Duality Bounds in Robustness Analysis

Jönsson, Ulf; Rantzer, Anders

1996

Document Version:
Publisher's PDF, also known as Version of record

Link to publication

Citation for published version (APA):
Jönsson, U., & Rantzer, A. (1996). *Duality Bounds in Robustness Analysis*. (Technical Reports TFRT-7544). Department of Automatic Control, Lund Institute of Technology (LTH).

Total number of authors:
2

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

ISSN 0280–5316
ISRN LUTFD2/TFRT--7544--SE

Duality Bounds in Robustness Analysis

Ulf Jönsson
Anders Rantzer

Department of Automatic Control
Lund Institute of Technology
January 1996

Department of Automatic Control
Lund Institute of Technology
P.O. Box 118
S-221 00 Lund Sweden

Document name
INTERNAL REPORT

Date of issue
January 1996

Document Number
ISRN LUTFD2/TFRT--7544--SE

Author(s)
Ulf Jönsson and Anders Rantzer

Supervisor

Sponsoring organisation

Title and subtitle
Duality Bounds in Robustness Analysis

Abstract

Multipliers are used in stability theory to reduce conservatism and exploit structural information about system components. For an important class of stability problems, which result in infinite-dimensional convex multiplier optimization, a corresponding dual problem is stated. The dual gives valuable information about the original problem, in particular error bounds for its finite dimensional approximations.

Key words
Robustness, Multipliers, Stability, Duality

Classification system and/or index terms (if any)

Supplementary bibliographical information

ISSN and key title
0280–5316

ISBN

Language
English

Number of pages
19

Recipient's notes

Security classification

1. Introduction

Absolute stability theory, including passivity and small gain theorems, is an important tool for analysis of systems with nonlinearities, time-variations and uncertainty. So called multipliers are used to reduce conservatism and exploit structural information about the system components. However, systematic methods for computation and optimization of such multipliers have not been available until recently.

The development of numerical methods for multiplier optimization started with the structured singular value in the early eighties, [21, 4]. The real breakthrough came with the polynomial time algorithms for convex optimization with constraints defined by linear matrix inequalities, [14, 3]. This was used in connection with multiplier optimization in Balakrishnan, et.al. [1] and Ly et.al. [11] and in full generality by Rantzer and Megretski [18, 12].

In general, the computation of multipliers becomes a convex optimization problem over an infinite-dimensional space. Such problems can be solved by considering a sequence of finite-dimensional approximations. More specifically, in the search for a rational function that satisfies certain constraints, the finite-dimensional approximation could mean that the denominator is fixed, while the search is restricted to the numerator coefficeints. For example, this approach was used in controller design by Boyd and Barratt [2].

Duality plays an important role in optimization theory, particularly in convex optimization. This paper aims to demonstrate that multiplier optimization is no exception. For an important class of stability problems, which results in infinite-dimensional convex optimization, we will state a corresponding infinite-dimensional dual problem. The dual gives valuable information about the original problem, in particular error bounds for the finite dimensional approximations.

A general and unified approach to the use of multipliers was introduced by Megretski and Rantzer [18, 12] based on the concept *integral quadratic constraint* (IQC). An operator Δ (possibly nonlinear) on $\mathbf{L}_2^m[0,\infty)$ is said to satisfy the IQC defined by the matrix function Π, called *multiplier*, if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{(\Delta v)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{(\Delta v)}(j\omega) \end{bmatrix} d\omega \quad \geq \quad 0 \qquad \text{for all } v \in \mathbf{L}_2^m[0,\infty)$$

Here $\widehat{v}$ denotes the Fourier transform of v. Based on this definition, each operator Δ can be described by a set $\widetilde{\Pi}_\Delta$ of multipliers Π, that define IQC:s satisfied by Δ. For example, a passive operator satisfies the IQC defined by

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

while a linear time-invariant operator with H_∞-norm less than one, satisfies any IQC defined by a matrix of the form

$$\begin{bmatrix} x(j\omega)I & 0 \\ 0 & -x(j\omega)I \end{bmatrix}$$

where $x(j\omega) \geq 0$ for $\omega \in \mathbf{R}$. Basically, all properties of an operator, that can be expressed by IQC's, can be exploited in stability analysis. This is demonstrated by the following result, that reduces stability analysis of the feedback loop in Figure 1, to a search for a matrix function Π in $\widetilde{\Pi}_\Delta$, that satisfies a certain matrix inequality.

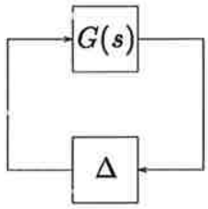

Figure 1. Feedback system with perturbation.

PROPOSITION 1—[12]
Let G be a linear causal operator with transfer function $G(s) \in \mathbf{RH}_\infty^{m\times m}$ and let Δ be a bounded causal operator on $\mathbf{L}_2^m[0,\infty)$. Assume that

(i) for any $\tau \in [0,1]$, the interconnection of G and $\tau\Delta$ is well-posed.

(ii) for any $\tau \in [0,1]$, the IQC defined by Π is satisfied by $\tau\Delta$.

(iii) there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon I \;\; \forall\, \omega \geq 0 \tag{1}$$

Then the feedback interconnection of G and Δ is stable. □

Robust stability and performance analysis based on IQC:s can be formulated as optimization problems on the form

$$\inf \gamma \quad \text{subject to} \tag{2}$$
$$P : \begin{cases} \exists \Pi \in \widetilde{\Pi}_\Delta(\gamma) \quad \text{such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \; \forall \omega \in [0,\infty] \end{cases}$$

where γ corresponds to the robustness criterion under consideration. The set $\widetilde{\Pi}_\Delta(\gamma)$ is generally infinite-dimiusional and solutions to (2) can be obtained by considering optimization over a finite-dimensional subset of $\widetilde{\Pi}_\Delta$. Conservativeness of this approach can be investigated by means of the dual optimization problem corresponding to (2). The dual was considered for general assumptions on the multipliers in $\widetilde{\Pi}_\Delta(\gamma)$ in [7] and [8]. In this paper we show that much stronger results can be obtained for the case of constant multipliers, frequency dependent multipliers defined by a frequency independent constraint and multipliers that are a combination of these two classes of multipliers.

2. Mathematical Preliminaries

This section presents the necessary mathematical preliminaries and notation needed in the paper. The following standard definitions and results from functional analysis are available in for example [10].

- Let X be a normed vector space. The dual of X is the normed space consisting of all bounded linear functionals on X and it is denoted by X^*. If $x \in X$ and $x^* \in X^*$, then $\langle x, x^* \rangle$ denotes the value of the linear functional x^* at x. The vector spaces considered in this paper are defined over the real scalar field and the linear functionals defined by functions from the dual space are real valued.

- The (Cartesian) product of two vector spaces X_1 and X_2, which are defined over the same field of scalars, is denoted $X_1 \times X_2$ and it consists of all ordered pairs $x = (x_1, x_2)$, with $x_1 \in X_1$ and $x_2 \in X_2$. x_1 and x_2 are said to be the coordinates of $X_1 \times X_2$. Addition and scalar multiplication is defined as $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$.
- The dual of $X_1 \times X_2$ is given as $X_1^* \times X_2^*$, where X_1^* and X_2^* are the duals of X_1 and X_2 respectively. Given $x = (x_1, x_2) \in X_1 \times X_2$ and $x^* = (x_1^*, x_2^*) \in X_1^* \times X_2^*$, we define $\langle x, x^* \rangle = \langle x_1, x_1^* \rangle + \langle x_2, x_2^* \rangle$.
- X^N denotes the Cartesian product of N copies of X.
- Let $H : X \mapsto Y$ be a bounded linear operator. Then the adjoint operator $H^\times : Y^* \mapsto X^*$ is defined by the equation

$$\langle Hx, y^* \rangle = \langle x, H^\times y^* \rangle$$

  for all $x \in X$ and $y^* \in Y^*$.

Next is a list of notation and function spaces used in this paper.

$\overline{M}$ Conjugation of a complex valued matrix.

M^* Hermitian conjugation of a matrix.

$\|\cdot\|_F$ The Frobenius norm of a real or complex matrix M is defined as $\|M\|_F = \sqrt{\mathrm{tr}(M^*M)}$.

$\mathbf{RL}_\infty^{n\times n}$ The space consisting of proper real rational matrix functions with no poles on the imaginary axis. $F \in \mathbf{RL}_\infty^{m\times m}$ satisfies $F(-j\omega) = \overline{F(j\omega)}$.

$\mathbf{RH}_\infty^{m\times m}$ The subspace of $\mathbf{RL}_\infty^{m\times m}$ consisting functions with no poles in the closed right half plane. Note that G^* generally means the Hilbert adjoint of $G(s) \in \mathbf{RH}_\infty^{m\times m}$, defined as $G^T(-s)$. The Hilbert adjoint reduces to the Hermitean conjugate of G when $s = i\omega$. We let $\|\cdot\|_\infty$ denote the usual norm on $\mathbf{RH}_\infty^{m\times m}$, defined as $\|G\|_\infty = \sup_\omega \overline{\sigma}(G)$.

$\mathcal{S}_{\mathrm{R}}^{m\times m}$ The subspace of $\mathbf{R}^{m\times m}$ consisting of symmetric matrices with the topology determined by the Frobenius norm. The dual space can be identified with $\mathcal{S}_{\mathrm{R}}^{m\times m}$ itself. The linear functionals are defined as $\langle X, Z \rangle_{\mathbf{R}} = \mathrm{tr}(XZ)$, where $X, Z \in \mathcal{S}_{\mathrm{R}}^{m\times m}$.

$\mathcal{S}_{\mathrm{C}}^{m\times m}$ The subspace of $\mathbf{C}^{m\times m}$ consisting of Hermitean matrices with the topology determined by the Frobenius norm. The dual space can be identified with $\mathcal{S}_{\mathrm{C}}^{m\times m}$ itself. The linear functionals are defined as $\langle X, Z \rangle_{\mathbf{C}} = \mathrm{tr}\{XZ\}$, where $X, Z \in \mathcal{S}_{\mathrm{C}}^{m\times m}$.

2.1 Some Results from Convex Analysis

We will next state some results and definitions from convex analysis. References for this material can be found in for example, [10], [20]

- A translated subspace is called an *affine set* (linear variety). The dimension of an affine set is defined as the dimension of this subspace.

- The *affine hull* of a nonempty set S, denoted aff S, is the unique smallest affine set containing S
- The *relative interior* of a nonempty set S, denoted ri S, is the collection of points in S, which are interior points of S relative to aff S. This means that for every x_0 in the relative interior of S, there exists $\varepsilon > 0$ such that all $x \in \text{aff } S$ satisfying $\|x - x_0\|$ are also members of S. Hence, the relative interior of S is an open subset of aff S.
- The dimension of a convex set C is defined as the dimension of the affine hull of S.
- A *convex cone* C is a convex subset of a vector space with the property that if $x \in C$, then $\alpha x \in C$ for all $\alpha \geq 0$.

The following separating hyperplane theorem will be a main tool in this paper

THEOREM 1—SEPARATING HYPERPLANE THEOREM
Let C_1 and C_2 be disjoint convex sets in a vector space X. Assume further that C_1 is open, then there exists $x^* \in X^*$ such that $\langle x_1, x^* \rangle < \langle x_2, x^* \rangle$ for all $x_1 \in C_1$ and $x_2 \in C_2$.

Proof: This is a minor reformulation of Theorem 3 on page 133 in [10]. □

THEOREM 2—HELLY
Let $\{C_i | i \in I\}$ be a collection of closed, bounded convex sets in $\mathbf{R}^n$. I is a set of indices with arbitrary cardinality. If $\cap_{i \in I} C_i = \emptyset$, then there exists a subcollection consisting of $n+1$ or fewer sets $\{C_{\alpha_i} | \alpha_i \in I,\ i = 1, \ldots, n+1\}$ such that $\cap_{i=1}^{n+1} C_{\alpha_i} = \emptyset$. If the index set I is finite then the result also holds when the C_i are not necessarily closed or bounded.

Proof: Follows from [20]. □

3. Frequency Dependent Multipliers

We will in this section study the case when the multipliers are defined by a frequency independent constraint. More precisely we consider the convex cone

$$\tilde{\Pi}_\Delta(\gamma) = \{\Pi \in \mathbf{RL}_\infty^{2m \times 2m} : \Pi(j\omega) \in \Pi_\Delta(\gamma),\ \forall \omega \in [0, \infty]\}$$

where $\Pi_\Delta(\gamma) \subset \mathcal{S}_{\mathbf{C}}^{2m \times 2m}$ is a closed convex cone for all $\gamma \in \mathbf{R}$, satisfying the following assumption

Assumption on $\Pi_\Delta(\gamma)$: If $\gamma_2 \geq \gamma_1$, then $\forall \Pi_1 \in \Pi_\Delta(\gamma_1)$, there exists $\Pi_2 \in \Pi_\Delta(\gamma_2)$ such that $\Pi_1 \geq \Pi_2$.

This means that we consider frequency dependent multipliers, where the values between different frequencies are independent except for the requirement that Π should be a rational function.

THEOREM 3

$$\inf_P \gamma = \sup_D \gamma$$

where primal and dual constraints are defined as

$$P: \begin{cases} \exists \Pi \in \mathbf{RL}_\infty^{2m\times 2m}, \text{ such that} \\ \Pi(j\omega) \in \Pi_\Delta(\gamma), \ \forall \omega \in [0,\infty] \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0,\infty] \end{cases}$$

$$D: \begin{cases} \exists \omega_0 \in [0,\infty], \ Z = Z^* \geq 0, \quad Z \neq 0, \text{ s.t.} \\ \begin{bmatrix} G(j\omega_0) \\ I \end{bmatrix} Z \begin{bmatrix} G(j\omega_0) \\ I \end{bmatrix}^* \in \Pi_\Delta(\gamma)^\oplus \end{cases}$$

and where

$$\Pi_\Delta(\gamma)^\oplus = \{Z \in \mathcal{S}_{\mathbf{C}}^{2m\times 2m} | \langle \Pi, Z\rangle_{\mathbf{C}} \geq 0, \quad \forall \Pi \in \Pi_\Delta(\gamma)\}$$

Proof: Introduce $\gamma^* = \inf_P \gamma$. If $\gamma < \gamma^*$ then there exists $\omega_0 \in [0,\infty]$ such that the convex sets

$$\mathcal{P} = \left\{ \begin{bmatrix} G(j\omega_0) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G(j\omega_0) \\ I \end{bmatrix} : \Pi \in \Pi_\Delta(\gamma) \right\}$$

$$\mathcal{Q} = \{X \in \mathcal{S}_{\mathbf{C}}^{m\times m} : X < 0\}$$

are disjoint. It follows from the assumption on $\Pi_\Delta(\gamma)$ and from Lemma 1 in the Appendix that this also a necessary condition for $\gamma < \gamma^*$. By the separating hyperplane theorem there exists a nonzero $Z \in \mathcal{S}_{\mathbf{C}}^{m\times m}$ such that

$$\langle X, Z\rangle_{\mathbf{C}} \geq 0, \quad \forall X \in \mathcal{P} \tag{3}$$
$$\langle X, Z\rangle_{\mathbf{C}} < 0, \quad \forall X \in \mathcal{Q} \tag{4}$$

For (4) to hold we need to have $Z \neq 0$ such that $Z \geq 0$. The condition in (3) can be reformulated in the following way

$$\begin{aligned} & \left\langle \begin{bmatrix} G(j\omega_0) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G(j\omega_0) \\ I \end{bmatrix}, Z \right\rangle_{\mathbf{C}} \geq 0, && \forall \Pi \in \Pi_\Delta(\gamma) \\ \Longleftrightarrow & \left\langle \Pi, \begin{bmatrix} G(j\omega_0) \\ I \end{bmatrix} Z \begin{bmatrix} G(j\omega_0) \\ I \end{bmatrix}^* \right\rangle_{\mathbf{C}} \geq 0, && \forall \Pi \in \Pi_\Delta(\gamma) \\ \Longleftrightarrow & \begin{bmatrix} G(j\omega_0) \\ I \end{bmatrix} Z \begin{bmatrix} G(j\omega_0) \\ I \end{bmatrix}^* \in \Pi_\Delta(\gamma)^\oplus \end{aligned}$$

This proves that $\inf_P \gamma \leq \sup_D \gamma$. The opposite inequality is obvious. □

We will next give a simple example, which illustrates the theorem. In particular, the example shows that the frequency $\omega = \infty$ needs to be included in the dual.

Example 1
Consider the system in Figure 1 with

$$G(s) = \frac{2s+1}{s^2+2s+1} - 2$$

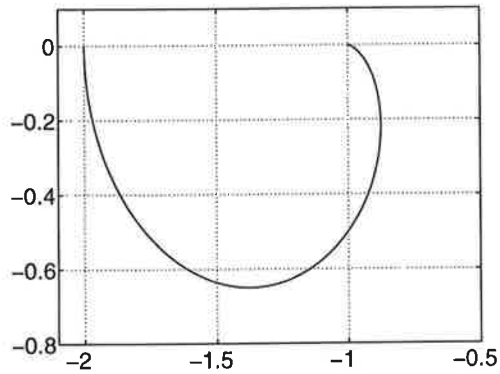

Figure 2. Nyquist curve for the system in Example 1.

and $\Delta = \delta$, where δ is an uncertain real valued parameter, which takes values in $[-\alpha, \alpha]$. We want to find a bound α^* such that the system is stable when $\alpha < \alpha^*$. We can obtain one such bound by considering the primal in Theorem 3 with

$$\Pi_\Delta(\gamma) = \left\{ \begin{bmatrix} x & jy \\ -jy & -\gamma x \end{bmatrix} : x \geq 0, y \in \mathbf{R} \right\}$$

and then use $\alpha^* = 1/\sqrt{\gamma^*}$, where γ^* is the primal objective. The primal optimization problem can be formulated as

$$\inf \gamma \quad \text{subject to}$$

$$P : \begin{cases} \exists x, y \in \mathbf{RL}_\infty^{1\times 1}, \text{ such that} \\ \sup\limits_{\omega \in [0,\infty]} x(j\omega)[|G(j\omega)|^2 - \gamma] + 2y(j\omega)\text{Im } G(j\omega) < 0 \\ x(j\omega) \geq 0, \quad \forall \omega \in [0,\infty] \\ y(j\omega) \in \mathbf{R}, \quad \forall \omega \in [0,\infty] \end{cases}$$

It is easy to see that the optimal solution is the maximal value of $|G(j\omega)|$, subject to the constraint that Im $G(j\omega) = 0$. From Figure 2 we see that $\gamma^* = \max(|G(0)|^2, |G(j\infty)|^2) = |G(j\infty)| = 2$. We will next see that the dual of Theorem 3 gives exactly this solution. It is easy to verify that

$$\Pi_\Delta(\gamma)^\oplus = \{Z \in S_{\mathbf{C}}^{2\times 2} : Z_{11} - \gamma Z_{22} \geq 0, \quad \text{Im } Z_{12} = 0\}$$

Hence, the dual can be simplified to

$$\sup \gamma \quad \text{subject to}$$

$$D : \begin{cases} \exists \omega_0 \in [0,\infty], \text{ such that} \\ |G(j\omega_0)|^2 - \gamma \geq 0 \\ \text{Im } G(j\omega_0) = 0 \end{cases}$$

which of course has the solution $\gamma^* = |G(i\infty)|^2 = 4$.

The next example is also simple but illustrative. It will will be continued in the next section

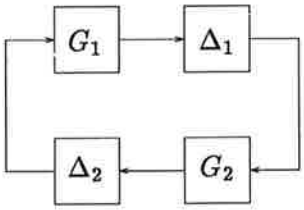

Figure 3. System in Example 1 and Example 2.

EXAMPLE 2
Consider the system in Figure 3, where $G_1, G_2 \in \mathbf{RH}_\infty$ and where Δ_1 and Δ_2 are linear time-invariant uncertainties satisfying $||\Delta_1||_\infty \leq 1$ and $||\Delta_2||_\infty \leq \alpha$. The system can equivalently be described as in Figure 1 with $\Delta = \text{diag}(\Delta_1, \Delta_2)$ and

$$G = \begin{bmatrix} 0 & G_1 \\ G_2 & 0 \end{bmatrix} \in \mathbf{RH}_\infty^{2\times 2}$$

We want to find a bound α^* such that the system is stable if $\alpha < \alpha^*$. Such a bound can be obtained by solving the primal in Theorem 3 with the following choice of $\Pi_\Delta(\gamma)$

$$\Pi_\Delta(\gamma) = \{\text{diag}(x_1, x_2, -x_1, -\gamma x_2)\,|x_k \geq 0, k = 1, 2\}$$

and then use $\alpha^* = 1/\sqrt{\gamma^*}$. It is easy to verify that

$$\Pi_\Delta(\gamma)^\oplus = \left\{Z \in \mathcal{S}_{\mathbf{C}}^{4\times 4}\,|z_{11} - z_{33} \geq 0, z_{22} - \gamma z_{44} \geq 0\right\}$$

The dual in Theorem 3 can be written

$$\sup \gamma \quad \text{subject to}$$

$$D : \begin{cases} \exists \omega_0 \in [0, \infty], \; z_{11}, z_{22} \geq 0, \quad z_{11} \neq 0 \text{ or } z_{22} \neq 0, \text{ s.t.} \\ z_{22}|G_1(j\omega_0)|^2 - z_{11} \geq 0 \\ z_{11}|G_2(j\omega_0)|^2 - \gamma z_{22} \geq 0 \end{cases}$$

which can be further simplified into

$$\sup_{\omega_0 \in [0,\infty]} |G_1(j\omega_0)G_2(j\omega_0)|^2$$

Hence the dual objective is equivalent to $||G_1G_2||_\infty^2$ This result is of course expected. We will see in the next section that when we consider the same problem with time-varying parameters δ_1 and δ_2 then two frequencies will be involved in the dual.

Remark If we allow Δ_1, Δ_2 to be time varying with arbitrary slow rate of of variation then the primal in Example 2 is also a necessary condition for stability, see [17].

4. Constant Multipliers

We will in this section give a similar result as Theorem 3, with the multiplier assumed to be a constant matrix from the closed convex cone

$$\widetilde{\Pi}_\Delta(\gamma) = \{\Psi : \Psi \in \Psi_\Delta(\gamma)\}$$

It is assumed that $\Psi_\Delta(\gamma)$ is a closed convex cone for all $\gamma \in \mathbf{R}$, which satisfies the following assumption.

Assumption on $\Psi_\Delta(\gamma)$: If $\gamma_2 \geq \gamma_1$, then $\forall \Psi_1 \in \Psi_\Delta(\gamma_1)$, there exists $\Psi_2 \in \Psi_\Delta(\gamma_2)$ such that $\Psi_1 \geq \Psi_2$.

THEOREM 4

$$\inf_P \gamma = \sup_D \gamma$$

where the primal and dual constraints are defined as

$$P : \begin{cases} \exists \Psi \in \Psi_\Delta(\gamma), \text{ such that} \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Psi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty] \end{cases}$$

$$D : \begin{cases} \exists \omega_1, \ldots, \omega_N \in [0, \infty], \; Z_k = Z_k^* \geq 0, \quad Z_k \neq 0, \text{ s.t.} \\ \sum_{k=1}^{N} \operatorname{Re} \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^* \in \Psi_\Delta(\gamma)^\oplus \end{cases}$$

where $N \leq \dim(\Psi_\Delta(\gamma)) + 1$, and

$$\Psi_\Delta(\gamma)^\oplus = \left\{ Z \in \mathcal{S}_{\mathbf{R}}^{2m \times 2m} \,|\, \langle \Psi, Z \rangle_{\mathbf{R}} \geq 0, \quad \forall \Psi \in \Psi_\Delta(\gamma) \right\}$$

Furthermore, if $0 \notin \operatorname{ri} \Psi_\Delta(\gamma)$, then $N \leq \dim(\Psi_\Delta(\gamma))$.

Proof This theorem is a special case of Theorem 5 in the next section. □

EXAMPLE 3
Consider again the system in Figure 3 but now when δ_1 and δ_2 are time-varying parameters satisfying $|\delta_1(t)| \leq 1$, $\forall t \geq 0$ and $|\delta_2(t)| \leq \alpha$, $\forall t \geq 0$ respectively. Again, we search for the smallest α^* such that the system is stable if $\alpha < \alpha^*$. By solving the primal in Theorem 4 with

$$\Psi_\Delta(\gamma) = \{\operatorname{diag}(x_1, x_2, -x_1, -\gamma x_2) \,|\, x_k \geq 0, k = 1, 2\}$$

we get the bound $\alpha^* = 1/\sqrt{\gamma^*}$. It is easy to verify that

$$\Psi_\Delta(\gamma)^\oplus = \left\{ Z \in \mathcal{S}_{\mathbf{R}}^{4 \times 4} \,|\, z_{11} - z_{33} \geq 0, z_{22} - \gamma z_{44} \geq 0 \right\}$$

Since $N = \dim(\Psi_\Delta(\gamma)) = 2$ and $0 \notin \operatorname{ri} \Psi_\Delta(\gamma)$, we get

$$\sup \gamma \quad \text{subject to}$$

$$D : \begin{cases} \exists \omega_1, \omega_2 \in [0, \infty], \; z_{11}^1, z_{22}^1, z_{11}^2, z_{22}^2 \geq 0, \text{ not all zero, s.t.,} \\ z_{22}^1 |G_1(j\omega_1)|^2 - z_{11}^1 + z_{22}^2 |G_1(j\omega_2)|^2 - z_{11}^2 \geq 0 \\ z_{11}^1 |G_2(j\omega_1)|^2 - \gamma z_{22}^1 + z_{11}^2 |G_2(j\omega_2)|^2 - \gamma z_{22}^2 \geq 0 \end{cases} \tag{5}$$

Now choose ω_1 and ω_2 such that $|G_1(j\omega_1)| = ||G_1||_\infty$ and $|G_2(j\omega_2)| = ||G_2||_\infty$, respectively. Then the constraint in (5) can formulated as

$$\gamma \leq \frac{||G_2||_\infty^2}{z_{22}^1 + z_{22}^2}\left[z_{22}^1||G_1||_\infty^2 + z_{22}^2|G_1(j\omega_2)|^2 - z_{11}^1[1 - \frac{|G_2(j\omega_1)|^2}{||G_2||_\infty^2}]\right]$$

This implies that $\gamma \leq ||G_1||_\infty^2||G_2||_\infty^2$. Furthermore, we obtain equality by choosing $z_{11}^1 = z_{22}^2 = 0$. Hence we have shown that the dual objective is $\gamma^* = ||G_1||_\infty^2||G_2||_\infty^2$, and furthermore that the dual optimization problem involves two frequencies unless the norms $||G_1||_\infty$ and $||G_2||_\infty$ are obtained at the same frequency.

Remark If we instead consider two time-varying operators Δ_1 and Δ_2 with arbitrary time-variation, which satisfies $|\Delta_1|| \leq 1$ and $||\Delta_2|| \leq \alpha$, then the primal in Example 3 is also a necessary condition for stability, see [13], [17].

5. Mixed Multipliers

We will in this section derive the dual of robustness problems involving both frequency dependent multipliers and constant multipliers. More precisely, the multipliers involved are from the closed convex cone

$$\tilde{\Pi}_\Delta(\gamma) = \left\{\Pi + \Psi \in \mathbf{RL}_\infty^{2m\times 2m} : \Pi(j\omega) \in \Pi_\Delta(\gamma), \forall\omega \in [0,\infty],\ \Psi \in \Psi_\Delta(\gamma)\right\}$$

where $\Pi_\Delta(\gamma) \subset \mathcal{S}_{\mathrm{C}}^{2m\times 2m}$ and $\Psi_\Delta(\gamma) \in \mathcal{S}_{\mathrm{R}}^{2m\times 2m}$ are closed convex cones for all $\gamma \in \mathbf{R}$, satisfying the following assumptions.

Assumptions on $\Pi_\Delta(\gamma)$ and $\Psi_\Delta(\gamma)$:

1. If $\gamma_2 \geq \gamma_1$, then $\forall\Pi_1 \in \Pi_\Delta(\gamma_1)$, there exists $\Pi_2 \in \Pi_\Delta(\gamma_2)$ such that $\Pi_1 \geq \Pi_2$.
2. If $\gamma_2 \geq \gamma_1$, then $\forall\Psi_1 \in \Psi_\Delta(\gamma_1)$, there exists $\Psi_2 \in \Psi_\Delta(\gamma_2)$ such that $\Psi_1 \geq \Psi_2$.

Next follows the main result in this paper.

THEOREM 5

$$\inf_P \gamma = \sup_D \gamma$$

where the primal and dual constraints are defined as

$$P: \begin{cases} \exists\Psi \in \Psi_\Delta(\gamma),\ \Pi \in \mathbf{RL}_\infty^{2m\times 2m}, \text{ such that} \\ \Pi(j\omega) \in \Pi_\Delta(\gamma),\ \forall\omega \in [0,\infty] \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Pi(j\omega) + \Psi) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\omega \in [0,\infty] \end{cases}$$

$$D : \begin{cases} \exists \omega_1, \ldots, \omega_N \in [0, \infty], \; Z_k = Z_k^* \geq 0, \; Z_k \neq 0, \text{ s.t.} \\ \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^* \in \Pi_\Delta(\gamma)^\oplus \\ \sum_{k=1}^{N} \mathrm{Re} \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^* \in \Psi_\Delta(\gamma)^\oplus \end{cases}$$

where $N \leq \dim(\Psi_\Delta(\gamma)) + 1$ and

$$\Pi_\Delta(\gamma)^\oplus = \left\{ Z \in \mathcal{S}_{\mathbf{C}}^{2m \times 2m} | \langle \Pi, Z \rangle_{\mathbf{C}} \geq 0, \quad \forall \Pi \in \Pi_\Delta(\gamma) \right\}$$
$$\Psi_\Delta(\gamma)^\oplus = \left\{ Z \in \mathcal{S}_{\mathbf{R}}^{2m \times 2m} | \langle \Psi, Z \rangle_{\mathbf{R}} \geq 0, \quad \forall \Psi \in \Psi_\Delta(\gamma) \right\}$$

Furthermore, if $0 \notin \mathrm{ri}\ \Psi_\Delta(\gamma)$, then $N \leq \dim(\Psi_\Delta(\gamma))$.

Proof The case when $\dim(\Psi_\Delta(\gamma)) = 0$ is treated in Theorem 3 so we may assume that $\dim(\Psi_\Delta(\gamma)) > 0$. The proof is based on an idea in [17] and [16]. For given $\omega \in [0, \infty]$ we define

$$C_\omega(\gamma) = \{ \Psi \in \Psi_\Delta(\gamma) : \exists \Pi \in \Pi_\Delta(\gamma), \text{ s.t. } M_G(\omega)(\Pi + \Psi) < 0 \}$$

where $M_G(\omega) : \mathcal{S}_{\mathbf{C}}^{2m \times 2m} \to \mathcal{S}_{\mathbf{C}}^{m \times m}$ denotes the linear operator defined by

$$M_G(\omega)\Pi = \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}$$

for any $\Pi \in \mathcal{S}_{\mathbf{C}}^{2m \times 2m}$ and $\omega \in [0, \infty]$. We note that $C_\omega(\gamma)$ is a convex set. The proof is based on the following two statements, which are proved in the appendix.

(i) We have

$$\gamma^* = \inf_{P} \gamma = \sup_{\cap_{\omega \in [0,\infty]} C_\omega(\gamma) = \emptyset} \gamma$$

(ii) $\cap_{\omega \in [0,\infty]} C_\omega(\gamma) = \emptyset$ if and only if there exists at most $N = \dim(\Psi_\Delta(\gamma)) + 1$ frequencies $\omega_1, \ldots, \omega_N \in [0, \infty]$ such that $\cap_{k=1}^{N} C_{\omega_k}(\gamma) = \emptyset$. Furthermore, if $0 \notin \mathrm{ri}\ \Psi_\Delta(\gamma)$, then $N \leq \dim(\Psi_\Delta(\gamma))$.

From statement (i) and statement (ii) above, it follows that $\gamma < \gamma^*$ iff there exists at most N frequencies $\omega_1, \ldots, \omega_N$ such that the convex sets

$$\mathcal{P} = \{ (M_G(\omega_1)(\Pi_1 + \Psi), \ldots, M_G(\omega_N)(\Pi_N + \Psi)) : \Pi_k \in \Pi_\Delta(\gamma), \Psi \in \Psi_\Delta(\gamma) \}$$
$$\mathcal{Q} = \left\{ X \in \mathcal{S}_{\mathbf{C}}^{m \times m} : X < 0 \right\}^N$$

are disjoint. By the separating hyperplane theorem this is equivalent to the existence of a nonzero N-tuple $(Z_1, \ldots, Z_N) \in (\mathcal{S}_{\mathbf{C}}^{m \times m})^N$, such that

$$\sum_{k=1}^{N} \langle X_k, Z_k \rangle_{\mathbf{C}} \geq 0, \quad \forall (X_1, \ldots, X_N) \in \mathcal{P} \tag{6}$$

$$\sum_{k=1}^{N} \langle X_k, Z_k \rangle_{\mathbf{C}} < 0, \quad \forall (X_1, \ldots, X_N) \in \mathcal{Q} \tag{7}$$

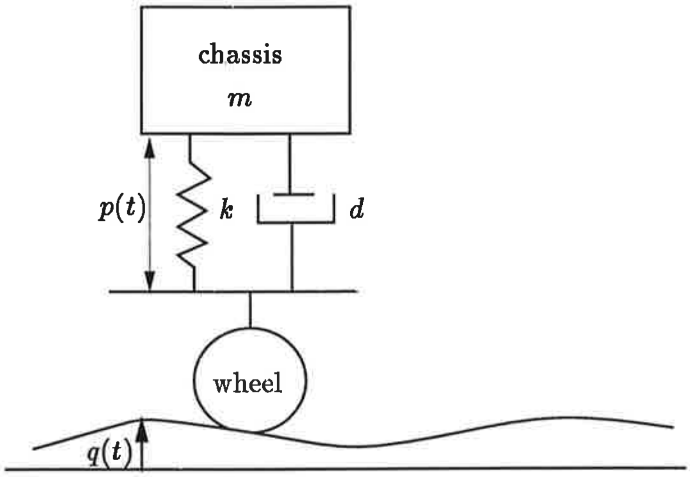

Figure 4. Process model of car suspension.

It is clear that for (7) to hold we need $Z_k \geq 0$, for $k = 1, \ldots, N$.

The condition in (6) gives

$$\sum_{k=1}^{N} \langle M_G(\omega_k)\Pi_k, Z_k \rangle_{\mathbf{C}} + \sum_{k=1}^{N} \langle M_G(\omega_k)\Psi, Z_k \rangle_{\mathbf{C}} =$$
$$\sum_{k=1}^{N} \langle \Pi_k, M_G(\omega_k)^{\times} Z_k \rangle_{\mathbf{C}} + \left\langle \Psi, \sum_{k=1}^{N} \mathrm{Re} M_G(\omega_k)^{\times} Z_k \right\rangle_{\mathbf{R}} \geq 0$$

for all $\Pi_k \in \Pi_\Delta(\gamma)$ and for all $\Psi \in \Psi_\Delta(\gamma)$. Hence,

$$\begin{cases} \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^* \in \Pi_\Delta(\gamma)^{\oplus}, & k = 1, \ldots, N \\ \sum_{k=1}^{N} \mathrm{Re} \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix} Z_k \begin{bmatrix} G(j\omega_k) \\ I \end{bmatrix}^* \in \Psi_\Delta(\gamma)^{\oplus} \end{cases}$$

and the theorem follows. □

6. Example

In this example we investigate the performance of the suspension of a simple car model. We will follow the approach in [6] for obtaining a simple model of the system. Figure 4 shows one fourth of a car with one wheel and the car suspension equipment consisting of a nonlinear spring with nonlinear spring constant $k(\cdot)$ and damping ratio d. Below follows a list of notation for this example

Notation	Explanation
$p(t)$	spring length
$q(t)$	road profile
p_0	unsprung length of spring
m	mass of car body
g	9.81m/s^2
k	nonlinear spring constant
d	damping ratio of the spring

The differential equation describing the length of the spring $p(t)$ due to the road profile $q(t)$ is

$$m\left[\ddot{p}(t)+\ddot{q}(t)\right]=-k(p(t)-p_0)-d\dot{p}(t)-mg$$

and is valid as long as the car has contact with the road. In order to obtain a state space equation we use the states $x_1(t)=p(t)-p^0$, and $x_2(t)=\frac{dx_1}{dt}$, where p^0 is the stationary value of $p(t)$ when $q(t)\equiv 0$. We assume that $k(p(t)-p_0)+mg=k(x_1(t)+p^0-p_0)+mg=k_l[x_1+\phi(x_1)]$, where ϕ is a nonlinear function satisfying $k_{\min}x^2\leq\phi(x)x\leq k_{\max}x^2$, $\forall x\in\mathbf{R}$.

The mass of the car is uncertain due to varying load. It is assumed that $m\in[\underline{m},\overline{m}]$. We let the nominal mass be m_0 and we define m by the relation

$$\frac{1}{m}=\frac{1}{m_0}+a\delta$$

where $m_0=2\underline{m}\overline{m}/(\underline{m}+\overline{m})$, $\delta\in[-1,1]$, and $a=(\overline{m}-\underline{m})/(2\underline{m}\overline{m})$.

With the state $x=\begin{bmatrix}x_1 & x_2\end{bmatrix}^T$ and output z as the normal force acting on the compartment, we get the following model for the system

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}&=\begin{bmatrix}x_2\\ \frac{1}{m}[-k_l[x_1+\phi(x_1)]-dx_2]\end{bmatrix}+\begin{bmatrix}0\\ -1\end{bmatrix}e(t)\\ z(t)&=k_l(x_1(t)+\phi(x_1(t)))+dx_2(t)\end{aligned}\tag{8}$$

where $e(t)=\frac{d^2q}{dt^2}$ is regarded as a disturbance to the system.

The system in (8) is equivalent to the system in Figure 5 where the nonlinearity ϕ and the uncertainty δ are collected in a perturbation matrix. This system is equivalent to the system in Figure 1 with

$$\begin{aligned}G&=\text{starp}(Q,H)\\ \Delta&=\begin{bmatrix}\phi & 0\\ 0 & \delta\end{bmatrix}\end{aligned}$$

where starp$(\cdot,\cdot)$, denotes the Redheffer star product defined as, [19] and [15]

$$\text{starp}(Q,H)=\begin{bmatrix}F_l(Q,H_{11}) & Q_{12}(I-H_{11}Q_{22})^{-1}H_{12}\\ H_{21}(I-Q_{22}H_{11})^{-1}Q_{21} & F_u(H,Q_{22})\end{bmatrix}$$

where

$$F_l(Q,H_{11})=Q_{11}+Q_{12}H_{11}(I-Q_{22}H_{11})^{-1}Q_{21}$$

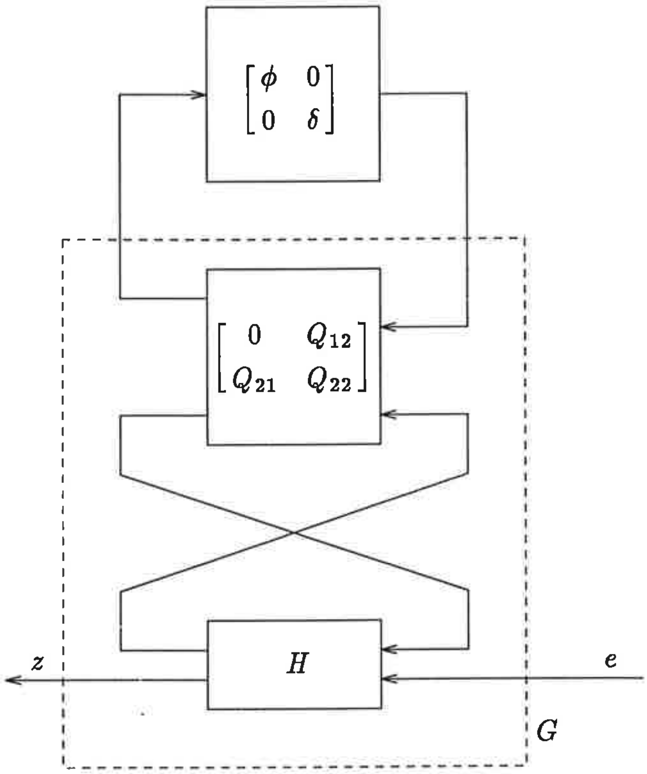

Figure 5. Transformed system.

and

$$F_u(H, Q_{22}) = H_{22} + H_{21}Q_{22}(I - H_{11}Q_{22})^{-1}H_{12}$$

H and Q are defined as

$$H = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -k_l & 0 & -k_l & -d & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ k_l & 0 & k_l & d & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 0_2 & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

$$Q_{12} = [\,I_2 \quad 0_2\,], \quad Q_{21} = \begin{bmatrix} 1 & 0 \\ 0 & a \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_{22} = \mathrm{diag}(0, 1/m_0, s^{-1}I_2)$$

We are interested in study worst case $\mathbf{L}_2$ performance of the system above subject to the nonlinearity ϕ and the mass uncertainty δ. We assume the following normalized parameter values, $k_l = 1$, $d = 0.2$, $k_{\min} = -0.1$, $k_{\max} = 0.1$ and $m \in [0.8, 1.2]$. In this case $G \in \mathbf{RH}_\infty^{3\times 3}$. A bound for the induced $\mathbf{L}_2$-norm of the system is given as $\sqrt{\gamma^*}$, where γ^* is the solution to the following

$R_0(s)$	$S_0(s)$	γ^*
1	1	UNFEASIBLE
1	Ritz(0.2, 1)	14.64
1	Ritz(5, 1)	14.64

Table 1. Numerical results for the primal optimization problem in the car suspension example.

convex optimization problem

$$\inf \gamma \quad \text{subject to}$$
$$P: \begin{cases} \exists \Psi \in \Psi_\Delta(\gamma), \Pi \in \mathbf{RL}_\infty^{2m\times 2m}, \text{ such that} \\ \Pi(j\omega) \in \Pi_\Delta(\gamma); \quad \forall \omega \in [0,\infty] \\ \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Pi(j\omega) + \Psi) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0,\infty] \end{cases}$$

where

$$\Pi_\Delta(\gamma) = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & jy & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -jy & 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} : \quad x, y \in \mathbf{R}, x \geq 0 \right\}$$

and

$$\Psi_\Delta(\gamma) = \left\{ \begin{bmatrix} 0.1^2 x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma x_2 \end{bmatrix} : \quad x_1, x_2 \geq 0 \right\}$$

A solution to the primal can be obtained as suggested in [9]. The idea is to restrict the search of the frequency dependent multiplier to a finite dimensional subspace. Let $x(j\omega) = R^*(j\omega) U R(j\omega)$ and $y(j\omega) = V S(j\omega) - S^*(j\omega) V^T$, where $R \in \mathbf{RH}_\infty^{N\times 1}$, $S \in \mathbf{H}_\infty^{M\times 1}$ are *basis multipliers* and where $U \in \mathbf{R}^{N\times N}$, satisfying $U = U^T \geq 0$ and $V \in \mathbf{R}^{1\times M}$ are the corresponding *coordinates*. The resulting optimization problem can be transformed into an LMI optimization problem. We obtained the solution in Table 1 using LMI-lab, [5], where

$$\text{Ritz}(p,n) = \begin{bmatrix} 1 & \frac{s-p}{s+p} & \cdots & \frac{(s-p)^n}{(s+p)^n} \end{bmatrix}^T$$

No higher order basis functions gave smaller primal objective value. Does this mean that the last two basis function in Table 1 are close to optimal? We will use the dual in Theorem 5 to investigate this question.

In order to solve the dual for the car suspension example we need to determine the cones $\Pi_\Delta(\gamma)^\oplus$ and $\Psi_\Delta(\gamma)^\oplus$. It is simple to verify that

$$\begin{aligned}
\Pi_\Delta(\gamma)^\oplus &= \{Z \in \mathcal{S}_{\mathbb{C}}^{6\times 6}: \quad Z_{22} - Z_{55} \geq 0, \quad \mathrm{Im}\, Z_{25} = 0\} \\
\Psi_\Delta(\gamma)^\oplus &= \{Z \in \mathcal{S}_{\mathbb{R}}^{6\times 6}: \quad 0.1^2 Z_{11} - Z_{44} \geq 0, \quad Z_{33} - \gamma Z_{66} \geq 0\}
\end{aligned}$$

We have $\dim(\Psi_\Delta(\gamma)) = 2$, which means that we can formulate the dual for the car suspension example as

$$\sup \gamma \quad \text{subject to}$$

$$D: \begin{cases}
\exists \omega_1, \omega_2 \in [0, \infty], Z_1, Z_2 \in \mathcal{S}_{\mathbb{C}}^{3\times 3}, \text{ such that} \\
Z_1, Z_2 \geq 0, Z_1 \neq 0 \\
H_2(j\omega_k) Z_k H_2^*(j\omega_k) - H_5 Z_k H_5^T \geq 0, \quad k = 1,2 \\
\mathrm{Im}\, \{H_2(j\omega_k) Z_k H_5^T\} = 0, \quad k = 1,2 \\
\sum_{k=1}^{2} \mathrm{Re}\{0.1^2 H_1(j\omega_k) Z_k H_1^*(j\omega_k) - H_4 Z_k H_4^T\} \geq 0 \\
\sum_{k=1}^{2} \mathrm{Re}\{H_3(j\omega_k) Z_k H_3^*(j\omega_k) - \gamma H_6 Z_k H_6^T\} \geq 0
\end{cases}$$

where

$$\begin{aligned}
H_1 &= [G_{11} \quad G_{12} \quad G_{13}] \\
H_2 &= [G_{21} \quad G_{22} \quad G_{23}] \\
H_3 &= [G_{31} \quad G_{32} \quad G_{33}] \\
H_4 &= [1 \quad 0 \quad 0] \\
H_5 &= [0 \quad 1 \quad 0] \\
H_6 &= [0 \quad 0 \quad 1]
\end{aligned}$$

This optimization problem can be transformed into an LMI optimization problem involving real matrices and N algebraic conditions, which corresponds to the constraints $\mathrm{Im}\, \{H_2(j\omega_k) Z_k H_5^T\} = 0$, $k = 1,2$.

With only the frequency $\omega_1 = 0.91$ in the dual optimization problem, we get $\gamma^* = 14.60$ and

$$Z_1 \approx \begin{bmatrix} 0.1491 & -0.1230 & 0.1020 \\ -0.1230 & 14.9706 & -0.1650 \\ 0.1020 & -0.1650 & 0.0704 \end{bmatrix} + i \begin{bmatrix} 0 & -1.4879 & 0.0081 \\ 1.4879 & 0 & 1.0122 \\ -0.0081 & -1.0122 & 0 \end{bmatrix}$$

It is easy to verify that Z_1 satisfies the conditions for the dual and we see that the dual objective $\gamma^* = 14.60$ is very close to the value of the primal objective $\gamma^* = 14.64$. This shows that the chosen basis are indeed close to optimal.

7. Conclusions

We have derived duality results for obtaning bounds for robustness analysis. Several examples have shown its applicability for simple cases.

8. Appendix

This section contains the proofs of statement (i) and (ii) in the proof for Theorem 5.

Proof of statement (i) The next lemma is the main tool for the proof of statement (i). We will use functions from the the Banach space $\mathbf{B}^{m\times m}$ defined as

DEFINITION 1
Let $\mathbf{B}^{m\times m}$ be the Banach space of $n \times n$ complex valued functions that are bounded on the extended imaginary axis and satisfy $X(-i\omega) = \overline{X(j\omega)}$ for all $\omega \in [0, \infty]$. We define the norm on $\mathbf{B}^{m\times m}$ as $\|X\|_{\mathrm{B}} = \sup_{\omega\in[0,\infty]} \sigma_{max}(X(j\omega))$. $\square$

Note that $\mathbf{RL}_\infty \subset \mathbf{B}$.

LEMMA 1
Given $G \in \mathbf{RH}_\infty^{m\times m}$, $\Psi \in \mathcal{S}_{\mathrm{R}}^{2m\times 2m}$, and $\gamma \in \mathbf{R}$, then the conditions

$$\begin{cases} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Pi(j\omega) + \Psi) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, & \forall\omega \in [0, \infty] \\ \Pi(j\omega) = \Pi(j\omega)^* \in \Pi_\Delta(\gamma), & \forall\omega \in [0, \infty] \end{cases} \tag{9}$$

are satisfied by some $\Pi \in \mathbf{B}^{2m\times 2m}$ if and only if they are satisfied by some $\Pi \in \mathbf{RL}_\infty^{2m\times 2m}$.

Proof: The sufficiency is trivial. For the necessity we transform the condition in (9) to the unit circle. For this we use the Möbius transform $\psi(z) = (z-1)/(z+1)$. Let $\widehat{G}(z) = G(\psi(z))$, $\widehat{\Pi}(z) = \Pi(\psi(z))$, and let the affine map H be defined as

$$H(\Pi)(e^{j\omega}) = \begin{bmatrix} \widehat{G}(e^{j\omega}) \\ I \end{bmatrix}^* (\widehat{\Pi}(e^{j\omega}) + \Psi) \begin{bmatrix} \widehat{G}(e^{j\omega}) \\ I \end{bmatrix}$$

for any $\Pi \in \mathbf{B}^{2m\times 2m}$. Consider the convex set

$$\mathcal{C} = \left\{ \Pi \in \mathbf{B}^{2m\times 2m} : H(\Pi)(e^{j\omega}) < 0,\ \widehat{\Pi}(e^{j\omega}) = \widehat{\Pi}(e^{j\omega})^* \in \Pi_\Delta(\gamma);\ \forall\omega \in [0, \pi] \right\}$$

By assumption there exists a function $\Pi \in \mathcal{C}$. We will show that it is possible to construct from Π a function $\widetilde{\Pi} \in \mathbf{RL}_\infty^{2m\times 2m}$, which is in $\mathcal{C}$. Then the lemma follows.

Since $\widehat{G}$ is continuous on the unit circle it follows that every $\widehat{\omega} \in [0, \pi]$ is contained in an open (as a subset of $[0, \pi]$) interval $I_{\widehat{\omega}}$ such that

$$\begin{bmatrix} \widehat{G}(e^{j\omega}) \\ I \end{bmatrix}^* (\widehat{\Pi}(e^{j\widehat{\omega}}) + \Psi) \begin{bmatrix} \widehat{G}(e^{j\omega}) \\ I \end{bmatrix} < 0, \quad \forall\omega \in I_{\widehat{\omega}}$$

Compactness of $[0, \pi]$ implies that there exists a finite number of such intervals $I_{\widehat{\omega}_1}, \ldots, I_{\widehat{\omega}_N}$ that cover $[0, \pi]$. It is no restriction to assume the following

1. $\widehat{\Pi}(e^{j\widehat{\omega}_k})$ is in the relative interior of $\Pi_\Delta(\gamma)$, for $k = 1, \ldots, N$.

2. $\widehat{\omega}_1 = 0$ and $\widehat{\omega}_N = \pi$. Since, $\widehat{G}(e^{j\widehat{\omega}_1}), \widehat{G}(e^{j\widehat{\omega}_N}) \in \mathbf{R}^{m\times m}$ it then follows that it is no restriction to use $\widehat{\Pi}(e^{j\widehat{\omega}_1}), \widehat{\Pi}(e^{j\widehat{\omega}_N}) \in \mathcal{S}_{\mathrm{R}}^{2m\times 2m}$.

3. $I_{\widehat{\omega}_k} \cap I_{\widehat{\omega}_{k+2}} = \emptyset$, for $k = 1, \dots, N-2$.

Let us now define the intervals I_k and J_k as

$$I_k = I_{\widehat{\omega}_k} \backslash (I_{\widehat{\omega}_{k+1}} \cup I_{\widehat{\omega}_{k-1}})$$
$$J_k = (\alpha_k, \beta_k) = I_{\widehat{\omega}_k} \cap I_{\widehat{\omega}_{k+1}}$$

It is then clear that the continuous function $\widehat{\Pi}_c$ defined as

$$\widehat{\Pi}_c(e^{j\omega}) = \begin{cases} \widehat{\Pi}(e^{j\widehat{\omega}_k}), & \omega \in I_k \\ \alpha\widehat{\Pi}(e^{j\widehat{\omega}_k}) + (1-\alpha)\widehat{\Pi}(e^{j\widehat{\omega}_{k+1}}), & \omega = \alpha\alpha_k + (1-\alpha)\beta_k \in J_k \\ \overline{\widehat{\Pi}_c(e^{-j\omega})}, & \omega \in [-\pi, 0) \end{cases}$$

is in the relative interior of $\Pi_\Delta(\gamma)$, for all $\omega \in [0, \pi]$.

Next, let us define

$$\widehat{\Pi}_m(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{\Pi}_c(e^{j\zeta}) K_m(e^{j(\omega-\zeta)}) d\zeta \tag{10}$$

where the Fejér kernel K_m is defined as

$$K_m(z) = \sum_{n=-m}^{m} \frac{m+1-n}{m+1} z^n$$

It is easy to verify that $\widehat{\Pi}_m(e^{-j\omega}) = \overline{\widehat{\Pi}_m(e^{j\omega})}$, and since $\Pi_\Delta(\gamma)$ is a closed convex cone, and by the properties of the convolution integral in (10), it follows that $\widehat{\Pi}_m(e^{j\omega}) = \widehat{\Pi}_m(e^{j\omega})^* \in \Pi_\Delta(\gamma)$, for all $\omega \in [0, \pi]$. We note that $\widehat{\Pi}_m$ corresponds to the rational function

$$\widehat{\Pi}_m(z) = \sum_{n=-m}^{m} \widetilde{\Pi}_n z^n$$

where

$$\widetilde{\Pi}_n = \frac{m+1-n}{2\pi(m+1)} \int_{-\pi}^{\pi} \widehat{\Pi}_c(e^{j\zeta}) e^{-jn\zeta} d\zeta \in \mathbf{R}^{2m\times 2m}$$

From Fejèrs Theorem it follows that $\widehat{\Pi}_m \to \widehat{\Pi}_c$ uniformly on the unit circle as $m \to \infty$, see for example [22]. In other words, since $\Pi_c = \widehat{\Pi}_c(\psi^{-1}(s))$ is in $\mathcal{C}$, it follows that the function $\widetilde{\Pi}(s) = \widehat{\Pi}_m(\psi^{-1}(s)) \in \mathbf{RL}_\infty^{2m\times 2m}$ also is in $\mathcal{C}$ if m is taken large enough. □

It is clear from Lemma 1 that $\gamma < \gamma^*$ implies that $\cap_{\omega\in[0,\infty]} C_\omega(\gamma) = \emptyset$. Furthermore, it follows from the assumptions on $\Pi_\Delta(\gamma)$ and $\Psi_\Delta(\gamma)$ that if $\gamma > \gamma^*$ then $\cap C_\omega \neq \emptyset$. This means that the condition $\cap C_\omega = \emptyset$ implies that $\gamma \leq \gamma^*$, and statement (i) is proved.

Proof of statement (ii) We first reduce the number of frequencies to $N = \dim(\Psi_\Delta(\gamma)) + 1$. This part is proven exactly as in [17]. We give the details for completeness. For any $\varepsilon > 0$ and $\rho > 0$, define the compact convex set $\widetilde{C}_\omega(\gamma, \varepsilon, \rho)$ as

$$\widetilde{C}_\omega(\gamma, \varepsilon, \rho) = \{\Psi \in \Psi_\Delta(\gamma) : M_G(\omega)(\Pi + \Psi) \leq -\varepsilon I, \Pi \in \Pi_\Delta(\gamma), |\Pi| \leq \rho, |\Psi| \leq \rho\}$$

Next assume that $\cap C_\omega(\gamma) = \emptyset$. Since $\widetilde{C}_\omega(\gamma, \varepsilon, \rho) \subset C_\omega(\gamma)$ it follows that $\cap \widetilde{C}_\omega(\gamma, \varepsilon, \rho) = \emptyset$. By Helly's Theorem there exists N distinct frequencies $\omega_1, \ldots, \omega_N \in [0, \infty]$ such that

$$\cap_{k=1}^N \widetilde{C}_{\omega_k}(\gamma, \varepsilon, \rho) = \emptyset \tag{11}$$

Take sequences $\{\varepsilon_i\} \to 0$, $\{\rho_i\} \to \infty$ and let ω_k^i be corresponding frequencies such that (11) holds for all i. By compactness of $[0, \infty]$ there exists subsequences $\{\varepsilon_{i_j}\} \searrow 0$, $\{\rho_{i_j}\} \nearrow \infty$ and $\{\omega_k^{i_j}\} \to \omega_k^0$.

Assume that $\cap_{k=1}^N C_{\omega_k^0}(\gamma)$ is nonempty, i.e. there exists $\Psi_0 \in \cap_{k=1}^N C_{\omega_k^0}(\gamma)$. By continuity of $G \in \mathbf{RH}_\infty^{m \times m}$, there exists $\varepsilon_0 > 0$, $\rho_0 < \infty$ and $\delta_0 > 0$ such that for $k = 1, \ldots, N$, $\Psi_0 \in \widetilde{C}_\omega(\gamma, \varepsilon_0, \rho_0)$, $\forall \omega \in [\omega_k^0 - \delta_0, \omega_k^0 + \delta_0]$.

Choose index j such that $\varepsilon_{i_j} \leq \varepsilon_0$, $\rho_{i_j} \geq \rho_0$ and $\omega_k^{i_j} \in [\omega_k^0 - \delta_0, \omega_k^0 + \delta_0]$, then $\Psi_0 \in \cap_{k=1}^N \widetilde{C}_{\omega_k^{i_j}}(\gamma, \varepsilon_{i_j}, \rho_{i_j}) = \emptyset$, which is a contradiction. Hence, it follows that $\cap_{k=1}^N C_{\omega_k^0}(\gamma) = \emptyset$. The reversed implication is trivial.

Next assume that $0 \notin \text{ri } \Psi_\Delta(\gamma)$. We will now use this assumption to show that the existence of $N + 1 = \dim(\Psi_\Delta(\gamma)) + 1$ frequencies ω_k such that $\cap_{k=1}^{N+1} C_{\omega_k}(\gamma) = \emptyset$ is equivalent to the existence of a subcollection consisting of N frequencies ω_{k_j} such that $\cap_{j=1}^N C_{\omega_{k_j}}(\gamma) = \emptyset$. The sufficiency is trivial. For the necessity we can assume that $C_{\omega_k}(\gamma) \neq \emptyset$, since otherwise the result is obvious. Note that if $\Psi \in C_{\omega_k}(\gamma)$, then $\alpha\Psi \in C_{\omega_k}(\gamma)$ for any $\alpha > 0$. By this and the assumption that $0 \notin \text{ri } \Psi_\Delta(\gamma)$ it follows it is no restriction to fix one element in $\Psi_\Delta(\gamma)$ to 1 or -1. We have now reduced the dimension on $\Psi_\Delta(\gamma)$ by one and the result follows from Helly's Theorem.

9. References

[1] V. Balakrishnan, Y. Huang, A. Packard, and J. Doyle. "Linear Matrix Inequalities in Analysis with Multipliers." In *Proceedings of the American Control Conference*, pp. 1228–1232, Baltimore, Maryland, 1994.

[2] S. Boyd and C. Barratt. *Linear Controller Design—Limits of Performance*. Prentice Hall, Englewood Cliffs, New Jersey 07632, 1991.

[3] S. Boyd, L. Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM Studies in Applied Mathematics, 1994.

[4] J. C. Doyle. "Analysis of feedback systems with structured uncertainties." In *IEE Proceedings*, volume D-129, pp. 242–251, 1982.

[5] P. Gahinet and A. Nemirovskii. *LMI-lab: A Package for Manipulating and Solving LMI's*, version 2.0 edition, 1993.

[6] A. Hansson. *Stochastic Control of Critical Processes.* PhD thesis, Department of Automatic Control, Lund Institute of Technology, 1995.

[7] U. Jönsson. "Duality in Analysis via Integral Quadratic Constraints." Technical Report TFRT–7543, Lund Institute of Technolgy, January 1996.

[8] U. Jönsson and A. Rantzer. "On Duality in Robustness Analysis." In *Proceedings of the IEEE Conference on Decision and Control, New Orleans*, pp. 1443–1448, New Orleans, Louisiana, 1995.

[9] U. Jönsson and A. Rantzer. "A Unifying Format for Multiplier Optimization." In *Proceedings of the American Control Conference*, pp. 3859–3860, Seattle, Wahington, 1995.

[10] D. Luenberger. *Optimization by Vector Space Methods.* Wiley, 1969.

[11] J. Ly, M. Safonov, and R. Chiang. "Real/Complex Multivariable Stability Margin Computation via Generalized Popov Multiplier–LMI Approach." In *Proceedings of the American Control Conference*, pp. 425–429, Baltimore, Maryland, 1994.

[12] A. Megretski and A. Rantzer. "System Analysis via Integral Quadratic Constraints: Part I." Technical Report TFRT–7531, Department of Automatic Control, Lund Institute of Technology, 1995.

[13] A. Megretski and S. Treil. "Power Distribution Inequalities in Optimization and Robustness of Uncertain Systems." *Journal of Mathematical Systems, Estimation, and Control*, **3:3**, pp. 301–319, 1993.

[14] Y. Nesterov and A. Nemirovski. *Interior point polynomial methods in convex programming*, volume 13 of *Studies in Applied Mathematics.* SIAM, Philadelphia, 1993.

[15] A. Packard and J. Doyle. "The complex structured singular value." *Automatica*, **29:1**, pp. 71–109, 1993.

[16] F. Paganini. "Analysis of Systems with Combined Time Invariant/Time Varying Structured Uncertainty." In *Proceedings of the American Control Conference*, pp. 3878–3883, Seattle, Washington, 1995.

[17] K. Poola and A. Tikku. "Robust Performance against Time-Varying Structured Perturbations." *IEEE Transactions on Automatic Control*, **40:9**, pp. 1589–1602, September 1995.

[18] A. Rantzer and A. Megretski. "System Analysis via Integral Quadratic Constraints." In *Proceedings of the Conference of Decision and Control*, Orlando, Florida, 1994.

[19] R. Redheffer. "Inequalities for a Matrix Equation." *Journal for Mathematics and Mechanics*, **8:3**, 1959.

[20] R. Rockafellar. *Convex Analysis.* Princeton, 1970.

[21] M. Safonov and M. Athans. "A multi-loop generalization of the circle criterion for stability margin analysis." *IEEE Transactions on Automatic Control*, **26**, pp. 415–422, 1981.

[22] N. Young. *An introduction to Hilbert Space.* Cambridge Mathematical Textbooks, 1988.