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Jönsson, Ulf

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PO Box 117  
221 00 Lund  
+46 46-222 00 00



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# Stability Analysis with Popov Multipliers and Integral Quadratic Constraints

Ulf Jönsson

Department of Automatic Control  
Lund Institute of Technology  
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<i>Title and subtitle</i> Stability Analysis with Popov Multipliers and Integral Quadratic Constraints			
<i>Abstract</i> <p>It will be shown how Popov multipliers can be used in stability analysis based on Integral Quadratic Constraints (IQC). The Popov multiplier is nonproper and a condition that the nominal plant is strictly proper will be imposed in order to ensure boundedness of the IQC corresponding to the Popov multiplier. The Popov multiplier is defined by a small number of parameters. This is advantageous for computational reasons in stability analysis of large systems. An example will indicate how recent results on stability analysis of systems with parametric uncertainty can be extended in the IQC framework.</p>			
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# 1. Introduction

We show how to use Popov multipliers in stability analysis based on Integral Quadratic Constraints (IQC). The framework for using IQCs in stability analysis was proposed by Megretskii in [5] and further refined by Megretskii and Rantzer in [6] and [8]. The system under consideration consists of a nominal linear, time-invariant, and stable operator  $G$  in a positive feedback interconnection with a causal and bounded perturbation  $\Delta$ . The idea is to find descriptions of  $\Delta$  in terms of bounded and Hermitian valued matrix function  $\Pi$  that define valid IQCs in the sense that

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{\Delta(y)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{y}(j\omega) \\ \widehat{\Delta(y)}(j\omega) \end{bmatrix} d\omega \geq 0,$$

for all square integrable inputs  $y$  to  $\Delta$ . The matrix function  $\Pi$  is called multiplier.

The IQC methodology gives a unified approach for multiplier based stability analysis that has several advantages compared to the classical framework in for example [2], [7], [10], [11], and [12]. It should be noted that the term multiplier is used in a somewhat different meaning in the classical papers where it is used to denote a device that is used to multiply the operators in the feedback loop in order to make them look passive or contractive. The multiplier  $\Pi$  in the IQC approach can be thought of as a way to collect the *classical multipliers* in a structure that is flexible for defining integral quadratic constraints.

Some of the most distinguished advantages with the IQC methodology are the following

- Noncausal multipliers are easy to use since there is no need for factorization conditions.
- It is easy to combine multipliers that describe a certain operator. This follows because convex combinations of multipliers that define valid IQCs for a certain operator still gives a valid IQC for this operator.
- There is generally no need to consider the multipliers in terms of loop transformations.

The cost for this increased flexibility is the need for a stronger but still very reasonable well-posedness assumption on the system.

The Popov multipliers considered in this paper are of the form

$$\Pi(j\omega) = \begin{bmatrix} 0 & -j\omega\Lambda \\ j\omega\Lambda & 0 \end{bmatrix},$$

where  $\Lambda$  is a real symmetric matrix. These multipliers are used to define constraints in terms of the integral

$$\int_0^{\infty} v^T \Lambda \dot{y} dt,$$

where  $v = \Delta(y)$  and where it is assumed that both  $\dot{y}$  and  $v$  are square integrable. It is thus necessary that the output signal from the nominal plant is differentiable.

We will derive a stability result along the lines of [6] and [8] by assuming that the nominal plant is strictly proper. This is a reasonable assumption since

the Popov multipliers are typically used to describe static nonlinearities, uncertain parameters and slowly time-varying parameters. The strict properness of the plant assures that there is no algebraic loop in the system for these three examples.

The Popov multiplier is defined by a small number of parameters. This is advantageous in stability analysis of large complex systems, since such multipliers are computationally cheap to determine. One consequence of this paper is that the recent results on stability for systems with parametric uncertainty in [4] and [1] can be extended into the IQC framework.

### Notation and Preliminaries

$M^*$	Hermitian conjugation of a matrix.
$ \cdot $	The Euclidean norm $ x  = \sqrt{x^T x}$ .
$\bar{\sigma}(M)$	The largest singular value of a real or complex matrix $M$ .
$\mathbf{RL}_\infty^{n \times n}$	The space consisting of proper real rational matrix functions with no poles on the imaginary axis. If $H \in \mathbf{RL}_\infty^{m \times m}$ then $H^*(s) = H(-s)^T$ .
$\mathbf{RH}_\infty^{m \times m}$	The subspace of $\mathbf{RL}_\infty^{m \times m}$ consisting of functions with no poles in the closed right half plane.
$P_T$	The projection operator defined by $P_T u(t) = u(t)$ when $t \leq T$ and $P_T u(t) = 0$ when $t > T$ .
$\mathbf{L}_2^m[0, \infty)$	The Lebesgue space of $\mathbf{R}^m$ valued signals with norm defined by

$$\|u\|^2 = \int_0^\infty |u(t)|^2 dt.$$

$\mathbf{L}_2^m(-\infty, \infty)$  is defined similarly as  $\mathbf{L}_2^m[0, \infty)$ .

$\mathbf{L}_{2e}^m[0, \infty)$	The vector space of functions $f$ satisfying the condition $P_T f \in \mathbf{L}_2^m[0, \infty)$ for all $T > 0$ .
$\theta$	The unit step function defined as $\theta(t) = 1$ for $t \geq 0$ and $\theta(t) = 0$ for $t < 0$ .

An operator  $H : \mathbf{L}_{2e}^m[0, \infty) \rightarrow \mathbf{L}_{2e}^m[0, \infty)$  is said to be causal if  $P_T H P_T = P_T H$  for all  $T \geq 0$ . This means that the output value at a certain time instant does not depend on future values of the input.

A causal operator  $H$  on  $\mathbf{L}_{2e}^m[0, \infty)$  is bounded if  $H(0) = 0$  and if the gain defined as

$$\|H\| = \sup_{\mathbf{L}_{2e}^m[0, \infty) \ni u \neq 0} \frac{\|Hu\|}{\|u\|} \quad (1)$$

is finite. Note that the gain is defined in terms of functions in  $\mathbf{L}_2^m[0, \infty)$  and not the corresponding extended space. However, the definition in (1) implies boundedness on  $\mathbf{L}_{2e}^m[0, \infty)$ , since  $\|P_T H u\| \leq \|H\| \cdot \|P_T u\|$  for all  $u \in \mathbf{L}_{2e}^m[0, \infty)$  and all  $T \geq 0$ . It can be shown that  $\|H\|$  is the smallest such bound, see [10].

Linear time invariant convolution operators with transfer function  $H \in \mathbf{RH}_\infty^{m \times m}$  are causal and bounded on  $\mathbf{L}_{2e}^m[0, \infty)$ . A bounded convolution operator with transfer function  $H \in \mathbf{RL}_\infty^{m \times m}$  defines a (possibly non-causal) map of  $\mathbf{L}_2[0, \infty)$  into  $\mathbf{L}_2(-\infty, \infty)$ .

Let  $H = H^* \in \mathbf{RL}_\infty^{m \times m}$  and  $u \in \mathbf{L}_2^m[0, \infty)$ , then the quadratic form  $\langle \cdot, \cdot \rangle$  is defined by

$$\langle u, Hu \rangle = \int_0^\infty u(t)^T (Hu)(t) dt = \int_{-\infty}^\infty \hat{u}(j\omega)^* H(j\omega) \hat{u}(j\omega) d\omega,$$

where  $Hu$  denotes the convolution  $h * u$  between  $u$  and the impulse response  $h$  corresponding to  $H$ . Furthermore,  $\hat{u}$  denotes the Fourier transform of  $u$ , which we define as

$$\hat{u}(j\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty u(t) e^{-j\omega t} dt.$$

We will finally make some remarks on absolute continuity, see [9]. Absolute continuity of a function  $x : \mathbf{R}^+ \rightarrow \mathbf{R}^m$  implies that the time derivative  $\dot{x} := \frac{d}{dt}x$  exists and is finite almost everywhere. Furthermore, an absolutely continuous function  $x$  is the indefinite integral of its derivative, i.e. the relation  $x(t) = x_0 + \int_0^t \dot{x}(\tau) d\tau$  holds for all  $t \geq 0$ . From now on it is assumed that  $x$  is absolutely continuous when making assumptions on the derivative  $\dot{x}$ .

The next lemma will be used frequently in the paper.

LEMMA 1

If a function satisfies  $x, \dot{x} \in \mathbf{L}_2^m[0, \infty)$ , then  $x$  is bounded and  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

**Proof:** The lemma is well known. A proof is given in Appendix 1 for completeness.  $\square$

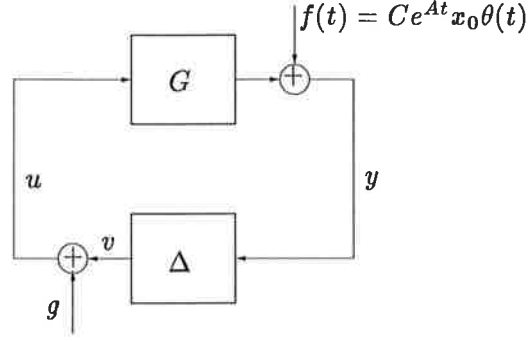
## 2. Stability Analysis Based on IQCs

We will consider stability of the system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \\ u &= \Delta(y) + g, \end{aligned} \tag{2}$$

where it is assumed that  $A \in \mathbf{R}^{n \times n}$ ,  $B \in \mathbf{R}^{n \times m}$ ,  $C \in \mathbf{R}^{m \times n}$  and that  $A$  is Hurwitz. It is further assumed that  $g \in \mathbf{L}_{2e}^m[0, \infty)$  and that  $\Delta$  is a causal and bounded operator on  $\mathbf{L}_{2e}^m[0, \infty)$ . The system in (2) can equivalently be represented as the feedback system in Figure 1, where the operator  $G(s) = C(sI - A)^{-1}B \in \mathbf{RH}_\infty^{m \times m}$  is assumed to be initially at rest, and where  $f(t) = Ce^{At}x_0\theta(t)$  represents the response of the initial condition.

The idea behind the IQC approach for stability analysis is to find descriptions of the operator  $\Delta$  in terms of multipliers that define valid IQCs. We will consider IQCs defined by combinations of bounded and Hermitian valued multipliers with Popov multipliers.



**Figure 1** Block diagram representation of the system in (2). Here  $f$  corresponds to the response of the initial condition and  $G(s) = C(sI - A)^{-1}B$ .

**DEFINITION 1**

Let  $\Pi_1 = \Pi_1^* \in \mathbf{RL}_\infty^{2m \times 2m}$  and let  $\Pi_2$  be the Popov multiplier

$$\Pi_2(j\omega) = \begin{bmatrix} 0 & -j\omega\Lambda \\ j\omega\Lambda & 0 \end{bmatrix},$$

where  $\Lambda = \Lambda^T \in \mathbf{R}^{m \times m}$ . We say that  $\Delta$  satisfies the IQC defined by  $\Pi = \Pi_1 + \Pi_2$  if there exists a positive constant  $\gamma$  such that

$$\begin{aligned} \left\langle \begin{bmatrix} y \\ v \end{bmatrix}, \Pi_1 \begin{bmatrix} y \\ v \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} \dot{y} \\ v \end{bmatrix}, \begin{bmatrix} 0 & \Lambda \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ v \end{bmatrix} \right\rangle = \\ \int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix}^* \Pi_1(j\omega) \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix} + \int_0^{\infty} 2v^T \Lambda \dot{y} dt \geq -\gamma|y_0|^2, \end{aligned}$$

for all  $y, \dot{y}, v \in \mathbf{L}_2^m[0, \infty)$  such that  $v = \Delta(y)$ . Here  $\hat{y}$  and  $\hat{v}$  denotes the Fourier transforms of  $y$  and  $v$  respectively. We used the notation  $y_0 = Cx_0$ .  $\square$

**REMARK 1**

Differentiability is only necessary for the components of the vector  $y$  that corresponds to nonzero colons of  $\Lambda$ . The condition  $\dot{y} \in \mathbf{L}_2^m[0, \infty)$  is thus not necessary if  $\Pi_2 = 0$ .

We will next give some examples of the use of Popov multipliers for describing nonlinearities, uncertain parameters and slowly time-varying parameters. The first example is adapted from [2] and [11].

**EXAMPLE 1**

Let  $\varphi : \mathbf{R} \rightarrow \mathbf{R}$  be a measurable function satisfying  $\varphi(0) = 0$  and a sector condition  $\alpha x^2 \leq \varphi(x)x \leq \beta x^2$ , where  $-\infty < \alpha < \beta < \infty$ . Then  $\varphi$  satisfies the IQC defined by the Popov multiplier

$$\Pi(j\omega) = \begin{bmatrix} 0 & -j\omega\lambda \\ j\omega\lambda & 0 \end{bmatrix},$$

where  $\lambda \in \mathbf{R}$ . This follows since

$$2\lambda \lim_{T \rightarrow \infty} \int_0^T \varphi(y)\dot{y}dt = 2\lambda \lim_{T \rightarrow \infty} \int_{y_0}^{y(T)} \varphi(\sigma)d\sigma = -2\lambda \int_0^{y_0} \varphi(\sigma)d\sigma \geq -\gamma|y_0|^2,$$



for all  $y, \dot{y} \in \mathbf{L}_2^m[0, \infty)$ , where we use  $\gamma = |\lambda| \max(|\alpha|, |\beta|)$ . The third equality follows since Lemma 1 implies that  $y(T) \rightarrow 0$  as  $T \rightarrow \infty$ .  $\square$

The next two examples are used to indicate that the  $\Lambda$  parameter of the Popov multiplier need not be a diagonal matrix.

#### EXAMPLE 2

Let  $\Delta = \delta I$ , where  $\delta$  is a constant but uncertain real-valued parameter with  $\delta \in [-1, 1]$ . Then  $\Delta$  satisfies the IQC defined by the Popov multiplier

$$\Pi(j\omega) = \begin{bmatrix} 0 & -j\omega\Lambda \\ j\omega\Lambda & 0 \end{bmatrix},$$

where  $\Lambda = \Lambda^T \in \mathbf{R}^{m \times m}$ . This follows since

$$\int_0^\infty \delta 2y^T \Lambda \dot{y} dt = \int_0^\infty \delta (y^T \Lambda y)' dt = \lim_{t \rightarrow \infty} \delta [y^T \Lambda y]_0^t = -\delta y_0^T \Lambda y_0 \geq -\gamma |y_0|^2,$$

for any  $y, \dot{y} \in \mathbf{L}_2^m[0, \infty)$  if we use  $\gamma = \bar{\sigma}(\Lambda)^2$ . This Popov multiplier combined with well known multipliers for uncertain parameters was used in [4] to obtain a computationally inexpensive criterion for stability analysis of systems with uncertain parameters.  $\square$

#### EXAMPLE 3

Let  $\Delta = \delta(t)I$ , where  $\delta$  is a slowly time-varying parameter with  $\delta(t) \in [-1, 1]$ ,  $\forall t \geq 0$ . If we assume that  $\delta$  is differentiable with derivative bounded as  $\frac{d}{dt}\delta(t) \in [-b, b]$ ,  $\forall t \geq 0$ , then  $\Delta$  satisfies the IQC defined by the multiplier  $\Pi = \Pi_1 + \Pi_2$ , where

$$\Pi_1(j\omega) = \begin{bmatrix} b\Lambda & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi_2(j\omega) = \begin{bmatrix} 0 & -j\omega\Lambda \\ j\omega\Lambda & 0 \end{bmatrix},$$

and where  $\Lambda = \Lambda^T \geq 0$ . This follows since integration by parts gives

$$\begin{aligned} \int_0^\infty 2\delta y^T \Lambda \dot{y} dt + \int_0^\infty b y^T \Lambda y dt &= \lim_{t \rightarrow \infty} [\delta y^T \Lambda y]_0^t + \int_0^\infty (b - \dot{\delta}) y^T \Lambda y dt \\ &\geq -\delta(0) y_0^T \Lambda y_0 \geq -\bar{\sigma}(\Lambda)^2 |y_0|^2, \end{aligned}$$

for any  $y, \dot{y} \in \mathbf{L}_2^m[0, \infty)$ . This multiplier can be used in combination with other multipliers for slowly time-varying parameters.  $\square$

We will next formulate the assumptions needed for our main stability result. We first define well-posedness and stability for the system in (2).

#### DEFINITION 2

The system in (2) is *well-posed* if for any initial condition  $x_0$  and for any input  $g \in \mathbf{L}_{2e}^m[0, \infty)$  there exists a unique solution  $(x, v) \in \mathbf{L}_{2e}^n[0, \infty) \times \mathbf{L}_{2e}^m[0, \infty)$ , where  $x$  is an absolutely continuous function. Furthermore, the map from  $g$  to  $(x, v)$  should be causal.  $\square$

**DEFINITION 3**

The system in (2) is *stable* if it is well-posed and if there exist constants  $c > 0$  and  $\rho > 0$  such that

$$\int_0^T (|y|^2 + |u|^2) dt \leq \rho |x_0|^2 + c \int_0^T |g|^2 dt,$$

for all  $T > 0$  and for arbitrary  $x_0 \in \mathbf{R}^n$  and  $g \in \mathbf{L}_{2e}^m[0, \infty)$ .  $\square$

We will make the following assumption on the system in (2).

**ASSUMPTION 1**

We assume that there is a continuous parametrization  $\Delta_\tau$ ,  $\tau \in [0, 1]$  of the operator  $\Delta$  and a corresponding parametrized system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \\ u &= \Delta_\tau(y) + g, \end{aligned} \tag{3}$$

such that

1.  $\Delta_\tau$  is a causal and bounded operator on  $\mathbf{L}_{2e}^m[0, \infty)$  for all  $\tau \in [0, 1]$ .
2.  $\Delta = \Delta_1$ .
3. For some  $\kappa > 0$  we have that

$$\|\Delta_{\tau_2}(y) - \Delta_{\tau_1}(y)\| \leq \kappa |\tau_2 - \tau_1| \cdot \|y\|,$$

for all  $y \in \mathbf{L}_2[0, \infty)$ .

4. For any  $\tau \in [0, 1]$ ,  $\Delta_\tau$  satisfies the IQC defined by the multiplier  $\Pi = \Pi_1 + \Pi_2$ , where  $\Pi_1 = \Pi_1^* \in \mathbf{RL}_{\infty}^{2m \times 2m}$  and  $\Pi_2$  is a Popov multiplier.
5. For any  $\tau \in [0, 1]$ , the system in (3) is well-posed.
6. The system in (3) is stable when  $\tau = 0$ .  $\square$

Next follows the main result of this paper. The theorem gives conditions for the parametrized system in (3) to be stable for all  $\tau \in [0, 1]$ . This in particular implies that the system in (2) is stable.

**THEOREM 1**

The system in (3) is stable for all  $\tau \in [0, 1]$  if Assumption 1 holds and if there exists  $\varepsilon > 0$  such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Pi_1(j\omega) + \Pi_2(j\omega)) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon I, \quad \forall \omega \geq 0,$$

where  $G(s) = C(sI - A)^{-1}B$ .

**Proof:** The proof is given in Appendix 2.  $\square$

**REMARK 2**

The parametrization of  $\Delta$  in Assumption 1 can often be taken as  $\Delta_\tau = \tau\Delta$ .

REMARK 3

It follows from the proof that we could allow the operator  $G$  to have a direct term, i.e.  $G(s) = C(sI - A)^{-1}B + D \in \mathbf{RH}_\infty^{m \times m}$ , under the condition that  $\Lambda D = 0$ , where  $\Lambda$  is the parameter in the Popov multiplier  $\Pi_2$ .

REMARK 4

We do not use the assumed boundedness of  $\Delta$  and  $\Delta_\tau$  in the proof. It is possible to use Theorem 1 also when we have systems with unbounded  $\Delta$ . However one needs to be careful and in general we need to make modifications in the assumptions to apply the results for unbounded operators. An example is when  $\Delta$  is not bounded in the sense that  $\Delta(0) \neq 0$  due to non-zero initial conditions of  $\Delta$ . In this case we can still use Theorem 1 given that the modifications below holds. Only small and obvious changes of the proof is needed to show this.

1. Let the continuity property in Assumption 1 be: For some  $\kappa > 0$  and  $\beta > 0$  we have that

$$\|\Delta_{\tau_2}(y) - \Delta_{\tau_1}(y)\| \leq \kappa|\tau_2 - \tau_1| \cdot \|y\| + \beta,$$

for all  $y \in \mathbf{L}_2^m[0, \infty)$ . The constant  $\beta$  can be used to bound the influence due to a non-zero initial condition in  $\Delta$ .

2. Generally, we need to allow a constraint on the form

$$\left\langle \begin{bmatrix} y \\ v \end{bmatrix}, \Pi_1 \begin{bmatrix} y \\ v \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} \dot{y} \\ v \end{bmatrix}, \begin{bmatrix} 0 & \Lambda \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ v \end{bmatrix} \right\rangle \geq -\zeta(v_0, y_0),$$

for all  $y, \dot{y}, v \in \mathbf{L}_2^m[0, \infty)$  in Definition 1, where the function  $\zeta : \mathbf{R}^{2m} \rightarrow \mathbf{R}^+$  could be a constant. We used the notation  $v_0 = v(0)$  and  $y_0 = y(0) = Cx_0$ .

3. Let the first term of the right-hand side of the inequality in Definition 3 be a function  $\rho : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}^+$  with the arguments  $v_0$  and  $x_0$ . Note that  $\rho$  will depend on the constant  $\beta$ .

Generally for a diagonal perturbation  $\Delta = \text{diag}(\Delta_1, \Delta_2)$ , where  $\Delta_2$  is unbounded, we use the parametrization  $\Delta_\tau = \text{diag}(\Delta_{1\tau}, \Delta_2)$ . The third condition in Assumption 1 is then satisfied if the parametrization of  $\Delta_1$  satisfies this continuity condition. The sixth condition in Assumption 1 is particularly easy to verify if  $\Delta_{10}$  is linear.

COROLLARY 1

If  $g \in \mathbf{L}_2^m[0, \infty)$  and if the conditions of Theorem 1 hold, then  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

**Proof:** The conditions of the corollary imply that the system is stable. Since  $g \in \mathbf{L}_2^m[0, \infty)$  and  $A$  is Hurwitz we conclude that  $x, \dot{x} \in \mathbf{L}_2^m[0, \infty)$ . Lemma 1 gives the desired result.  $\square$

REMARK 5

It would be more useful to verify exponential convergence to zero of the state vector  $x$  than just asymptotic convergence with no guaranteed rate of decay as in Corollary 1. If for example  $g = 0$ , then the conditions in Theorem 1 imply exponential convergence if the operator  $\Delta$  is memoryless and bounded. This follows from a result in [6].

### 3. The Stability Analysis in Practice

We will here shortly discuss how the stability analysis can be performed in practice. The first step is to find an IQC description of the operator  $\Delta$  in terms of multipliers. If  $\Delta$  satisfies the IQC defined by  $\Pi_i$ , for  $i = 1, \dots, n$ , then  $\Delta$  also satisfies the IQC defined by the multiplier  $\Pi = \sum \alpha_i \Pi_i$ , where  $\alpha_i \geq 0$  for  $i = 1, \dots, n$ . We can in this way obtain a description of  $\Delta$  in terms of a convex cone  $\Pi_\Delta$  of multipliers. The stability analysis now involves a search for a suitable multiplier in  $\Pi_\Delta$  such that the last condition in Theorem 1 is satisfied.

We will finally continue Example 2 with some important remarks.

#### EXAMPLE 4—EXAMPLE 2 CONT'D

Consider the case when  $\Delta = \delta I$ , where  $\delta$  is an uncertain parameter with  $\delta \in [-1, 1]$ . Then we can combine the Popov description in Example 2 with the IQC that can be obtained from [3]. We get

$$\Pi(j\omega) = \begin{bmatrix} X(i\omega) & Y(j\omega)^* - j\omega\Lambda \\ Y(\omega) + j\omega\Lambda & -X(j\omega) \end{bmatrix},$$

where  $X(j\omega) = X(j\omega)^* \geq 0$ ,  $Y(j\omega) = -Y(j\omega)^*$  for all  $\omega \in \mathbf{R}$  and  $\Lambda = \Lambda^T$ . The first step in the stability analysis could now be to search for a  $\Lambda$  and constant matrices  $X = X^T \geq 0$  and  $Y = -Y^T$  such that the last condition in Theorem 1 is satisfied. If this is not possible then use more general  $X$  and  $Y$ . This example can be generalized to include diagonal structures of uncertain parameters as in [4] or to more general parametric uncertainty as in [1].

### 4. Conclusions

We showed how Popov multipliers can be used in the IQC methodology for stability analysis.

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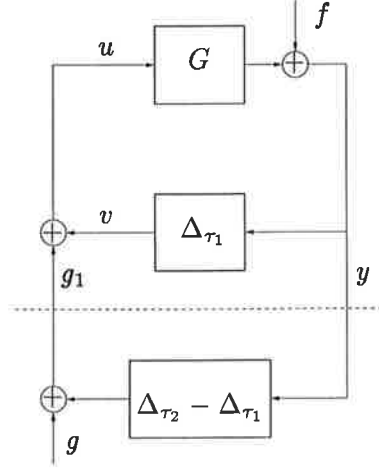
### Appendix 1: Proof of Lemma 1

The boundedness is trivial since

$$|x(t)|^2 = |x_0|^2 + \int_0^t 2\dot{x}^T x d\tau \leq |x_0|^2 + 2\|x\| \cdot \|\dot{x}\|, \quad \forall t \geq 0.$$

We will next prove convergence to zero. Assume that this is not the case. Then there exists  $\varepsilon > 0$  and a sequence  $\{t_i\}_{i=1}^\infty$  such that,  $t_i \rightarrow \infty$  as  $i \rightarrow \infty$  and  $|x(t_i)| > 2\varepsilon$  for all  $i$ . We will show that this contradicts the condition that  $x \in \mathbf{L}_2^m[0, \infty)$ . Let  $I_i = [t_i - \delta, t_i + \delta]$ , where  $\delta < \varepsilon^2/\|\dot{x}\|^2$ . Then

$$|x(t) - x(t_i)| = \left| \int_{t_i}^t \dot{x}(\tau) d\tau \right| \leq \sqrt{\delta} \|\dot{x}\| < \varepsilon, \quad \forall t \in I_i.$$



**Figure 2** Block diagram for illustration of the proof of Theorem 1: If the upper loop is stable when it is disconnected from the lower branch, then also the total system is stable if  $|\tau_2 - \tau_1|$  is small enough.

Hence,  $|x(t)| > \varepsilon$ ,  $\forall t \in I_i$ . It is no restriction to assume that the intervals  $I_i$  are disjoint. This implies that  $\|x\| = \infty$ , since

$$\int_0^\infty |x(\tau)|^2 d\tau \geq \sum_{i=1}^n \int_{I_i} |x(\tau)|^2 d\tau \geq n2\delta\varepsilon^2,$$

and the right hand side tends to infinity as  $n \rightarrow \infty$ .

## Appendix 2: Proof of Theorem 1

The following inequalities will be used in the proof

$$\langle x, y \rangle \leq \|y\| \cdot \|x\|, \quad (4)$$

$$\|x + y\|^2 \leq 2(\|x\|^2 + \|y\|^2), \quad (5)$$

for arbitrary  $x, y \in \mathbf{L}_2^m[0, \infty)$ . We will also use that

$$\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}, \quad (6)$$

for arbitrary  $x, y \geq 0$ .

The idea for the proof is to use a similar homotopy argument as that in [6]. Consider the system in Figure 2. We will show that if the upper loop is stable when it is disconnected from the lower branch, then also the whole system is stable if  $|\tau_2 - \tau_1|$  is small enough. Iterative use of this from  $\tau = 0$  up to  $\tau = 1$  in small steps of equal size will finish the proof.

*Part 1:* Assume that the system

$$\begin{aligned} y &= Gu + f, & f(t) &= Ce^{At}\theta(t)x_0, \\ u &= v + g, & v &= \Delta_\tau(y) \end{aligned} \quad (7)$$

is stable for some  $\tau \in [0, 1]$ . Then for any  $x_0 \in \mathbf{R}^n$  and  $g \in \mathbf{L}_2^m[0, \infty)$  we have  $y, v \in \mathbf{L}_2^m[0, \infty)$  and we will show that there exist constants  $c_0 > 0$  and  $\rho_0 > 0$ , which both are independent of  $\tau$ , such that  $y$  is bounded as

$$\|y\| \leq \rho_0 \|x_0\| + c_0 \|g\|. \quad (8)$$

To prove that (8) holds, we notice that  $g, v \in \mathbf{L}_2^m[0, \infty)$  implies that also  $\dot{y} \in \mathbf{L}_2^m[0, \infty)$ . Will use that the Fourier transform of the time derivative of  $Gv$ , where  $G(s) = C(sI - A)^{-1}B \in \mathbf{RH}_\infty^{m \times m}$ , is  $j\omega G(j\omega)\hat{v}(j\omega)$ . Using this together with the relation  $y = Gv + Gg + f$ , we obtain

$$\begin{aligned} \left\langle \begin{bmatrix} \dot{y} \\ v \end{bmatrix}, \begin{bmatrix} 0 & \Lambda \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ v \end{bmatrix} \right\rangle &= \left\langle v, \begin{bmatrix} G \\ I \end{bmatrix}^* \Pi_2 \begin{bmatrix} G \\ I \end{bmatrix} v \right\rangle + 2 \int_0^\infty v^T \Lambda \dot{f} dt \\ &\quad + 2 \operatorname{Re} \int_{-\infty}^\infty \hat{v}(j\omega)^* \Lambda j\omega G(j\omega) \hat{g}(j\omega) d\omega. \end{aligned}$$

Hence, the IQC constraint corresponding to the multiplier  $\Pi = \Pi_1 + \Pi_2$  gives

$$\begin{aligned} -\gamma |y_0|^2 &\leq \left\langle \begin{bmatrix} y \\ v \end{bmatrix}, \Pi_1 \begin{bmatrix} y \\ v \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} \dot{y} \\ v \end{bmatrix}, \begin{bmatrix} 0 & \Lambda \\ \Lambda^T & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ v \end{bmatrix} \right\rangle \\ &= \left\langle v, \begin{bmatrix} G \\ I \end{bmatrix}^* (\Pi_1 + \Pi_2) \begin{bmatrix} G \\ I \end{bmatrix} v \right\rangle + 2 \operatorname{Re} \left\langle \begin{bmatrix} Gv \\ v \end{bmatrix}, \Pi_1 \begin{bmatrix} Gg + f \\ 0 \end{bmatrix} \right\rangle \\ &\quad + \left\langle \begin{bmatrix} Gg + f \\ 0 \end{bmatrix}, \Pi_1 \begin{bmatrix} Gg + f \\ 0 \end{bmatrix} \right\rangle + 2 \int_0^\infty v^T \Lambda \dot{f} dt \\ &\quad + 2 \operatorname{Re} \int_{-\infty}^\infty \hat{v}(j\omega)^* \Lambda j\omega G(j\omega) \hat{g}(j\omega) d\omega \\ &\leq -\varepsilon \|v\|^2 + 2(c_1 \|x_0\| + c_2 \|g\|) \|v\| + c_3 \|x_0\|^2 + c_4 \|g\|^2. \end{aligned}$$

The first term in the last inequality follows from the frequency inequality in the theorem statement. The other terms follows by use of (4) and (5). If we let  $\|\Pi_{1i}\| = \sup_\omega \bar{\sigma}(\Pi_{1i}(j\omega))$  for  $i = 1, 2, 3$  denote the norms for the blocks of the matrix operator

$$\Pi_1 = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{13} \end{bmatrix},$$

then the constants can be taken as

$$\begin{aligned} c_1 &= (\|\Pi_{11}\| \cdot \|G\| + \|\Pi_{12}\|) \gamma_1 + \gamma_2, \\ c_2 &= \|\Pi_{11}\| \cdot \|G\|^2 + \|\Pi_{12}\| \cdot \|G\| + \|\Lambda j\omega G\|, \\ c_3 &= 2\|\Pi_{11}\| \gamma_1^2, \\ c_4 &= 2\|G\|^2 \cdot \|\Pi_{11}\|. \end{aligned}$$

The constants  $\gamma_1$  and  $\gamma_2$  defined as

$$\begin{aligned} \gamma_1^2 &= \int_0^\infty \bar{\sigma}(Ce^{At})^2 dt, \\ \gamma_2^2 &= \int_0^\infty \bar{\sigma}(\Lambda C A e^{At})^2 dt \end{aligned}$$

are bounds on  $\|f\|$  and  $\|\Lambda f\|$ , respectively. After some work with use of (6) we obtain the bound

$$\|v\| \leq \alpha_1 |x_0| + \alpha_2 \|g\|, \quad (9)$$

where

$$\alpha_1 = \frac{1}{\varepsilon} \left( c_1 + \sqrt{2c_1^2 + \varepsilon(c_3 + \gamma\bar{\sigma}(C)^2)} \right),$$

$$\alpha_2 = \frac{1}{\varepsilon} \left( c_2 + \sqrt{2c_2^2 + \varepsilon c_4} \right).$$

Hence, using (9) we get

$$\|y\| = \|G(v + g) + f\| \leq \rho_0 |x_0| + c_0 \|g\|,$$

where  $\rho_0 = \|G\|\alpha_1 + \gamma_1$  and  $c_0 = \|G\|(\alpha_2 + 1)$ . This is the bound in (8).

*Part 2:* Assume now that the system in (7) is stable when  $\tau = \tau_1$ , and consider the case when  $\tau = \tau_2$ . The system equations can be written as

$$y = Gu + f,$$

$$u = \underbrace{\Delta_{\tau_1}(y) + (\Delta_{\tau_2} - \Delta_{\tau_1})(y)}_v + g.$$

The well posedness assumption implies that for arbitrary  $x_0 \in \mathbf{R}^n$  and  $g \in \mathbf{L}_{2e}^n[0, \infty)$  there exists a solution  $y, v \in \mathbf{L}_{2e}^m[0, \infty)$ . Let  $y_T = P_T y$ ,  $g_T = P_T g$ , and define  $g_1 = (\Delta_{\tau_2} - \Delta_{\tau_1})(y_T) + g_T$ . It follows from the assumption on the parametrization of  $\Delta$  that  $g_1 \in \mathbf{L}_2^m[0, \infty)$ . If we let  $f$  and  $g_1$  be input signals to the system in (7) when  $\tau = \tau_1$ , then we get the system equations (where the loop signals are denoted  $y_1$ ,  $u_1$  and  $v_1$ )

$$y_1 = Gu_1 + f,$$

$$u_1 = \underbrace{\Delta_{\tau_1}(y_1)}_{v_1} + \underbrace{(\Delta_{\tau_2} - \Delta_{\tau_1})(y_T) + g_T}_{g_1}.$$

We notice that the causality assumption implies that  $y_T = P_T y_1$ . Furthermore, the assumed stability of the system in (7) when  $\tau = \tau_1$  implies that  $y_1, v_1 = \Delta_{\tau_1}(y_1) \in \mathbf{L}_2^m[0, \infty)$ . We obtain from (8) and the assumption on the parametrization of  $\Delta$  that

$$\begin{aligned} \|y_T\| &= \|P_T y_1\| \leq \|y_1\| \leq \rho_0 |x_0| + c_0 \|(\Delta_{\tau_2} - \Delta_{\tau_1})(y_T) + g_T\| \\ &\leq \rho_0 |x_0| + c_0 \kappa |\tau_2 - \tau_1| \cdot \|y_T\| + c_0 \|g_T\|. \end{aligned}$$

Let  $|\tau_2 - \tau_1| < 1/c_0 \kappa$ , then we get

$$\|y_T\| \leq \tilde{\rho}_0 |x_0| + \tilde{c}_0 \|g_T\|, \quad (10)$$

where  $\tilde{\rho}_0 = \rho_0 / (1 - c_0 \kappa |\tau_2 - \tau_1|)$  and  $\tilde{c}_0 = c_0 / (1 - c_0 \kappa |\tau_2 - \tau_1|)$  are positive.

A similar bound on  $u_T = P_T u$  can be obtained as follows. By causality we have  $u_T = P_T u_1 = P_T(v_1 + (\Delta\tau_1 - \Delta\tau_2)(y_T)) + g_T$ . Hence

$$\begin{aligned} \|u_T\| &\leq \|v_1\| + \kappa|\tau_2 - \tau_1| \cdot \|y_T\| + \|g_T\| \\ &\leq \alpha_1\|x_0\| + \alpha_2\|g_1\| + \kappa|\tau_2 - \tau_1| \cdot \|y_T\| + \|g_T\| \\ &\leq (\alpha_1 + (1 + \alpha_2)\kappa|\tau_2 - \tau_1|\tilde{\rho}_0) \cdot \|x_0\| + \\ &\quad (1 + \alpha_2)(1 + \kappa|\tau_2 - \tau_1|\tilde{c}_0) \cdot \|g_T\|, \end{aligned} \tag{11}$$

where the second inequality follows from (9) with  $g$  replaced by  $g_1$  and the last inequality follows by use of (10).

The bounds in (10) and (11) hold for any  $T \geq 0$ , which implies that the system in (7) is stable when  $\tau = \tau_1 + \Delta\tau$ , if  $\Delta\tau < 1/(c_0\kappa)$ . Iterative application of this conclusion from  $\tau = 0$ , where the system is assumed to be stable, up to  $\tau = 1$  in steps  $\Delta\tau < 1/(c_0\kappa)$  shows that the system in (7) is stable for all  $\tau \in [0, 1]$ .

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