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Duality in Analysis via Integral Quadratic Constraints

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Abstract

Frequency domain conditions involving multipliers is a powerful tool for robustness analysis. The resulting analysis problem is generally convex, but infinite dimensional and numerical solutions restricted to finite-dimensional subspaces need to be considered. The finite-dimensional problem can be transformed to a linear matrix inequality, which can be solved with efficient algorithms. This paper presents a format for the dual of the infinite-dimensional problem. The dual can be used to estimate the conservatism of a particular finite-dimensional subspace of the primal.

Key words
Robustness Analysis, Multiplier, Stability, Duality, LMI

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1. Introduction

Many practical systems can be modeled as a feedback interconnection of a linear time-invariant (LTI) plant G and a perturbation Δ. The perturbation contains everything in the system that cannot be modeled as an LTI plant. For example, it can contain nonlinear elements, time-varying elements, and uncertain elements with various assumptions on the uncertainty.

Several classical results from 1960–1975 give sufficient conditions for stability in terms of the Nyquist curve in the case when G is a single-input single-output (SISO) plant for various nonlinear and/or time-varying perturbations, see for example [3]. Since the early 1980s much progress has been made on computational methods for robustness analysis in the case of MIMO plants. For example, Doyle introduced μ-analysis, which can be used for robustness test of a large class of systems with structured LTI perturbations by solving an optimization problem at a preselected grid of frequencies, see [4], [15]. However, in the case of nonlinear and/or time-varying perturbations there exists coupling between frequencies and the optimization problems at different frequencies can not be treated separately. One way to overcome this problem is to parametrize the multipliers involved in the optimization problem in terms of a basis of rational transfer functions, [11] and [2]. The corresponding optimization problem can then be transformed into an equivalent Linear Matrix Inequality (LMI) which can be solved by efficient numerical algorithms. The effectiveness of this approach is generally dependent on the choice of basis.

The objective of this paper is to derive a dual optimization problem which can be used to investigate the quality of a particular basis.

We will consider robustness analysis in the unified framework based on Integral Quadratic Constraints (IQC) that was suggested by Megretski and Rantzer, [12], [17]. The computation of a robustness criterion γ is an optimization problem of the following type

$$\inf \gamma \qquad \text{subject to} \tag{1}$$
$$\begin{cases} \exists \Pi \in \Pi_\Delta(\gamma), \text{ such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0,\infty] \end{cases}$$

where $\Pi_\Delta(\gamma)$ is a convex cone of rational transfer functions for every value of the parameter γ. $\Pi_\Delta(\gamma)$ is generally infinite-dimensional and we obtain solutions to (1) by introducing a finite-dimensional rational basis for the multipliers in $\Pi_\Delta(\gamma)$ and then solve an equivalent optimization problem which has LMI conditions in the constraint. The solutions that are obtained by this method are generally suboptimal.

In order to estimate the conservativeness of a particular basis we derive the dual optimization problem. The dual is generally an infinite-dimensional optimization problem which may be hard to solve. However, by considering finite-dimensional restrictions, we can obtain solutions to it for a large number of problems that are of interest in practice. We will consider one such approach that leads to coupled LMI tests for a preselected set of frequencies. Similar results as in this paper was obtained in a somewhat different framework in [6].

2. Mathematical Preliminaries

This section presents the necessary mathematical preliminaries and notation needed in the paper. The following standard definitions and results from functional analysis and convex analysis can be found in [10].

- Let X be a normed vector space. The dual of X is the normed space consisting of all bounded linear functionals on X and it is denoted by X^*. If $x \in X$ and $x^* \in X^*$, then $\langle x, x^* \rangle$ denotes the value of the linear functional x^* at x.
- Let $H : X \mapsto Y$ be a bounded linear operator. Then the adjoint operator $H^\times : Y^* \mapsto X^*$ is defined by the equation

$$\langle Hx, y^* \rangle = \langle x, H^\times y^* \rangle$$

  for all $x \in X$ and $y^* \in Y^*$.
- A *convex cone* C is a convex subset of a vector space with the property that if $x \in C$, then $\alpha x \in C$ for all $\alpha \geq 0$.
- We will use the following notation for optimization problems with constraints

$$\inf_P \gamma \stackrel{\text{def}}{=} \inf \gamma \quad \text{subject to } P$$

  where P denotes a constraint definition.

The following separating hyperplane theorem will be a main tool in this paper

Proposition 1—Separating Hyperplane Theorem
Let C_1 and C_2 be disjoint convex sets in a vector space X. Assume further that C_1 is open, then there exists $z \in X^*$ such that $\langle x_1, z \rangle < \langle x_2, z \rangle$ for all $x_1 \in C_1$ and $x_2 \in C_2$.

Proof: This is essentially Theorem 3 on page 133 in [10]. In fact: $C = C_1 - C_2$ is an open set such that $0 \notin C$. By the Geometric Hahn-Banach there exists an element $z \in X^*$ such that $\langle x, z \rangle < 0$ for all $x \in C$, from which the proposition follows. □

Next is a list of notation and function spaces used in this paper.

$\overline{M}$	Denotes conjugation of a complex valued matrix.
M^T	Denotes the transpose of a complex valued matrix.
M^*	$M^* = (\overline{M})^T$ denotes Hermitian conjugation of a complex valued matrix.
$\lvert \cdot \rvert_F$	The Frobenius norm of a real or complex matrix M is defined as $\lvert M \rvert_F = \sqrt{\text{tr}(M^*M)}$.
$\mathbf{RL}_\infty^{n\times n}$	The space consisting of proper real rational matrix functions with no poles on the imaginary axis. Note that $F \in \mathbf{RL}_\infty^{m\times m}$ satisfy $F(-j\omega) = \overline{F(j\omega)}$.

$\mathbf{RH}_\infty^{m\times m}$ The subspace of $\mathbf{RL}_\infty^{m\times m}$ consisting of functions with no poles in the closed right half plane. Note that G^* generally means the Hilbert adjoint of $G(s)$, defined as $G^T(-s)$. The Hilbert adjoint reduces to the Hermitean conjugate of G when $s = i\omega$.

$\mathcal{S}_{\mathbf{R}}^{m\times m}$ The subspace of $\mathbf{R}^{m\times m}$ consisting of symmetric $m \times m$ matrices with the topology determined by the Frobenius norm. The dual space can be identified with $\mathcal{S}_{\mathbf{R}}^{m\times m}$ itself and the linear functionals are defined as $\langle X, Z\rangle_{\mathbf{R}} = \mathrm{tr}(XZ)$, where $X, Z \in \mathcal{S}_{\mathbf{R}}^{m\times m}$.

$\mathcal{S}_{\mathbf{C}}^{m\times m}$ The subspace of $\mathbf{C}^{m\times m}$ consisting of Hermitean $m \times m$ matrices with the topology determined by the Frobenius norm. The dual space can be identified with $\mathcal{S}_{\mathbf{C}}^{m\times m}$ itself and the linear functionals are defined as $\langle X, Z\rangle_{\mathbf{C}} = \mathrm{tr}\{XZ\}$, where $X, Z \in \mathcal{S}_{\mathbf{C}}^{m\times m}$.

$\mathcal{S}_{\infty}^{m\times m}$ The subspace of $\mathbf{RL}_\infty^{m\times m}$ consisting of functions satisfying $x(j\omega) = x(j\omega)^*$ for all $\omega \in [0,\infty]$. We define the norm on $\mathcal{S}_{\infty}^{m\times m}$ as $\|x\| = \max_{\omega\in[0,\infty]} |x(j\omega)|_F$. Note that this is not the usual norm on $\mathbf{RL}_\infty^{m\times m}$.

$\mathcal{P}_{\infty}^{m\times m}$ The positive cone defined as

$$\mathcal{P}_{\infty}^{m\times m} = \left\{x \in \mathcal{S}_{\infty}^{m\times m} : x(j\omega) \geq 0,\ \forall\omega \in [0,\infty]\right\}$$

$\mathcal{S}_{\mathbf{NBV}}$ The normalized Banach space of functions $\mathbf{R} \cup \{\infty\} \to \mathcal{S}_{\mathbf{C}}^{m\times m}$ of bounded variation. Every $z \in \mathcal{S}_{\mathbf{NBV}}$ satisfies the following properties.

1. $z(-\omega) = -\overline{z(\omega)}$ for all $\omega \in [0,\infty]$.
2. z is continuous to the right on $(0,\infty)$ and it satisfies $z(0) = 0$.
3. The norm of z is defined as $\|z\| = T.V.(z)$, where

$$T.V.(z) = 2\sup\sum_{k=1}^{N} |z(\omega_k) - z(\omega_{k-1})|_F$$

where $0 = \omega_0 \leq \omega_1 \leq \omega_2 \leq \ldots \leq \omega_N = \infty$ is a partition of $[0,\infty]$, and the supremum is taken with respect to all such partitions.

$\mathcal{P}_{\mathbf{NBV}}^{m\times m}$ The positive cone defined as

$$\mathcal{P}_{\mathbf{NBV}}^{m\times m} = \left\{z \in \mathcal{S}_{\mathbf{NBV}}^{m\times m} : z(\omega_1) \geq z(\omega_2), \forall\omega_1 > \omega_2 \geq 0\right\}$$

PROPOSITION 2
The dual of $\mathcal{S}_{\infty}^{m\times m}$ can be identified with $\mathcal{S}_{\mathbf{NBV}}^{m\times m}$ and if $x \in \mathcal{S}_{\infty}^{m\times m}$ and $z \in \mathcal{S}_{\mathbf{NBV}}^{m\times m}$, then the linear functional is defined by

$$\begin{aligned}\langle x, z\rangle &= 2\int_0^\infty \mathrm{tr}[x(j\omega)dz(\omega)] \\ &:= 2\lim_{N\to\infty}\sum_{k=1}^{N}\mathrm{tr}[x(j\omega_{k-1})[z(\omega_k) - z(\omega_{k-1})]] \end{aligned} \tag{2}$$

where $0 = \omega_0 \leq \omega_1 \leq \omega_2 \leq \ldots \leq \omega_N = \infty$ is a partition of $[0,\infty]$, and the limit is considered as $\max_{k\in\{1,\ldots,N-1\}} |\omega_k - \omega_{k-1}| \to 0$ and as $\omega_{N-1} \to \infty$.

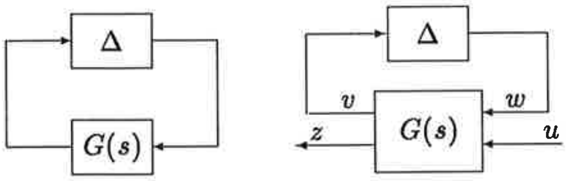

Figure 1. System setup for stability analysis and performance analysis respectively.

Proof: We use the Möbius transform $\psi(z) = (z-1)/(z+1)$ to transform to the imaginary axis to the unit circle. We denote the unit circle $\partial\mathbf{D}$. For any $x \in \mathcal{S}_\infty^{m\times m}$ let $\tilde{x} = x \circ \psi$. The restriction to the unit circle of the vector space consisting of the functions $\tilde{x}$ obtained in this way is dense in the Banach space of continuous functions $\tilde{x} : \partial\mathbf{D} \mapsto \mathcal{S}_\mathrm{C}^{m\times m}$, satisfying

1. $\tilde{x}(e^{-j\omega}) = \overline{\tilde{x}(e^{j\omega})}$, for all $\omega \in [0,\pi]$.
2. The norm is defined as $\|\tilde{x}\| = \max_{\omega\in[0,\pi]} |\tilde{x}(e^{j\omega})|$.

It follows from Theorem 1, on page 113 in [10] that the dual space consists of functions $z : \partial\mathbf{D} \mapsto \mathcal{S}_\mathrm{C}^{m\times m}$ of bounded variation. The proposition follows after transformation with the inverse Möbius map ψ^{-1}. Note that the symmetry around $\omega = 0$ for the spaces $\mathcal{S}_\infty^{m\times m}$ and $\mathcal{S}_\mathrm{NBV}^{m\times m}$ implies that we can define the functionals only in terms of positive frequencies. □

PROPOSITION 3
For any $x \in \mathcal{S}_\infty^{m\times m}$ and $z \in \mathcal{S}_\mathrm{NBV}^{m\times m}$, we have $\langle x, z\rangle \geq 0$.

Proof: This is obvious from the definition of the linear functional in (2). □

3. The Primal Robustness Test

A general and unified approach to the use of multipliers was introduced in [17, 14]. The method is based on the concept *integral quadratic constraint* (IQC). An operator Δ (possibly nonlinear) on $\mathbf{L}_2^m[0,\infty)$ is said to satisfy the IQC defined by the matrix function Π, called *multiplier*, if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ (\widehat{\Delta(v)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \widehat{\Delta(v)}(j\omega) \end{bmatrix} d\omega \;\geq\; 0 \quad \text{for all } v \in \mathbf{L}_2^m[0,\infty)$$

Here $\hat{v}$ and $\widehat{\Delta(v)}$ denotes the Fourier transform of v and $\Delta(v)$. Based on this definition, each operator Δ can be described by a set Π_Δ of multipliers Π, that define IQC:s satisfied by Δ.
IQC:s can be used in the analysis of the systems in Figure 1. Here G is a causal, linear time invariant operator with transfer function in $\mathbf{RH}_\infty^{m\times m}$ and Δ is a bounded and causal operator on $\mathbf{L}_2[0,\infty)$. It is possible to analyze the system with respect to either robust stability or robust performance. The first step in the analysis consists of finding a description of the perturbation Δ in terms of IQC:s. The following properties are convenient when deriving an IQC description of Δ.

Property 1 The set of all Π that describes Δ in terms of IQC:s is a convex cone. Hence, if Δ is described by the convex cones $\Pi_{1\Delta}$ and $\Pi_{2\Delta}$, then Δ is also described by $\Pi_\Delta = \Pi_{1\Delta} + \Pi_{2\Delta} = \{\Pi_1 + \Pi_2 : \Pi_1 \in \Pi_{1\Delta}, \Pi_2 \in \Pi_{2\Delta}\}$.

Property 2 Assume Δ has the block-diagonal structure $\Delta = \text{diag}[\Delta_1, \Delta_2]$, and that Δ_i satisfies the IQC defined by Π_i, $i = 1, 2$. Then Δ satisfies the IQC defined by $\Pi = \text{daug}[\Pi_1, \Pi_2]$, where the operation daug is defined as follows. If

$$\Pi_i = \begin{bmatrix} \Pi_{i1} & \Pi_{i2} \\ \Pi_{i2}^* & \Pi_{i3} \end{bmatrix}, \quad i = 1, 2,$$

where the block structures are consistent with the size of Δ_1 and Δ_2, respectively, then

$$\text{daug}(\Pi_1, \Pi_2) = \left[\begin{array}{cc|cc} \Pi_{11} & 0 & \Pi_{12} & 0 \\ 0 & \Pi_{21} & 0 & \Pi_{22} \\ \hline \Pi_{12}^* & 0 & \Pi_{13} & 0 \\ 0 & \Pi_{22}^* & 0 & \Pi_{23} \end{array}\right]$$

If Δ_i is described by the convex cones $\Pi_{i\Delta}$, i=1,2, then Δ is described by the convex cone $\text{daug}(\Pi_{1\Delta}, \Pi_{2\Delta}) = \{\text{daug}(\Pi_1, \Pi_2) : \Pi_1 \in \Pi_{1\Delta}, \Pi_2 \in \Pi_{2\Delta}\}$.

In robust performance analysis we also need IQC descriptions of the performance specification and the characteristics of the input signal. These should be augmented to the IQC description of Δ. This is discussed in more detail in Section 8. The robustness analysis can be formulated as an optimization problem on the following form

The Primal optimization problem

$$\inf \gamma \quad \text{subject to} \tag{3}$$

$$P : \begin{cases} \exists \Pi \in \Pi_\Delta(\gamma), \text{ such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty] \end{cases}$$

The parameter γ corresponds to the robustness criteria which is investigated. This could for example be a stability margin. We make the following assumptions

Assumptions on $\Pi_\Delta(\gamma)$***:***

1. $\Pi_\Delta(\gamma) \subset \mathcal{S}_\infty^{2m \times 2m}$ is a convex cone for any given $\gamma \in \mathbf{R}$.
2. If $\gamma_2 \geq \gamma_1$, then $\forall \Pi_1 \in \Pi_\Delta(\gamma_1)$ there exists $\Pi_2 \in \Pi_\Delta(\gamma_2)$ such that $\Pi_1 \geq \Pi_2$. This ensure that the primal constraint is satisfied for all $\gamma > \inf_P \gamma$.

The constraint P in the primal optimization problem (3) corresponds to an infinite-dimensional convex feasibility test. The following computational algorithm for obtaining a, possibly suboptimal, solution to (3) can be used. The first two steps are explained in more detail in [7].

1. Restrict the primal optimization problem to a finite dimensional subspace by considering a subset of $\Pi_\Delta(\gamma)$ consisting of matrix functions on the form

$$\Pi(j\omega) = \Psi(j\omega)^* M(\lambda, \gamma)\Psi(j\omega) \tag{4}$$

where $\Psi \in \mathbf{RL}_\infty^{N\times m}$ is a *basis multiplier* and where $M : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathcal{S}_\mathrm{R}^{N\times N}$ is linear in the parameters $\lambda \in \mathbf{R}^n$ for fixed γ. The range of λ is constrained by LMI:s on the form $\Phi_k^T M(\lambda,\gamma)\Phi_k \leq 0$, $k = 1, \ldots, K$.

2. It follows from the Kalman-Yakubovich-Popov lemma that the constraint of the restricted optimization problem is equivalent with a finite dimensional LMI test P_{LMI} on the form

$$P_{LMI} : \begin{cases} \exists\lambda \in \mathbf{R}^n, P_0 = P_0^T, \text{ such that} \\ \mathcal{N}^T \mathcal{A}(P_0, \lambda, \gamma)\mathcal{N} < 0 \\ \Phi_k^T M(\lambda,\gamma)\Phi_k \leq 0, \; k = 1, \ldots, K \end{cases}$$

where $\mathcal{N}$ and $\mathcal{A}$ are obtained from a state space realization of

$$\Phi_0 = \Psi \begin{bmatrix} G \\ I \end{bmatrix}$$

$\mathcal{A}$ is linear in P_0 and for fixed γ it is also linear in λ. The dimension of P_0 corresponds to the dimension of the state space realization of Φ_0.

3. We can now obtain a solution to the restricted primal by either

   a. Bisection on γ. This follows from the second assumption on $\Pi_\Delta(\gamma)$.

   b. It is often possible to transform the restricted primal to a *generalized eigenvalue problem.* There is support in LMI-lab for solving such problems, [5].

   c. For some problems it is possible to fix one element in the finite dimensional cone obtained in (4) such that the restricted primal becomes an *linear objective minimization problem*, which has support in LMI-lab.

The potential of the computational method described above is dependent on the choice of a good finite-dimensional subspace for the restricted primal. It is desirable to keep the dimension of this subspace as low as possible since the speed of the LMI computations depends critically on it. In order to evaluate the quality of a particular subspace we would like to have a method for obtaining upper bounds on the primal optimization problem in (3). The dual optimization problem derived in the next section can be used for exactly this purpose. The dual turns out to be infinite-dimensional. However, in many applications we may obtain, possibly suboptimal, solutions by restricting attention to finite-dimensional subspaces, which results in finite-dimensional convex optimization problems. We can then consider the primal optimization problem solved when we have obtained suboptimal solutions of the primal and the dual with a small gap between their corresponding objective values. We illustrate with an example.

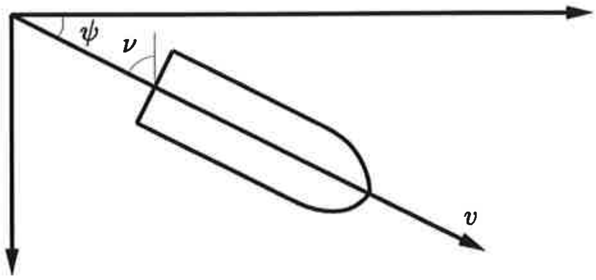

Figure 2. Notation used to describe the motion of ships.

Example 1—Ship Steering Dynamics

We will consider ship steering dynamics as in Example 9.6 in [1]. The dynamics for the ship can, with notation as in Figure 2, be approximated by the Nomoto model

$$\begin{aligned} \dot{x}(t) &= v(t)(-ax(t) + bv(t)\nu(t)) \\ \dot{\psi}(t) &= x(t) \end{aligned}$$

where ψ denotes the heading of the ship, ν denotes the rudder angle and v is the speed of the ship. It is assumed that $v(t) \geq 0$. We will as in [1] study stability of the ship dynamics for an unstable tanker, which is controlled by a PD regulator

$$\begin{aligned} \nu &= -K\psi \\ K(s) &= k(1 + sT_d) \end{aligned}$$

where $k = 2.5$ and $T_d = 0.86$. It is also assumed that $a = -0.3$ and $b = 0.8$. We will investigate the particular case when $v(t) = v_0 + A\cos(\omega_0 t)$, where $v_0 > 0$ and $A > 0$. It is possible to represent the system as in Figure 1, with $\Delta = a\cos(\omega_0 t)I_2$ and a transfer function G, which will be in $\mathbf{RH}_\infty^{2\times 2}$ when $v_0 > 0.1744$. Let $\gamma = a^{-2}$. We can then describe Δ above with the convex cone $\Pi_\Delta(\gamma)$ consisting of matrices of the form, [14]

$$\Pi(j\omega) = \begin{bmatrix} \frac{1}{2}[X(j(\omega + \omega_0)) + X(j(\omega - \omega_0))] & 0 \\ 0 & -\gamma X(j\omega) \end{bmatrix} \tag{5}$$

where $X \in P_\infty^{2\times 2}$. Let γ_{opt} be the solution to the primal optimization problem in (3) with this G and $\Pi_\Delta(\gamma)$. We can then guarantee stability for the ship dynamics when $a < 1/\sqrt{\gamma_{opt}}$. We solved the restricted primal for six choices of finite-dimensional subspace for the case when $v_0 = 0.5$ and $\omega_0 = 0.5$. Figure 3 shows how the optimal value of a increases for increasing subspaces. The upper bound given by the dual assure that the two largest finite-dimensional subspaces are close to being optimal. We refer to Section 6 for details on the computations and the choice of multipliers.

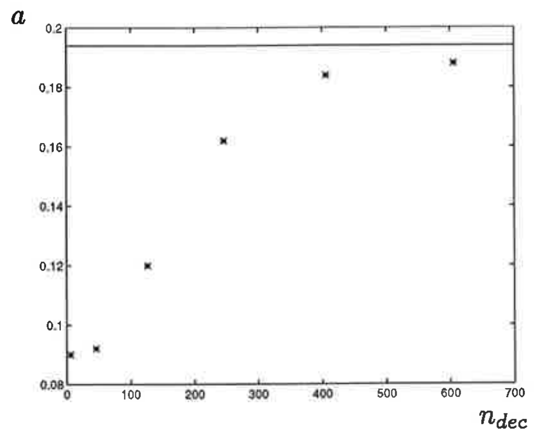

Figure 3. The primal optimization problem is solved for six different choices of finite dimensional subspace (*). The size of the subspace is given in terms of the number of decision variables in the corresponding LMI constraint. The solid line corresponds to an upper bound obtained from the dual optimization problem.

4. The Dual Robustness Test

We will in this section investigate the following optimization problem.

The Dual Optimization Problem

$$\sup \gamma \quad \text{subject to} \qquad (6)$$
$$D : \begin{cases} \exists Z \in P_{\text{NBV}}^{m\times m},\ Z \neq 0, \text{ such that} \\ M_G^\times Z \in \Pi_\Delta(\gamma)^\oplus \end{cases}$$

where the conjugate cone $\Pi_\Delta(\gamma)^\oplus$ is defined as

$$\Pi_\Delta(\gamma)^\oplus = \left\{ Z \in \mathcal{S}_{\text{NBV}}^{2m\times 2m} : \langle \Pi, Z\rangle \geq 0,\ \forall \Pi \in \Pi_\Delta(\gamma)\right\}$$

and where $M_G^\times : \mathcal{S}_{\text{NBV}}^{m\times m} \to \mathcal{S}_{\text{NBV}}^{2m\times 2m}$ is the adjoint of the operator $M_G : \mathcal{S}_\infty^{2m\times 2m} \to \mathcal{S}_\infty^{m\times m}$ defined as

$$M_G \Pi = \begin{bmatrix} G \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I \end{bmatrix}$$

for any $\Pi \in \mathcal{S}_\infty^{2m\times 2m}$. The following more suggestive formulation of the second constraint in the dual will sometimes be used

$$\begin{bmatrix} G \\ I \end{bmatrix} dZ \begin{bmatrix} G \\ I \end{bmatrix}^* \in d\Pi_\Delta(\gamma)^\oplus$$

where $M_G^\times Z$ is replaced by its corresponding measure and where $d\Pi_\Delta(\gamma)^\oplus = \{dZ : Z \in \Pi_\Delta(\gamma)^\oplus\} = \{dZ : \int \text{tr}(\Pi dZ) \geq 0,\ \forall \Pi \in \Pi_\Delta(\gamma)\}$

We have the following result

THEOREM 1
The primal optimization problem in (3) and the dual optimization problem in (6) have the same objective value, i.e

$$\inf_P \gamma = \sup_D \gamma$$

The primal and dual constraints P and D are defined in (3) and (6) respectively.

Proof: Let $\gamma^* = \inf_P \gamma$. From the second assumption on $\Pi_\Delta(\gamma)$ it follows that $\gamma < \gamma^*$ in the primal optimization problem if and only if the convex sets

$$\mathcal{P} = \{M_G\Pi : \Pi \in \Pi_\Delta(\gamma)\}$$
$$\mathcal{Q} = \left\{X \in \mathcal{S}_\infty^{m\times m} : X(j\omega) < 0, \forall\omega \in [0,\infty]\right\}$$

are disjoint. By the separating hyperplane theorem and the property that $0 \in \Pi_\Delta(\gamma)$ there exists a nonzero $Z \in \mathcal{S}_{\text{NBV}}^{m\times m}$ such that

$$\langle X, Z\rangle \geq 0, \quad \forall X \in \mathcal{P} \tag{7}$$
$$\langle X, Z\rangle < 0, \quad \forall X \in \mathcal{Q} \tag{8}$$

For (8) to hold we need $Z \in P_{\text{NBV}}^{m\times m}$. Condition (7) gives

$$\begin{aligned}&\langle M_G\Pi, Z\rangle \geq 0, \quad \forall\Pi \in \Pi_\Delta(\gamma) \quad \Longleftrightarrow\\ &\langle \Pi, M_G^\times Z\rangle \geq 0, \quad \forall\Pi \in \Pi_\Delta(\gamma) \quad \Longleftrightarrow\\ &M_G^\times Z \in \Pi_\Delta(\gamma)^\oplus\end{aligned}$$

Hence $\inf_P \gamma \leq \sup_D \gamma$. For the opposite direction we note that $\gamma > \sup_D \gamma$ implies that there is no hyperplane separating $\mathcal{P}$ and $\mathcal{Q}$. Hence, there is a $\Pi \in \Pi_\Delta(\gamma)$ such that the primal constraint P is satisfied. □

Remark The second assumption on $\Pi_\Delta(\gamma)$ implies that the primal constraint P is satisfied for all $\gamma > \gamma^* = \inf_P \gamma = \sup_D \gamma$. Furthermore, it follows from the proof of theorem that the dual constraint D is satisfied for all $\gamma < \gamma^*$. In fact: If $\exists Z \in P_{\text{NBV}}^{m\times m}$ such that $M_G^\times Z \in \Pi_\Delta(\gamma_2)^\oplus$ or equivalently $\langle M_G\Pi, Z\rangle \geq 0$, $\forall\Pi \in \Pi_\Delta(\gamma_2)$, then for all $\gamma_1 \leq \gamma_2$ and for all $\Pi_1 \in \Pi_\Delta(\gamma_1)$, there exist $\Pi_2 \in \Pi_\Delta(\gamma_2)$ such that $\langle M_G\Pi_1, Z\rangle \geq \langle M_G\Pi_2, Z\rangle \geq 0$. Hence, $M_G^\times Z \in \Pi_\Delta(\gamma_1)^\oplus$ for all $\gamma_1 \leq \gamma_2$.
The left half of Figure 4 illustrates that the primal and dual constraints are satisfied above and below the optimal value γ^*, respectively. In applications we generally find solutions to the primal and dual optimization problem by considering restrictions to finite-dimensional subspaces. The right half of Figure 4 illustrates that the resulting primal and dual generally gives suboptimal solutions γ_p and γ_d, respectively. The size of the duality gap $\gamma_p - \gamma_d$ gives an indication on the quality of these solutions.

Remark It follows from the proof of Theorem 1 that the existence of a $Z \in P_{\text{NBV}}^{m\times m}$ such that $M_G^\times Z \in \Pi_\Delta^\oplus$ indicates that there is no solution to the following feasibility problem: Find a $\Pi \in \Pi_\Delta$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \ \forall\omega \in [0,\infty]$$

where Π_Δ is a convex cone.

Refinement of the Multiplier: We will next derive the dual in case the multiplier specification is refined to be $\Pi_\Delta(\gamma) = \text{daug}(\Pi_{\Delta 1}(\gamma), \ldots, \Pi_{\Delta n}(\gamma))$, where

$$\Pi_{\Delta i}(\gamma) = \sum_{j=1}^{m_i} \Pi_{\Delta ij}(\gamma)$$

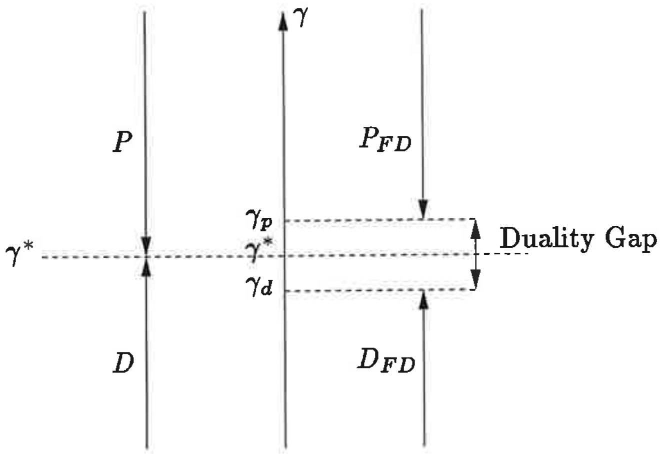

Figure 4. The primal constraint P is satisfied for all $\gamma > \gamma^*$ and the dual constraint D is satisfied for all $\gamma < \gamma^*$. The size of the duality gap indicates the quality of the suboptimal solutions that are obtained when finite-dimensional restrictions are considered.

It is assumed that $\Pi_{\Delta ij}(\gamma)$ satisfies the assumptions on $\Pi_\Delta(\gamma)$ in Section 3 . In this case

$$\begin{aligned}\Pi_\Delta(\gamma)^\oplus &= \left\{ Z | \langle \mathrm{daug}(\Pi_1,\ldots,\Pi_n), Z\rangle \geq 0,\ \forall \Pi_i = \sum_{j=1}^{m_i} \Pi_{ij},\ \Pi_{ij} \in \Pi_{\Delta ij}(\gamma) \right\} \\ &= \left\{ Z | \sum_{i=1}^{n}\sum_{j=1}^{m_i} \langle \Pi_{ij}, \mathcal{P}_i Z\rangle \geq 0,\ \forall \Pi_{ij} \in \Pi_{\Delta ij}(\gamma) \right\} \\ &= \left\{ Z | \mathcal{P}_i Z \in \cap_{j=1}^{m_i} \Pi_{\Delta ij}(\gamma)^\oplus,\ i = 1,\ldots,n \right\}\end{aligned}$$

where $\mathcal{P}_i$ denotes the projection on Π_i, i.e. $\mathcal{P}_i\Pi = \Pi_i$, if $\Pi = \mathrm{daug}(\Pi_1,\ldots,\Pi_n)$. The corresponding dual becomes

$$\sup \gamma \quad \text{subject to} \qquad (9)$$

$$D: \begin{cases} \exists Z \in P_{\mathrm{NBV}}^{m\times m},\ Z \neq 0, \text{ such that} \\ \mathcal{P}_i M_G^\times Z \in \cap_{j=1}^{m_i} \Pi_{\Delta ij}(\gamma)^\oplus,\ i = 1,\ldots,n \end{cases}$$

5. Computational Issues

The dual optimization problem in (6) is defined in terms of functions in $\mathcal{S}_{\mathrm{NBV}}$. This class of functions is very large and the corresponding optimization problem is therefore not tractable for computations. The main purpose of this section is to show how to restrict the dual to a subspace such that the resulting optimization problem involves only a finite number of matrix constraints. This approach for obtaining suboptimal solutions to the dual optimization problem is useful in a large number of practical applications.
We will use the following notation

$\mathcal{S}_{\mathrm{AM}}^{m\times m}$ The normed space consisting of step functions of the form

$$z(\omega) = \sum_{k=1}^{N} z_k\theta(\omega - \omega_k) - \overline{z_k}\theta(-\omega - \omega_k)$$

where the unit step function is defined as

$$\theta(\omega) = \begin{cases} 0, & \omega \leq 0 \\ I, & \omega > 0 \end{cases}$$

and where $z_k \in \mathcal{S}_{\mathrm{C}}^{m\times m}$, $\sum_{k=1}^{\infty} |z_k|_F < \infty$, N is any finite integer, and $\omega_k \in [0, \infty]$, $k = 1, \ldots, N$. The norm on $\mathcal{S}_{\mathrm{AM}}$ is defined as $\|z\| = 2\sum_{k=1}^{N} |z_k|_F < \infty$.

$P_{\mathrm{AM}}^{m\times m}$ The positive cone of functions $z \in \mathcal{S}_{\mathrm{AM}}^{m\times m}$ having coefficients satisfying $z_k \geq 0$ for all k.

The functions in $\mathcal{S}_{\mathrm{AM}}^{m\times m}$ can be identified with the subset of $\mathcal{S}_{\mathrm{NBV}}^{m\times m}$ consisting of functions where the variation only corresponds to step discontinuities. We can formulate a dual optimization problem restricted to this subspace as

$$\sup \gamma \quad \text{subject to} \qquad (10)$$
$$D_{AM}: \begin{cases} \exists Z \in P_{\mathrm{AM}}^{m\times m},\ Z \neq 0, \text{ such that} \\ M_G^{\times} Z \in \Pi_\Delta(\gamma)^{\oplus} \end{cases}$$

The restricted adjoint $M_G^{\times} : \mathcal{S}_{\mathrm{AM}}^{m\times m} \to \mathcal{S}_{\mathrm{AM}}^{2m\times 2m}$ is defined as the function in $\mathcal{S}_{\mathrm{AM}}^{2m\times 2m}$ with coefficients

$$\begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^*$$

for any $Z \in \mathcal{S}_{\mathrm{AM}}^{m\times m}$. This means that the last constraint in (10) can be formulated as (we neglect the negative frequencies in order to save space).

$$\sum_{k=1}^{N} \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^* \delta_{\omega_k}(\cdot) \in d\Pi\Delta(\gamma)^{\oplus}$$

where δ_{ω_k} denotes the purely atomic measure on $\mathbf{R} \cup \{\infty\}$ with support at ω_k.

Hence, for any given choice of frequency grid $\Omega = \{\omega_1, \ldots, \omega_N\}$, which defines the frequencies for the discontinuities of the step functions, the constraint definition D_{AM} in (10) involves only a finite number of complex valued matrices. In a large number of applications it turns out that for any given $\gamma \in \mathbf{R}$, the constraint in D_{AM} consists of only linear matrix equalities and inequalities. We use the next proposition to transform these conditions into equivalent conditions involving only real valued matrices.

PROPOSITION 4
Let $z = z_r + iz_i \in \mathbf{C}^{m\times m}$ be a complex valued matrix with $z_r, z_i \in \mathbf{R}^{m\times m}$. We can represent z as a matrix in $\mathbf{R}^{2m\times 2m}$ as

$$Z = \begin{bmatrix} z_r & z_i \\ -z_i & z_r \end{bmatrix}$$

We then have the following properties.

1. The conditions for z to be Hermitian can be stated as $z = z^* \Leftrightarrow Z = Z^T$, which implies that $z_r = z_r^T$ and $z_i = -z_i^T$.
2. If z is Hermitian, then $z \geq 0 \Leftrightarrow Z \geq 0$.
3. Multiplication and addition of complex matrices corresponds to multiplication and addition of the corresponding real valued matrices. Hence, we have $z_1 + z_2 \Leftrightarrow Z_1 + Z_2$ and $z_1 z_2 \Leftrightarrow Z_1 Z_2$.

Proof: This is a well known fact from algebra. □

For any choice of frequency grid $\Omega = \{\omega_1, \ldots, \omega_N\}$, we solve the restricted dual with the bisection algorithm below. We use the following notation

1. The notation $\gamma \in D_{AM}$ means that there exists a solution to the finite dimensional convex feasibility test D_{AM} defined in (10).
2. Let γ_u be an upper bound for γ. We assume that $\gamma_u \notin D_{AM}$
3. Let γ_l be a lower bound for γ. We assume that $\gamma_l \in D_{AM}$.

For a given precision $\varepsilon > 0$ solve for γ_{opt} by the following bisection algorithm

Bisection Algorithm for the Dual

while $\gamma_u - \gamma_l < \varepsilon$
  $\gamma = (\gamma_u + \gamma_l)/2$
  if $\gamma \in D_{AM}$ *then* $\gamma_l = \gamma$ *else* $\gamma_u = \gamma$
end
$\gamma_{opt} = \gamma$

There are cases when the computational approach described above is not successful. For example, there does not always exist a function with a finite number of step discontinuities such that the constraint in the dual is satisfied. Note also that for our computational approach to be successful it is necessary that every conjugate cone $\Pi_{\Delta_{ij}}(\gamma)^{\oplus}$ in (9) is suited for the approach. If this is not the case then another basis for the restriction of $\mathcal{S}_{\mathrm{NBV}}$ than $\mathcal{S}_{\mathrm{AM}}$ should be considered. In, [8] and [9] it is shown that the computational approach discussed above is successful with a small number for frequencies in the frequency grid Ω for a large class of problems that are of practical interest.

It often happens that there are algebraic constraints in the dual, which are hard to treat numerically. The following simple example is illustrative

EXAMPLE 2
Consider robust stability for a system in Figure 1. Let Δ be described in terms of $\Pi_{\Delta 1}(\gamma)$ and $\Pi_{\Delta 2}(\gamma)$, where

$$\Pi_{\Delta 1}(\gamma) = \left\{ \begin{bmatrix} X & 0 \\ 0 & -\gamma X \end{bmatrix} : X \in P_{\infty}^{m \times m} \right\}$$

and

$$\Pi_{\Delta 2}(\gamma) = \left\{ \begin{bmatrix} 0 & Y \\ Y^* & 0 \end{bmatrix} : Y \in \mathbf{RL}_{\infty}^{m \times m},\ Y(j\omega)^* = -Y(j\omega),\ \forall \omega \in [0, \infty] \right\}$$

The conjugate cones are easily shown to be $\Pi_{\Delta 1}(\gamma)^{\oplus} = \{Z \in \mathcal{S}_{\text{NBV}}^{2m\times 2m} : Z_{11} - \gamma Z_{22} \in P_{\text{NBV}}^{m\times m}\}$ and $\Pi_{\Delta 2}(\gamma) = \{Z \in \mathcal{S}_{\text{NBV}}^{2m\times 2m} : Z_{12} - Z_{12}^* \equiv 0\}$. It follows from (9) that the corresponding dual is

$$\sup \gamma \quad \text{subject to}$$
$$D : \begin{cases} \exists Z \in P_{\text{NBV}}^{m\times m},\ Z \neq 0, \text{ such that} \\ GZG^* - \gamma Z \in P_{\text{NBV}}^{m\times m} \\ GZ - ZG^* \equiv 0 \end{cases}$$

where the last constraint is algebraic. For this dual we can use the computational method described above with only one frequency in the grid Ω without any conservativity, [8]. We get

$$\sup \gamma \quad \text{subject to} \tag{11}$$
$$D : \begin{cases} \exists \omega \in [0,\infty],\ Z \in \mathcal{S}_{\text{C}}^{m\times m},\ Z \geq 0,\ Z \neq 0, \text{ s.t.} \\ G(j\omega)ZG(j\omega)^* - \gamma Z \geq 0 \\ G(j\omega)Z - ZG(j\omega)^* = 0 \end{cases}$$

It can be shown that the last constraint in (11) corresponds to finding a frequency where $G(j\omega)$ has a real valued eigenvalue. Finding such frequencies is generally a hopeless computational problem. It can actually be shown that (11) is equivalent to finding the frequency where $G(j\omega)$ has a real-valued eigenvalue of maximal modulus. More precisely we need to solve for $\rho = \max\{|\lambda(G(j\omega))| : \lambda(Gj\omega) \in \mathbf{R},\ \omega \in [0,\infty]\}$. Then the corresponding objective of the dual is $1/\rho^2$. □

We can often avoid duals with algebraic constraints of the type in this example by considering a different primal with *harder* constraints on the multiplier. This will give a corresponding dual with *softer* constraints. For example, let the primal be

Hard Primal

$$\inf \gamma \quad \text{such that} \tag{12}$$
$$P_H : \begin{cases} \exists \Pi \in \Pi_\Delta(\gamma), \text{ such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < -\varepsilon I, \quad \forall \omega \in [0,\infty] \\ -cI < \Pi(j\omega) < cI,\ \forall \omega \in [0,\infty] \end{cases}$$

where $\varepsilon > 0$ is small and where $c > 0$ is large. This restriction of the original primal is reasonble in a computations perspective since only bounded entities can be treated in the computer and the constraints will always be obtained with some marginal ε, if they are obtained at all.

The second assumption on $\Pi_\Delta(\gamma)$ is no longer enough to ensure that the primal constraint P_H is satisfied for all $\gamma > \inf_{P_H} \gamma$. The following two alternative assumptions will ensure this, where the second is the weakest

Second assumption on $\Pi_\Delta(\gamma)$***:***

2'. If $\gamma_2 \geq \gamma_1$ then for all $-cI < \Pi_1 \in \Pi_\Delta(\gamma_1)$ there exists $\Pi_2 \in \Pi_\Delta(\gamma_2)$ such that $-cI < \Pi_2 \leq \Pi_1$.

2". $\gamma_2 \geq \gamma_1$ then for all $\Pi_1 \in \Pi_\Delta(\gamma_1)$ with $-cI \leq \Pi_1 \leq cI$ there exists $\Pi_2 \in \Pi_\Delta(\gamma_2)$ with $-cI \leq \Pi_2 \leq cI$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* [\Pi_2(j\omega) - \Pi_1(j\omega)] \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq 0$$

Under any of these two assumptions on $\Pi_\Delta(\gamma)$ we derive the following dual of (12)

Soft Dual

$$\sup \gamma \quad \text{such that} \tag{13}$$

$$D_S : \begin{cases} \exists Z_0 \in P_{\text{NBV}}^{m\times m}, Z_1, Z_2 \in P_{\text{NBV}}^{2m\times 2m}, Z_0 \neq 0, \text{ s.t.} \\ M_G^\times Z_0 + Z_1 - Z_2 \in \Pi_\Delta(\gamma)^\oplus \\ \langle I, Z_1 + Z_2 \rangle \leq \frac{\varepsilon}{c} \langle I, Z_0 \rangle \end{cases}$$

and we have

PROPOSITION 5

$$\inf_{P_H} \gamma = \sup_{D_S} \gamma$$

Proof: The proof is similar to the proof of Theorem 1 and it is given in the appendix. □

It is also shown in the appendix that with our computational approach the soft dual can be formulated as

Soft Dual for $\mathcal{S}_{AM}$

$$\sup \gamma \quad \text{such that} \tag{14}$$

$$D_S : \begin{cases} \exists Z \in P_{\text{AM}}^{m\times m}, Z \neq 0, \text{ such that} \\ M_G^\times Z \in \Pi_\Delta(\gamma)^\oplus + \mathcal{B}(\varepsilon, c, Z) \end{cases}$$

where

$$\mathcal{B}(\varepsilon, c, Z) = \left\{ X \in \mathcal{S}_{\text{AM}}^{2m\times 2m} : \sum \text{tr}|X_k| \leq \frac{\varepsilon}{c} \sum \text{tr} Z_k \right\}$$

Here $|X|$ denotes the absolute value of a matrix defined as $|X| = (X^2)^{1/2}$.

If we consider Example 2 again we see that for the second constraint to be satisfied in (11) it is enough that the algebraic constraints corresponding to $\Pi_\Delta(\gamma)^\oplus$ are satisfied with a precision which is dependent of ε. This can be formulated as an LMI condition. The suggested soft dual has very much in common with the idea of adding a small complex perturbation to each real parameter in the computations for the mixed real/complex singular value, see [16]. There are examples when the gap between the objective values for the soft dual in (13) and the dual in (6) is large even when ε is small and c is large. This is often an indicatation that the system model with G and Δ needs more attention.

6. The Ship Steering Example, cont'd

We will here describe the computations for the ship steering example in Section 3. We first discuss the primal optimization problem.

The Primal Optimization Problem We use the finite dimensional convex cone consisting of the functions

$$\Pi = \left[\begin{array}{c|c} \widetilde{R} & 0 \\ \hline 0 & R \end{array}\right]^* \left[\begin{array}{cc|c} U & 0 & 0 \\ 0 & U & 0 \\ \hline 0 & 0 & -U \end{array}\right] \left[\begin{array}{c|c} \widetilde{R} & 0 \\ \hline 0 & R \end{array}\right]$$

subject to the constraint that $U = U^T \geq 0$. Here $\widetilde{R}$ is obtained by spectral factorization of $[R^*(s+j\omega_0)UR(s+j\omega_0) + R^*(s-j\omega_0)UR(s-j\omega_0))]/2$. Its state space realization is given as

$$\widehat{R} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} sI - A & -\omega_0 I \\ \omega_0 I & sI - A \end{pmatrix}^{-1} \begin{pmatrix} B \\ 0 \end{pmatrix} + \begin{pmatrix} D \\ 0 \end{pmatrix} \in \mathbf{RH}_\infty^{2N\times m}$$

when R has the realization $R(s) = C(sI-A)^{-1}B + D \in \mathbf{RH}_\infty^{N\times 2}$. Table 1 shows numerical results obtained by LMI-lab, [5] when we use R on the form

$$\text{Ritz}(p,n) = \left[\, I_2, \;\; \tfrac{s-p}{s+p} I_2, \ldots, \tfrac{s-p}{s+p} I_2 \,\right]^T$$

The results are given in terms of the obtained bound on a.

The Dual Optimization Problem In order to solve the corresponding dual optimization problem we need the conjugate cone $\Pi_\Delta(\gamma)^\oplus$. In other words we need the set of $Z \in \mathcal{S}_{\text{NBV}}^{4\times 4}$ such that for any $\Pi \in \Pi_\Delta(\gamma)$

$$0 \leq \langle \Pi, Z \rangle = \left\langle X(j\omega), \frac{1}{2}[Z_{11}(\omega+\omega_0) + Z_{11}(\omega-\omega_0)] - \gamma Z_{22}(\omega) \right\rangle \qquad (15)$$

Since (15) shall hold for any $X(j\omega) \geq 0, \forall\omega$, we get

$$\Pi_\Delta(\gamma)^\oplus = \left\{ Z \in \mathcal{S}_{\text{NBV}}^{4\times 4} : \frac{1}{2}[Z_{11}(\omega+\omega_0) + Z_{11}(\omega-\omega_0)] - \gamma Z_{22}(\omega) \in P_{\text{NBV}}^{2\times 2} \right\}$$

If we let S_{ω_0} denote the shift operator defined by $S_{\omega_0}Z(\omega) = Z(\omega+\omega_0)$, then the dual can be formulated as

$$\begin{aligned} &\sup \gamma \quad \text{subject to} \\ &D: \begin{cases} \exists Z \in P_{\text{NBV}}^{2\times 2},\; Z \neq 0, \text{ such that} \\ S_{\omega_0} GdZG^* + S_{-\omega_0} GdZG^* - \gamma Z \in P_{\text{NBV}}^{2\times 2} \end{cases} \end{aligned} \qquad (16)$$

The computational ideas in Section 5 and the form of the second constraint in (16) suggests that we choose a frequency grid $\Omega = \{\omega_1, \ldots, \omega_N\}$ satisfying

$$\begin{aligned} &\omega_l = \omega_{L+l} - \omega_0 \\ &\omega_{(k-1)L+l} + \omega_0 = \omega_{kL+l} = \omega_{(k+1)L+l} - \omega_0 \\ &\omega_{(K-1)L+l} + \omega_0 = \omega_{KL+l} \end{aligned}$$

$R(s)$	a
I_2	0.090
Ritz(1,1)	0.092
Ritz(1,2)	0.12
Ritz(1,3)	0.162
Ritz(1,4)	0.184
Ritz(1,5)	0.188

K	L	ω_1	a
2	1	0.185	0.195
4	1	0.185	0.194
8	1	0.185	0.194

Table 1. Table with results for the primal and dual optimization problem respectively.

for $k = 1, \ldots, K-1$ and $l = 1, \ldots, L$. Let

$$Z_k = \begin{bmatrix} Z_{kR} & Z_{kI} \\ -Z_{kI} & Z_{kR} \end{bmatrix}$$

for $k = 1, \ldots, N$, where $N = (K+1)L$. Here $Z_{kR} = Z_{kR}^T \in \mathbf{R}^{2\times 2}$ and $Z_{kI} = -Z_{kI}^T \in \mathbf{R}^{2\times 2}$. Further let

$$G_k = \begin{bmatrix} \text{Re } G(j\omega_k) & \text{Im } G(j\omega_k) \\ -\text{Im } G(j\omega_k) & \text{Re } G(j\omega_k) \end{bmatrix} \in \mathbf{R}^{4\times 4}$$

We can formulate the restricted dual optimization problem as

$$\sup_{D_{AM}} \gamma$$

$$D_{AM} : \begin{cases} \exists Z_{kM+l} \geq 0, \; k = 0, \ldots, K, l = 1, \ldots, L \text{ s.t.} \\ \frac{1}{2} G_{L+l} Z_{(L+l)} G_{L+l}^T - \gamma Z_l \geq 0 \\ \frac{1}{2} [G_{(k+1)L+l} Z_{(k+1)L+l} G_{(k+1)L+l}^T \\ + G_{(k-1)M+l} Z_{(k-1)L+l} G_{(k-1)L+l}^T] - \gamma Z_{kL+l} \geq 0 \\ \frac{1}{2} G_{(K-1)L+l} Z_{(K-1)L+l} G_{(K-1)L+l} - \gamma Z_{KL+l} \geq 0 \\ \text{for } k = 1, \ldots, K-1, \; l = 1, \ldots, L \end{cases}$$

Remark Note that the constraint set is an LMI condition.

Numerical results obtained with LMI-lab are given in Table 1.

7. Slope Restricted Nonlinearities

We will in this section consider a class of IQC:s which gives a particularly complicated dual optimization problem. Let $\phi : \mathbf{R} \to \mathbf{R}$ be a slope restricted nonlinearity. We assume that ϕ satisfies the following properties

(i) $\phi(0) = 0$

(ii) ϕ is odd.

(iii) $0 \leq \frac{\phi(x_1)-\phi(x_2)}{x_1-x_2} \leq k, \quad x_1 \neq x_2$.

where $0 \leq k < \infty$. If $k = \infty$ then we assume that ϕ satisfies the boundedness condition $\phi(x) \leq c|x|$, $\forall x \in \mathbf{R}$ for some $c > 0$. This class of nonlinearities satisfies the IQC:s from the following convex cone, [20]

$$\Pi_\Delta = \left\{ \begin{bmatrix} 0 & h_0 + H(j\omega) \\ h_0 + H(j\omega)^* & -\frac{2}{k}\text{Re}\,[h_0 + H(j\omega)] \end{bmatrix} : \|h\|_1 \leq h_0 \right\} \tag{17}$$

where $\|h\|_1$ denotes the L_1 norm of the impulse response of H, i.e.

$$h(t) = \int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega$$
$$\|h\|_1 = \int_{-\infty}^{\infty} |h(t)|dt$$

Lemma 1

Let $k = \infty$, then the dual cone $\Pi_\Delta^\oplus \cap \mathcal{S}_{\text{AM}}^{2\times 2}$ is defined as

$$\Pi_\Delta^\oplus \cap \mathcal{S}_{\text{AM}}^{2\times 2} = \left\{ \begin{bmatrix} z_a & z^* \\ z & z_b \end{bmatrix} \in \mathcal{S}_{\text{AM}}^{2\times 2} : \text{such that (18) holds} \right\}$$

where the condition (18) is defined as

$$\text{Re} \sum_{k=1}^{N} z_k \geq \sup_{t\in\mathbf{R}} \left| \text{Re} \sum_{k=1}^{N} z_k e^{-j\omega_k} \right| \tag{18}$$

Proof: For any $\Pi \in \Pi_\Delta$ we need

$$\langle \Pi, Z \rangle = 4 \sum_{k=1}^{N} \text{Re}\; z_k[h_0 + H(j\omega_k)] \geq 0. \tag{19}$$

We have that

$$H(j\omega_k) = \int_{-\infty}^{\infty} h(t)e^{-j\omega_k t}dt.$$

Hence, the inequality

$$\sum_{k=1}^{N} z_k H(j\omega_k) = \int_{-\infty}^{\infty} \text{Re}\; \{\sum_{k=1}^{N} z_k e^{-j\omega_k}\} h(t)dt \geq -\sup_{t\in\mathbf{R}} |\text{Re} \sum_{k=1}^{N} z_k e^{-j\omega_k t}| h_0, \tag{20}$$

holds since $||h||_1 \leq h_0$. A sufficient condition that $Z \in \Pi_\Delta^\oplus$ is thus

$$\sum_{k=1}^{N} \text{Re } z_k \geq \sup_{t \in \mathbf{R}} |\text{Re } \sum_{k=1}^{N} z_k e^{-j\omega_k t}|$$

In order to see that this condition is also necessary we notice that $f(t) = \text{Re } \sum_{k=1}^{N} z_k e^{-j\omega_k t} \in \mathbf{L}_\infty(-\infty,\infty)$. Since $\mathbf{L}_1(-\infty,\infty)^* = \mathbf{L}_\infty(-\infty,\infty)$, we have, by the definition of the norm for the dual space, that

$$\inf_{||h||_1 \leq h_0} \langle h, f \rangle = -||f||_\infty h_0 \tag{21}$$

where $\langle h, f \rangle$ denotes the integral in (20). In our application we only consider $H \in \mathbf{RL}_\infty^{1\times 1}$. The exponentials $\{t^k e^{-t}\theta(t)\}_{k=0}^\infty$ and $\{t^k e^{t}\theta(-t)\}_{k=0}^\infty$ are dense in $\mathbf{L}_1[0,\infty)$ and $\mathbf{L}_1(-\infty, 0]$ respectively, see [19]. This means that we can approximate an arbitrary $h \in \mathbf{L}_1(-\infty,\infty)$ with any accuracy with a suitable finite linear combination of such exponential functions. The corresponding transfer function will be in $\mathbf{RL}_\infty^{1\times 1}$ and it follows that (21) also holds when we consider the optimization over impulse responses corresponding to this class of rational functions. This proves the necessity. □

It is clear from (18) that the frequencies for the step discontinuities must be chosen with care. Actually, for choices $(\omega_1, \ldots, \omega_N)$ in a dense subset of $[0,\infty]^N$ the right hand side of (18) will be $\sum_{k=1}^{N} |z_k|$.

However if we for example chose only rational frequencies then the right hand side of (18) will be periodic and it possible that the condition is satisfied.

We will next consider a stability test for a system with a linear time invariant plant G in the forward loop and a slope restricted nonlinearity in the feedback loop.

THEOREM 2
There is no solution to the feasibility test: Find $H \in \mathbf{RL}_\infty^{1\times 1}$ with corresponding impulse response h such that

a. $||h||_1 \leq 1$

b. $\text{Re } [G(j\omega) - k][1 + H(j\omega)^*] < -\varepsilon, \ \forall \omega \in [0, \infty]$

if there exists $\omega_1, \ldots, \omega_N \in [0,\infty]$ and $z_1, \ldots, z_N \geq 0$ such that

$$\sum_{k=1}^{N} \text{Re } z_k[G(j\omega_k) - \frac{1}{k}] \geq \sup_{t \in \mathbf{R}} |\sum_{k=1}^{N} \text{Re } z_k[G(j\omega_k) - \frac{1}{k}]e^{j\omega_k}| - \frac{\varepsilon}{2}\sum_{k=1}^{N} z_k \tag{22}$$

Proof: The feasibility test can be formulated as: Find $\Pi \in \Pi_\Delta$ such that $M_G\Pi < -2\varepsilon I$, where $M_G : \mathcal{S}_\infty^{2m\times 2m} \to \mathcal{S}_\infty^{m\times m}$ is defined as

$$M_G \Pi = \begin{bmatrix} G - \frac{1}{k} \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G - \frac{1}{k} \\ I \end{bmatrix}$$

and where Π_Δ corresponds to the convex set in (17) when $k = \infty$ and $h_0 = 1$. Unfeasibility means that the convex sets

$$\mathcal{P} = \{M_G\Pi + 2\varepsilon I : \Pi \in \Pi_\Delta(\gamma)\}$$
$$\mathcal{Q} = \{X \in \mathcal{S}_\infty^{1\times 1} : X(j\omega) < 0, \forall \omega\}$$

are disjoint, which is the case if there exists a nonzero $Z \in P_{\text{AM}}^{1\times 1}$ such that $\langle \Pi, M_G^\times Z \rangle \geq -2\varepsilon \langle I, Z \rangle$. Hence, the proof follows from lemma 1. □

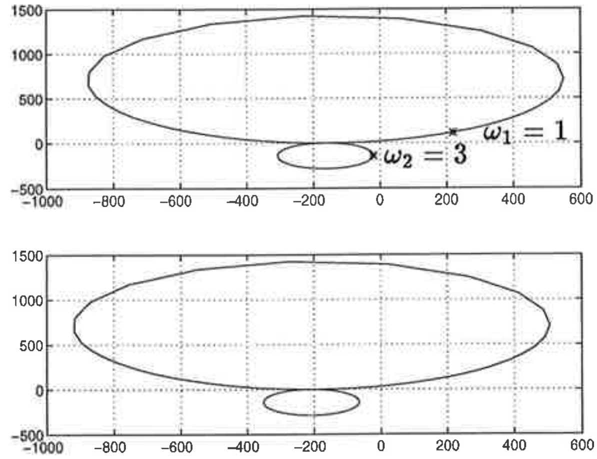

Figure 5. The upper plot shows the Nyquist diagram of $G(j\omega)-1/k$, $k = 0.0061$. There is no solution to the feasibility test in Theorem 2 when $\varepsilon = 0$ for this value of k. The lower plot shows the Nyquist diagram of $G(j\omega) - 1/k$, $k = 0.0048$. The multiplier $H = 6.25/(6 + 2.5)^2$ can be used to prove stability for this value of k.

It will in most applications will be very hard to find a suitable frequency grid for application of Theorem 2. However the next example show that it is possible to use it. A different way of treating unfeasibility of the stability test in the theorem has been reported in [13].

Example 3
We will consider the system in Figure 1 when Δ is a slope restricted nonlinearity with slope in $[0, k]$ and when G has transfer function

$$G(s) = \frac{s^2}{(s^2+\alpha)(s^2+\beta)+10^{-4}(14s^3+21s)}$$

where $\alpha = 0.9997$ and $\beta = 9.0039$. This is a system with two very distinct resonances at $\omega \approx 1$ and $\omega \approx 3$. The purpose of the example is to find a bound on k such that stability of the system is guaranteed. For $\varepsilon = 0$ in Theorem 2 the simple multiplier $H(s) = -\frac{6.25}{(s+2.5)^2}$ can be used to prove stability for $k = 0.0048$. If we use the dual with $\omega_1 = 1$ and $\omega_2 = 3$, then the condition in (22) is satisfied if $k = 0.0061$. Hence, the duality gap is reasonable small despite the low order of the multiplier H. Figure 5 shows the Nyquist curves for $G(j\omega) - \frac{1}{k}$ for $k = 0.0061$ and $k = 0.0048$ respectively.

□

8. Robust Performance Analysis

We will here investigate the dual that appears in robust performance analysis of the second system in Figure 1. We assume that the transfer function has structure

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \in \mathbf{RH}_\infty^{(m+q)\times(m+q)}$$

and that the input signal is in the class of $u \in \mathbf{L}_2^q[0,\infty)$ satisfying the IQC defined by the closed convex cone $\Psi_{inp} \subset \mathcal{S}_\infty^{q\times q}$, i.e.

$$\int_{-\infty}^{\infty} \hat{u}(j\omega)^* \Psi(j\omega) \hat{u}(j\omega) d\omega \geq 0, \ \forall \Psi \in \Psi_{inp}$$

It follows from [18] that robust performance analysis gives optimization problems on the form

$$\inf \gamma \quad \text{subject to}$$
$$P: \begin{cases} \exists \Pi_1 \in \Pi_\Delta, \Pi_2 \in \Pi_{perf}(\gamma), \Psi \in \Psi_{inp}, \text{ such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \text{daug}\left(\Pi_1, \Pi_2 + \begin{bmatrix} 0 & 0 \\ 0 & \Psi \end{bmatrix}\right) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0,\infty] \end{cases}$$

where $\Pi_{perf}(\gamma)$ defines the performance criterion. We assume that $\Pi_{perf}(\gamma)$ is a convex cone for any given $\gamma \in \mathbf{R}$, and if $\gamma_2 \geq \gamma_1$, then $\forall \Pi_1 \in \Pi_{perf}(\gamma_1)$ there exists $\Pi_2 \in \Pi_{perf}(\gamma_2)$ such that $\Pi_1 \geq \Pi_2$.

It follows from (9) that the corresponding dual is

$$\sup \gamma \quad \text{subject to}$$
$$D: \begin{cases} Z \in P_{\text{NBV}}^{(m+q)\times(m+q)}, \ Z \neq 0, \text{ such that} \\ \begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix} dZ \begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix}^* \in d\Pi_\Delta^\oplus \\ \begin{bmatrix} G_{21} & G_{22} \\ 0 & I \end{bmatrix} dZ \begin{bmatrix} G_{21} & G_{22} \\ 0 & I \end{bmatrix}^* \in d\Pi_{perf}(\gamma)^\oplus \cap d\Psi^\oplus \end{cases}$$

where $\Psi^\oplus$ is defined as

$$\Psi^\oplus = \left\{ \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \in \mathcal{S}_{\text{NBV}}^{2q\times 2q} : Z_{22} \in \Psi_{inp}^\oplus \right\}$$

and where

$$\Psi_{inp}^\oplus = \left\{ Z \in \mathcal{S}_{\text{NBV}}^{q\times q} : \langle \Psi, Z \rangle \geq 0, \ \forall \Psi \in \Psi_{inp} \right\}$$

We can also formulate this dual as

$$\sup \gamma \quad \text{subject to}$$
$$D: \begin{cases} Z \in P_{\text{NBV}}^{(m+q)\times(m+q)}, \ Z \neq 0, \text{ such that} \\ \begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix} dZ \begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix}^* \in d\Pi_\Delta^\oplus \\ \begin{bmatrix} G_{21} & G_{22} \\ 0 & I \end{bmatrix} dZ \begin{bmatrix} G_{21} & G_{22} \\ 0 & I \end{bmatrix}^* \in d\Pi_{perf}(\gamma)^\oplus \\ Z_{22} \in \Psi_{inp}^\oplus \end{cases}$$

We will apply this to a simple example

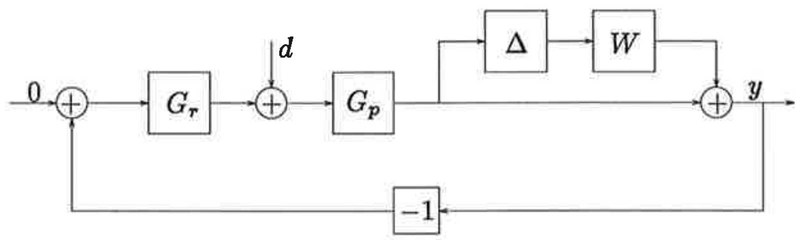

Figure 6. Control system for Example 4

EXAMPLE 4
Figure 4 shows a feedback system consisting of a plant G_p and a controller G_r. There is also a multiplicative uncertainty represented as $W(s)\Delta(s)$, where W denotes a weighting filter and where Δ is a linear time invariant uncertainty satisfying $\|\Delta(j\omega)\|_\infty \leq 1$ ($\|\cdot\|_\infty$ is the usual H_∞-norm). The load disturbance d is assumed to be a low-frequency signal satisfying supp $\widehat{d}(j\omega) \subset [-0.1, 0.1]$, where supp $\widehat{d}$ denotes the support of the Fourier transform of d. The purpose of the example is to compute the worst case induced $\mathbf{L}_2$-norm of the system. The system can be transformed into the normal form for robust performance problems in Figure 1, with

$$G(s) = \frac{1}{1 + G_p G_r} \begin{bmatrix} -G_p G_r W & G_p \\ W & G_p \end{bmatrix}$$

Assume that the plant has transfer function

$$G_p(s) = \frac{10}{(s+1)^2(s+10)}$$

and that we use a P controller designed with the Ziegler-Nichols frequency method, which gives $G_r(s) = 12.2$. Further assume that the weighting filter is the constant $W(s) = 0.1$. For this example we use the IQC descriptions

$$\Pi_\Delta = \left\{ \begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix} : X \in P_\infty^{1\times 1} \right\}$$

$$\Pi_{perf}(\gamma) = \left\{ \begin{bmatrix} xI & 0 \\ 0 & -\gamma x I \end{bmatrix} : x \geq 0 \right\}$$

and

$$\Psi_{inp} = \left\{ \Psi \in \mathcal{S}_\infty^{1\times 1} : \Psi(j\omega) = \begin{cases} \geq 0, & \omega \in [-a, a] \\ \leq 0, & \omega \notin [-a, a] \end{cases} \right\}$$

It is easy to see that

$$\Pi_\Delta^\oplus = \left\{ Z \in \mathcal{S}_{\mathbf{NBV}}^{2\times 2} : Z_{11} - Z_{22} \in P_{\mathbf{NBV}}^{1\times 1} \right\}$$

$$\Pi_{perf}(\gamma)^\oplus = \left\{ Z \in \mathcal{S}_{\mathbf{NBV}}^{2\times 2} : \langle I, Z_{11} - \gamma Z_{22} \rangle \geq 0 \right\}$$

and that

$$\Psi_{inp}^{\oplus} = \left\{ Z \in \mathcal{S}_{\text{NBV}}^{1\times 1} : Z(\omega_1) - Z(\omega_2) = \begin{cases} \geq 0, \ \forall a \geq \omega_1 > \omega_2 \geq 0 \\ \leq 0, \ \forall \infty \geq \omega_1 > \omega_2 \geq a \end{cases} \right\}$$

It follows from the duality constraint that only one frequency need to be considered in the dual optimization problem. The dual can therefore be formulated as the following complex valued LMI optimization problem

$$\gamma^* = \sup \gamma \quad \text{subject to}$$
$$D : \begin{cases} \exists \omega \in [0, \infty], \ Z \in \mathbf{C}^{2\times 2}, \ Z = Z^* \geq 0, Z \neq 0, \ \text{s.t.} \\ \left[G_{11}(j\omega) \quad G_{12}(j\omega) \right] Z \left[G_{11}(j\omega) \quad G_{12}(j\omega) \right]^* - Z_{11} \geq 0 \\ \left[G_{21}(j\omega) \quad G_{22}(j\omega) \right] Z \left[G_{21}(j\omega) \quad G_{22}(j\omega) \right]^* - \gamma Z_{22} \geq 0 \\ Z_{22} = 0 \text{ if } \omega \notin [0, a] \end{cases}$$

A numerical solution is obtained as follows

(i) If there exists $\omega \in (a, \infty]$ such that $|G_{11}(j\omega)| \geq 1$, then $\gamma^* = \infty$. In this case the closed loop system is unstable.

(ii) Otherwise introduce

$$Z = \begin{bmatrix} Z_r & Z_i \\ -Z_i & Z_r \end{bmatrix}, \ Z_r = Z_r^T \in \mathbf{R}^{2\times 2}, \ Z_i = -Z_i^T \in \mathbf{R}^{2\times 2}$$

and

$$G_1(j\omega) = \begin{bmatrix} \text{Re } G_{11}(j\omega) & \text{Re } G_{12}(j\omega) & \text{Im } G_{11}(j\omega) & \text{Im } G_{12}(j\omega) \\ -\text{Im } G_{11}(j\omega) & -\text{Im } G_{12}(j\omega) & \text{Re } G_{11}(j\omega) & \text{Re } G_{12}(j\omega) \end{bmatrix}$$
$$G_2(j\omega) = \begin{bmatrix} \text{Re } G_{21}(j\omega) & \text{Re } G_{22}(j\omega) & \text{Im } G_{21}(j\omega) & \text{Im } G_{22}(j\omega) \\ -\text{Im } G_{21}(j\omega) & -\text{Im } G_{22}(j\omega) & \text{Re } G_{21}(j\omega) & \text{Re } G_{22}(j\omega) \end{bmatrix}$$
$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
$$E_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then solve the LMI optimization problem

$$\gamma^* = \sup \gamma \quad \text{subject to}$$
$$D_{AM} : \begin{cases} \exists \omega \in [0, a], \ Z \geq 0, \ Z \neq 0, \ \text{such that} \\ G_1(j\omega) Z G_1(j\omega)^T - E_1 Z E_1^T \geq 0 \\ G_2(j\omega) Z G_2(j\omega)^T - \gamma E_2 Z E_2^T \geq 0 \end{cases}$$

With $a = 0.1$ we obtained the solution $\gamma^* = 0.07635$.

The primal optimization problem can be solved in the following way. We obtain a finite dimensional description of $\Pi_\Delta(\gamma)$ as

$$\Pi_\Delta = \left\{ \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}^* \begin{bmatrix} U & 0 \\ 0 & -U \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} : U = U^T \geq 0 \right\}$$

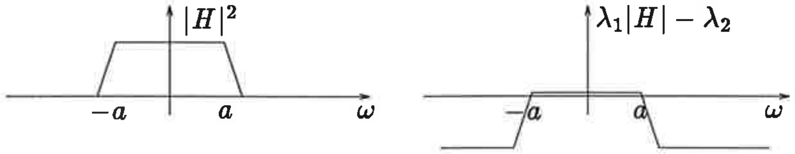

Figure 7. Filter specification for low-pass signals

H	R	γ^*
$\frac{1}{s+1}$	1	0.105
$\frac{1}{s+1}$	Ritz(3, 1)	0.0919
$\frac{1}{s+1}$	Ritz(3, 2)	0.0919
$\frac{10}{s+10}$	1	0.0879
$\frac{10}{s+10}$	Ritz(3, 1)	0.0778
$\frac{10}{s+10}$	Ritz(3, 2)	0.0778

Table 2. Numerical Results

for some basis multiplier $R \in \mathbf{RH}_\infty^{N\times 1}$. The $\mathbf{L}_2$-performance specification is finite dimensional and as a finite dimensional description of Ψ_{inp}, we use

$$\Psi_{inp} = \left\{\lambda_1|H(j\omega)|^2 - \lambda_2 : \lambda_1|H(ja)|^2 - \lambda_2 \geq 0, \lambda_1, \lambda_2 \geq 0\right\}$$

Here H denotes any rational low pass filter with monotonically decreasing amplitude function. This is illustrated in Figure 7.

The results in Table 2 were obtained by using LMI-lab. γ^* denotes the optimal value of the primal optimization problem and the corresponding $\mathbf{L}_2$-performance is $\sqrt{\gamma^*}$. The multiplier Ritz(p, n) is defined as

$$\text{Ritz}(p, n) = \begin{bmatrix} 1 & \frac{s-p}{s+p} & \cdots & \frac{(s-p)^n}{(s+p)^n} \end{bmatrix}$$

The duality gap 0.014 is mainly due to the low order of the filter H. However, it seems that a filter of very high order is needed in order to obtain a smaller duality gap.

9. Conclusions

We have derived the format for the dual to optimization problems that appear in robustness analysis based on IQC:s. The purpose of the dual is to give upper bounds to infinite-dimensional robustness tests, which correspond to finding an optimal multiplier in an infinite-dimensional convex set. We have shown how solutions to the dual in many cases can be obtained by solving a finite dimensional optimization problem at a preselected frequency grid. It is shown in [8] that this approach is successful with a small number of frequencies in the grid when constant multipliers are combined with frequency dependent multipliers, which take independent values at different frequencies.

10. Appendix: Proof of Proposition 5

From the second assumption on $\Pi_\Delta(\gamma)$ it follows that $\gamma < \inf_{P_H} \gamma$ if and only if the convex sets

$$\mathcal{P} = \{(M_G\Pi + \varepsilon I, \Pi - cI, -\Pi - cI) : \Pi \in \Pi_\Delta(\gamma)\}$$
$$\mathcal{Q} = \left\{X \in \mathcal{S}_\infty^{m\times m} : X(j\omega) < 0, \forall\omega\right\} \times \left\{X \in \mathcal{S}_\infty^{2m\times 2m} : X(j\omega) < 0,\ \forall\omega\right\}^2$$

are disjoint. Since the second set is open it follows from the separating hyperplane theorem that there exists a nonzero triple $Z = (Z_0, Z_1, Z_3) \in \mathcal{S}_{\mathrm{NBV}}^{m\times m} \times (\mathcal{S}_{\mathrm{NBV}}^{2m\times 2m})^2$ such that

$$\langle X, Z\rangle \geq 0, \quad \forall X \in \mathcal{P} \tag{23}$$
$$\langle X, Z\rangle < 0, \quad \forall X \in \mathcal{Q} \tag{24}$$

For (24) to hold we need $Z \in P_{\mathrm{NBV}}^{m\times m} \times (P_{\mathrm{NBV}}^{2m\times 2m})^2$. Condition (23) can be reformulated as

$$\langle \Pi, M_G^\times Z_0 + Z_1 - Z_2\rangle + c\left\langle I, \frac{\varepsilon}{c} Z_0 - Z_1 - Z_2\right\rangle \geq 0, \quad \forall \Pi \in \Pi_\Delta(\gamma) \tag{25}$$

Since $\Pi_\Delta(\gamma)$ is a cone containing 0 it required that both terms need to be positive. Hence, the following constraints need to be satisfied

$$M_G^\times Z_0 + Z_1 - Z_2 \in \Pi_\Delta(\gamma)^\oplus$$
$$\langle I, Z_1 + Z_2\rangle \leq \frac{\varepsilon}{c}\langle I, Z_0\rangle \tag{26}$$

from which it also follows that $Z_0 \neq 0$, since otherwise $Z = (Z_0, Z_1, Z_2) = 0$. Hence $\inf_{P_H} \gamma \leq \sup_{D_S} \gamma$. For the opposite direction we note that $\gamma > \sup_{D_S} \gamma$ implies that there is no hyperplane separating $\mathcal{P}$ and $\mathcal{Q}$. Hence, there is a $\Pi \in \Pi_\Delta(\gamma)$ such that the primal constraint P is satisfied.

For the reformulation with step functions we notice that if we let $\widetilde{Z} = \widetilde{Z}_1 - \widetilde{Z}_2$ for some $\widetilde{Z}_1, \widetilde{Z}_2 \in P_{\mathrm{AM}}^{2m\times 2m}$, then we may choose $Z_1 = P\widetilde{Z}$ and $Z_2 = (I - P)\widetilde{Z}$, where P is the projection $\mathcal{S}_{\mathrm{AM}}^{2m\times 2m} \to P_{\mathrm{AM}}^{2m\times 2m}$ and we have

$$Z_1 - Z_2 = \widetilde{Z}$$
$$Z_1 + Z_2 = |\widetilde{Z}| \leq \widetilde{Z}_1 + \widetilde{Z}_2$$

It is now clear that the soft dual can be formulated as in (14) when it is considered over the step functions.

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