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Abstract

It is shown that it is sometimes possible to construct better controllers that H_∞ controllers when having more knowledge of the disturbances than just a bound on the norm.

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Preface

This report is the result of a project in a short-course on H_∞ control held by Michael Green 1990. I would like to express my gratitude to Michael Green for his interesting seminars and to Per Hagander for his inspiring guidance during the project.

Lund September 1990

Anders Hansson

Contents

1. Introduction

The aim of this project in H_∞ control has been to compare H_∞ controllers with other optimal controllers when adding more structure to the problem.

In Section 2 the problem is described, and in Section 3 the solutions are presented. Then in Section 4 the behavior of the controllers are examined by simulations. Section 5 contains the conclusions.

2. Problem

The model of interest is taken from [1] pp 45 and can approximately be described by

$$\begin{cases} \dot{y} = f(u,w) = K(u+w) \\ w(t) = Fe^{\mu t} \\ y(0) = y_0 \end{cases},$$

where $F \in [F_0, F_1]$, $\mu \in [\mu_0, \mu_1]$, $F_0, \mu_0 > 0$, y is the variable to be controlled, and where w can be interpreted as load-disturbance. The problem can then be formulated as minimizing the loss-function

$$J(u) = \int_0^T (\rho^2 y^2 + u^2) dt$$

over u for the worst case load-disturbance.

3. Solution

The different solutions are derived using the necessary condition of the Euler-Lagrange equation.

In the first subsection a necessary and sufficient condition for a H_∞ solution is given. In the second subsection a controller is derived using the necessary condition without utilizing the structure of the load-disturbance. In the third subsection it is shown for a specific example that the control signal is bounded although the gain is infinite. In the fourth subsection the structure of the load-disturbance is utilized to derive another H_∞ controller. In the fifth subsection an optimal controller is designed by fully utilizing the known structure of the load-disturbance.

3.1 H_∞ control

The problem can be formulated as finding a u such that for $y_0 = 0$

$$J(u) < \gamma^2 ||w||_2^2, \quad \forall w \neq 0,$$

where $||w||_2^2 = \int_0^T w^2 dt$. It is shown in [3] pp 5–10 that the problem has a solution if and only if the Euler-Lagrange equation

$$\begin{cases} \dot{y} = f \\ \dot{\lambda} = -H_y \\ 0 = H_u \\ 0 = H_w \end{cases},$$

where

$$\begin{cases} H = L + \lambda^T f \\ L = \rho^2 y^2 + u^2 - \gamma^2 w^2 \end{cases}$$

under $y(0) = y_0 = 0$ and $\lambda(T) = 0$ has a solution, and that one of the solutions is given by the solution to this equation.

3.2 Simple H_∞ solution

The condition of the Euler-Lagrange equation can be summarized in

$$\begin{cases} \begin{pmatrix} \dot{y} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & \frac{K^2}{2}(\gamma^{-2} - 1) \\ -2\rho^2 & 0 \end{pmatrix} \begin{pmatrix} y \\ \lambda \end{pmatrix} \\ \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -\frac{K}{2} \\ \frac{K}{2\gamma^2} \end{pmatrix} \lambda \end{cases}$$

with $\lambda(T) = 0$ and $y(0) = y_0 = 0$. Solving for $\gamma < 1$ will give the solution

$$y(s) = A \sin(rs) + B \cos(rs),$$

where $r = K\rho\sqrt{\gamma^{-2} - 1}$ and A and B is found by solving

$$\begin{pmatrix} \sin(rt) & \cos(rt) \\ r\cos(rT) & -r\sin(rT) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} y(t) \\ 0 \end{pmatrix}.$$

Then

$$u(t) = -\frac{K}{2}\lambda(t) = -\frac{\rho}{\sqrt{\gamma^{-2} - 1}} \tan(r(T - t))y(t).$$

If $\gamma > 1/\sqrt{1 + (\frac{\pi}{2K\rho T})^2}$, then $u(t)$ is defined $\forall t \in [0, T]$. It is interesting to note that the gain of the controller approaches infinity for $t = 0$ as γ approaches its limit.

3.3 An example

It is tempting to believe that infinite gain implies infinite control signal. In the solution in Subsection 2 let $T = \pi/2 - \epsilon$, $K = 1$, $\rho^2 = \pi/(\pi - 2\epsilon)$ and $\gamma^{-2} = 1 + (\pi - 2\epsilon)/\pi$. It is easily seen that $r = 1$ and that these values satisfy the inequality condition on γ for all $\epsilon > 0$. Let

$$w(t) = \begin{cases} 1/h, & 0 \le t \le h^2 \\ 0, & h^2 < t \le T \end{cases},$$

which implies $||w||_2 = 1$. Since

$$u = -\frac{1}{\tan(t + \epsilon)} y,$$

the differential equation describing the behavior of the controlled system is

$$\dot{y} + \frac{1}{\tan(t + \epsilon)} y = w,$$

which, if $y_0 = 0$, has the solution

$$y(t) = \begin{cases} \dfrac{\cos\epsilon - \cos(t+\epsilon)}{h\sin(t+\epsilon)}, & 0 \le t \le h^2 \\ \dfrac{\cos\epsilon - \cos(h^2+\epsilon)}{h\sin(t+\epsilon)}, & h^2 < t \le T \end{cases}.$$

Expanding $y(t)$, $\quad t \le h^2$ for $\epsilon = 0$ in a Taylor series will give

$$y(t) = \frac{t^2/2 + \mathcal{O}(t^3)}{ht + \mathcal{O}(t^3)}.$$

Since the gain for $\epsilon = 0$ can be expanded as

$$\frac{1}{\tan(t)} = \frac{\cos(t)}{t + \mathcal{O}(t^3)},$$

the control signal is bounded for $h > 0$ and all values of ϵ and t.

3.4 H_∞ solution for modeled load-disturbance

Model the load-disturbance as

$$\dot{v} = \mu_b v$$

with $v(0) = F_b \in [F_0, F_1]$ and $\mu_b \in [\mu_0, \mu_1]$. Then w is what is unmodelled. Modify f to

$$f = \begin{pmatrix} K(u+v+w) \\ \mu_b v \end{pmatrix}$$

and y to $\begin{pmatrix} y & v \end{pmatrix}^T$. As in Subsection 2 the Euler-Lagrange equation can be summarized in

$$\begin{cases} \begin{pmatrix} \dot{y} \\ \dot{v} \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} 0 & K & \frac{K}{2}(\gamma^{-2}-1) & 0 \\ 0 & \mu_b & 0 & 0 \\ -2\rho^2 & 0 & 0 & 0 \\ 0 & 0 & -K & -\mu_b \end{pmatrix} \begin{pmatrix} y \\ v \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \\ \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -\frac{K}{2} \\ \frac{K}{2\gamma^2} \end{pmatrix} \lambda_1 \end{cases}$$

with $\lambda_1(T) = \lambda_2(T) = 0$, $y(t) = y_0 = 0$ and $v(t) = F_b e^{\mu_b t}$. Solving for $\gamma < 1$ will give

$$y(s) = A\sin(rs) + B\cos(rs) + Ce^{\mu_b s},$$

where $r = K\rho\sqrt{\gamma^{-2}-1}$, $C = KF_b\mu_b/(\mu_b^2 + r^2)$, and where A and B are found by solving

$$\begin{pmatrix} \sin(rt) & \cos(rt) \\ r\cos(rT) & -r\sin(rT) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} y(t) - Ce^{\mu_b t} \\ (KF_b - C\mu_b)e^{\mu_b T} \end{pmatrix}.$$

Then

$$\begin{aligned}
u(t) &= -\frac{K}{2}\lambda_1(t) \\
&= -\frac{1}{K(\gamma^{-2}-1)}\left(\dot{y}(t) - KF_b e^{\mu_b t}\right) \\
&= -\frac{1}{K(\gamma^{-2}-1)}[r(y(t) - Ce^{\mu_b t})\tan(r(T-t)) \\
&+ \frac{KF_b r^2}{(\mu_b^2 + r^2)\cos(r(T-t))}e^{\mu_b T} - \frac{KF_b r^2}{\mu_b^2 + r^2}e^{\mu_b t}]
\end{aligned}$$

It is interesting to note the feed-forward-terms. The constraint for γ is the same as in the previous subsection. Also for this controller the gain approaches infinity, and so does the feed-forward-terms for $t = 0$ as γ approaches its limit.

3.5 Optimal solution

Suppose that the worst-case disturbance for the problem in Section 2 is $w = F_1 e^{\mu_1 t}$, and find the optimal control for that disturbance. Define

$$\begin{cases} H = L + \lambda f \\ L = \rho^2 + u^2 \end{cases},$$

where f is defined as in Section 2. Then the Euler-Lagrange equation

$$\begin{cases} \dot{y} = f \\ \dot{\lambda} = -H_y \\ 0 = H_u \end{cases}$$

is a necessary condition for minimizing $J(u)$ in Section 2, [2] p 43. The Euler-Lagrange equation is summarized in

$$\begin{cases} \dot{y} = K(u + w) \\ \dot{\lambda} = -2\rho^2 y \\ u = -\dfrac{K}{2}\lambda \end{cases}$$

with $\lambda(T) = 0$ and $y(0) = y_0$. If $\mu_1 \neq \rho K$, then the solution is given by

$$x(s) = Ae^{rs} + Be^{-rs} + Ce^{\mu_1 s},$$

where $r = \rho K$, $C = \frac{KF_1\mu_1}{\mu_1^2 - r^2}$, $D = \frac{KF_1}{\mu_1}$ and A and B as the solution of

$$\begin{pmatrix} e^{rt} & e^{-rt} \\ re^{rT} & -re^{-rT} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} y(t) - Ce^{\mu_1 t} \\ (KF_1 - C\mu_1)e^{\mu_1 T} \end{pmatrix}.$$

Then

$$\begin{aligned}
u(t) &= \frac{1}{K}\dot{x}(t) - w(t) \\
&= -\frac{1}{K}[r(y(t) - Ce^{\mu_1 t})\tanh(r(T-t)) \\
&- \frac{KF_1 - C\mu_1}{\cosh(r(T-t))}e^{\mu_1 T} + (KF_1 - C\mu_1)e^{\mu_1 t}]
\end{aligned}$$

This controller also introduces feed-forward, but it does not suffer from the problem of having high gain for small values of t.

4. Simulations

To compare the different controllers with respect to load-disturbances of the type described in Section 2, simulations in Simnon have been performed. The values of the parameters has been $K = \rho = T = 1$, $\mu_1 = 1.1$, $F_1 = 1.1$, $\mu_b = 0.8$ and $F_b = 0.7$. Three different load-disturbances has been used: $(F, \mu) = (0.1, 0.1), (0.7, 0.8), (1.1, 1.1)$. The value of γ has been chosen 0.01 above its limit. The initial value has been $y(0) = 0$. The results are presented in figures 1–3.

It is obvious that the strategy of the H_∞ controllers is to control most at the beginning, knowing that errors in the beginning will be integrated over the remaining time. The H_∞ controller with modeled load-disturbance is not as good as the first one. This is perhaps due to $v(0) \neq 0$. The strategy of the optimal controller is not to control too much in the beginning, because there will be a big load-disturbance at the end. The values of $J(u)$ are roughly equal for the best H_∞ controller and the optimal controller. The H_∞ controller is better for small disturbances, while the optimal controller is better for large disturbances.

If the initial values are not equal to zero, there will be problems with infinite gain for optimal values of γ of H_∞ controllers. One way to overcome this is to choose a larger value of γ. Since there are no good rules for how to choose γ, it is not easy to design good H_∞ controllers.

5. Conclusions

It has been shown that utilizing known structure of load-disturbances can improve control. Also the drawback of finding a good value of γ in H_∞ control has been exemplified.

Since the solution to the Euler-Lagrange equation with $y_0 \neq 0$ will give

$$J(u) \leq y_0^T \lambda(0) y_0 + \gamma^2 ||w||_2^2,$$

where $\lambda(0)$ is depending on γ, it would be interesting to try to make the right hand side in the inequality as small as possible by varying γ under for example the assumption that $y_0^T y_0 + ||w||_2^2 \leq 1$.

6. References

[1] Axelsson, J.P. (1989): "Modelling and Control of Fermentation Processes," Ph.D thesis CODEN: LUTFD2/TFRT-1030, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

[2] Sage, A.P. and C.C. White, III (1977): *Optimum systems control*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, USA.

[3] Limebeer, D.J.N., B.D.O. Anderson, P.P. Khargonekar and M. Green (1990): "A game theoretic approach to H^∞ control for time varying systems," to be published.

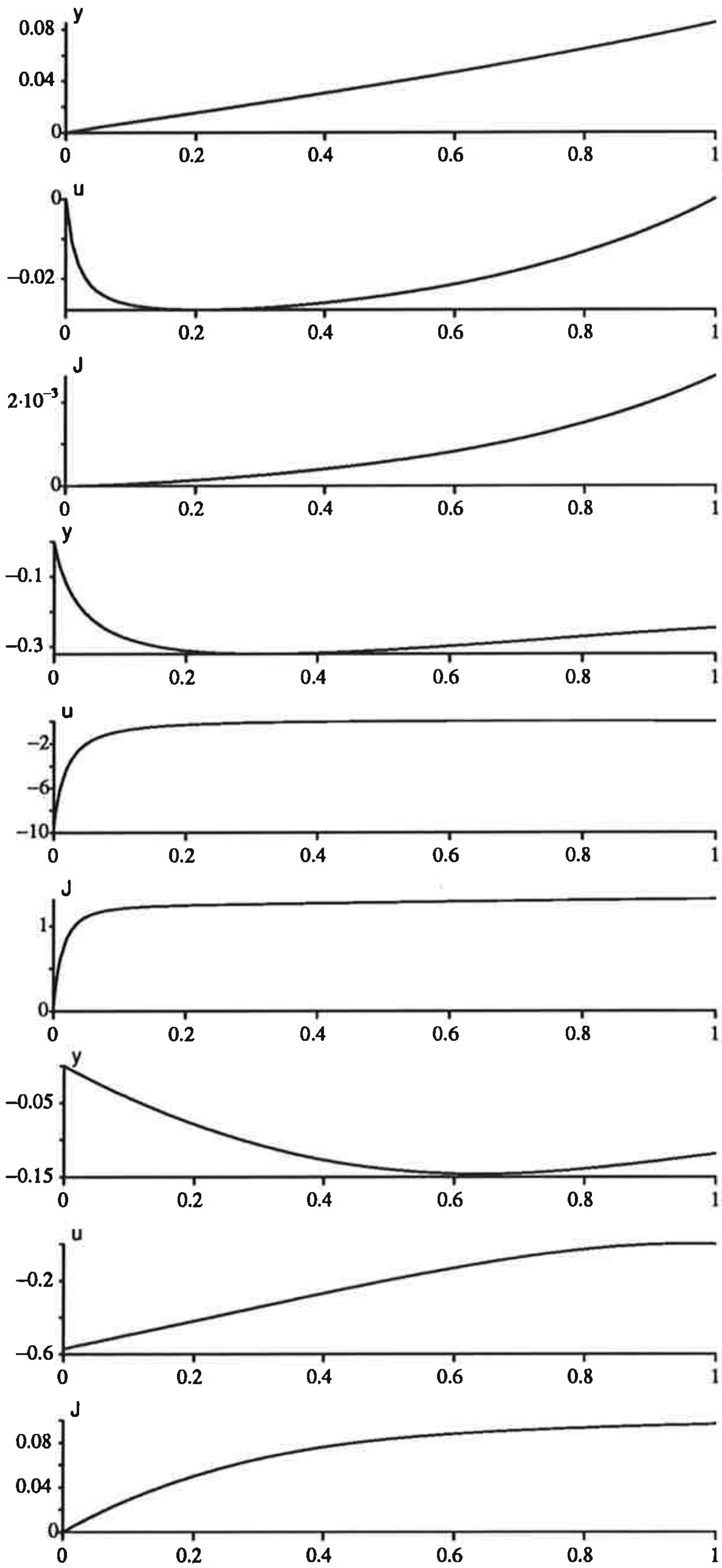

Figure 1. Simulation of H_∞ control with no disturbance model (top), with disturbance model (middle) and optimal control (bottom), $F = 0.1, \mu = 0.1$

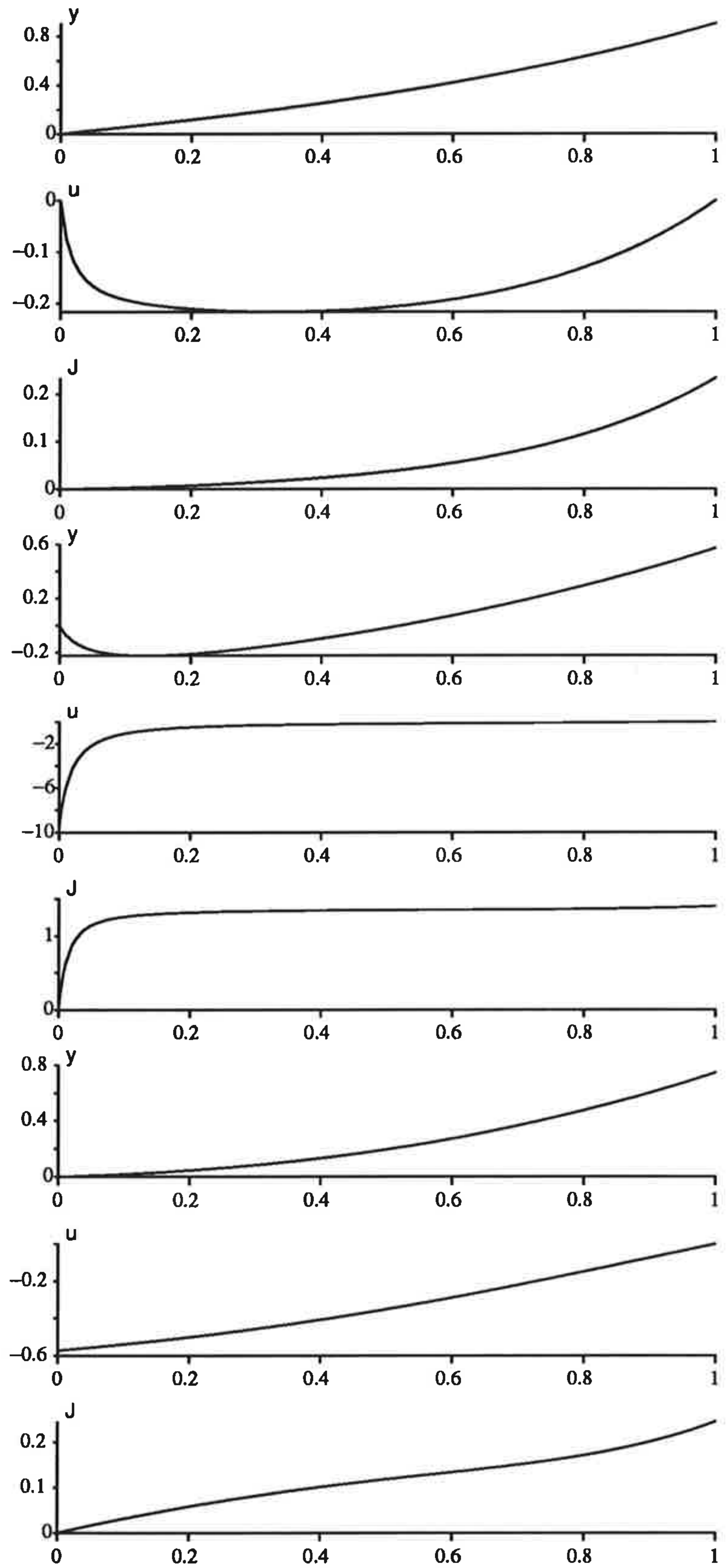

Figure 2. Simulation of H_∞ control with no disturbance model (top), with disturbance model (middle) and optimal control (bottom), $F = 0.7, \mu = 0.8$

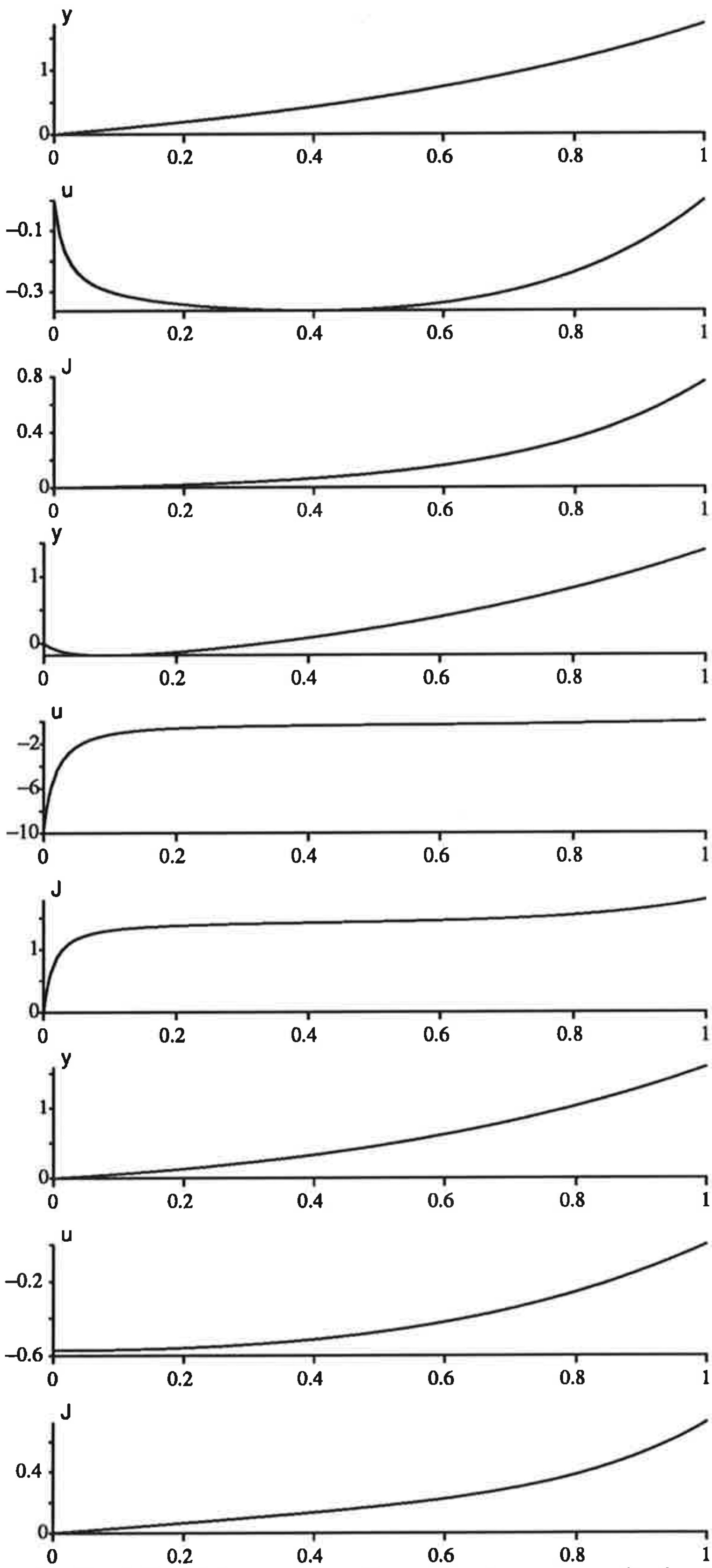

Figure 3. Simulation of H_∞ control with no disturbance model (top), with disturbance model (middle) and optimal control (bottom), $F = 1.1, \mu = 1.1$

A. Simnon-code

```
continuous system PLANT
"simulates an integrator

input u w
output y
state x v
der dx dv

dx=K*(u+w)
y=x

dv=w^2

K:1
END

continuous system REG
"A H-infinity controller with optimal gamma for
"[dx/dt]    [A  B1  B2 ][x]
"[  z  ] = [C1 D11 D12][w]
"[  y  ]    [C2 D21 D22][u]
"A=0; B1=B2=K
"C1=[ 0 ]; D11=[0]; D12=[1]
"    [rho]       [0]       [0]
"C2=1; D21=D22=0

time t
input y
output u
state J
der dJ

u=-rho/sqrt(1/gamma^2-1)*tan(K*rho*sqrt(1/gamma^2-1)*(t1-t))*y

gamma=1/sqrt(1+(pi/(2*rho*K*t1))^2)+0.01

dJ=rho^2*y^2+u^2

rho:1
K:1
t1:1 "final horizon-time
pi:3.141592654
END

continuous system REG
"An H-infinity-controller for an integrator and a modell of an exponentially
"growing load-disturbance

time t
input y
output u
state J
der dJ

u=-1/K/(1/gamma^2-1)*(term1+term2-term3)

term1=r*(y-C*exp(myb*t))*tan(r*(t1-t))
term2=(K*Fb*r^2)/(myb^2+r^2)/cos(r*(t1-t))*exp(myb*t1)
term3=K*Fb*r^2/(myb^2+r^2)*exp(myb*t)
C=K*myb*Fb/(myb^2+r^2)
r=rho*K*sqrt(1/gamma^2-1)
gamma=1/sqrt(1+(pi/2/K/rho/t1)^2)+0.01

dJ=rho^2*y^2+u^2

rho:1
myb:0.8
Fb:0.7
```

```
t1:1 "final horizon-time
K:1
pi:3.141592654
END


continuous system REG
"An optimal controller for an integrator with respect to an exponentially
"growing load-disturbance

time t
input y
output u
state J
der dJ

u=-1/K*(term1-term2+term3)

term1=r*(y-C*exp(my1*t))*tanh(r*(t1-t))
term2=(K*F1-C*my1)/cosh(r*(t1-t))*exp(my1*t1)
term3=(K*F1-C*my1)*exp(my1*t)
C=K*F1*my1/(my1^2-r^2)
r=rho*K

dJ=rho^2*y^2+u^2

rho:1
my1:1.1
F1:1.1
t1:1 "final horizon-time
K:1
END


connecting system CON
"connecting system for H-infinity-control and optimal control of an
"integrator with 'exponentially' growing load-disturbance

TIME t

u[PLANT]=u[REG]
y[REG]=y[PLANT]
w[PLANT]=F*exp(my*t)

F:1.1
my:1.1
END
```