a short conclusion in Section 7 and several useful appendices.

2 The null-field approach to a collection of scatterers

We study a collection of N different scatterers, where each scatterer is centered at the location $\boldsymbol{r}_p$, defining the position of the local origin O_p, $p = 1, 2, \ldots, N$, relative the global origin O, see Figure 1. The radii of the maximum inscribed and minimum circumscribed spheres, both centered at the local origin, of each scatterer are denoted a_p and A_p, $p = 1, 2, \ldots, N$, respectively. The scatterers are located in a lossless, homogeneous, isotropic media with permittivity ϵ and permeability μ. The wave number and relative wave impedance are k and η, respectively — both real numbers. We assume that no circumscribing spheres intersect. Each scatterer has its own material properties, which don't have to be the same for all scatterers. The entire collection of scatterers is enclosed by a sphere (circumscribed sphere of

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[1] Also known as Beer's law, the Beer-Lambert law, the Lambert-Beer law, or the Beer-Lambert-Bouguer law *etc.*, but it is incorrect to accredit Lambert to this law, since Bouguer made the original contributions [4].
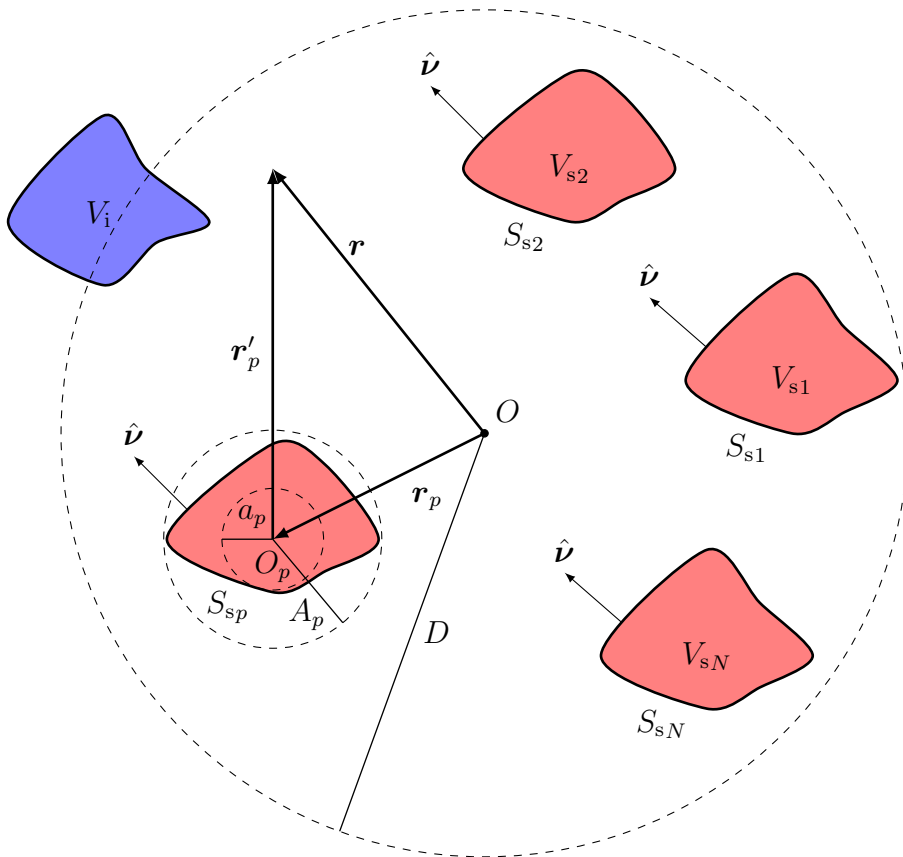
Figure 1: The geometry of a collection of the N scatterers and the region of prescribed sources V_i. The positions of the local origins are $\boldsymbol{r}_p$, $p = 1, \ldots, N$, and the radii of the maximum inscribed and the minimum circumscribed spheres of each local scatterer are a_p and A_p, respectively.

the collection) with radius D. The prescribed sources are located in the region V_i, which is a region disjoint to all scatterers,[2] and these sources generate the field $\boldsymbol{E}_i(\boldsymbol{r})$ everywhere outside V_i.

We start with a collection of perfectly conducting obstacle, and then generalize to more complex scatterers below. As we see below, this assumption is no loss of generality. It is only to make the derivation of the results more simple. The final result holds for a collection of scatterers with arbitrary materials.

Each scatterer is characterized by the boundary condition — the total electric field, $\boldsymbol{E} = \boldsymbol{E}_i(\boldsymbol{r}) + \boldsymbol{E}_s(\boldsymbol{r})$, satisfies $\hat{\boldsymbol{\nu}} \times \boldsymbol{E} = \boldsymbol{0}$ on all S_{sp}, $p = 1, 2, \ldots, N$. Our

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[2]More precisely, the circumscribing sphere of the source region must not include any local origin $\boldsymbol{r}_p$, $p = 1, 2, \ldots, N$. Of course, an incident plane wave fulfil these restrictions.

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starting point is the integral representation for the electric field [41].

$$\mathrm{i}\frac{\eta_0\eta}{k}\nabla\times\left\{\nabla\times\sum_{p=1}^{N}\iint_{S_{\mathrm{s}p}}\mathbf{G}_\mathrm{e}(k,|\boldsymbol{r}-\boldsymbol{r}'|)\cdot\boldsymbol{J}_S(\boldsymbol{r}')\,\mathrm{d}S'\right\}$$

$$=\begin{cases}\boldsymbol{E}_\mathrm{s}(\boldsymbol{r}), & \boldsymbol{r}\text{ outside all }S_{\mathrm{s}p}\\ -\boldsymbol{E}_\mathrm{i}(\boldsymbol{r}), & \boldsymbol{r}\text{ inside any }S_{\mathrm{s}p}\end{cases}\quad p=1,2,\dots,N \tag{2.1}$$

where the boundary condition $\hat{\boldsymbol{\nu}}\times\boldsymbol{E}=\boldsymbol{0}$ on each $S_{\mathrm{s}p}$ has been used, and we have, for convenience, denoted the surface current density $\boldsymbol{J}_S=\hat{\boldsymbol{\nu}}\times\boldsymbol{H}$. The wave impedance of vacuum is denoted η_0. The Green's dyadic $\mathbf{G}_\mathrm{e}(k,|\boldsymbol{r}-\boldsymbol{r}'|)$ is

$$\mathbf{G}_\mathrm{e}(k,|\boldsymbol{r}-\boldsymbol{r}'|)=\left(\mathbf{I}_3+\frac{1}{k^2}\nabla\nabla\right)g(k,|\boldsymbol{r}-\boldsymbol{r}'|)$$

where the Green's function $g(k,|\boldsymbol{r}-\boldsymbol{r}'|)$ in free space is

$$g(k,|\boldsymbol{r}-\boldsymbol{r}'|)=\frac{\mathrm{e}^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

The top row in (2.1) holds for all points outside all scatterers. This field, the scattered field $\boldsymbol{E}_\mathrm{s}(\boldsymbol{r})$, is due to the sources inside the scatterers, and it is source-free outside the scatterers, and it satisfies appropriate radiation conditions at infinity. This is the formal definition of the scattered field $\boldsymbol{E}_\mathrm{s}(\boldsymbol{r})$.

The lower row in (2.1) holds for all points inside a particular scatterer, say the pth, and represents the prescribed field (more precisely, $-\boldsymbol{E}_\mathrm{i}(\boldsymbol{r})$) at the location of the pth scatterer.

The incident field is specialized to a plane wave[3] impinging along the direction $\hat{\boldsymbol{k}}_\mathrm{i}$ with an expansion in regular spherical vector waves, $\boldsymbol{v}_n(k\boldsymbol{r})$, see Appendix A, i.e.,

$$\boldsymbol{E}_\mathrm{i}(\boldsymbol{r})=\boldsymbol{E}_0\mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}}=\sum_n a_n\boldsymbol{v}_n(k\boldsymbol{r}) \tag{2.2}$$

where the index n is a multi-index,[4] and where $\boldsymbol{E}_0$ is the field value of the plane wave at the common origin O, and the expansion coefficients, a_n, are given in terms of the vector spherical harmonics, $\boldsymbol{A}_{\tau n}(\hat{\boldsymbol{r}})$, see Appendix A, by

$$a_{\tau n}=4\pi\mathrm{i}^{l-\tau+1}\boldsymbol{A}_{\tau n}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0 \tag{2.3}$$

In particular, if the incident direction is along the positive z-direction, i.e., $\hat{\boldsymbol{k}}_\mathrm{i}=\hat{\boldsymbol{z}}$, we get (the index $\sigma=\mathrm{e}$ corresponds to the upper line and $\sigma=\mathrm{o}$ to the lower line)

$$\begin{cases}a_{1n}=4\pi\mathrm{i}^l\boldsymbol{A}_{1n}(\hat{\boldsymbol{z}})\cdot\boldsymbol{E}_0=-\mathrm{i}^l\delta_{m1}\sqrt{2\pi(2l+1)}\left(\hat{\boldsymbol{z}}\times\begin{Bmatrix}\hat{\boldsymbol{x}}\\\hat{\boldsymbol{y}}\end{Bmatrix}\right)\cdot\boldsymbol{E}_0\\ a_{2n}=-4\pi\mathrm{i}^{l+1}\boldsymbol{A}_{2n}(\hat{\boldsymbol{z}})\cdot\boldsymbol{E}_0=-\mathrm{i}^{l+1}\delta_{m1}\sqrt{2\pi(2l+1)}\begin{Bmatrix}\hat{\boldsymbol{x}}\\\hat{\boldsymbol{y}}\end{Bmatrix}\cdot\boldsymbol{E}_0\end{cases}\quad\hat{\boldsymbol{k}}_\mathrm{i}=\hat{\boldsymbol{z}} \tag{2.4}$$

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[3]More general incident fields can be handled with the present approach.

[4]Depending on the context, the index n consists of three or four different indices, i.e., $n=\sigma ml$ or $n=\tau\sigma ml$, where $\tau=1,2$, $\sigma=\mathrm{e},\mathrm{o}$, $m=0,1,2,\dots,l$, and $l=1,2,3,\dots$. Both conventions are used in this paper.

We decompose the Green's dyadic in spherical vector waves, see [3].

$$\mathbf{G}_{\mathrm{e}}(k, |\boldsymbol{r} - \boldsymbol{r}'|) = \mathrm{i}k \sum_n \boldsymbol{v}_n(k\boldsymbol{r}_<)\boldsymbol{u}_n(k\boldsymbol{r}_>) \tag{2.5}$$

where the summation in the index n is over $\tau = 1, 2$, $\sigma = \mathrm{e,o}$, $m = 0, 1, 2, \ldots, l$, and $l = 1, 2, 3, \ldots$, and where $\boldsymbol{r}_<$ ($\boldsymbol{r}_>$) is the position vector with the smallest (largest) distance to the origin, i.e., if $r < r'$ then $\boldsymbol{r}_< = \boldsymbol{r}$ and $\boldsymbol{r}_> = \boldsymbol{r}'$. This expansion is uniformly convergent in finite domains, provided $r \neq r'$ in the domain.

For an observation point $\boldsymbol{r}$ outside the circumscribing sphere (centered at the global origin O) of the entire collection of scatterers, we obtain a representation of the scattered field from the integral representation (2.1).

$$\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) = \sum_n f_n \boldsymbol{u}_n(k\boldsymbol{r}), \quad r > D \tag{2.6}$$

where the expansion coefficients f_n are

$$f_n = -k^2 \eta_0 \eta \sum_{p=1}^{N} \iint_{S_{\mathrm{s}p}} \boldsymbol{v}_n(k\boldsymbol{r}) \cdot \boldsymbol{J}_S(\boldsymbol{r}) \, \mathrm{d}S$$

The linear relation between the expansion coefficients of the scattered field, f_n, in terms of outgoing spherical vector waves, $\boldsymbol{u}_n(k\boldsymbol{r})$, and the expansion coefficients, a_n, of the incident field in terms of the regular spherical vector waves, $\boldsymbol{v}_n(k\boldsymbol{r})$, is given by the transition matrix, $T_{nn'}$, of the entire collection of scatterers.

$$f_n = \sum_{n'} T_{nn'} a_{n'}$$

The null-field approach (T-matrix method) is a powerful technique to solve electromagnetic scattering problems. The method is well-documented, see the comprehensive database [30–34].

The expression in (2.6) gives the scattered field in terms of the transition matrix of the entire collection of scatterer. There is, however, another procedure which stresses the effect of each scatterer more clearly. This analysis resembles the analysis by Peterson and Ström [38], which emphasizes expansions w.r.t. the local origins, O_p, $p = 1, 2, \ldots, N$, instead of the global origin, O.

Assume that the observation point lies outside all circumscribing spheres of the scatterer, i.e., $\boldsymbol{r}$ satisfies $|\boldsymbol{r} - \boldsymbol{r}_p| > A_p$ for all $p = 1, 2, \ldots, N$. The Green's dyadic has the expansion, see (2.5)

$$\mathbf{G}_{\mathrm{e}}(k, |\boldsymbol{r} - \boldsymbol{r}'|) = \mathbf{G}_{\mathrm{e}}(k, |\boldsymbol{r} - \boldsymbol{r}_p - (\boldsymbol{r}' - \boldsymbol{r}_p)|) = \mathrm{i}k \sum_n \boldsymbol{u}_n(k(\boldsymbol{r} - \boldsymbol{r}_p))\boldsymbol{v}_n(k(\boldsymbol{r}' - \boldsymbol{r}_p)) \tag{2.7}$$

where $\boldsymbol{r}' \in S_{\mathrm{s}p}$, $p = 1, 2, \ldots, N$. Notice that $|\boldsymbol{r}' - \boldsymbol{r}_p| < |\boldsymbol{r} - \boldsymbol{r}_p|$ for all $p = 1, 2, \ldots, N$. The integral representation in (2.1) implies

$$\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) = \sum_{p=1}^{N} \sum_n f_n^p \boldsymbol{u}_n(k(\boldsymbol{r} - \boldsymbol{r}_p)), \quad |\boldsymbol{r} - \boldsymbol{r}_p| > A_p, \ p = 1, 2, \ldots, N \tag{2.8}$$

where the expansion coefficients, f_n^p, are

$$f_n^p = -k^2 \eta_0 \eta \iint\limits_{S_{\mathrm{s}p}} \boldsymbol{v}_n(k(\boldsymbol{r}' - \boldsymbol{r}_p)) \cdot \boldsymbol{J}_S(\boldsymbol{r}') \, \mathrm{d}S', \quad p = 1, 2, \ldots, N$$

Thus, the scattered field, $\boldsymbol{E}_{\mathrm{s}}$, has a representation in terms of a contribution from each scatterer.

$$\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) = \sum_{p=1}^{N} \boldsymbol{E}_{\mathrm{s}p}(\boldsymbol{r})$$

where

$$\boldsymbol{E}_{\mathrm{s}p}(\boldsymbol{r}) = \sum_{n} f_n^p \boldsymbol{u}_n(k(\boldsymbol{r} - \boldsymbol{r}_p)), \quad |\boldsymbol{r} - \boldsymbol{r}_p| > A_p, \ p = 1, 2, \ldots, N$$

Similarly, for an observation point inside the inscribed sphere of a particular scatterer, say the pth scatterer, we have $|\boldsymbol{r} - \boldsymbol{r}_p| < a_p$, and, see (2.5)

$$\mathbf{G}_{\mathrm{e}}(k, |\boldsymbol{r} - \boldsymbol{r}'|) = \mathbf{G}_{\mathrm{e}}(k, |\boldsymbol{r} - \boldsymbol{r}_p - (\boldsymbol{r}' - \boldsymbol{r}_p)|) = \mathrm{i}k \sum_{n} \boldsymbol{v}_n(k(\boldsymbol{r} - \boldsymbol{r}_p)) \boldsymbol{u}_n(k(\boldsymbol{r}' - \boldsymbol{r}_p))$$

for this particular scatterer. For all other scatterers the expansion in (2.7) holds. The integral representation in (2.1) implies ($p = 1, 2, \ldots, N$)

$$\boldsymbol{E}_{\mathrm{i}}(\boldsymbol{r}) = \sum_{n} \alpha_n^p \boldsymbol{v}_n(k(\boldsymbol{r} - \boldsymbol{r}_p)) - \sum_{\substack{q=1 \\ q \neq p}}^{N} \sum_{n} f_n^q \boldsymbol{u}_n(k(\boldsymbol{r} - \boldsymbol{r}_q)), \quad |\boldsymbol{r} - \boldsymbol{r}_p| < a_p \quad (2.9)$$

where the expansion coefficients, α_n^p, are defined as

$$\alpha_n^p = k^2 \eta_0 \eta \iint\limits_{S_{\mathrm{s}p}} \boldsymbol{u}_n(k(\boldsymbol{r}' - \boldsymbol{r}_p)) \cdot \boldsymbol{J}_S(\boldsymbol{r}') \, \mathrm{d}S', \quad p = 1, 2, \ldots, N$$

Equation (2.9) can be written as

$$\boldsymbol{E}_{\mathrm{exc}p}(\boldsymbol{r}) = \sum_{n} \alpha_n^p \boldsymbol{v}_n(k(\boldsymbol{r} - \boldsymbol{r}_p)) = \boldsymbol{E}_{\mathrm{i}}(\boldsymbol{r}) + \sum_{\substack{q=1 \\ q \neq p}}^{N} \boldsymbol{E}_{\mathrm{s}q}(\boldsymbol{r}), \quad |\boldsymbol{r} - \boldsymbol{r}_p| < a_p \quad (2.10)$$

where $\boldsymbol{E}_{\mathrm{exc}p}(\boldsymbol{r})$ is the exciting field at the position of the scatterer located at $\boldsymbol{r}_p$, which consists of the incident field $\boldsymbol{E}_{\mathrm{i}}(\boldsymbol{r})$ plus all scattered fields from the scatterers except the contribution from the pth scatterer.[5]

To obtain a relation in terms of the unknown coefficients, f_n^p, we need to compare this relation using coordinates referring to the same origin, in this case O_p.

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[5]This field is not the total field at the pth scatterer with the pth scatterer removed, since the presence of the pth is needed to generate the correct sources at the locations of all other scatterers.

The translation properties of the spherical vector waves are now engaged [3]. In particular, the translation properties of Appendix B gives

$$\begin{aligned} \boldsymbol{u}_n(k(\boldsymbol{r} - \boldsymbol{r}_q)) &= \boldsymbol{u}_n(k(\boldsymbol{r} - \boldsymbol{r}_p) + k(\boldsymbol{r}_p - \boldsymbol{r}_q)) \\ &= \sum_{n'} \mathcal{P}_{nn'}(k(\boldsymbol{r}_p - \boldsymbol{r}_q))\boldsymbol{v}_{n'}(k(\boldsymbol{r} - \boldsymbol{r}_p)), \quad |\boldsymbol{r} - \boldsymbol{r}_p| < |\boldsymbol{r}_p - \boldsymbol{r}_q| \end{aligned}$$

The incident field has also an expansion in terms of the local origin, O_p. From above, we get by a simple translation of the origin of the incident plane wave

$$\boldsymbol{E}_\mathrm{i}(\boldsymbol{r}) = \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i} \cdot \boldsymbol{r}_p} \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i} \cdot (\boldsymbol{r} - \boldsymbol{r}_p)} = \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i} \cdot \boldsymbol{r}_p} \sum_n a_n \boldsymbol{v}_n(k(\boldsymbol{r} - \boldsymbol{r}_p))$$

Orthogonality of the vector spherical harmonics inside the inscribed sphere of the pth scatterer applied to (2.10) implies [42, 53]

$$\alpha_n^p = \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i} \cdot \boldsymbol{r}_p} a_n + \sum_{\substack{q=1 \\ q \neq p}}^{N} \sum_{n'} f_{n'}^q \mathcal{P}_{n'n}(k(\boldsymbol{r}_p - \boldsymbol{r}_q)), \quad p = 1, 2, \ldots, N \qquad (2.11)$$

This set of equations is equivalent to the result (4.1) in Peterson and Ström [38].

The transition matrix of the pth scatterer, $T_{nn'}^p$, connects the expansion coefficients α_n^p and f_n^p to each other, *viz.*,

$$f_n^p = \sum_{n'} T_{nn'}^p \alpha_{n'}^p, \quad p = 1, 2, \ldots, N$$

The transition matrix $T_{nn'}^p$ is the linear relation between the expansion coefficients of the scattered field, f_n^p, in terms of outgoing spherical vector waves, $\boldsymbol{u}_n(k\boldsymbol{r})$, at the local origin $\boldsymbol{r}_p$, see (2.8), and the expansion coefficients, α_n^p, of the local excitation in terms of the regular spherical vector waves, $\boldsymbol{v}_n(k\boldsymbol{r})$, at the local origin $\boldsymbol{r}_p$, see (2.10).

Equation (2.11) then becomes (use also the relation $\mathcal{P}_{n'n}(-k\boldsymbol{r}) = \mathcal{P}_{nn'}(k\boldsymbol{r})$)

$$\alpha_n^p = \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i} \cdot \boldsymbol{r}_p} a_n + \sum_{\substack{q=1 \\ q \neq p}}^{N} \sum_{n'n''} \mathcal{P}_{nn'}(k(\boldsymbol{r}_q - \boldsymbol{r}_p)) T_{n'n''}^q \alpha_{n''}^q, \quad p = 1, 2, \ldots, N \qquad (2.12)$$

This set of linear equations is well-known and has been derived in the literature many times [5, 42, 44]. Equation (2.12) determines the unknown expansion coefficients α_n^p in terms of the excitation characterized by the expansion coefficients, a_n. This relation is identical to the relation found in the literature, see *e.g.*, [23].

At this stage, we notice that if the scatterer is not a perfectly conducting scatterer as assumed above, the result can be generalized. The main reason for the assumption of perfectly conducting scatterers was to simplify the theoretical work. If a more general scatterer is present, the only change that has to be made is to replace the transition matrix of the scatterer with the appropriate one. Therefore, the results above hold for any set of scatterers — single or multiple, transparent or

not, homogeneous or not — only the individual transition matrices of the scatterers (nonintersecting circumscribing spheres) are known.

An alternative form of the relation (2.12) is obtained by multiplying the relation with $T_{nn'}^p$ followed by a summation over the free index. The result is ($p = 1, 2, \ldots, N$)

$$f_n^p = \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}_p} \sum_{n'} T_{nn'}^p a_{n'} + \sum_{\substack{q=1\\q\neq p}}^{N} \sum_{n'n''} T_{nn'}^p \mathcal{P}_{n'n''}(k(\boldsymbol{r}_q - \boldsymbol{r}_p)) f_{n''}^q \qquad (2.13)$$

This is a very useful relation between the unknown coefficients f_n^p, which we exploit in this paper.

The scattered field is therefore possible to compute by (2.8), once f_n^p are known, and these coefficients are obtained from the solution of (2.13). In principle, the field everywhere outside the scatterers (more precisely, outside the circumscribing spheres of the scatterers) can be obtained.

The coefficients of the scattered field of the entire collection of scatterers, f_n, see (2.6), can be expressed in terms of the expansion coefficients of the local scatterers, f_n^p, by the use of the translation matrices, see Appendix B. The result is, see (2.8)

$$\boldsymbol{E}_\mathrm{s}(\boldsymbol{r}) = \sum_{p=1}^{N} \boldsymbol{E}_{\mathrm{s}p}(\boldsymbol{r}) = \sum_{p=1}^{N} \sum_{n} f_n^p \boldsymbol{u}_n(k(\boldsymbol{r} - \boldsymbol{r}_p))$$

$$= \sum_{p=1}^{N} \sum_{nn'} f_n^p \mathcal{R}_{nn'}(-k\boldsymbol{r}_p) \boldsymbol{u}_{n'}(k\boldsymbol{r}), \quad |\boldsymbol{r}| > D \qquad (2.14)$$

which implies (use the relation $\mathcal{R}_{nn'}(-k\boldsymbol{r}) = \mathcal{R}_{n'n}(k\boldsymbol{r})$, and use (2.6))

$$f_n = \sum_{p=1}^{N} \sum_{n'} f_{n'}^p \mathcal{R}_{n'n}(-k\boldsymbol{r}_p) = \sum_{p=1}^{N} \sum_{n'} \mathcal{R}_{nn'}(k\boldsymbol{r}_p) f_{n'}^p \qquad (2.15)$$

2.1 Iterative solution

The infinite set of equations in the unknown f_n^p, (2.13), can be solved by iteration (or repeated insertion), where each terms can be interpreted as multiple scattering contributions. We obtain ($p = 1, 2, \ldots, N$)

$$f_n^p = \sum_{l=1}^{\infty} f_n^p(l) \qquad (2.16)$$

where

$$\begin{cases} f_n^p(1) = \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}_p} \sum_{n'} T_{nn'}^p a_{n'} \\ f_n^p(l+1) = \sum_{\substack{q=1\\q\neq p}}^{N} \sum_{n'n''} T_{nn'}^p \mathcal{P}_{n'n''}(k(\boldsymbol{r}_q - \boldsymbol{r}_p)) f_{n''}^q(l), \quad l = 1, 2, \ldots \end{cases} \qquad (2.17)$$

The first term, $f_n^p(1) = \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}_p}\sum_{n'}T_{nn'}^p a_{n'}$, is the single scattering term, since this term produces a scattered field of the form

$$\boldsymbol{E}_\mathrm{s}^1(\boldsymbol{r}) = \sum_{p=1}^N\sum_n f_n^p(1)\boldsymbol{u}_n(k(\boldsymbol{r}-\boldsymbol{r}_p)) = \sum_{p=1}^N \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}_p}\sum_{nn'}\boldsymbol{u}_n(k(\boldsymbol{r}-\boldsymbol{r}_p))T_{nn'}^p a_{n'}$$

which is the sum of scattering contributions with the exciting field evaluated at the local origin. The next term gives a scattered field of the form

$$\begin{aligned}\boldsymbol{E}_\mathrm{s}^2(\boldsymbol{r}) &= \sum_{p=1}^N\sum_n f_n^p(2)\boldsymbol{u}_n(k(\boldsymbol{r}-\boldsymbol{r}_p)) \\ &= \sum_{p=1}^N\sum_{\substack{q=1\\q\neq p}}^N \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i}\cdot\boldsymbol{r}_q}\sum_{nn'n''n'''}\boldsymbol{u}_n(k(\boldsymbol{r}-\boldsymbol{r}_p))T_{nn''}^p\mathcal{P}_{n''n'}(k(\boldsymbol{r}_q-\boldsymbol{r}_p))T_{n'n'''}^q a_{n'''}\end{aligned}$$

This term contains a product of two transition matrices, and represents the double scattering contributions. Higher order terms represent more complex scattering contributions.

2.2 Cross sections

The extinction cross section $\sigma_\mathrm{ext}(\hat{\boldsymbol{k}}_\mathrm{i})$ for a given incident plane wave (direction $\hat{\boldsymbol{k}}_\mathrm{i}$), see (2.2), can be expressed in the transition matrix of the scatterers. The result is [10, 27, 29]

$$\begin{aligned}\sigma_\mathrm{ext}(\hat{\boldsymbol{k}}_\mathrm{i}) &= -\frac{8\pi^2}{k^2\left|\boldsymbol{E}_0\right|^2}\sum_{nn'}\mathrm{i}^{-l+\tau-1}\boldsymbol{E}_0^*\cdot\boldsymbol{A}_n(\hat{\boldsymbol{k}}_\mathrm{i})\left(T_{nn'}+T_{nn'}^\dagger\right)\mathrm{i}^{l'-\tau'+1}\boldsymbol{A}_{n'}(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0 \\ &= -\frac{1}{2k^2|\boldsymbol{E}_0|^2}\sum_n a_n^*\left(T_{nn'}+T_{nn'}^\dagger\right)a_{n'} = -\frac{1}{k^2|\boldsymbol{E}_0|^2}\,\mathrm{Re}\sum_n a_n^* f_n\end{aligned}$$

since, by (2.3), $a_n = 4\pi\mathrm{i}^{l-\tau+1}\boldsymbol{A}_n(\hat{\boldsymbol{k}}_\mathrm{i})\cdot\boldsymbol{E}_0$. Insert (2.15), and we obtain

$$\sigma_\mathrm{ext}(\hat{\boldsymbol{k}}_\mathrm{i}) = -\frac{1}{k^2|\boldsymbol{E}_0|^2}\sum_{p=1}^N\mathrm{Re}\sum_{nn'}a_n^*\mathcal{R}_{nn'}(k\boldsymbol{r}_p)f_{n'}^p = -\frac{1}{k^2|\boldsymbol{E}_0|^2}\sum_{p=1}^N\mathrm{Re}\sum_n a_n^{p*}f_n^p$$

where the coefficient $a_n^p = \sum_{n'}a_{n'}\mathcal{R}_{n'n}(k\boldsymbol{r}_p)$ is the expansion coefficient of the incident field w.r.t. the local origin at O_p, i.e.,

$$\begin{aligned}\boldsymbol{E}_\mathrm{i}(\boldsymbol{r}) = \sum_n a_n\boldsymbol{v}_n(k\boldsymbol{r}) &= \sum_n a_n\boldsymbol{v}_n(k(\boldsymbol{r}-\boldsymbol{r}_p)+k\boldsymbol{r}_p) \\ &= \sum_{nn'}a_n\mathcal{R}_{nn'}(k\boldsymbol{r}_p)\boldsymbol{v}_{n'}(k(\boldsymbol{r}-\boldsymbol{r}_p)) = \sum_n a_n^p\boldsymbol{v}_n(k(\boldsymbol{r}-\boldsymbol{r}_p))\end{aligned}$$

If we define the extinction cross section of each individual scatterer as

$$\widetilde{\sigma}_\mathrm{ext}^p(\hat{\boldsymbol{k}}_\mathrm{i}) = -\frac{1}{k^2|\boldsymbol{E}_0|^2}\,\mathrm{Re}\sum_n a_n^{p*}f_n^p$$
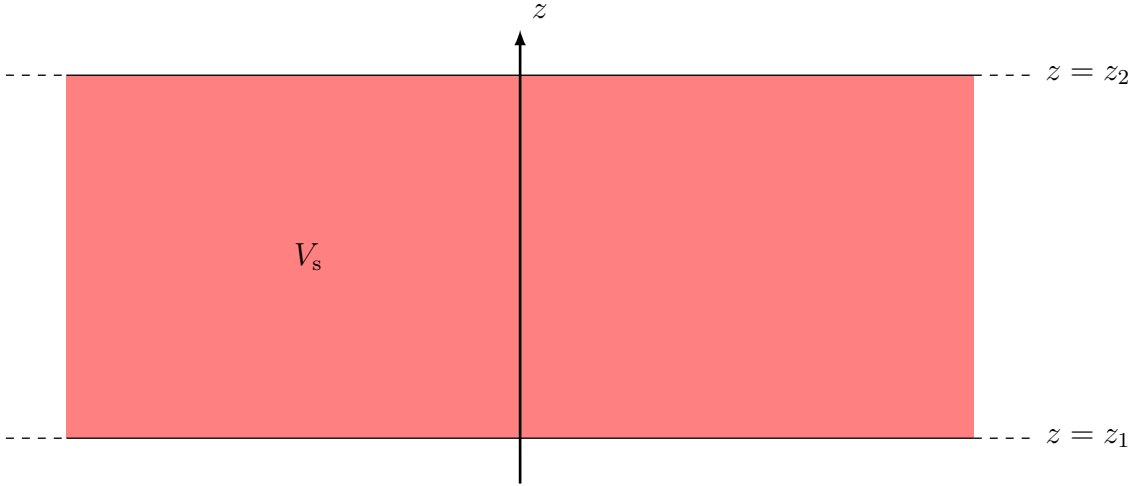
Figure 2: The geometry of the slab region.

we get

$$\sigma_{\text{ext}}(\hat{\boldsymbol{k}}_{\text{i}}) = \sum_{p=1}^{N} \widetilde{\sigma}_{\text{ext}}^{p}$$

At first glance, this result is somewhat deceptive. It does not state that the total extinction cross section is the sum of the extinction cross sections of the individual scatterer, *e.g.*, the total extinction cross section of several identical spherical objects is not just the extinction cross sections of a single sphere times the number of spheres. The difference is that the scattered field in each term, $\widetilde{\sigma}_{\text{ext}}^{p}$, contains the contribution from all other scatterers. Notice that the sum in $\widetilde{\sigma}_{\text{ext}}^{p}$ contains the coefficients f_{n}^{p}, which contain all multiple scattering contributions. Another difference is that the scattering cross section for the entire ensemble of scatterers refers to a single plane wave, and the scattering cross sections for the individual are due to a much more complex incident field. This is also the reason for a separate notion with a tilde on the extinction cross section of the individual scatterers.

2.3 Half space or slab confinement

If the origins of the scatterers are confined to a half space or a slab, *i.e.*, $V_{\text{s}} = \{\boldsymbol{r} \in \mathbb{R}^3 : z > z_1\}$ or $V_{\text{s}} = \{\boldsymbol{r} \in \mathbb{R}^3 : z_1 < z < z_2\}$ with a suitable orientation of the z-axis,[6] the scattered field of these configurations is naturally described in terms of planar vector waves. The pertinent geometry is depicted in Figure 2. The planar

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[6]In general, the volume V_{s} is the region of possible positions of the local origins $\boldsymbol{r}_p$. Of course, these infinite volumes have to be taken as proper limit processes from the finite volume formulation above.

vector waves are defined as [3]

$$\begin{cases} \boldsymbol{\varphi}_1^\pm(\boldsymbol{k}_{\mathrm{t}};\boldsymbol{r}) = \dfrac{\hat{\boldsymbol{z}} \times \boldsymbol{k}_{\mathrm{t}}}{4\pi\mathrm{i}k_{\mathrm{t}}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\mathrm{t}}\cdot\boldsymbol{\rho}\pm\mathrm{i}k_z z} \\[2mm] \boldsymbol{\varphi}_2^\pm(\boldsymbol{k}_{\mathrm{t}};\boldsymbol{r}) = \dfrac{\mp\boldsymbol{k}_{\mathrm{t}}k_z + k_{\mathrm{t}}^2\hat{\boldsymbol{z}}}{4\pi k k_{\mathrm{t}}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{\mathrm{t}}\cdot\boldsymbol{\rho}\pm\mathrm{i}k_z z} \end{cases}$$

where the lateral wave vector $\boldsymbol{k}_{\mathrm{t}} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}$, and where the lateral wave number, $k_{\mathrm{t}} = |\boldsymbol{k}_{\mathrm{t}}|$, and the longitudinal wave number k_z is defined by

$$k_z = \left(k^2 - k_{\mathrm{t}}^2\right)^{1/2} = \begin{cases} \sqrt{k^2 - k_{\mathrm{t}}^2} \text{ for } k_{\mathrm{t}} < k \\[2mm] \mathrm{i}\sqrt{k_{\mathrm{t}}^2 - k^2} \text{ for } k_{\mathrm{t}} > k \end{cases}$$

The outgoing spherical vector waves, $\boldsymbol{u}_n(k\boldsymbol{r})$, can be expressed in the planar vector waves, $\boldsymbol{\varphi}_j^\pm(\boldsymbol{k}_{\mathrm{t}};\boldsymbol{r})$, $j = 1,2$. This transformation reads, see [3, p. 183]

$$\boldsymbol{u}_n(k\boldsymbol{r}) = 2\sum_{j=1,2}\iint_{\mathbb{R}^2} B_{nj}^\pm(\boldsymbol{k}_{\mathrm{t}})\boldsymbol{\varphi}_j^\pm(\boldsymbol{k}_{\mathrm{t}};\boldsymbol{r})\frac{k}{k_z}\frac{\mathrm{d}k_x\,\mathrm{d}k_y}{k^2}, \quad z \gtrless 0 \qquad (2.18)$$

where

$$B_{nj}^+(\boldsymbol{k}_{\mathrm{t}}) = \mathrm{i}^{-l}C_{lm}\left\{-\mathrm{i}\delta_{\tau j}\Delta_l^m(k_z/k)\begin{Bmatrix} \cos m\beta \\ \sin m\beta \end{Bmatrix} - \delta_{\tau\bar{j}}\pi_l^m(k_z/k)\begin{Bmatrix} -\sin m\beta \\ \cos m\beta \end{Bmatrix}\right\}$$

and

$$B_{nj}^-(\boldsymbol{k}_{\mathrm{t}}) = (-1)^{l+m+\tau+j+1}B_{nj}^+(\boldsymbol{k}_{\mathrm{t}})$$

where the index $\bar{j}$ is the dual index of j, defined by $\bar{1} = 2$ and $\bar{2} = 1$, and where $\boldsymbol{k}_{\mathrm{t}} = k_{\mathrm{t}}(\hat{\boldsymbol{x}}\cos\beta + \hat{\boldsymbol{y}}\sin\beta)$, and

$$C_{lm} = \sqrt{\frac{\varepsilon_m}{2\pi}}\sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}$$

The Neumann factor is defined as $\varepsilon_m = 2 - \delta_{m0}$, and

$$\Delta_l^m(t) = -\frac{(1-t^2)^{1/2}}{\sqrt{l(l+1)}}P_l^{m\prime}(t), \quad \pi_l^m(t) = \frac{m}{\sqrt{l(l+1)}(1-t^2)^{1/2}}P_l^m(t)$$

Notice that the argument t can take complex values.

Below, we need to evaluate these functions at $t = 1$. The recursion relations of the associated Legendre functions give

$$\Delta_l^m(1) = \pi_l^m(1) = \frac{1}{2}\sqrt{l(l+1)}\delta_{m1}$$

Applying the transformation (2.18) to the scattered field in (2.8) gives an expression of the scattered field outside the half space or the slab geometry. The result for the half space is

$$\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) = \sum_{j=1,2}\iint_{\mathbb{R}^2} f_j(\boldsymbol{k}_{\mathrm{t}})\boldsymbol{\varphi}_j^-(\boldsymbol{k}_{\mathrm{t}};\boldsymbol{r})\frac{\mathrm{d}k_x\,\mathrm{d}k_y}{k^2}, \quad z < z_1 - \max_p A_p$$

where

$$f_j(\boldsymbol{k}_{\mathrm{t}}) = 2\frac{k}{k_z}\sum_{p=1}^{N}\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_{\mathrm{t}}-k_z\hat{\boldsymbol{z}})\cdot\boldsymbol{r}_p}\sum_n f_n^p B_{nj}^-(\boldsymbol{k}_{\mathrm{t}})$$

For the slab we have

$$\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) = \sum_{j=1,2}\iint_{\mathbb{R}^2} f_j^\pm(\boldsymbol{k}_{\mathrm{t}})\boldsymbol{\varphi}_j^\pm(\boldsymbol{k}_{\mathrm{t}};\boldsymbol{r})\frac{\mathrm{d}k_x\,\mathrm{d}k_y}{k^2}, \quad z \gtrless \begin{matrix} z_2 + \max_p A_p \\ z_1 - \max_p A_p \end{matrix} \qquad (2.19)$$

where

$$f_j^\pm(\boldsymbol{k}_{\mathrm{t}}) = 2\frac{k}{k_z}\sum_{p=1}^{N}\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_{\mathrm{t}}\pm k_z\hat{\boldsymbol{z}})\cdot\boldsymbol{r}_p}\sum_n f_n^p B_{nj}^\pm(\boldsymbol{k}_{\mathrm{t}})$$

An approximative value of the field inside the slab or half space is attained if an infinitesimal plane, $z = z^* = \mathrm{constant}$, is inserted in the material such that the material is separated in a part $z < z^*$ and one part $z > z^*$. This arrangement slightly changes the configuration of the slab, but in the case of many scatterers the effect can be made negligible. The field inside the material is

$$\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) = \sum_{j=1,2}\iint_{\mathbb{R}^2} f_j^+(\boldsymbol{k}_{\mathrm{t}})\boldsymbol{\varphi}_j^+(\boldsymbol{k}_{\mathrm{t}};\boldsymbol{r})\frac{\mathrm{d}k_x\,\mathrm{d}k_y}{k^2} + \sum_{j=1,2}\iint_{\mathbb{R}^2} f_j^-(\boldsymbol{k}_{\mathrm{t}})\boldsymbol{\varphi}_j^-(\boldsymbol{k}_{\mathrm{t}};\boldsymbol{r})\frac{\mathrm{d}k_x\,\mathrm{d}k_y}{k^2}$$

where the sum in p in the two terms in $f_j^\pm(\boldsymbol{k}_{\mathrm{t}})$ is only over the particles in $z < z^*$ and $z > z^*$, respectively. The result in (2.8), however, is still valid inside the material, provided the observation point lies outside all circumscribing spheres.

3   Randomly distributed scatterers

To solve the set of equations in (2.13) for the unknowns, f_n^p, is a formidable task, if the number of scatterers is large. Fortunately, there are statistical methods that apply in this case — especially if the location and state (material properties, size, shape, and orientation *etc.*) of the scatterers are randomly distributed. Moreover, with a large number of scatterers, we rarely have complete information about the position and the state of each scatterer, and we are often not interested in the physical quantities of a particular configuration, but ensemble averages suffice. The presentation in this section follows to some extent the one presented in [26,42,52,55], but deviates in the method of solving the problem.

The explicit assumptions made about the scatterers in this section are:

1. the number of scatterers N is large, so that statistical methods are appropriate to apply

2. the N scatterers are characterized by a common probability density function, and the numbering of the scatterers is arbitrary

3. the state variables are independent between different scatterers and of the position variables

4. no circumscribing spheres of the individual scatterers intersect

The position variables cannot be statistically independent variables for finite size scatterers, due to the assumption of non-intersecting circumscribing spheres. The assumption of independence in Item 3 makes the averaging of scatterer position separate from the averaging over their states.

3.1 Probability density functions

The statistical distribution of the N scatterers, positioned at $\boldsymbol{r}_p$, $p = 1, 2, \ldots, N$, and the state, which is collected in one symbol ξ_p, $p = 1, 2, \ldots, N$, is described by the N-particle probability density function (PDF) $P(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \xi_1, \ldots, \xi_N)$. This function quantifies the probability of finding the first scatterer in a volume element $\mathrm{d}v_1$ centered at $\boldsymbol{r}_1$ with its state in $\mathrm{d}\xi_1$ centered at ξ_1, the second scatterer in a volume element $\mathrm{d}v_2$ centered at $\boldsymbol{r}_2$ with its state in $\mathrm{d}\xi_2$ centered at ξ_2, and, in general, the pth scatterer within a volume element $\mathrm{d}v_p$ centered at position $\boldsymbol{r}_p$ with its state in $\mathrm{d}\xi_p$ centered at ξ_p as

$$P(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \xi_1, \ldots, \xi_N) \prod_{p=1}^{N} \mathrm{d}v_p \, \mathrm{d}\xi_p$$

The assumption of the independence of the state and position variables, see Item 3 above, implies that the N-particle probability density function takes the form

$$P(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \xi_1, \ldots, \xi_N) = P(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \prod_{p=1}^{N} P_{\mathrm{s}}(\xi_p)$$

where $P(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$ denotes the probability density function depending on the location of the individual scatterer, and $P_{\mathrm{s}}(\xi_p)$ the probability density function depending on the state of the pth scatterer. Conservation of probability implies

$$\int_{V_{\mathrm{s}}^N} P(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \prod_{p=1}^{N} \mathrm{d}v_p = 1$$

and

$$\int_{\Omega_{\mathrm{s}}} P_{\mathrm{s}}(\xi) \, \mathrm{d}\xi = 1$$

where the integration is taken over V_{s}, which is the entire range of the scatterers' positions,[7] and the states is assumed to take values in the abstract space Ω_{s}.

---

[7]To be exact, the volume that the local origins, $\boldsymbol{r}_p$, can occupy. This volume is not the same as the convex hull of all scatterer.

The probability density function $P(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$ is expressed in terms of conditional probability density functions according to[8,9]

$$P(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = P(\boldsymbol{r}_1)P(\boldsymbol{r}_2, \ldots, \boldsymbol{r}_N|\boldsymbol{r}_1)$$
$$= P(\boldsymbol{r}_1)P(\boldsymbol{r}_2|\boldsymbol{r}_1)P(\boldsymbol{r}_3, \ldots, \boldsymbol{r}_N|\boldsymbol{r}_1, \boldsymbol{r}_2)$$

where $P(\boldsymbol{r}_1)$ denotes the probability density of finding scatterer 1 in a volume element $\mathrm{d}v_1$ centered at $\boldsymbol{r}_1$, and where the function $P(\boldsymbol{r}_2, \ldots, \boldsymbol{r}_N|\boldsymbol{r}_1)$ is the conditional probability density of finding scatterer 2 in a volume element $\mathrm{d}v_2$ centered at $\boldsymbol{r}_2$ and scatterer 3 in a volume element $\mathrm{d}v_3$ centered at $\boldsymbol{r}_3$ *etc.*, given scatterer 1 in a volume element $\mathrm{d}v_1$ centered at $\boldsymbol{r}_1$. $P(\boldsymbol{r}_2|\boldsymbol{r}_1)$ denotes the conditional probability density of finding scatterer 2 in a volume element $\mathrm{d}v_2$ centered at $\boldsymbol{r}_2$ provided scatterer 1 is known to be in a volume element $\mathrm{d}v_1$ centered at $\boldsymbol{r}_1$, and the function $P(\boldsymbol{r}_3, \ldots, \boldsymbol{r}_N|\boldsymbol{r}_1, \boldsymbol{r}_2)$ is the conditional probability density of finding scatterer 3 in a volume element $\mathrm{d}v_3$ centered at $\boldsymbol{r}_3$ and scatterer 4 in a volume element $\mathrm{d}v_4$ centered at $\boldsymbol{r}_4$ *etc.*, given scatterer 1 in a volume element $\mathrm{d}v_1$ centered at $\boldsymbol{r}_1$ and scatterer 2 in a volume element $\mathrm{d}v_2$ centered at $\boldsymbol{r}_2$.

The single-scatterer and two-scatterer probability density functions $P(\boldsymbol{r}_p)$ and $P(\boldsymbol{r}_p, \boldsymbol{r}_q)$ are computed as

$$P(\boldsymbol{r}_p) = \int_{V_{\mathrm{s}}^{N-1}} P(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) \prod_{\substack{r=1 \\ r\neq p}}^{N} \mathrm{d}v_r, \qquad \iiint_{V_{\mathrm{s}}} P(\boldsymbol{r})\,\mathrm{d}v = 1 \qquad (3.1)$$

and

$$P(\boldsymbol{r}_p, \boldsymbol{r}_q) = \int_{V_{\mathrm{s}}^{N-2}} P(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_N) \prod_{\substack{r=1 \\ r\neq p,q}}^{N} \mathrm{d}v_r, \qquad \int_{V_{\mathrm{s}}^2} P(\boldsymbol{r}_1, \boldsymbol{r}_2)\,\mathrm{d}v_1\,\mathrm{d}v_2 = 1 \quad (3.2)$$

respectively. Moreover, we have by definition

$$\iiint_{V_{\mathrm{s}}} P(\boldsymbol{r}_p, \boldsymbol{r}_q)\,\mathrm{d}v_q = P(\boldsymbol{r}_p)$$

From these definitions and the definition of conditional probability density function we also have

$$P(\boldsymbol{r}_1) \int_{V_{\mathrm{s}}^{N-1}} P(\boldsymbol{r}_2, \ldots, \boldsymbol{r}_N|\boldsymbol{r}_1) \prod_{r=2}^{N} \mathrm{d}v_r = \int_{V_{\mathrm{s}}^{N-1}} P(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) \prod_{r=2}^{N} \mathrm{d}v_r = P(\boldsymbol{r}_1)$$

---

[8]The conditional probability density function are defined as [37, Sec. 7.2]

$$P(\boldsymbol{r}_{k+1}, \ldots, \boldsymbol{r}_N|\boldsymbol{r}_1, \ldots, \boldsymbol{r}_k)) = \frac{P(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)}{P(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_k)}$$

[9]Since the order of numbering is arbitrary, we specialize to scatterers 1 and 2. Any other combination of scatterers follows with a similar notation.

and therefore

$$\int\limits_{V_{\mathrm{s}}^{N-1}} P(\boldsymbol{r}_2,\ldots,\boldsymbol{r}_N|\boldsymbol{r}_1)\prod_{r=2}^{N}\mathrm{d}v_r = 1$$

The two-scatterer probability density functions $P(\boldsymbol{r}_p,\boldsymbol{r}_q)$ and the conditional probability density $P(\boldsymbol{r}_q|\boldsymbol{r}_p)$ are related by

$$P(\boldsymbol{r}_p,\boldsymbol{r}_q) = P(\boldsymbol{r}_p)P(\boldsymbol{r}_q|\boldsymbol{r}_p)$$

and

$$\iiint\limits_{V_{\mathrm{s}}} P(\boldsymbol{r}_q|\boldsymbol{r}_p)\,\mathrm{d}v_q = 1 \tag{3.3}$$

The average of a function $f(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N;\xi_1,\ldots,\xi_N)$ over all position variables is denoted

$$\langle f\rangle = \int\limits_{\Omega_{\mathrm{s}}^N}\int\limits_{V_{\mathrm{s}}^N} P(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)\prod_{p=1}^{N}P_{\mathrm{s}}(\xi_p)f(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N;\xi_1,\ldots,\xi_N)\prod_{p=1}^{N}\mathrm{d}v_p\,\mathrm{d}\xi_p$$

If the position and state of the first scatterer is held fixed and all other scatterers are averaged over, we use the notion

$$\langle f\rangle\,(\boldsymbol{r}_1;\xi_1) = \int\limits_{\Omega_{\mathrm{s}}^{N-1}}\int\limits_{V_{\mathrm{s}}^{N-1}} P(\boldsymbol{r}_2,\ldots,\boldsymbol{r}_N|\boldsymbol{r}_1)\prod_{p=2}^{N}P_{\mathrm{s}}(\xi_p)$$

$$\times f(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N;\xi_1,\ldots,\xi_N)\prod_{p=2}^{N}\mathrm{d}v_p\,\mathrm{d}\xi_p$$

and

$$\langle f\rangle\,(\boldsymbol{r}_1) = \int\limits_{\Omega_{\mathrm{s}}^N}\int\limits_{V_{\mathrm{s}}^{N-1}} P(\boldsymbol{r}_2,\ldots,\boldsymbol{r}_N|\boldsymbol{r}_1)\prod_{p=1}^{N}P_{\mathrm{s}}(\xi_p)$$

$$\times f(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N;\xi_1,\ldots,\xi_N)\prod_{p=2}^{N}\mathrm{d}v_p\prod_{p=1}^{N}\mathrm{d}\xi_p$$

if we also include the average over the state variable ξ_1.

With scatterer 1 and 2 held fixed, we use the notion

$$\langle f\rangle\,(\boldsymbol{r}_1,\boldsymbol{r}_2;\xi_1,\xi_2) = \int\limits_{\Omega_{\mathrm{s}}^{N-2}}\int\limits_{V_{\mathrm{s}}^{N-2}} P(\boldsymbol{r}_3,\ldots,\boldsymbol{r}_N|\boldsymbol{r}_1,\boldsymbol{r}_2)\prod_{p=3}^{N}P_{\mathrm{s}}(\xi_p)$$

$$\times f(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N;\xi_1,\ldots,\xi_N)\prod_{p=3}^{N}\mathrm{d}v_p\,\mathrm{d}\xi_p$$

and a similar notation for any other combination of scatterers.

3.2   Number density functions

Of special interest is (single-scatterer) number density function $n_1(\boldsymbol{r})$ defined by

$$n_1(\boldsymbol{r}) = \sum_{p=1}^{N} \delta(\boldsymbol{r} - \boldsymbol{r}_p)$$

This definition implies

$$\iiint_{V_{\mathrm{s}}} n_1(\boldsymbol{r}) \, \mathrm{d}v = N$$

which gives the total number of scatterers in V_{s}. The average number density is (use (3.1))

$$\langle n_1(\boldsymbol{r}) \rangle = \int_{V_{\mathrm{s}}^N} P(\boldsymbol{r}_1, \dots, \boldsymbol{r}_N) n_1(\boldsymbol{r}) \prod_{p=1}^{N} \mathrm{d}v_p = \sum_{p=1}^{N} P(\boldsymbol{r}) = N P(\boldsymbol{r})$$

If the medium is statistically homogenous, the single-particle probability is simple (denote the volume of V_{s} by $|V_{\mathrm{s}}|$)

$$P(\boldsymbol{r}) = \begin{cases} \dfrac{1}{|V_{\mathrm{s}}|} = \dfrac{n_0}{N}, & \boldsymbol{r} \in V_{\mathrm{s}} \\ 0, & \boldsymbol{r} \notin V_{\mathrm{s}} \end{cases}$$

where the number density is[10]

$$n_0 = \langle n_1(\boldsymbol{r}) \rangle = \frac{N}{|V_{\mathrm{s}}|}$$

The two-scatterer number density $n_2(\boldsymbol{r}, \boldsymbol{r}')$ is defined as

$$n_2(\boldsymbol{r}, \boldsymbol{r}') = \sum_{p=1}^{N} \sum_{\substack{q=1 \\ q \neq p}}^{N} \delta(\boldsymbol{r} - \boldsymbol{r}_p) \delta(\boldsymbol{r}' - \boldsymbol{r}_q)$$

The definition implies

$$\iiint_{V_{\mathrm{s}}} n_2(\boldsymbol{r}, \boldsymbol{r}') \, \mathrm{d}v' = (N-1) n_1(\boldsymbol{r})$$

The average two-particle number density is (use (3.2))

$$\langle n_2(\boldsymbol{r}, \boldsymbol{r}') \rangle = \int_{V_{\mathrm{s}}^N} P(\boldsymbol{r}_1, \dots, \boldsymbol{r}_N) n_2(\boldsymbol{r}, \boldsymbol{r}') \prod_{p=1}^{N} \mathrm{d}v_p$$

$$= \sum_{p=1}^{N} \sum_{\substack{q=1 \\ q \neq p}} P(\boldsymbol{r}, \boldsymbol{r}') = N(N-1) P(\boldsymbol{r}, \boldsymbol{r}')$$

---

[10]More exactly, the density or concentration of local origins.

The pair distribution function $g(\boldsymbol{r}, \boldsymbol{r}')$ is now defined in terms of the one- and two-scatterer number densities. We have the definition

$$\langle n_2(\boldsymbol{r}, \boldsymbol{r}')\rangle = \langle n_1(\boldsymbol{r})\rangle \langle n_1(\boldsymbol{r}')\rangle g(\boldsymbol{r}, \boldsymbol{r}')$$

We expect the two-scatterer probability density function $P(\boldsymbol{r}, \boldsymbol{r}') \to P(\boldsymbol{r})P(\boldsymbol{r}')$ (independent random variables) as $|\boldsymbol{r} - \boldsymbol{r}'| \to \infty$, and we therefore predict $g(\boldsymbol{r}, \boldsymbol{r}') \to 1$ in this limit. We can express the two-scatterer probability density function $P(\boldsymbol{r}, \boldsymbol{r}')$ in terms of the pair distribution function $g(\boldsymbol{r}, \boldsymbol{r}')$ as

$$P(\boldsymbol{r}, \boldsymbol{r}') = \frac{\langle n_1(\boldsymbol{r})\rangle \langle n_1(\boldsymbol{r}')\rangle g(\boldsymbol{r}, \boldsymbol{r}')}{N(N-1)} = \frac{NP(\boldsymbol{r})P(\boldsymbol{r}')g(\boldsymbol{r}, \boldsymbol{r}')}{N-1}$$

or in terms of the conditional probability density function $P(\boldsymbol{r}_q|\boldsymbol{r}_p)$ as

$$P(\boldsymbol{r}'|\boldsymbol{r}) = \frac{NP(\boldsymbol{r}')g(\boldsymbol{r}, \boldsymbol{r}')}{N-1}$$

For a homogeneous medium, we get

$$\langle n_2(\boldsymbol{r}, \boldsymbol{r}')\rangle = n_0^2 g(\boldsymbol{r}, \boldsymbol{r}')$$

and

$$P(\boldsymbol{r}, \boldsymbol{r}') = \frac{Ng(\boldsymbol{r}, \boldsymbol{r}')}{|V_{\mathrm{s}}|^2(N-1)} = \frac{n_0^2 g(\boldsymbol{r}, \boldsymbol{r}')}{N(N-1)}, \qquad P(\boldsymbol{r}'|\boldsymbol{r}) = \frac{n_0 g(\boldsymbol{r}, \boldsymbol{r}')}{N-1}$$

As the number of scatterers becomes large, the two-scatterer probability density function $P(\boldsymbol{r}, \boldsymbol{r}') \to g(\boldsymbol{r}, \boldsymbol{r}')/|V_{\mathrm{s}}|^2$ and the conditional probability density function $P(\boldsymbol{r}'|\boldsymbol{r}) \to g(\boldsymbol{r}, \boldsymbol{r}')/|V_{\mathrm{s}}|$.

4 The coherent field

The general expression of the scattered electric field is given in (2.14), *i.e.*,

$$\boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) = \sum_{p=1}^{N} \sum_{nn'} \boldsymbol{u}_{n'}(k\boldsymbol{r})\mathcal{R}_{n'n}(k\boldsymbol{r}_p)f_n^p, \quad |\boldsymbol{r} - \boldsymbol{r}_p| > A_p, \ p = 1, 2, \ldots, N$$

The ensemble average and the use of definition of conditional probability density function imply

$$\langle \boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r})\rangle = \sum_{p=1}^{N} \sum_{nn'} \boldsymbol{u}_{n'}(k\boldsymbol{r}) \iiint_{V_{\mathrm{s}}} P(\boldsymbol{r}_p)\mathcal{R}_{n'n}(k\boldsymbol{r}_p) \langle f_n^p\rangle (\boldsymbol{r}_p) \, \mathrm{d}v_p, \quad \boldsymbol{r} \notin \widetilde{V}_{\mathrm{s}}$$

where $\widetilde{V}_{\mathrm{s}}$ is the volume V_{s} plus the region of all circumscribing spheres.[11] Under the assumption of ergodicity, the ensemble and time averages are related, for a more detailed description of ergodicity, see [29].

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[11]An approximate value of the field inside the material at $\boldsymbol{r}$ is attained by excluding a small ball of radius $a = \max_p A_p$ around $\boldsymbol{r}$ such that no scatterer are allowed inside this ball. This arrangement alters the statistics of the problem slightly, but the error becomes negligible as the number of scatterers in V_{s} grows.

The coherent field reflected and transmitted by the slab is, see (2.19)

$$\langle \boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) \rangle = \sum_{p=1}^{N} \sum_{j=1,2} \sum_{n} \iint_{\mathbb{R}^2} 2\frac{k}{k_z} B_{nj}^{\pm}(\boldsymbol{k}_{\mathrm{t}}) \boldsymbol{\varphi}_j^{\pm}(\boldsymbol{k}_{\mathrm{t}}; \boldsymbol{r})$$

$$\left(\iiint_{V_{\mathrm{s}}} P(\boldsymbol{r}_p) \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_{\mathrm{t}} \pm k_z \hat{\boldsymbol{z}}) \cdot \boldsymbol{r}_p} \langle f_n^p \rangle (\boldsymbol{r}_p) \, \mathrm{d}v_p \right) \frac{\mathrm{d}k_x \, \mathrm{d}k_y}{k^2}, \quad z \gtrless \begin{array}{c} z_2 + a \\ z_1 - a \end{array}$$

where $a = \max_p A_p$, the maximum radius of the circumscribing spheres.[12]

With a large number of scatterers N, which are randomly distributed (statistically homogenous and isotropic), the position of all scatterers are equally probable in the volume occupied by the scatterers. We have $P(\boldsymbol{r}) = n_0/N$. Since the numbering of the scatterers is arbitrary, see Item 2 on page 12, all integrals over V_{s} in the two averages above are identical. As a consequence, the label p is superfluous, and we get

$$\langle \boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) \rangle = n_0 \sum_{nn'} \boldsymbol{u}_{n'}(k\boldsymbol{r}) \iiint_{V_{\mathrm{s}}} \mathcal{R}_{n'n}(k\boldsymbol{r}') \langle f_n \rangle (\boldsymbol{r}') \, \mathrm{d}v', \quad \boldsymbol{r} \notin \widetilde{V}_{\mathrm{s}}$$

and the result for the slab confinement is

$$\langle \boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) \rangle = n_0 \sum_{j=1,2} \sum_{n} \iint_{\mathbb{R}^2} 2\frac{k}{k_z} B_{nj}^{\pm}(\boldsymbol{k}_{\mathrm{t}}) \boldsymbol{\varphi}_j^{\pm}(\boldsymbol{k}_{\mathrm{t}}; \boldsymbol{r})$$

$$\left(\iiint_{V_{\mathrm{s}}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_{\mathrm{t}} \pm k_z \hat{\boldsymbol{z}}) \cdot \boldsymbol{r}'} \langle f_n \rangle (\boldsymbol{r}') \, \mathrm{d}v' \right) \frac{\mathrm{d}k_x \, \mathrm{d}k_y}{k^2}, \quad z \gtrless \begin{array}{c} z_2 + a \\ z_1 - a \end{array} \tag{4.1}$$

4.1 Computation of the unknowns $\langle f_n \rangle (\boldsymbol{r})$

To complete the calculations of the scattered field, we need to find $\langle f_n \rangle (\boldsymbol{r})$. This is done in two stages, first we determine $\langle f_n \rangle (\boldsymbol{r}; \xi)$, which then is used to compute $\langle f_n \rangle (\boldsymbol{r})$ by averaging over the state variable ξ.

To proceed, equation (2.13) is averaged over the possible positions and states except the pth position and state, which are kept fixed. The transition matrix $T_{nn'}$ does not depend on the position $\boldsymbol{r}_p$ but on the stage variable ξ_p. However, the coefficients f_n^q depend on all positions $\boldsymbol{r}_r$, and all state variables ξ_r, $r = 1, 2, \ldots, N$. The result of this average is

$$\langle f_n^p \rangle (\boldsymbol{r}_p; \xi_p) = \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}} \cdot \boldsymbol{r}_p} \sum_{n'} T_{nn'}^p(\xi_p) a_{n'} + \sum_{\substack{q=1 \\ q \neq p}}^{N} \sum_{n'n''} T_{nn''}^p(\xi_p) \langle \mathcal{P}_{n''n'}(k(\boldsymbol{r}_q - \boldsymbol{r}_p)) f_{n'}^q \rangle (\boldsymbol{r}_p; \xi_p)$$

---

[12]As already pointed out in Footnote 11, the field inside the material can be attained if we introduce a small excluding slab at z. The thickness of the slab is $2a$, and this arrangement alters the statistics of the problem slightly, but the error becomes negligible as the number of scatterers in V_{s} grows.

Notice that the conditional average does not include the transition matrix $T^p_{nn''}(\xi_p)$, since this factor only depends on the fixed state variable ξ_p. This term can therefore be taken outside the conditional average, which involve all positions and state variables except the pth ones. The conditional average on the right-hand side reads

$$
\langle \mathcal{P}_{n''n'}(k(\boldsymbol{r}_q - \boldsymbol{r}_p))f^q_{n'}\rangle (\boldsymbol{r}_p; \xi_p)
$$
$$
= \int\limits_{\Omega_{\text{s}}^{N-1}} \int\limits_{V_{\text{s}}^{N-1}} P(\boldsymbol{r}_1, \overset{'}{\ldots}, \boldsymbol{r}_N | \boldsymbol{r}_p) \prod_{r=1}^{N}{}' P_{\text{s}}(\xi_r) \mathcal{P}_{n''n'}(k(\boldsymbol{r}_q - \boldsymbol{r}_p))
$$
$$
\times f^q_{n'}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \xi_1, \ldots, \xi_N) \prod_{r=1}^{N}{}' \mathrm{d}v_r \, \mathrm{d}\xi_r
$$

where the prime in the probability density function and in the product of differentials means that the pth variable is excluded. Using definition of conditional probability density function, we obtain with the notation above

$$
\langle \mathcal{P}_{n''n'}(k(\boldsymbol{r}_q - \boldsymbol{r}_p))f^q_{n'}\rangle (\boldsymbol{r}_p; \xi_p)
$$
$$
= \int\limits_{\Omega_{\text{s}}^{N-1}} \int\limits_{V_{\text{s}}^{N-1}} P(\boldsymbol{r}_q|\boldsymbol{r}_p) P(\boldsymbol{r}_1, \overset{''}{\ldots}, \boldsymbol{r}_N | \boldsymbol{r}_p, \boldsymbol{r}_q) \prod_{r=1}^{N}{}' P_{\text{s}}(\xi_r) \mathcal{P}_{n''n'}(k(\boldsymbol{r}_q - \boldsymbol{r}_p))
$$
$$
\times f^q_{n'}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N; \xi_1, \ldots, \xi_N) \prod_{r=1}^{N}{}'' \mathrm{d}v_r \, \mathrm{d}v_q \prod_{r=1}^{N}{}' \mathrm{d}\xi_r
$$

where the double prime in the probability density function and in the product of differentials means that the pth and q^{th} variable are excluded. Collecting terms implies

$$
\langle f^p_n\rangle (\boldsymbol{r}_p; \xi_p) = \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\text{i}}\cdot\boldsymbol{r}_p} \sum_{n'} T^p_{nn'}(\xi_p) a_{n'}
$$
$$
+ \sum_{\substack{q=1 \\ q\neq p}}^{N} \sum_{n'n''} T^p_{nn''}(\xi_p) \iiint\limits_{V_{\text{s}}} P(\boldsymbol{r}_q|\boldsymbol{r}_p) \mathcal{P}_{n''n'}(k(\boldsymbol{r}_q - \boldsymbol{r}_p)) \langle f^q_{n'}\rangle (\boldsymbol{r}_p, \boldsymbol{r}_q; \xi_p) \, \mathrm{d}v_q
$$

$$(4.2)$$

Notice that the average procedure with one particle fixed leads to a hierarchy requiring higher and higher order probability densities. It is possible to continue this process replacing the integrand with higher and higher conditional probability densities, but at some stage we have to stop, and this calls for a way to truncate the procedure. In the next section, we outline one of the most popular truncation procedures.

4.2 Quasi Crystalline Approximations

In practice, higher order density functions are hard to obtain. Lax [22] introduced a way to truncate the hierarchy introduced in (4.2) — the Quasi Crystalline Approx-

imation (QCA).

$$\langle f_{n'}^q \rangle (\boldsymbol{r}_p, \boldsymbol{r}_q; \xi_p) \approx \langle f_{n'}^q \rangle (\boldsymbol{r}_q; \xi_p)$$

This approximation simply states that the conditional average with two positions held fixed, $\langle f_{n'}^q \rangle (\boldsymbol{r}_p, \boldsymbol{r}_q; \xi_p)$, is replaced with the conditional average with one position held fixed, $\langle f_{n'}^q \rangle (\boldsymbol{r}_q; \xi_p)$. This approximation has been successful in a range of concentrations from tenuous to dense media, and from the low-frequency to intermediate frequency range [59].

This approximation leads to a set of relations in the unknowns, $\langle f_n^p \rangle (\boldsymbol{r}_p; \xi_p)$, where $p = 1, 2, \ldots, N$, *viz.*,

$$\langle f_n^p \rangle (\boldsymbol{r}_p; \xi_p) = \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}} \cdot \boldsymbol{r}_p} \sum_{n'} T_{nn'}^p(\xi_p) a_{n'}$$

$$+ \sum_{\substack{q=1 \\ q \neq p}}^{N} \sum_{n'n''} T_{nn''}^p(\xi_p) \iiint_{V_{\mathrm{s}}} P(\boldsymbol{r}_q | \boldsymbol{r}_p) \mathcal{P}_{n''n'}(k(\boldsymbol{r}_q - \boldsymbol{r}_p)) \langle f_{n'}^q \rangle (\boldsymbol{r}_q; \xi_p)) \, \mathrm{d}v_q$$

(4.3)

The equation (4.3) is a system of integral equations for the unknown $\langle f_n^p \rangle (\boldsymbol{r}_p; \xi_p)$, $p = 1, 2, \ldots, N$, which can be simplified under the assumption made in this paper. The state variable ξ_p acts as a parameter in this system of integral equations.

If the circumscribing spheres of the scatterers are non-overlapping, then

$$P(\boldsymbol{r}_q | \boldsymbol{r}_p) = \begin{cases} \dfrac{Ng(\boldsymbol{r}_p, \boldsymbol{r}_q)}{(N-1)|V_{\mathrm{s}}|} = g(\boldsymbol{r}_p, \boldsymbol{r}_q) \dfrac{n_0}{N-1}, & |\boldsymbol{r}_p - \boldsymbol{r}_q| > 2a \\ 0, & |\boldsymbol{r}_p - \boldsymbol{r}_q| < 2a \end{cases}$$

(4.4)

The pair distribution function $g(\boldsymbol{r}_p, \boldsymbol{r}_q)$ for symmetric scatterers depends only on the distance between the two scatterers, *i.e.*, $|\boldsymbol{r}_p - \boldsymbol{r}_q|$, $g(\boldsymbol{r}_p, \boldsymbol{r}_q) = g(|\boldsymbol{r}_p - \boldsymbol{r}_q|)$. We define $g(r) = 0$ for $r < 2a$, and we have that, see (3.3)

$$\iiint_{V_{\mathrm{s}}} g(r) n_0 \, \mathrm{d}v = N - 1$$

There exist several theories for determining the pair distribution function $g(r)$, *e.g.*, hole-corrections (HC) $g(r) = 1$, $r > 2a$, the hypernetted-chain equation, the Percus-Yevick approximation (P-YA), the self-consistent approximation, and Monte Carlo calculations [6, 46, 58]. In this paper, we confine ourselves to a treatment of the hole-corrections, and leave the others, more accurate models for dense media, to a forthcoming paper.

Since the numbering of the scatterers is arbitrary, see Item 2 on page 12, all terms in the sum in (4.3) are identical, and we get (the variables $\boldsymbol{r}_p$ and $\boldsymbol{r}_q$ are dummy variables)

$$\langle f_n \rangle (\boldsymbol{r}; \xi) = \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_{\mathrm{i}} \cdot \boldsymbol{r}} \sum_{n'} T_{nn'}(\xi) a_{n'}$$

$$+ n_0 \sum_{n'n''} T_{nn''}(\xi) \iiint_{V_{\mathrm{s}}} g(|\boldsymbol{r}' - \boldsymbol{r}|) \mathcal{P}_{n''n'}(k(\boldsymbol{r}' - \boldsymbol{r})) \langle f_{n'} \rangle (\boldsymbol{r}'; \xi) \, \mathrm{d}v', \quad \boldsymbol{r} \in V_{\mathrm{s}}, \xi \in \Omega_{\mathrm{s}}$$

where the expansion coefficients a_n are given by (2.3). The state variable ξ acts as a parameter, and the system of integral equations is of convolution type, *i.e.*,

$$
\begin{aligned}
\langle f_n \rangle (\boldsymbol{r}; \xi) = {}& \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}_\mathrm{i} \cdot \boldsymbol{r}} \sum_{n'} T_{nn'}(\xi) a_{n'} \\
& + k^3 \iiint\limits_{V_\mathrm{s}} \sum_{n'} \mathcal{K}_{nn'}(\boldsymbol{r} - \boldsymbol{r}'; \xi) \langle f_{n'} \rangle (\boldsymbol{r}'; \xi) \, \mathrm{d}v', \quad \boldsymbol{r} \in V_\mathrm{s} \, \xi \in \Omega_\mathrm{s}
\end{aligned}
\tag{4.5}
$$

with kernel

$$
\mathcal{K}_{nn'}(\boldsymbol{r}; \xi) = \frac{n_0}{k^3} \sum_{n''} T_{nn''}(\xi) g(|\boldsymbol{r}|) \mathcal{P}_{n''n'}(-k\boldsymbol{r}))
$$

The state variable ξ, which acts as a parameter, is often suppressed in the analysis below, provided it is not essential for the analysis.

5 Integral equation — slab confinement

Suppose the spheres are confined by two parallel planes at $z = 0$ and $z = d > 0$. This case constitutes an important geometry in many applications.[13] The integration of the $\boldsymbol{r}'$ variable in (4.5) is then constrained to all locations inside the planes $z_1 = a$ and $z_2 = d - a$, which is the possible locations of local origins, *i.e.*, the volume V_s, see Figure 3. We suppress the state variable ξ in this section, since it is only a parameter.

If the scatterers are randomly distributed within the volume between the planes, and the incidence direction $\hat{\boldsymbol{k}}_\mathrm{i} = \hat{\boldsymbol{z}}$, see Figure 3, the spatial behavior of the coefficients $\langle f_n \rangle (\boldsymbol{r})$ can only depend on the depth z of the slab. The unknown coefficients $\langle f_n \rangle (\boldsymbol{r}) = \langle f_n \rangle (z)$, and (4.5) then simplify to[14]

$$
\begin{aligned}
\langle f_n \rangle (z) = {}& \mathrm{e}^{\mathrm{i}kz} \sum_{n'} T_{nn'} a_{n'} \\
& + n_0 \sum_{n'n''} T_{nn''} \iiint\limits_{V_\mathrm{s}} g(|\boldsymbol{r}' - \boldsymbol{r}|) \mathcal{P}_{n''n'}(k(\boldsymbol{r}' - \boldsymbol{r})) \langle f_{n'} \rangle (z') \, \mathrm{d}v', \quad z_1 < z < z_2
\end{aligned}
$$

Translational invariance in the lateral variables, shows that it suffices to compute the integral along the z axis $\boldsymbol{r} = z\hat{\boldsymbol{z}}$, *i.e.*,

$$
\begin{aligned}
\langle f_n \rangle (z) = {}& \mathrm{e}^{\mathrm{i}kz} \sum_{n'} T_{nn'} a_{n'} \\
& + n_0 \sum_{n'n''} T_{nn''} \iiint\limits_{V_\mathrm{s}} g(|\boldsymbol{r}' - z\hat{\boldsymbol{z}}|) \mathcal{P}_{n''n'}(k(\boldsymbol{r}' - z\hat{\boldsymbol{z}})) \langle f_{n'} \rangle (z') \, \mathrm{d}v', \quad z_1 < z < z_2
\end{aligned}
$$

---

[13]The half-space confinement is obtained by appropriately letting $d \to \infty$.

[14]We require the integrand to vanish in an appropriate way as the lateral variables, x' and y', approach infinity. One way to accomplish this is to assume an infinitely small positive imaginary part of the wave number k.
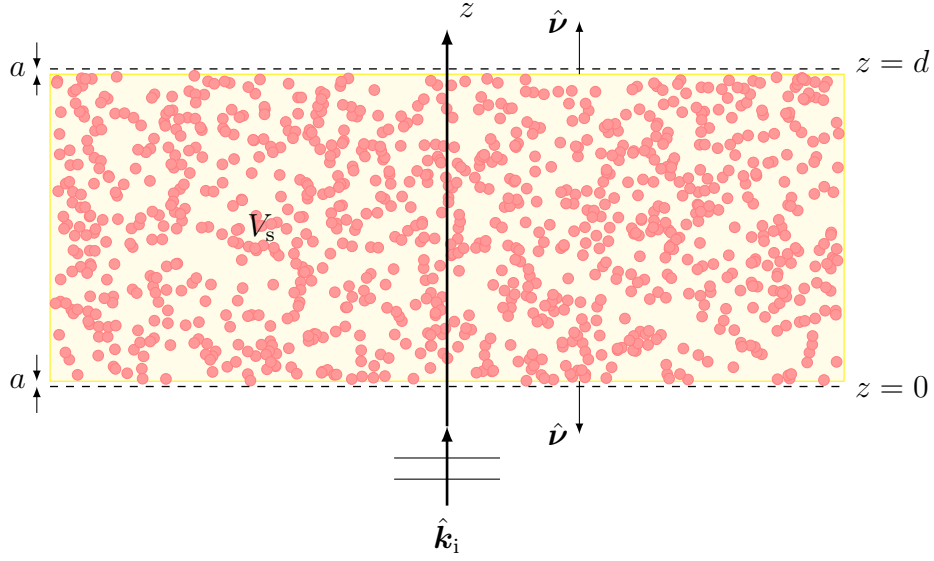
Figure 3: The geometry of the stratified scattering region. Note the direction of the unit normals on the bounding surfaces. In this two-dimensional drawing it looks like the spheres intersect, but this is not the case in three dimensions. However, in this two-dimensional graph some of the projections of the spheres overlap. The yellow region denotes the region V_s, which is the domain of possible locations of local origins, $i.e.$, the interval $[z_1, z_2]$.

where the expansion coefficients a_n are given by (2.4).

The integral equation is a one-dimensional integral equation in z, $viz.$,

$$\langle f_n \rangle (z) = \mathrm{e}^{\mathrm{i}kz} \sum_{n'} T_{nn'} a_{n'} + k \int_{z_1}^{z_2} \sum_{n'} K_{nn'}(z - z') \langle f_{n'} \rangle (z') \,\mathrm{d}z', \quad z_1 < z < z_2 \quad (5.1)$$

where

$$K_{nn'}(z) = \frac{n_0}{k} \sum_{n''} T_{nn''} \iint_{\mathbb{R}^2} g(|\boldsymbol{\rho} - z\hat{\boldsymbol{z}}|) \mathcal{P}_{n''n'}(k(\boldsymbol{\rho} - z\hat{\boldsymbol{z}})) \,\mathrm{d}x \,\mathrm{d}y, \quad |z| < z_2 - z_1 \quad (5.2)$$

5.1   Reflected and transmitted field

The average scattered field (reflected and transmitted), see (4.1), now reads

$$\langle \boldsymbol{E}_s(\boldsymbol{r}) \rangle = n_0 \sum_{j=1,2} \sum_n \iint_{\mathbb{R}^2} 2\frac{k}{k_z} B_{nj}^{\pm}(\boldsymbol{k}_t) \boldsymbol{\varphi}_j^{\pm}(\boldsymbol{k}_t; \boldsymbol{r})$$

$$\times \left(\iiint_{V_s} \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}_t \pm k_z \hat{\boldsymbol{z}}) \cdot \boldsymbol{r}'} \langle f_n \rangle (z') \,\mathrm{d}v' \right) \frac{\mathrm{d}k_x \,\mathrm{d}k_y}{k^2}, \quad z \gtrless \begin{matrix} d \\ 0 \end{matrix}$$

Integration w.r.t. the lateral variables produces a delta function $\delta(\boldsymbol{k}_{\mathrm{t}})$, followed by an integration in the lateral wave number $\boldsymbol{k}_{\mathrm{t}}$ implies

$$\langle \boldsymbol{E}_{\mathrm{s}}(\boldsymbol{r}) \rangle = \frac{8\pi^2 n_0}{k^2} \sum_{j=1,2} \sum_n B_{nj}^{\pm}(\boldsymbol{0}) \boldsymbol{\varphi}_j^{\pm}(\boldsymbol{0}; \boldsymbol{r}) \int_{z_1}^{z_2} \mathrm{e}^{\mp \mathrm{i}kz'} \langle f_n \rangle (z') \, \mathrm{d}z', \quad z \gtrless \genfrac{}{}{0pt}{}{d}{0}$$

The explicit expressions of the planar vector waves

$$\begin{cases} \boldsymbol{\varphi}_1^{\pm}(\boldsymbol{0}; \boldsymbol{r}) = \dfrac{\hat{\boldsymbol{z}} \times \hat{\boldsymbol{e}}_{\parallel}}{4\pi \mathrm{i}} \mathrm{e}^{\pm \mathrm{i}kz} \\[2mm] \boldsymbol{\varphi}_2^{\pm}(\boldsymbol{0}; \boldsymbol{r}) = \mp \dfrac{\hat{\boldsymbol{e}}_{\parallel}}{4\pi} \mathrm{e}^{\pm \mathrm{i}kz} \end{cases}$$

and the transformation functions

$$B_{nj}^+(\boldsymbol{0}) = \frac{\mathrm{i}^{-l} \delta_{m,1}}{2\sqrt{2\pi}} \sqrt{2l+1} \left\{ -\mathrm{i}\delta_{\tau,j} \begin{Bmatrix} \cos\beta \\ \sin\beta \end{Bmatrix} - \delta_{\tau,\bar{j}} \begin{Bmatrix} -\sin\beta \\ \cos\beta \end{Bmatrix} \right\}$$

and

$$B_{nj}^-(\boldsymbol{0}) = (-1)^{l+m+\tau+j+1} B_{nj}^+(\boldsymbol{0})$$

formally give the fields for a normally incident plane wave on both sides of the slab. The result is

$$\langle \boldsymbol{E}(\boldsymbol{r}) \rangle = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}kz} + \boldsymbol{E}_{\mathrm{r}} \mathrm{e}^{-\mathrm{i}kz}, \quad z < 0$$

and

$$\langle \boldsymbol{E}(\boldsymbol{r}) \rangle = \boldsymbol{E}_{\mathrm{t}} \mathrm{e}^{\mathrm{i}kz}, \quad z > d$$

where the transmitted amplitude is

$$\boldsymbol{E}_{\mathrm{t}} = \boldsymbol{E}_0 - \frac{2\pi n_0}{k^2} \sum_n \left(\mathrm{i}\hat{\boldsymbol{z}} \times \hat{\boldsymbol{e}}_{\parallel} B_{n1}^+(\boldsymbol{0}) + \hat{\boldsymbol{e}}_{\parallel} B_{n2}^+(\boldsymbol{0}) \right) \int_{z_1}^{z_2} \mathrm{e}^{-\mathrm{i}kz'} \langle f_n \rangle (z') \, \mathrm{d}z'$$

and where the reflected amplitude is

$$\boldsymbol{E}_{\mathrm{r}} = -\frac{2\pi n_0}{k^2} \sum_n \left(\mathrm{i}\hat{\boldsymbol{z}} \times \hat{\boldsymbol{e}}_{\parallel} B_{n1}^-(\boldsymbol{0}) - \hat{\boldsymbol{e}}_{\parallel} B_{n2}^-(\boldsymbol{0}) \right) \int_{z_1}^{z_2} \mathrm{e}^{\mathrm{i}kz'} \langle f_n \rangle (z') \, \mathrm{d}z'$$

where $\hat{\boldsymbol{e}}_{\parallel} = \boldsymbol{k}_{\mathrm{t}}/k_{\mathrm{t}}$. These expression can further be simplified to

$$\boldsymbol{E}_{\mathrm{t}} = \boldsymbol{E}_0 + \frac{2\pi n_0}{k^2} \sum_n \mathrm{i}^{-l+\tau-1} \boldsymbol{A}_n(\hat{\boldsymbol{z}}) \int_{z_1}^{z_2} \mathrm{e}^{-\mathrm{i}kz'} \langle f_n \rangle (z') \, \mathrm{d}z'$$

and

$$\boldsymbol{E}_{\mathrm{r}} = \frac{2\pi n_0}{k^2} \sum_n \mathrm{i}^{l-\tau+1} \boldsymbol{A}_n(\hat{\boldsymbol{z}}) \int_{z_1}^{z_2} \mathrm{e}^{\mathrm{i}kz'} \langle f_n \rangle (z') \, \mathrm{d}z'$$

Insertion of the explicit form of $\boldsymbol{A}_n(\hat{\boldsymbol{z}})$, see (A.1), gives the final expression. We have

$$\boldsymbol{E}_{\mathrm{t}} = \boldsymbol{E}_0 + \frac{2\pi n_0}{k^2} \sum_{l=1}^{\infty} \mathrm{i}^{-l} \sqrt{\frac{2l+1}{8\pi}} \left(\hat{\boldsymbol{x}} \int_{z_1}^{z_2} \mathrm{e}^{-\mathrm{i}kz'} \left(\langle f_{1\mathrm{o}1l} \rangle (z') + \mathrm{i} \langle f_{2\mathrm{e}1l} \rangle (z') \right) \mathrm{d}z' \right.$$
$$\left. - \hat{\boldsymbol{y}} \int_{z_1}^{z_2} \mathrm{e}^{-\mathrm{i}kz'} \left(\langle f_{1\mathrm{e}1l} \rangle (z') - \mathrm{i} \langle f_{2\mathrm{o}1l} \rangle (z') \right) \mathrm{d}z' \right) \quad (5.3)$$

and

$$\boldsymbol{E}_{\mathrm{r}} = \frac{2\pi n_0}{k^2} \sum_{l=1}^{\infty} \mathrm{i}^l \sqrt{\frac{2l+1}{8\pi}} \left(\hat{\boldsymbol{x}} \int_{z_1}^{z_2} \mathrm{e}^{\mathrm{i}kz'} \left(\langle f_{1\mathrm{o}1l} \rangle (z') - \mathrm{i} \langle f_{2\mathrm{e}1l} \rangle (z') \right) \mathrm{d}z' \right.$$
$$\left. - \hat{\boldsymbol{y}} \int_{z_1}^{z_2} \mathrm{e}^{\mathrm{i}kz'} \left(\langle f_{1\mathrm{e}1l} \rangle (z') + \mathrm{i} \langle f_{2\mathrm{o}1l} \rangle (z') \right) \mathrm{d}z' \right) \quad (5.4)$$

The RTE does not predict any coherent contribution in the reflected direction [7, 16, 29]. However, the present analysis shows a reflected field given by (5.4). Moreover, the coherent contribution (5.3) in the forward direction (transmitted field) generalizes the Bouguer-Beer law, since no assumptions of far field conditions between the scatterer are made.

Under the assumptions discussed in Footnote 12 on page 18, the approximate field inside the slab is

$$\langle \boldsymbol{E}(z) \rangle = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}kz}$$
$$+ \frac{2\pi n_0}{k^2} \sum_{l=1}^{\infty} \mathrm{i}^{-l} \sqrt{\frac{2l+1}{8\pi}} \left(\hat{\boldsymbol{x}} \int_{z_1}^{z-a} \mathrm{e}^{-\mathrm{i}kz'} \left(\langle f_{1\mathrm{o}1l} \rangle (z') + \mathrm{i} \langle f_{2\mathrm{e}1l} \rangle (z') \right) \mathrm{d}z' \right.$$
$$\left. - \hat{\boldsymbol{y}} \int_{z_1}^{z-a} \mathrm{e}^{-\mathrm{i}kz'} \left(\langle f_{1\mathrm{e}1l} \rangle (z') - \mathrm{i} \langle f_{2\mathrm{o}1l} \rangle (z') \right) \mathrm{d}z' \right) \mathrm{e}^{\mathrm{i}kz}$$
$$+ \frac{2\pi n_0}{k^2} \sum_{l=1}^{\infty} \mathrm{i}^l \sqrt{\frac{2l+1}{8\pi}} \left(\hat{\boldsymbol{x}} \int_{z+a}^{z_2} \mathrm{e}^{\mathrm{i}kz'} \left(\langle f_{1\mathrm{o}1l} \rangle (z') - \mathrm{i} \langle f_{2\mathrm{e}1l} \rangle (z') \right) \mathrm{d}z' \right.$$
$$\left. - \hat{\boldsymbol{y}} \int_{z+a}^{z_2} \mathrm{e}^{\mathrm{i}kz'} \left(\langle f_{1\mathrm{e}1l} \rangle (z') + \mathrm{i} \langle f_{2\mathrm{o}1l} \rangle (z') \right) \mathrm{d}z' \right) \mathrm{e}^{-\mathrm{i}kz}, \quad z \in [z_1 + a, z_2 - a]$$

The transmissivity T and the reflectivity R of the slab then is

$$T = \frac{|\boldsymbol{E}_{\mathrm{t}}|^2}{|\boldsymbol{E}_0|^2}, \qquad R = \frac{|\boldsymbol{E}_{\mathrm{r}}|^2}{|\boldsymbol{E}_0|^2}$$

5.2 Alternative integral equation

The unknown $\langle f_n \rangle (z)$ is obtained by solving the system of integral equations (5.1). To find the transmitted and reflected amplitudes in (5.3) and (5.4), the finite Fourier transform

$$\int_{z_1}^{z_2} \mathrm{e}^{\pm \mathrm{i}kz'} \langle f_n \rangle (z') \, \mathrm{d}z'$$

has to be computed. However, there is also an alternative way of solving the problem that aims for the finite Fourier transforms directly. We define

$$y_n^{\pm}(z) = k \int_{z_1}^{z} \mathrm{e}^{\pm \mathrm{i}kz'} \langle f_n \rangle (z') \, \mathrm{d}z', \quad z_1 \leq z \leq z_2$$
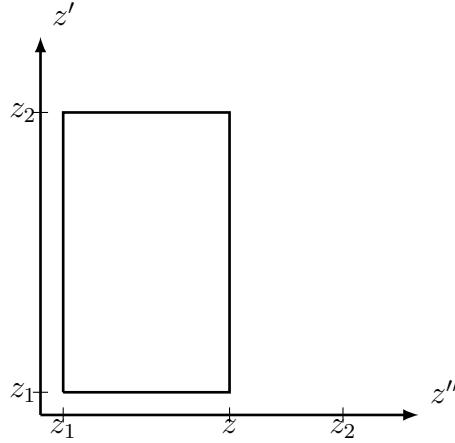
Figure 4: The domain of integration in z' and z''.

which also has the limit value $y_n^\pm(z_1) = 0$. With these definitions, we see that the unknown in (5.3) and (5.4) are $y_n^\pm(z_2)$. The explicit expressions for the transmitted amplitude (lower sign) and reflected amplitude (upper sign) are

$$
\boldsymbol{E}_\mathrm{t} = \boldsymbol{E}_0
$$
$$
+ \frac{2\pi n_0}{k^3} \sum_{l=1}^{\infty} \mathrm{i}^{-l} \sqrt{\frac{2l+1}{8\pi}} \left(\hat{\boldsymbol{x}} \left(y_{1\mathrm{o}1l}^-(z_2) + \mathrm{i}y_{2\mathrm{e}1l}^-(z_2) \right) - \hat{\boldsymbol{y}} \left(y_{1\mathrm{e}1l}^-(z_2) - \mathrm{i}y_{2\mathrm{o}1l}^-(z_2) \right) \right) \quad (5.5)
$$

and

$$
\boldsymbol{E}_\mathrm{r} = \frac{2\pi n_0}{k^3} \sum_{l=1}^{\infty} \mathrm{i}^{l} \sqrt{\frac{2l+1}{8\pi}} \left(\hat{\boldsymbol{x}} \left(y_{1\mathrm{o}1l}^+(z_2) - \mathrm{i}y_{2\mathrm{e}1l}^+(z_2) \right) - \hat{\boldsymbol{y}} \left(y_{1\mathrm{e}1l}^+(z_2) + \mathrm{i}y_{2\mathrm{o}1l}^+(z_2) \right) \right)
$$
$$
(5.6)
$$

The functions $y_n^\pm(z)$ satisfy, see (5.1)

$$
y_n^-(z) = k(z - z_1) \sum_{n'} T_{nn'} a_{n'}
$$
$$
+ k \sum_{n'} \int_{z_1}^{z} \int_{z_1}^{z_2} K_{nn'}(z'' - z') \mathrm{e}^{-\mathrm{i}k(z''-z')} \frac{\mathrm{d}y_{n'}^-(z')}{\mathrm{d}z'} \, \mathrm{d}z' \, \mathrm{d}z'', \quad z_1 \leq z \leq z_2
$$

and

$$
y_n^+(z) = \frac{1}{2\mathrm{i}} \left(\mathrm{e}^{2\mathrm{i}kz} - \mathrm{e}^{2\mathrm{i}kz_1} \right) \sum_{n'} T_{nn'} a_{n'}
$$
$$
+ k \sum_{n'} \int_{z_1}^{z} \int_{z_1}^{z_2} K_{nn'}(z'' - z') \mathrm{e}^{\mathrm{i}k(z''-z')} \frac{\mathrm{d}y_{n'}^+(z')}{\mathrm{d}z'} \, \mathrm{d}z' \, \mathrm{d}z'', \quad z_1 \leq z \leq z_2
$$

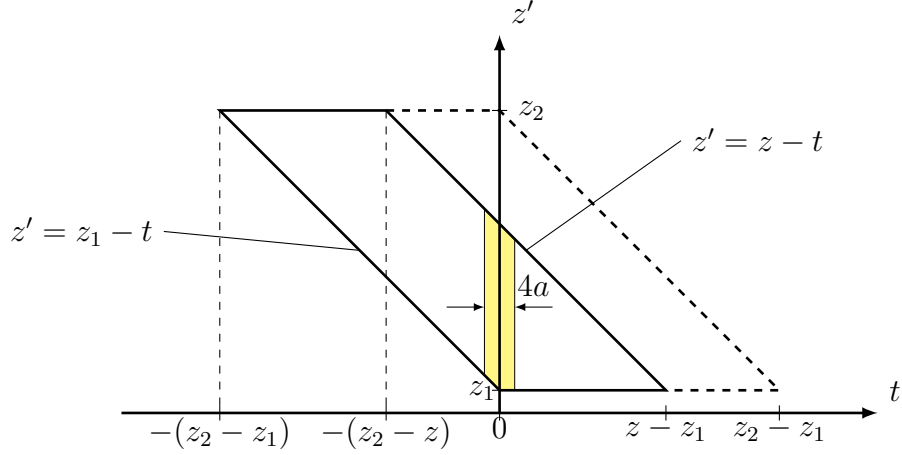where the domain of integration in the integral is depicted in Figure 4.

Figure 5: The domain of integration in t and z'. In the yellow area the integrand depicts the area with hole correction.

Make a change in variables $z'' \to t = z'' - z'$, and the new domain of integration is illustrated in Figure 5. The result is

$$y_n^-(z) = k(z - z_1) \sum_{n'} T_{nn'} a_{n'} + \sum_{n'} \int_{z_1}^{z_2} \widehat{K}_{nn'}^-(z, z') \frac{\mathrm{d}y_{n'}^-(z')}{\mathrm{d}z'} \, \mathrm{d}z', \quad z_1 \leq z \leq z_2$$

and

$$y_n^+(z) = \frac{1}{2\mathrm{i}} \left(\mathrm{e}^{2\mathrm{i}kz} - \mathrm{e}^{2\mathrm{i}kz_1} \right) \sum_{n'} T_{nn'} a_{n'}$$
$$+ \sum_{n'} \int_{z_1}^{z_2} \widehat{K}_{nn'}^+(z, z') \frac{\mathrm{d}y_{n'}^+(z')}{\mathrm{d}z'} \, \mathrm{d}z', \quad z_1 \leq z \leq z_2$$

where the function $\widehat{K}_{nn'}^\pm(z, z')$ (indefinite Fourier transform) is defined as

$$\widehat{K}_{nn'}^\pm(z, z') = k \int_{z_1 - z'}^{z - z'} K_{nn'}(t) \mathrm{e}^{\pm \mathrm{i}kt} \, \mathrm{d}t, \quad z_1 \leq z, z' \leq z_2 \tag{5.7}$$

We see that $\widehat{K}_{nn'}^\pm(z_1, z') = 0$, $z_1 \leq z' \leq z_2$ and in particular $\widehat{K}_{nn'}^\pm(z_1, z_1) = 0$.

Integrate these new equations by parts, and we obtain (use $y_n^\pm(z_1) = 0$)

$$y_n^-(z) = k(z - z_1) \sum_{n'} T_{nn'} a_{n'} + \sum_{n'} \widehat{K}_{nn'}^-(z, z_2) y_{n'}^-(z_2)$$
$$- \sum_{n'} \int_{z_1}^{z_2} \frac{\mathrm{d}\widehat{K}_{nn'}^-(z, z')}{\mathrm{d}z'} y_{n'}^-(z') \, \mathrm{d}z', \quad z_1 \leq z \leq z_2 \tag{5.8}$$

and

$$y_n^+(z) = \frac{1}{2\mathrm{i}} \left(\mathrm{e}^{2\mathrm{i}kz} - \mathrm{e}^{2\mathrm{i}kz_1} \right) \sum_{n'} T_{nn'} a_{n'} + \sum_{n'} \widehat{K}_{nn'}^+(z, z_2) y_{n'}^+(z_2)$$

$$- \sum_{n'} \int_{z_1}^{z_2} \frac{\mathrm{d}\widehat{K}_{nn'}^+(z, z')}{\mathrm{d}z'} y_{n'}^+(z') \, \mathrm{d}z', \quad z_1 \le z \le z_2 \quad (5.9)$$

The derivative of the $\widehat{K}_{nn'}^+(z, z')$ can be explicitly expressed as

$$\frac{\mathrm{d}\widehat{K}_{nn'}^\pm(z, z')}{\mathrm{d}z'} = k K_{nn'}(z_1 - z') \mathrm{e}^{\pm \mathrm{i}k(z_1 - z')} - k K_{nn'}(z - z') \mathrm{e}^{\pm \mathrm{i}k(z - z')}, \quad z_1 \le z, z' \le z_2$$

The evaluation of the kernel $\widehat{K}_{nn'}^\pm(z, z')$ involves the indefinite Fourier transform of the kernel $K_{nn'}(z - z')$. This is a subject of further development in Section 5.3.

5.3 Evaluation of the kernels $K_{nn'}(z)$ and $\widehat{K}_{nn'}^\pm(z, z')$

The translation matrix $\mathcal{P}_{nn'}(k\boldsymbol{r})$ has the form, see (B.2) in Appendix B

$$\mathcal{P}_{nn'}(k\boldsymbol{r}) = \sum_{\lambda=|l-l'|+|\tau-\tau'|}^{l+l'} h_\lambda^{(1)}(kr) \left(A_{nn'\lambda}(\phi) P_\lambda^{m-m'}(\cos\theta) + B_{nn'\lambda}(\phi) P_\lambda^{m+m'}(\cos\theta) \right)$$

where ϕ is the azimuthal angle of the translation vector $\boldsymbol{r}$. The summation in the index λ in $A_{nn'\lambda}$ and $B_{nn'\lambda}$ is effectively only over a limited interval of integers. All other components of $A_{nn'\lambda}$ and $B_{nn'\lambda}$ vanish. Note that the last index in $A_{nn'\lambda}$ and $B_{nn'\lambda}$ can take the value $\lambda = 0$.

Integration w.r.t. the lateral variables x and y in the definition of the kernel $K_{nn'}(z)$ in (5.2) implies that the following dimensionless integral is of interest in the computation: ($\boldsymbol{\rho} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$)

$$I_l(z) = k^2 \int_{\rho(z)}^\infty g(|\boldsymbol{\rho} + z\hat{\boldsymbol{z}}|) h_l^{(1)}(k\sqrt{\rho^2 + z^2}) P_l(z/\sqrt{\rho^2 + z^2}) \rho \, \mathrm{d}\rho, \quad z \in \mathbb{R}$$

where

$$\rho(z) = \begin{cases} \sqrt{4a^2 - z^2}, & -2a \le z \le 2a \\ 0, & |z| > 2a \end{cases}$$

The integral is continuous as a function of z provided $a > 0$. The lower limit $\rho(r)$ cuts out the origin where the spherical Hankel function is singular.

The kernel $K(z)$ of the integral equation in (5.2) then becomes

$$K_{nn'}(z) = \frac{n_0}{k^3} \sum_{n''} T_{nn''} \sum_{\lambda=|l'-l''|+|\tau'-\tau''|}^{l'+l''} I_\lambda(-z) \mathcal{A}_{nn'\lambda}, \quad |z| \le z_2 - z_1 \quad (5.10)$$

where $\mathcal{A}_{nn'\lambda}$ is the azimuthal average of the translation matrix $\mathcal{P}_{nn'}(k\boldsymbol{r})$, which are explicitly evaluated in Appendix B.1.
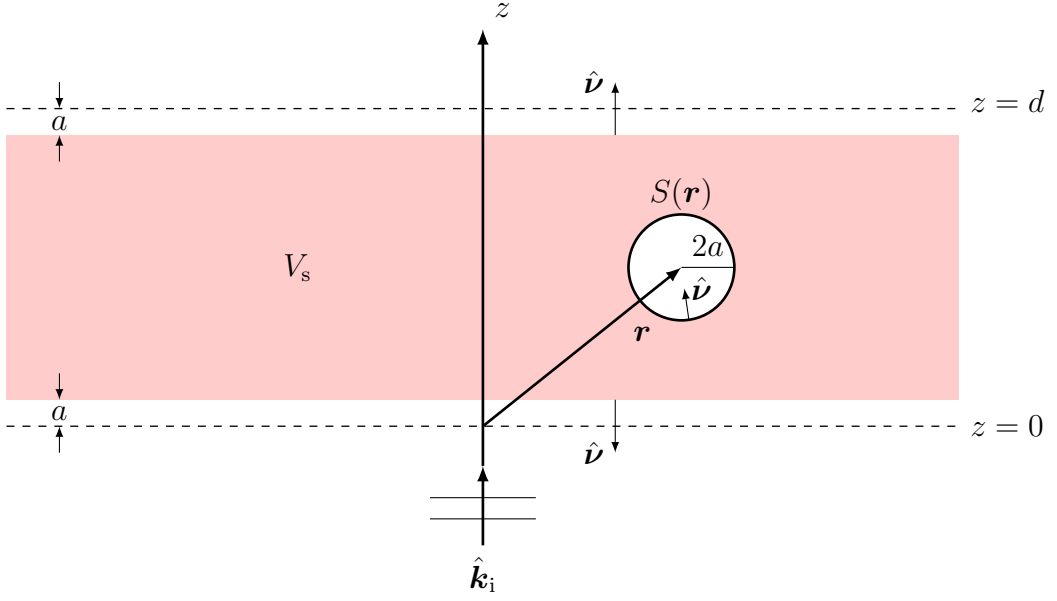
Figure 6: The geometry of the stratified scattering region, and the exclusion volume bounded by $S(\boldsymbol{r})$. Note the direction of the unit normals on the bounding surfaces. It suffices, due to translational invariance in the lateral variables, to restrict the exclusion volume to the z axis.

We now try to evaluating the integral $I_l(z)$. Due to parity of the associated Legendre functions, we have

$$I_l(-z) = (-1)^l I_l(z)$$

The pair distribution function $g(r)$ models the probability of finding the sphere at a distance r provided one particle is located at the origin. As discussed above, several models for this function exist. All have the property that $g(r) = 0$ when $0 < r < 2a$. This is the impenetrability condition. It is convenient to divide the integral $I_l(z)$ in a sum of two parts

$$I_{1l}(z) = k^2 \int_{\rho(z)}^{\infty} h_l^{(1)}(k\sqrt{\rho^2 + z^2})P_l(z/\sqrt{\rho^2 + z^2})\rho \; \mathrm{d}\rho, \quad z \in \mathbb{R}$$

and

$$I_{2l}(z) = k^2 \int_{\rho(z)}^{\infty} \left(g(|\boldsymbol{\rho} + z\hat{\boldsymbol{z}}|) - 1\right) h_l^{(1)}(k\sqrt{\rho^2 + z^2})P_l(z/\sqrt{\rho^2 + z^2})\rho \; \mathrm{d}\rho, \quad z \in \mathbb{R}$$

For simplicity we restrict ourselves to the important case of hole-corrections, $g(r) = H(r - 2a)$, where $H(x)$ is the Heaviside function, see Figure 6. The appropriate integrals then becomes $I_{2l}(z) = 0$ and $I_l(z) = I_{1l}(z)$.

$$I_l(z) = k^2 \int_{\rho(z)}^{\infty} h_l^{(1)}\left(k\sqrt{\rho^2 + z^2}\right) P_l\left(z/\sqrt{\rho^2 + z^2}\right) \rho \; \mathrm{d}\rho \tag{5.11}$$

In Ref. [20] it is shown that the integral in the interval $z \leq -2a$ is

$$I_l(z) = i^l e^{-ikz}$$

and in the interval $z \in (-2a, 2a)$ it can be expressed as a finite sum of spherical waves, $i.e.$,

$$I_l(z) = i^{1-l} 2ka h_0^{(1)}(2ka) P_{l-2[l/2]}(z/2a)$$
$$+ \sum_{n=0}^{[l/2]-1} (-1)^n 2ka h_{l-2n-1}^{(1)}(2ka) \left(P_{l-2n}(z/2a) - P_{l-2n-2}(z/2a) \right)$$

In the interval $z \geq 2a$, the integral is

$$I_l(z) = i^{-l} e^{ikz}$$

The explicit values of $I_l(z)$, $l = 0, 1, 2$ are [20]

$$I_0(z) = \begin{cases} e^{-ikz}, & z \leq -2a \\ e^{2ika}, & -2a < z < 2a \\ e^{ikz}, & z \geq 2a \end{cases}$$

$$I_1(z) = \begin{cases} i e^{-ikz}, & z \leq -2a \\ -i e^{2ika} \dfrac{z}{2a}, & -2a < z < 2a \\ -i e^{ikz}, & z \geq 2a \end{cases}$$

$$I_2(z) = \begin{cases} -e^{-ikz}, & z \leq -2a \\ e^{2ika} \dfrac{4(3i + 2ka) - 3(i + 2ka)(z/a)^2}{16ka}, & -2a < z < 2a \\ -e^{ikz}, & z \geq 2a \end{cases}$$

The kernel $K_{nn'}(z)$ in (5.10) then becomes

$$K_{nn'}(z) = \frac{n_0}{k^3} \sum_{n''} T_{nn''} \sum_{\lambda = |l-l'|+|\tau-\tau'|}^{l+l'} \left((1 - H(-z+2a)) i^\lambda e^{ikz} + H(-z-2a) i^{-\lambda} e^{-ikz} \right.$$
$$\left. + (H(-z+2a) - H(-z-2a)) I_\lambda(-z) \right) \mathcal{A}_{nn'\lambda}, \quad |z| \leq z_2 - z_1$$

The kernel $\widehat{K}_{nn'}^{\pm}(z, z')$ in (5.7) has also a solution in terms of spherical waves [20]. The non-trivial part of the evaluation of this kernel can be expressed in the indefinite Fourier transform of the function $I_l(z)$, $i.e.$,

$$\widehat{I}_l^{\pm}(z) = k \int_{-(z_2-z_1)}^{z} I_l(t) e^{\pm ikt} \, dt, \quad |z| \leq z_2 - z_1, \quad l = 0, 1, 2, \ldots$$

In order to evaluate this integral, the interval $[-(z_2 - z_1), z_2 - z_1]$ is divided in three parts.[15] In the interval $-(z_2 - z_1) \leq z < -2a$, we have

$$\widehat{I}_l^{\pm}(z) = \mathrm{i}^l k \int_{-(z_2-z_1)}^{z} \mathrm{e}^{\mathrm{i}(\pm 1 - 1)kt} \, \mathrm{d}t = \mathrm{i}^l \begin{cases} k(z + z_2 - z_1) \\ \frac{1}{2\mathrm{i}} \left(\mathrm{e}^{2\mathrm{i}k(z_2-z_1)} - \mathrm{e}^{-2\mathrm{i}kz} \right) \end{cases}$$

and the interval $-2a < z < 2a$, we have

$$\widehat{I}_l^{\pm}(z) = \mathrm{i}^l \begin{cases} k(-2a + z_2 - z_1) \\ \frac{1}{2\mathrm{i}} \left(\mathrm{e}^{2\mathrm{i}k(z_2-z_1)} - \mathrm{e}^{4\mathrm{i}ka} \right) \end{cases} + \mathrm{i}^{1-l} 2ka h_0^{(1)}(2ka) h_{l-2[l/2]}^{\pm}(z)$$

$$+ \sum_{n=0}^{[l/2]-1} (-1)^n 2ka h_{l-2n-1}^{(1)}(2ka) \left(h_{l-2n}^{\pm}(z) - h_{l-2n-2}^{\pm}(z) \right)$$

and the interval $2a < z < z_2 - z_1$, we have

$$\widehat{I}_l^{\pm}(z) = \mathrm{i}^l \begin{cases} k(-2a + z_2 - z_1) \\ \frac{1}{2\mathrm{i}} \left(\mathrm{e}^{2\mathrm{i}k(z_2-z_1)} - \mathrm{e}^{4\mathrm{i}ka} \right) \end{cases} + \mathrm{i}^{1-l} 8(ka)^2 h_0^{(1)}(2ka)(\pm\mathrm{i})^{l-2[l/2]} j_{l-2[l/2]}(2ka)$$

$$+ 8(ka)^2 (\pm\mathrm{i})^l \sum_{n=0}^{[l/2]-1} h_{l-2n-1}^{(1)}(2ka) \left(j_{l-2n}(2ka) + j_{l-2n-2}(2ka) \right)$$

$$+ \mathrm{i}^{-l} \begin{cases} \frac{1}{2\mathrm{i}} \left(\mathrm{e}^{2\mathrm{i}kz} - \mathrm{e}^{4\mathrm{i}ka} \right) \\ k(z - 2a) \end{cases}$$

where the function

$$h_l^{\pm}(z) = k \int_{-2a}^{z} P_l(t/(2a)) \mathrm{e}^{\pm\mathrm{i}kt} \, \mathrm{d}t = 2ka h_l(z/(2a), \pm 2ka), \quad |z| \leq 2a$$

is expressed in terms of the indefinite Fourier transform of the Legendre polynomials, i.e.,[16]

$$h_l(\eta, \varsigma) = \int_{-1}^{\eta} P_l(t) \mathrm{e}^{\mathrm{i}\varsigma t} \, \mathrm{d}t, \quad |\eta| \leq 1$$

For $z = 2a$ the integral is a spherical Bessel function, *viz.*,

$$h_l^{\pm}(2a) = 2ka \int_{-1}^{1} P_l(t) \mathrm{e}^{\pm\mathrm{i}2kat} \, \mathrm{d}t = 4ka(\pm\mathrm{i})^l j_l(2ka)$$

The functions $h(\eta, \varsigma)$ can be generated from the iteration scheme [20]

$$h_{l+1}(\eta, \varsigma) = \frac{1}{\mathrm{i}\varsigma} \left(P_{l+1}(\eta) - P_{l-1}(\eta) \right) \mathrm{e}^{\mathrm{i}\varsigma\eta} - \frac{2l+1}{\mathrm{i}\varsigma} h_l(\eta, \varsigma) + h_{l-1}(\eta, \varsigma), \quad l = 1, 2, 3, \dots$$

---

[15]We always assume $z_2 - z_1 > 2a$. The slab is otherwise so thin that there is no room for any scatterers.

[16]This quantity is also useful in the evaluation of the reflected and transmitted fields, if the method presented in Appendix E is used to solve the system of integral equations in (5.1).

with starting values

$$h_0(\eta, \zeta) = \eta h_0^{(1)}(\zeta\eta) + h_0^{(2)}(\zeta)$$

and

$$h_1(\eta, \zeta) = \mathrm{i}\left(\eta^2 h_1^{(1)}(\zeta\eta) + h_1^{(2)}(\zeta)\right)$$

The explicit solution is

$$h_l(\eta, \zeta) = f_l(\eta, \zeta)\mathrm{e}^{\mathrm{i}\zeta\eta} + \mathrm{i}^l h_l^{(2)}(\zeta), \quad l = 0, 1, 2, 3, \ldots$$

where

$$f_l(\eta, \zeta) = \mathrm{i}^l h_l^{(1)}(\zeta)\left\{\sum_{k=1}^{l}\frac{1}{\zeta h_{k-1}^{(1)}(\zeta)h_k^{(1)}(\zeta)}\left(-\sum_{n=0}^{k}\mathrm{i}^{-n+1}(2n+1)\frac{h_n^{(1)}(\zeta)}{\zeta}P_n(\eta)\right.\right.$$
$$\left.\left. + \mathrm{i}^{-k+2}h_k^{(1)}(\zeta)P_{k-1}(\eta) + \mathrm{i}^{-k+1}h_{k+1}^{(1)}(\zeta)P_k(\eta)\right) - \mathrm{i}\frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)}\right\}, \quad l = 0, 1, 2, \ldots$$

5.4   Integral equation — half space

If the medium is confined in a half space, $z \geq 0$, the analysis is very similar and follows from the slab result by taking the limit $d \to \infty$ in an appropriate way. The reflected field is ($z_1 = a$)

$$\boldsymbol{E}_\mathrm{r} = \frac{2\pi n_0}{k^2}\sum_n \mathrm{i}^{l-\tau+1}\boldsymbol{A}_n(\hat{\boldsymbol{z}})\int_{z_1}^{\infty}\mathrm{e}^{\mathrm{i}kz'}\langle f_n\rangle(z')\,\mathrm{d}z', \quad z < 0$$

and where the integral equation is

$$\langle f_n\rangle(z) = \mathrm{e}^{\mathrm{i}kz}\sum_{n'}T_{nn'}a_{n'} + k\int_{z_1}^{\infty}\sum_{n'}K_{nn'}(z-z')\langle f_{n'}\rangle(z')\,\mathrm{d}z', \quad z_1 < z$$

6   Approximations

In this section we exploit two different approximations of the final expression of the transmission and reflection coefficients in Section 3, *viz.*, the tenuous (sparse media) and low frequency approximations.

6.1   Tenuous media

In a tenuous media we assume the interaction is only due to single scattering, *i.e.*, first order term in (5.1)

$$\langle f_n\rangle(z) = \mathrm{e}^{\mathrm{i}kz}\sum_{n'}T_{nn'}a_{n'}, \quad z_1 < z < z_2$$

The transmitted and reflected fields are given by (5.3) and (5.4), which implies using (2.3)

$$\boldsymbol{E}_\text{t} = \boldsymbol{E}_0 + \frac{2\pi n_0 \left(z_2 - z_1\right)}{k^2} \sum_{n,n'} \mathrm{i}^{-l+\tau-1} \boldsymbol{A}_n(\hat{\boldsymbol{z}}) T_{nn'} a_{n'}$$

$$= \left(\mathbf{I} + \frac{8\pi^2 n_0 \left(z_2 - z_1\right)}{k^2} \sum_{n,n'} \mathrm{i}^{l'-l+\tau-\tau'} \boldsymbol{A}_n(\hat{\boldsymbol{z}}) T_{nn'} \boldsymbol{A}_{n'}(\hat{\boldsymbol{z}}) \right) \cdot \boldsymbol{E}_0$$

and

$$\boldsymbol{E}_\text{r} = \frac{2\pi n_0}{k^2} \frac{\left(\mathrm{e}^{2\mathrm{i}kz_2} - \mathrm{e}^{2\mathrm{i}kz_1}\right)}{2\mathrm{i}k} \sum_{n,n'} \mathrm{i}^{l-\tau+1} \boldsymbol{A}_n(\hat{\boldsymbol{z}}) T_{nn'} a_{n'}$$

$$= -\frac{8\pi^2 n_0}{k^2} \frac{\left(\mathrm{e}^{2\mathrm{i}kz_2} - \mathrm{e}^{2\mathrm{i}kz_1}\right)}{2\mathrm{i}k} \sum_{n,n'} \mathrm{i}^{l'+l-\tau-\tau'} \boldsymbol{A}_n(\hat{\boldsymbol{z}}) T_{nn'} \boldsymbol{A}_{n'}(\hat{\boldsymbol{z}}) \cdot \boldsymbol{E}_0$$

respectively. Since the scattering dyadic for the single scatterer in the forward and backward directions for a plane wave with incident propagation $\hat{\boldsymbol{k}}_\text{i}$ are

$$\mathbf{S}(\hat{\boldsymbol{k}}_\text{i}, \hat{\boldsymbol{k}}_\text{i}) = \frac{4\pi}{\mathrm{i}k} \sum_{n,n'} \mathrm{i}^{l'-l+\tau-\tau'} \boldsymbol{A}_n(\hat{\boldsymbol{k}}_\text{i}) T_{nn'} \boldsymbol{A}_{n'}(\hat{\boldsymbol{k}}_\text{i}),$$

and

$$\mathbf{S}(-\hat{\boldsymbol{k}}_\text{i}, \hat{\boldsymbol{k}}_\text{i}) = -\frac{4\pi}{\mathrm{i}k} \sum_{n,n'} \mathrm{i}^{l'+l-\tau-\tau'} \boldsymbol{A}_n(\hat{\boldsymbol{k}}_\text{i}) T_{nn'} \boldsymbol{A}_{n'}(\hat{\boldsymbol{k}}_\text{i})$$

respectively, we get

$$\boldsymbol{E}_\text{t} = \left(\mathbf{I} + \frac{2\mathrm{i}\pi n_0}{k^3} k \left(z_2 - z_1\right) k \mathbf{S}(\hat{\boldsymbol{z}}, \hat{\boldsymbol{z}}) \right) \cdot \boldsymbol{E}_0$$

and

$$\boldsymbol{E}_\text{r} = \frac{2\mathrm{i}\pi n_0}{k^3} \frac{\left(\mathrm{e}^{2\mathrm{i}kz_2} - \mathrm{e}^{2\mathrm{i}kz_1}\right)}{2\mathrm{i}} k \mathbf{S}(-\hat{\boldsymbol{z}}, \hat{\boldsymbol{z}}) \cdot \boldsymbol{E}_0$$

respectively. The derivation of the transmitted field above agrees with the result in the literature [1,19,35]. It is also in agreement with the tenuous media approximation of Bouguer-Beer law [16,29] or with the transmission result (coherent part) obtained by the RTE [16], *i.e.*,

$$T = \mathrm{e}^{-n_0 \sigma_\text{ext}(z_2 - z_1)} = \mathrm{e}^{-2\frac{2\pi n_0 k(z_2 - z_1)}{k^3} k \operatorname{Im} \boldsymbol{p}_\text{e}^* \cdot \mathbf{S}(\hat{\boldsymbol{z}}, \hat{\boldsymbol{z}}) \cdot \boldsymbol{p}_\text{e}} = \left| \mathrm{e}^{\mathrm{i}\frac{2\pi n_0 k(z_2 - z_1)}{k^3} k \boldsymbol{p}_\text{e}^* \cdot \mathbf{S}(\hat{\boldsymbol{z}}, \hat{\boldsymbol{z}}) \cdot \boldsymbol{p}_\text{e}} \right|^2$$

where we in the third equality have used the optical theorem [19], and where the polarization state of the incident field is $\boldsymbol{p}_\text{e} = \boldsymbol{E}_0 / \left| \boldsymbol{E}_0 \right|$.

If we restrict to spherical dielectric obstacles so that $T_{nn'}$ is diagonal in its indices (we adopt the notion $t_{\tau l}$), we get

$$\boldsymbol{E}_\text{t} = \boldsymbol{E}_0 \left(1 + \frac{3fkd}{4(ka)^3} \sum_{l=1}^{\infty} (2l+1) \left(t_{1l} + t_{2l}\right) \right) = t\boldsymbol{E}_0$$

and

$$\boldsymbol{E}_{\mathrm{r}} = \boldsymbol{E}_0 \frac{3f}{4(ka)^3(1 - 2a/d)} \frac{\left(\mathrm{e}^{2\mathrm{i}kz_2} - \mathrm{e}^{2\mathrm{i}kz_1}\right)}{2\mathrm{i}} \sum_{l=1}^{\infty} (-1)^l (2l+1) \left(t_{1l} - t_{2l}\right) = r\boldsymbol{E}_0$$

$$\approx \boldsymbol{E}_0 \frac{3fkd}{4(ka)^3} \sum_{l=1}^{\infty} (-1)^l (2l+1) \left(t_{1l} - t_{2l}\right)$$

where we used the dimensionless volume fraction, $f = 4\pi a^3 (1 - 2a/d) n_0/3$,[17] and where the last approximation holds for the thin slab approximation, and where we identified the co-polarized scattering dyadic component in the forward and backward directions for a single scatterer characterized by the transition matrix $t_{\tau l}$, i.e.,

$$\mathbf{S}(\hat{\boldsymbol{z}}, \hat{\boldsymbol{z}}) = \frac{1}{2\mathrm{i}k} \mathbf{I} \sum_{l=1}^{\infty} (2l+1) \left(t_{1l} + t_{2l}\right)$$

and

$$\mathbf{S}(-\hat{\boldsymbol{z}}, \hat{\boldsymbol{z}}) = \frac{1}{2\mathrm{i}k} \mathbf{I} \sum_{l=1}^{\infty} (2l+1)(-1)^l \left(t_{1l} - t_{2l}\right)$$

Moreover, the transmission and reflection coefficients t and r for the tenuous media are

$$t = 1 + \frac{3fkd}{4(ka)^3} \sum_{l=1}^{\infty} (2l+1) \left(t_{1l} + t_{2l}\right)$$

and

$$r = \frac{3f}{4(ka)^3(1 - 2a/d)} \frac{\left(\mathrm{e}^{2\mathrm{i}kz_2} - \mathrm{e}^{2\mathrm{i}kz_1}\right)}{2\mathrm{i}} \sum_{l=1}^{\infty} (-1)^l (2l+1) \left(t_{1l} - t_{2l}\right)$$

The result in this section is illustrated in Figure 7.

6.2 Low-frequency approximation

The aim of this section is to solve the integral equations (5.8) and (5.9) for $y_n^{\pm}(z_2)$ at low frequencies. We proceed by keeping only the dominant power in ka in all expressions. We also restrict ourselves to spherical dielectric obstacles so that $T_{nn'}$ is diagonal in all its indices and independent of m and σ (we adopt the notion $t_{\tau l}$). The dominant contribution in powers of ka of each diagonal entry of the transition matrix is then $(ka)^{2l+1}$.

We need to find the dominant contribution in powers of ka in the integral kernels $K_{nn'}(z)$ and $\widehat{K}_{nn'}^{\pm}(z, z_2)$. The most singular term in the integral $I_l(z)$ is of the order $(ka)^{1-l}$ $(I_0(z) = O(1))$. This leads to a dominant contribution of the kernel $K_{nn'}(z)$, see (5.10), of $(ka)^{-3+2l+1+(1-l-l')} = (ka)^{l-l'-1}$. Since the source terms in (5.8) and (5.9) have a dominant contribution $(ka)^3$ when $l = 1$, we conclude that the dominant contribution to the solutions $y_n^{\pm}(z)$ is $(ka)^3$, which occurs for $l = 1$.

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[17]Note the difference in volume between the material confinement in $[0, d]$ and the volume of the potential local origins in $[a, d-a]$.
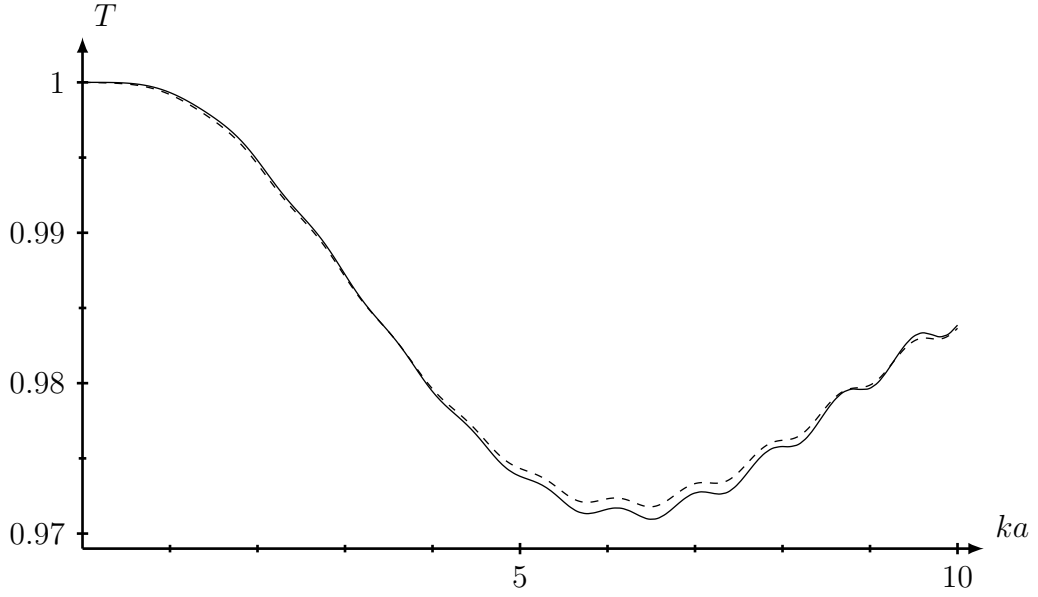
Figure 7: The transmissivity $T = |t|^2$ as a function of the electrical size ka for a slab of thickness $d/a = 100$ and constant volume fraction $f = 0.0001$ consisting of dielectric spheres. The tenuous approximation of the transmissivity is displayed. The material parameters are $\epsilon_1/\epsilon = 1.33^2$ (solid curve), $\epsilon_1/\epsilon = 1.33^2(1 + 0.01\mathrm{i})$ (dashed curve) and $\mu_1/\mu = 1$.

For a plane wave of normal incidence, the expansion coefficients of the plane wave are given by (2.4), *i.e.*, (only $l = 1$ and $m = 1$ contribute, the index $\sigma = \mathrm{e}$ corresponds to the upper line and $\sigma = \mathrm{o}$ to the lower line)

$$\begin{cases} a_{1\sigma11} = -\mathrm{i}\sqrt{6\pi} \left(\hat{\boldsymbol{z}} \times \begin{Bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{Bmatrix} \right) \cdot \boldsymbol{E}_0 = -\mathrm{i}\sqrt{6\pi} \begin{Bmatrix} \hat{\boldsymbol{y}} \\ -\hat{\boldsymbol{x}} \end{Bmatrix} \cdot \boldsymbol{E}_0 \\ a_{2\sigma11} = \sqrt{6\pi} \begin{Bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{Bmatrix} \cdot \boldsymbol{E}_0 \end{cases}$$

With only $l = 1$ and $m = 1$ contributing, we get, to leading order in powers of ka, the kernel $K_{nn'}(z)$, see (5.10) and (D.1) in Appendix D (we adopt the dual index convention $\bar{1} = 2, \bar{2} = 1$ and $\bar{\mathrm{e}} = \mathrm{o}, \bar{\mathrm{o}} = \mathrm{e}$)

$$\begin{cases} K_{\tau\sigma11,\tau\sigma11}(z) = -\dfrac{3ft_{\tau1}}{2(ka)^3(1 - 2a/d)} \begin{cases} O(1), & |z| \geq 2a \\ \mathrm{i}\dfrac{12 - 3(z/a)^2}{32ka} + O(1), & |z| < 2a \end{cases} \\ K_{\tau\sigma11,\overline{\tau\sigma}11}(z) = O(1) \end{cases}$$

where we used the dimensionless volume fraction, $f = 4\pi a^3(1 - 2a/d)n_0/3$. The kernels $\widehat{K}_{nn'}^{\pm}(z_2, z_2)$ and $\partial_{z'}\widehat{K}_{nn'}^{\pm}(z_2, z')$ have to be evaluated. We have to leading

order in powers of ka, $\widehat{K}_{\tau\sigma11,\overline{\tau\sigma}11}(z) = O(ka)$ and (we always assume $z_2 - z_1 > 2a$)

$$\widehat{K}^{\pm}_{\tau\sigma11,\tau\sigma11}(z_2, z_2) = k \int_{z_1-z_2}^{0} K_{\tau\sigma11,\tau\sigma11}(t) \mathrm{e}^{\pm ikt}\, \mathrm{d}t$$

$$= -\frac{3\mathrm{i}ft_{\tau1}}{2(ka)^3} k \int_{-2a}^{0} \frac{12 - 3(t/a)^2}{32ka} \mathrm{e}^{\pm ikt}\, \mathrm{d}t + O(ka) = -\frac{3\mathrm{i}ft_{\tau1}}{4(ka)^3} + O(ka)$$

and

$$\frac{\mathrm{d}\widehat{K}^{\pm}_{\tau\sigma11,\tau\sigma11}(z_2, z')}{\mathrm{d}z'} = kK_{\tau\sigma11,\tau\sigma11}(z_1 - z')\mathrm{e}^{\pm ik(z_1-z')} - kK_{\tau\sigma11,\tau\sigma11}(z_2 - z')\mathrm{e}^{\pm ik(z_2-z')}$$

$$= \frac{3ft_{\tau1}}{2(ka)^3}
\begin{cases}
-\mathrm{i}k\dfrac{12 - 3(z_1 - z')^2 a^{-2}}{32ka}\mathrm{e}^{\pm ik(z_1-z')} + O(1), & z_1 \leq z' \leq z_1 + 2a \\[2mm]
O(1), & z_1 + 2a \leq z' \leq z_2 - 2a \\[2mm]
\mathrm{i}k\dfrac{12 - 3(z_2 - z')^2 a^{-2}}{32ka}\mathrm{e}^{\pm ik(z_2-z')} + O(1), & z_2 - 2a \leq z' \leq z_2
\end{cases}$$

To leading order in ka, the integrals in (5.8) and (5.9) contribute as

$$\sum_{n'} \int_{z_1}^{z_2} \frac{\mathrm{d}\widehat{K}^{\pm}_{nn'}(z_2, z')}{\mathrm{d}z'} y^{\pm}_{n'}(z')\, \mathrm{d}z'$$

$$= y^{\pm}_n(z_2)\frac{3\mathrm{i}ft_{\tau1}}{2(ka)^3} \int_{z_2-2a}^{z_2} k\frac{12 - 3(z_2 - z')^2 a^{-2}}{32ka}\mathrm{e}^{\pm ik(z_2-z')}\, \mathrm{d}z' + O(ka)$$

$$= y^{\pm}_n(z_2)\frac{3\mathrm{i}ft_{\tau1}}{4(ka)^3} + O(ka)$$

For a spherical scatterer, the integral equations (5.8) and (5.9) decouple in two blocks, see Appendix C. Evaluated at $z = z_2$, the integral equations are

$$\begin{pmatrix} y^{-}_{1\mathrm{e}11}(z_2) \\ y^{-}_{2\mathrm{o}11}(z_2) \end{pmatrix} = \sqrt{6\pi}\boldsymbol{E}_0 \cdot \hat{\boldsymbol{y}}k(z_2 - z_1)\begin{pmatrix} -\mathrm{i}t_{11} \\ t_{21} \end{pmatrix} - \frac{3\mathrm{i}f}{2(ka)^3}\begin{pmatrix} t_{11}y^{-}_{1\mathrm{e}11}(z_2) \\ t_{21}y^{-}_{2\mathrm{o}11}(z_2) \end{pmatrix}$$

$$\begin{pmatrix} y^{-}_{1\mathrm{o}11}(z_2) \\ y^{-}_{2\mathrm{e}11}(z_2) \end{pmatrix} = \sqrt{6\pi}\boldsymbol{E}_0 \cdot \hat{\boldsymbol{x}}k(z_2 - z_1)\begin{pmatrix} \mathrm{i}t_{11} \\ t_{21} \end{pmatrix} - \frac{3\mathrm{i}f}{2(ka)^3}\begin{pmatrix} t_{11}y^{-}_{1\mathrm{o}11}(z_2) \\ t_{21}y^{-}_{2\mathrm{e}11}(z_2) \end{pmatrix}$$

and

$$\begin{pmatrix} y^{+}_{1\mathrm{e}11}(z_2) \\ y^{+}_{2\mathrm{o}11}(z_2) \end{pmatrix} = \sqrt{6\pi}\boldsymbol{E}_0 \cdot \hat{\boldsymbol{y}}\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{2\mathrm{i}kz_2} - \mathrm{e}^{2\mathrm{i}kz_1}\right)\begin{pmatrix} -\mathrm{i}t_{11} \\ t_{21} \end{pmatrix} - \frac{3\mathrm{i}f}{2(ka)^3}\begin{pmatrix} t_{11}y^{+}_{1\mathrm{e}11}(z_2) \\ t_{21}y^{+}_{2\mathrm{o}11}(z_2) \end{pmatrix}$$

$$\begin{pmatrix} y^{+}_{1\mathrm{o}11}(z_2) \\ y^{+}_{2\mathrm{e}11}(z_2) \end{pmatrix} = \sqrt{6\pi}\boldsymbol{E}_0 \cdot \hat{\boldsymbol{x}}\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{2\mathrm{i}kz_2} - \mathrm{e}^{2\mathrm{i}kz_1}\right)\begin{pmatrix} \mathrm{i}t_{11} \\ t_{21} \end{pmatrix} - \frac{3\mathrm{i}f}{2(ka)^3}\begin{pmatrix} t_{11}y^{+}_{1\mathrm{o}11}(z_2) \\ t_{21}y^{+}_{2\mathrm{e}11}(z_2) \end{pmatrix}$$

respectively. Solve for $y^{\pm}_n(z_2)$. We get

$$y^{\pm}_{\tau\sigma11}(z_2) = \left\{ \begin{array}{l} \dfrac{1}{2\mathrm{i}}\left(\mathrm{e}^{2\mathrm{i}kz_2} - \mathrm{e}^{2\mathrm{i}kz_1}\right) \\[2mm] k(z_2 - z_1) \end{array} \right\} A_{\tau\sigma}$$

where

$$A_{\tau\sigma} = \frac{2\sqrt{6\pi}(ka)^3 t_{\tau 1}}{2(ka)^3 + 3\mathrm{i}ft_{\tau 1}} \boldsymbol{E}_0 \cdot \{-\mathrm{i}\delta_{\tau,1}\left(\delta_{\sigma,\mathrm{e}}\hat{\boldsymbol{y}} - \delta_{\sigma,\mathrm{o}}\hat{\boldsymbol{x}}\right) + \delta_{\tau,2}\left(\delta_{\sigma,\mathrm{e}}\hat{\boldsymbol{x}} + \delta_{\sigma,\mathrm{o}}\hat{\boldsymbol{y}}\right))\}$$

The transmitted and reflected amplitudes are finally obtained by (5.5) and (5.6), respectively. We get

$$\boldsymbol{E}_{\mathrm{t}} = \boldsymbol{E}_0 - \frac{3\mathrm{i}f}{2(ka)^3}\sqrt{\frac{3}{8\pi}}\left(\hat{\boldsymbol{x}}\left(y_{\mathrm{1o11}}^-(z_2) + \mathrm{i}y_{\mathrm{2e11}}^-(z_2)\right) - \hat{\boldsymbol{y}}\left(y_{\mathrm{1e11}}^-(z_2) - \mathrm{i}y_{\mathrm{2o11}}^-(z_2)\right)\right)$$

and

$$\boldsymbol{E}_{\mathrm{r}} = \frac{3\mathrm{i}f}{2(ka)^3}\sqrt{\frac{3}{8\pi}}\left(\hat{\boldsymbol{x}}\left(y_{\mathrm{1o11}}^+(z_2) - \mathrm{i}y_{\mathrm{2e11}}^+(z_2)\right) - \hat{\boldsymbol{y}}\left(y_{\mathrm{1e11}}^+(z_2) + \mathrm{i}y_{\mathrm{2o11}}^+(z_2)\right)\right)$$

which gives the low-frequency expressions of the transmitted and reflected amplitudes, *viz.*,

$$\boldsymbol{E}_{\mathrm{t}} = \left(1 + 9fk\left(z_2 - z_1\right)\left(\frac{t_{11}}{4(ka)^3 + 6\mathrm{i}ft_{11}} + \frac{t_{21}}{4(ka)^3 + 6\mathrm{i}ft_{21}}\right)\right)\boldsymbol{E}_0$$

and

$$\boldsymbol{E}_{\mathrm{r}} = \frac{9\mathrm{i}f}{2}\left(\mathrm{e}^{2\mathrm{i}kz_2} - \mathrm{e}^{2\mathrm{i}kz_1}\right)\left(\frac{t_{11}}{4(ka)^3 + 6\mathrm{i}ft_{11}} - \frac{t_{21}}{4(ka)^3 + 6\mathrm{i}ft_{21}}\right)\boldsymbol{E}_0$$

respectively. The transmission and reflection coefficients t and r in the low-frequency approximation are

$$t = 1 + 9fk\left(z_2 - z_1\right)\left(\frac{t_{11}}{4(ka)^3 + 6\mathrm{i}ft_{11}} + \frac{t_{21}}{4(ka)^3 + 6\mathrm{i}ft_{21}}\right)$$

and

$$r = \frac{9\mathrm{i}f}{2}\left(\mathrm{e}^{2\mathrm{i}kz_2} - \mathrm{e}^{2\mathrm{i}kz_1}\right)\left(\frac{t_{11}}{4(ka)^3 + 6\mathrm{i}ft_{11}} - \frac{t_{21}}{4(ka)^3 + 6\mathrm{i}ft_{21}}\right)$$

respectively.

If also the volume fraction f is low, we get

$$t = 1 + \frac{9f}{4(ka)^3}k\left(z_2 - z_1\right)\left(t_{11} + t_{21}\right)$$

and

$$r = \frac{9\mathrm{i}f}{8(ka)^3}\left(\mathrm{e}^{2\mathrm{i}kz_2} - \mathrm{e}^{2\mathrm{i}kz_1}\right)\left(t_{11} - t_{21}\right)$$

respectively. These expressions agree with corresponding expressions in Section 6.1.

The transmission coefficient t can be compared with the transmission coefficient of a dielectric slab of thickness $z_2 - z_1$ with permittivity ϵ' and permeability μ'. The thin thickness approximation is [19]

$$t = 1 + \mathrm{i}\left(\frac{\epsilon' - \epsilon}{2\epsilon} + \frac{\mu' - \mu}{2\mu}\right)kd + O\left((kd)^2\right) \tag{6.1}$$

6.2.1   Non-magnetic dielectric sphere

If the spherical particles are non-magnetic, $\mu = 1$, and with a permittivity ϵ_1, then $t_{11} = 0$ and to leading order in powers of ka for the real and imaginary parts (lossless materials assumed)

$$t_{21} = \frac{2\mathrm{i}(ka)^3}{3} y \left(1 + \frac{2\mathrm{i}(ka)^3}{3} y \right)$$

where

$$y = \frac{\epsilon_1 - \epsilon}{\epsilon_1 + 2\epsilon}$$

This form is due to the conservation relation of the transition matrix $\mathrm{Re}\, T + T^\dagger T = 0$ [56]. To leading order in powers of ka the transmission coefficients t is

$$t = 1 + \frac{3\mathrm{i}fy}{2} k\, (z_2 - z_1) \frac{1 + \frac{2\mathrm{i}y}{3}(ka)^3}{1 - fy \left(1 + \frac{2\mathrm{i}y}{3}(ka)^3 \right)}$$

Comparison with the thin thickness approximation of the transmission coefficient t in (6.1) implies

$$\frac{\epsilon'}{\epsilon} = \frac{1 + 2fy \left(1 + \frac{2\mathrm{i}y}{3}(ka)^3 \right)}{1 - fy \left(1 + \frac{2\mathrm{i}y}{3}(ka)^3 \right)}$$

which is the famous relation by Clausius-Mossotti [19].

The transmissivity in the low-frequency limit is illustrated in Figure 8.

7   Conclusion and discussion

In this paper we treat scattering of electromagnetic waves by an ensemble of randomly distributed, non-intersecting objects. The Quasi Crystalline Approximation is employed to break the hierarchy of increasing conditional probability densities, but otherwise the treatment is general and exact. In particular, the slab geometry is analyzed and a system of one-dimensional integral equations is obtained. The solution of this integral equation then determines the coherent reflection and transmission characteristics of the slab. No assumptions on far field conditions between the scatterers are made, and, in this respect, the present analysis generalizes the existing results, such as the Bouguer-Beer law. These characteristics can then be used to determine the bulk properties of the material in the slab, which gives a means to determine the validity of the homogenization procedure, by a comparison of the reflection and transmission properties obtained with the present formulation and the corresponding results obtained by a homogeneous slab.

The two natural approximations — tenuous media and low frequency approximations — show consistency, and, moreover, the low-frequency approximation gives results consistent with the law of Clausius-Mossotti. This last observation shows that the integral equation for the slab generalizes the Lorentz depolarization factor — analytically soluble in the low-frequency limit [8, 51].

There are several obvious extensions of the results presented in this paper. Oblique incidence and the incoherent or diffuse contribution are currently under
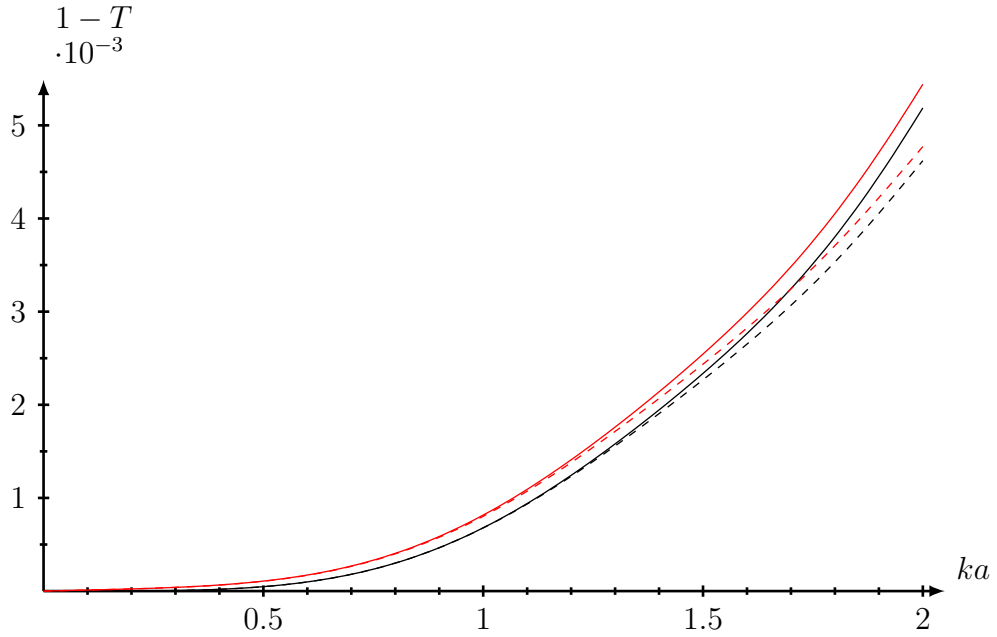
Figure 8: The low-frequency behaviour of the transmissivity $1 - T$ as a function of the electrical size ka for a slab of thickness $d/a = 100$ and constant volume fraction $f = 0.0001$ consisting of dielectric spheres. Solid curve depicts the tenuous media approximation and the dashed curve the low-frequency approximation. The material parameters are $\epsilon_1/\epsilon = 1.33^2$ (black curves), $\epsilon_1/\epsilon = 1.33^2(1 + 0.01\mathrm{i})$ (red curves) and $\mu_1/\mu = 1$.

investigation. Another background material in slab and more general pair distribution functions, $g(\boldsymbol{r}, \boldsymbol{r'})$, are straightforward extensions of the present analysis. The connection of the Bouguer-Beer law to the present approach is of interest. Moreover, an intriguing implication of the results in this paper is the possibility to characterize the coherent backscattering more closely. In recent years, the passive properties of a scatterer have been utilized to obtain several intriguing sum rules [14, 39, 40]. In particular, the slab geometry analyzed in this paper has been investigated. All these extensions are under investigation and will be reported elsewhere.

Acknowledgement

Appendix A   Spherical vector waves

The vector spherical harmonics are defined as [3]

$$
\begin{cases}
\boldsymbol{A}_{1n}(\hat{\boldsymbol{r}}) = \dfrac{1}{\sqrt{l(l+1)}} \nabla \times (\boldsymbol{r} Y_n(\hat{\boldsymbol{r}})) = \dfrac{1}{\sqrt{l(l+1)}} \nabla Y_n(\hat{\boldsymbol{r}}) \times \boldsymbol{r} \\[3mm]
\boldsymbol{A}_{2n}(\hat{\boldsymbol{r}}) = \dfrac{1}{\sqrt{l(l+1)}} x \nabla Y_n(\hat{\boldsymbol{r}}) \\[3mm]
\boldsymbol{A}_{3n}(\hat{\boldsymbol{r}}) = \hat{\boldsymbol{r}} Y_n(\hat{\boldsymbol{r}})
\end{cases}
$$

where the spherical harmonics are denoted by $Y_n(\hat{\boldsymbol{r}})$. The index n is a multi-index for the integer indices $l = 1, 2, 3, \ldots$, $m = 0, 1, \ldots, l$, and $\sigma = $ e,o (even and odd in the azimuthal angle).[18] From these definitions we see that the first two vector spherical harmonics, $\boldsymbol{A}_{1n}(\hat{\boldsymbol{r}})$ and $\boldsymbol{A}_{2n}(\hat{\boldsymbol{r}})$, are tangential to the unit sphere Ω in $\mathbb{R}^3$ and they are related as

$$
\begin{cases}
\hat{\boldsymbol{r}} \times \boldsymbol{A}_{1n}(\hat{\boldsymbol{r}}) = \boldsymbol{A}_{2n}(\hat{\boldsymbol{r}}) \\
\hat{\boldsymbol{r}} \times \boldsymbol{A}_{2n}(\hat{\boldsymbol{r}}) = -\boldsymbol{A}_{1n}(\hat{\boldsymbol{r}})
\end{cases}
$$

The vector spherical harmonics form an orthonormal set over the unit sphere Ω in $\mathbb{R}^3$, *i.e.*,

$$
\iint_{\Omega} \boldsymbol{A}_{\tau n}(\hat{\boldsymbol{r}}) \cdot \boldsymbol{A}_{\tau' n'}(\hat{\boldsymbol{r}}) \, \mathrm{d}\Omega = \delta_{nn'} \delta_{\tau\tau'}
$$

where $\mathrm{d}\Omega$ is the surface measure on the unit sphere.

The explicit values for $\hat{\boldsymbol{r}} = \hat{\boldsymbol{z}}$ are

$$
\begin{cases}
\boldsymbol{A}_{1\sigma ml}(\hat{\boldsymbol{z}}) = -\delta_{m1} \sqrt{\dfrac{2l+1}{8\pi}} \, \hat{\boldsymbol{z}} \times \begin{Bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{Bmatrix} \\[4mm]
\boldsymbol{A}_{2\sigma ml}(\hat{\boldsymbol{z}}) = \delta_{m1} \sqrt{\dfrac{2l+1}{8\pi}} \begin{Bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \end{Bmatrix}
\end{cases}
\tag{A.1}
$$

The radiating solutions to the Maxwell equations in vacuum are defined as (outgoing spherical vector waves)

$$
\begin{cases}
\boldsymbol{u}_{1n}(k_0 \boldsymbol{r}) = \dfrac{\xi_l(k_0 r)}{k_0 r} \boldsymbol{A}_{1n}(\hat{\boldsymbol{r}}) \\[3mm]
\boldsymbol{u}_{2n}(k_0 \boldsymbol{r}) = \dfrac{1}{k_0} \nabla \times \left(\dfrac{\xi_l(k_0 r)}{k_0 r} \boldsymbol{A}_{1n}(\hat{\boldsymbol{r}}) \right)
\end{cases}
$$

Here, we use the Riccati-Bessel functions $\xi_l(x) = x h_l^{(1)}(x)$, where $h_l^{(1)}(x)$ is the spherical Hankel function of the first kind [36]. These vector waves satisfy

$$
\nabla \times (\nabla \times \boldsymbol{u}_{\tau n}(k_0 \boldsymbol{r})) - k_0^2 \boldsymbol{u}_{\tau n}(k_0 \boldsymbol{r}) = \boldsymbol{0}, \qquad \tau = 1, 2
$$

---

[18]The index set at several places in this paper also denotes a four index set, and includes the τ index. That is, the index n can denote $n = \{\sigma, l, m\}$ or $n = \{\tau, \sigma, l, m\}$.

and they also satisfy the Silver-Müller radiation condition [9]. Another representation of the definition of the vector waves is

$$
\begin{cases}
\boldsymbol{u}_{1n}(k_0\boldsymbol{r}) = \dfrac{\xi_l(k_0r)}{k_0r}\boldsymbol{A}_{1n}(\hat{\boldsymbol{r}}) \\[2mm]
\boldsymbol{u}_{2n}(k_0\boldsymbol{r}) = \dfrac{\xi_l'(k_0r)}{k_0r}\boldsymbol{A}_{2n}(\hat{\boldsymbol{r}}) + \sqrt{l(l+1)}\dfrac{\xi_l(k_0r)}{(k_0r)^2}\boldsymbol{A}_{3n}(\hat{\boldsymbol{r}})
\end{cases}
$$

A simple consequence of these definitions is

$$
\begin{cases}
\boldsymbol{u}_{1n}(k_0\boldsymbol{r}) = \dfrac{1}{k_0}\nabla \times \boldsymbol{u}_{2n}(k_0\boldsymbol{r}) \\[2mm]
\boldsymbol{u}_{2n}(k_0\boldsymbol{r}) = \dfrac{1}{k_0}\nabla \times \boldsymbol{u}_{1n}(k_0\boldsymbol{r}).
\end{cases}
$$

In a similar way, the regular spherical vector waves $\boldsymbol{v}_{\tau n}(k_0\boldsymbol{r})$ are defined [3].

$$
\begin{cases}
\boldsymbol{v}_{1n}(k_0\boldsymbol{r}) = j_l(k_0r)\boldsymbol{A}_{1n}(\hat{\boldsymbol{r}}) \\[2mm]
\boldsymbol{v}_{2n}(k_0\boldsymbol{r}) = \dfrac{1}{k_0}\nabla \times (j_l(k_0r)\boldsymbol{A}_{1n}(\hat{\boldsymbol{r}}))
\end{cases}
$$

where $j_l(x)$ is the spherical Bessel function of the first kind [36].

Appendix B   The translation matrices

The translation properties of the vector spherical waves are instrumental for the formulation and the solution of the scattering problem of many individual scatterers. These translation properties are well know, and we refer to, *e.g.*, [3] for details.

Let $\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{d}$, see Figure 9. Then

$$
\begin{cases}
\boldsymbol{v}_n(k\boldsymbol{r}') = \displaystyle\sum_{n'}\mathcal{R}_{nn'}(k\boldsymbol{d})\boldsymbol{v}_{n'}(k\boldsymbol{r}), & \text{for all } \boldsymbol{d} \\[4mm]
\boldsymbol{u}_n(k\boldsymbol{r}') = \displaystyle\sum_{n'}\mathcal{R}_{nn'}(k\boldsymbol{d})\boldsymbol{u}_{n'}(k\boldsymbol{r}), & r > d \\[4mm]
\boldsymbol{u}_n(k\boldsymbol{r}') = \displaystyle\sum_{n'}\mathcal{P}_{nn'}(k\boldsymbol{d})\boldsymbol{v}_{n'}(k\boldsymbol{r}), & r < d
\end{cases}
\tag{B.1}
$$

Translation in the opposite direction is identical to the transpose of the translation matrices, *i.e.*, [19]

$$
\mathcal{R}^t(k\boldsymbol{d}) = \mathcal{R}(-k\boldsymbol{d}), \quad \mathcal{P}^t(k\boldsymbol{d}) = \mathcal{P}(-k\boldsymbol{d})
$$

---

[19]Multiple use of the translation properties and the completeness of the regular spherical vector waves imply

$$
\boldsymbol{v}_n(k\boldsymbol{r}) = \boldsymbol{v}_n(k(\boldsymbol{r} - \boldsymbol{d} + \boldsymbol{d})) = \sum_{n'}\mathcal{R}_{nn'}(-k\boldsymbol{d})\boldsymbol{v}_{n'}(k(\boldsymbol{r} + \boldsymbol{d}))
$$

$$
= \sum_{n'n''}\mathcal{R}_{nn'}(-k\boldsymbol{d})\mathcal{R}_{n'n''}(k\boldsymbol{d})\boldsymbol{v}_{n''}(k\boldsymbol{r})
$$

and consequently, $\mathcal{R}^{-1}(k\boldsymbol{d}) = \mathcal{R}(-k\boldsymbol{d})$.
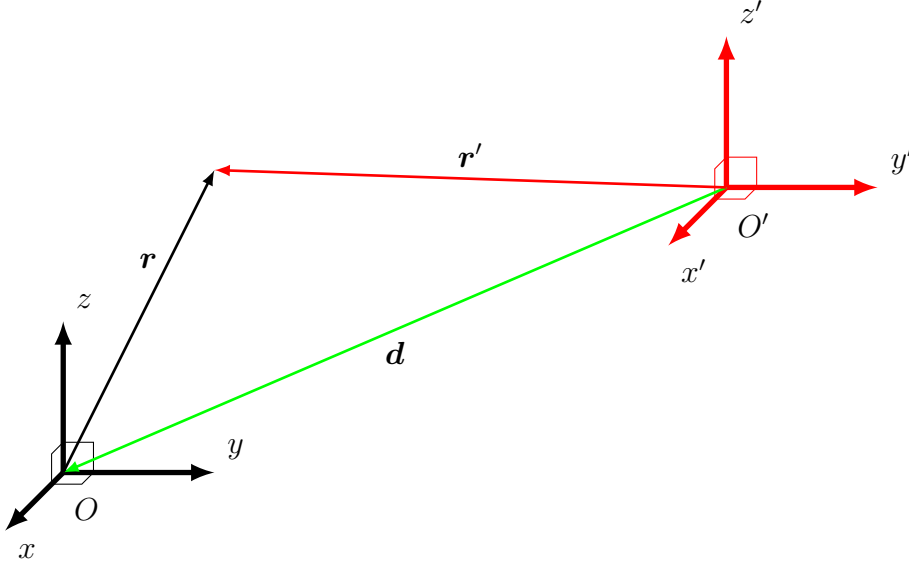
Figure 9: The relation between the translated origins O and O' and the position vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ at the different origins.

Denote the spherical coordinates of $\boldsymbol{r}$, $\boldsymbol{r}'$, and $\boldsymbol{d}$ by (r, θ, ϕ), (r', θ', ϕ'), and (d, η, ψ), respectively. The translation matrices for a translation $\boldsymbol{d}$ $(d \leq 0)$ are [3]

$$
\begin{aligned}
\mathcal{P}_{1\sigma ml, 1\sigma m'l'}(k\boldsymbol{d}) = {}& (-1)^{m'} C_{ml,m'l'}(d, \eta) \cos(m - m')\psi \\
& + (-1)^{\sigma} C_{ml,-m'l'}(d, \eta) \cos(m + m')\psi
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}_{1\sigma ml, 1\sigma' m'l'}(k\boldsymbol{d}) = {}& (-1)^{m'+\sigma'} C_{ml,m'l'}(d, \eta) \sin(m - m')\psi \\
& + C_{ml,-m'l'}(d, \eta) \sin(m + m')\psi, \quad \sigma \neq \sigma'
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}_{1\sigma ml, 2\sigma' m'l'}(k\boldsymbol{d}) = {}& (-1)^{m'+\sigma} D_{ml,m'l'}(d, \eta) \cos(m - m')\psi \\
& - D_{ml,-m'l'}(d, \eta) \cos(m + m')\psi, \quad \sigma \neq \sigma'
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}_{1\sigma ml, 2\sigma m'l'}(k\boldsymbol{d}) = {}& (-1)^{m'} D_{ml,m'l'}(d, \eta) \sin(m - m')\psi \\
& + (-1)^{\sigma} D_{ml,-m'l'}(d, \eta) \sin(m + m')\psi
\end{aligned}
$$

$$
\mathcal{P}_{2\sigma ml, \tau\sigma' m'l'}(k\boldsymbol{d}) = \mathcal{P}_{1\sigma ml, \bar{\tau}\sigma' m'l'}(k\boldsymbol{d}), \quad \tau = 1, 2
$$

where

$$
\begin{aligned}
C_{ml,m'l'}(d, \eta) = {}& \frac{(-1)^{m+m'}}{2} \sqrt{\frac{\varepsilon_m \varepsilon_{m'}}{4}} \\
& \times \sum_{\lambda=|l-l'|}^{l+l'} \mathrm{i}^{l'-l+\lambda}(2\lambda + 1) \sqrt{\frac{(2l+1)(2l'+1)(\lambda - (m - m'))!}{l(l+1)l'(l'+1)(\lambda + (m - m'))!}} \\
& \times \begin{pmatrix} l & l' & \lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & \lambda \\ m & -m' & m' - m \end{pmatrix} [l(l+1) + l'(l'+1) - \lambda(\lambda+1)] \\
& \times h_\lambda^{(1)}(kd) P_\lambda^{m-m'}(\cos \eta)
\end{aligned}
$$

$$D_{ml,m'l'}(d,\eta) = \frac{(-1)^{m+m'}}{2}\sqrt{\frac{\varepsilon_m\varepsilon_{m'}}{4}}$$

$$\times \sum_{\lambda=|l-l'|+1}^{l+l'} i^{l'-l+\lambda+1}(2\lambda+1)\sqrt{\frac{(2l+1)(2l'+1)(\lambda-(m-m'))!}{l(l+1)l'(l'+1)(\lambda+(m-m'))!}}$$

$$\times \begin{pmatrix} l & l' & \lambda-1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l' & \lambda \\ m & -m' & m'-m \end{pmatrix}\sqrt{\lambda^2-(l-l')^2}$$

$$\times \sqrt{(l+l'+1)^2-\lambda^2}h_\lambda^{(1)}(kd)P_\lambda^{m-m'}(\cos\eta)$$

where $\varepsilon_m = 2 - \delta_{m,0}$ is the Neumann factor, and where $\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ denotes Wigner's 3j symbol [11], and

$$(-1)^\sigma = \begin{cases} 1, & \sigma = \mathrm{e} \\ -1, & \sigma = \mathrm{o} \end{cases}$$

Note that the factors $i^{l'-l+\lambda}$ in $C_{ml,m'l'}(d,\eta)$ and $i^{l'-l+\lambda+1}$ in $D_{ml,m'l'}(d,\eta)$ are always real numbers, due to the conditions on the Wigner's 3j symbol.

The translation matrix $\mathcal{R}_{nn'}(k\boldsymbol{d})$ is identical to $\mathcal{P}_{nn'}(k\boldsymbol{d})$ but with $h_\lambda^{(1)}(kd)$ replaced with $j_\lambda(kd)$.

We notice that the translation matrices have the form

$$\mathcal{P}_{nn'}(k\boldsymbol{d}) = \sum_{\lambda=|l-l'|+|\tau-\tau'|}^{l+l'} h_\lambda^{(1)}(kd)\left(A_{nn'\lambda}(\psi)P_\lambda^{m-m'}(\cos\eta)+B_{nn'\lambda}(\psi)P_\lambda^{m+m'}(\cos\eta)\right)$$

$$\tag{B.2}$$

B.1 Average w.r.t. the azimuthal angle ψ

The integral of the translational matrix w.r.t. the azimuthal variable ψ is relevant. Explicitly, the non-zero contributions for a general $m \geq 0$ are

$$\begin{cases} \displaystyle\int_0^{2\pi} A_{1eml,1em'l'\lambda}(\psi)\,\mathrm{d}\psi = 2\pi(-1)^m\delta_{mm'}C_{mll'\lambda} \\[2mm] \displaystyle\int_0^{2\pi} A_{1eml,2om'l'\lambda}(\psi)\,\mathrm{d}\psi = 2\pi(-1)^m\delta_{mm'}D_{mll'\lambda} \\[2mm] \displaystyle\int_0^{2\pi} A_{2oml,1em'l'\lambda}(\psi)\,\mathrm{d}\psi = \int_0^{2\pi} A_{1oml,2em'l'\lambda}(\psi)\,\mathrm{d}\psi \\[2mm] \displaystyle\int_0^{2\pi} A_{2oml,2om'l'\lambda}(\psi)\,\mathrm{d}\psi = \int_0^{2\pi} A_{1oml,1om'l'\lambda}(\psi)\,\mathrm{d}\psi \end{cases}$$

$$\begin{cases} \displaystyle\int_0^{2\pi} A_{1oml,1om'l'\lambda}(\psi) \ \mathrm{d}\psi = 2\pi(-1)^m \delta_{mm'} C_{mll'\lambda} \\[2ex] \displaystyle\int_0^{2\pi} A_{1oml,2em'l'\lambda}(\psi) \ \mathrm{d}\psi = -2\pi(-1)^m \delta_{mm'} D_{mll'\lambda} \\[2ex] \displaystyle\int_0^{2\pi} A_{2eml,1om'l'\lambda}(\psi) \ \mathrm{d}\psi = \int_0^{2\pi} A_{1eml,2om'l'\lambda}(\psi) \ \mathrm{d}\psi \\[2ex] \displaystyle\int_0^{2\pi} A_{2eml,2em'l'\lambda}(\psi) \ \mathrm{d}\psi = \int_0^{2\pi} A_{1eml,1em'l'\lambda}(\psi) \ \mathrm{d}\psi \end{cases}$$

and the average over the function B becomes

$$\begin{cases} \displaystyle\int_0^{2\pi} B_{1eml,1em'l'\lambda}(\psi) \ \mathrm{d}\psi = 2\pi\delta_{mm'}\delta_{m,0} C_{mll'\lambda} \\[2ex] \displaystyle\int_0^{2\pi} B_{1eml,2om'l'\lambda}(\psi) \ \mathrm{d}\psi = -2\pi\delta_{mm'}\delta_{m,0} D_{mll'\lambda} \\[2ex] \displaystyle\int_0^{2\pi} B_{2oml,1em'l'\lambda}(\psi) \ \mathrm{d}\psi = \int_0^{2\pi} B_{1oml,2em'l'\lambda}(\psi) \ \mathrm{d}\psi \\[2ex] \displaystyle\int_0^{2\pi} B_{2oml,2om'l'\lambda}(\psi) \ \mathrm{d}\psi = \int_0^{2\pi} B_{1oml,1om'l'\lambda}(\psi) \ \mathrm{d}\psi \end{cases}$$

$$\begin{cases} \displaystyle\int_0^{2\pi} B_{1oml,1om'l'\lambda}(\psi) \ \mathrm{d}\psi = -2\pi\delta_{mm'}\delta_{m,0} C_{mll'\lambda} \\[2ex] \displaystyle\int_0^{2\pi} B_{1oml,2em'l'\lambda}(\psi) \ \mathrm{d}\psi = -2\pi\delta_{mm'}\delta_{m,0} D_{ml,ml'\lambda} \\[2ex] \displaystyle\int_0^{2\pi} B_{2eml,1om'l'\lambda}(\psi) \ \mathrm{d}\psi = \int_0^{2\pi} B_{1eml,2om'l'\lambda}(\psi) \ \mathrm{d}\psi \\[2ex] \displaystyle\int_0^{2\pi} B_{2eml,2em'l'\lambda}(\psi) \ \mathrm{d}\psi = \int_0^{2\pi} B_{1eml,1em'l'\lambda}(\psi) \ \mathrm{d}\psi \end{cases}$$

The explicit form of the average is

$$\mathcal{A}_{nn'\lambda} = \int_0^{2\pi} A_{nn'\lambda}(\phi) \ \mathrm{d}\phi + \int_0^{2\pi} B_{nn'\lambda}(\phi) \ \mathrm{d}\phi$$

$$= 2\pi(-1)^m \delta_{mm'} \begin{array}{c} \\ \begin{array}{c} 1e \\ 2o \\ 1o \\ 2e \end{array} \begin{pmatrix} \overset{\displaystyle 1e}{C} & \overset{\displaystyle 2o}{D} & \overset{\displaystyle 1o}{0} & \overset{\displaystyle 2e}{0} \\ -D & C & 0 & 0 \\ 0 & 0 & C & -D \\ 0 & 0 & D & C \end{pmatrix} \end{array}, \quad m = 1, 2, \ldots$$

and

$$\mathcal{A}_{nn'\lambda} = \int_0^{2\pi} A_{nn'\lambda}(\phi) \, \mathrm{d}\phi + \int_0^{2\pi} B_{nn'\lambda}(\phi) \, \mathrm{d}\phi$$

$$= 2\pi\delta_{mm'} \begin{array}{c} \\ {}^{1e} \\ {}^{2o} \\ {}^{1o} \\ {}^{2e} \end{array} \begin{pmatrix} \overset{1e}{C} & \overset{2o}{0} & \overset{1o}{0} & \overset{2e}{0} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C \end{pmatrix}, \quad m = 0$$

where

$$C = C_{mll'\lambda} = \frac{\mathrm{i}^{l'-l+\lambda}(2\lambda+1)}{2}\sqrt{\frac{(2l+1)(2l'+1)}{l(l+1)l'(l'+1)}}$$

$$\times \begin{pmatrix} l & l' & \lambda \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l' & \lambda \\ m & -m & 0 \end{pmatrix} [l(l+1) + l'(l'+1) - \lambda(\lambda+1)]$$

$$D = D_{mll'\lambda} = \frac{\mathrm{i}^{l'-l+\lambda+1}(2\lambda+1)}{2}\sqrt{\frac{(2l+1)(2l'+1)}{l(l+1)l'(l'+1)}}$$

$$\times \begin{pmatrix} l & l' & \lambda-1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l' & \lambda \\ m & -m & 0 \end{pmatrix} \sqrt{\lambda^2 - (l-l')^2}\sqrt{(l+l'+1)^2 - \lambda^2}$$

In particular, if $m = l = l' = 1$

$$C_{111\lambda} = \frac{3}{4}\mathrm{i}^{\lambda}(2\lambda+1)\begin{pmatrix} 1 & 1 & \lambda \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & \lambda \\ 1 & -1 & 0 \end{pmatrix} [4 - \lambda(\lambda+1)]$$

$$D_{111\lambda} = \frac{3}{4}\mathrm{i}^{\lambda+1}(2\lambda+1)\begin{pmatrix} 1 & 1 & \lambda-1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & \lambda \\ 1 & -1 & 0 \end{pmatrix} \lambda\sqrt{9 - \lambda^2}$$

Edmonds shows [11]

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\frac{1}{\sqrt{3}}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 0, \quad \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \sqrt{\frac{2}{15}}$$

and

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{3}}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{6}}, \quad \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{30}}$$

This implies

$$C_{111\lambda} = -\delta_{\lambda,0} + \frac{1}{2}\delta_{\lambda,2}$$

$$D_{111\lambda} = \frac{3}{2}\delta_{\lambda,1}$$

Appendix C   Spherical scatterers

If the scatterers are dielectric spheres of radius a, the transition matrix is diagonal in all indices (we adopt the notation $T_{nn'} = t_{\tau l}\delta_{\tau,\tau'}\delta_{l,l'}\delta_{m,m'}\delta_{\sigma,\sigma'}$).

The two sets of indices, $\{1e, 2o\}$ and $\{1o, 2e\}$, do not couple. This is a consequence of the diagonal structure of the transition matrix and the form of the azimuthal averages in Appendix B.1. For a plane wave incidence, only $m = 1$ contributes, see (2.4), and we obtain two generic systems of equations, which both have the form

$$\begin{pmatrix} y_{1l}^{\pm}(z) \\ y_{2l}^{\pm}(z) \end{pmatrix} = \begin{pmatrix} g_{1l}^{\pm}(z) \\ g_{2l}^{\pm}(z) \end{pmatrix} + k \sum_{l'=1}^{\infty} \int_{z_1}^{z_2} \begin{pmatrix} h_{1l,1l'}^{\pm}(z,z') & h_{1l,2l'}^{\pm}(z,z') \\ h_{2l,1l'}^{\pm}(z,z') & h_{2l,2l'}^{\pm}(z,z') \end{pmatrix} \begin{pmatrix} y_{1l'}^{\pm}(z') \\ y_{2l'}^{\pm}(z') \end{pmatrix} \, \mathrm{d}z'$$

where (m index suppressed)

$$\begin{cases} g_{1l}^{-}(z) = k(z-z_1)t_{1l}a_{1eml} \\ g_{2l}^{-}(z) = k(z-z_1)t_{2l}a_{2oml} \end{cases} \quad \text{or} \quad \begin{cases} g_{1l}^{-}(z) = k(z-z_1)t_{1l}a_{1oml} \\ g_{2l}^{-}(z) = k(z-z_1)t_{2l}a_{2eml} \end{cases}$$

$$\begin{cases} g_{1l}^{+}(z) = \dfrac{1}{2i}\left(\mathrm{e}^{2\mathrm{i}kz} - \mathrm{e}^{2\mathrm{i}kz_1}\right)t_{1l}a_{1eml} \\ g_{2l}^{+}(z) = \dfrac{1}{2i}\left(\mathrm{e}^{2\mathrm{i}kz} - \mathrm{e}^{2\mathrm{i}kz_1}\right)t_{2l}a_{2oml} \end{cases} \quad \text{or} \quad \begin{cases} g_{1l}^{+}(z) = \dfrac{1}{2i}\left(\mathrm{e}^{2\mathrm{i}kz} - \mathrm{e}^{2\mathrm{i}kz_1}\right)t_{1l}a_{1oml} \\ g_{2l}^{+}(z) = \dfrac{1}{2i}\left(\mathrm{e}^{2\mathrm{i}kz} - \mathrm{e}^{2\mathrm{i}kz_1}\right)t_{2l}a_{2eml} \end{cases}$$

and

$$\begin{cases} h_{1l,1l}^{\pm}(z,z') = -\dfrac{\mathrm{d}\widehat{K}_{1eml,1eml}^{\pm}(z,z')}{\mathrm{d}z'} + \widehat{K}_{1eml,1eml}^{\pm}(z,z_2)\delta(z'-z_2) \\[2mm] h_{1l,2l}^{\pm}(z,z') = -\dfrac{\mathrm{d}\widehat{K}_{1eml,2oml}^{\pm}(z,z')}{\mathrm{d}z'} + \widehat{K}_{1eml,2oml}^{\pm}(z,z_2)\delta(z'-z_2) \\[2mm] h_{2l,1l}^{\pm}(z,z') = -\dfrac{\mathrm{d}\widehat{K}_{2oml,1eml}^{\pm}(z,z')}{\mathrm{d}z'} + \widehat{K}_{2oml,1eml}^{\pm}(z,z_2)\delta(z'-z_2) \\[2mm] h_{2l,2l}^{\pm}(z,z') = -\dfrac{\mathrm{d}\widehat{K}_{2oml,2oml}^{\pm}(z,z')}{\mathrm{d}z'} + \widehat{K}_{2oml,2oml}^{\pm}(z,z_2)\delta(z'-z_2) \end{cases}$$

or

$$\begin{cases} h_{1l,1l}^{\pm}(z,z') = -\dfrac{\mathrm{d}\widehat{K}_{1oml,1oml}^{\pm}(z,z')}{\mathrm{d}z'} + \widehat{K}_{1oml,1oml}^{\pm}(z,z_2)\delta(z'-z_2) \\[2mm] h_{1l,2l}^{\pm}(z,z') = -\dfrac{\mathrm{d}\widehat{K}_{1oml,2eml}^{\pm}(z,z')}{\mathrm{d}z'} + \widehat{K}_{1oml,2eml}^{\pm}(z,z_2)\delta(z'-z_2) \\[2mm] h_{2l,1l}^{\pm}(z,z') = -\dfrac{\mathrm{d}\widehat{K}_{2eml,1oml}^{\pm}(z,z')}{\mathrm{d}z'} + \widehat{K}_{2eml,1oml}^{\pm}(z,z_2)\delta(z'-z_2) \\[2mm] h_{2l,2l}^{\pm}(z,z') = -\dfrac{\mathrm{d}\widehat{K}_{2eml,2eml}^{\pm}(z,z')}{\mathrm{d}z'} + \widehat{K}_{2eml,2eml}^{\pm}(z,z_2)\delta(z'-z_2) \end{cases}$$

Appendix D   Asymptotic evaluation of the kernel

The kernel $K_{nn'}(z)$ is evaluated in terms of a power series of ka. The lowest order term is used in the low frequency expansion of the transmission and reflection coef-

ficient of the slab in Section 6.2. One key element in the asymptotic evaluation of the kernel in (5.2) and (5.10) is

$$M_{nn'}(z) = \iint\limits_{\mathbb{R}^2} g(|\boldsymbol{\rho} - z\hat{\boldsymbol{z}}|)\mathcal{P}_{nn'}(k(\boldsymbol{\rho} - z\hat{\boldsymbol{z}})) \, \mathrm{d}x \, \mathrm{d}y$$

$$= \sum_{\lambda = |l - l'| + |\tau - \tau'|}^{l+l'} I_\lambda(-z) \int_0^{2\pi} A_{nn'\lambda}(\phi) \, \mathrm{d}\phi, \quad z \in \mathbb{R}$$

The azimuthal average of $A_{nn'\lambda}(\phi)$ is analyzed in Appendix B.1, and the most singular term in the integral $I_l(z)$ is of the order $(ka)^{1-l}$ ($I_0(z) = O(1)$). The relevant contribution in low-frequency expansions of the kernel $K_{nn'}(z)$ therefore is for $l = l' = 1$. We get, see Appendix B.1

$$\begin{cases} M_{\tau\sigma11,\tau\sigma11}(z) = -2\pi\left(-I_0(-z) + \dfrac{1}{2}I_2(-z)\right) \\ M_{\tau\sigma11,\overline{\tau\sigma}11}(z) = -2\pi(-1)^\sigma\dfrac{3}{2}I_1(-z) \end{cases} \quad \tau = 1, 2, \sigma = \mathrm{e}, \mathrm{o}$$

where we used the dual index of τ and σ ($\overline{1} = 2, \overline{2} = 1, \overline{\mathrm{e}} = \mathrm{o}, \overline{\mathrm{o}} = \mathrm{e}$), and the explicit values of $I_l(z)$, $l = 0, 1, 2$ are [20]

$$I_0(z) = \begin{cases} \mathrm{e}^{-\mathrm{i}kz}, & z \leq -2a \\ \mathrm{e}^{2\mathrm{i}ka}, & -2a < z < 2a = O(1) \\ \mathrm{e}^{\mathrm{i}kz}, & z \geq 2a \end{cases}$$

$$I_1(z) = \begin{cases} \mathrm{i}\mathrm{e}^{-\mathrm{i}kz}, & z \leq -2a \\ -\mathrm{i}\mathrm{e}^{2\mathrm{i}ka}\dfrac{z}{2a}, & -2a < z < 2a = O(1) \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}kz}, & z \geq 2a \end{cases}$$

$$I_2(z) = \begin{cases} -\mathrm{e}^{-\mathrm{i}kz}, & z \leq -2a \\ \mathrm{e}^{2\mathrm{i}ka}\dfrac{4(3\mathrm{i} + 2ka) - 3(\mathrm{i} + 2ka)(z/a)^2}{16ka}, & -2a < z < 2a \\ -\mathrm{e}^{\mathrm{i}kz}, & z \geq 2a \end{cases}$$

$$= \begin{cases} O(1), & |z| \geq 2a \\ \mathrm{i}\dfrac{12 - 3(z/a)^2}{16ka} + O(1), & |z| < 2a \end{cases}$$

To leading order in powers of ka, the kernels are

$$\begin{cases} M_{\tau\sigma11,\tau\sigma11}(z) = -2\pi\begin{cases} O(1), & |z| \geq 2a \\ \mathrm{i}\dfrac{12 - 3(z/a)^2}{32ka} + O(1), & |z| < 2a \end{cases} \quad \tau = 1, 2, \sigma = \mathrm{e}, \mathrm{o} \\ M_{\tau\sigma11,\overline{\tau\sigma}11}(z) = O(1) \end{cases}$$

(D.1)

and we observe that the kernel to leading order in powers of ka has support in $z - z' \in [-2a, 2a]$.

Appendix E   Solution of the integral equation in terms of Legendre polynomials

We expect the solution to be close to a polynomial in z, and therefore we expand the solution in Legendre polynomials $P_l(t)$, were $t = t(z) = (2z - z_2 - z_1)/(z_2 - z_1)$. We write the generic integral equation on the form (this is a scalar version of the system of equations in the analysis, but it illustrates the idea)

$$f(z) = g(z) + k \int_{z_1}^{z_2} K(z - z')f(z') \, \mathrm{d}z', \quad z_1 \leq z \leq z_2$$

and explicitly we expand

$$f(z) = \sum_{l=0}^{\infty} f_l P_l((2z - z_2 - z_1)/(z_2 - z_1))$$

The coefficients f_l, due to orthogonality of the Legendre polynomials, are

$$f_l = \frac{2l + 1}{z_2 - z_1} \int_{z_1}^{z_2} f(z) P_l((2z - z_2 - z_1)/(z_2 - z_1)) \, \mathrm{d}z$$

The integral equation then reads

$$\sum_{l=0}^{\infty} f_l P_l((2z - z_2 - z_1)/(z_2 - z_1)) = g(z)$$

$$+ k \sum_{l'=0}^{\infty} f_{l'} \int_{z_1}^{z_2} K(z - z') P_{l'}((2z' - z_2 - z_1)/(z_2 - z_1)) \, \mathrm{d}z', \quad z_1 \leq z \leq z_2$$

Multiply with $P_l((2z - z_2 - z_1)/(z_2 - z_1))$ and integrate over $z \in [z_1, z_2]$. Orthogonality of the Legendre polynomials implies

$$f_l \frac{z_2 - z_1}{2l + 1} = g_l \frac{z_2 - z_1}{2l + 1}$$
$$+ k \frac{(z_2 - z_1)^2}{4} \sum_{l'=0}^{\infty} f_{l'} \int_{-1}^{1} \mathrm{d}t \int_{-1}^{1} \mathrm{d}t' P_l(t) K \left((t - t')(z_2 - z_1)/2\right) P_{l'}(t')$$

or

$$f_l = g_l + \sum_{l'=0}^{\infty} M_{ll'} f_{l'}$$

where the vector g_l is

$$g_l = \frac{2l + 1}{z_2 - z_1} \int_{z_1}^{z_2} g(z) P_l((2z - z_2 - z_1)/(z_2 - z_1)) \, \mathrm{d}z$$

and the matrix $M_{ll'}$ is

$$M_{ll'} = \frac{(z_2 - z_1)(2l + 1)}{4} \int_{-1}^{1} \mathrm{d}t \int_{-1}^{1} \mathrm{d}t' P_l(t) K \left((t - t')(z_2 - z_1)/2\right) P_{l'}(t')$$

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