Stability of Uncertain Systems with Hysteresis Nonlinearities

Jönsson, Ulf

1996

ISSN 0280–5316
ISRN LUTFD2/TFRT--7548--SE

Stability of Uncertain Systems with Hysteresis Nonlinearities

Ulf Jönsson

Department of Automatic Control
Lund Institute of Technology
May 1996

Department of Automatic Control
Lund Institute of Technology
Box 118
S-221 00 Lund Sweden

Document name
INTERNAL REPORT

Date of issue
May 1996

Document Number
ISRN LUTFD2/TFRT--7548--SE

Author(s)
Ulf Jönsson

Supervisor

Sponsoring organisation

Title and subtitle
Stability of Uncertain Systems with Hysteresis Nonlinearities

Abstract

Stability of systems with hysteresis nonlinearities, parametric uncertainty and finite dimensional unmodeled dynamics is considered. Conditions for exponential decay of the signals in the system to an equilibrium position are given. The equilibrium is generally not unique. The stability condition is given in terms of a frequency domain inequality, which involves the transfer function of the nominal system and a multiplier. The search for a suitable multiplier can be performed by use of convex optimization in terms of linear matrix inequalities. A simple example will be given that illustrates our result.

Key words
Exponential Stability, Hysteresis, Nonlinearities,Uncertainty

Classification system and/or index terms (if any)

Supplementary bibliographical information

ISSN and key title
0280–5316

ISBN

Language
English

Number of pages
16

Recipient's notes

Security classification

The report may be ordered from the Department of Automatic Control or borrowed through the University Library 2, Box 1010, S-221 03 Lund, Sweden, Fax +46 46 110019, Telex: 33248 lubbis lund.

1. Introduction

We consider stability of systems with hysteresis nonlinearities. A simple form of such systems is described by the equations

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\
y(t) &= Cx(t), \\
u(t) &= \varphi_h(y_{[0,t]}, t, u_0),
\end{aligned}
$$

where $\varphi_h(y_{[0,t]}, t, u_0)$ denotes the hysteresis function. Its value at time t is generally dependent on the time history of y, on t, and on the initial value $u(0) = u_0$. Stability of systems on this form has been studied in early work by Yakubovich, [12], [13], and by Barabanov and Yakubovich in [3]. We will extend a result in [13] to systems including hysteresis nonlinearities, parametric uncertainty and finite dimensional unmodeled dynamics. We use Integral Quadratic Constraints (IQC, see [9]) to obtain conditions that ensure exponential convergence of the signals in the system to an equilibrium point.

Hysteresis is common in mechanical components such valves and transmission mechanisms but also in electro mechanical components such as relays. Several frequently appearing hysteresis nonlinearities can be modeled as multi-valued functions. An example is given in Figure 1, which illustrates the nonlinear characteristic for a valve with backlash, see [1]. The multi-valued nature

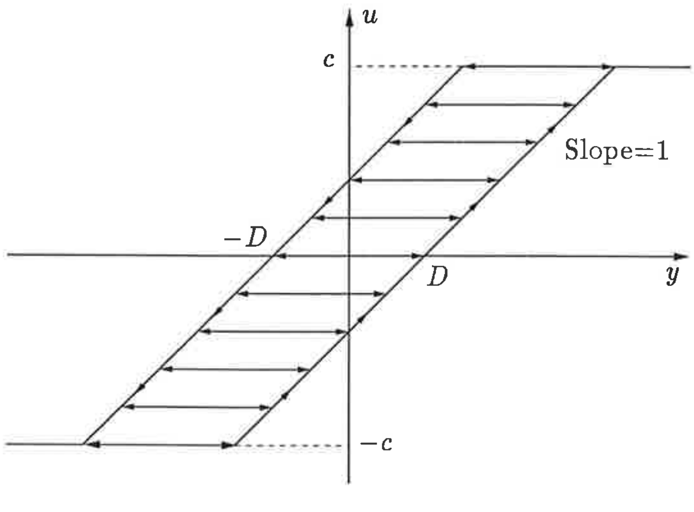

Figure 1 The graph for a valve with backlash.

of the nonlinearity can lead to energy storage, which may cause oscillations in the system. Limit cycles caused by hysteresis nonlinearities are sometimes desirable, as for example in auto-tuning, see [2]. However, in many cases the oscillations are undesirable and lead to unnecessary wear of the system components and to poor control performance. There are classical methods such as describing function analysis that can be used to predict limit cycles in systems with hysteresis. The result in this paper can be used to obtain conditions that guarantee that there cannot appear any oscillations in the system.

Notation and Preliminaries

I_n	An $n \times n$ identity matrix. The size of I is not always stated explicitly.
$\lvert \cdot \rvert$	The Euclidean norm $\lvert x \rvert = \sqrt{x^T x}$.
$\overline{\sigma}(M)$	The largest singular value of a real or complex matrix M.
$\mathbf{RL}_{\infty}^{n \times n}$	The space consisting of proper real rational matrix functions with no poles on the imaginary axis.
$\mathbf{RH}_{\infty}^{m \times m}$	The subspace of $\mathbf{RL}_{\infty}^{m \times m}$ consisting of functions with no poles in the closed right half plane. The norm is the usual H-infinity norm defined as $\lVert H \rVert_{\infty} = \sup_{\omega \in [0,\infty]} \overline{\sigma}(H)$.
$\mathbf{L}_2^m[0,\infty)$	The Lebesgue space of $\mathbf{R}^m$ valued signals with norm defined by $$\lVert u \rVert^2 = \int_0^{\infty} \lvert u(t) \rvert^2 dt.$$
$\mathbf{L}_{2e}^m[0,\infty)$	The vector space of functions f satisfying the condition that f_T defined as $$f_T = \begin{cases} f, & t \in [0,T] \\ 0, & t > T \end{cases}$$ is in $\mathbf{L}_2^m[0,\infty)$ for all $T > 0$.
$\theta(t)$	The unit step function defined as $$\theta(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0. \end{cases}$$
$\operatorname{diag}(\cdot,\cdot)$	If $\Delta_i : \mathbf{L}_{2e}^{m_i}[0,\infty) \to \mathbf{L}_{2e}^{m_i}[0,\infty)$, for $i = 1,2$, then the operator $\operatorname{diag}(\Delta_1, \Delta_2) : \mathbf{L}_{2e}^{m_1+m_2}[0,\infty) \to \mathbf{L}_{2e}^{m_1+m_2}[0,\infty)$ is defined by the input/output relation $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \Delta_1(u_1) \\ \Delta_2(u_2) \end{bmatrix}$$ where $y_i, u_i \in \mathbf{L}_{2e}^{m_i}[0,\infty)$ for $i = 1,2$.

If $y : \mathbf{R}^+ \to \mathbf{R}^m$ is absolutely continuous, then the time derivative $\dot{y} := \frac{d}{dt}y$ exists as a measurable function that is bounded almost everywhere. Furthermore, an absolutely continuous function is the indefinite integral of its derivative, i.e. $y(t) = \int_0^t \dot{y} d\tau + y(0)$, see [10]. It is easy to see that y is absolutely continuous on every finite interval $[0,T]$ if and only if $y, \dot{y} \in \mathbf{L}_{2e}^m[0,\infty)$.

The following obvious fact will be used frequently in the paper.

Fact: Let H be a convolution operator with transfer function realization $H(s) = C(sI - A)^{-1}B + D \in \mathbf{RH}_{\infty}^{m \times m}$, and let $u(t)$ be an absolutely continuous input signal defined for $t \geq 0$. Then the following two system representations are equivalent in the sense that they give the same absolutely continuous output signals $y(t)$ for $t \geq 0$.

1. The operator representation

$$y(t) = (Hu)(t) + Ce^{At}x_0\theta(t), \tag{1}$$

where $(Hu)(t) = (h * u)(t)$. Here $h * u$ denotes convolution of u and the kernel $h(t) = Ce^{At}B\theta(t) + D\delta(t)$, where $\delta(t)$ is the usual impulse distribution.

2. The state space representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{2}$$

2. Hysteresis Nonlinearities

The hysteresis functions used in this paper are defined as in [12], [13], and [3]. The input-output relation of the hysteresis function is denoted $u(t) = \phi(y_{[t_0,t]}, t, u_0)$, where we use the notation $y_{[t_0,T]}$ for the truncated signal

$$y_{[t_0,T]}(t) = \begin{cases} y(t), & t \in [t_0, T]. \\ 0, & t > T. \end{cases}$$

The initial values of the input and output are denoted $y_0 = y(t_0)$ and $u_0 = u(t_0)$. The output of the hysteresis nonlinearity is thus dependent on the time history of y, on t, and on the initial output. The hysteresis nonlinearity is *static* if there is no time dependence, i.e. if the second argument of φ_h can be omitted. The hysteresis functions are assumed to satisfy the following conditions:

1. The initial output value $u_0 \in \mathbf{R}$ belongs to a closed and bounded set $E(y_0)$, which depends on the initial input value y_0. This means that E is a set-valued function.

2. For every absolutely continuous input y and initial condition $u_0 \in E(y_0)$, the output u is an absolutely continuous function with $u(t) \in E(y(t))$ for all $t \geq t_0$.

3. The *semi-group* condition

$$\varphi_h(y_{[t_0,t]}, t, u_0) = \varphi_h(y_{[t_1,t]}, t, \varphi_h(y_{[t_0,t_1]}, t_1, u_0)).$$

4. The slope condition

$$\alpha(\frac{dy}{dt})^2 \leq \frac{d\varphi_h(y_{[t_0,t]}, t, u_0)}{dt}\frac{dy}{dt} \leq \beta(\frac{dy}{dt})^2 \tag{3}$$

holds for almost all $t \geq t_0$, where it is assumed that $0 < \alpha \leq 0 < \beta < \infty$.

5. The continuity condition: If $y(t) \to y_\infty$, and $\varphi_h(y_{[t_0,t]}, t, u_0) \to u_\infty$ as $t \to \infty$, then $u_\infty \in E(y_\infty)$ and $\varphi_h(y_\infty, t, u_\infty) = u_\infty$ for all $t \geq t_0$.

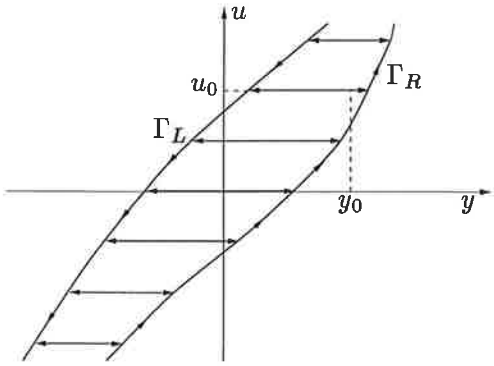

Figure 2 The nonlinear characteristic of a generalized play.

We note that all the conditions above hold for a static nonlinearity $\varphi : \mathbf{R} \to \mathbf{R}$, which satisfies the slope condition

$$\alpha \leq \frac{\varphi(y_1) - \varphi(y_2)}{y_1 - y_2} \leq \beta, \quad y_1 \neq y_2,$$

where $-\infty < \alpha \leq 0 < \beta < \infty$. Our result is therefore applicable to systems with such nonlinearities.

Multi-valued nonlinearities can ofted be used to define hysteresis functions. An example is the *generalized play* with the graph in Figure 2, see [7]. The generalized play is characterized by two continuous nondecreasing functions $\Gamma_L(y)$ and $\Gamma_R(y)$, defined on the intervals $(-\infty, a_l)$ and (a_r, ∞) respectively, where $\Gamma_L(y) \geq \Gamma_R(y)$ for all $y \in (a_r, a_l)$. The signal $(y(t), u(t))$ moves along a horizontal line until it reaches any of these two curves. If it hits for example Γ_R then it proceeds along Γ_R until $y(t)$ is decreasing. Then the motion continues along a horizontal line again. The generalized play is static and it satisfies property 1-5 under weak conditions on the functions Γ_L and Γ_R, see [7].

DEFINITION 1
We say that a hysteresis nonlinearity is bounded if there exists $\kappa > 0$ such that $|\varphi_h(y_{[t_0,t]}, t, u_0)| \leq \kappa |y(t)|$ for all $t \geq t_0$, $u_0 \in \mathbf{R}$ and y. □

The generalized play in Figure 2 is unbounded since the hysteresis loop includes a neighborhood of the origin. It is for this reason not a bounded operator on $\mathbf{L}_2[0, \infty)$.

The valve with the graph in Figure 1 defines an unbounded hysteresis nonlinearity that satisfies property 1-5.

3. Problem Formulation

We consider stability of the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \\ u(t) &= \Delta(y_{[0,t]}, t, u_0) + g(t), \end{aligned} \tag{4}$$

where it is assumed that $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, and $C \in \mathbf{R}^{m\times n}$ and $D \in \mathbf{R}^{m\times m}$. The matrix A is assumed to be Hurwitz and the operator Δ has the diagonal structure

$$\Delta = \operatorname{diag}(\varphi_h, \Delta_1, \Delta_2), \tag{5}$$

where

1. $\varphi_h = \operatorname{diag}(\varphi_{h1}, \dots, \varphi_{hL})$ consists of hysteresis nonlinearities. At least one is unbounded and they are assumed to satisfy the slope condition in (3) with $-\infty < \alpha_l \le 0 < \beta_l < \infty$.
2. Δ_1 is a real parametric uncertainty block with the diagonal structure $\Delta_1 = \operatorname{diag}(\delta_1 I_{m_1}, \dots, \delta_M I_{m_M})$, where $\delta_i \in [-1,1]$, $i = 1, \dots, M$ are constant.
3. Δ_2 is used to represent finite dimensional unmodeled dynamics with the structure

$$\Delta_2 = \operatorname{diag}(\Delta_{21}, \dots, \Delta_{2N}), \tag{6}$$

where we assume that $\Delta_{2i} \in \{\Delta \in \mathbf{RH}_\infty^{n_i\times n_i} : \|\Delta\|_\infty \le 1\}$ for $i = 1, \dots, N$. We can assume that Δ_2 has a state space realization $\Delta_2(s) = C_\Delta(sI - A_\Delta)^{-1}B_\Delta + D_\Delta$, where the structure of $A_\Delta, B_\Delta, C_\Delta, D_\Delta$ is consistent with the diagonal structure in (6). Note that A_Δ is Hurwitz and of unknown size. It is no restriction to assume that Δ_2 is initially at rest since the response of an initial condition can be contained in the input signal g. If the initial state is $x_{\Delta 0}$, then we let $g(t) = (\,0 \quad (C_\Delta e^{A_\Delta t} x_{\Delta 0})^T\,)^T\,\theta(t)$.

For consistency of the dimensions in the definition of Δ we need

$$L + \sum_{i=1}^{M} m_i + \sum_{i=1}^{N} n_i = m.$$

We finally assume that the direct term of the nominal system in (4) has the structure

$$D = (\,0_{m\times L} \quad D_0\,), \quad \text{where } D_0 \in \mathbf{R}^{m\times(m-L)}. \tag{7}$$

This ensures that there are no algebraic loop around the nonlinearities in Δ.

We will next give an equivalent representation for the system in (4). Assume that $g(t) = (\,0 \quad (C_\Delta e^{A_\Delta t} x_{\Delta 0})^T\,)^T\,\theta(t)$. Then it follows that the system representation in (4) is equivalent with the representation

$$\begin{aligned}
\begin{bmatrix} \dot{x}(t) \\ \dot{x}_\Delta(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & A_\Delta \end{bmatrix} \begin{bmatrix} x(t) \\ x_\Delta(t) \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B_\Delta \end{bmatrix} \begin{bmatrix} u(t) \\ y_3(t) \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ x_\Delta(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_{\Delta 0} \end{bmatrix}, \\
y(t) &= Cx(t) + Du(t), \\
u(t) &= \begin{bmatrix} \varphi_h(y_{1[0,t]}, t, u_{10}) \\ \Delta_1 y_2(t) \\ C_\Delta x_\Delta(t) + D_\Delta y_3(t) \end{bmatrix},
\end{aligned} \tag{8}$$

where the partition $y = (\,y_1^T, y_2^T, y_3^T\,)$ is consistent with the block structure in (5) and where $u_{10} = u_1(0)$ is the component of u that corresponds to the hysteresis nonlinearity.

We assume that the system in (8) satisfies the following well-posedness condition

Definition 2—Well-posedness

The system in (8) is *well-posed* if for every initial condition $(x_0, x_{\Delta 0}, u_0)$, there exists a unique absolutely continuous solution x, x_Δ, u on every finite interval $[0, T]$. □

The hysteresis nonlinearities in φ_h are assumed to be unbounded as explained above. This means that we cannot expect to prove stability in the sense that finite energy input signals gives finite energy output signals or in terms of convergence of the signals to the origin. In fact: There is in general no unique stationary point for the system in (8) but rather a stationary set. This set may contain an infinite number of points, as for example in a system containing the generalized play in Figure 2. The stationary set $\mathcal{S}$ for the system in (8) is defined as

$$\mathcal{S} = \{(x_0, x_{\Delta 0}, u_0) : \text{ such that (10) holds}\}, \tag{9}$$

where (10) is the equation system

$$\begin{aligned} y_0 &= [CA^{-1}B + D]u_0, \\ u_0 &= \begin{bmatrix} \varphi_h(y_{10}, t, u_{10}) \\ \Delta_1 y_{20} \\ \Delta_2(0) y_{30} \end{bmatrix}, \quad \forall t \geq 0, \\ \begin{bmatrix} x_0 \\ x_{\Delta 0} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & A_\Delta \end{bmatrix}^{-1} \begin{bmatrix} B & 0 \\ 0 & B_\Delta \end{bmatrix} \begin{bmatrix} u_0 \\ y_{30} \end{bmatrix}. \end{aligned} \tag{10}$$

The partition $y_0 = [\, y_{10}^T, y_{20}^T, y_{30}^T \,]$ is consistent with the block structure in (5) and where $u_{10} = u_1(0)$ is the component of u_0 that corresponds to the hysteresis nonlinearity. We note that $\mathcal{S}$ cannot be defined in advance since Δ_1 and Δ_2 are not known.

The best we can hope for is to derive conditions for exponential convergence of the state vector to the stationary set $\mathcal{S}$.

Definition 3—Exponential Stability

The system in (8) is *exponentially stable* to $\mathcal{S}$ if for every initial condition $(x_0, x_{\Delta 0}, u_0)$ there are constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and $(x_\infty, x_{\Delta_\infty}, u_\infty) \in \mathcal{S}$ such that

$$\begin{aligned} |\tilde{x}(t) - \tilde{x}_\infty| &\leq \alpha_1 e^{-\beta_1 t} |\tilde{x}_0|, \\ |u(t) - u_\infty| &\leq \alpha_2 e^{-\beta_2 t} |u_0|, \end{aligned}$$

for all $t \geq 0$, where $\tilde{x} = (\, x^T, x_\Delta^T \,)^T$. □

This definition of stability implies in particular that the signals are bounded and that there cannot be any limit cycles.

4. Main Result

We derive conditions for stability of the system (8) in this section. Our means for doing this is to give conditions for exponential decay to zero of the time derivative of the state vector. This idea was first suggested by Yakubovich in [13] where he applied it to a system with one hysteresis nonlinearity.

We need to find a description of the relationship between the derivative of the input signal and the derivative of the output signal of the operator Δ in (5). We use descriptions in terms of IQCs.

The notation $\mathcal{G}_{D\Delta}$ is used to denote the set of all possible pairs of differentiated input and output signals for the operator Δ. We define

$$\mathcal{G}_{D\Delta} = \{(\nu, \sigma) : \nu = \dot{u},\ \sigma = \dot{y},\ \text{where } u(t) = \Delta(y_{[0,t]}, t, u_0),\ y, \dot{y} \in \mathbf{L}_{2e}^m[0,\infty)\}.$$

We note that given the initial condition (y_0, u_0), then $\dot{y}$ and $\dot{u}$ are related through a linear and time-varying operator $D\Delta$.

The following definition of IQC for $\mathcal{G}_{D\Delta}$ is similar to the definition in [9].

DEFINITION 4
We say that $\mathcal{G}_{D\Delta}$ satisfies the IQC defined by the multiplier $\Pi = \Pi^* \in \mathbf{RL}_\infty^{2m\times 2m}$ if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{\sigma}(j\omega) \\ \widehat{\nu}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{\sigma}(j\omega) \\ \widehat{\nu}(j\omega) \end{bmatrix} \geq 0,$$

for all $(\nu, \sigma) \in \mathcal{G}_{D\Delta} \cap \mathbf{L}_2^m[0,\infty) \times \mathbf{L}_2^m[0,\infty)$. Here $\widehat{\sigma}$ and $\widehat{\nu}$ denotes the Fourier transforms of σ and ν, respectively. □

Next follows multiplier descriptions for the block components of Δ.

Hysteresis Nonlinearity: Let φ_h be a hysteresis nonlinearity with

$$\alpha(\frac{dy}{dt})^2 \leq \frac{d\varphi_h(y_{[0,t]}, t, u_0)}{dt}\frac{dy}{dt} \leq \beta(\frac{dy}{dt})^2,$$

almost everywhere for some $-\infty < \alpha \leq 0 < \beta < \infty$. Then $\mathcal{G}_{D\varphi_h}$ can be described by the multiplier

$$\Pi(j\omega) = \begin{bmatrix} -2\beta\alpha & \beta+\alpha \\ \beta+\alpha & -2 \end{bmatrix}. \tag{11}$$

Parametric uncertainty: If $u = \delta y$, where δ is an uncertain real-valued parameter with $\delta \in [-1,1]$, then $\dot{u} = \delta\dot{y}$. This means that $\mathcal{G}_{D\delta I}$ satisfies the IQC defined by the multiplier

$$\Pi(j\omega) = \begin{bmatrix} X(j\omega) & Y(j\omega) \\ Y(j\omega)^* & -X(j\omega) \end{bmatrix},$$

where $X(j\omega) = X(j\omega)^* \geq 0$ and $Y(j\omega) = -Y(j\omega)^*$ for all $\omega \in \mathbf{R}$.

Unmodeled LTI Dynamics: If $u = \Delta y$, where $\Delta \in \{\Delta \in \mathbf{RH}_\infty^{m\times m} : \|\Delta\|_\infty \leq 1\}$, then $\dot{u} = \Delta\dot{y}$. Hence, $\mathcal{G}_{D\Delta}$ satisfies the IQC defined by the multiplier

$$\Pi(j\omega) = \begin{bmatrix} x(j\omega)I_m & 0 \\ 0 & -x(j\omega)I_m \end{bmatrix},$$

where $x(j\omega) = \overline{x(j\omega)} \geq 0$ for all $\omega \in \mathbf{R}$.

Combination of Multipliers: We can now obtain a multiplier description of $\mathcal{G}_{D\Delta}$ by combining the above multiplier descriptions. We use the following rule: Assume Δ has the block-diagonal structure $\Delta = \text{diag}(\Delta_1, \Delta_2)$, and that $\mathcal{G}_{D\Delta_i}$ satisfies the IQC defined by Π_i, $i = 1, 2$. Then $\mathcal{G}_{D\Delta}$ satisfies the IQC defined by $\Pi = \text{daug}(\gamma_1\Pi_1, \gamma_2\Pi_2)$, where $\gamma_1, \gamma_2 \geq 0$, and where the operation daug is defined as follows: If

$$\Pi_i = \begin{bmatrix} \Pi_{i1} & \Pi_{i2} \\ \Pi_{i2}^* & \Pi_{i3} \end{bmatrix}, \quad i = 1, 2,$$

where the block structures are consistent with the size of Δ_1 and Δ_2, respectively, then

$$\text{daug}(\Pi_1, \Pi_2) = \left[\begin{array}{cc|cc} \Pi_{11} & 0 & \Pi_{12} & 0 \\ 0 & \Pi_{21} & 0 & \Pi_{22} \\ \hline \Pi_{12}^* & 0 & \Pi_{13} & 0 \\ 0 & \Pi_{22}^* & 0 & \Pi_{23} \end{array}\right].$$

This rule holds for any finite number of diagonal blocks in Δ.

We can now formulate our main result

THEOREM 1
Assume that

1. The system in (8) is well posed.
2. $\mathcal{G}_{D\Delta}$ satisfies the IQC defined by Π.
3. The inequality

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0$$

holds for all $\omega \in [0, \infty]$, where $G(s) = C(sI - A)^{-1} + D$.

Then the system in (8) is exponentially stable as defined in Definition 3.

Proof: The proof is given in the Appendix. □

REMARK 1
Note that there is no guaranteed rate of exponential convergence. It is in general dependent on the particular unmodeled dynamics part Δ_2 of the system in (4). Further constraints in terms of for example pole locations of Δ_2 is needed to obtain a worst case convergence rate.

REMARK 2
The search for a suitable multiplier such that the third condition of Theorem 1 holds can be performed by use of convex optimization in terms linear matrix inequalities. To do this we parametrize a finite dimensional convex set of multipliers. The corresponding frequency domain condition can be transformed to an equivalent Linear Matrix Inequality. Numerical search for a suitable multiplier can then be done by use of efficient algorithms for solution of linear matrix inequalities, see e.g. [5] and [4]. A suitable format for the parametrization of the multipliers was proposed in for example, [6].

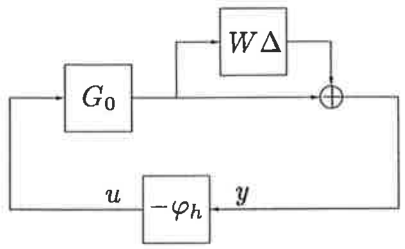

Figure 3 System with nominal plant G, unmodeled dynamics $W\Delta$, and a hysteresis nonlinearity φ_h.

5. Example

We will in this section apply Theorem 1 to a simple example and discuss a possible improvement of the theorem.

EXAMPLE 1
Consider the system in Figure 3 where the nominal plant and the weighting function are

$$G_0(s) = \frac{s+1}{s^2+0.2s+1}, \quad W(s) = \frac{s}{s+10},$$

and where $\Delta \in \{\Delta \in \mathbf{RH}_\infty : \|\Delta\|_\infty \leq 1\}$. The hysteresis nonlinearity φ_h represents a valve with the characteristic in Figure 1, see [1].

We can collect φ_h and Δ in the block diagonal operator $\mathrm{diag}(\varphi_h, \Delta)$. Theorem 1 can then be applied with

$$G(s) = \begin{bmatrix} -G_0(s) & W(s) \\ -G_0(s) & 0 \end{bmatrix},$$

and

$$\Pi(j\omega) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & x(j\omega) & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & -x(j\omega) \end{bmatrix},$$

where $x(j\omega) \geq 0$ for all $\omega \in \mathbf{R}$. It is easy to see that the third condition of the theorem reduces to the existence of $x \in \mathbf{RL}_\infty$ with $x(j\omega) > 0$, $\forall \omega \in \mathbf{R}$ such that

$$\mathrm{Re}\; G_0(j\omega) - \frac{1}{2}\left(|G_0(j\omega)|^2 x(j\omega) + \frac{|W(j\omega)|^2}{x(j\omega)}\right) > -1, \quad \forall \omega \in [0, \infty].$$

This can be shown to be equivalent with the condition that

$$\mathrm{Re}\; G_0(j\omega)\,(1 + W(j\omega)\Delta(j\omega)) > -1, \quad \forall \omega \in [0, \infty] \qquad (12)$$

holds for all $\Delta \in \{\Delta \in \mathbf{RH}_\infty : \|\Delta\|_\infty \leq 1\}$, which of course is expected.

Figure 4 shows the Nyquist curve of G_0 with uncertainty circles that corresponds to the unmodeled dynamics. We see that the stability criterion in (12)

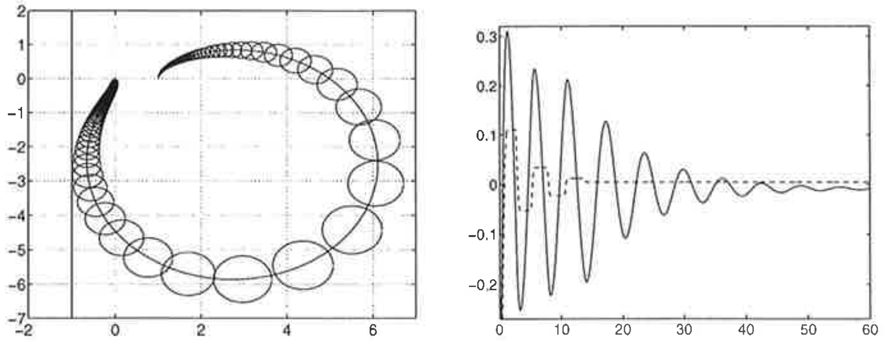

Figure 4 The left plot shows the Nyquist curve for the nominal system G_0 together with uncertainty circles with radius $|G_0(j\omega)W(j\omega)|$. The right diagram shows a simulation of the system when $\Delta = -1$. The solid line corresponds to y and the dashed line is u.

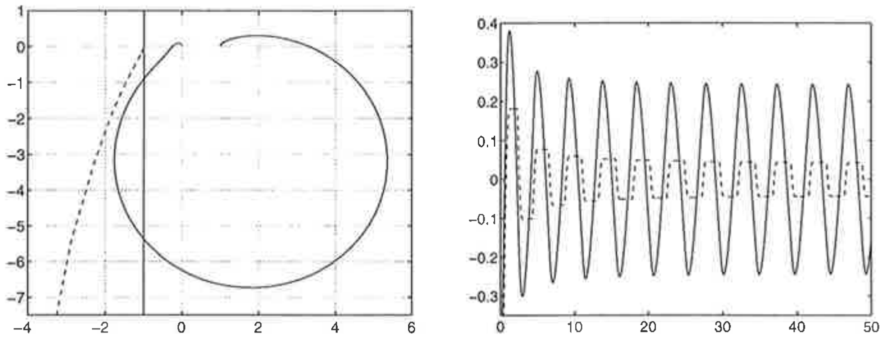

Figure 5 The left plot shows the Nyquist curve for $G_0(1 + W\Delta)$ when $\Delta = -3.2$ in solid line. The describing function for φ_h represented with the dashed curve. The right diagram shows a simulation of the system when $\Delta = -3.2$. The solid line corresponds to y and the dashed line is u.

is satisfied. The simulation in the same figure shows that the signals y and u defined in Figure 3 converges. We used $\Delta = -1$, $c = 5$, $D = 0.2$, and minimal state space representations of G_0 and W for the simulation.

The stability criterion in (12) is conservative. We can allow a larger uncertainty block before the system becomes unstable. Simulations show that the uncertainty used for generation of Figure 4 can be increased to $\Delta = -3.2$ before the system becomes unstable. Figure 5 shows that there appears a stable limit cycle with this Δ. The describing function for φ_h is also given in the Figure.

The describing function method is an approximate method that is used to predict limit cycles in nonlinear systems, see for example [11]. Intersection (or as in our case close to intersection) of the describing function and the Nyquist curve indicate that there may appear a limit cycle in the system. □

The example showed that our stability criterion in general is conservative. This is to be expected since we use a minimum of information in the description of the hysteresis nonlinearity. An improvement would be to include the information that the hysteresis nonlinearity circulates the origin counter clockwise. This can be described by the unbounded multiplier

$$\Pi(j\omega) = \begin{bmatrix} 0 & \frac{1}{j\omega} \\ -\frac{1}{j\omega} & 0 \end{bmatrix}.$$

This multiplier would not contribute to the analysis in our example, since it only gives negative phase delay. However, the multiplier is useful in general. It has been used for analysis of systems with a single hysteresis nonlinearity in [3].

6. Concluding Remarks

There are several possible ways to improve the result in this paper. The IQC description of the hysteresis nonlinearity is generally crude since we only use the sector condition for the slopes of the hysteresis nonlinearity. One extension would be to use the information on the circulation of the hysteresis loop as was discussed in Section 5, see also [3]. Another possible extension would be to allow more general dynamic uncertainty blocks. It may also be possible to consider hysteresis nonlinearities that are discontinuous in the sense that the slope may be infinitely steep. This was treated in [13] and [3].

Appendix: Proof of Theorem 1

The next lemma will be used in the proof. A similar lemma was used in [8].

LEMMA 1
Consider the system

$$\dot{x} = (A + \tau B\Lambda(t)C)x, \quad x(0) = x_0. \tag{13}$$

where $A \in \mathbf{R}^{n\times n}$ is Hurwitz and where $\Lambda : \mathbf{R}^+ \to \mathbf{R}^{m\times m}$ is measurable with $\Lambda(t) \in \Omega$, $\forall t \geq 0$ for some bounded set $\Omega \in \mathbf{R}^{m\times m}$.

Assume that there exists a positive constant c such that

$$\int_t^\infty |x(\tau)|^2 d\tau \leq c|x(t)|^2 \tag{14}$$

for every $\tau \in [0,1]$ such that the solution to (13) is in $\mathbf{L}_2^n[0,\infty)$ for arbitrary initial condition $(t, x(t)) \in \mathbf{R}^+ \times \mathbf{R}^n$. Then the system in (13) is exponentially stable to the origin for all $\tau \in [0,1]$.

Proof: Assume that we have exponential stability for some $\tau_0 \in [0,1]$. Then clearly $x \in \mathbf{L}_2^n[0,\infty)$ and we can define the Lyapunov function candidate

$$V(t,x) = x^T P_0(t) x = \int_t^\infty |x(\tau)|^2 d\tau. \tag{15}$$

It is clear that $P_0(t)$ is symmetric and differentiable. The next two properties of P_0 show that (15) defines a Lyapunov function:

1. P_0 satisfies the Lyapunov equation

$$(A + \tau_0 B\Lambda(t)C)^T P_0(t) + P_0(t)(A + \tau_0 B\Lambda(t)C) + \dot{P}_0(t) < -I,$$

for all $t \geq 0$. This follows from differentiation of (15).

2. P_0 is bounded from below and above. In other words there exist constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 I \leq P_0(t) \leq c_2 I, \quad \forall t \geq 0.$$

From (14) it follows that we can take $c_2 = c$. For the lower bound we notice that the boundedness of Ω implies that there is a constant $c_1 > 0$ such that $c_1 \frac{d}{dt}|x(t)|^2 \geq -|x(t)|^2$. Hence

$$\int_t^\infty |x(\tau)|^2 d\tau \geq c_1|x(t)|^2 - \lim_{\tau\to\infty} |x(\tau)|^2 = c_1|x(t)|^2,$$

since x decays exponentially to zero.

We can in a standard way use the Lyapunov function in (15) to show that $|x(t)| \leq \alpha e^{-\beta t}|x_0|$ for all $t \geq 0$, where $\alpha^2 = c_2/c_1$ and $\beta = 1/(2c_2)$.

The boundedness of Ω and P_0 implies that there exists $\delta > 0$ such that

$$\delta |P_0(t)B\Lambda(t)C| < \frac{1}{3}, \quad \forall t \geq 0.$$

If we consider the system in (13) for a $\tau \in [0, 1]$ such that $|\tau - \tau_0| \leq \delta$, then we obtain

$$\begin{aligned}\frac{d}{dt}V(t,x) =& 2x^T P_0(t)(A + \tau B\Lambda(t)C)x + x^T \dot{P}_0(t)x \\ =& 2x^T P_0(t)(A + \tau_0 B\Lambda(t)C)x + x^T \dot{P}_0(t)x + \\ & 2(\tau - \tau_0)x^T P_0(t)B\Lambda(t)Cx \leq -\frac{1}{3}|x|^2.\end{aligned}$$

Hence the system is exponentially stable for all $\tau \in [0, 1]$ such that $|\tau - \tau_0| \leq \delta$.

The system is exponentially stable when $\tau = 0$. Iterative application of the above argument in steps $\Delta\tau \leq \delta$ shows that the system is exponentially stable for all $\tau \in [0, 1]$. □

Now consider the system in (8). The sub-multiplicativity of φ_h implies that we can regard any $t_0 \geq 0$ as the initial time. We will next differentiate the signals in (8). Let $\chi = \dot{x}$, $\chi_\Delta = \dot{x}_\Delta$, $\sigma = \dot{y}$ and $\zeta = \dot{u}$. Then we obtain

$$\begin{aligned}&\begin{bmatrix}\dot{\chi} \\ \dot{\chi}_\Delta\end{bmatrix} = \begin{bmatrix}A & 0 \\ 0 & A_\Delta\end{bmatrix}\begin{bmatrix}\chi \\ \chi_\Delta\end{bmatrix} + \begin{bmatrix}B & 0 \\ 0 & B_\Delta\end{bmatrix}\begin{bmatrix}\zeta \\ \sigma_3\end{bmatrix}, \quad \begin{bmatrix}\chi(t_0) \\ \chi_\Delta(t_0)\end{bmatrix} = \begin{bmatrix}Ax(t_0) \\ A_\Delta x_\Delta(t_0)\end{bmatrix}, \\ &\sigma = C\chi + D\zeta, \\ &\zeta = \begin{bmatrix}\nu_1 \\ \Delta_1\sigma_2 \\ C_\Delta\chi_\Delta + D_\Delta\sigma_3\end{bmatrix},\end{aligned} \tag{16}$$

where the decomposition $\sigma = (\sigma_1^T, \sigma_2^T, \sigma_3^T)^T$ is consistent with the block structure in (5), and where $(\nu_1, \sigma_1) \in \mathcal{G}_{D\varphi_h}$.

The idea is to use Lemma 1 to show exponential decay of the state vector $(\chi^T \quad \chi_\Delta^T)^T$ in (16) to zero. To do this we will reformulate the system equations in (16). We first note that a pair $(\nu_1, \sigma_1) \in \mathcal{G}_{D\varphi_h}$ can be represented as $\nu_1(t) = \mu_1(t)\sigma_1(t)$, where $\mu_1 = \mathrm{diag}(\mu_{11}, \dots, \mu_{1L})$ is measurable on $\mathbf{R}^+$ such that $\mu_{1l}(t) \in [\alpha_l, \beta_l]$ for all $t \geq 0$.

If we use this representation and also scale the right hand side of the last equation in (16) with a parameter $\tau \in [0,1]$, then we obtain

$$\begin{bmatrix} \dot{\chi} \\ \dot{\chi}_\Delta \end{bmatrix} = \left(\begin{bmatrix} A & 0 \\ 0 & A_\Delta \end{bmatrix} + \tau \begin{bmatrix} B & 0 \\ 0 & B_\Delta \end{bmatrix} \Lambda(t) \widetilde{C} \right) \begin{bmatrix} \chi \\ \chi_\Delta \end{bmatrix},$$

where the initial condition is as in (16), and where $\widetilde{C}$ and $\Lambda(t)$ are defined as

$$\Lambda(t) = \begin{bmatrix} \Upsilon(t)(I - D\Upsilon)^{-1} & I \\ E(I - D\Upsilon)^{-1} & 0 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} C & DC_{\Delta 0} \\ 0 & C_{\Delta 0} \end{bmatrix},$$

and where

$$\begin{aligned} \Upsilon(t) &= \mathrm{diag}(\mu_1(t), \Delta_1, D_\Delta), \\ C_{\Delta 0}^T &= (0 \quad 0 \quad C_\Delta^T)^T, \\ E &= (0 \quad 0 \quad I). \end{aligned}$$

The partitions of $C_{\Delta 0}$ and E are consistent with the block structure of Δ in (5). We note that $(I - D\Upsilon)^{-1}$ is a constant and bounded matrix. This follows from the structure of D defined in (7) and the well-posedness assumption. For $\tau = 1$ this system is equivalent with (16) and for $\tau = 0$ it is exponentially stable since A and A_Δ are Hurwitz. If we show that condition (14) in Lemma 1 holds, then we can infer that (16) is exponentially stable. In order to do this we rewrite (16) in an equivalent operator form. We scale with τ also in this case.

$$\begin{aligned} \sigma &= G\zeta + Ce^{A(t-t_0)}\chi(t_0)\theta(t - t_0), \\ \zeta &= \tau\widehat{\Delta}\sigma + \begin{bmatrix} 0 \\ C_\Delta e^{A_\Delta(t-t_0)}\chi_\Delta(t_0)\theta(t - t_0) \end{bmatrix}, \end{aligned} \tag{17}$$

where $\widehat{\Delta}$ is the linear time-varying operator

$$\widehat{\Delta} = \mathrm{diag}(\mu_1, \Delta_1, \Delta_2),$$

which satisfies the IQC defined by Π since $\mathcal{G}_{D\Delta}$ do so.

The next step is to use a similar argument as in [9] to obtain the desired bound in (14). We use the notation $f(t) = Ce^{A(t-t_0)}\chi(t_0)\theta(t - t_0)$ and $g(t) = (0 \quad (C_\Delta e^{A_\Delta(t-t_0)}\chi_\Delta(t_0)\theta(t - t_0))^T)^T$. Furthermore, the quadratic form $Q(\sigma, \nu)$ is defined as

$$Q(\sigma, \nu) = \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{\sigma}(j\omega) \\ \widehat{\nu}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{\sigma}(j\omega) \\ \widehat{\nu}(j\omega) \end{bmatrix} d\omega.$$

Let $\nu = \widehat{\Delta}\sigma$. Then for every $\tau \in [0,1]$ such that the solution (ζ, σ) to (17) is in $\mathbf{L}_2^m[t_0, \infty) \times \mathbf{L}_2^m[t_0, \infty)$ we obtain

$$\begin{aligned} 0 \leq Q(\sigma, \tau\nu) &= Q(G\tau\nu, \tau\nu) + Q(\sigma, \tau\nu) - Q(G\tau\nu, \tau\nu) \\ &\leq -\varepsilon\|\tau\nu\|^2 + 2(c_1\|f\| + c_2\|g\|) \cdot \|\tau\nu\| + c_3\|f\|^2 + c_4\|g\|^2, \end{aligned}$$

where we used that $\sigma = G(\tau\nu + g) + f$. The first inequality follows since $\tau\widehat{\Delta}$ satisfies the IQC defined by Π for every $\tau \in [0, 1]$. This is a consequence of the sector condition on μ_1. The first term on the right hand side of the last inequality is due to the third condition of the theorem statement. The other terms give a bound on $|Q(\sigma, \tau\nu) - Q(G\tau\nu, \tau\nu)|$. If $\|\Pi_{ij}\| = \sup_\omega \overline{\sigma}(\Pi_{ij}(j\omega))$ denote the norms of the blocks in the partitioning

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$$

of Π into $m \times m$ blocks, then the constants can be taken as

$$\begin{aligned} c_1 &= \|\Pi_{11}\| \cdot \|G\| + \|\Pi_{12}\|, & c_3 &= 2\|\Pi_{12}\| \\ c_2 &= \|\Pi_{11}\| \cdot \|G\|^2 + \|\Pi_{12} \cdot \|G\|, & c_4 &= 2\|\Pi_{12}\| \cdot \|G\|^2. \end{aligned}$$

We obtain the bound

$$\|\tau\nu\| \leq \frac{1}{\varepsilon}(c_1 + \sqrt{2c_1^2 + \varepsilon c_3})\|f\| + \frac{1}{\varepsilon}(c_2 + \sqrt{2c_2^2 + \varepsilon c_4})\|g\|,$$

which implies that there exists $\gamma_0 > 0$ such that

$$\begin{aligned} \|\zeta\| &= \|\tau\nu + g\| \leq \gamma_0(\|f\| + \|g\|), \\ \|\sigma\| &= \|G\zeta + f\| \leq \gamma_0(\|f\| + \|g\|). \end{aligned}$$

Next we notice that $\|f\| \leq \gamma_1|\chi(t_0)|$ and $\|g\| \leq \gamma_2|\chi_\Delta(t_0)|$, where

$$\begin{aligned} \gamma_1^2 &= \int_0^\infty \overline{\sigma}(Ce^{At})^2 dt, \\ \gamma_2^2 &= \int_0^\infty \overline{\sigma}(C_\Delta e^{A_\Delta t})^2 dt. \end{aligned}$$

If we let

$$\widetilde{\chi} = \begin{bmatrix} \chi \\ \chi_\Delta \end{bmatrix}, \quad \widetilde{A} = \begin{bmatrix} A & 0 \\ 0 & A_\Delta \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B & 0 \\ 0 & B_\Delta \end{bmatrix},$$

then the above bounds on $\|\sigma\|$ and $\|\zeta\|$ can be used to derive the bound

$$\begin{aligned} \|\widetilde{\chi}\| &= \|(sI - \widetilde{A})^{-1}\widetilde{B} \begin{bmatrix} \sigma \\ \zeta \end{bmatrix} + e^{\widetilde{A}(t-t_0)}\widetilde{\chi}(t_0)\| \\ &\leq 2\|(sI - \widetilde{A})^{-1}\widetilde{B}\|\gamma_0 \max(\gamma_1, \gamma_2)(|\chi(t_0)| + |\chi_\Delta(t_0)|) + \gamma_3|\widetilde{\chi}(t_0)| \\ &\leq (2\sqrt{2}\|(sI - \widetilde{A})^{-1}\widetilde{B}\|\gamma_0 \max(\gamma_1, \gamma_2) + \gamma_3)|\widetilde{\chi}(t_0)| = c^{1/2}|\widetilde{\chi}(t_0)|, \end{aligned}$$

where we used

$$\gamma_3^2 = \int_0^\infty \overline{\sigma}(e^{\widetilde{A}t})^2 dt.$$

Hence

$$\int_{t_0}^\infty |\widetilde{\chi}(\tau)|^2 d\tau \leq c|\widetilde{\chi}(t_0)|^2,$$

for every pair $(t_0, \widetilde{\chi}(t_0))$, where $t_0 \geq 0$.

This is the desired inequality and we infer from Lemma 1 that there are constants $\alpha, \beta > 0$ such that $|\widetilde{\chi}(t)| \leq \alpha e^{-\beta t}|\widetilde{\chi}(0)|$ for all $t \geq 0$. Let $\widetilde{x} = (\, x^T \quad x_\Delta^T \,)^T$. The exponential decay of $\widetilde{\chi}$ implies that the following limit exists

$$\widetilde{x}_\infty = \lim_{t\to\infty} x(t) = x_0 + \int_0^\infty \widetilde{\chi}(\tau)d\tau.$$

From (16) we infer that there exists $\kappa > 0$ such that $|\zeta| \leq \kappa|\widetilde{\chi}|$. Hence, the existence of the limit $u_\infty = \lim_{t\to\infty} u(t)$ follows as above. The continuity property for φ_h (property 5 in Section 2) implies that $(x_\infty, x_{\Delta\infty}, u_\infty)$ is in the stationary set $\mathcal{S}$. Furthermore,

$$|\widetilde{x}(t) - \widetilde{x}_\infty| = |\int_t^\infty \widetilde{\chi}(\tau)d\tau| \leq \frac{\alpha}{\beta}e^{-\beta t}\overline{\sigma}(\widetilde{A})|\widetilde{x}_0|.$$

The corresponding bound for the exponential decay of u is obtained similarly. This concludes the proof.

7. References

[1] K. Åström and T. Hägglund. *PID Control: Theory Design and Tuning.*

[2] K. Åström and T. Hägglund. *Automatic Tuning of PID Controllers.* Research Triangle Park, N.C.: Instrument Society of America, 1988.

[3] N. Barabanov and V. Yakubovich. "Absolute Stability of Control Systems with one Hysteresis Nonlinearity." *Avtomatika i Telemekhanika*, **No 12**, pp. 5–12, December 1979. (Russian). English translation in Automatic and Remote Control.

[4] L. Elghaoui. *LMITOOL: An Interface to Solve LMI Problems.* ENSTA, 1995.

[5] P. Gahinet, A. Nemirovskii, A. Laub, and M. Chilali. *LMI Control Toolbox.* The Math Works Inc, 1995.

[6] U. Jönsson and A. Rantzer. "A unifying format for multiplier optimization." In *Proceedings of the American Control Conference*, pp. 3859–3860, Seattle, Wahington, 1995.

[7] M. Krasnosel'skiĭ and A. Pokrovskiĭ. *Systems with Hysteresis.* Springer-Verlag, 1989.

[8] A. Megretski. "Frequency-domain criteria of robust stability for slowly time-varying systems." *IEEE Transactions on Automatic Control*, **40:1**, pp. 153–155, 1995.

[9] A. Megretski and A. Rantzer. "System analysis via integral quadratic constraints: Part I." Technical Report TFRT–7531, Department of Automatic Control, Lund Institute of Technology, 1995.

[10] H. Royden. *Real Analysis.* Collier Macmillan, third edition, 1988.

[11] D. Šiljak. *Nonlinear Systems.* Wiley, 1969.

[12] V. Yakubovich. "Frequency conditions for absolute stability of control systems with hysteresis nonlinearities." *Dokl. Akad. Nauk SSSR*, **149:2**, pp. 288–291, 1963.

[13] V. Yakubovich. "The method of matrix inequalities in the theory of stablity of nonlinear controlled systems III. Absolute stability of systems with hysteresis nonlinearities." *Avtomatika i Telemekhanika*, **26:5**, pp. 753–763, 1965.