Steady waves in local and nonlocal models for water waves

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Tien Truong

LUND UNIVERSITY

DOCTORAL THESIS
Thesis advisor: Senior Lecturer Erik Wahlén
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Abstract
We study the steady Euler equations for inviscid, incompressible, and irrotational water waves of constant density. The thesis consists of three papers. The first paper approaches the Euler equations through a famous nonlocal model equation for gravity waves, namely the Whitham equation. We prove the existence of a highest gravity solitary wave which reaches the largest amplitude and forms a $C^{1/2}$ cusp at its crest. This confirms a 50-year-old conjecture by Whitham in the case of solitary waves, that the fully linear dispersion in the Whitham equation would allow for high-frequency phenomena such as highest waves. In the second paper, we use a recently developed center manifold theorem for nonlocal and nonlinear equations to study small-amplitude gravity–capillary generalized and modulated solitary waves in a Whitham equation with small surface tension. The last paper treats the steady Euler equations directly. Here, the gravity and capillary coefficients are fixed but arbitrary, and for simplicity we place a non-resonance condition on the problem. We address the transverse dynamics of two-dimensional gravity–capillary periodic waves using a spatial dynamics technique, followed by a perturbation argument.

Key words
The steady Euler equations, the Whitham equation, solitary waves, highest waves, periodic waves, transverse dynamics

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Contents

Populärvetenskaplig sammanfattning

Vattenvågor på öppna ytor är ett välbekant naturfenomen för många från barndomens havsutflykter, och det finns utan tvekan ett intresse för samhället att förstå samt förvalta dess krafter. Denna avhandling studerar rörelser hos sådana vågor. Den matematiska beskrivningen för vågrörelsen ges av Leonhard Euler för drygt 300 år sen, och består av partiella differentialekvationer med två okända variabler: den fria vattenytan och vattenpartiklarnas hastighet. Ekvationerna kallas för Eulers ekvationer. Eftersom variablerna förekommer icke-linjärt är Eulers ekvationer utmanande att studera och ett fortsatt aktivt forskningsområde trots sin rika matematiska historia. För enkelhetens skull antar vi att havsbottnen är en plan yta som inte har någon lutning. Vi antar också att vattnet har konstant densitet, att det inte virvlar, samt att det inte är trögflytande. Dessa antaganden passar bl.a. väl in på havsvågor som inte är alltför nära stranden. I denna avhandling undersöks fortskridande gravitations- och kapillär-gravitationsvågor. En våg är fortskridande om den fortplantas längs en horisontell riktning med konstant hastighet utan att dess form ändras. Med gravitationsvågor menas lösningar till Eulers ekvationer där jordens tyngdkraft är den dominerande återställande kraften. Med kapillär-gravitationsvågor menas lösningar till Eulers ekvationers där både ytspänningen och tyngdkraften har en påverkan. Sådana vågor är vanligt förekommande i naturen.

På grund av komplexiteten hos Eulers ekvationer är det vanligt att studera dess approximationer, som är enklare och som ger en god överblick över både lösningar och kvalitativa egenskaper hos Eulers ekvationer. Ett exempel är KdV-ekvationen, som lyckas fånga ett sällsynt vågfenomen, nämligen solitära vågor. Dessa är fortskridande vågor där vattnets massa koncentreras kring en punkt. Ett annat exempel är Whithamekvationen, där man använder den fullständiga linjära dispersionen från Eulers ekvationer. Dispersion är ett fenomen där vågens fortplantningshastiget beror på våglängden. Den fullständiga linjära dispersionen gör att Whithamekvationen blir icke-lokal, vilket innebär att lokal information nära en punkt är otillräcklig för att räkna ut termerna i ekvationen. Det förmodas att denna icke-lokala och icke-linjära struktur hos Whithamekvationen kan fånga högsta vågor, som uppnår den högsta möjliga amplituden och som förlorar glatthet. Förmodan är bekräftad för periodiska vågor och, i denna avhandling, för solitära vågor. Vår analys är baserad på nya matematiska verktyg för icke-lokala och icke-linjära ekvationer, och är en utgångspunkt för fortsatt matematisk forskning kring andra icke-lokala och icke-linjära ekvationer på liknande form. I den första artikeln ges ett bevis för existens av högsta solitära vågor för

Whithamekvationen. Här behövs en ny konstruktion av glatta och små solitära vågor för Whithamekvationen, d.v.s. en lokal bifurkation från det triviala jämviktsläget där vattnet är i vila. Därefter studeras global bifurkation, där vågamplituden inte längre är liten. Tar vi gränsvärdet av dessa "stora" solitära vågor kan ett antal fall inträffa. Exempelvis kan gränsvärdet bli noll eller saknas. Genom att studera hur solitära vågor uppför sig i allmänhet kan vi utesluta dessa oönskade fall. Slutligen kvarstår ett enda fall, där vågorna konvergerar mot en av dessa högsta solitära vågor.

Artikel II fördjupar sig i ett verktyg från artikel I. Målet är att konstruera små vågor för kapillär-gravitationsvågor i en Whithamekvation med svag ytspänning. Ytspänningseffekten medför nya och intressanta matematiska komplikationer, som delvis studeras med hjälp av ett klassiskt verktyg för ändligtdimensionella system. Artikel II ger ett nytt bevis för existensen av generaliserade solitära vågor. Dessa har också en koncentrerad massa kring en punkt. Dock tillför den kapillära effekten oscillerande svansar längre ut från masskoncentrationen. Ett annat resultat, som är helt nytt, är existensen av modulerade solitära vågor. Båda vågfenomen existerar och är välstuderade för Eulers ekvationer. Artikel II lägger fram ett ramverk för att behandla godtycklig ytspänning och fortplantningshastighet. Dessutom illustrerar artikeln vilken roll den fullständiga linjära dispersionen spelar för vågformation.

I den tredje och sista artikeln undersöks uppkomsten av tredimensionella periodiska kapillär-gravitationsvågor i Eulers ekvation genom en instabilitetsegenskap hos tvådimensionella vågor. Fenomenet är känt som dimensionsbrytande bifurkation på grund av den ökade dimensionen, där tredimensionella vågor förgrenas från en tvådimensionell våg. Tidigare arbeten har studerat detta fenomen för både periodiska och solitära vågor med ytspänning och fortplantningshastighet nära kritiska värden. Artikel III kompletterar dessa arbeten genom att undersöka fallet där ytspänning och fortplantningshastighet inte nödvändigtvis är nära något kritiskt värde. Utmaningen här är att ta fram spektralegenskaper hos en linjär operator, vars koefficienter är funktioner som saknar explicita uttryck. För att lösa detta betraktar vi den linjära operatorn i frågan som en störning av en enklare linjär operator med konstanta koefficienter.

Avhandlingen undersöker alltså vågformation utifrån olika perspektiv, och innehåller därför ett utbud av både moderna och klassiska matematiska verktyg.

List of publications

[**Paper I**] T. Truong, E. Wahlén and M. H. Wheeler, *Global bifurcation of solitary waves for the Whitham equation*, Mathematische Annalen, (2021), in print.

[**Paper II**] M. A. Johnson, T. Truong and M. H. Wheeler, *Solitary waves in a Whitham equation with small surface tension*, Studies in Applied Mathematics, (2021), in print.

[**Paper III**] M. Haragus, T. Truong and E. Wahlén, *Transverse dynamics of two-dimensional traveling periodic gravity–capillary water waves*, (2022), preprint.

My contribution is summarized below for each of the above papers.

[**Paper I**] I carried out the analysis of Sections 2 & 3 under the guidance of my collaborators. I also contributed to some of the analysis in Section 4. I verified the extensions and made adaptations of various results, including the center manifold theorem in Sobolev spaces of higher regularity and the augmented center manifold theorem. I applied these to the Whitham equation, wrote a draft of the article and edited it together with my collaborators.

[**Paper II**] Together with my collaborators, I computed the center manifold coefficients for the generalized solitary waves. I was responsible for the normal form computations. Later, I extended our framework to include another bifurcation phenomenon, carried out the analysis and the computations to establish the existence of modulated solitary waves. I authored the article and edited it according to the feedback from my collaborators and the reviewers.

[**Paper III**] Together with my collaborators, I carried out, verified and simplified the computations. Using Maple, I proposed a characterization of the transverse linear instability in question. My collaborators and I eventually proved this analytically, and I contributed to our analytical proof with an important observation. I wrote a draft of the article and edited parts of it under the guidance of my collaborators.

Preface

Preface

1 The water wave problem

It is perhaps not surprising that the list of water deities from Wikipedia is impressively long with a staggering contribution of 45 different deities from the Greek mythologies alone, whereas only five are listed for fire. This is a reflection of the wide variety of water waves, as these permeate our daily life in the most fundamental way, and their impact ranges from being a necessity of life to an absolute life-threatening catastrophe. There is without doubt an interest for the society to gain more understanding of water waves, and instead of inventing new deities we have shifted to studying water waves systematically. Often, "water" is modeled as an inviscid fluid, the motion of which owes its mathematical description to Leonhard Euler in 1757 [29]. This description is known as the *Euler equations* in the literature. Despite extensive research effort, the intriguing 300-year-old equations for water motions are not fully understood. Thus, the objective of this thesis is to improve our knowledge on this subject. The other category of fluids — viscous[1] fluids — can be modeled by the Navier–Stokes equations, and will not be treated in this thesis.

Let us now give an account of the Euler equations. The motion of a fluid can be described by a fluid velocity function $\mathbf{u}$ which depends on Cartesian coordinates $\mathbf{x} = (x, y, z)$ and time t. To be consistent with our presentation in Paper III, the vertical direction is y, whereas x and z are horizontal directions. There are numerous physical aspects of water-wave modeling: density homogeneity, compressibility[2], vorticity[3], forces acting on the fluid and the region in which the fluid occupies. We consider here an incompressible fluid with constant density and irrotational flow. Assume further that the fluid surface is free and can be expressed as a graph of the horizontal coordinates x, z. Thus, the fluid occupies a three-dimensional region

$$D_\eta = \{(x, y, z) \in \mathbb{R}^3 \, : \, 0 < y < h + \eta(x, z, t)\},$$

where we have taken a flat bottom $\{y \equiv 0\}$, a depth h at which the fluid is at rest, and surface profile $\eta(x, z, t) > -h$ relative to the depth h; see an illustration in Figure 1. In

[1] Roughly speaking, viscosity is a measure of the internal friction in a material. For example, honey is more viscous than water and water is more viscous than air.

[2] A fluid's ability to significantly change its density

[3] This is curl $\mathbf{u}$, which measures the local rotation of the fluid.

this case, the Euler equations have the form

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla P + \mathbf{F}, \end{aligned} \tag{1}$$

where $\mathbf{u}\colon (\mathbf{x}, t) \mapsto (u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$ is the fluid velocity function, $\mathbf{F}(\mathbf{x}, t)$ is the resultant external force, and $P(\mathbf{x}, t)$ is the pressure acting on the fluid. In addtion, $\mathbf{u}$ and P must satisfy the boundary conditions

$$\begin{aligned} &u_2 = 0 && \text{on } y = 0, \\ &u_2 = \eta_t + u_1 \eta_x + u_3 \eta_z && \text{on } y = h + \eta, \\ &P = P_{\text{atm}} - \frac{T}{\rho} \mathcal{K} && \text{on } y = h + \eta. \end{aligned} \tag{2}$$

Here, P_{atm} is the atmospheric pressure, T is a surface tension constant, ρ is a fluid density constant, and

$$\mathcal{K} := \left[\frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_x + \left[\frac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_z$$

is twice the mean curvature of the surface profile η. Since $\operatorname{curl} \mathbf{u} \equiv 0$, there exists a so-called velocity potential ϕ such that $\mathbf{u} = \nabla \phi$. Assume that the gravity $\mathbf{F} = (0, -g, 0)$ is the only external force present. Then, the nonlinear equations (1)–(2) reduce to

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{for } 0 < y < h + \eta \tag{3}$$

where ϕ satisfies the boundary conditions

$$\begin{aligned} &\phi_y = 0 && \text{on } y = 0, \\ &\phi_y = \eta_t + \eta_x \phi_x + \eta_z \phi_z && \text{on } y = h + \eta, \\ &\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta - \frac{T}{\rho} \mathcal{K} = B && \text{on } y = h + \eta. \end{aligned} \tag{4}$$

Here, B is the Bernoulli constant. The unknowns in equations (3)–(4) are both ϕ and η, posing a challenge to study as both appear nonlinearly in the boundary conditions. A solution pair (η, ϕ) to (3)–(4) is called a *wave*, and with the free-surface domain D_η being part of the unknowns, equations (3)–(4) constitute *the water wave problem*. Research topics for equations (3)–(4) include for example existence theory, stability, qualitative properties, well-posedness and asymptotic models. Each of these topics is further divided into three categories of *gravity*, *gravity–capillary* and *capillary* waves. In the gravity case, the surface tension effect is negligible. In the capillary case, the gravity effect is negligible. In the gravity–capillary case, none of these effects is negligible. Equations (3)–(4) describe water waves in *finite depth*. We mention that there is a version of (3)–(4) for *infinite depth*, which will not be covered here but can be found in for example [60]. This thesis investigates gravity and gravity–capillary waves with the following topic division:

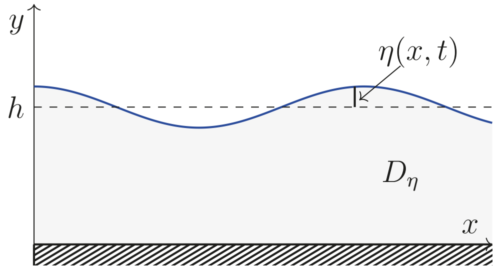


Figure 1: A sketch of the domain D_η, which is shaded gray, for the two-dimensional water wave problem. The bottom is flat, that is $\{y \equiv 0\}$, and impermeable. The top surface is free, and is described by $h + \eta(x,t)$.

1. the existence of two-dimensional traveling small-amplitude waves as local bifurcation phenomena through a model equation,
2. the existence of two-dimensional traveling large-amplitude waves as global bifurcation phenomena through a model equation,
3. the existence of three-dimensional traveling doubly periodic waves induced by an instability phenomenon.

A *traveling* wave by definition maintains a permanent form when propagating with constant velocity along a horizontal direction, which is taken as x throughout this thesis. A *two-dimensional* wave in this thesis is a wave that is constant along the z-direction. In the literature, these might also be referred to as one-dimensional waves[4]. Special attention will be paid to two-dimensional *solitary* waves, which are traveling waves that possess a localized profile, tending to 0 as $|x| \to \infty$. Two-dimensional *periodic* waves on the other hand are periodic in x, thus never localized. A *three-dimensional* wave is not constant in any horizontal direction. Finally, a *doubly periodic* wave is three-dimensional and periodic along x and z. Topics 1 & 2 are investigated for a famous model equation, namely the Whitham equation, and topic 3 is examined for the water wave problem (3)–(4). We shall motivate these concepts and topics below. Local and global bifurcation are however best discussed in connection with mathematical tools for them, and will therefore be postponed to Section 2. For research topics which are not covered here, see for example [59, 45, 21] on the well-posedness of (3)–(4) as an initial-value problem, and for example [60, 51, 15, 21] on asymptotic models. An overview of different versions of the water wave problems and more references can be found in [60, 50].

[4]Two-dimensional in this thesis refers to the dimension of the fluid domain D_η, whereas one-dimensional in the literature refers to the dimension of the fluid surface η.

1.1 Traveling waves For simplicity, consider the two-dimensional gravity water wave problem. This problem can be approached via approximations obtained from a number of additional assumptions. One such is the long-wave approximation, that is, equations (3)–(4) under the assumption that the wavelength is substantially larger than the fluid depth h. To aid our discussion, we perform a nondimensionalization

$$\begin{gathered} x = \lambda \tilde{x}, \quad y = h\tilde{y}, \quad t = \frac{\lambda}{\sqrt{gh}}\tilde{t}, \\ \eta = a\tilde{\eta}, \quad \phi = a\lambda\sqrt{\frac{g}{h}}\tilde{\phi}, \end{gathered} \tag{5}$$

where $\lambda > 0$ is a characteristic wavelength, and $a > 0$ is a characteristic wave amplitude. The point is that the new variables are independent of the measuring units. Define

$$\delta = \frac{h}{\lambda} \quad \text{and} \quad \varepsilon = \frac{a}{h}. \tag{6}$$

These are referred to as the shallowness parameter and the amplitude parameter, respectively. The long-wave approximation of equations (3)–(4) is obtained by letting $\delta \to 0$. Then, a linearization of this approximation gives

$$\tilde{\eta}_{\tilde{t}\tilde{t}} - \tilde{\eta}_{\tilde{x}\tilde{x}} = 0,$$

which by the method of characterics possesses d'Alembert's solutions

$$\tilde{\eta}(\tilde{x}, \tilde{t}) = \varphi(\tilde{x} - \tilde{t}) + \zeta(\tilde{x} + \tilde{t}),$$

where φ and ζ are determined from suitable initial data at a time $\tilde{t}_0$. This means that $\tilde{\eta}$ is a superposition of a right-traveling profile φ of unit constant velocity $c = 1$, and a left-traveling profile ζ. Taking for example a compactly supported initial wave profile ζ, the right-traveling profile φ will not be affected by ζ after some time. Hence, it suffices to study one of these, say $\tilde{\eta}(\tilde{x}, \tilde{t}) = \varphi(\tilde{x} - \tilde{t})$. This motivates the *traveling-wave Ansatz* $\tilde{\eta}(\tilde{x}, \tilde{t}) = \varphi(\tilde{x} - c\tilde{t})$, where we allow an arbitrary constant velocity $c > 0$. Finally, solving equations (3)–(4) using this Ansatz would rigorously confirm the existence of traveling waves.

1.2 Gravity solitary waves Traveling waves which are periodic in one horizontal variable and constant in the other are well observed in nature. A more elusive wave occurence includes solitary waves, the first observation of which was in 1834 by John Scott Russell in the Edinburgh–Glasgow canal. Russell referred to these as "the great waves of translations" and described them in his 1844 publication [69] as follows

> "a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed [...]."

See an illustration in Figure 4. This caused much debate among the water wave community, including initial sceptic opinions from acknowledged contributors within the field, such as Stokes and Airy. The existence of solitary waves was finally settled in 1877 by Boussinesq [8] and in 1895 by Korteweg and de Vries [57]. After a nondimensionalizing change of variables as in (5), taking

$$\delta^2 = \varepsilon, \quad \varepsilon \to 0, \tag{7}$$

followed by another scaling, equations (3)–(4) for two-dimensional gravity waves can be approximated by

$$\tilde{\eta}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \frac{3}{2}\tilde{\eta}\tilde{\eta}_{\tilde{x}} + \frac{1}{6}\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}} = 0, \tag{8}$$

which is called the *Korteweg–de Vries equation*, or shortly the KdV equation. Here, we use the same notations for the new variables for convenience. A derivation of (8) is found in [50], and the assumptions (7) are often referred to as *the KdV regime* when it comes to asymptotic model equations. Note that (8) only depends on the wave profile $\tilde{\eta}$. With the traveling-wave Ansatz $\tilde{\eta}(\tilde{x},\tilde{t}) = \varphi(\tilde{x}-\tilde{t})$ and a slow time scale $\tilde{t} = \varepsilon\tau$, equation (8) belongs to one of the rare nonlinear partial differential equations which can be solved explicitly, admitting a solitary-wave solution of the sech^2 form as well as periodic waves. Its success in capturing the solitary wave lies in a balance between the dispersion and the nonlinearity. Dispersion is the phenomenon that waves with different frequencies travel at different phase speeds. This is in general demonstrated by considering the linearized equation, for instance the linearized KdV equation

$$\tilde{\eta}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \frac{1}{6}\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}} = 0. \tag{9}$$

By making the Ansatz $\tilde{\eta}(\tilde{x},\tilde{t}) = \exp(\mathrm{i}\xi(\tilde{x} - c\tilde{t}))$, it is found that $\tilde{\eta}$ is a solution if

$$c = 1 - \frac{1}{6}\xi^2. \tag{10}$$

Here, ξ is the wavenumber and c is the phase speed of $\tilde{\eta}$. Equation (10) clearly shows that the phase speed c depends on the wavenumber ξ, and two waves each with different ξ_1 and ξ_2 travel at different speeds $c(\xi_1)$ and $c(\xi_2)$, respectively. Thus, after some time, the waves will disperse. Equation (8) is therefore called *dispersive*, and the special algebraic relation (10) is called *the linear dispersion relation* of (8). Roughly speaking, the nonlinearity $\tilde{\eta}\tilde{\eta}_{\tilde{x}}$ is responsible for steepening the wave profile which eventually leads to loss of smoothness, while the dispersive term $\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}}$ has a smoothing effect. Once these two elements are in balance, a solitary wave arises as a result and this balance manifests itself in the solution formula

$$\eta(x,t) = \varepsilon h \operatorname{sech}^2\left[\frac{\sqrt{\varepsilon}}{\sqrt{2h}}\left(x - \sqrt{gh}\left(1 + \frac{\varepsilon h}{3}\right)t\right)\right],$$

where $\varepsilon > 0$ is the amplitude parameter introduced in (6) and ε is small. Here, we have switched back to the original variables η, x and t. The strength of the dispersion is determined by the shallowness parameter $\delta = \sqrt{\varepsilon}$, whereas the strength of the nonlinear effect is determined by the amplitude parameter ε. Since both parameters are small, the wave η is called weakly dispersive and weakly nonlinear. It is also referred to as a *small-amplitude* wave due to the smallness of ε. In Section 2, we will depict the allure of solitary waves from the technical point of view and discuss some mathematical challenges of working with these waves.

1.3 The Whitham equation for large-amplitude gravity waves The contribution of Korteweg and de Vries was significant, but more work needed to be done. This was noted in a 1967 publication [77] by Whitham, according to whom the polynomial linear dispersion relation of the KdV equation could not feature *large-amplitude* waves, or *highest* waves. These were conjectured in 1880 by Stokes [74], who also argued that the highest wave was sharp-crested and included an angle of $2\pi/3$ at the crest. Indeed, 100 years later, the existence of highest water waves was rigorously confirmed by Amick & Toland [3, 4], Amick *et al.* [2] and Craig & Sternberg [16]. However, during the 60s, this was far from being proved. In an attempt to find highest waves, Whitham proposed replacing the polynomial linear dispersion in the KdV equation with that in equations (3)–(4) for unidirectional propagating waves, which is

$$c = \sqrt{\frac{\tanh(\xi)}{\xi}}. \tag{11}$$

Here, we have nondimensionalized as in (5), linearized (3)–(4) and used the Ansätze $\tilde{\eta}(\tilde{x}, \tilde{t}) = \exp(\mathrm{i}\xi(\tilde{x} - c\tilde{t}))$ and $\tilde{\phi}(\tilde{x}, \tilde{y}, \tilde{t}) = \hat{\phi}(\tilde{y}) \exp(\mathrm{i}\xi(\tilde{x} - c\tilde{t}))$. For small frequency $\xi \ll 1$, we have

$$c = \sqrt{\frac{\tanh(\xi)}{\xi}} = 1 - \frac{1}{6}\xi^2 + \mathcal{O}(\xi^4).$$

This shows that the polynomial linear dispersion (10) in the KdV equation is a second-order approximation of the full linear dispersion (11) in (3)–(4). The proposal of Whitham leads to the equation

$$\tilde{\eta}_{\tilde{t}} + \left[m(\mathrm{D})\tilde{\eta} + \tilde{\eta}^2\right]_{\tilde{x}} = 0, \quad \text{where} \quad m(\mathrm{D}) = \sqrt{\frac{\tanh(\mathrm{D})}{\mathrm{D}}}, \tag{12}$$

which is now known as *the Whitham equation*. Here, the Fourier multiplier operator $m(\mathrm{D})$ acts on the spatial variable $\tilde{x}$ as follows

$$m(\mathrm{D})\tilde{\eta}(\tilde{x}, \tilde{t}) = \frac{1}{2\pi} \int_{\mathbb{R}} \sqrt{\tfrac{\tanh(\xi)}{\xi}}\, \mathcal{F}\tilde{\eta}(\xi, \tilde{t}) \exp(\mathrm{i}\tilde{x}\xi)\, \mathrm{d}\xi,$$

where $\mathcal{F}$ denotes the Fourier transform in $\tilde{x}$. Because (12) uses the full linear dispersion rather than an approximation for small ξ, it is a *fully dispersive* equation in the water wave context. Expressing $m(\mathrm{D})$ as a convolution operator, the convolution kernel is found to be integrable near the origin and exponentially decaying as $|\tilde{x}| \to \infty$. Thus, the Whitham equation is a *nonlocal* equation, as values $\tilde{\eta}(\tilde{x}, \tilde{t})$ locally near a point $(\tilde{x}_0, \tilde{t}_0)$ are not sufficient to determine the left-hand side in (12) at $(\tilde{x}_0, \tilde{t}_0)$. Whitham conjectured that (12) would admit a highest wave with a $C^{1/2}$ cusp at its crest, and thus this crest could not include an angle of $2\pi/3$. Although the Whitham equation was introduced as a toy model, it has recently been justified rigorously as an accurate approximation of (3)–(4) in a certain asymptotic limit by Emerald [27]. An earlier contribution includes the work of Klein *et al.* [56], which validates the Whitham equation as an approximation of the KdV equation in the KdV regime (7). Another contribution is the work of Moldabayev *et al.* [65], which formally derives the Whitham equation from a Hamiltonian formulation of (3)–(4) in the regime $\varepsilon = \mathcal{O}(\exp(-A_1/\delta^{A_2}))$ for some constants $A_1, A_2 > 0$. These works [27, 56, 65] also investigate the modeling capacity of (12). The expectation has been that since the Whitham equation is fully dispersive, it should be able to describe both long waves in the KdV regime and some regime for shorter waves. This is numerically demonstrated in [65] and confirmed in [27]. However, experiments in [65] indicate that the accuracy of the Whitham equation is not better than that of the KdV equation in the KdV regime. At the same time, there is a strong theoretical interest in understanding the effect of the full linear dispersion of (3)–(4) in an equation.

Indeed, even before Emerald's contribution and despite that Stokes' conjecture has already been confirmed in the 80s, the two past decades have seen extensive research effort for equation (12): qualitative properies [9, 24, 25, 28], local well-posedness [23], existence of solutions [25, 6, 73, 42], modeling capacity [26, 56, 65], and stability [43, 70]; see also references therein. In particular, the work [25] by Ehrnström & Wahlén established the existence of a highest periodic wave for (12) with a $C^{1/2}$ cusp at the maximum point, thus proving Whitham's conjecture. See Figure 2 for a comparison between the extreme forms of two highest waves conjectured by Stokes and Whitham. We point out the works [42, 71] on wave breaking, which according to [77] also belongs to high-frequency wave phenomena. Paper I contributes to this line of research by proving the existence of a highest solitary wave for the Whitham equation. As we will see in Section 2, the construction is different to that for highest periodic waves. A cornerstone of Paper I consists in an application of a recently developed center manifold theorem for nonlocal and nonlinear equations by Faye & Scheel [30, 31]. The rich qualitative properties of this equation allow us to use an analytic global bifurcation theorem adapted for solitary waves, and analyze its outcome. These important theorems will be discussed in Section 2.

1.4 Gravity–capillary waves in finite depth Let us return to equations (3)–(4) for two-dimensional right-traveling gravity–capillary waves with constant speed $c > 0$. A revolutionary work of Kirchgässner in 1982 [54] shows that these can be studied as an evolutionary system of equations, where an unbounded spatial variable plays the role of time. This technique is known as *spatial dynamics* in modern literature. The process of

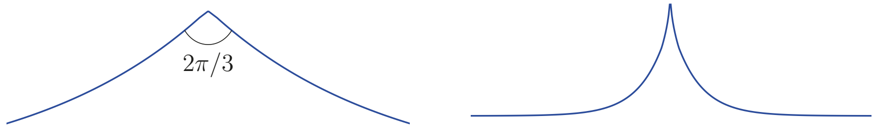


Figure 2: (Left) A highest wave of the gravity water wave problem (3)–(4) conjectured by Stokes. The crest of this wave includes an angle of $2\pi/3$. (Right) A highest wave of the Whitham equation (12) conjectured by Whitham. The crest of this wave is a $C^{1/2}$ cusp.

deriving a spatial dynamics formulation ultimately results in a system of the form

$$\frac{\mathrm{d}U}{\mathrm{d}\tilde{x}} = \mathbf{L}U + \mathbf{R}(U, \lambda), \tag{13}$$

where $\tilde{x} = x - ct$, U is a vector-valued function, λ is a small parameter modeling the gravity and surface tension effects, and $\mathbf{R}$ satisfies a tangency condition $\mathbf{R}(0, \lambda_0) = 0$ and $\mathrm{D}_U\mathbf{R}(0, \lambda_0) = 0$ at the trivial wave $(0, \lambda_0)$. There are several spatial dynamics formulations (13) of equations (3)–(4), for instance by Kirchgässner [54], Levi-Civita [61], and Groves & Toland [36]. Cast as (13), equations (3)–(4) have been studied using a center manifold reduction, which we will discuss in Section 2. To apply this technique, the linear operator $\mathbf{L}$ and its purely imaginary eigenvalues play an important role. In particular, $\mathrm{i}\xi$ is an eigenvalue of $\mathbf{L}$ if and only if

$$1 = \sqrt{(\alpha + \beta\xi^2)\frac{\tanh(\xi)}{\xi}}, \quad \text{where} \quad \beta = \frac{T}{\rho h c^2} \quad \text{and} \quad \alpha = \frac{gh}{c^2}. \tag{14}$$

The parameters β and α are dimensionless, they are the Weber number and the inverse square of the Froude number, respectively. We note that equation (14) is the linear dispersion relation of equations (3)–(4). The linear operator $\mathbf{L}$ has four eigenvalues near the origin counting algebraic multiplicity. Using symmetries in equations (3)–(4), one finds that these are symmetric with respect to the origin. Depending on their locations in the complex plane, the study of (3)–(4) further divides into different parameter regimes. One simple division is by weak/strong surface tension ($0 < \beta < 1/3$ or $\beta > 1/3$, respectively) and super-/subcritical wave speed ($0 < \alpha < 1$ or $\alpha > 1$, respectively). Another division is by bifurcation phenomena near special parameter curves, when two eigenvalues collide. These are

- C_1, along which $\mathbf{L}$ has a pair of real eigenvalues $\pm\kappa$ near the origin, each of double algebraic multiplicity. Near $(\beta, \alpha) = (1/3, 1)$, an 0^{4+} bifurcation occurs;

- C_2, along which $\mathbf{L}$ has a pair of purely imaginary eigenvalues $\pm i\kappa$, each of double algebraic multiplicity. These induce an $(i\kappa)^2$ bifurcation;

- C_3, given by $\beta < 1/3$ and $\alpha \equiv 1$. Along this curve, $\mathbf{L}$ has one eigenvalue 0 of double algebraic multiplicity, and a pair of simple purely imaginary eigenvalues $\pm i\kappa$. These induce an $0^{2+}(i\kappa)$ bifurcation;

- C_4, given by $\beta > 1/3$ and $\alpha \equiv 1$. Along this curve, $\mathbf{L}$ has one eigenvalue 0 of double algebraic multiplicity, and two real eigenvalues near the origin. These induce an 0^{2+} bifurcation.

The parameter curves above are a discovery by Kirchgässner [55]; see Figure 3. Roughly speaking, a pair of purely imaginary eigenvalues will generate an oscillatory wave profile. The $0^{2+}(i\kappa)$ bifurcation near C_2 of the gravity–capillary water wave problem manifests in a generalized solitary wave, that has a localized crest but limits to periodic functions as $|\tilde{x}| \to \infty$. The $(i\kappa)^2$ bifurcation takes the form of a modulated solitary wave, that has a localized envelope about an oscillating function. When approaching the critical point $(\beta, \alpha) = (1/3, 1)$, one finds either multipulse solitary waves along C_3, or multitrough solitary waves along C_1. On the other hand, the 0^{2+} bifurcation near C_4 results in solitary waves of depression, lacking oscillatory feature. A catalogue of these waves is found in Figure 4. These existence results are due to [5, 11, 12, 47]; see also the references therein. An important ingredient in their proofs lies in the symmetries of equations (3)–(4).

Along the same lines of Section 1.3, one can derive a gravity–capillary Whitham equation. Equations (3)–(4) with surface tension have a KdV-equation approximation, where only a coefficient in front of the third derivative term in (8) is affected by the surface tension T. Replacing the polynomial linear dispersion in this KdV equation with the full linear dispersion in (3)–(4) gives a gravity–capillary Whitham equation. Unlike (12), it is not clear whether this artificial replacement yields an actual approximation of (3)–(4). However, it is interesting to investigate whether it at least is able to capture small-amplitude solutions of the full problem (3)–(4). Paper II delivers a partial answer to this question, as it confirms the existence of generalized solitary waves and modulated solitary waves in the regime of small surface tension. In fact, solitary waves of depression can be found using similar analysis as in Paper I. The main technique of Paper II is a version of the Faye–Scheel center manifold theorem from Paper I. However, in Paper II, the application of this tool is different as the capillary effect adds two more dimensions. Thus, the analysis as well as the computations involved are much more requiring. Paper II also illustrates the role of the full linear dispersion in wave formation, and how the Faye–Scheel center manifold theorem serves as a bridge between the problem (3)–(4) and its nonlocal toy model. We mention an earlier work by Johnson & Wright [48], who establish the existence of both solitary waves of depression and generalized solitary waves using an implicit function theorem. The work by Arnesen [6] employs a variational approach and provides an existence theory of solitary waves for a class of equations, which covers the gravity–capillary Whitham

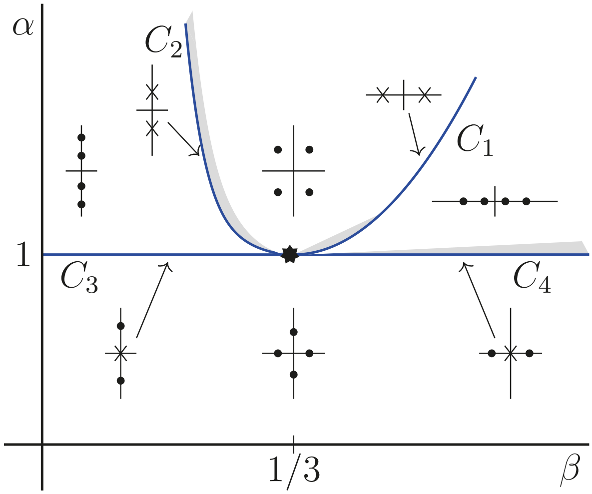
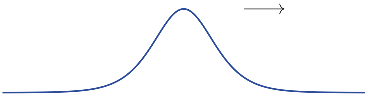


Figure 3: A disvision of the gravity–capillary water wave problem in finite depth by the spectrum of $\mathbf{L}$ near the origin. Here, dots indicate a simple eigenvalue, whereas crosses indicate an eigenvalue of algebraic multiplicity two. The grey areas are parameter regions in which modulated solitary waves, multitrough solitary waves and solitary waves of depression are found.

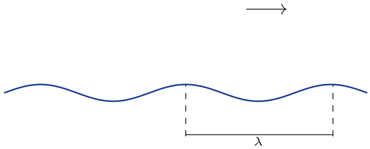
equation. Hur & Johnson [44] give an existence result for periodic waves and an instability result. The work of Ehrnström *et al.* [22] investigates global bifurcation of periodic waves and provides many important qualitative properties of the equation in question. Numerical investigations of the Whitham equation with surface tension can be found in [20]. To end this section, we comment that another potential tool is from the paper [33] by Groves, who re-establishes the gravity–capillary waves mentioned here using a rigorously derived nonlocal formulation for (3)–(4) and Fourier analysis.

1.5 Three-dimensional gravity–capillary waves Our search for waves extends to three-dimensional ones in this section. The trivial wave $(0, \lambda_0)$ in previous sections is replaced by a two-dimensional traveling wave $(U_\star, \lambda_\star)$. Here, x continues to be the direction of propagation, whereas z is the other horizontal direction — the *transverse* direction to x. Further, we consider a full three-dimensional *transverse spatial dynamics formulation* of equations (3)–(4), in which z plays the role of time. To contrast, the previous sections study two-dimensional waves homogeneous in z. Thus, all derivatives in z disappear and the only spatial variable available is the direction of propagation x. The transverse spatial dynamics formulation in a traveling frame $\tilde{x} = x - ct$ has the form

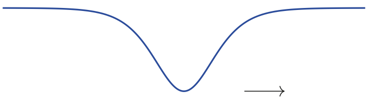
$$\frac{\mathrm{d}U}{\mathrm{d}z} = \mathbf{D}U_t + \mathbf{L}_\star U + \mathbf{R}_\star(U, \lambda) \tag{15}$$



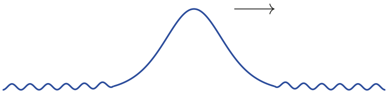
(a) A gravity solitary wave. It has supercritical wave speed ($\alpha < 1$).



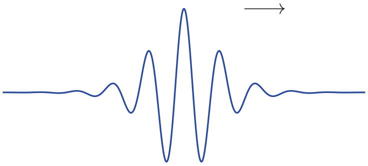
(b) A periodic wave with wavelength λ. Here, the wavenumber is $\xi = 2\pi/\lambda$.



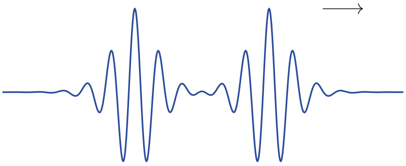
(c) A solitary wave of depression found near C_4. It has subcritical wave speed ($\alpha > 1$).



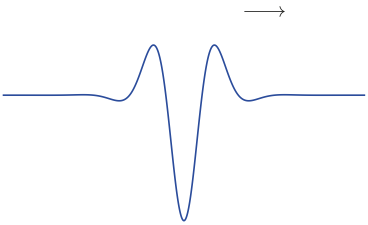
(d) A generalized solitary wave found near C_3. It has supercritical wave speed ($\alpha < 1$).



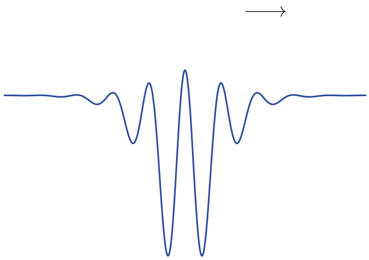
(e) A modulated solitary wave found near C_2. It has subcritical wave speed ($\alpha > 1$).



(f) A multipulse solitary wave near $(\beta, \alpha) = (1/3, 1)$ along C_2. It has subcritical wave speed ($\alpha > 1$).



(g) A multitrough solitary wave with one trough near $(\beta, \alpha) = (1/3, 1)$ along C_1. It has subcritical wave speed ($\alpha > 1$).



(h) A multitrough solitary wave with two troughs

Figure 4: A catalogue of small-amplitude waves, both gravity and gravity–capillary.

where $\mathbf{D}, \mathbf{L}_\star$ are linear operators, and $\mathbf{R}_\star$ satisfies the tangency condition $\mathbf{R}(U_\star, \lambda_\star) = 0$ and $\mathrm{D}_U \mathbf{R}_\star(U_\star, \lambda_\star) = 0$. A traveling-wave solution U with the same constant speed $c > 0$ satisfies

$$\frac{\mathrm{d}U}{\mathrm{d}z} = \mathbf{L}_\star U + \mathbf{R}_\star(U, \lambda). \tag{16}$$

Thus, $\mathbf{L}_\star$ is the linearization of the right-hand side in (16) at $(U_\star, \lambda_\star)$. We are interested in traveling-wave solutions to (16) that are not constant in any horizontal direction. As before, the purely imaginary spectrum of $\mathbf{L}_\star$ is of fundamental importance. In this case, it poses a major challenge to study. Its coefficients are no longer constant, as in the case of linearization at trivial waves. These now depend on $(U_\star, \lambda_\star)$ which lacks explicit formulas. If $\mathbf{L}_\star$ possesses exactly a pair of purely imaginary eigenvalues $\pm \mathrm{i}k_\star$ with $k_\star > 0$ and equation (16) satisfies a number of other requirements, there will be a solution curve of three-dimensional waves emerging from a two-dimensional wave. This phenomenon is called a *dimension-breaking bifurcation*.

A related topic to dimension-breaking bifurcation is the *transverse instability* of two-dimensional traveling waves. The most classical stability definition is by Lyapunov. A solution $U_\star$ to (13) with parameter $\lambda_\star$ is said to be stable if any other solution U with initial value close to $U_\star$ remains close to $U_\star$ for all future time. It is customary that the stability of $(U_\star, \lambda_\star)$ is studied first via the linearized equation of (15) at $(U_\star, \lambda_\star)$. The solution $(U_\star, \lambda_\star)$ is called transversely linearly unstable if

$$\frac{\mathrm{d}U}{\mathrm{d}z} = \mathbf{D}U_t + \mathbf{L}_\star U \tag{17}$$

has a solution exponentially growing in time t as $t \to \infty$ but bounded in x, z. A result by Godey [32] gives a simple criterion for transverse linear instability: if $\mathbf{L}_\star$ has a pair of purely imaginary eigenvalues $\pm \mathrm{i}k_\star$ with $k_\star \neq 0$ and equation (17) satisfies a number of other requirements, then $(U_\star, \lambda_\star)$ is transversely linearly unstable.

Both phenomena relate to the *transverse dynamics* of a solution and they boil down to the same investigation of the purely imaginary spectrum of $\mathbf{L}_\star$. Before attempting them, one might consult simpler model equations which describe the transverse dynamics of (3)–(4) well. An example is the Kadomtsev–Petviashvili equations, for short the KP equations. These are two-dimensional versions of the KdV equations when the transverse effect is weak. To clarify, the nondimensionalization (5) in the three-dimensional equations is completed with the scaling $\tilde{z} = \lambda_z z$, where λ_z is a characteristic wavelength in the z-direction. An additional parameter is defined to measure the transverse effect, namely $\gamma = \lambda/\lambda_z$. The KP equations are valid in the regime

$$\delta^2 = \varepsilon, \quad \gamma^2 = \varepsilon, \quad \text{and} \quad \varepsilon \to 0.$$

These are further grouped into the KP-I equation for strong surface tension ($\beta > 1/3$), and the KP-II equation for weak or zero surface tension ($0 \leq \beta < 1/3$). In other words, the KP-I equation is valid in the parameter region near C_4, and the KP-II equation is valid in the parameter region near C_3. The transverse dynamics of periodic and solitary waves in the parameter region near C_4 has been extensively studied in the literature, first for the KP-I equation [1, 49, 37] and then for the full gravity–capillary water wave problem [34, 38, 68]; see also references therein. The transverse dynamics of modulated solitary waves in the parameter region near C_2 has been studied by Groves *et al.* [35] using another model equation, namely the Davey–Stewartson equation. The case of C_3 has not received as much attention. However, there are predictions provided by Haragus & Wahlén [40] for periodic and generalized solitary waves from studying a fifth-order improvement of the KP-II equation. These predictions are not confirmed for the full problem (3)–(4).

In this exciting field of research, the contribution of Paper III consists in establishing the transverse dynamics of two-dimensional traveling periodic waves with fixed parameters (β, α), not near any parameter curves C_i, $i = 1, 2, 3, 4$. It fills in a gap in the above line of research, and is an initial step to confirm the instability predictions for two-dimensional periodic and generalized solitary waves near C_3 in [40].

2 Mathematical tools

Let X, Λ and Y be real Banach spaces. Consider an abstract parameter-dependent equation

$$F(U, \lambda) = 0, \tag{18}$$

where $F\colon X \times \Lambda \to Y$, $U\colon \mathbb{R}^m \to \mathbb{R}^n$ is a function and $\lambda \in \mathbb{R}^l$ is a parameter. Assume that the trivial state is a solution of (18) for each $\lambda \in \Lambda$, that is $F(0, \lambda) = 0$. Bifurcation is a phenomenon that when λ varies and hits a critical value λ_0, there are solutions (U, λ) accumulating at $(0, \lambda_0)$ and they are qualitatively different to the trivial solution $(0, \lambda_0)$. Another scenario is the dimension-breaking bifurcation, in which $(q + 1)$-dimensional solutions emerge from a family of q-dimensional solutions. For the water wave problem in Section 1, U could be the fluid surface or the couple (η, ϕ), λ could be the gravity parameter α or the surface tension parameter β, and the trivial state corresponds to the fluid at rest, that is, $\eta \equiv 0$ and $\Phi \equiv 0$. Bifurcation phenomena are generally divided into two topics: local and global. In local bifurcation, one studies nontrivial solutions accumulating at $(0, \lambda_0)$, which might form a continuous curve of solutions near $(0, \lambda_0)$. For example, the small-amplitude waves in Sections 1.2 and 1.4 bifurcate locally from $(0, \lambda_0)$ and they constitute a local bifurcation curve. In global bifurcation, one studies what happens with a continuation of this local bifurcation curve, for instance how it connects to other solution curves or bifurcation points. The highest waves discussed in Section 1.3 are a result of global bifurcation. We present some mathematical tools to study these phenomena below.

2.1 Techniques for local bifurcation We are interested in small solutions U to equation (18) for each λ sufficiently near λ_0. According to the implicit function theorem, if the linearization $\mathrm{D}_U F(0, \lambda_0)$ is invertible, there is a unique solution curve $\{(U(\lambda), \lambda)\}_\lambda$

parametrized by the parameter λ near $(0, \lambda_0)$. From the assumption, this must be the trivial solution curve. As a consequence, the trivial solution curve cannot branch out, or equivalently no bifurcation occurs. A more exciting scenario arises when $\mathrm{D}_U F(0, \lambda_0)$ is not invertible. Depending on the degeneracy of $\mathrm{D}_U F(0, \lambda_0)$, one might employ a reduction technique, which is a collective term for techniques that transform (18) into an equation in fewer dimensions, preferably finite. Below is a short excursion into the realm of these techniques and we assume without loss of generality that $(U_0, \lambda_0) = (0, 0)$.

Our first example is the Lyapunov–Schmidt reduction, due to Lyapunov [62, 63] and Schmidt [72]. Its setup starts with a C^2 mapping F in a neighborhood of $(0, 0)$. Further, it requires that $\mathrm{D}_U F(0, 0)$ is a Fredholm operator. By definition, this means that $\mathrm{D}_U F(0, 0)$ is bounded with closed range, and that its kernel and cokernel have finite dimensions. Its lack of invertibility is measured by a so-called Fredholm index, which is the difference between the kernel dimension and the cokernel dimension. As a consequence, $\mathrm{D}_U F(0, 0)$ becomes invertible once these finite-dimensional subspaces are taken care of. The Lyapunov–Schmidt reduction does this as follows. First, decompose the spaces

$$X = \ker \mathrm{D}_U F(0, 0) \oplus \tilde{X} \quad \text{and} \quad Y = \tilde{Y} \oplus \operatorname{coker} \mathrm{D}_U F(0, 0).$$

Let Q be a projection operator onto $\tilde{Y}$. Equation (18) can be decomposed into

$$QF(U, \lambda) = 0 \quad \text{and} \quad (\mathrm{Id} - Q)F(U, \lambda) = 0.$$

Further, $U = U_1 + U_2$ with $U_1 \in \ker \mathrm{D}_U F(0, 0)$ and $U_2 \in \tilde{X}$. The first equation can be solved using the implicit function theorem, as $Q\mathrm{D}_{U_2} F(0, 0) \colon \tilde{X} \to \tilde{Y}$ is now invertible. The solutions can be written as $U = U_1 + U_2(U_1, \lambda)$ in a neighborhood of $(0, 0)$ in $X \times \Lambda$. Inserting this representation into the second equation, we obtain a finite-dimensional equation

$$(\mathrm{Id} - Q)F(U_1 + U_2(U_1, \lambda), \lambda) = 0.$$

This is a reduction and the finite-dimensional equation aboved is called a reduced equation. The Lyapunov–Schmidt reduction is a basis of many local bifurcation results, such as the celebrated Crandall–Rabinowitz local bifurcation theorem; see for instance [17, 18]. It lists sufficient conditions for (18) to have a local bifurcation point at $(0, 0)$. In particular, if F is C^k with $k \geq 2$, then the local bifurcation curve is C^{k-1}.

Unfortunately, working with solitary waves often means that $\mathrm{D}_U F(0, 0)$ has an essential spectrum at 0, thus the Fredholm property cannot be fulfilled. A technique which handles this and provides in addition a systematic way to analyze the solutions is the center manifold reduction. The most established version treats equation (18) with F of the form

$$F(U, \lambda) = \frac{\mathrm{d}U}{\mathrm{d}\tau} - \mathbf{L}U - \mathbf{R}(U, \lambda), \tag{19}$$

where $U \colon \tau \mapsto U(\tau) \in Y$ is a real-valued function, $\mathbf{L} \colon Z \to Y$, Z is a Banach space and $Z \hookrightarrow X \hookrightarrow Y$ are continuous embeddings. Equation (18) in this case is an evolutionary equation in τ, τ is called the time variable, and Y is the phase space of (18). Assume that $\mathbf{R}$ is at least C^2 in a neighborhood of $(0, 0)$, and that it satisfies a tangency condition $\mathbf{R}(0, 0) = 0$ and $\mathrm{D}_U\mathbf{R}(0, 0) = 0$, which ensures that $U \mapsto \mathrm{d}U/\mathrm{d}\tau - \mathbf{L}U$ is the linearization of F at $(0, 0)$. The center manifold reduction requires that the purely imaginary spectrum of $\mathbf{L}$ consists of finitely many eigenvalues counting multiplicities, and that it is well-separated from the rest of the spectrum. One consequence of this requirement is that the spectral subspace corresponding to these purely imaginary eigenvalues is finite-dimensional. Using a spectral projection operator $\mathbf{P}_\mathrm{c}$ given by Dunford's integral formula, equation (18) can be decomposed into an equivalent system

$$\frac{\mathrm{d}U_\mathrm{c}}{\mathrm{d}\tau} = \mathbf{L}U_\mathrm{c} + \mathbf{P}_\mathrm{c}\mathbf{R}(U, \lambda) \quad \text{and} \quad \frac{\mathrm{d}U_h}{\mathrm{d}\tau} = \mathbf{L}U_h + (\mathrm{Id} - \mathbf{P}_0)\mathbf{R}(U, \lambda).$$

Here, $U = U_\mathrm{c} + U_h$, $U_\mathrm{c} = \mathbf{P}_\mathrm{c}U$, $U_h = (\mathrm{Id} - \mathbf{P}_\mathrm{c})U$, and $\mathbf{P}_\mathrm{c}\mathbf{L} = \mathbf{L}\mathbf{P}_\mathrm{c}$. Note that U_c belongs to the finite-dimensional spectral subspace corresponding to imaginary eigenvalues of $\mathbf{L}$. Suppose U_c solves the first equation. By cutting off in the phase space Y, one obtains a modification of (18). In particular, this modified equation is the same as (18) if U_c is sufficiently small. A fixed point scheme involving a contraction with parameter U_c can be set up for the modified equation. As a result, there exists a unique solution to the modified equation in a neighborhood of $(0, 0)$ for each such U_c for all τ. The solution set is parametrized by U_c and this parametrization has the same smoothness as $\mathbf{R}$. We write $U = U_\mathrm{c} + \Psi(U_\mathrm{c}, \lambda)$, where U_c belongs to the finite-dimensional spectral subspace corresponding to the purely imaginary eigenvalues of $\mathbf{L}$. The set given by $U = U_\mathrm{c} + \Psi(U_\mathrm{c}, \lambda)$ is called the center manifold, the function Ψ is called a reduction function. In particular, the center manifold captures all sufficiently small solutions to (18). Finally, a reduction is obtained by inserting the parametrization into the first equation

$$\frac{\mathrm{d}U_\mathrm{c}}{\mathrm{d}\tau} = \mathbf{L}U_\mathrm{c} + \mathbf{P}_\mathrm{c}\mathbf{R}(U_\mathrm{c} + \Psi(U_\mathrm{c}, \lambda), \lambda),$$

just as for the Lyapunov–Schmidt reduction. This defines a finite-dimensional system, the solution set to which contains sufficiently small solutions of (18). Moreover, by finding sufficiently small solutions U_c to the finite-dimensional reduction, one finds sufficiently small solutions to (18). This correspondence has proved extremely useful in finding small solutions to (18) *and* in understanding their qualitative properties. The center manifold reduction is due to Pliss [66], Kelley [52] for finite-dimensional systems, Mielke [64], and Vanderbauwhede & Iooss [75] for infinite-dimensional systems. A modern presentation of this theorem can be found in [39], along with examples and more references.

A recent development has combined the above techniques to study nonlocal equations, where F is of the form

$$F(U, \lambda) = U + K * U + \mathbf{R}(U, \lambda). \tag{20}$$

This work is due to Faye & Scheel [30, 31], and we refer to the following reduction technique as the Faye–Scheel center manifold theorem. Here, $U\colon \mathbb{R} \to \mathbb{R}^n$, $\mathbf{R}$ satisfies the tangency condition $\mathbf{R}(0,0) = 0$, $\mathrm{D}_U\mathbf{R}(0,0) = 0$ and several other technical conditions on smoothness. The convolution kernel $K = (K_{ij})_{i,j=1}^n \colon \mathbb{R} \to \mathbb{R}^{n\times n}$ is matrix-valued with integrable and exponentially localized entries K_{ij}, which can be motivated as follows. Suppose that $K_{ij}(x) \simeq \exp(-\eta_0 x)$ as $x \to \infty$. It follows that the linearization of (18) at $(0,0)$, that is, $\mathcal{T} := U \mapsto U + K * U$, will be bounded in an exponentially weighted Sobolev space

$$H^1_{-\eta} := \{U \in H^1_{\mathrm{loc}} \, : \, \omega_{-\eta} U^{(j)} \in L^2 \text{ for } j = 0,1\},$$

where $\eta \in (0, \eta_0)$ and $\omega_{-\eta} \colon \mathbb{R} \to \mathbb{R}$ is a positive, smooth and exponentially decaying function with rate $-\eta$. In addition, $\mathcal{T} \colon H^1_{-\eta} \to H^1_{-\eta}$ is a pseudodifferential operator with multiplier $\mathrm{Id} + \mathcal{F}(K)$. To determine the nullspace of this linearization, the roots of $\det(\mathrm{Id} + \mathcal{F}(K))$ in the complex strip $|\mathrm{Im}\, z| < \eta$ containing the real line are considered. Using Fourier analysis, these conditions on K ensure that $\det(\mathrm{Id} + \mathcal{F}(K))$ in this strip is analytic and has isolated roots, each of finite multiplicity. In particular, each real root corresponds to a nullspace element of at most algebraic growth. Thus, by taking a sufficiently thin strip, non-real roots (corresponding to an exponentially growing nullspace element as $x \to \infty$ or $x \to -\infty$) can be excluded. This bears resemblance to the spectral condition placed on the linearization $\mathbf{L}$ in (19). The authors of [30, 31] impose further conditions on K to guarantee that $\mathcal{T}$ is a Fredholm operator that is onto. This means in particular that $\mathcal{T}U = V$ is uniquely solvable for each V up to a nullspace element of $\mathcal{T}$. By performing a Fredholm bordering using a projection operator $\mathcal{Q} \colon H^1_{-\eta} \to \ker \mathcal{T}$, one obtains an extended system $(\mathcal{T}, \mathcal{Q})$ which is invertible. Equation (18) can now be decomposed into

$$\mathcal{Q}U = U_{\mathrm{c}} \quad \text{and} \quad \mathcal{T}U = -\mathbf{R}(U, \lambda).$$

Applying an appropriate cut-off operator in $H^1_{-\eta}$, a modified equation is obtained. As before, one sets up a fixed-point scheme involving a contraction to solve the modified equation uniquely for each parameter $U_{\mathrm{c}} \in \ker \mathcal{T}$. One can then express solutions to the modified equation as a manifold $U_{\mathrm{c}} + \Psi(U_{\mathrm{c}}, \lambda)$ parametrized by $U_{\mathrm{c}} \in \ker \mathcal{T}$. So far, this is just the Lyapunov–Schmidt reduction. One ingenuity of [30, 31] lies in the process of constructing a reduced finite-dimensional evolutionary system. For this purpose, it is required that equation (18) possesses the translation symmetry $S_\tau \colon U \mapsto U(\,\cdot\, + \tau)$ for all $\tau \in \mathbb{R}$. Choosing a projection $\mathcal{Q}$ that commutes with all translations, the resulting center manifold is invariant under translations as well, which means U is a center manifold element if and only if $S_\tau U$ is for any $\tau \in \mathbb{R}$. A finite-dimensional flow in the translation parameter τ is obtained by projecting $S_\tau U = S_\tau(U_{\mathrm{c}} + \Psi(U_{\mathrm{c}}, \lambda))$ onto $\ker \mathcal{T}$ for U in the center manifold. By differentiating in τ and evaluating at $\tau = 0$, one obtains an autonomous evolutionary system in U_{c}. This becomes a system of ODEs in U when a special choice of projection $\mathcal{Q}$ is made, namely $\mathcal{Q}$ with projection coefficients that are linear combinations of $\{U^{(k)}(0)\}_{k=0}^N$. Here, N is the dimension of $\ker \mathcal{T}$. As before, all sufficiently small solutions of (18) in the uniform local Sobolev norm are captured in the center manifold, for which there is a one-to-one correspondence to the reduced system of ODEs. Thus, an analysis of sufficiently small solutions to this reduced system of ODEs gives an approximate description of sufficiently small solutions to (18). At this step, one has access to the generous toolbox for nonlinear ODEs. Paper I extends the Faye–Scheel center manifold to kernels K that might lack one required integrability property but still give rise to a Fredholm linearization with desired properties. The reference [14] offers another version for quasilinear problems. Finally, the work of Bakker & Scheel [7] provides a Hamitonian structure for a class of nonlocal equations.

There are several other techniques which can be used to establish solutions near a local bifurcation point, for instance the implicit function theorem and variational methods. However, the center manifold is able to capture all sufficiently small solutions near this point. Moreover, it provides a description of these solutions via the corresponding reduced finite-dimensional system of ODEs. To illuminate its role in our analysis, we move to the topic of global bifurcation.

2.2 Global bifurcation For our convenience, we define

$$\begin{aligned}\mathcal{S} &= \{(U,\lambda) \in X \times \Lambda \,:\, F(U,\lambda) = 0\},\\ \mathcal{C}_T &= \{(0,\lambda) \,:\, \lambda \in \Lambda\} \subset \mathcal{S},\end{aligned}$$

which are the full solution set and the trivial solution curve, respectively. Let $U\colon s \mapsto U(s) \in X$ and $\lambda\colon s \mapsto \lambda(s) \in \Lambda$ be continuous functions in $s \in (0,\epsilon)$, such that $F(U(s),\lambda(s)) = 0$. In addition, we suppose $U(s) \to 0$ in X and $\lambda(s) \to \lambda_0$ in Λ as $s \to 0$. Define a local bifurcation curve

$$\mathcal{C}_{\lambda_0,\mathrm{loc}} = \{(U(s),\lambda(s)) \in \mathcal{S} \,:\, s \in (0,\epsilon)\}.$$

One of the earliest and most well-known contributions to the theory of global bifurcation was made by Rabinowitz [67] in 1971 for F of the form

$$F(U,\lambda) = U - \lambda \mathbf{L} U - \mathbf{R}(U,\lambda),$$

where $\Lambda = \mathbb{R}$, $\mathbf{L}\colon X \to X$ is a compact linear operator, $\mathbf{R}$ is a completely continuous mapping such that $\mathbf{R}(U,\lambda) = \mathcal{O}(\|U\|^2)$ as $U \to 0$ and uniformly bounded on compact intervals of λ. Suppose $\lambda_0 \in \mathbb{R} \setminus \{0\}$ is such that λ_0^{-1} is an eigenvalue of $\mathbf{L}$. A value λ_0 with this property is called a characteristic value of $\mathbf{L}$. A result by Krasnosel'skii [58] gives that if λ_0 has odd algebraic multiplicity, then $(0,\lambda_0)$ is a bifurcation point, and that $\mathcal{C}_{\lambda_0,\mathrm{loc}}$ exists. The Rabinowitz global bifurcation theorem now states that the closure $\overline{\mathcal{S}} \subset X \times \Lambda$ possesses a maximum subcontinuum $\mathcal{C}_{\lambda_0}$ that contains $\mathcal{C}_{\lambda_0,\mathrm{loc}}$ and which

(1) meets infinity in $X \times \Lambda$, or

(2) meets another bifurcation point $(\lambda_\star, 0)$, where $\lambda_\star \neq \lambda_0$ and $\lambda_\star$ is a characteristic value of $\mathbf{L}$.

Alternative (1) is called blowup in the modern literature. The proof of Rabinowitz' theorem consists in two main ingredients: the Leray–Schauder degree and point-set topological methods. The assumption on $\mathbf{L}$ guarantees that the Leray–Schauder degree is well-defined. The assumptions on $\mathbf{L}$ and $\mathbf{R}$ imply that $\mathrm{Id} - \lambda\mathbf{L} - \mathbf{R}(\,\cdot\,,\lambda)$ is a proper mapping, which in turn gives that bounded solution sets are compact. This makes point-set topological methods accessible. Unfortunately, these assumptions leave out many interesting mathematical models. Observing that these two ingredients above could be accessible without such strong assumptions, the works [53, 41] and references therein provide similar global bifurcation theorems for other mappings.

Another area of global bifurcation theory deals with analytic mappings. We summarize a version for one-dimensional global bifurcation from [12]. Here, $\Lambda \subset \mathbb{R}$ and $F\colon X \times \Lambda \to Y$ is real analytic on an open subset $\mathcal{U} \subset X \times \Lambda$ containing the curve $\mathcal{C}_T$. Further, it is assumed that $\mathrm{D}_U F(U,\lambda)$ is Fredholm of index 0 for all $(U,\lambda) \in \mathcal{S} \cap \mathcal{U}$. Lastly, for some critical value λ_0, the kernel $\mathrm{D}_U F(0,\lambda_0)$ is one-dimensional and satisfies a transversality condition, which guarantees that the local bifurcation curve branches out from $\mathcal{C}_T$ in a transversal fashion. Then, there exists a local bifurcation curve $\mathcal{C}_{\lambda_0,\mathrm{loc}}$ analytically parametrized by a real parameter $s \in (0,\epsilon)$. Suppose that the parametrization $\lambda(s)$ is injective, and that bounded and closed subsets in $\mathcal{S} \subset X \times \Lambda$ are compact. Then $\mathcal{C}_{\lambda_0,\mathrm{loc}}$ has a continuous-curve extension $\mathcal{C}_{\lambda_0}$ parametrized by $s \in (0,\infty)$, such that

$$\mathcal{C}_{\lambda_0,\mathrm{loc}} \subset \mathcal{C}_{\lambda_0} \subset \mathcal{S}.$$

Along this extension $\mathcal{C}_{\lambda_0}$ there might be points $(U_\star,\lambda_\star)$ such that $\ker \mathrm{D}_U F(U_\star,\lambda_\star)$ is nontrivial. These are potential secondary bifurcation points of $\mathcal{C}_{\lambda_0}$, and they will not have any accumulation point by analyticity. Even though the curve $\mathcal{C}_{\lambda_0}$ might not be smooth, it has a reparametrization that is analytic locally near each point. The analytic global bifurcation theorem gives three different alternatives for $\mathcal{C}_{\lambda_0}$:

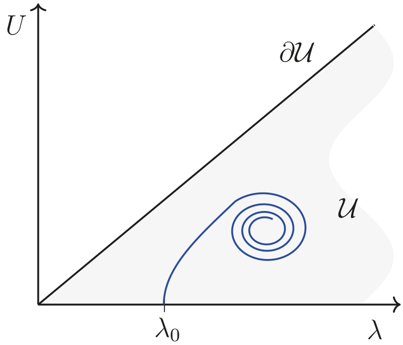
(1a) $\|(U(s),\lambda(s))\|_{X\times\Lambda} \to \infty$ as $s \to \infty$,

(2a) $(U(s),\lambda(s))$ tends to the boundary of $\mathcal{U}$ as $s \to \infty$,

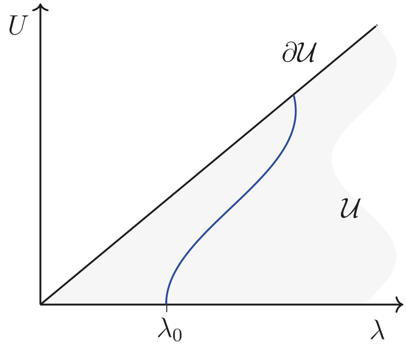
(3a) $\mathcal{C}_{\lambda_0}$ is a closed loop.

These are not mutually exclusive. The global bifurcation picture here is much more detailed, its success consists in the assumption on the analytic structure, which unlocks results on analytic varieties and which is well-studied in complex analysis.

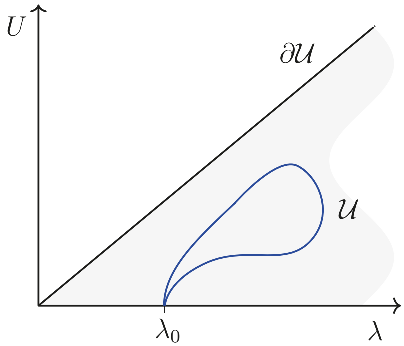
The Euler equations indeed have the analytic structure required by the above analytic global bifurcation theorem. However, working with solitary waves still presents two major challenges. First, the linearization $\mathrm{D}_U F(0,\lambda_0)$ is not Fredholm and therefore finding a local bifurcation curve of solitary waves might prove difficult. Second, useful compactness properties are not available for Sobolev spaces on unbounded intervals, which are common analytic settings for solitary waves. None of these issues are present when working with



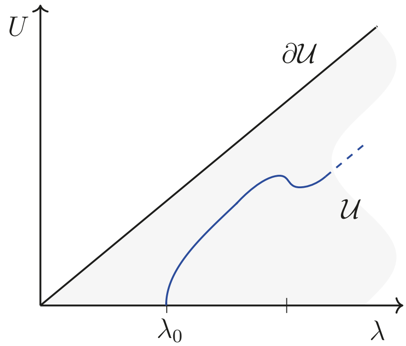
(a) Loss of compactness



(b) A blowup scenario, in which $\mathcal{C}_{\lambda_0}$ approaches $\partial\mathcal{U}$



(c) A blowup scenario, in which $\mathcal{C}_{\lambda_0}$ approaches $\partial\mathcal{U}$ by returning to the bifurcation point



(d) A blowup scenario, in which a sequence of solutions along $\mathcal{C}_{\lambda_0}$ tends to infinity

Figure 5: An illustration of some alternatives given by the analytic global bifurcation theorem adapted for solitary waves by Chen *et al.* [13]. Here, the open set $\mathcal{U}$ is the gray area. The trivial solution $U = 0$ with parameter $\lambda = \lambda_0$ belongs to $\partial\mathcal{U}$. Thus, the blowup alternative (1a-s) includes the closed-loop alternative (3a).

periodic functions, for example. The first issue might not hinder a construction of the local bifurcation curve. Thus, we might replace assumptions that give a local bifurcation curve by the existence of $\mathcal{C}_{\lambda_0,\mathrm{loc}}$ with desired properties. Given a local bifurcation curve, the Fredholm property of $\mathrm{D}_U F(U,\lambda)$ for $(U,\lambda) \in \mathcal{C}_{\lambda_0,\mathrm{loc}} \setminus \{(0,\lambda_0)\}$ can be used to extend this curve a little further. To address the second issue, we simply include a loss of compactness alternative and exclude it later using qualitative properties. This was first recognized by Wheeler in [76] and an analytic version of this is formulated in [13], which is used in Paper I. In this version, $F\colon X \times \Lambda \to Y$ and $\mathcal{U} \subset X \times \Lambda$ is an open subset, on which F is an analytic mapping. Assume that $(0, \lambda_0) \in \partial\mathcal{U}$ is a bifurcation point. Further, $\mathcal{C}_{\lambda_0,\mathrm{loc}}$ is a local bifurcation curve with a continuous parametrization by $s \in (0, \epsilon)$, with $U(s) \to 0$ and $\lambda(s) \to \lambda_0$ as $s \to 0$. For each $(U, \lambda) \in \mathcal{C}_{\lambda_0,\mathrm{loc}}$, the linearization $\mathrm{D}_U F(U, \lambda)$ is invertible. Lastly, for each $(U, \lambda) \in \mathcal{U} \cap \mathcal{S}$, $\mathrm{D}_U F(U, \lambda)$ is Fredholm of index 0. Then, $\mathcal{C}_{\lambda_0,\mathrm{loc}}$ has an extension $\mathcal{C}_{\lambda_0}$ which is continuously parametrized by $s \in (0, \infty)$. The extension $\mathcal{C}_{\lambda_0}$ has an analytic reparametrization locally near each point $(U, \lambda) \in \mathcal{C}_{\lambda_0}$. Define

$$M(s) := \|U(s)\|_X + \|\lambda(s)\|_\Lambda + \frac{1}{\mathrm{dist}((U(s), \lambda(s)), \partial\mathcal{U})}.$$

The alternatives for $\mathcal{C}_{\lambda_0}$ are now

(1a-s) the quantity $M(s)$ tends to ∞ as $s \to \infty$,

(2a-s) there exists a sequence $s_n \to \infty$ as $n \to \infty$, for which $\sup_n M(s_n) < \infty$ but $\{U(s_n)\}_n$ has no convergent subsequence in X.

The authors of [13] refer to (1a-s) as the blowup alternative. Note that it includes (1-a)–(3-a) in the previous analytic global bifurcation theorem. Alternative (2a-s) is called the loss of compactness alternative. An illustration is provided in Figure 5.

In Paper I, the aim is to prove the existence of a highest wave using the analytic global bifurcation theorem adapted for solitary waves. A lot of effort is dedicated to qualitative properties of the solutions, which are used in the application and then in the analysis of the global bifurcation theorem. In particular, by studying the highest wave, we see that it reaches the highest amplitude only by losing regularity. This corresponds to one specific scenario, namely $\|U(s)\|_X \to \infty$, in the blowup alternative if $X = H^k(\mathbb{R})$ with $k \geq 2$. However, the blowup alternative contains more scenarios, for instance $\mathcal{C}_{\lambda_0}$ returning to $(0, \lambda_0)$; see Figure 5. It is precisely here that the center manifold reduction finds its competitive edge. Through a qualitative description of solutions along the local bifurcation curve, we can exclude one of the undesired scenarios.

3 Summary of the research papers

Paper I The paper is concerned with the Whitham equation

$$c\varphi - K * \varphi - \varphi^2 = 0, \tag{21}$$

where K is the inverse Fourier transform of the Whitham symbol $m\colon \mathbb{R} \to \mathbb{R}$ given by

$$m(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}},$$

$c > 0$ is the wave speed and $c = 1$ is the critical wave speed. We establish the existence of a highest solitary-wave solution to equation (21). A local bifurcation curve emanating from $(\varphi, c) = (0, 1)$ is found using a version of the Faye–Scheel center manifold reduction. This theorem provides a second-order ODE, which is equivalent to (21) near the point $(\varphi, c) = (0, 1)$ in the $H^3_{\mathrm{u}} \times \mathbb{R}$ norm, and which reads

$$\begin{aligned}\varphi'' &= -6\varphi^2 + \frac{19}{5}(\varphi')^2 + 6(c-1)\varphi \\ &\quad + \mathcal{O}\left(|(\varphi, \varphi')| \left((c-1)^2 + |\varphi|^2 + |\varphi'|^2\right)\right).\end{aligned} \tag{22}$$

Here, H^3_{u} is a space of uniformly local H^3 functions defined on $\mathbb{R}$. After a rescaling, equation (22) is a perturbation of the KdV equation for each $c - 1 > 0$ sufficiently small. Using reversibility of (21), we find a family of symmetric solitary-wave solutions of the sech^2 form parametrized by $c - 1$, which defines a local bifurcation curve $\mathcal{C}_{\mathrm{loc}}$ emanating from $(\varphi, c) = (0, 1)$. Using non-negativity of solutions to (21), we show that any nontrivial, small-amplitude, even solitary-wave solution to (21) near $(0, 1) \in H^3_{\mathrm{u}} \times \mathbb{R}$ belongs to $\mathcal{C}_{\mathrm{loc}}$. This is precisely the advantage of a center manifold reduction. The local curve is extended using an analytic global bifurcation theorem adapted for solitary waves, where an additional possibility of loss of compactness is included. This possibility is ruled out by an integral identity for (21) and qualitative properties of solutions. After excluding other alternatives, we show that a sequence along the global bifurcation curve limits to a highest, even solitary-wave solution, which has supercritical wave speed $c > 1$, exponential decay as $|x| \to \infty$, and a crest of exactly $C^{1/2}$ Hölder regularity at the origin. This aligns with the findings of paper [25] for periodic waves.

Paper II The paper studies a gravity–capillary Whitham equation

$$\varphi - cK_\tau * \varphi + K_\tau * \varphi^2 = 0. \tag{23}$$

Here, K_τ is the inverse Fourier transform of

$$\ell_\tau(\xi) = \sqrt{\frac{1}{1 + \tau\xi^2} \frac{\xi}{\tanh(\xi)}},$$

$c > 0$ is a wave speed parameter and $\tau > 0$ is a surface tension parameter. Equation (23) is divided into two regimes — that of small surface tension ($\tau < 1/3$) and that of large surface tension ($\tau > 1/3$). Bifurcation phenomena of (23) are determined by complex solutions of the algebraic equation

$$1 - c\ell_\tau(z) = 0, \tag{24}$$

close to the real line. The linear dispersion relation for the water wave problem emerges after a rearrangement, suggesting

1. an $0^{2+}(\mathrm{i}k_{0,\tau})$ reversible bifurcation when crossing the curve

$$C_3 = \left\{ (\tau_0, c_0) \,:\, \tau_0 < \frac{1}{3} \text{ and } c_0 = 1 \right\}.$$

Here, for each fixed $(\tau, c) = (\tau_0, c_0) \in C_3$, the solutions to (24) consist precisely of 0, 0, $-k_{0,\tau}$ and $k_{0,\tau}$ counting multiplicity, where $k_{0,\tau} > 0$.

2. an $(\mathrm{i}s)^2$ reversible bifurcation when crossing the curve

$$C_2 = \left\{ (\tau_0, c_0) \,:\, \tau_0 = c_0^2 \cdot \left(-\frac{1}{2\sinh^2(s)} + \frac{1}{2s\tanh(s)} \right), \right.$$
$$\left. c_0^2 = \left(\frac{s^2}{2\sinh^2(s)} + \frac{s}{2\tanh(s)} \right)^{-1}, \text{ for } s > 0 \right\}$$

from below. Here, for each fixed $(\tau, c) = (\tau_0(s), c_0(s)) \in C_2$, the solutions to (24) are precisely $-s, -s, s, s$ counting multiplicity.

These parameter curves C_2 and C_3 are in fact the ones discussed in Section 1.4 with $\alpha = 1/c_0^2$ and $\beta = \tau_0/c_0^2$. We use a version of the Faye–Scheel center manifold theorem from Paper I and compute the reduced system of ODEs, which in this case is four-dimensional. Normal form theory is then applied to analyze this system. The bifurcation when crossing C_3 gives rise to a family of small-amplitude generalized solitary waves

$$\varphi(x) = \frac{3}{2}|\mu|\rho^{1/2}\operatorname{sech}^2\left(\frac{\rho^{1/4}\sigma^{1/2}|\mu|^{1/2}x}{\sqrt{2}} \right) + \frac{\mu}{2}(1 - \operatorname{sgn}(\mu)\rho^{1/2})$$
$$+ |\mu|k^{1/2}\cos\Big((k_0 + \mathcal{O}(\mu))x + \Theta_* + \mathcal{O}(\mu) \Big) + \mathcal{O}(\mu^2\rho^{1/2}),$$

where $\mu = c - 1$ is small, $\Theta_* \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ is arbitrary, $\sigma = (1/3 - \tau)^{-1}$, $\rho = 1 + 24k$ and $k = \mathcal{O}(|\mu|^{-1-2\kappa})$ for some $\kappa \in [0, 1/2)$. The bifurcation when crossing C_2 gives two distinct families of modulated solitary waves

$$\varphi(x) = \sqrt{\frac{-8q_0\mu}{q_1}}\operatorname{sech}\left(\sqrt{q_0\mu}x\right)\cos\left(sx + \Theta_* + \mathcal{O}(|\mu|^{1/2})\right) + \mathcal{O}(\mu^2),$$

of elevation ($\Theta_* = 0$) and depression ($\Theta_* = \pi$), respectively. Here, $\mu < 0$ is small, the coefficients q_0, q_1 are negative. These results align with the rich literature for the irrotational water wave problem, for example [5, 10, 11, 19, 47, 46, 54, 55], dating back to the pioneering work of Kirchgässner in 1982.

Paper III We consider the gravity–capillary water wave problem, that is, the Euler equations (3)–(4). Here, the capillary and gravity parameters, respectively β and α, are fixed. Traveling waves, that are periodic along the direction of propagation and constant along the other horizontal direction, are known to exist in

1. Region I, that satisfies $\beta > 0$ and $\alpha < 1$,
2. Region II, that satisfies $\alpha > 1$ and lies to the left of the curve

$$C_2 = \Big\{(\beta, \alpha) \,:\, \beta = -\frac{1}{2\sinh^2(s)} + \frac{1}{2s\tanh(s)} \quad \text{and} \\ \alpha = \frac{s^2}{2\sinh^2(s)} + \frac{s}{2\tanh(s)}, \quad \text{for } s \in [0, \infty)\Big\}.$$

These waves will be referred to as two-dimensional traveling periodic waves. In Region I, the linear dispersion relation (14) has precisely a pair of solutions $\pm k_*$, where $k_* > 0$. This implies that a two-dimensional spatial dynamics formulation of (3)–(4) with $\tilde{x}$ as time linearized at a trivial flow possesses a pair of simple purely imaginary eigenvalues $\pm \mathrm{i}k_*$, each of algebraic multiplicity one. Consequently, for each (β, α) in Region I, there is a one-parameter family of two-dimensional traveling periodic waves. In Region II, (14) has two pairs of simple solutions $\pm k_{*,1}$ and $\pm k_{*,2}$, with $0 < k_{*,1} < k_{*,2}$. For simplicity, we impose in addition a non-resonance condition that $k_{*,2}/k_{*,1} = q \notin \mathbb{N}$. Equivalently, parameters β and α are not allowed to lie on the curves

$$C_{2,q} = \Big\{(\beta, \alpha) \,:\, \beta = \frac{1}{(1-q^2)s\tanh(s)} - \frac{q}{(1-q^2)s\tanh(qs)} \text{ and} \\ \alpha = -\frac{q^2 s}{(1-q^2)\tanh(s)} + \frac{qs}{(1-q^2)\tanh(qs)}, \text{ for } s \in [0, \infty)\Big\},$$

for $q \in \mathbb{N}$. As a result, the linearization at a trivial flow has two pairs of simple purely imaginary eigenvalues $\pm \mathrm{i}k_{*,1}$ and $\pm \mathrm{i}k_{*,2}$, each gives rise to a distinct one-parameter family of two-dimensional traveling periodic waves. Our interest lies in the transverse dynamics of these periodic waves, which in this paper includes two phenomena — that of transverse linear instability and that of dimension-breaking bifurcation. The first step is to provide a transverse spatial dynamics formulation, which is a spatial dynamics formulation that has the horizontal direction transverse to the direction of propagation as "time". Linearizing the equations at these two-dimensional traveling periodic waves results in a linear operator. The following step consists in a spectral analysis of this linear operator as a closed and relatively bounded perturbation of the linearization at a trivial flow. It turns out that the linear operator in question has a pair of simple purely imaginary eigenvalues in Region I. This allows us to apply a simple criterion for transverse linear instability from [32], which readily establishes this transverse dynamics phenomenon of periodic waves in Region I.

The same spectral information enables an application of a Lyapunov center theorem, which establishes dimension-breaking bifurcations at each of these two-dimensional waves. The analysis for Region II is the same, but the outcome is slightly different. The family that consists of periodic waves with big wavenumbers is shown to feature these two transverse dynamics phenomena. However, this might not be the case for the other family of periodic waves. We give a characterization using wavenumbers for the case when both families exhibit transverse linear instability and dimension-breaking bifurcations.

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Paper I

Global bifurcation of solitary waves for the Whitham equation

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Abstract

The Whitham equation is a nonlocal shallow water-wave model which combines the quadratic nonlinearity of the KdV equation with the linear dispersion of the full water wave problem. Whitham conjectured the existence of a highest, cusped, traveling-wave solution, and his conjecture was recently verified in the periodic case by Ehrnström and Wahlén. In the present paper we prove it for solitary waves. Like in the periodic case, the proof is based on global bifurcation theory but with several new challenges. In particular, the small-amplitude limit is singular and cannot be handled using regular bifurcation theory. Instead we use an approach based on a nonlocal version of the center manifold theorem. In the large-amplitude theory a new challenge is a possible loss of compactness, which we rule out using qualitative properties of the equation. The highest wave is found as a limit point of the global bifurcation curve.

1 Introduction

In this paper, we continue the story of singular wave phenomena featured in the Whitham equation. The equation was proposed by G. B. Whitham in 1967 [33], in an attempt to remedy the failure of the KdV equation in capturing wave breaking and peaking. He proposed that the linear dispersion in the KdV equation with Fourier symbol $1 - \frac{1}{6}\xi^2$ should be replaced by the exact linear dispersion in the Euler equation with Fourier symbol

$$m(\xi) := \sqrt{\frac{\tanh(\xi)}{\xi}}.$$

Note that the dispersion in the KdV equation is the second-order approximation of m at $\xi = 0$. This leads to the nonlinear nonlocal evolution equation

$$u_t + (K * u + u^2)_x = 0,$$

known as the *Whitham equation*. Here, $u(x, t)$ describes the one-dimensional wave profile and the integral kernel K is given by

$$K(x) = (\mathcal{F}^{-1}m)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} m(\xi)\exp(\mathrm{i}x\xi)\ \mathrm{d}\xi.$$

The function K will be referred to as the *Whitham kernel*, and the function m as the *Whitham symbol*. Specializing to traveling waves $u = \varphi(x - ct)$ where $c > 0$ is the wave speed, integrating and performing a Galilean change of variables, the Whitham equation reduces to the nonlinear integral equation

$$c\varphi - K * \varphi - \varphi^2 = 0. \tag{1}$$

We are interested in functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ which satisfy (1) pointwise on $\mathbb{R}$, and which we refer to as solutions of (1) with wave speed c. More specifically, the results of this paper will concern *solitary* solutions, also called *solitary-wave* solutions. These are solutions $\varphi\colon \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{|x|\to\infty} \varphi(x) = 0$.

Despite its simple form, the nonlocal and nonlinear nature of the Whitham equation has made it challenging to study. Recent years have seen a large amount of existence and qualitative results on the solutions of the equation. Traveling small-amplitude periodic solutions were found by Ehrnström and Kalisch [21] using the Crandall–Rabinowitz bifurcation theorem. Then, Ehrnström, Groves and Wahlén [19] proved the existence of solitary waves using a variational method for a class of Whitham-type equations. This was followed up by Arnesen [4] where a class covering the Whitham equation with surface tension was considered. By applying a different technique — the implicit function theorem — Stefanov and Wright [31] achieved the same result. Ehrnström and Wahlén [22] showed the existence of a traveling cusped periodic wave φ using global bifurcation theory, and proved that $\frac{c}{2} - \varphi(x) \sim |x|^{1/2}$ near the origin. This wave attains the highest amplitude possible and is referred to as an *extreme wave solution*. They also conjectured that φ is convex and $\varphi = \frac{c}{2} - \sqrt{\pi/8}|x|^{1/2} + o(x)$ as $x \to 0$. Convexity of the extreme wave was shown by Encisco, Goméz-Serrano and Vergara [23] using a computer assisted proof.

The goal of this paper is to prove the existence of an extreme solitary-wave solution of (1) and our plan is to use a global bifurcation theorem appearing in [11]; see also [15, 16, 10]. The first main step is the construction of a local bifurcation curve, emanating from the point $(\varphi, c) = (0, 1)$, and the second is the application of the global bifurcation theorem.

A key to our success is the fact that a lot of qualitative properties have been shown for the Whitham kernel, the Whitham symbol and the solutions of (1), thanks to [22], [9] and [21]. These guide us in choosing a convenient function space to study (1) and have been extremely useful in the application of the global bifurcation theorem. In Section 2, we list the relevant properties and prove an integral identity. We also study how sequences of solutions converge and the Fredholm properties of important linear operators.

Another key is the recently developed center manifold theorem for nonlocal equations in [24]. This result states that nonlocal equations with exponentially decaying convolution kernels are essentially local equations near an equilibrium. It also provides a method to derive the local equation, which can then be studied using familiar ODE tools. In our case, the equilibrium is $(\varphi, c) = (0, 1)$. Although the Whitham kernel has the required exponential decay, it fails a local integrability condition. Seeing that this condition is only for proving Fredholm properties of linear operators, we directly prove these properties instead. All necessary changes for the general center manifold theorem are listed in Appendix B. In Section 3, we state the center manifold theorem for the Whitham equation and compute the corresponding local equation. More specifically, we prove the following.

Lemma 1.1. *There exist a neighborhood $\mathcal{V}' \subset \mathbb{R}$ of $c = 1$ and a number $\delta' > 0$ such that if $\varphi\colon \mathbb{R} \to \mathbb{R}$ satisfies $\sup_{y\in\mathbb{R}} \|\varphi(\,\cdot\, + y)\|_{H^3([0,1])} \le \delta'$, then φ solves* (1) *with wave speed $c \in \mathcal{V}'$ if and only if φ solves the second-order ODE*

$$\varphi'' = -6\varphi^2 + \frac{19}{5}(\varphi')^2 + 6(c-1)\varphi + \mathcal{O}(|(\varphi,\varphi')|((c-1)^2 + |\varphi|^2 + |\varphi'|^2)). \quad (2)$$

The ODE in this lemma is a c-dependent family of perturbed KdV equations. Restricting to $c > 1$ with c sufficiently close to 1, it features a unique positive even solitary-wave solution φ with exponential decay for each fixed c. Using equation (2), we show that $\sup_{y\in\mathbb{R}} \|\varphi(\,\cdot\, + y)\|_{H^3([0,1])} \lesssim c - 1$. So for c sufficiently close to 1, φ is also a solution to equation (1). We thus arrive at the first main result of this paper (repeated as Theorem 3.3 in Section 3).

Theorem 1.2. *There exists a unique local bifurcation curve $\mathcal{C}_{\text{loc}}$ which emanates from $(\varphi, c) = (0, 1)$ and consists of the non-trivial even solitary-wave solutions φ to* (1) *with wave speeds $c \in \mathcal{V}'$ satisfying $\sup_{y\in\mathbb{R}} \|\varphi(\,\cdot\, + y)\|_{H^3([0,1])} < \delta'$.*

While both [19] and [31] contain existence results for supercritical solitary waves, the additional information provided by the center manifold approach concerning uniqueness is crucial in the subsequent analysis. To end Section 3, we use the center manifold theorem to prove that the linearization of the left-hand side of (1) along $\mathcal{C}_{\text{loc}}$ is invertible. This is in preparation for the global bifurcation theorem.

The global bifurcation theory in [11] can now be applied to extend $\mathcal{C}_{\text{loc}}$ and this extension is referred to as the global bifurcation curve $\mathcal{C}$. The theory dictates several possible behaviors for $\mathcal{C}$ and the content of Section 4 is the exclusion of unwanted behaviors. We rule out the loss of compactness alternative using qualitative properties of the solutions, how sequences of solutions converge and an integral identity for (1). Then, we show that the blowup alternative happens as the Sobolev norm blows up and that an extreme solitary-wave solution is obtained in the limit. More precisely, we have the following result (repeated as Theorem 4.8 in Section 4).

Theorem 1.3. *There exists a sequence of elements (φ_n, c_n) on the global bifurcation curve $\mathcal{C}$ such that $\lim_{n\to\infty} \|\varphi_n\|_{H^3} = \infty$ and $(\varphi_n, c_n) \to (\varphi, c)$ locally uniformly, where φ is a solitary-wave solution of* (1) *with supercritical wave speed $c > 1$. The solitary solution φ is even, bounded, continuous, exponentially decaying, smooth everywhere except at the origin and*

$$C_1|x|^{1/2} \le \frac{c}{2} - \varphi(x) \le C_2|x|^{1/2} \quad \textit{as} \quad |x| \to 0,$$

for some constants $0 < C_1 < C_2$.

The function φ in the above theorem is the extreme solitary-wave solution we set out to find.

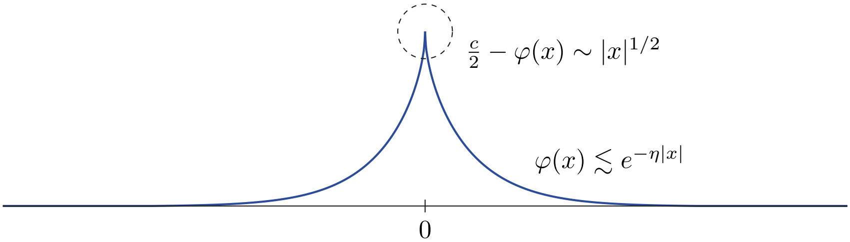

Figure 1: An extreme solitary-wave solution found by taking a limit of elements along the global bifurcation curve in Theorem 4.8. The wave speed c is supercritical, that is, $c > 1$. The wave profile φ is even and smooth on $\mathbb{R} \setminus \{0\}$. It has exponential decay as $|x| \to \infty$ and behaves like $\frac{c}{2} - C|x|^{1/2}$ as $|x| \to 0$. Ehrnström and Wahlén conjecture in [22] that $C = \sqrt{\pi/8}$.

By demonstrating the use of recent spatial-dynamics tools, this paper serves as an example to studies of other nonlocal nonlinear evolution equations. In particular, these results will likely extend to a larger class of equation, such as in [5] and [20].

Finally, it is interesting to compare our results with the global bifurcation theory for the water wave problem. The existence of an unbounded, connected set of solitary water waves, including a highest wave in a certain limit, was proved by Amick & Toland [2, 3] following several earlier small-amplitude results. Around the same time, Amick, Fraenkel & Toland [1] verified Stokes' conjecture for both periodic and solitary water waves, showing in particular that the limiting solitary wave is Lipschitz continuous at the crest with a corner enclosing a $120°$ angle. Thus, the behavior at the crest is different from the extreme Whitham wave, which has no corner due to the $C^{1/2}$ cusp in Theorem 1.3. The construction of the global solution continua in [2, 3] is also different from ours. While both proofs are based on nonlinear integral equations, the common approach in [2, 3] is to first apply global bifurcation theory to a regularized problem and then pass to the limit. On the other hand, we use global bifurcation theory directly on the solitary Whitham problem. A similar approach has in fact recently been used for solitary water waves with vorticity and stratification, but based on a PDE formulation [32, 11]. For the water wave problem with vorticity and stratification, the limiting behavior of large-amplitude waves is more complex and there is numerical and some analytical evidence of overhanging waves; see for example [14, 17, 18, 28] and references therein.

Notation We use the following notations for function spaces.

– The space of p^{th} power integrable functions on an interval $I \subset \mathbb{R}$ with respect to a measure μ is denoted by

$$L^p(I,\mu) := \left\{ f\colon \mathbb{R} \to \mathbb{R} \,\middle|\, \|f\|_{L^p(\mu)} < \infty \right\},$$

where

$$\|f\|_{L^p(I,\mu)} := \left(\int_I |f|^p \, \mathrm{d}\mu \right)^{1/p} \quad \text{if } p \in [1,\infty)$$

and

$$\|f\|_{L^\infty(I,\mu)} := \mu\text{-ess-sup}_{x\in I} |f(x)| \ \text{ if } p = \infty.$$

For $\sigma \in \mathbb{R}$, we write

$$L^p(I) := L^p(I, \mathrm{d}x), \qquad L^p := L^p(\mathbb{R}, \mathrm{d}x) \quad \text{ and } \quad L^p_\sigma := L^p(\mathbb{R}, \omega_\sigma^p \cdot \mathrm{d}x),$$

where $\mathrm{d}x$ is the Lebesgue measure and $\omega_\sigma\colon \mathbb{R} \to \mathbb{R}$ is a positive and smooth function, which equals $\exp(\sigma|x|)$ for $|x| > 1$. In particular, functions in L^p_η when $\eta > 0$ are necessarily exponentially decaying while functions in $L^p_{-\eta}$ can grow exponentially.

– The Sobolev space is denoted by

$$W^{j,p}(I) := \left\{ f\colon I \to \mathbb{R} \,\middle|\, f^{(n)} \in L^p(I), \ \text{for } 0 \le n \le j \right\}$$

and the weighted Sobolev space is

$$W^{j,p}_\sigma := \left\{ f\colon \mathbb{R} \to \mathbb{R} \,\middle|\, f^{(n)} \in L^p_\sigma, \ \text{for } 0 \le n \le j \right\},$$

where $f^{(n)}$ are weak derivatives of f for $1 \le n \le j$. These spaces are equipped with the norms

$$\|f\|_{W^{j,p}} := \left(\sum_{n=0}^{j} \|f^{(n)}\|^p_{L^p} \right)^{1/p} \quad \text{and} \quad \|f\|_{W^{j,p}_\sigma} := \left(\sum_{n=0}^{j} \|f^{(n)}\|^p_{L^p_\sigma} \right)^{1/p}.$$

We have the natural inclusions $W^{j,p}_{\sigma_2} \subset W^{j,p}_{\sigma_1}$ whenever $\sigma_1 < \sigma_2$. For $p = 2$, we denote the Hilbert spaces $W^{j,2}(I)$ and $W^{j,2}_\sigma$ by $H^j(I)$ and H^j_σ, respectively. As before, when $I = \mathbb{R}$, we omit writing $\mathbb{R}$.

– We define the space of uniformly local H^j functions

$$H^j_{\mathrm{u}} := \left\{ f \in H^j_{\mathrm{loc}} \,\middle|\, \|f\|_{H^j_{\mathrm{u}}} < \infty \right\}, \quad \text{where} \quad \|f\|_{H^j_{\mathrm{u}}} := \sup_{y\in\mathbb{R}} \|f(\,\cdot\, + y)\|_{H^j([0,1])}.$$

- C^k denotes the space of k times continuously differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$. $BUC^k \subset C^k$ denotes the space of functions with bounded and uniformly continuous derivatives of order up to and including k. $C^{k,\alpha}$ denotes the Hölder spaces

$$C^{k,\alpha} := \left\{ f \in BUC^k \,\middle|\, \sup_{h\neq 0} \frac{|f^{(k)}(x+h) - f^{(k)}(x)|}{|h|^\alpha} < \infty \right\}.$$

- The Besov space is denoted by $B^s_{p,q}$, where $s \in \mathbb{R}$, $p, q \in [1, \infty]$. We have

$$B^s_{2,2} = H^s, \;\; s \in \mathbb{R} \quad \text{and} \quad B^s_{\infty,\infty} = C^{\lfloor s \rfloor, s - \lfloor s \rfloor}, \;\; s \in \mathbb{R}_+ \setminus \mathbb{N}.$$

- $\mathscr{C}^k(\mathcal{X}, \mathcal{Y})$ denotes the space of k times Fréchet differentiable mappings between two normed spaces $\mathcal{X}$ and $\mathcal{Y}$.

We use the following scaling of the Fourier transform:

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}} f(x) \exp(-\mathrm{i}x\xi) \,\mathrm{d}x$$

and

$$\mathcal{F}^{-1}g(x) := \frac{1}{2\pi}\mathcal{F}g(-x).$$

2 Qualitative properties

2.1 The Whitham kernel and the Whitham symbol

The Whitham kernel K is given by $(\mathcal{F}^{-1}m)(x)$, where

$$m(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}} = 1 - \frac{1}{6}\xi^2 + \frac{19}{360}\xi^4 + \mathcal{O}(\xi^6). \tag{3}$$

Since $m(0) = 1$, we have $\int_{\mathbb{R}} K \,\mathrm{d}x = 1$. However, since $m \notin L^1$, K is singular at the origin. More specifically,

$$K(x) = \frac{1}{\sqrt{2\pi|x|}} + K_{\mathrm{reg}}(x), \tag{4}$$

where K_{reg} is real analytic on $\mathbb{R}$; see Proposition 2.4 in [22]. In addition, as $|x| \to \infty$,

$$K(x) = \frac{\sqrt{2}}{\pi\sqrt{|x|}} \exp\Big(-\frac{\pi}{2}|x|\Big) + \mathcal{O}\Big(|x|^{-3/2} \exp\Big(-\frac{\pi}{2}|x|\Big)\Big), \tag{5}$$

by Corollary 2.26 in [22]. Since m is an even function, so is K. The fact that K is a positive function has been shown in Proposition 2.23 in [22].

Because the Whitham symbol m satisfies

$$|m^{(j)}(\xi)| \leq C_j(1+|\xi|)^{-\frac{1}{2}-j}, \qquad \text{for } j \in \mathbb{N},$$

the linear operator

$$\varphi \mapsto m(\mathrm{D})\varphi := K * \varphi, \qquad B^s_{p,q} \to B^{s+1/2}_{p,q} \tag{6}$$

is bounded; see for example Proposition 2.78 in [6].

2.2 Properties of solutions When choosing appropriate function spaces for (1), we will rely on the following qualitative properties of solutions.

Proposition 2.1. *Let φ be a continuous and bounded solution to* (1) *with wave speed $c \geq 1$. We have*

(i) (non-negativity) $\varphi \geq 0$;

(ii) (exponential decay) if φ is solitary and $c > 1$, there exists $\eta > 0$ such that

$$\exp(\eta|x|)\varphi(x) \in L^\infty(\mathbb{R}),$$

that is, φ has exponential decay;

(iii) (symmetry) if φ is solitary, $c > 1$ and $\sup_{x\in\mathbb{R}} \varphi(x) < c/2$, there exists $x_0 \in \mathbb{R}$ such that $\varphi(\,\cdot\, - x_0)$ is an even function which is non-increasing on $[0,\infty)$;

(iv) (regularity) φ is smooth on any open set where $\varphi < c/2$;

(v) (boundedness) if φ is of class BUC^1, even, non-constant and non-increasing on $(0,\infty)$, then $\varphi' < 0$ and $\varphi < c/2$ on $(0,\infty)$. If φ in addition is of class BUC^2, then $\varphi < c/2$ everywhere;

(vi) (singularity) if φ is even, non-constant, non-increasing on $(0,\infty)$, $\sup_{x\in\mathbb{R}} \varphi(x) \leq c/2$ and $\varphi(0) = c/2$, then as $|x| \to 0$,

$$C_1|x|^{\frac{1}{2}} \leq \frac{1+\nu}{2} - \varphi(x) \leq C_2|x|^{\frac{1}{2}};$$

(vii) (lower bound on the wave speed) if φ is non-trivial and has finite limits $\lim_{x\to\pm\infty} \varphi(x)$, then $c > 1$;

(viii) (upper bound on the wave speed) if φ is non-constant and $\varphi \leq c/2$, then $c \leq 2$.

Item (i) is stated in Lemma 4.1 in [22]. Items (ii) and (iii) are Proposition 3.13 and Theorem 4.4 in [9] respectively. The optimal exponent $\eta = \eta_c$ depends on c and is given implicitly by $\sqrt{\tan(\eta_c)/\eta_c} = c$, with $\eta_c \in (0, \pi/2)$; see [5], Theorem 6.2. We remark that the requirement $\sup_{x\in\mathbb{R}} \varphi(x) < c/2$ in (iii) is not mentioned in [9] despite its importance in the proof; see the introduction in [30] for a detailed discussion. Items (iv), (v) and (vi) can be found in Theorem 4.9, Theorem 5.1 and Theorem 5.4 in [22]. The upper bound in (viii) comes from [22], equation (6.9). The lower bound in (vii) comes from non-negativity and the following proposition.

Proposition 2.2. *Let φ be a bounded and continuously differentiable solution to* (1) *with wave speed c, such that the limits $\lim_{x\to\pm\infty} \varphi(x)$ exist. Then*

$$\lim_{R\to\infty} \int_{-R}^{R} \varphi(\varphi - (c-1)) \, \mathrm{d}x = 0. \tag{7}$$

In particular, $\inf \varphi < c-1 < \sup \varphi$*,* $\varphi \equiv c-1$ *or* $\varphi \equiv 0$ *if* $c \geq 1$*. The latter statement is in fact true for bounded and continuous solutions φ with finite limits* $\lim_{x\to\pm\infty} \varphi(x)$*.*

Proof. In general, if φ is any bounded and continuously differentiable function, the limits $\lim_{x\to\pm\infty} \varphi(x)$ exist, and $\mathcal{K}$ is any non-negative even function with $\int_{\mathbb{R}} \mathcal{K} \, \mathrm{d}x = 1$, then

$$\lim_{R\to\infty} \int_{-R}^{R} (\varphi - \mathcal{K} * \varphi) \, \mathrm{d}x = 0.$$

A proof of this can be found in [8], pp. 113–114.

Since φ solves (1) with wave speed c,

$$K * \varphi = c\varphi - \varphi^2,$$

and since the Whitham kernel K is a non-negative even function with $\int_{\mathbb{R}} K \, \mathrm{d}x = 1$,

$$\begin{aligned} 0 &= \lim_{R\to\infty} \int_{-R}^{R} (\varphi - K * \varphi) \, \mathrm{d}x \\ &= \lim_{R\to\infty} \int_{-R}^{R} (\varphi - c\varphi + \varphi^2) \, \mathrm{d}x \\ &= \lim_{R\to\infty} \int_{-R}^{R} \varphi(\varphi - (c-1)) \, \mathrm{d}x. \end{aligned}$$

By non-negativity of bounded and continuous solutions with $c \geq 1$, $\varphi - (c-1)$ must be sign-changing, $\varphi \equiv 0$, or $\varphi \equiv c-1$, otherwise the generalized integral cannot converge to zero. This proves the claim for bounded and continuously differentiable functions φ.

If φ is only a bounded and continuous solution, convolution with a non-zero smooth and compactly supported test function $\phi \geq 0$ gives

$$\phi * (\varphi - K * \varphi) = \phi * \varphi(\varphi - (c-1)).$$

The left-hand side equals

$$\phi * \varphi - K * (\phi * \varphi),$$

due to associativity and commutativity of convolution. By Lebesgue's dominated convergence theorem, the function $\phi * \varphi$ is bounded, continuously differentiable and the limits as $x \to \pm\infty$ exist. It follows that

$$\lim_{R\to\infty} \int_{-R}^{R} [\phi * \varphi - K * (\phi * \varphi)] \,\mathrm{d}x = 0,$$

which implies

$$\lim_{R\to\infty} \int_{-R}^{R} \phi * (\varphi(\varphi - (c-1))) \,\mathrm{d}x = 0.$$

Again, we must have $\varphi \equiv 0$, $\varphi \equiv c - 1$, or $\varphi - (c-1)$ is sign-changing. □

2.3 Convergence of solution sequences Modes of convergence of solution sequences will be important in ruling out alternatives from the global bifurcation theorem. We start with pointwise convergence, using the Arzelà–Ascoli theorem and the smoothing property of convolution with K.

Proposition 2.3. *Let $(\varphi_n)_{n=1}^{\infty}$ be a sequence of continuous and bounded solutions to* (1) *such that each φ_n has wave speed $c_n \in [1, 2]$ and $\sup_{x\in\mathbb{R}} \varphi_n(x) \leq c_n/2$. Then, there exists a subsequence $(\varphi_{n_k})_{k=1}^{\infty}$ satisfying*

$$\lim_{k\to\infty} c_{n_k} = c \in [1, 2], \qquad \lim_{k\to\infty} \varphi_{n_k}(x) = \varphi(x),$$

for every $x \in \mathbb{R}$. The convergence is uniform on every bounded interval of $\mathbb{R}$. The limit φ is a continuous, bounded and non-negative solution of (1) *with wave speed c, and $\sup_{x\in\mathbb{R}} \varphi(x) \leq c/2$.*

Proof. We can without loss of generality assume that $\lim_{n\to\infty} c_n = c \in [1, 2]$. For each n, we have

$$\left(\frac{c_n}{2} - \varphi_n\right)^2 - \frac{c_n^2}{4} = -K * \varphi_n.$$

A rearrangement gives

$$\varphi_n = \frac{c_n}{2} - \sqrt{\left(\frac{c_n}{2}\right)^2 - K * \varphi_n}.$$

We claim that the right-hand side forms an equicontinuous sequence. Indeed, $\varphi \mapsto K * \varphi$ is a bounded map from $L^\infty \subset B^0_{\infty,\infty}$ to $B^{1/2}_{\infty,\infty} = C^{1/2}$ according to (6). Because $c_n \in [1, 2]$, this gives

$$\|K * \varphi_n\|_{C^{1/2}} \lesssim \sup_{x\in\mathbb{R}} \varphi_n(x) \leq \frac{c_n}{2} \leq 1.$$

Hence, $(K * \varphi_n)_{n=1}^{\infty}$ is an equicontinuous sequence of functions. The square root of a non-negative equicontinuous sequence is an equicontinuous sequence. So, $(\varphi_n)_{n=1}^{\infty}$ is equicontinuous. The Arzelà–Ascoli theorem gives a subsequence $(\varphi_{n_k})_{k=1}^{\infty}$, which converges uniformly to a function φ on each bounded interval of $\mathbb{R}$. Also, φ is continuous and bounded by $c/2$.

Finally, since $\sup_{x\in\mathbb{R}} \varphi_n(x) \leq 1$ and $\|K\|_{L^1} = 1$, Lebesgue's dominated convergence theorem gives $K * \varphi_n(x) \to K * \varphi(x)$ as $n \to \infty$ for all x. It follows that φ is a solution to (1) with wave speed c. □

Here are several immediate consequences.

Corollary 2.4.

(i) If φ solves (1) with wave speed $c \in [1, 2]$, $\sup_{x\in\mathbb{R}} \varphi(x) \leq c/2$, and $\lim_{x\to\infty} \varphi(x) = a$, then a solves (1) with wave speed c. In particular, the constant a is either zero or $c - 1$.

(ii) Let φ_n, c_n, φ and c be as in Proposition 2.3. If φ_n is even and monotone on $[0, \infty)$ for each n, its locally uniform limit φ inherits evenness and monotonicity on $[0, \infty)$. If in addition $\lim_{|x|\to\infty} \varphi(x) = 0$, then φ_n converges to φ uniformly on $\mathbb{R}$.

(iii) Let $(\varphi_n)_{n=1}^{\infty}$ be a sequence of even solutions which are decreasing on $[0, \infty)$. Define $\tau_{x_n}\varphi_n := \varphi_n(\,\cdot\, + x_n)$ for a sequence of real numbers x_n with $\lim_{n\to\infty} x_n = \infty$. Then, the sequence of translated solutions $\tau_{x_n}\varphi_n$ is a sequence of solutions to (1). It has a non-increasing locally uniform limit $\tilde{\varphi}$.

Proof. Item (iii) is straightforward. We only prove items (i) and (ii).

For (i), we define

$$\tau_n\varphi := \varphi(\,\cdot\, + n), \qquad n \in \mathbb{N}.$$

Each $\tau_n\varphi$ is a solution of (1) with wave speed $c \in [1, 2]$. We have

$$\lim_{n\to\infty} \tau_n\varphi(x) = \lim_{n\to\infty} \varphi(x + n) = \lim_{x\to\infty} \varphi(x) = a, \qquad \text{for each } x \in \mathbb{R}.$$

By Proposition 2.3, a is a constant solution to (1) with wave speed c and hence by Proposition 2.2, we have $a = 0$ or $a = c - 1$.

The evenness and monotonicity in (ii) are clear, so assume that $\lim_{|x|\to\infty} \varphi(x) = 0$ and fix $\epsilon > 0$. Then, there exists $R > 0$ such that

$$|\varphi(x)| \leq \epsilon \quad \text{for} \quad |x| \geq R. \tag{8}$$

Due to $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ locally uniformly, there exists an $N_\epsilon > 0$ such that

$$\sup_{|x|\leq R} |\varphi_n(x) - \varphi(x)| \leq \epsilon \quad \text{for} \quad n > N_\epsilon.$$

In particular, we have $|\varphi_n(R) - \varphi(R)| \leq \epsilon$, which in turn implies $|\varphi_n(R)| \leq 2\epsilon$ by (8). Since φ_n is non-increasing, we have $|\varphi_n(x)| \leq 2\epsilon$ for $|x| \geq R$. But then, again by (8), $|\varphi_n(x) - \varphi(x)| \leq 3\epsilon$ for $|x| \geq R$ and $n > N_\epsilon$, and the claim about uniform convergence is proved. $\square$

Now, we consider convergence in H^j for $j > 0$. Combining the smoothing property of convolution with K and (1), we use a bootstrap argument to increase the regularity of the solutions, starting with convergence in $L^2 = H^0$.

Proposition 2.5. *Let φ_n, c_n, φ and c be as in Proposition 2.3. If*

(a) $\varphi_n \to \varphi$ *uniformly and* $\varphi_n \to \varphi$ *in* L^2*,*

(b) $\sup_{x\in\mathbb{R}} \varphi_n(x) < c_n/2$ *and* $\sup_{x\in\mathbb{R}} \varphi(x) < c/2$*,*

then $\varphi_n \to \varphi$ *in* H^j *for any* $j > 0$.

Proof. Since φ_n and φ solve (1) with wave speed c_n and c respectively, we can write

$$\varphi_n - \varphi = f(K * \varphi_n, c_n) - f(K * \varphi, c), \text{ where } f(\omega, c) = \frac{c}{2} - \sqrt{\frac{c^2}{4} - \omega}. \tag{9}$$

Letting $\omega_n = K * \varphi_n$ and $\omega = K * \varphi$, the assumptions imply

(a') $\omega_n \to \omega$ in $H^{1/2}$ and $\omega_n \to \omega$ uniformly,

(b') $\inf_{n,x} \left(\frac{c_n^2}{4} - \omega_n\right) > \varepsilon/3$,

(c') $\inf_{n,x} \left(\frac{c^2}{4} - \omega_n\right) > \varepsilon/3$,

for some $\varepsilon > 0$ and for sufficiently large n. Without loss of generality, (b') and (c') are assumed to hold for all n. We claim that if (a'), (b') and (c') are met, then

$$f(\omega_n, c_n) \to f(\omega, c) \text{ in } H^j, \text{ for some } j \in (0, 1/2).$$

Consider

$$\|f(\omega_n, c_n) - f(\omega, c)\|_{H^j} \leq \|f(\omega_n, c_n) - f(\omega_n, c)\|_{H^j} + \|f(\omega_n, c) - f(\omega, c)\|_{H^j}.$$

A quick calculation gives

$$f(\omega_n, c_n) - f(\omega_n, c) = \frac{c_n - c}{2} \cdot \left(1 - \frac{1}{2} \cdot \frac{c_n + c}{\sqrt{\frac{c_n^2}{4} - \omega_n} + \sqrt{\frac{c^2}{4} - \omega_n}}\right).$$

Define

$$g_n(x) = 1 - \frac{1}{2} \cdot \frac{c_n + c}{\sqrt{\frac{c_n^2}{4} - x} + \sqrt{\frac{c^2}{4} - x}}, \text{ with domain } D_{g_n} = \left[0, \min\left\{\frac{c^2}{4}, \frac{c_n^2}{4}\right\} - \frac{\varepsilon}{3}\right).$$

For each n, g_n is smooth and $g_n(0) = 0$. Moreover, the range of ω_n belongs to the domain of g_n. A standard result in the theory of paradifferential operators, for instance Theorem 2.87 in [6], gives

$$\|g_n(\omega_n)\|_{H^j} \leq C\|\omega_n\|_{H^j},$$

where $C > 0$ depends on $j, \sup_{x\in\mathbb{R}} |\omega_n|$ and $\sup_{x\in D_{g_n}} |g_n'(x)|$. A computation shows that $|g_n'|$ is uniformly bounded in n for $c_n, c \in [1,2]$ and $x \in D_{g_n}$. Also, $\sup_n \|\omega_n\|_{H^{1/2}} < \infty$, as well as $\sup_{n,x} |\omega_n(x)| < \infty$ by (a'). It follows that $\|g_n(\omega_n)\|_{H^j}$ is uniformly bounded in n and

$$\|f(\omega_n, c_n) - f(\omega_n, c)\|_{H^j} = \frac{1}{2}|c_n - c| \cdot \|g_n(\omega_n)\|_{H^j} \to 0, \qquad \text{as } n \to \infty.$$

To deal with the second term, let c be fixed and define

$$h(x) = f'(x,c) - f'(0,c), \text{ with domain } D_h = \left[0, \frac{c^2}{4} - \frac{\varepsilon}{3}\right).$$

Then, we can write

$$f(\omega_n, c) - f(\omega, c) = (\omega_n - \omega) \int_0^1 h(\omega_n + \tau(\omega - \omega_n) \, \mathrm{d}\tau + f'(0,c)(\omega_n - \omega).$$

Due to (a') and (b'), we have $\omega_n(s) + \tau(\omega(s) - \omega_n(s)) \in D_h$ for all $s \in \mathbb{R}$. Note that h is smooth and $h(0) = 0$. Theorem 2.87 in [6] gives an estimate for the integral

$$\begin{aligned} \left\| \int_0^1 h(\omega_n + \tau(\omega - \omega_n)) \, \mathrm{d}\tau \right\|_{H^{1/2}} &\leq \sup_{\tau\in[0,1]} \|h(\omega_n + \tau(\omega - \omega_n))\|_{H^{1/2}} \\ &\leq C \sup_{\tau\in[0,1]} \|\omega_n + \tau(\omega - \omega_n)\|_{H^{1/2}}, \end{aligned}$$

where C depends on s, $\sup_x |f''(x,c)|$, c and ε. Another standard result in paradifferential calculus, for example Theorem 8.3.1 in [29], gives

$$\begin{aligned} \|f(\omega_n, c) - f(\omega, c)\|_{H^j} \lesssim \|\omega_n - \omega\|_{H^{1/2}} &\left\| \int_0^1 h(\omega_n + \tau(\omega - \omega_n)) \, \mathrm{d}\tau \right\|_{H^{1/2}} \\ &+ |f'(0,c)| \cdot \|\omega_n - \omega\|_{H^j}, \end{aligned}$$

for all $j \in (0, 1/2)$, which tends to 0 as $n \to \infty$ by (a').

So, the right-hand side of (9) tends to 0 in H^j for all $j \in (0, 1/2)$. Hence, $\varphi_n \to \varphi$ in H^j. Then, convolution with K increases the regularity of φ_n and φ by $1/2$. Choosing $j = 1/4$ and replacing (a') with

$$\omega_n \to \omega \text{ in } H^{1/4+1/2},$$

the critical case of Theorem 8.3.1 in [29] is no longer relevant and the convergence is in $H^{j'}$ for all $j' \in (0, 1/4 + 1/2]$. By iterating as many times as needed, the claim of the proposition is proved. $\square$

Remark 2.6. Since Theorem 2.87 in [6] and Theorem 8.3.1 in [29] are valid for the Besov spaces $B^s_{p,q}$, we can replace the L^2 space in (a) with $B^0_{p,q}$ and obtain $\varphi_n \to \varphi$ in $B^s_{p,q}$ for $s > 0$ using the same proof idea.

2.4 Fredholmness of linear operators Let $j > 0$ be an integer. As a preparation for future bifurcation results, we study the operators

$$\mathcal{T} \colon \varphi \mapsto \varphi - K * \varphi, \qquad H^j_{-\eta} \to H^j_{-\eta}$$

and

$$L[\varphi^*, c^*] \colon \phi \mapsto c^*\phi - K * \phi - 2\varphi^*\phi, \qquad H^j \to H^j,$$

where φ^* is a solitary-wave solution with wave speed $c^* > 1$, satisfying $\sup_{x\in\mathbb{R}} \varphi^*(x) < c^*/2$. These are linearizations of the left-hand side in (1) at $(0, 1)$ and (φ^*, c^*), respectively. We show that $\mathcal{T}$ and $L[\varphi^*, c^*]$ are Fredholm with Fredholm index two and zero respectively, using results from [27]. The central idea is to relate a pseudodifferential operator $t(x, \mathrm{D})$ acting on H^j to a positively homogeneous function A via the symbol $t(x, \xi)$. By studying the boundary value and the winding number of A around the origin, the Fredholm property of $t(x, \mathrm{D})$ can be determined. Appendix A summarizes the relevant theorems from [27].

Up until now, the weight $\eta > 0$ has remained somewhat mysterious. Since our interest lies in the Fredholm properties of $\mathcal{T}$, η should be chosen so that $\mathcal{T}$ is at least bounded. By (4), K is locally L^1 around the origin. From (5), we deduce that $K \in L^1_{\eta_0}$ for $\eta_0 \in (0, \pi/2)$. It follows that $\mathcal{T} \colon H^j_{-\eta} \to H^j_{-\eta}$ is bounded for any $\eta \in (0, \pi/2)$. Indeed, since $\omega_{-\eta}$ is 1 on $[-1, 1]$ and equals $\exp(-\eta|x|)$ as $|x| \to \infty$, we have

$$\begin{aligned}
\|K * \varphi\|^2_{L^2_{-\eta}} &\lesssim \int_{\mathbb{R}} \left(\int_{\mathbb{R}} K(y)\varphi(x-y)\,\mathrm{d}y \right)^2 \exp(-2\eta|x|)\,\mathrm{d}x \\
&\le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} K(y)\varphi(x-y)\,\mathrm{d}y \right)^2 \exp\left(-2\eta|x-y| + 2\eta|y|\right)\mathrm{d}x \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} K(y)\exp(\eta|y|) \cdot \varphi(x-y)\exp(-\eta|x-y|)\,\mathrm{d}y \right)^2 \mathrm{d}x \\
&= \|[K \cdot \exp(\eta|\cdot|)] * [\varphi \cdot \exp(-\eta|\cdot|)]\|^2_{L^2} \\
&\le \|K\|^2_{L^1_\eta} \|\varphi\|^2_{L^2_{-\eta}},
\end{aligned}$$

where Young's inequality for the L^p norms of convolutions is used in the last step. Noting that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(K * \varphi) = K * \varphi^{(n)}$$

and applying the above estimates on $\varphi^{(n)}$,

$$\|\mathcal{T}\varphi\|_{H^j_{-\eta}} \lesssim \|\varphi\|_{H^j_{-\eta}},$$

for η in the range $(0, \pi/2)$.

2.4.1 Fredholmness of $\mathcal{T}$ Multiplication with $\cosh(\eta\,\cdot\,)$

$$M_{\cosh}\colon \varphi \mapsto \cosh(\eta\,\cdot\,)\varphi, \qquad H^j \to H^j_{-\eta}.$$

is an invertible linear operator. Its inverse is multiplication with $1/\cosh(\eta\,\cdot\,)$, mapping $H^j_{-\eta}$ to H^j. Conjugating $\mathcal{T}$ with these gives

$$\tilde{\mathcal{T}} = M^{-1}_{\cosh} \circ \mathcal{T} \circ M_{\cosh}, \qquad H^j \to H^j,$$

and more explicitly

$$\tilde{\mathcal{T}}\varphi(x) = \mathrm{Id} - (\tilde{K}(\,\cdot\,, x) * \varphi)(x), \qquad \text{where} \qquad \tilde{K}(z, x) = K(z)\frac{\cosh(\eta(x-z))}{\cosh(\eta x)}.$$

Setting

$$\phi_\pm(x) = \frac{\exp(\pm\eta x)}{2\cosh(\eta x)} \qquad \text{and} \qquad K_\pm(z) = K(z)\exp(\pm\eta z),$$

$\tilde{\mathcal{T}}$ can be rewritten as

$$\tilde{\mathcal{T}}\varphi(x) = \varphi(x) - \phi_+(x)(K_- * \varphi)(x) - \phi_-(x)(K_+ * \varphi)(x),$$

with the symbol

$$\tilde{t}(x, \xi) = 1 - \phi_+(x)m(\xi - \mathrm{i}\eta) - \phi_-(x)m(\xi + \mathrm{i}\eta).$$

Lemma 2.7. *The conjugated pseudodifferential operator $\tilde{\mathcal{T}} = \tilde{t}(x, \mathrm{D})\colon H^j \to H^j$ is a Fredholm operator.*

Proof. The idea is to apply Proposition A.1. We define a positively homogeneous function A by

$$A(x_0, x, \xi_0, \xi) := \tilde{t}\left(\frac{x}{x_0}, \frac{\xi}{\xi_0}\right)$$

for $x, \xi \in \mathbb{R}$ and $x_0, \xi_0 > 0$. In order to apply Proposition A.1, we need to check that A is smooth in

$$\overline{\mathbb{S}} := \{(x_0, x, \xi_0, \xi) \in \mathbb{R}^4 \,\big|\, x_0^2 + x^2 = \xi_0^2 + \xi^2 = 1, x_0 \geq 0, \xi_0 \geq 0\},$$

and that $A(x_0, x, \xi_0, \xi) \neq 0$ on Γ, where Γ is the boundary of $\overline{\mathbb{S}}$. Γ can be decomposed into the arcs

$$\begin{aligned}
\Gamma_1 &= \{\ (0, 1, \xi_0, \xi)\ \ |\, \xi_0^2 + \xi^2 = 1, \xi_0 \geq 0\},\\
\Gamma_2 &= \{(0, -1, \xi_0, \xi)\,|\, \xi_0^2 + \xi^2 = 1, \xi_0 \geq 0\},\\
\Gamma_3 &= \{\ (x_0, x, 0, 1)\ \ |\, x_0^2 + x^2 = 1, x_0 \geq 0\},\\
\Gamma_4 &= \{(x_0, x, 0, -1)|\, x_0^2 + x^2 = 1, x_0 \geq 0\}.
\end{aligned}$$

We compute the value of A along each arc Γ_i and show that A is nowhere vanishing. From the computations, it will be apparent that A is smooth on $\overline{\mathbb{S}}$.

On $\Gamma_1^* := \Gamma_1 \setminus \{(0,1,0,1), (0,1,0,-1)\}$, we have $\xi_0 = \sqrt{1-\xi^2} > 0$ and

$$\begin{aligned}
A(x_0, x, \xi_0, \xi)\Big|_{\Gamma_1^*} &= \lim_{x_0 \to 0^+} \tilde{t}\left(\frac{1}{x_0}, \frac{\xi}{\sqrt{1-\xi^2}}\right)\\
&= 1 - m\Big(\frac{\xi}{\sqrt{1-\xi^2}} - \mathrm{i}\eta\Big),
\end{aligned}$$

since $\lim_{y\to\infty} \phi_+(y) = 1$ and $\lim_{y\to\infty} \phi_-(y) = 0$. Let $\theta = \xi(1-\xi^2)^{-1/2}$. As $\xi \in (-1,1)$, $\theta \in (-\infty, \infty)$. To compute the values at the endpoint $(0,1,0,1)$, we note that taking the limit as $\xi \to 1^-$ corresponds to taking the limit as $\theta \to \infty$. A calculation shows that

$$|m(\theta \pm \mathrm{i}\eta)|^4 = \frac{\sinh^2(2\theta) + \sin^2(2\eta)}{(\theta^2+\eta^2)(\cosh(2\theta) + \cos(2\eta))^2}$$

which gives

$$\lim_{\theta \to \pm\infty} |m(\theta \pm \mathrm{i}\eta)| = 0.$$

So, the value of A at $(0,1,0,1)$ is 1. Similarly, A at $(0,1,0,-1)$ corresponds to taking the limit as $\theta \to -\infty$ and the value is 1. Along Γ_1^*, if $1 - m(\theta - \mathrm{i}\eta) = 0$ for some $\theta \in \mathbb{R}$, then

$$\mathrm{Re}(m(\theta - \mathrm{i}\eta)^2) = 1 \quad \text{and} \quad \mathrm{Im}(m(\theta - \mathrm{i}\eta)^2) = 0,$$

where

$$\mathrm{Re}(m(\theta - \mathrm{i}\eta)^2) = \frac{\theta \sinh(2\theta) + \eta \sin(2\eta)}{(\theta^2+\eta^2)(\cosh(2\theta) + \cos(2\eta))}$$

and

$$\mathrm{Im}(m(\theta - \mathrm{i}\eta)^2) = \frac{\eta \sinh(2\theta) - \theta \sin(2\eta)}{(\theta^2+\eta^2)(\cosh(2\theta) + \cos(2\eta))}.$$

The second equation is satisfied only when the numerator $\eta\sinh(2\theta)-\theta\sin(2\eta)$ is zero. Since this is an odd and smooth function in θ, a trivial solution is $\theta=0$. Since

$$\frac{\sinh(2\theta)}{2\theta}>1 \quad \text{and} \quad \frac{\sin(2\eta)}{2\eta}<1,$$

for $\theta\neq 0$ and $\eta\in(0,\pi/2)$, there are no other solutions. When $\theta=0$, $\mathrm{Re}(m(\theta-\mathrm{i}\eta)^2)$ is $\tan(\eta)/\eta>1$ for all $\eta\in(0,\pi/2)$. We can thus conclude that $A\neq 0$ on Γ_1.

Similar computations yield

$$A\Big|_{\Gamma_2}=\begin{cases}1-m(\theta+\mathrm{i}\eta), & \xi_0>0,\\ 1, & \xi_0=0,\end{cases}$$

which is nowhere vanishing by the same argument. Along the other arcs,

$$A\Big|_{\Gamma_3}=A\Big|_{\Gamma_4}=1.$$

So, A along Γ is nowhere vanishing and Proposition A.1 gives the desired conclusion. $\square$

The next result is about the Fredholm index of $\tilde{\mathcal{T}}$.

Lemma 2.8. *$\tilde{\mathcal{T}}=\tilde{t}(x,\mathrm{D})\colon H^j\to H^j$ has Fredholm index two.*

Proof. According to Proposition A.1, the total increase of the argument of A as Γ is traversed with the counter-clockwise orientation

$$\begin{tikzcd}
(0,-1,0,1) \arrow[d, "\Gamma_2"] & (0,1,0,1) \arrow[l, "\Gamma_3"'] \\
(0,-1,0,-1) \arrow[r, "\Gamma_4"] & (0,1,0,-1) \arrow[u, "\Gamma_1"']
\end{tikzcd}$$

determines the Fredholm index of $t(x,\mathrm{D})$. We begin with Γ_1 from $(0,1,0,-1)$ to $(0,1,0,1)$ and consider the total increase of the argument of $1-m(\theta-\mathrm{i}\eta)$ from $\theta=-\infty$ to $\theta=\infty$; see the proof of Lemma 2.7. As before, it is easier to deal with $m(\theta-\mathrm{i}\eta)^2$. The sign of the real part of $m(\theta-\mathrm{i}\eta)^2$ is determined by the sign of

$$\theta\sinh(2\theta)+\eta\sin(2\eta)$$

which is positive for $\theta\in\mathbb{R}$ and $\eta\in(0,\pi/2)$. This means that m^2 stays in the first and fourth quadrant of $\mathbb{C}$. The sign of the imaginary part equals the sign of

$$\eta\sinh(2\theta)-\theta\sin(2\eta),$$

which is a strictly increasing function in θ taking the value zero at $\theta=0$. This means that at $\theta=0$, m^2 enters the first quadrant from the fourth. By computing the value of m^2

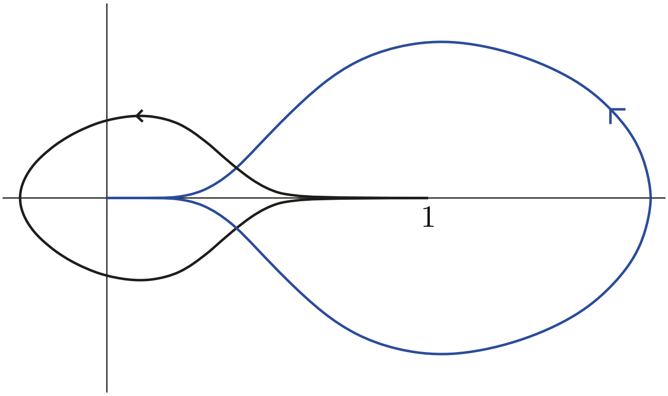

Figure 2: Graphs of m^2 (blue line) and $1-m$ (black line) along Γ_1. The increase of the argument of $1-m$ about the origin equals the increase of the argument of m^2 around 1, which is 2π.

as $\theta \to -\infty$, at $\theta = 0$ and as $\theta \to \infty$, we can conclude that m^2 along Γ_1 makes one counter-clockwise revolution about 1. Taking the square root of m^2 preserves the signs of the real and imaginary part. Then, multiplication with -1 flips the signs and addition with 1 corresponds to horizontal translation to the right by 1; see Figure 2. Finally, we arrive at the conclusion that the increase of the argument of A along Γ_1 from $(0, 1, 0, -1)$ to $(0, 1, 0, 1)$ is 2π.

A similar analysis shows that an additional increase of 2π is gained along Γ_2. On Γ_3 and Γ_4, $A \equiv 1$, so there is no contribution from these arcs. In total, the increase of the argument along Γ is 4π. Proposition A.1 now gives that the Fredholm index is two. $\square$

Conjugation with the invertible linear operator $M_{\cosh}$ preserves Fredholmness and the Fredholm index. Hence, $\mathcal{T} = M_{\cosh} \circ \tilde{\mathcal{T}} \circ M_{\cosh}^{-1} \colon H^j_{-\eta} \to H^j_{-\eta}$ is Fredholm with Fredholm index two. We have proved the first part of the main result of this section, which is the following.

Proposition 2.9. *$\mathcal{T} \colon H^j_{-\eta} \to H^j_{-\eta}$ is Fredholm with Fredholm index two and*

$$\operatorname{Ker} \mathcal{T} = \operatorname{span}\{1, x\}.$$

Proof. The statement concerning the Fredholm properties of $\mathcal{T}$ is already proved. Note that solving $\mathcal{T}\varphi = 0$ for $\varphi \in L^2_{-\eta}$ using the Fourier transform is problematic because $\mathcal{F}\varphi$ is not necessarily a tempered distribution. Thus, we consider the L^2-adjoint of $\mathcal{T}$ acting on $L^2_{-\eta}$, which is $\mathcal{T} \colon L^2_{\eta} \to L^2_{\eta}$, and determine its range. The equation $\mathcal{T}\psi = g$ in L^2_{η} corresponds to $(1-m)\mathcal{F}\psi = \mathcal{F}g$ on the Fourier side, where $\mathcal{F}\psi$ and $\mathcal{F}g$ are analytic functions bounded on the strip $|\operatorname{Im} z| < \eta$. In view of (3), $1 - m(\xi)$ vanishes to second order at $\xi = 0$ and is bounded away from zero if ξ is. As a consequence, the range of $\mathcal{T}$ on

L^2_η consists of functions g satisfying $\mathcal{F}g(0) = (\mathcal{F}g)'(0) = 0$, or equivalently $\int_{\mathbb{R}} g(x)\,\mathrm{d}x = \int_{\mathbb{R}} xg(x)\,\mathrm{d}x = 0$. This immediately implies that $\operatorname{Ker}\mathcal{T}$ in $L^2_{-\eta}$ is $\operatorname{span}\{1, x\} \subset H^j_{-\eta}$ and the claim is established. □

2.4.2 Fredholmness of $L[\varphi^*, c^*]\colon H^j \to H^j$ The application of Proposition A.1 is simpler for $L[\varphi^*, c^*]\colon H^j \to H^j$, as conjugation does not take place.

Proposition 2.10. *Let* φ^* *be a solution with wave speed* $c^* > 1$, *satisfying* $\sup_{x\in\mathbb{R}} \varphi^*(x) < c^*/2$. *Then* $L[\varphi^*, c^*]\colon H^j \to H^j$ *is Fredholm with Fredholm index zero.*

Proof. Proposition A.1 is employed once again. The linear operator

$$L[\varphi^*, c^*]\colon \phi \mapsto c^*\phi - K * \phi - 2\varphi^*\phi$$

has the symbol

$$l(x, \xi) = c^* - m(\xi) - 2\varphi^*(x) \in C^\infty(\mathbb{R} \times \mathbb{R}),$$

where smoothness of φ^* is from Proposition 2.1(iv). The corresponding positively homogeneous function B is

$$B(x_0, x, \xi_0, \xi) = l\left(\frac{x}{x_0}, \frac{\xi}{\xi_0}\right), \qquad \text{where } x_0 > 0 \text{ and } \xi_0 > 0.$$

As before, we verify that B does not vanish at any point along $\Gamma = \cup_{1\le i\le 4}\Gamma_i$; see the proof of Lemma 2.7. Along Γ_1 and Γ_2,

$$\begin{aligned} B(x_0, x, \xi_0, \xi)\Big|_{\Gamma_{1,2}\setminus\{\xi_0=0\}} &= \lim_{x_0\to 0\pm} B(x_0, 1, \xi_0, \xi) \\ &= \lim_{x_0\to 0\pm} l\left(\frac{1}{x_0}, \frac{\xi}{\sqrt{1-\xi^2}}\right) \\ &= c^* - m\left(\frac{\xi}{\sqrt{1-\xi^2}}\right), \end{aligned}$$

as φ^* is smooth and $\lim_{|t|\to\infty} \varphi^*(t) = 0$. Since $m \le 1$, B cannot attain the value zero. Along Γ_3 and Γ_4, B is $c^* - 2\varphi^*(x/\sqrt{1-x^2})$. Since $\sup_{x\in\mathbb{R}} \varphi^*(x) < c^*/2$ by assumption, B cannot take the value zero. Moreover, the argument of B is constant along Γ as B is real-valued and the claim is proved. □

3 Local bifurcation

We apply an adaptation of the nonlocal center manifold theorem in [24] to (1) in order to construct a small-amplitude solitary-solution curve emanating from $(\varphi, c) = (0, 1)$. For convenience, we work with a different bifurcation parameter $\nu := c-1$ which will be small and positive along the local curve. In the notation of [24], equation (1) becomes

$$\mathcal{T}\varphi + \mathcal{N}(\varphi, \nu) = 0, \tag{10}$$

where $\mathcal{T}$ is defined in Section 2.4, and

$$\mathcal{N}\colon (\varphi, \nu) \mapsto \nu\varphi - \varphi^2.$$

Equation (10) will be studied for $\nu \in (0, \infty)$, in the Sobolev space of even functions H^3_{even} and the weighted Sobolev spaces $H^3_{-\eta}$. This regularity choice $j = 3$ is with regard to Proposition 2.1. A solution $\varphi \in H^3_{\text{even}}$ with wave speed $c > 1$, such that $\sup_{x\in\mathbb{R}} \varphi(x) < c/2$, is smooth on $\mathbb{R}$. Moreover, $\varphi' < 0$ on $(0, \infty)$ and φ has exponential decay.

The center-manifold reduction technique gives a reduced equation equivalent to the nonlocal equation (10) near the bifurcation point $(\varphi, \nu) = (0, 0)$ in $H^3_{\text{u}} \times \mathbb{R}$, where H^3_{u} is the space of functions which are uniformly local H^3. Since the reduced equation is an ODE, standard arguments give the existence of small-amplitude solitary-wave solutions in $H^3_{-\eta}$. Hence, by the exponential decay of solitary-wave solutions with supercritical wave speed, these are of class $H^3 \subset H^3_{\text{u}}$. We also prove that the bifurcation curve of non-trivial even solitary solutions is locally unique in $H^3_{\text{u}} \times (0, \infty)$, and refer to it as $\mathcal{C}_{\text{loc}}$.

The global bifurcation theorem demands $L[\varphi^*, \nu^*]$ to be invertible in H^3_{even} where $(\varphi^*, \nu^*) \in \mathcal{C}_{\text{loc}}$ and $\nu^* = c^* - 1$. Seeing that the Fredholm index of $L[\varphi^*, \nu^*]$ is zero, it suffices to show that the nullspace of $L[\varphi^*, \nu^*]$ is trivial. We consider equation (10), together with the linearized equation $L[\varphi^*, \nu^*]\phi = 0$, and formulate a center manifold theorem for this system. Exploiting the previous reduction for (10), we simplify the reduced equation for the linearized problem and are able to solve it completely in H^3_{even}.

3.1 Center manifold reductions Two center manifold reductions are presented: one for the nonlinear problem (10) and the other for the linearized problem $L[\varphi^*, \nu^*]\phi = 0$.

For (10), we use an adaptation of the center manifold theorem in [24]. In this reference, it is assumed that the convolution kernel belongs to $W^{1,1}_\eta$, which is not the case for the Whitham kernel K, as K' is not locally L^1 according to (4). Seeing that this requirement is only used for proving the Fredholm properties of the linear part $\mathcal{T}$, we replace it with requirements on the Fredholm properties on $\mathcal{T}$; see Hypothesis B.1(ii) in Appendix B.1. The rest of the proof of the center manifold theorem in [24] remains the same.

We consider (10) together with the modified equation

$$\mathcal{T}\varphi + \mathcal{N}(\chi^\delta(\varphi), \nu) = 0, \tag{11}$$

where $\chi^\delta(\varphi)$ is a nonlocal and translation invariant cutoff operator defined in Appendix B. We have $\chi^\delta(\varphi) = \varphi$ if $\|\varphi\|_{H^3_{\text{u}}} \leq C_0\delta$ and $\chi^\delta(\varphi) = 0$ if $\|\varphi\|_{H^3_{\text{u}}}$ is sufficiently large. Since $H^3_{\text{u}} \subset H^3_{-\eta}$ for all $\eta > 0$, the operator χ^δ is a cutoff in the $H^3_{-\eta}$ norm. More details are provided in Appendix B.

We have shown that $\operatorname{Ker}\mathcal{T}$ has dimension two in $H^3_{-\eta}$ and equals $\operatorname{span}\{1, x\}$. Hence, elements $A + Bx \in \operatorname{Ker}\mathcal{T}$ will often be identified with $(A, B) \in \mathbb{R}^2$. We define a projection on $\operatorname{Ker}\mathcal{T}$,

$$\mathcal{Q}\colon \varphi \mapsto \varphi(0) + \varphi'(0)x, \qquad H^3_{-\eta} \to \operatorname{Ker}\mathcal{T}, \tag{12}$$

which could also be considered as a mapping from $H^3_{-\eta}$ to $\mathbb{R}^2$. Finally, the shift $\varphi \mapsto \varphi(\,\cdot\, + \xi)$ will be denoted by τ_ξ.

Theorem 3.1. *For equation* (10)*, there exist a neighborhood $\mathcal{V}$ of $0 \in \mathbb{R}$, a cutoff radius δ, a weight $\eta^* \in (0, \pi/2)$ and a map*

$$\Psi\colon \mathbb{R}^2 \times \mathcal{V} \to \operatorname{Ker}\mathcal{Q} \subset H^3_{-\eta^*}$$

with the center manifold

$$\mathcal{M}_0^\nu = \{A + Bx + \Psi(A, B, \nu) \mid A, B \in \mathbb{R}, \nu \in \mathcal{V}\} \subset H^3_{-\eta^*}$$

as its graph. We have

- *(i) (smoothness) $\Psi \in \mathscr{C}^3$;*
- *(ii) (tangency) $\Psi(0, 0, 0) = 0$ and $\mathrm{D}_{(A,B)}\Psi(0, 0, 0) = 0$;*
- *(iii) (global reduction) $\mathcal{M}_0^\nu$ consists precisely of φ such that $\varphi \in H^3_{-\eta^*}$ is a solution of the modified equation* (11) *with parameter ν;*
- *(iv) (local reduction) any φ solving* (10) *with parameter ν and $\|\varphi\|_{H^3_{\mathrm{u}}} < C_0\delta$ is contained in $\mathcal{M}_0^\nu$;*
- *(v) (correspondence) any element $\varphi = A + Bx + \Psi(A, B, \nu) \in \mathcal{M}_0^\nu$ solves the local equation*

  $$\varphi''(x) = f(\varphi(x), \varphi'(x), \nu), \quad \textit{where} \quad f(A, B, \nu) = \Psi''(A, B, \nu)(0), \tag{13}$$

  and conversely, any solution of this equation is an element in $\mathcal{M}_0^\nu$. The Taylor expansion of Ψ gives

  $$\varphi'' = -6\varphi^2 + \frac{19}{5}(\varphi')^2 + 6\nu\varphi + \mathcal{O}\Big(|(\varphi, \varphi')|(\nu^2 + |\varphi|^2 + |\varphi'|^2)\Big); \tag{14}$$

- *(vi) (equivariance) besides the translations τ_ξ, equations* (10) *and* (11) *possess a reflection symmetry $R\varphi(x) := \varphi(-x)$, meaning $\mathcal{T}R\varphi = R\mathcal{T}\varphi$, $\mathcal{N}(R\varphi, \nu) = R\mathcal{N}(\varphi, \nu)$ and $\chi^\delta(R\varphi) = R\chi^\delta(\varphi)$. It is hence reversible. The function f in (v) commutes with all translations and anticommutes with the reflection symmetry.*

Proof. We use Theorem B.5. Proposition 2.9 shows that Hypothesis B.1 is met. Also, the fact that $\mathcal{N}$ is a Nemytskii operator verifies Hypothesis B.3. In particular, $\mathcal{N} \in \mathscr{C}^\infty$ and $\mathcal{N}$ commutes with the translations τ_ξ for all $\nu \in (0, \infty)$. This means that in Hypothesis B.3, we can choose any regularity $k \geq 2$, possibly at the price of a smaller cutoff radius δ and weight η^*. Since a quadratic-order Taylor expansion of Ψ suffices for our purposes, $k = 3$ is chosen. Statement (vi) concerning the reflection symmetry R follows directly from K being an even function. Hence, Theorem B.5 applies and gives items (i)–(vi).

Equation (14) in (v) is given by Theorem B.5(vii). We use $\mathcal{Q}$ defined in (12) to compute the reduced vector field. Let $\varphi \in \mathcal{M}_0^\nu$. According to Theorem B.5(viii), $\mathcal{M}_0^\nu$ is invariant under translation symmetries. Hence, $\tau_\xi\varphi$ is also an element of $\mathcal{M}_0^\nu$ for all $\xi \in \mathbb{R}$. Applying Theorem B.5(vii) on $\tau_\xi\varphi$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Q}(\tau_t\tau_\xi\varphi)\Big|_{t=0} = \varphi'(\xi) + \varphi''(\xi)x.$$

We compute $\varphi''(\xi)$ by noting that

$$\varphi''(\xi) = \varphi''(x+\xi)\big|_{x=0} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\tau_\xi\varphi)\Big|_{x=0},$$

and since $\tau_\xi\varphi \in \mathcal{M}_0^\nu$,

$$\tau_\xi\varphi(x) = \varphi(\xi) + \varphi'(\xi)x + \Psi(\varphi(\xi), \varphi'(\xi), \nu)(x).$$

Hence,

$$\varphi''(\xi) = f(\varphi(\xi), \varphi'(\xi), \nu), \qquad \text{where} \qquad f(A, B, \nu) = \Psi''(A, B, \nu)(0),$$

which is (13).

To prove equation (14), we compute the Taylor expansion of Ψ. In view of $\Psi(0,0,0) = 0$ and $\mathrm{D}_{(A,B)}\Psi(0,0,0) = 0$, the Taylor expansion of $\Psi\colon \mathbb{R}^2 \times \mathcal{V} \to \operatorname{Ker} \mathcal{Q}$ is

$$\begin{aligned}\Psi(A, B, \nu) = g(\nu)\Psi_{001} &+ A\nu\Psi_{101} + B\nu\Psi_{011} \\ &+ A^2\Psi_{200} + AB\Psi_{110} + B^2\Psi_{020} \\ &+ \mathcal{O}((|A| + |B|)(\nu^2 + |A|^2 + |B|^2)),\end{aligned}$$

where $\Psi_{ijk} \in \operatorname{Ker} \mathcal{Q}$ for $1 \leq i + j + k \leq 2$. Since $\mathcal{N}(0, \nu) = 0$, we have $\Psi(0, 0, \nu) = 0$ and consequently $g(\nu) = 0$ for all $\nu \in \mathbb{R}$. So, elements φ in the center manifold $\mathcal{M}_0^\nu$ have the form

$$\begin{aligned}\varphi(x) &= A + Bx + \Psi(A, B, \nu)(x) \\ &= A + Bx + A\nu\Psi_{101}(x) + B\nu\Psi_{011}(x) \\ &\quad + A^2\Psi_{200}(x) + AB\Psi_{110}(x) + B^2\Psi_{020}(x) \\ &\quad + \mathcal{O}((|A| + |B|)(\nu^2 + |A|^2 + |B|^2)).\end{aligned}$$

Substituting $\varphi = A + Bx + \Psi(A, B, \nu)$ into (10), then identifying coefficients of orders $\mathcal{O}(A\nu)$, $\mathcal{O}(B\nu)$, $\mathcal{O}(A^2)$, $\mathcal{O}(AB)$ and $\mathcal{O}(B^2)$, we are led to the linear equations

$$\begin{aligned}\mathcal{T}\Psi_{200} &= -\mathcal{T}\Psi_{101} = 1,\\ \mathcal{T}\Psi_{110} &= -2\mathcal{T}\Psi_{011} = 2x,\\ \mathcal{T}\Psi_{020} &= x^2,\end{aligned}$$

uniquely determined by the condition $\mathcal{Q}(\Psi_{ijk}) = 0$, $i + j + k > 1$. Indeed, if there are two solutions Ψ_{ijk} and $\tilde{\Psi}_{ijk}$, then $\Psi_{ijk} - \tilde{\Psi}_{ijk}$ lies in $\operatorname{Ker}\mathcal{T} \cap \operatorname{Ker}\mathcal{Q}$, and hence is zero. The linear equations are solved in Appendix C. This gives

$$\begin{aligned}\Psi(A, B, \nu)(x) = \Big(-3A^2 + \frac{19}{10}B^2 + 3A\nu\Big)x^2 + \Big(-2AB + B\nu\Big)x^3\\ - \frac{B^2}{2}x^4 + \mathcal{O}((|A| + |B|)(\nu^2 + |A|^2 + |B|^2)).\end{aligned}$$

Differentiating Ψ twice with respect to x and evaluating at $x = 0$ shows equation (14). $\square$

When solving the linearized problem $L[\varphi^*, \nu^*]\phi = 0$, we want to take advantage of the assumption that $\varphi^* \in \mathcal{M}_0^{\nu^*}$ is a solution of (10) with parameter ν^*. Hence, we consider (10) and $L[\varphi^*, \nu^*]\phi = 0$ simultaneously:

$$\mathbf{T}(\varphi, \phi) + \mathbf{N}(\varphi, \phi, \nu) = 0, \tag{15}$$

where $\mathbf{T} : (H^3_{-\eta})^2 \to (H^3_{-\eta})^2$ is an onto Fredholm operator with Fredholm index four given by

$$\mathbf{T}\colon (\varphi, \phi) \mapsto (\mathcal{T}\varphi, \mathcal{T}\phi),$$

and

$$\mathbf{N}\colon (\varphi, \phi, \nu) \mapsto (\mathcal{N}(\varphi, \nu), \mathrm{D}_\varphi\mathcal{N}(\varphi, \nu)\phi).$$

The modified system is

$$\mathbf{T}(\varphi, \phi) + \mathbf{N}^\delta(\varphi, \phi, \nu) = 0,$$

with the nonlinearity

$$\mathbf{N}^\delta(\varphi, \phi, \nu) = (\mathcal{N}^\delta(\varphi, \nu), \mathrm{D}_\varphi\mathcal{N}^\delta(\varphi, \nu)\phi).$$

Since we only cut off in φ, the modified linearized equation coincides with the original one, as long as $\varphi \in \mathcal{M}_0^\nu$ is sufficiently small in the H^3_{u} topology. Hence, all solutions to the linearized equation will be captured. The downside of this scheme is that our previous adaptation of the results in [24] cannot be applied directly. We replace the contraction principle with a fiber contraction principle to achieve the following result; see Appendix B.2.

Theorem 3.2. *For* (15)*, there exist a cutoff radius* δ*, a neighborhood* $\mathcal{V}$ *of* $0 \in \mathbb{R}$*, a weight* $\eta^* \in (0, \pi/2)$*, two mappings* Ψ_1 *and* Ψ_2*, where*

$$\Psi_1 \colon \mathbb{R}^2 \times \mathcal{V} \to \operatorname{Ker} \mathcal{Q} \subset H^3_{-\eta^*},$$

with the center manifold

$$\mathbf{M}^\nu_{0,1} \coloneqq \{A + Bx + \Psi_1(A, B, \nu) \mid A, B \in \mathbb{R}, \nu \in \mathcal{V}\}$$

as its graph, and at each fixed element $\varphi \in \mathbf{M}^\nu_{0,1}$ *uniquely determined by* (A, B)*,*

$$\Psi_2[A, B, \nu] : \operatorname{Ker} \mathcal{T} \to \operatorname{Ker} \mathcal{Q},$$

with graph

$$\mathbf{M}_{0,2}[A, B, \nu] \coloneqq \{C + Dx + \Psi_2[A, B, \nu](C, D) \mid C, D \in \mathbb{R}\}.$$

The following statements hold.

(i) $\mathbf{M}^\nu_{0,1}$ *coincides with* $\mathcal{M}^\nu_0$ *in Theorem 3.1 and all statements in this theorem hold for* $\mathbf{M}^\nu_{0,1}$*;*

(ii) $\Psi_2[A, B, \nu] = \mathrm{D}_{(A,B)}\Psi_1(A, B, \nu)$*, so* $\Psi_2[A, B, \nu]$ *is a bounded linear operator from* $\operatorname{Ker} \mathcal{T}$ *to* $\operatorname{Ker} \mathcal{Q}$*. Also,* Ψ_2 *is* $\mathscr{C}^{k-1}$ *in* (A, B, ν)*.*

Suppose that $\varphi^* \in \mathbf{M}^{\nu^*}_{0,1}$ *is sufficiently small in the* H^3_{u} *norm, so that* φ^* *is a solution of* (10) *with parameter* ν^**, uniquely determined by* (A^*, B^*)*. Then,*

(iii) $\mathbf{M}_{0,2}[A^*, B^*, \nu^*]$ *consists precisely of solutions* $\phi \in H^3_{-\eta^*}$ *of the linearized equation* $\mathcal{T}\phi + \mathrm{D}_\varphi\mathcal{N}(\varphi^*, \nu^*)\phi = 0$*;*

(iv) every $\phi \in \mathbf{M}_{0,2}[A^*, B^*, \nu^*]$ *is a solution of*

$$\phi''(x) = g(\varphi^*(x), (\varphi^*)'(x), \phi(x), \phi'(x), \nu^*),$$

where

$$g(A^*, B^*, C, D, \nu^*) = \mathrm{D}_A f(A^*, B^*, \nu^*)C + \mathrm{D}_B f(A^*, B^*, \nu^*)D,$$

and $f(A, B, \nu) = \Psi_1''(A, B, \nu)(0)$*. Conversely, any solution of the above second-order ODE is an element in* $\mathbf{M}_{0,2}[A^*, B^*, \nu^*]$*. The Taylor expansion of* g *gives*

$$\begin{aligned} \phi'' = {} & -12\varphi^*\phi + \frac{38}{5}(\varphi^*)'\phi' + 6\nu^*\phi \\ & + \mathcal{O}((|\varphi^*\phi| + |(\varphi^*)'\phi'|)((\nu^*)^2 + |\varphi^*| + |(\varphi^*)'|). \end{aligned} \tag{16}$$

Proof. Theorem B.6 applies and gives (i)–(iv). The cutoff radii δ given by Theorem 3.1 and Theorem 3.2 are not necessarily the same, but the smallest one can be chosen to have (i). Arguing along the same lines as the proof of Theorem 3.1 gives equation ((iv)) and differentiating the Taylor expansion of f in (14) gives equation (16). □

3.2 Local bifurcation curve The center manifold theorem states that a solution φ sufficiently small in the H^3_{u} norm solves the reduced ODE (14), which is local in nature and allows spatial dynamics tools. Let $\varphi = P$, $\varphi' = Q$ and regard the spatial variable x as "time" t. Equation (14) defines the following system of ODEs

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = Q \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = -6P^2 + \dfrac{19}{5}Q^2 + 6\nu P + \mathcal{O}((|P| + |Q|)(\nu^2 + |P|^2 + |Q|^2)), \end{cases} \tag{17}$$

which is reversible by Theorem 3.1(vi). We aim to rescale (17) into a KdV equation when $\nu = 0$, that is

$$\begin{cases} \dfrac{\mathrm{d}\tilde{P}}{\mathrm{d}T} = \tilde{Q}(T) \\ \dfrac{\mathrm{d}\tilde{Q}}{\mathrm{d}T} = \tilde{P}(T) - \dfrac{3}{2}\tilde{P}(T)^2. \end{cases}$$

Hence, we set

$$T = \alpha t, \qquad P(t) = \beta \tilde{P}(T), \qquad Q(t) = \gamma \tilde{Q}(T).$$

Differentiating, substituting into (17) and identifying coefficients yield

$$\frac{\gamma}{\beta\alpha} = 1, \qquad 6\nu\frac{\beta}{\alpha\gamma} = 1, \qquad \frac{6\beta^2}{\alpha\gamma} = \frac{3}{2},$$

which are satisfied by

$$\alpha = \sqrt{6\nu}, \qquad \beta = \frac{3}{2}\nu, \qquad \gamma = \sqrt{\frac{3^3\nu^3}{2}}.$$

The resulting rescaled system is

$$\begin{cases} \dfrac{\mathrm{d}\tilde{P}}{\mathrm{d}T} = \tilde{Q}(T) \\ \dfrac{\mathrm{d}\tilde{Q}}{\mathrm{d}T} = \tilde{P}(T) - \dfrac{3}{2}\tilde{P}(T)^2 + \dfrac{57}{10}\nu\tilde{Q}(T)^2 \\ \qquad\qquad + \mathcal{O}\left(\nu(|\tilde{P}| + \nu^{1/2}|\tilde{Q}|)(1 + |\tilde{P}|^2 + \nu|\tilde{Q}|^2)\right). \end{cases} \tag{18}$$

For $\nu = 0$, (18) is the KdV equation with the explicitly known pair of solutions

$$\tilde{P}(T) = \operatorname{sech}^2\left(\frac{T}{2}\right), \qquad \tilde{Q}(T) = -\operatorname{sech}^2\left(\frac{T}{2}\right)\tanh\left(\frac{T}{2}\right),$$

which corresponds to a symmetric and homoclinic orbit. For $\nu > 0$, the symmetric homoclinic orbit persists by the same argument as in [26], p. 955. Undoing the rescaling and switching back to $P = \varphi$ as well as $Q = \varphi'$ give

$$\begin{aligned} \varphi(t) &= \frac{3}{2}\nu \operatorname{sech}^2\left(\frac{\sqrt{6\nu}t}{2}\right) + \mathcal{O}(\nu^2) \\ \varphi'(t) &= -\frac{3^{3/2}\nu^{3/2}}{2^{1/2}}\operatorname{sech}^2\left(\frac{\sqrt{6\nu}t}{2}\right)\tanh\left(\frac{\sqrt{6\nu}t}{2}\right) + \mathcal{O}(\nu^{5/2}), \end{aligned} \tag{19}$$

for $\nu > 0$. The supercritical solitary-wave solution φ is exponentially decaying, so both φ and φ' belong to $H^3 \subset H^3_{\mathrm{u}}$. Also, they depend continuously on the parameter ν. We denote this solution as $\varphi^*_{\nu^*}$ with parameter ν^* and define

$$\mathcal{C}_{\mathrm{loc}} = \{(\varphi^*_{\nu^*}, \nu^*) \,|\, 0 < \nu^* < \nu'\},$$

for some $\nu' > 0$. The main result of this section is reached.

Theorem 3.3. *There exists a neighborhood of* $(\varphi, \nu) = (0, 0)$ *in* $H^3_{\mathrm{u}} \times (0, \infty)$, *for which* $\mathcal{C}_{\mathrm{loc}}$ *is the unique* ν*-dependent family of non-trivial even small-amplitude solitary solutions to* (10) *emanating from* $(0, 0)$. *We refer to* $\mathcal{C}_{\mathrm{loc}}$ *as the local bifurcation curve.*

Proof. The function $\varphi^*_{\nu^*}$ belongs to $\mathcal{M}_0^{\nu^*}$ by the one-to-one correspondence between (17) and $\mathcal{M}_0^{\nu^*}$ in Theorem 3.1(v). From (19) combined with the fact that $\varphi^*_{\nu^*}$ is exponentially decaying, we have $\|\varphi^*_{\nu^*}\|_{H^1_{\mathrm{u}}} \lesssim \nu^*$. Since the reduced vector field f in (13) is superlinear in $\varphi^*_{\nu^*}(x)$ and $(\varphi^*_{\nu^*})'(x)$ by Theorem 3.1(ii), the bound by ν^* in (19) is carried over to $(\varphi^*_{\nu^*})''$. Differentiating (13) gives

$$(\varphi^*_{\nu^*})^{(3)} = \mathrm{D}_1 f(\varphi^*_{\nu^*}, (\varphi^*_{\nu^*})', \nu^*) \cdot (\varphi^*_{\nu^*})' + \mathrm{D}_2 f(\varphi^*_{\nu^*}, (\varphi^*_{\nu^*})', \nu^*) \cdot (\varphi^*_{\nu^*})'',$$

where $\mathrm{D}_1 f$ and $\mathrm{D}_2 f$ are bounded in view of Theorem 3.1(i). Hence, $(\varphi^*_{\nu^*})^{(3)}$ is also bounded by ν^*. We obtain the improvement $\|\varphi^*_{\nu^*}\|_{H^3_{\mathrm{u}}} \lesssim \nu^*$ and then by choosing ν^* sufficiently small, $\varphi^*_{\nu^*}$ is indeed a solution of (10) according to Theorem 3.1(iv). The existence of $\mathcal{C}_{\mathrm{loc}}$ in H^3_{even} is now established since $\varphi^*_{\nu^*} \in H^3$ is an even function.

Our argument for the uniqueness of $\mathcal{C}_{\mathrm{loc}}$ is similar to the one in [11], Lemma 5.10. Suppose that (φ, ν) is a non-trivial even solitary wave solution which is small enough in $H^3_{\mathrm{u}} \times (0, \infty)$ that φ lies on the center manifold $\mathcal{M}_0^{\nu}$. Then $(P, Q) = (\varphi, \varphi')$ is a reversible homoclinic solution of the ODE (17), whose phase portrait is qualitatively the same as in Figure 3. The homoclinic orbit in the right half plane corresponds to the case $\varphi = \varphi_\nu$ and hence $(\varphi, \nu) \in \mathcal{C}_{\mathrm{loc}}$. Any other solution must therefore approach the origin along the portions of its stable and unstable manifolds lying in the left half plane. But this would force $P(t) = \varphi(t) < 0$ for sufficiently large $|t|$, contradicting (i) in Proposition 2.1. $\square$

Remark 3.4. Since the amplitude of $\varphi^*_{\nu^*}$ is $\mathcal{O}(\nu^*)$ as $\nu^* \to 0$, we can find a $\nu' > 0$, such that

$$\sup_{x\in\mathbb{R}} \varphi^*_{\nu^*}(x) < \frac{1+\nu^*}{2}, \qquad \text{for all } \nu^* \in (0, \nu').$$

From Proposition 2.1, the solutions $\varphi^*_{\nu^*}$ are everywhere smooth and strictly decreasing on $(0, \infty)$.

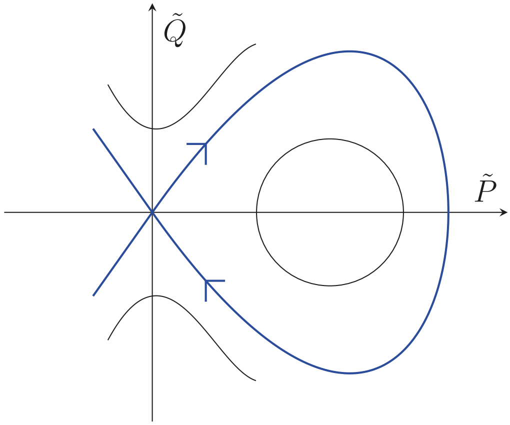

Figure 3: Phase portrait for (18) when $\nu = 0$. The homoclinic orbit persists for $\nu > 0$.

3.3 Invertibility of $L[\varphi^*, \nu^*]$ Let $\varphi^* := \varphi^*_{\nu^*} \in \mathcal{C}_{\mathrm{loc}}$ and the corresponding parameter ν^* be fixed. The linear operator $L[\varphi^*, \nu^*]$ is the linearization of the left-hand side of (10) with respect to the φ-component. Note that $L[\varphi^*, \nu^*]\colon H^3_{\mathrm{even}} \to H^3_{\mathrm{even}}$ is Fredholm with Fredholm index zero. Hence, we only need to show that the nullspace of $L[\varphi^*, \nu^*]$ is trivial. The invertibility of $L[\varphi^*, \nu^*]$ has already been shown in [31]. In this section, we showcase an alternative approach exploiting Theorem 3.2 and are able to make quantitative statements for elements in the nullspace of $L[\varphi^*, \nu^*]$ in $H^3_{-\eta^*}$, where η^* is as in Theorem 3.2. This approach is inspired by Lemmas 4.14 and 4.15 in [32].

Proposition 3.5. *The nullspace* $\operatorname{Ker} L$ *of* $L[\varphi^*, \nu^*]\colon H^3_{-\eta^*} \to H^3_{-\eta^*}$ *is two-dimensional, spanned by the exponentially decaying function* $(\varphi^*)'$ *and an exponentially growing function. Seeing that* $(\varphi^*)'$ *is odd,* $\operatorname{Ker} L$ *restricted on* H^3_{even} *is trivial and* $L[\varphi^*, \nu^*]$ *is thus invertible in* H^3_{even}.

Proof. We use Theorem 3.2(iv), that is, elements $\phi \in \operatorname{Ker} L \subset H^3_{-\eta^*}$ have a one-to-one correspondence to the solutions of (16). Letting $\phi = U$, $\phi' = V$ and regarding x as a time variable t, we cast (16) into a system

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = V \\ \dfrac{\mathrm{d}V}{\mathrm{d}t} = 6\nu^* U - 12\varphi^* U + \dfrac{38}{5}(\varphi^*)' V \\ \qquad\qquad + \mathcal{O}((|\varphi^* U| + |(\varphi^*)' V|)((\nu^*)^2 + |\varphi^*| + |(\varphi^*)'|)). \end{cases} \tag{20}$$

This can be considered as a perturbation problem of the form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = (M + R(t))\mathbf{u},$$

where $\mathbf{u}\colon \mathbb{R}^2 \to \mathbb{R}^2$, $M \in \mathbb{R}^{2\times 2}$ is a matrix with constant coefficients and $R\colon \mathbb{R} \to \mathbb{R}^2$ is an integrable remainder term. In this case,

$$M = \begin{pmatrix} 0 & 1 \\ 6\nu^* & 0 \end{pmatrix},$$

so the eigenvalues for M are $\sqrt{6\nu^*}$ and $-\sqrt{6\nu^*}$. Moreover, the exponential decay of φ^* and $(\varphi^*)'$ guarantees the integrability condition; see the discussion after (19). Applying for example Problem 29, Chapter 3 in [13], as well as switching back to ϕ and ϕ', the statements concerning $\operatorname{Ker} L$ are immediate. In particular, $\operatorname{Ker} L$ is spanned by a function ϕ_1 behaving as $\exp(t\sqrt{6\nu^*})$ and ϕ_2 behaving as $\exp(-t\sqrt{6\nu^*})$ as $t \to \infty$. It is a straightforward calculation to show that one exponentially decaying function in $\operatorname{Ker} L$ is $(\varphi^*)'$, which is an odd function. Since even functions in H^3 cannot be written as linear combinations of an odd exponentially decaying function and an exponentially growing function, $\operatorname{Ker} L$ is trivial in H^3_{even}. □

4 Global bifurcation

We use a global bifurcation theorem from [11] in a slightly modified form because the open set in our case is not a product set; see Appendix D. For (1), we take

$$\mathcal{X} = \mathcal{Y} = H^3_{\mathrm{even}}, \quad F(\varphi,\nu) = \mathcal{T}\varphi + \mathcal{N}(\varphi,\nu),$$

and

$$\mathcal{U} = \left\{ (\varphi,\nu) \in \mathcal{X} \times (0,\infty) \,\middle|\, \sup_{x\in\mathbb{R}} \varphi(x) < \frac{1+\nu}{2} \right\}.$$

Since $H^3 \subset BUC^2$, the supremum norm is controlled by the H^3 norm and $\mathcal{U}$ is thus an open set in $H^3_{\mathrm{even}} \times \mathbb{R}$. We aim to use Theorem D.1. Proposition 2.10 verifies Hypothesis A in this theorem, while Section 3.2 and Proposition 3.5 together verify Hypothesis B. Here, the local curve $\mathcal{C}_{\mathrm{loc}}$ bifurcates from $(0,0) \in \partial\mathcal{U}$. We have thus the following global bifurcation theorem for (1) in H^3_{even} and $\mathcal{U}$.

Theorem 4.1. *The local bifurcation curve $\mathcal{C}_{\mathrm{loc}}$ in Section 3.2 is contained in a curve of solutions $\mathcal{C}$, which is parametrized as*

$$\mathcal{C} = \{(\varphi_s,\nu_s) \,|\, 0 < s < \infty\} \subset \mathcal{U} \cap F^{-1}(0)$$

for some continuous map $(0,\infty) \ni s \mapsto (\varphi_s,\nu_s)$. We have

(a) One of the following alternatives holds:

(i) (blowup) as $s \to \infty$,

$$M(s) := \|\varphi_s\|_{H^3} + \nu_s + \frac{1}{\operatorname{dist}((\varphi_s,\nu_s),\partial\mathcal{U})} \to \infty;$$

(ii) (loss of compactness) there exists a sequence $s_n \to \infty$ as $n \to \infty$ such that $\sup_n M(s_n) < \infty$ but $(\varphi_{s_n})_n$ has no subsequence convergent in $\mathcal{X}$.

(b) Near each point $(\varphi_{s_0}, \nu_{s_0}) \in \mathcal{C}$, we can reparametrize $\mathcal{C}$ so that $s \mapsto (\varphi_s, \nu_s)$ is real analytic.

(c) $(\varphi_s, \nu_s) \notin \mathcal{C}_{\mathrm{loc}}$ for s sufficiently large.

In this section, we use the integral identity in Proposition 2.2 to exclude the loss of compactness scenario. An alternative route is to employ the Hamiltonian structures for nonlocal problems in [7]. Even though the Whitham kernel does not fit into this framework, a direct differentiation confirms that equation (43) in [7] indeed gives a Hamiltonian for the Whitham equation. We also study how $M(s)$ blows up as $s \to \infty$.

4.1 Preservation of nodal structure We begin by showing that the nodal structure is preserved along the global bifurcation curve.

Theorem 4.2. *If $(\varphi, \nu) \in \mathcal{C} \subset \mathcal{U}$, then φ is smooth on $\mathbb{R}$ and strictly decreasing on the interval* $(0, \infty)$.

Proof. The property $\sup_{x\in\mathbb{R}} \varphi(x) < (1+\nu)/2$ for $(\varphi, \nu) \in \mathcal{U}$ implies smoothness on $\mathbb{R}$ by Proposition 2.1(iv). Because in addition $\varphi \in H^3_{\text{even}}$ is a solitary-wave solution with parameter $\nu > 0$, we have that φ is non-increasing by Proposition 2.1(iii). In order to apply Proposition 2.1(v) to conclude that φ is strictly decreasing on $(0, \infty)$ we only need to establish that φ is non-constant.

The only constant solutions are 0 and $\nu > 0$ and the latter is excluded by the fact that $\varphi \in H^3$ is a solitary-wave solution. To show that $\varphi(x) \not\equiv 0$, note that the linearization $L[0,\nu] \colon \phi \mapsto (1+\nu)\phi - K * \phi$ on H^3_{even} is Fredholm of index zero for all $\nu \in \mathcal{I}$ by Proposition 2.10 and Ker $L[0,\nu]$ is trivial. So, $L[0,\nu]$ is invertible. The implicit function theorem applies and prevents $\mathcal{C}$ from intersecting the trivial solution line. Hence, this alternative cannot occur. ☐

4.2 Compactness of the global curve $\mathcal{C}$ The following result rules out alternative (ii) in Theorem 4.1(a).

Theorem 4.3. *Every sequence $(\varphi_n, \nu_n)_{n=1}^{\infty} \coloneqq (\varphi_{s_n}, \nu_{s_n})_{n=1}^{\infty} \subset \mathcal{C}$ satisfying*

$$\sup_n M(s_n) < \infty$$

has a convergent subsequence in $H^3_{\text{even}} \times (0, \infty)$.

Proof. Proposition 2.3 gives a locally uniform limit φ for a subsequence of functions φ_n. The idea is to use Proposition 2.5 to show that φ_n converges to φ in H^3. We observe that the assumption $\sup_n M(s_n) < \infty$ implies

$$\sup_n \|\varphi_n\|_{H^3} < \infty \quad \text{and} \quad \inf_n \nu_n > 0,$$

where $\nu_n = c_n - 1$. Also, according to Corollary 2.4, φ inherits non-negativity, continuity, evenness, boundedness and monotonicity from φ_n. More precisely, we have shown in Theorem 4.2 that φ_n is strictly decreasing on $(0, \infty)$. So, φ is at least non-increasing on $(0, \infty)$.

First, we verify that $\varphi_n \to \varphi$ uniformly. Because the sequence of functions φ_n is uniformly bounded in H^3, it has a weak limit which coincides with the locally uniform limit φ. So $\varphi \in H^3$. Since φ is in addition monotone on the real half-lines, we have $\lim_{|x|\to\infty} \varphi(x) = 0$. Corollary 2.4(ii) now confirms the desired uniform convergence of φ_n to φ.

Next, for the L^2 convergence, we use the integral identity in Proposition 2.2. For each n, $\varphi_n \in BUC^2$ and $\lim_{|x|\to\infty} \varphi_n(x) = 0$. Hence,

$$\lim_{R\to\infty} \int_{-R}^{R} \varphi_n(\varphi_n - \nu_n)\,\mathrm{d}x = 0.$$

Also, since $\nu_n > 0$, the solitary-wave solution φ_n has exponential decay according to Proposition 2.1(ii) and we are allowed to write

$$\int_{\mathbb{R}} \varphi_n^2\,\mathrm{d}x = \nu_n \int_{\mathbb{R}} \varphi_n\,\mathrm{d}x.$$

Since $\sup_n \|\varphi_n\|_{H^3} < \infty$, the L^2 integral on the left-hand side is uniformly bounded in n. Because $\inf_n \nu_n > 0$, the L^1 integral on the right-hand side is uniformly bounded as well. Taking into account that $\lim_{|x|\to\infty} \varphi_n(x) = 0$ uniformly in n, we obtain

$$\begin{aligned}\int_{\mathbb{R}} \varphi_n^2\,\mathrm{d}x &= \int_{|x|<R_\epsilon} \varphi_n^2\,\mathrm{d}x + \int_{|x|>R_\epsilon} \varphi_n^2\,\mathrm{d}x \\ &\le \int_{|x|<R_\epsilon} \varphi_n^2\,\mathrm{d}x + \epsilon \cdot \sup_n \|\varphi_n\|_{L^1}\end{aligned}$$

As $n \to \infty$, we have $\int_{|x|<R_\epsilon} \varphi_n^2\,\mathrm{d}x \to \int_{|x|<R_\epsilon} \varphi^2\,\mathrm{d}x$. Letting $\epsilon \to 0$ confirms that $\varphi_n \to \varphi$ in L^2.

Finally, we observe that $\sup_n M(s_n) < \infty$ also implies

$$\sup_{x\in\mathbb{R}} \varphi_n(x) < (1+\nu_n)/2 \quad \text{and} \quad \sup_{x\in\mathbb{R}} \varphi(x) < (1+\nu)/2.$$

All prerequisites of Proposition 2.5 are now checked and we have $\varphi_n \to \varphi$ in H^3. $\square$

4.3 Analysis of the blowup Having excluded the loss of compactness alternative, we examine the blowup alternative

$$\lim_{s\to\infty}\left(\|\varphi_s\|_{H^3}+\nu_s+\frac{1}{\operatorname{dist}((\varphi_s,\nu_s),\partial\mathcal{U})}\right)=\infty,$$

where $(\varphi_s,\nu_s)\in\mathcal{C}$. In this case, for any sequence $s_n\to\infty$ we can extract a subsequence (also denoted $\{s_n\}$) for which at least one of the following four possibilities holds:

(P1) $\|\varphi_{s_n}\|_{H^3}\to\infty$,

(P2) $\nu_{s_n}\to\infty$,

(P3) $\nu_{s_n}\to 0$,

(P4) $(1+\nu_{s_n})/2-\sup_{x\in\mathbb{R}}\varphi_{s_n}(x)\to 0$,

where (P3) and (P4) belong to the case when $\operatorname{dist}((\varphi_{s_n},\nu_{s_n}),\partial\mathcal{U})\to 0$.

Theorem 4.4. *The alternatives (P2) and (P3) cannot occur.*

Proof. Alternative (P2) cannot occur since the definition of $\mathcal{U}$ and Proposition 2.1(viii) imply that $\nu_{s_n}\leq 1$.

To exclude alternative (P3), we assume $\nu_{s_n}\to 0$ as $n\to\infty$. Any locally uniform limit $(\varphi,0)$ solves (1). Moreover, φ is bounded, continuous, and monotone; see Proposition 2.3. Proposition 2.2 gives

$$\inf_x\varphi(x)<0<\sup_x\varphi(x)\qquad\text{or}\qquad\varphi\equiv 0.$$

Because φ is non-negative, we must have $\varphi\equiv 0$. In particular, $\lim_{|x|\to\infty}\varphi(x)=0$. In virtue of Corollary 2.4(ii), $\varphi_{s_n}\to 0$ uniformly and now according to Remark 2.6, $\varphi_{s_n}\to 0$ in C^k for any k, which implies that $\varphi_{s_n}\to\varphi$ in $H^3_{-\eta}$ for all $\eta>0$ and that $(\varphi_{s_n},\nu_{s_n})$ reenters any small neighborhood of $(0,0)$ in $H^3_{\mathrm{u}}\times(0,\infty)$. This cannot happen in light of Theorem 4.1(c) and the uniqueness of $\mathcal{C}_{\mathrm{loc}}$ given by Theorem 3.3. □

Next, we show a useful characterization for when the H^3 norm stays bounded.

Lemma 4.5. *By possibly taking a subsequence of $(\varphi_{s_n},\nu_{s_n})_{n=1}^{\infty}$, we have $\nu_{s_n}\to\nu>0$ and $\varphi_{s_n}\to\varphi$ locally uniformly as $n\to\infty$. Then, $\inf_n\nu_{s_n}>0$. Moreover,*

$$\sup_n\|\varphi_{s_n}\|_{H^3}<\infty\quad\textit{if and only if}\quad\sup_{x\in\mathbb{R}}\varphi(x)<\frac{1+\nu}{2}\ \textit{and}\ \lim_{|x|\to\infty}\varphi(x)=0.$$

Proof. The existence of such a subsequence is given by Proposition 2.3. Since Theorem 4.4 has excluded (P3), we must have $\nu_{s_n}\to\nu>0$, which also implies that $\inf_n\nu_{s_n}>0$. We focus on proving the last statement. The proof of Theorem 4.3 already gives that

$\sup_n \|\varphi_{s_n}\|_{H^3} < \infty$ implies $\varphi \in H^3$, thus $\sup_{x\in\mathbb{R}} \varphi(x) < (1+\nu)/2$ by Proposition 2.1 (v) and $\lim_{|x|\to\infty} \varphi(x) = 0$. Conversely, assume on the contrary that there exists a subsequence of functions φ_{s_n} such that $\|\varphi_{s_n}\|_{H^3} \to \infty$ as $n \to \infty$, yet its locally uniform limit φ satisfies $\sup_{x\in\mathbb{R}} \varphi(x) < (1+\nu)/2$ and $\lim_{|x|\to\infty} \varphi(x) = 0$. Then, φ is smooth by (iv) in Proposition 2.1. Also, Corollary 2.4(ii) gives $\varphi_{s_n} \to \varphi$ uniformly, so $\varphi_{s_n}(x) \to 0$ as $|x| \to \infty$ uniformly in n. Similar to the proof of Theorem 4.3, we get

$$\nu_{s_n} \int_{\mathbb{R}} \varphi_{s_n} \, \mathrm{d}x = \int_{\mathbb{R}} \varphi_{s_n}^2 \, \mathrm{d}x \le \int_{|x|<R_\epsilon} \varphi_{s_n}^2 \, \mathrm{d}x + \epsilon \int_{\mathbb{R}} \varphi_{s_n} \, \mathrm{d}x,$$

where ϵ and R_ϵ are independent of n. Rearranging gives

$$\int_{\mathbb{R}} \varphi_{s_n} \, \mathrm{d}x \le \frac{1}{\nu_{s_n} - \epsilon} \int_{|x|<R_\epsilon} \varphi_{s_n}^2 \, \mathrm{d}x \le \frac{2R_\epsilon}{\nu_{s_n} - \epsilon} \cdot \sup_{x\in\mathbb{R}} \varphi_{s_n}^2(x) \le \frac{2R_\epsilon}{\nu_{s_n} - \epsilon},$$

where $\sup_{x\in\mathbb{R}} \varphi_{s_n}^2(x) < 1$ because $\nu_{s_n} \in (0,1]$. Recall that $\inf_n \nu_{s_n} > 0$. Choosing $\epsilon = \inf_n \nu_{s_n}/2$, this shows $\sup_n \|\varphi_{s_n}\|_{L^1} < \infty$. It follows that the sequence of functions φ_{s_n} is uniformly bounded in L^2 and arguing as in the proof of Proposition 2.5 gives the uniform boundedness in H^3, which is a contradiction to the assumption. □

We can now establish the following equivalence.

Theorem 4.6. *(P4) and (P1) are equivalent.*

Proof. Let $(\varphi_{s_n}, \nu_{s_n})_{n=1}^\infty$ be a sequence satisfying (P4). By possibly taking a subsequence, Proposition 2.3 gives that $\varphi_{s_n} \to \varphi$ locally uniformly and $\nu_{s_n} \to \nu$, where $\nu > 0$ as we have excluded (P3). Since each φ_{s_n} is even and strictly decreasing, (P4) is the same as

$$\lim_{n\to\infty} \left| \frac{1+\nu_{s_n}}{2} - \varphi_{s_n}(0) \right| = 0,$$

which is equivalent to

$$\varphi(0) = \frac{1+\nu}{2}.$$

By Lemma 4.5, this implies (P1). For the other implication, let $(\varphi_{s_n}, \nu_{s_n})_{n=1}^\infty$ be a sequence satisfying (P1). We also have that $\varphi_{s_n} \to \varphi$ locally uniformly and $\nu_{s_n} \to \nu > 0$. Once again by Proposition 2.3 and Corollary 2.4, φ solves (1) with parameter $\nu > 0$ and is continuous, bounded, even, and φ is non-increasing on $(0,\infty)$. Then, the limit $\lim_{|x|\to\infty} \varphi(x)$ exists. According to Corollary 2.4(i), this can take the value

$$\text{(a)} \lim_{|x|\to\infty} \varphi(x) = 0 \quad \text{or} \quad \text{(b)} \lim_{|x|\to\infty} \varphi(x) = \nu > 0.$$

In addition, Proposition 2.2 says

$$\text{(A)} \inf_{x\in\mathbb{R}} \varphi(x) < \nu < \sup_{x\in\mathbb{R}} \varphi(x), \quad \text{(B)}\, \varphi \equiv 0 \quad \text{or} \quad \text{(C)}\, \varphi \equiv \nu > 0.$$

The combinations (aC) and (bB) are quickly excluded. If $\varphi \equiv 0$, then $\sup_{x\in\mathbb{R}} \varphi(x) < (1+\nu)/2$ and Lemma 4.5 gives that $\sup_n \|\varphi_{s_n}\|_{H^3} < \infty$, which contradicts the blowup alternative. This rules out (aB). The fact that φ is non-increasing on $(0,\infty)$ rules out (bA). Assume (bC), which is just (C). Then, we arrive at a contradiction as follows. Consider the sequence of translated solutions $\tau_{x_n}\varphi_{s_n} = \varphi_{s_n}(\,\cdot\, + x_n)$, where each x_n is chosen in such a way that

$$\varphi_{s_n}(x_n) = \tilde{\nu}, \text{ where } 0 < \tilde{\nu} < \inf_n \frac{\nu_{s_n}}{2}.$$

Such a number $\tilde{\nu}$ exists because $\nu_{s_n} > 0$ cannot limit to 0. Moreover, $\lim_{n\to\infty} x_n = \infty$, or we cannot have $\varphi \equiv \nu > 0$ for every x while each φ_n has exponential decay. Corollary 2.4(iii) applied to $(\tau_{x_n}\varphi_{s_n})_n$ gives a bounded, continuous and non-increasing locally uniform limit $\tilde{\varphi}$. The function $\tilde{\varphi}$ is a solution to (10) with parameter ν. Its limits $\tilde{\varphi}(x)$ as $x \to \pm\infty$ are guaranteed to exist and these can take the value zero or $\nu > 0$. By construction,

$$\tilde{\varphi}(0) = \lim_n \varphi_{s_n}(0 + x_n) = \tilde{\nu} \in (0, \nu),$$

which implies

$$\lim_{x\to-\infty} \tilde{\varphi}(x) = \nu \quad \text{and} \quad \lim_{x\to\infty} \tilde{\varphi}(x) = 0.$$

On the other hand, this also shows that $\tilde{\varphi}$ is not a constant function, implying

$$0 = \inf_{x\in\mathbb{R}} \tilde{\varphi}(x) < \nu < \sup_{x\in\mathbb{R}} \tilde{\varphi}(x) = \nu,$$

which is a contradiction. We conclude that (C) cannot occur. Hence, we must have

$$\lim_{|x|\to\infty} \varphi(x) = 0 \quad \text{and} \quad \inf_{x\in\mathbb{R}} \varphi(x) < \nu < \sup_{x\in\mathbb{R}} \varphi(x),$$

where the first condition is the same as (P4). Applying Lemma 4.5 gives the desired implication. ☐

Remark 4.7. This in fact implies that $\mathcal{C}$ satisfies $\|\varphi_s\|_{H^3} \to \infty$ and $(1+\nu_s)/2 - \sup_x \varphi_s(x) \to 0$ as $s \to \infty$, without the need for considering subsequences.

Finally, since (P1) and (P4) are equivalent, (P2) and (P3) cannot happen and the blowup alternative must take place, we must have a sequence $(\varphi_{s_n}, \nu_{s_n})_{n=1}^{\infty} \subset \mathcal{C}$ such that

$$\lim_{n\to\infty} \|\varphi_{s_n}\|_{H^3} = \infty \text{ and } \lim_{n\to\infty} \nu_{s_n} = \nu > 0.$$

By taking the limit of a subsequence, an extreme solitary-wave solution φ attaining the highest possible amplitude $(1+\nu)/2$ is found; see Figure 1.

Theorem 4.8. *There exists a sequence of elements $(\varphi_{s_n}, \nu_{s_n}) \in \mathcal{C}$, such that*

$$\lim_{n\to\infty} \|\varphi_{s_n}\|_{H^3} = \infty \quad \textit{and} \quad \lim_{n\to\infty} \nu_{s_n} = \nu > 0.$$

The sequence of solutions φ_{s_n} has a locally uniform limit φ. We have *(i)* φ *is continuous, bounded, even and non-increasing on the positive real half-line;*

(ii) φ *is a non-trivial solitary-wave solution to* (10) *with parameter* $\nu > 0$*;*

(iii) $\varphi(0) = (1+\nu)/2$ *and more precisely*

$$C_1|x|^{\frac{1}{2}} \leq \frac{1+\nu}{2} - \varphi(x) \leq C_2|x|^{\frac{1}{2}}, \tag{21}$$

near the origin and for some constants $0 < C_1 < C_2$*;*

(iv) φ *is smooth everywhere except at* $x = 0$*;*

(v) φ *has exponential decay.*

Proof. Existence has already been shown. According to Proposition 2.3, there is such a locally uniform limit φ. Statement (i) is immediate from Corollary 2.4(ii). Statement (ii) follows from the proof of Theorem 4.6, as we have shown that

$$\lim_{|x|\to\infty} \varphi(x) = 0 \quad \text{and} \quad \inf_{x\in\mathbb{R}} \varphi(x) < \nu < \sup_{x\in\mathbb{R}} \varphi(x).$$

Statement (iii) is Theorem 4.6 and the estimate (21) for φ near the origin is Proposition 2.1(vi). Together with φ being non-increasing on $(0,\infty)$, we have $\varphi(x) < (1+\nu)/2$ if $x \neq 0$. Proposition 2.1(iv) applies and gives statement (iv). Statement (v) follows from Proposition 2.1(ii). □

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A Fredholmness of pseudodifferential operators

Let $x^* = (x_0, x) \in \mathbb{R}^2$ and

$$X^* = \{x^* \in \mathbb{R}^2 \,|\, x_0 \geq 0, x^* \neq 0\}.$$

Similarly, let $\xi^* = (\xi_0, \xi) \in \mathbb{R}^2$ and

$$E^* = \{\xi^* \in \mathbb{R}^2 \,|\, \xi_0 \geq 0, \xi^* \neq 0\}.$$

We define

$$\mathbb{S} = \{(x_0, x, \xi_0, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x_0^2 + |x|^2 = \xi_0^2 + |\xi|^2 = 1, x_0 > 0, \xi_0 > 0\},$$

and denote the relative closure of $\mathbb{S}$ in $X^* \times E^*$ by $\overline{\mathbb{S}}$ and the boundary of $\overline{\mathbb{S}}$ by Γ.

Let $\mathcal{A}$ be the class of functions $A(x^*, \xi^*) \in C^\infty(X^* \times E^*)$ such that A is positively homogeneous of degree 0 in x^* and ξ^*, that is,

$$A(\lambda x^*, \xi^*) = A(x^*, \lambda \xi^*) = A(x^*, \xi^*), \qquad \lambda > 0.$$

Clearly, each $A \in \mathcal{A}$ is uniquely determined by its values on $\mathbb{S}$. Conversely, each function $\tilde{A} \in C^\infty(\overline{\mathbb{S}})$ can be uniquely homogeneously extended to $X^* \times E^*$. So, $\mathcal{A} \cong C^\infty(\overline{\mathbb{S}})$. By $S^0_{\mathcal{A}}$, we denote the set of symbols $p_A(x, \xi)$ which are given by

$$p_A(x, \xi) = A(1, x, 1, \xi),$$

for some $A \in \mathcal{A}$. For $p_A \in S^0_{\mathcal{A}}$, we have the following result, which combines Theorems 4.1 and 4.2 in [27].

Proposition A.1. *Let $j \in \mathbb{R}$. If $p_A(x, \xi) \in S^0_{\mathcal{A}}$ and $A(x^*, \xi^*) \neq 0$ on Γ, then*

$$p_A(x, \mathrm{D})\colon H^j \to H^j$$

is Fredholm with Fredholm index

$$\operatorname{ind} p_A(x, \mathrm{D}) = \frac{1}{2\pi}\Big(\arg\, A(x^*, \xi^*)\Big|_{\Gamma}\Big),$$

where $\arg A(x^*, \xi^*)|_\Gamma$ is the increase in the argument of $A(x^*, \xi^*)$ around Γ as Γ is traversed with the counter-clockwise orientation.

We comment that a version of Proposition A.1 is available for matrix-valued symbols $p_A(x, \xi)$; see [27, Section 4].

B Center manifold theorems

B.1 A nonlocal version of the center manifold theorem Nonlocal nonlinear parameter-dependent problems of the form

$$\mathcal{T} v + \mathcal{N}(v, \mu) = 0, \tag{22}$$

where

$$\mathcal{T} v = v + \mathcal{K} * v,$$

are considered in the weighted Sobolev spaces $H^j_{-\eta}(\mathbb{R}^n)$ for some $\eta > 0$, positive integers n and $j = 1$. The following presentation focuses on dimension $n = 1$ and arbitrary regularity $j \geq 1$. $\mathcal{T}$ will be referred to as the linear part and $\mathcal{N}$ as the nonlinear part of (22).

To obtain an appropriate modification of equation (22), we introduce the space of uniformly local H^j functions,

$$H^j_{\mathrm{u}} = \left\{ v \in H^j_{\mathrm{loc}} \,\middle|\, \|v\|_{H^j_{\mathrm{u}}} < \infty \right\} \qquad \text{where} \qquad \|v\|_{H^j_{\mathrm{u}}} = \sup_{y\in\mathbb{R}} \|v(\,\cdot\, + y)\|_{H^j([0,1])}.$$

Note that the embeddings $H^j \subset H^j_{\mathrm{u}} \subset H^j_{-\eta}$ are continuous for all $\eta > 0$. Next, let $\underline{\chi} : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function satisfying

$$\underline{\chi}(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 2, \end{cases} \qquad \text{and} \qquad \sup_{x\in\mathbb{R}} |\underline{\chi}'(x)| \leq 2.$$

In addition, let $\theta : \mathbb{R} \to \mathbb{R}$ be a smooth and even function with

$$\sum_{j\in\mathbb{Z}} \theta(x-j) = 1, \qquad \operatorname{supp}\theta \subset [-1,1], \qquad \theta(x) \geq 0, \qquad \theta\left(\left[0,\frac{1}{2}\right]\right) \subset \left[\frac{1}{2},1\right],$$

so that

$$\int_{y\in\mathbb{R}} \theta(x-y)\,\mathrm{d}y = \int_{y=0}^{1} \sum_{j\in\mathbb{Z}} \theta(x-y-j)\,\mathrm{d}y = 1.$$

We define

$$\chi \colon v \mapsto \int_{y\in\mathbb{R}} \underline{\chi}(\|\theta(\,\cdot\, - y)v\|_{H^j})\theta(x-y)v(x)\,\mathrm{d}y,$$

and then

$$\chi^\delta : v \mapsto \delta \cdot \chi\left(\frac{v}{\delta}\right), \qquad \delta > 0.$$

The cutoff operator $\chi : H^j_{-\eta} \to H^j_{\mathrm{u}}$ is well-defined, Lipschitz continuous and $\|\chi(v)\|_{H^j_{\mathrm{u}}} \leq C_0$ for all $v \in H^j_{-\eta}$. Lemma 2 in [25] proves these claims for $j = 1$ and an asymmetric function θ with $\operatorname{supp}\theta \subset [-1/4, 5/4]$. Generalizing to higher regularity $j \geq 1$ and verifying the results in [25] with our choice of θ are straightforward. The scaled cutoff operator $\chi^\delta : H^j_{-\eta} \to H^j_{\mathrm{u}}$ naturally inherits these properties, in particular $\|\chi^\delta(v)\| \leq C_0\delta$ for all $v \in H^j_{-\eta}$. The modification of equation (22) is

$$\mathcal{T}v + \mathcal{N}^\delta(v,\mu) = 0, \tag{23}$$

where

$$\mathcal{N}^\delta(v,\mu) : (v,\mu) \mapsto \mathcal{N}(\chi^\delta(v),\mu).$$

For $\xi \in \mathbb{R}$, $\tau_\xi v = v(\,\cdot\, + \xi)$ denotes the shift by ξ. Regarded as an operator on $H^j_{-\eta}$ or H^j, τ_ξ is bounded with $\|\tau_\xi\|_{H^j_{-\eta}\to H^j_{-\eta}} \lesssim \exp(\eta\xi)$ and $\|\tau_\xi\|_{H^j\to H^j} = 1$.

Also, we let $\mathcal{Q}\colon H^j_{-\eta} \to H^j_{-\eta}$ be a bounded projection on the nullspace Ker $\mathcal{T}$ of $\mathcal{T}$ with a continuous extension to $H^{j-1}_{-\eta}$, such that $\mathcal{Q}$ commutes with the inclusion map from $H^j_{-\eta}$ to $H^j_{-\eta'}$, for all $0 < \eta' < \eta$.

Hypothesis B.1 (The linear part $\mathcal{T}$).

(i) There exists $\eta_0 > 0$ *such that* $\mathcal{K} \in L^1_{\eta_0}$.

(ii) The operator

$$\mathcal{T}\colon v \mapsto v + \mathcal{K} * v, \qquad H^j_{-\eta} \to H^j_{-\eta}$$

is Fredholm for $\eta \in (0, \eta_0)$, *its nullspace* Ker $\mathcal{T}$ *is finite-dimensional and*

$$\operatorname{ind} \mathcal{T} = \mathit{dim}\ \mathrm{Ker}\ \mathcal{T},$$

where $\operatorname{ind} \mathcal{T}$ *is the Fredholm index of* $\mathcal{T}$. *In other words,* $\mathcal{T}$ *is onto.*

Remark B.2. Hypothesis B.1(i) gives boundedness of $\mathcal{T}$ on $H^j_{-\eta}$ for $\eta \in (0, \eta_0)$; see Section 2.4. The original assumption $\mathcal{K} \in W^{1,1}_\eta$ in [24] is only used to guarantee item (ii). Since the Whitham kernel $K \notin W^{1,1}_\eta$, we instead require (ii) directly.

Hypothesis B.3 (The nonlinear part $\mathcal{N}$). *There exist* $k \geq 2$, *a neighborhood* $\mathcal{U}$ *of* $0 \in H^j_{-\eta}$ *and* $\mathcal{V}$ *of* $0 \in \mathbb{R}$, *such that for all sufficiently small* $\delta > 0$, *we have*

(i) $\mathcal{N}^\delta\colon H^j_{-\eta} \times \mathcal{V} \to H^j_{-\eta}$ *is* $\mathscr{C}^k$. *In addition, for all non-negative pairs* (ζ, η) *satisfying* $0 < k\zeta < \eta < \eta_0$, $\mathrm{D}^l_v \mathcal{N}^\delta\colon (H^j_{-\zeta})^l \to H^j_{-\eta}$ *is bounded for all* $0 < l\zeta \leq \eta < \eta_0$ *and* $0 \leq l \leq k$, *and Lipschitz continuous in* v *for* $1 \leq l \leq k-1$ *uniformly in* $\mu \in \mathcal{V}$;

(ii) $\mathcal{N}^\delta(\tau_\xi v, \mu) = \tau_\xi \mathcal{N}^\delta(v, \mu)$ *for all* $\mu \in \mathcal{V}$ *and* $\xi \in \mathbb{R}$;

(iii) $\mathcal{N}^\delta(0,0) = 0, \mathrm{D}_v \mathcal{N}^\delta(0,0) = 0$ *and as* $\delta \to 0$,

$$\delta_1(\delta) := \mathit{Lip}_{H^j_{-\eta} \times \mathcal{V}} \mathcal{N}^\delta = \mathcal{O}(\delta + |\mu|).$$

In dimension $n \geq 1$, a symmetry is a triple $(\rho, \tau_\xi, \kappa) \in \mathbf{O}(n) \times (\mathbb{R} \times \mathbf{O}(1))$, where the orthogonal linear transformation $\rho \in \mathbf{O}(n)$ acts on $v(x) \in \mathbb{R}^n$ while τ_ξ and κ act on the variable $x \in \mathbb{R}$. In particular, a symmetry (ρ, τ_ξ, κ) is called equivariant if $(\rho, \tau_\xi, \kappa) \in \mathbf{O}(n) \times (\mathbb{R} \times \{\mathrm{Id}\})$, and reversible otherwise. Lemma 2 in [25] and the fact that θ is even give the invariance of χ^δ under the whole group $\mathbf{O}(n) \times (\mathbb{R} \times \mathbf{O}(1))$ for $n = 1$, that is, $\chi^\delta(\gamma v) = \gamma \chi^\delta(v)$ for all $\gamma \in \mathbf{O}(1) \times (\mathbb{R} \times \mathbf{O}(1))$.

Hypothesis B.4 (Symmetries). *There exists a symmetry group* $S \subset \mathbf{O}(1) \times (\mathbb{R} \times \mathbf{O}(1))$ *which contains all translations on the real line and which commutes with the linear part* $\mathcal{T}$ *as well as the nonlinear part* $\mathcal{N}$, *that is,*

$$\mathcal{T}(\gamma v) = \gamma(\mathcal{T} v) \ \textit{ and } \ \mathcal{N}(\gamma v, \mu) = \gamma \mathcal{N}(v, \mu), \qquad \textit{for all } \gamma \in S.$$

We state the equivariant parameter-dependent center manifold theorem. Even though the hypothesis on $\mathcal{K}$ is changed, the proof is the same.

Theorem B.5. *Assume Hypotheses B.1, B.3 and B.4 are met for equation* (22). *Then, by possibly shrinking the neighborhood $\mathcal{V}$ of $0 \in \mathbb{R}$, there exist a cutoff radius $\delta > 0$, a weight $\eta^* \in (0, \eta_0)$ and a map*

$$\Psi \colon \operatorname{Ker} \mathcal{T} \times \mathcal{V} \subset H^j_{-\eta^*} \times \mathbb{R} \to \operatorname{Ker} \mathcal{Q} \subset H^j_{-\eta^*}$$

with the center manifold

$$\mathcal{M}_0^\mu := \{v_0 + \Psi(v_0, \mu) \,|\, v_0 \in \operatorname{Ker} \mathcal{T}, \mu \in \mathcal{V}\} \subset H^j_{-\eta^*},$$

as its graph. The following statements hold:

(i) *(smoothness) $\Psi \in \mathscr{C}^k$, where k is as in Hypothesis B.3;*

(ii) *(tangency) $\Psi(0, 0) = 0$ and $\mathrm{D}_{v_0}\Psi(0, 0) = 0$;*

(iii) *(global reduction) $\mathcal{M}_0^\mu$ consists precisely of functions v such that $v \in H^j_{-\eta^*}$ is a solution of the modified equation* (23) *with parameter μ;*

(iv) *(local reduction) any function $v \in H^j_{\mathrm{u}}$ solving* (22) *with $\|v\|_{H^j_{\mathrm{u}}} < C_0\delta$ is contained in $\mathcal{M}_0^\mu$;*

(v) *(translation invariance) the shift τ_ξ by $\xi \in \mathbb{R}$ acting on $\mathcal{M}_0^\mu$ induces a μ-dependent flow*

$$\Phi_\xi \colon \operatorname{Ker} \mathcal{T} \to \operatorname{Ker} \mathcal{T}$$

through $\Phi_\xi = \mathcal{Q} \circ \tau_\xi \circ (\mathrm{Id} + \Psi)$;

(vi) *(reduced vector field) the reduced flow Φ_ξ is of class $\mathscr{C}^k$ in v_0, μ, ξ and is generated by a reduced parameter-dependent vector field f of class $\mathscr{C}^{k-1}$ on the finite-dimensional* $\operatorname{Ker} \mathcal{T}$*;*

(vii) *(correspondence) any element $v = v_0 + \Psi(v_0, \mu)$ of $\mathcal{M}_0^\mu$ corresponds one-to-one to a solution of*

$$\frac{\mathrm{d}v_0}{\mathrm{d}x} = f(v_0, \mu) := \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{Q}(\tau_x v)\Big|_{x=0};$$

(viii) *(equivariance) $\operatorname{Ker} \mathcal{T}$ is invariant under S and $\mathcal{Q}$ can be chosen to commute with all $\gamma \in S$. Consequently, Ψ commutes with $\gamma \in S$ and $\mathcal{M}_0^\mu$ is invariant under S. Finally, the reduced vector field f in item (vi) commutes with all equivariant symmetries and anticommutes with the reversible ones in S.*

B.2 Center manifold theorem for an augmented problem In this section, we prove a center manifold theorem for a system consisting of the nonlocal equation (22) and its linearization at (v, μ), that is,

$$\mathcal{T} w + \mathrm{D}_v \mathcal{N}(v, \mu) w = 0. \tag{24}$$

More precisely, we study

$$\mathbf{T}(v, w) + \mathbf{N}(v, w, \mu) = 0, \tag{25}$$

where the linear part is

$$\mathbf{T}\colon (v, w) \mapsto (\mathcal{T} v, \mathcal{T} w) = (v + \mathcal{K} * v, w + \mathcal{K} * w),$$

and the nonlinear part is

$$\mathbf{N}\colon (v, w, \mu) \mapsto (\mathcal{N}(v, \mu), \mathrm{D}_v \mathcal{N}(v, \mu) w).$$

We consider the modified system

$$\mathbf{T}(v, w) + \mathbf{N}^\delta(v, w, \mu) = 0, \tag{26}$$

with

$$\mathbf{N}^\delta(v, w, \mu) = (\mathcal{N}^\delta(v, \mu), \mathrm{D}_v \mathcal{N}^\delta(v, \mu) w),$$

where $\mathcal{N}^\delta(v, \mu)$ is defined in the previous section. Observe that we only cut off in v, which allows capturing all solutions of the linearized equation (24). This requires yet another adaptation of the center manifold theorem, where the usual contraction principle is replaced with a fiber contraction principle.

Theorem B.6. *Assume Hypotheses B.1, B.3, B.4 and let $\mathcal{Q}$ be the same projection as in Theorem B.5. For (25), there exist a cutoff radius $\delta > 0$, a weight $\eta^* \in (0, \eta_0)$, a neighborhood $\mathcal{V}$ of $0 \in \mathbb{R}$ and two mappings*

$$\Psi_1\colon \operatorname{Ker} \mathcal{T} \times \mathcal{V} \to \operatorname{Ker} \mathcal{Q} \subset H^j_{-\eta^*},$$

with the center manifold

$$\mathbf{M}^\mu_{0,1} := \{ v_0 + \Psi_1(v_0, \mu) \mid v_0 \in \operatorname{Ker} \mathcal{T}, \mu \in \mathcal{V} \}$$

as its graph, and at each fixed element $v = v_0 + \Psi_1(v_0, \mu) \in \mathbf{M}^\mu_{0,1}$,

$$\Psi_2[v_0, \mu]\colon \operatorname{Ker} \mathcal{T} \to \operatorname{Ker} \mathcal{Q},$$

with graph

$$\mathbf{M}_{0,2}[v_0, \mu] := \{ w_0 + \Psi_2[v_0, \mu](w_0) \mid w_0 \in \operatorname{Ker} \mathcal{T} \}.$$

The following statements hold.

(i) $\mathbf{M}_{0,2}[v_0, \mu]$ *consists precisely of the solutions to the modified linearized equation* $\mathcal{T}w + \mathrm{D}_v\mathcal{N}^\delta(v,\mu)w = 0$ *at* $v = v_0 + \Psi_1(v_0,\mu) \in \mathbf{M}^\mu_{0,1}$ *and the problem*

$$\mathcal{T}w + \mathrm{D}_v\mathcal{N}^\delta(v,\mu)w = 0 \quad \textit{with} \quad \mathcal{Q}w = w_0$$

has a unique solution for each $w_0 \in \operatorname{Ker}\mathcal{T}$.

(ii) $\mathbf{M}^\mu_{0,1}$ *coincides with* $\mathcal{M}^\mu_0$ *in Theorem B.5 and all statements in this theorem hold for* $\mathbf{M}^\mu_{0,1}$. *In particular, if* $v = v_0 + \Psi_1(v_0,\mu) \in \mathbf{M}^\mu_{0,1}$ *is sufficiently small in the* H^j_{u} *norm, so that* v *solves the original equation* (22) *with parameter* μ, *then* $w \in \mathbf{M}_{0,2}[v_0,\mu]$ *solves the linearized equation* (24) *at* v.

(iii) We have $\Psi_2[v_0,\mu] = \mathrm{D}_{v_0}\Psi_1(v_0,\mu)$. *Consequently,* $\Psi_2[v_0,\mu]$ *is a bounded linear map,* $\Psi_2[0,0] = 0$ *and* Ψ_2 *is* $\mathscr{C}^{k-1}$ *in* (v_0,μ).

(iv) The shift τ_ξ *acting on* $\mathbf{M}_{0,2}[\,\cdot\,,\mu]$ *induces a flow*

$$\Phi_{2,\xi}\colon \operatorname{Ker}\mathcal{T} \to \operatorname{Ker}\mathcal{T}$$

through $\Phi_{2,\xi} \coloneqq \mathcal{Q} \circ \tau_\xi \circ (\mathrm{Id} + \Psi_2[\tau_\xi v_0,\mu])$.

(v) The reduced flow $\Phi_{2,\xi}$ *is of class* $\mathscr{C}^{k-1}$ *in* v_0, μ, ξ *and boundedly linear in* w_0. *It is generated by a reduced parameter-dependent vector field* g *of class* $\mathscr{C}^{k-2}$ *in* (v_0,μ) *and boundedly linear in* w_0.

(vi) Any element $w = w_0 + \Psi_2[v_0,\mu]w_0$ *of* $\mathbf{M}_{0,2}[v_0,\mu]$ *corresponds one-to-one to a solution of*

$$\frac{\mathrm{d}w_0}{\mathrm{d}x} = g(v_0,w_0,\mu) \coloneqq \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{Q}(\tau_x w_0 + \Psi_2[\tau_x v_0,\mu](\tau_x w_0))\Big|_{x=0}.$$

We have $g(v_0,w_0,\mu) = \mathrm{D}_{v_0} f(v_0,\mu)w_0$.

We will use the following fiber contraction theorem in the proof; see Section 1.11.8 in [12].

Proposition B.7. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be complete metric spaces. Consider a continuous map* $\Lambda\colon \mathcal{X}\times\mathcal{Y} \to \mathcal{X}\times\mathcal{Y}$ *of the form*

$$\Lambda(x,y) = (\lambda_1(x), \lambda_2(x,y)),$$

where $\lambda_1\colon \mathcal{X} \to \mathcal{X}$ *and* $\lambda_2\colon \mathcal{X}\times\mathcal{Y} \to \mathcal{Y}$. *If* λ_1 *is a contraction in* $\mathcal{X}$, *and* $y \mapsto \lambda_2(x,y)$ *is a contraction in* $\mathcal{Y}$ *for every fixed* $x \in \mathcal{X}$, *then* Λ *has a unique fixed point* $(x_0,y_0) \in \mathcal{X}\times\mathcal{Y}$.

Proof of Theorem B.6. We present the necessary changes in the proof of the center manifold theorem in [24] without the parameter μ. The transition to the parameter-dependent version is the same as in [24].

In view of Hypothesis B.1(ii), the operator $\mathbf{T} = (\mathcal{T}, \mathcal{T})$ is Fredholm and its Fredholm index is twice the index of $\mathcal{T}$. We define the Fredholm-bordered operator

$$\tilde{\mathbf{T}} \colon (H^j_{-\eta})^2 \to (H^j_{-\eta} \times \operatorname{Ker} \mathcal{T})^2$$

with

$$\tilde{\mathbf{T}} \colon (v, w) \mapsto (\tilde{\mathcal{T}} v, \tilde{\mathcal{T}} w) := (\mathcal{T} v, \mathcal{Q} v, \mathcal{T} w, \mathcal{Q} w).$$

By Lemma 3.2 in [24], $\tilde{\mathbf{T}}$ is invertible for any $\eta \in (0, \eta_0)$ with

$$\tilde{\mathbf{T}}^{-1} = (\tilde{\mathcal{T}}^{-1}, \tilde{\mathcal{T}}^{-1}), \quad \text{ and } \quad \|\tilde{\mathbf{T}}^{-1}\|_{H^j_{-\eta} \to H^j_{-\eta}} \le C(\eta),$$

where C is a continuous function of η. We define the bordered nonlinearity

$$\tilde{\mathbf{N}}^\delta \colon (v, w, v_0, w_0) \mapsto (\mathcal{N}^\delta(v), -v_0, \mathrm{D}_v \mathcal{N}^\delta(v) w, -w_0).$$

Due to Hypothesis B.3, $\tilde{\mathbf{N}}^\delta$ is continuous in $(v, w) \in (H^j_{-\eta})^2$. The bordered equation becomes

$$\tilde{\mathbf{T}}(v, w) + \tilde{\mathbf{N}}^\delta(v, w, v_0, w_0) = 0.$$

Applying $\tilde{\mathbf{T}}^{-1}$ on both sides, then moving the nonlinear term to the right-hand side, we obtain a fixed point equation

$$\begin{aligned}(v, w) &= -\tilde{\mathbf{T}}^{-1} \left(\tilde{\mathbf{N}}^\delta(v, w, v_0, w_0) \right) \\ &=: \mathcal{S}^\delta(v, w, v_0, w_0).\end{aligned}$$

Let v_0 and w_0 be fixed. We apply Proposition B.7 with $\Lambda = \mathcal{S}^\delta$ and $\mathcal{X} = \mathcal{Y} = H^j_{-\eta}$. Continuity of $\mathcal{S}^\delta$ in (v, w) is clear in view of Hypothesis B.3(i) and Theorem B.5 already shows that the first component of $\mathcal{S}^\delta$ is a contraction mapping for appropriately chosen cutoff radii $\delta > 0$ and weights $\eta \in (0, \eta_0)$. Now, let v be fixed in the second component of $\mathcal{S}^\delta$. Hypothesis B.3(iii) together with $\mathcal{N}^\delta \colon H^j_{-\eta} \to H^j_{-\eta}$ being $\mathscr{C}^k$ for $k \ge 2$ imply that

$$\sup_{v \in H^j_{-\eta}} \|\mathrm{D}_v \mathcal{N}^\delta(v)\|_{H^j_{-\eta} \to H^j_{-\eta}} = \mathcal{O}(\delta), \quad \text{ as } \delta \to 0.$$

This gives

$$\|\mathrm{D}_v \mathcal{N}^\delta(v) w_1 - \mathrm{D}_v \mathcal{N}^\delta(v) w_2\|_{H^j_{-\eta}} \lesssim \delta \|w_1 - w_2\|_{H^j_{-\eta}}.$$

Choosing δ sufficiently small, the second component is a contraction mapping on $H^j_{-\eta}$ for each fixed v. By Theorem B.7, there exists a unique fixed point $(v, w) := \tilde{\Psi}(v_0, w_0)$ for each prescribed v_0 and w_0:

$$(v, w) = \tilde{\Psi}(v_0, w_0) := (v_0 + \Psi_1(v_0), w_0 + \Psi_2[v_0](w_0)),$$

where $\mathcal{Q}v = v_0, \mathcal{Q}w = w_0$ and $\Psi_i \colon \operatorname{Ker} \mathcal{T} \to \operatorname{Ker} \mathcal{Q}$ for $i = 1, 2$. Item (i) is thus established.

Item (ii) follows from the uniqueness of $\mathcal{M}_0$ and the fact that $\chi^\delta(v) = v$ if the H^j_{u} norm of v is sufficiently small. Since elements of $\mathbf{M}_{0,1}$ are precisely solutions of the modified equation $\mathcal{T}v + \mathcal{N}^\delta(v) = 0$, we insert $v = \mathcal{Q}v + \Psi_1(\mathcal{Q}v)$ into the equation and differentiate with respect to v. This gives that

$$w = \mathcal{Q}w + \mathrm{D}\Psi_1(\mathcal{Q}v)\mathcal{Q}w = w_0 + \mathrm{D}_{v_0}\Psi_1(v_0)w_0$$

is a solution of the modified linearized equation $\mathcal{T}w + \mathrm{D}_v\mathcal{N}^\delta(v)w = 0$. Since

$$\mathrm{D}_{v_0}\Psi_1(v_0)w_0 \in \operatorname{Ker} \mathcal{Q},$$

we get $\mathcal{Q}(w_0 + \mathrm{D}_{v_0}\Psi(v_0)w_0) = w_0$. By uniqueness of a solution w for each $\mathcal{Q}w = w_0$, we obtain $\Psi_2[v_0] = \mathrm{D}_{v_0}\Psi_1(v_0)$. The remaining claims in (iii) are straightforward in view of (ii) and Theorem B.5. It follows from Hypothesis B.4 that $\tau_\xi \circ \mathcal{N}^\delta = \mathcal{N}^\delta \circ \tau_\xi$, which after a differentiation gives

$$\tau_\xi \left(\mathrm{D}_v\mathcal{N}^\delta(v)w\right) = \mathrm{D}_v\mathcal{N}^\delta(\tau_\xi v)\tau_\xi w.$$

Then, reasoning as in [24] validates statements (iv)–(vi), except for the last claim $g(v_0, w_0) = \mathrm{D}_{v_0}f(v_0)w_0$. This is shown by plugging $\tau_x w = (\mathrm{Id} + \mathrm{D}_{v_0}\Psi_1(\tau_x v_0))\tau_x w_0$ into the reduced vector field g in (vi), and then identifying the result with

$$\mathrm{D}_{v_0}f = \mathrm{D}_{v_0}(\mathcal{Q} \circ \tau_x \circ (\mathrm{Id} + \Psi_1)).$$

□

C Computation of the coefficients Ψ_{ijk}

We wish to solve

$$\mathcal{T}\Psi_{200} = -\mathcal{T}\Psi_{101} = 1, \tag{27}$$
$$\mathcal{T}\Psi_{110} = -2\mathcal{T}\Psi_{011} = 2x, \tag{28}$$
$$\mathcal{T}\Psi_{020} = x^2, \tag{29}$$

subjected to the condition $\mathcal{Q}(\Psi_{ijk}) = 0$ for all $i + j + k \geq 1$. This condition is imposed for unique solvability of Ψ_{ijk}; see the proof of Theorem 3.1.

Using the fact that multiplication by x^n corresponds to n-times differentiation on the Fourier side, we have

$$\int_{\mathbb{R}} K(x)x^n \,\mathrm{d}x = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}m^{(n)}(0) & \text{if } n \text{ is even.} \end{cases}$$

Now, we can compute the convolution of K and monomials x^n, where $n \in \mathbb{N}$. For instance, $K * 1 = 1$ and

$$\int_{\mathbb{R}} K(y)(x-y)\,\mathrm{d}y = x\int_{\mathbb{R}} K(y)\,\mathrm{d}y - \int_{\mathbb{R}} yK(y)\,\mathrm{d}y = x\cdot(K*1) - 0 = x,$$

where $\int_{\mathbb{R}} yK(y)\,\mathrm{d}y = 0$ because the integrand is odd. Utilizing the binomial theorem to expand $(x-y)^n$ together with symmetries of the integrands, we arrive at

$$x^2 - \int_{\mathbb{R}} K(y)(x-y)^2\,\mathrm{d}y = m''(0), \tag{30}$$

$$x^3 - \int_{\mathbb{R}} K(y)(x-y)^3\,\mathrm{d}y = 3m''(0)x, \tag{31}$$

$$x^4 - \int_{\mathbb{R}} K(y)(x-y)^4\,\mathrm{d}y = 6m''(0)x^2 - m^{(4)}(0). \tag{32}$$

To solve (27), we are motivated by (30) and make the Ansatz $\Psi_{200} = \alpha x^2 - \mathcal{Q}(\alpha x^2)$, where subtraction by $\mathcal{Q}(\alpha x^2)$ is to make sure that $\mathcal{Q}(\Psi_{200}) = 0$. Since $\mathcal{Q}(\alpha x^2) = 0$ for all $\alpha \in \mathbb{R}$, it can be removed. Plugging the Ansatz into (27) yields

$$\mathcal{T}\Psi_{200} = \alpha m''(0) = 1.$$

Since $m''(0) = -1/3$, $\alpha = -3$ necessarily. In conclusion, $\Psi_{200} = -3x^2$. The linear equations (28) and (29) are solved in a similar way. We summarize the results below.

$$\begin{aligned} \Psi_{101} &= 3x^2, & \Psi_{200} &= -3x^2,\\ \Psi_{110} &= -2x^3, & \Psi_{020} &= -\frac{1}{2}x^4 + \frac{19}{10}x^2,\\ \Psi_{011} &= x^3. \end{aligned}$$

D A GLOBAL BIFURCATION THEOREM

Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces and $\mathcal{U} \subset \mathcal{X} \times \mathbb{R}$ an open set. Consider the abstract operator equation

$$F(\varphi, \nu) = 0,$$

where $F\colon \mathcal{U} \to \mathcal{Y}$ is an analytic mapping. The following is a version of Theorem 6.1 in [11] which has been slightly modified to better fit the situation in the present paper. The proof remains the same.

Theorem D.1. *Assume*

(A) for all $(\varphi^*, \nu^*) \in \mathcal{U} \cap F^{-1}(0)$, the Fréchet derivative $\mathrm{D}_\varphi F(\varphi^*, \nu^*)$ is Fredholm of index zero;

(B) *there exists a local curve of solutions* $\mathcal{C}_{\mathrm{loc}}$ *with a continuous parametrization* $(0, \nu') \ni \nu^* \mapsto (\varphi^*_{\nu^*}, \nu^*)$, *so that*

$$\mathcal{C}_{\mathrm{loc}} = \{(\varphi^*_{\nu^*}, \nu^*) \,|\, 0 < \nu^* < \nu'\} \subset \mathcal{U} \cap F^{-1}(0)$$

and

$$\lim_{\nu^* \to 0^+} (\varphi^*_{\nu^*}, \nu^*) \in \partial\mathcal{U},$$

as well as

$$\mathrm{D}_\varphi F(\varphi^*_{\nu^*}, \nu^*) \colon \mathcal{X} \to \mathcal{Y} \text{ is invertible for all } (\varphi^*_{\nu^*}, \nu^*) \in \mathcal{C}_{\mathrm{loc}}.$$

Then, $\mathcal{C}_{\mathrm{loc}}$ *is contained in a curve of solutions* $\mathcal{C}$, *which is parametrized as*

$$\mathcal{C} = \{(\varphi_s, \nu_s) \,|\, 0 < s < \infty\} \subset \mathcal{U} \cap F^{-1}(0)$$

for some continuous map $(0, \infty) \ni s \mapsto (\varphi_s, \nu_s)$. *The global curve* $\mathcal{C}$ *has the following properties.*

(a) *One of the following alternatives holds:*

  (i) *(blowup) as* $s \to \infty$,

$$M(s) := \|\varphi_s\|_{\mathcal{X}} + |\nu_s| + \frac{1}{\mathrm{dist}((\varphi_s, \nu_s), \partial\mathcal{U})} \to \infty;$$

  (ii) *(loss of compactness) there exists a sequence* $s_n \to \infty$ *as* $n \to \infty$ *such that* $\sup_n M(s_n) < \infty$ *but* $(\varphi_{s_n})_n$ *has no subsequence convergent in* $\mathcal{X}$.

(b) *Near each point* $(\varphi_{s_0}, \nu_{s_0}) \in \mathcal{C}$, *we can reparametrize* $\mathcal{C}$ *so that* $s \mapsto (\varphi_s, \nu_s)$ *is real analytic.*

(c) $(\varphi_s, \nu_s) \notin \mathcal{C}_{\mathrm{loc}}$ *for* s *sufficiently large.*

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Paper II

Solitary waves in a Whitham equation with small surface tension

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Abstract

Using a nonlocal version of the center manifold theorem and a normal form reduction, we prove the existence of small-amplitude generalized solitary-wave solutions and modulated solitary-wave solutions to the steady gravity–capillary Whitham equation with weak surface tension. Through the application of the center manifold theorem, the nonlocal equation for the solitary wave profiles is reduced to a four-dimensional system of ODEs inheriting reversibility. Along particular parameter curves, relating directly to the classical gravity–capillary water wave problem, the associated linear operator is seen to undergo either a reversible $0^{2+}(\mathrm{i}k_0)$ bifurcation or a reversible $(\mathrm{i}s)^2$ bifurcation. Through a normal form transformation, the reduced system of ODEs along each relevant parameter curve is seen to be well approximated by a truncated system retaining only second-order or third-order terms. These truncated systems relate directly to systems obtained in the study of the full gravity–capillary water wave equation and, as such, the existence of generalized and modulated solitary waves for the truncated systems is guaranteed by classical works, and they are readily seen to persist as solutions of the gravity–capillary Whitham equation due to reversibility. Consequently, this work illuminates further connections between the gravity–capillary Whitham equation and the full two-dimensional gravity–capillary water wave problem.

1 Introduction

In this paper, we consider the existence of small-amplitude solitary-wave solutions of the gravity–capillary Whitham equation

$$u_t + (\mathcal{M}_{g,d,T} u + u^2)_x = 0, \tag{1}$$

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where here $\mathcal{M}_{g,d,T}$ is a Fourier-multiplier operator, acting on the spatial variable x, defined via its symbol

$$m_{g,d,T}(\xi) = \left((g + T\xi^2) \frac{\tanh(d\xi)}{\xi} \right)^{1/2} .$$

Here, $u(x,t)$ corresponds to the height of the fluid surface at position $x \in \mathbb{R}$ and time t, g is the gravitational constant, d is the undisturbed depth of the fluid, and $T \geq 0$ is the coefficient of surface tension. This symbol is precisely the phase speed for uni-directional waves in the full gravity–capillary water wave problem in [27, 35]. In the absence of surface tension, that is when $T = 0$, equation (1) is referred to as *the gravity Whitham equation*, or simply *the Whitham equation*, and was introduced by Whitham in [35, 34] as a full-dispersion generalization of the standard KdV equation. In the case $T = 0$, the bifurcation and dynamics of both periodic and solitary solutions of (1) have been studied intensively over the last decade by many authors. It has been found that many high-frequency phenomena in water waves, such as breaking, peaking and the famous Benjamin–Feir instability, which do not manifest in the KdV or other shallow water, are indeed manifested the Whitham equation. See for example [6, 14, 12, 11, 15, 21, 20] and references therein.

Given the success of the Whitham equation it is thus natural to consider the existence and dynamics of solutions when additional physical effects are incorporated. In this work, we will concentrate on the existence of solitary-wave solutions[1] of (1) with non-zero surface tension $T > 0$.

It is straightforward to see that the properties of $m_{g,d,T}$ depend on the non-dimensional ratio

$$\tau = \frac{T}{gd^2}, \tag{2}$$

which is referred to as the Bond number. In the full gravity–capillary water wave problem, it is known that the existence of solutions depends sensitively on whether $\tau \in (0, 1/3)$ or $\tau > 1/3$, referred to as the weak- and strong-surface tension cases, respectively. Indeed, in the case of strong surface tension the full gravity–capillary water wave problem admits subcritical solitary waves of depression, i.e. asymptotically constant traveling wave solutions with a unique critical point corresponding to an absolute minimum. See, for instance, [1, 2]. Here, "subcritical" means the speed of the traveling wave is strictly less than the long-wave speed $m_{g,d,T}(0) = \sqrt{gd}$. If the traveling wave's speed is greater than $\sqrt{gd}$, it is said to be supercritical. In the small surface tension case, however, considerably less is known about the existence of truly localized (e.g. integrable) solitary waves. It is known, however, that for small surface tension there exist generalized solitary waves, sometimes referred to as *solitary waves with ripples*. These correspond to bounded solutions of (4) which are (roughly) a superposition of a solitary wave and a co-propagating periodic wave with

[1] Note that the bifurcation and dynamics of periodic waves has been previously studied in [13, 22, 26].

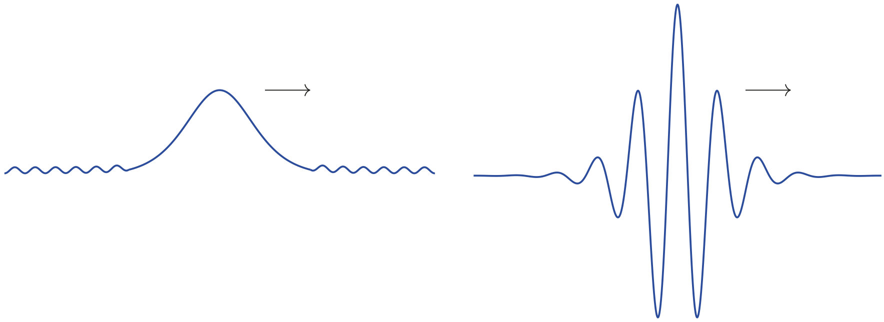

Figure 1: Illustration of a generalized solitary-wave solution (left) and a modulated solitary-wave (of elevation) solution (right). The arrow indicates the direction of travel, that is, $c > 0$.

significantly smaller amplitude[2]. See [31, 5, 32, 29, 30]. In particular, note that generalized solitary waves are not, in fact, *solitary waves* in the traditional sense since they are not asymptotically constant at $x = \pm\infty$. It is also known that there exist *modulated solitary waves*, which are bounded solutions of (4) with a solitary-wave envelope multiplying a complex exponential. See, for instance, [25, 23, 7]. Specifically, we note that [7] proves the existence of geometrically distinct multipulse modulated solitary waves with exponential decay. Illustrations of both generalized and modulated solitary waves can be seen in Figure 1.

Unfortunately, many of the existence proofs described above for the full gravity–capillary water wave problem rely fundamentally on classical dynamical systems techniques, requiring, in particular, that the equation governing the profile of the traveling wave be recast as a first-order system of ordinary differential equations. Such techniques seem at first glance to not be applicable to the gravity–capillary Whitham equation (1) due to the nonlocal operator $\mathcal{M}_{g,d,T}$. However, [16, 17] recently derived a generalization of the classical center-manifold theory that is applicable to a wide class of nonlocal problems, and this was further extended in [33] to an even wider class of nonlocal problems which, as we will show, includes (1). With this in mind, the primary goal of this paper is to use a nonlocal version of the center manifold theorem and a corresponding normal form reduction to establish the existence of small amplitude generalized solitary and modulated solitary-wave solutions to the gravity–capillary Whitham equation (1) in the small surface tension case. While such solutions were recently shown to exist in [26], this work relies on direct implicit function theorem techniques. Our goal is to attempt to establish similar

[2]We allow for the possibility of an asymptotic phase shift in the periodic wave between $x = -\infty$ and $x = +\infty$.

results using a center-manifold reduction technique.

To begin our search for solitary waves, we note that a straightforward nondimensionalization converts (1) to

$$u_t + \Big(\mathcal{M}_\tau u + u^2\Big)_x = 0 \tag{3}$$

where now $\mathcal{M}_\tau$ is a Fourier multiplier with symbol

$$m_\tau(\xi) = \left((1+\tau\xi^2)\,\frac{\tanh(\xi)}{\xi}\right)^{1/2},$$

and $\tau > 0$ is the Bond number defined in (2). Making the traveling wave Ansatz $u(x,t) = \varphi(x-ct)$ in (3) and integrating yields the (nonlocal) stationary profile equation[3]

$$\mathcal{M}_\tau\varphi - c\varphi + \varphi^2 = 0. \tag{4}$$

The profile equation (4) has received several treatments in recent years and theoretical frameworks for studying them are expanding. Existence results for (4) include periodic waves by Hur & Johnson [22] in 2015 and Ehrnström, Johnson, Maehlen & Remonato [13] in 2019, solitary (e.g. integrable) waves for both strong and weak surface tension by Arnesen [3] in 2016, solitary waves of depression for strong surface tension $\tau > 1/3$ and subcritical wave speed $c < 1$ by Johnson & Wright [26] in 2018, as well as generalized solitary waves for weak surface tension $\tau \in (0, 1/3)$ and supercritical wave speed $c > 1$ also by [26]. Each of these known results use either the implicit function theorem and a Lyapunov–Schmidt reduction or appropriate variational methods.

In this work, we utilize instead an approach based on the recent nonlocal center manifold reduction technique developed by Faye & Scheel [16, 17] and further refined by Truong, Wahlén & Wheeler [33]. As we will see, this set of techniques provides a unified approach for proving existence of both periodic and solitary waves for (4). The nonlocal center manifold theorem bears resemblance to its classical local counterpart, that there exists a neighborhood in a uniform locally Sobolev space where the nonlocal equation is equivalent to a local finite-dimensional system of ODEs. After this reduction, tools for ODEs can be applied to find an approximate solution and then to investigate its persistence. Provided the solution is sufficiently small in the uniform local Sobolev norm it qualifies as a true solution to the original nonlocal equation. So, previously mentioned small-amplitude waves are likely to be included in the center manifold. However, this framework does not fit nonlocal equations from hydrodynamics. To remedy this, Truong, Wahlén & Wheeler [33] have extended this result to a larger class of nonlocal equations. They also demonstrate the strength of this reduction technique and exemplify how to extract qualitative information on the solutions from the reduced ODE, which they use to construct an extreme solitary wave for the gravity Whitham equation. This reduction technique is also available for local quasilinear problems [9].

[3]Note that thanks to Galilean invariance, one can without loss of generality take the constant of integration to be zero. See Remark 5.2 for more details.

Remark 1.1. As noted above, the nonlocal center manifold theorem developed in [16, 17] does not directly apply to the profile equation (4). Indeed, one of the hypotheses of Faye & Scheel's result is that both of the functions[4] $\mathcal{F}^{-1}(m_\tau^{-1})$ and $\partial_x \mathcal{F}^{-1}(m_\tau^{-1})$ are integrable and exhibit exponential decay, which are highly non-trivial properties. While the exponential decay and integrability of $\mathcal{F}^{-1}(m_\tau^{-1})$ was recently established in [13], this reference also unfortunately shows that $\partial_x \mathcal{F}^{-1}(m_\tau^{-1})$ is not an integrable function. By carefully considering the methodologies used in [16, 17], Truong, Wahlén and Wheeler were recently able to circumvent this difficulty in [33], where they present a refinement of the result in [16, 17] which does not rely on the integrability of $\mathcal{F}^{-1}(m_\tau^{-1})'$. The authors explain the purpose of this hypothesis is to establish the Fredholmness of the linearized operator obtained by linearizing (4) about $\varphi = 0$. Fortunately, this can be checked directly by other means for equations of the form (4), which is one of the achievements of the refinement [33]. It is technically this refinement which we use in our analysis. For comparison, we note that such properties of $\mathcal{F}^{-1}(m_\tau^{-1})$ are not needed in the implicit function theorem approach used by Johnson & Wright [26].

We now provide an outline of the paper, as well as state the main results. We begin in Section 2 by inverting the operator $\mathcal{M}_\tau$ in (4), thereby recasting the profile equation into the form studied in [16, 17, 33]. We then study the equation

$$m_\tau(\xi) - c = \left((1+\tau\xi^2)\frac{\tanh(\xi)}{\xi}\right)^{1/2} - c = 0,$$

which gives solutions to the linearized equation of (4) about the trivial solution $\varphi = 0$. By writing $c^{-2} = \alpha$, $\tau c^{-2} = \beta$ and rearranging the terms, the above equation is recognized as the well-known linear dispersion relation for purely imaginary eigenvalues in two-dimensional capillary-gravity water wave equations, modeling the motion of a perfect unit-density fluid with irrotational flow under the influence of gravity and surface tension in finite depth: see, for example, the works of Kirchgässner [28], Buffoni, Groves & Toland [8], Amick & Kirchgässner [2] and Diass & Iooss [10]. These classical bifurcation curves in the (β, α)-plane naturally guide us in selecting two parameter curves where, restricting ourselves now to the case of small surface tension $\tau \in (0, 1/3)$, we expect generalized solitary-wave and modulated solitary-wave solutions could be found: see Figure 2 below. We further establish a key Fredholm property for the associated linearized operator in Section 3 which is required for the application of the center manifold theorem.

With this preliminary linear analysis completed, we then turn towards applying the nonlocal center manifold theorems from [16, 17, 33] to the profile equation (4). These results, as mentioned before, reduce the nonlocal profile equation considered here to a local ODE near the equilibrium and provide an algorithmic method of approximating the local ODE. We often refer to this local ODE as the reduced ODE. For completeness, we

[4]Here and throughout, $\mathcal{F}$ denotes the Fourier transform. For the specific normalization used here, see equation (5) below.

state the general center manifold theorem from [33] in Appendix B, and we apply it in Section 4 along both bifurcation curves in the (β, α)-plane of interest.

In Section 5 we approximate the reduced ODE near the bifurcation curve where generalized solitary-wave solutions are expected to be found as a result of an $0^{2+}(\mathrm{i}k_0)$ reversible bifurcation.

Up to a standard normal form reduction, rescaling and truncating the nonlinearity, this ends up being almost identical to the normal form equation obtained by Iooss & Kirchgässner in [24] in their analysis of the full gravity–capillary water wave equations. In particular, the truncated reduced ODE in this case admits an explicit family of small amplitude generalized solitary-wave solutions which are then shown to persist as solutions of (4) by a reversibility argument. Putting this all together establishes our first main result.

Theorem 1.2 (Existence of Generalized Solitary Waves). *For each sufficiently small $\mu \in \mathbb{R}$, there exists a family of generalized solitary waves to the gravity–capillary Whitham equation with wave speed $c = 1 + \mu$ and $\tau < 1/3$, given by*

$$\varphi(x) = \frac{3}{2}|\mu|\rho^{1/2}\operatorname{sech}^2\left(\frac{\rho^{1/4}\sigma^{1/2}|\mu|^{1/2}x}{\sqrt{2}}\right) + \frac{\mu}{2}(1 - \operatorname{sgn}(\mu)\rho^{1/2})$$
$$+ |\mu|k^{1/2}\cos\Big((k_0 + \mathcal{O}(\mu))x + \Theta_* + \mathcal{O}(\mu)\Big) + \mathcal{O}(\mu^2\rho^{1/2}),$$

where $\Theta_* \in \mathbb{R}/2\pi\mathbb{Z}$ is arbitrary, $\sigma = (1/3 - \tau)^{-1}$, $\rho = 1 + 24k$, $k_0 > 0$ is such that $m_\tau(k_0) = 1$, and $k = \mathcal{O}(|\mu|^{-1-2\kappa})$ for any $\kappa \in [0, 1/2)$.

It is interesting to note that the above allows for an asymptotic phase shift in the cosinus term between $x = -\infty$ and $x = \infty$ of order $\mathcal{O}(\rho^{1/4}|\mu|^{1/2})$. Further, after a Galilean change of variables, all the generalized solitary-wave solutions found above may be seen to have supercritical wave speed $c > 1$: see Remark 5.2 below for details. Note, however, this result does not establish that some waves have asymptotic oscillations which are exponentially small in relation to the solitary term as in Johnson & Wright [26]. On the other hand, we are able to allow for a more general asymptotic phase shift between $x = \pm\infty$.

In Section 6 we analogously treat the bifurcation curve in the (β, α)-plane where modulated solitary waves are expected to be found as a result of a Hamiltonian-Hopf bifurcation, also known as an $(\mathrm{i}s)^2$ bifurcation. By computing the necessary center manifold coefficients and performing the appropriate normal form reduction, we again find the results from [25] applicable, thus establishing our second main result.

Theorem 1.3 (Existence of Modulated Solitary Waves). *Fix $s > 0$ and set*

$$c_0^2 = \left(\frac{s^2}{2\sinh^2(s)} + \frac{s}{2\tanh(s)}\right)^{-1}, \quad \tau_0 = c_0^2\left(-\frac{1}{2\sinh^2(s)} + \frac{1}{2s\tanh(s)}\right),$$

so that $c_0^{-2}(1+\tau_0)\sinh(s) = s\cosh(s)$*. Then, for* $\mu < 0$ *sufficiently small, there exist two distinct modulated solitary-wave solutions to the gravity–capillary Whitham equation with amplitude of order* $\mathcal{O}(|\mu|^{1/2})$*, surface tension* τ_0 *and subcritical wave speed* $c_0+\mu<1$*. More precisely, the modulated solitary-wave solutions are described asymptotically via*

$$\varphi(x) = \sqrt{\frac{-8q_0\mu}{q_1}}\,\operatorname{sech}(\sqrt{q_0\mu}\,x)\cos\left(sx+\Theta_*+\mathcal{O}(|\mu|^{1/2})\right)+\mathcal{O}(\mu^2),\quad \Theta_*\in\{0,\pi\},$$

which have an asymptotic phase shift between $x=-\infty$ *and* $x=\infty$ *of order* $\mathcal{O}(|\mu|^{1/2})$*. Here, the coefficients* q_0 *and* q_1 *are*

$$q_0 = -\frac{2}{m_\tau''(s)},\quad q_1 = -\frac{4(-c_0+m_\tau(2s))^{-1}+8(1-c_0)^{-1}}{m_\tau''(s)},$$

and are both negative.

The solutions constructed in Theorem 1.3 correspond to (distinct) modulated solitary waves of elevation ($\Theta_* = 0$) and depression ($\Theta_* = \pi$). Note that Figure 1 depicts a modulated solitary-wave solution of elevation.

Remark 1.4. As mentioned above, the center manifold methodology used here provides a unified approach for proving existence of both periodic and solitary waves for (4). Consequently, one could continue the above line of investigation to establish the existence of other classes of solutions as well including, for instance, the subcritical, small amplitude solitary waves of depression in the case of large surface tension $\tau > 1/3$ constructed in [26]. This specific case is very similar to the gravity Whitham equation studied by Truong, Wahlén and Wheeler [33] and is thus excluded here.

This paper provides further connections between the model equation (1) and the two-dimensional gravity–capillary water wave problem. It also exemplifies the application of nonlocal center manifold reduction in existence theory. A natural continuation of this paper could be to investigate the existence of multipulse modulated solitary-wave solutions as in [7], as well as bifurcation phenomena in other parameter regions.

Notation The following notation will be used throughout this work.

- For $\sigma\in\mathbb{R}$, we define the *σ-weighted L^p spaces*

$$L^p_\sigma := \left\{ f\colon \mathbb{R}\to\mathbb{R} \,\middle|\, \int_{\mathbb{R}} |f|^p\omega_\sigma^p\,\mathrm{d}x \right\}.$$

  Here the *weight function* $\omega_\sigma\colon\mathbb{R}\to\mathbb{R}$ is positive and smooth. Also, ω_σ is constantly 1 on $[-1,1]$ and equals $\exp(\sigma|x|)$ for $|x|\geq 2$.

– Similarly, we define the *weighted Sobolev spaces*

$$W^{m,p}_\sigma := \left\{ f\colon \mathbb{R} \to \mathbb{R} \,\middle|\, f^{(n)} \in L^p_\sigma, \text{ for } 0 \le n \le m \right\}.$$

We have the natural inclusions $W^{m,p}_{\sigma_2} \subset W^{m,p}_{\sigma_1}$ whenever $\sigma_1 < \sigma_2$. For $p = 2$, we denote the Hilbert space $W^{m,2}_\sigma$ by H^m_σ.

– The non-weighted Sobolev spaces are denoted by $W^{m,p}$ and the special case $W^{m,2}$ is denoted by H^m.

– The uniformly local H^m space is

$$H^m_{\mathrm{u}} := \left\{ f\colon \mathbb{R} \to \mathbb{R} \,\middle|\, \|f\|_{H^m_{\mathrm{u}}} < \infty \right\} \quad \text{with} \quad \|f\|_{H^m_{\mathrm{u}}} := \sup_{y \in \mathbb{R}} \|f(\,\cdot\, + y)\|_{H^m([0,1])}.$$

– We use the following scaling of the Fourier transform:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}} f(x) \exp(-\mathrm{i}x\xi)\,\mathrm{d}x \quad \text{and} \quad \mathcal{F}^{-1}g(x) = \frac{1}{2\pi}\mathcal{F}g(-x). \tag{5}$$

2 The operator equation

In this section, we begin our study of the nonlocal profile equation (4). Observe that since m_τ is strictly positive on $\mathbb{R}$, the operator $\mathcal{M}_\tau$ is invertible on any Fourier based space. We denote the inverse of $\mathcal{M}_\tau$ by $\mathcal{L}_\tau$, defined via

$$\widehat{\mathcal{L}_\tau f}\,(\xi) = \ell_\tau(\xi)\hat{f}(\xi), \quad \ell_\tau(\xi) := m_\tau(\xi)^{-1}.$$

In particular, the profile equation (4) can be written in the "smoothing" form

$$\varphi - cK_\tau * \varphi + K_\tau * \varphi^2 = 0, \tag{6}$$

where here $K_\tau := \mathcal{F}^{-1}\ell_\tau$ denotes the convolution kernel corresponding to the operator $\mathcal{L}_\tau$. Observe that (6) is similar to the profile equation for the gravity Whitham equation (i.e. (1) with $T = 0$), but now with a *nonlocal* nonlinearity.

As we seek small amplitude solutions of (6), we begin by linearizing (6) about $\varphi = 0$ which, after applying the Fourier transform, yields the equation

$$(1 - c\ell_\tau(\xi))\widehat{v}(\xi) = 0,$$

which we seek to solve for non-trivial $v \in L^2(\mathbb{R})$. This motivates considering the equation

$$1 - c\ell_\tau(\xi) = 0, \quad \text{i.e.} \quad \xi\cosh(\xi) = \left(\frac{1}{c^2} + \frac{\tau}{c^2}\xi^2\right)\sinh(\xi). \tag{7}$$

By setting

$$\alpha = \frac{1}{c^2} \quad \text{and} \quad \beta = \frac{\tau}{c^2},$$

equation (7) is recognized as the well-known linear dispersion relation for purely imaginary eigenvalues in the two-dimensional water wave equations in finite depth: see, for example, [27, 35]. The importance of (7) in finding solutions to the gravity–capillary water wave equations was recognized by Kirchgässner [28], followed by a multitude of other papers (see e.g. [2, 10, 24, 8]). Looking at the bifurcation curves in the (β, α)-planes for the classical gravity–capillary water wave problem, it is natural to expect the following:

- that modulated solitary-wave solutions may be found as a result of a Hamiltonian–Hopf bifurcation, when crossing the curve

$$C_2 = \left\{ (\beta, \alpha) \,\middle|\, \beta = -\frac{1}{2\sinh^2(s)} + \frac{1}{2s\tanh(s)}, \right.$$
$$\left. \alpha = \frac{s^2}{2\sinh^2(s)} + \frac{s}{2\tanh(s)}, \; s \in [0, \infty) \right\}$$

  from below;

- that generalized solitary-wave solutions may be found as a result of an $0^{2+}(\mathrm{i}k_0)$ bifurcation, when crossing the curve

$$C_3 = \left\{ (\beta, \alpha) \,\middle|\, \beta \leq \frac{1}{3} \text{ and } \alpha = 1 \right\}$$

  either from above or below. Here, $k_0 \in \mathbb{R}$ satisfies equation (7) for a fixed β along C_3;

- and that solitary-wave solutions of depression may be found as a result of an 0^{2+} bifurcation, when crossing the curve

$$C_4 = \left\{ (\beta, \alpha) \,\middle|\, \beta \geq \frac{1}{3} \text{ and } \alpha = 1 \right\}$$

  from above.

There is an additional curve C_1 in the (β, α)-plane along which one may expect the existence of multipulse solitary waves [8]. The argument in [8] uses the Hamiltonian structure of the full water wave problem. While equation (1) exhibits a variational formulation in the form investigated by Bakker & Scheel [4], its smoothing form (10) below does not. As m_τ does not have an L^1 Fourier transform, using results in [4] would therefore call for a careful examination and adaptation. Thus, it is more appropriate to consider this bifurcation phenomenon in a separate paper and we will not comment further regarding C_1. See Figure 2 for depictions of the curves C_1, C_2, C_3 and C_4 in the (β, α)-plane.

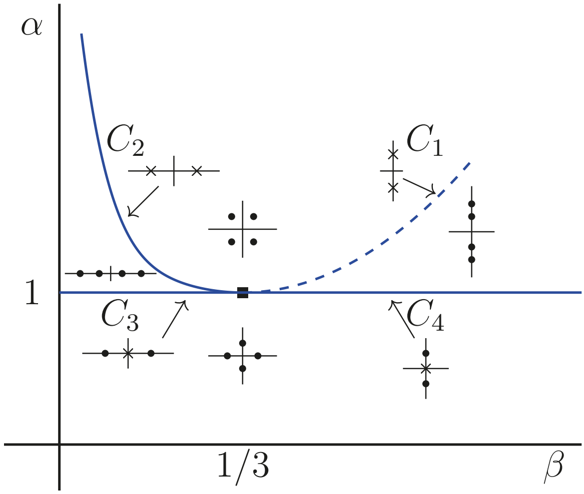

Figure 2: Sketch of the bifurcation curves C_1, C_2, C_3 and C_4 along with zeros of the function $1 - c\ell_\tau(\xi)$, which is the same as those of $(\alpha + \beta\xi^2)\sinh(\xi) - \xi\cosh(\xi)$. Here, dots and crosses represent algebraic multiplicity one and two, respectively.

It is illustrative to understand what these curves mean in terms of the physical (τ, c) parameters. For example, crossing $(\beta_0, \alpha_0) = (\beta_0, 1) \in C_3$ corresponds to studying (6) for $\beta = \beta_0$ and $\alpha = \alpha_0 - \tilde{\mu} = 1 - \tilde{\mu}$ for some $|\tilde{\mu}| \ll 1$ which, in terms of τ and c, gives

$$\tau = \beta_0 c_0^2 < \frac{1}{3} \quad \text{and} \quad c = c_0 + \mu \ \text{ with } \ c_0 = 1 \ \text{ and } \ |\mu| \ll 1. \tag{8}$$

Thus, crossing the curve C_3 corresponds to waves with weak surface tension, while the wave speed is nearly critical. Likewise, crossing a point $(\beta_0, \alpha_0) \in C_2$ from below corresponds to studying (6) with $\beta = \beta_0$ and $\alpha = \alpha_0 + \tilde{\mu}$ with $0 < \tilde{\mu} \ll 1$ which, in terms of τ and c gives

$$\tau = \beta_0 c_0^2 \ \text{ and } \ c = c_0 + \mu, \ \text{ with } \ c_0 = \alpha_0^{-1/2} \ \text{ and } \ 0 < -\mu \ll 1. \tag{9}$$

Since $\alpha_0 > 1$, it follows that $\tau \in (0, 1/3)$, i.e. the surface tension is again weak, and the speed c is subcritical. By similar reasoning, crossing the curve C_4 from above corresponds to strong surface tension, i.e. $\tau > 1/3$, and subcritical speeds. Note that in this work, we focus only on bifurcation phenomena connected to the curves C_2 and C_3. The 0^{2+} bifurcation along C_4, as one might expect, resembles the one already covered in [33] for the gravity Whitham equation and is thus excluded here.

As mentioned previously, we approach the above bifurcation phenomena for the nonlocal profile equation (6) by following the center manifold reduction strategy in [16, 17, 33].

To this end, we rewrite (6) as

$$\mathcal{T}\varphi + \mathcal{N}(\varphi, \mu) = 0, \tag{10}$$

which is now of the structural form studied in Faye & Scheel [16, 17], where here

$$\mathcal{T}\colon \varphi \mapsto \varphi - c_0 K_\tau * \varphi \quad \text{and} \quad \mathcal{N}\colon (\varphi, \mu) \mapsto K_\tau * (\varphi^2 - \mu\varphi).$$

Since τ is fixed, the subscript τ will be dropped for notational convenience. Our goal is to study the operator equation (10) for parameters τ, c_0 satisfying (8) corresponding to crossing C_3, as well as for τ, c_0 satisfying (9) corresponding to crossing C_2. The first step of this analysis is to understand the linear operator $\mathcal{T}$, which we now turn to studying.

3 The linear operator $\mathcal{T}$

Fix $\tau \in (0, 1/3)$. As preparation for our forthcoming bifurcation analysis, and following the general strategy in [16, 17, 33], in this section we study the linear operator[5]

$$\mathcal{T}\colon \varphi \mapsto \varphi - c_0 K * \varphi, \qquad H^5_{-\eta} \to H^5_{-\eta},$$

for $\eta > 0$ along the two parameter curves (8) and (9). Note that $\mathcal{T}$ is precisely the linearization of (10) about the trivial solution $\varphi = 0$ and, as such, it is crucial to understand the Fredholm and invertibility properties of $\mathcal{T}$ along the curves C_2 and C_3. Key to this analysis is an understanding of convolution kernel K. The relevant properties are detailed in the following result.

Proposition 3.1. *The convolution kernel K is even. Moreover, we have*

(i) the singularity of K as $|x| \to 0$ is

$$\lim_{x\to 0} \sqrt{|x|}K(x) = \frac{1}{\sqrt{2\pi\tau}};$$

(ii) K has exponential decay as $|x| \to \infty$, that is

$$|K(x)| \lesssim \exp(-\eta|x|) \quad \textit{for} \quad |x| > 1,$$

where $0 < \eta < \eta^* := \min\{1/\sqrt{\tau}, \pi/2\}$.

For a proof, see Ehrnström, Johnson, Maehlen & Remonato [13, Theorem 2.7]. An immediate consequence is that $K \in L^1_\eta$ for $\eta \in (0, \eta^*)$ and hence, by a straightforward application of Young's inequality, that for such η the linear operator

$$\mathcal{T}\colon H^5_{-\eta} \to H^5_{-\eta}$$

[5] Recall that since τ is fixed, for convenience the corresponding subscript will be dropped from K_τ and ℓ_τ.

is bounded regardless of the choice of c_0. A proof of this claim is found in [33] but is repeated here for the readers' convenience. We estimate

$$\begin{aligned}
\|K * \varphi\|_{L^2_{-\eta}}^2 &\lesssim \int_{\mathbb{R}} \left(\int_{\mathbb{R}} K(y)\varphi(x-y)\,\mathrm{d}y \right)^2 \exp(-2\eta|x|)\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} K(y)\varphi(x-y)\,\mathrm{d}y \right)^2 \exp(-2\eta|x-y| + 2\eta|y|)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} K(y)\exp(\eta|y|) \cdot \varphi(x-y)\exp(-\eta|x-y|)\,\mathrm{d}y \right)^2 \mathrm{d}x \\
&= \left\| (K \cdot \exp(\eta|\,\cdot\,|)) * (\varphi \cdot \exp(-\eta|\cdot|)) \right\|_{L^2}^2 \\
&\leq \|K\|_{L^1_\eta}^2 \cdot \|\varphi\|_{L^2_{-\eta}}^2.
\end{aligned}$$

This establishes that $\mathcal{T}$ is bounded on $L^2_{-\eta}$. Using that

$$\frac{\mathrm{d}}{\mathrm{d}x^n}(K * \varphi) = K * \left(\frac{\mathrm{d}}{\mathrm{d}x^n}\varphi \right), \quad \text{for all } n \geq 0,$$

the boundedness of $\mathcal{T}$ on $H^5_{-\eta}$ readily follows. Note that the works [16, 17] additionally require $K' \in L^1_\eta$ which, by Proposition 3.1, does not hold in this case. Next, we follow Truong, Wahlén & Wheeler [33] and study the Fredholm properties of $\mathcal{T}$ using theory for pseudodifferential operators in non-weighted Sobolev spaces H^5 from Grushin [18] (see also Appendix A).

To this end, we fix $\eta \in (0, \eta^*)$ and consider the conjugated operator

$$\tilde{\mathcal{T}} := M^{-1} \circ \mathcal{T} \circ M, \qquad H^5 \to H^5,$$

where $M : H^5 \to H^5_{-\eta}$ is multiplication with the strictly positive function $\cosh(\eta\,\cdot\,)$. Noting that conjugation by M preserves Fredholmness and the Fredholm index, we may establish the desired Fredholm properties of $\mathcal{T} = M \circ \tilde{\mathcal{T}} \circ M^{-1}$ acting on the weighted space $H^5_{-\eta}$ by studying the operator $\tilde{\mathcal{T}}$ acting on the non-weighted H^5. These latter properties are established by following the work [18], where the author relates the pseudodifferential operator $\tilde{\mathcal{T}}$ acting on H^5 to a positively homogeneous function A and determining the winding number of A around the origin. The relevant details are summarized in Appendix A.

By direct calculation, the symbol of $\tilde{\mathcal{T}}$ is seen to be

$$\tilde{t}(x, \xi) = 1 - c_0\phi_+(x)\ell(\xi - \mathrm{i}\eta) - c_0\phi_-(x)\ell(\xi + \mathrm{i}\eta),$$

where $\phi_\pm(x) = \exp(\pm\eta x)/(2\cosh(\eta x))$. In particular, note that

$$\begin{cases} \lim\limits_{x\to\infty} \phi_+(x) = 1 \;\text{ and }\; \lim\limits_{x\to-\infty} \phi_+(x) = 0 \\ \lim\limits_{x\to\infty} \phi_-(x) = 0 \;\text{ and }\; \lim\limits_{x\to-\infty} \phi_-(x) = 1. \end{cases} \tag{11}$$

Following [18], we define the positive, homogeneous degree-zero function

$$A(x_0, x, \xi_0, \xi) := \tilde{t}\left(\frac{x}{x_0}, \frac{\xi}{\xi_0}\right)$$

for $x, \xi \in \mathbb{R}$ and $x_0, \xi_0 > 0$ and study A acting on[6]

$$\overline{\mathbb{S}^1_+} \times \overline{\mathbb{S}^1_+} := \left\{(x_0, x, \xi_0, \xi) \in \mathbb{R}^4 \,\middle|\, x_0^2 + x^2 = \xi_0^2 + \xi^2 = 1, \;\; x_0 \geq 0, \;\; \xi_0 \geq 0\right\}.$$

According to Proposition A.1, the linear opertaor $\tilde{\mathcal{T}}$ is Fredholm provided that the function A is smooth in $\overline{\mathbb{S}^1_+} \times \overline{\mathbb{S}^1_+}$ and nowhere vanishing along the boundary Γ of $\overline{\mathbb{S}^1_+} \times \overline{\mathbb{S}^1_+}$, which can be decomposed into the arcs

$$\begin{aligned}
\Gamma_1 &= \{\quad (0, 1, \xi_0, \xi) \mid \xi_0^2 + \xi^2 = 1, \, \xi_0 \geq 0\} \\
\Gamma_2 &= \{\, (0, -1, \xi_0, \xi) \mid \xi_0^2 + \xi^2 = 1, \, \xi_0 \geq 0\} \\
\Gamma_3 &= \{\quad (x_0, x, 0, 1) \mid x_0^2 + x^2 = 1, x_0 \geq 0\} \\
\Gamma_4 &= \{(x_0, x, 0, -1) \mid x_0^2 + x^2 = 1, x_0 \geq 0\}.
\end{aligned}$$

Further, the Fredholm index of $\tilde{\mathcal{T}}$ is precisely the winding number of A as Γ is traversed in the counter-clockwise direction, that is

$$\begin{tikzcd}
(0,-1,0,1) \arrow[d, "\Gamma_2"] & (0,1,0,1) \arrow[l, "\Gamma_3"'] \\
(0,-1,0,-1) \arrow[r, "\Gamma_4"] & (0,1,0,-1). \arrow[u, "\Gamma_1"']
\end{tikzcd}$$

As such, it is important to locate the roots of the function $1 - c_0\ell$ when the parameters τ, c_0 correspond to crossing the bifurcation curves C_1 and C_2. This motivates the following Lemma.

Lemma 3.2. *The multiplier $\ell\colon \mathbb{C} \to \mathbb{C}$ is analytic in the complex strip $|\mathrm{Im}\, z| < \eta^*$, with η^* as in Proposition 3.1. Moreover, there exists a possibly smaller strip $|\mathrm{Im}\, z| < \tilde{\eta}$ in which the function $1 - c_0\ell\colon \mathbb{C} \to \mathbb{C}$ has precisely the zeros*

(i) $k_0, -k_0, 0,$ *and* 0 *counting multiplicities for some* $k_0 > 0$*, when* τ *and* c_0 *are as in* (8),

(ii) $s, s, -s$ *and* $-s$ *counting multiplicities for some* $s > 0$*, when* τ *and* c_0 *are as in* (9).

See Figure 3.

Proof. See Corollary 2.2 in [13] for the analyticity of ℓ. Item (i) can be found as Lemma 2 in [2] and item (ii) can be found in Section IV in [28]. □

[6]Note A may be extended by continuity down to $x_0 = 0$ and $\xi_0 = 0$.

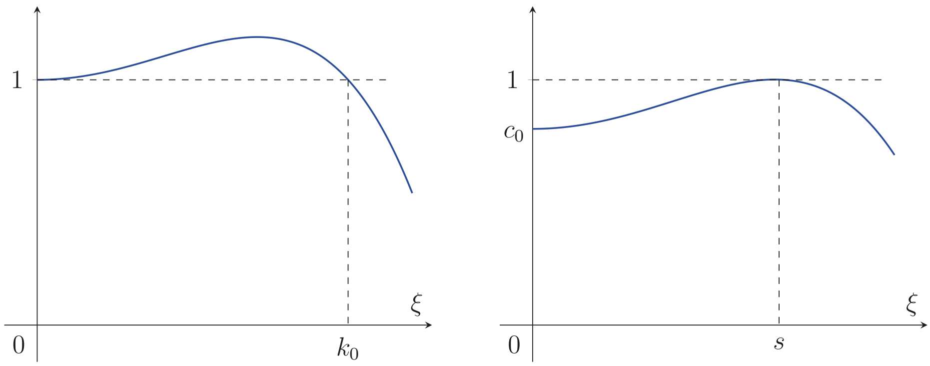

Figure 3: The multiplier $c_0\ell$ for (8) (left) and for (9) (right).

With this preliminary result, we are now ready to prove the main result of this section.

Theorem 3.3. *For each $\eta \in (0, \min\{\eta^*, \tilde{\eta}\})$ and choice of parameters τ, c_0 satisfying either* (8) *or* (9)*, the linear operator $\mathcal{T} : H^5_{-\eta} \to H^5_{-\eta}$ is Fredholm with Fredholm index four. For each set of parameters, its nullspace* $\operatorname{Ker}\mathcal{T}$ *is four-dimensional, given by*

$$\operatorname{Ker}\mathcal{T} = \operatorname{span}\{1, x, \cos(k_0 x), \sin(k_0 x)\} \tag{12}$$

if τ, c_0 satisfy (8)*, and*

$$\operatorname{Ker}\mathcal{T} = \operatorname{span}\{\cos(sx), x\cos(sx), \sin(sx), x\sin(sx)\} \tag{13}$$

if τ, c_0 satisfy (9).

Proof. Let $\eta \in (0, \min\{\eta^*, \tilde{\eta}\})$ be fixed. Following the outline above, we first verify the Fredholmness of $\tilde{\mathcal{T}}$ by showing that A is non-vanishing on Γ. To this end, recall that

$$\ell(z) = \left(\frac{1}{1+\tau z^2}\,\frac{z}{\tanh(z)}\right)^{1/2}, \qquad z \in \mathbb{C}.$$

Along Γ_1 we have $\xi_0 = \sqrt{1-\xi^2}$ and hence for $(0,1,\xi_0,\xi) \in \Gamma_1$ with $\xi_0 \neq 0$, i.e. $\xi \neq \pm 1$, we have, recalling (11),

$$A(0,1,\xi_0,\xi) = \lim_{x_0 \to 0^+} \tilde{t}\left(\frac{1}{x_0}, \frac{\xi}{\sqrt{1-\xi^2}}\right) = 1 - c_0\ell\left(\frac{\xi}{\sqrt{1-\xi^2}} - \mathrm{i}\eta\right).$$

To evaluate at the end points $(0,1,0,1)$ and $(0,1,0,-1)$, it is equivalent to compute the limit of $\ell(\xi' - \mathrm{i}\eta)$ as $\xi' \to \infty$ and $\xi' \to -\infty$, respectively. A calculation gives

$$|\ell(\xi' \pm \mathrm{i}\eta)|^4 = \frac{4(\xi')^2 + 4\eta^2}{(1+\tau(\xi')^2 \mp \tau\eta^2)^2 + (2\tau\xi'\eta)^2} \cdot \frac{(\cosh^2(\xi') - 1 + \cos^2(\eta))^2}{\cosh^2(2\xi') - \cos^2(2\eta)},$$

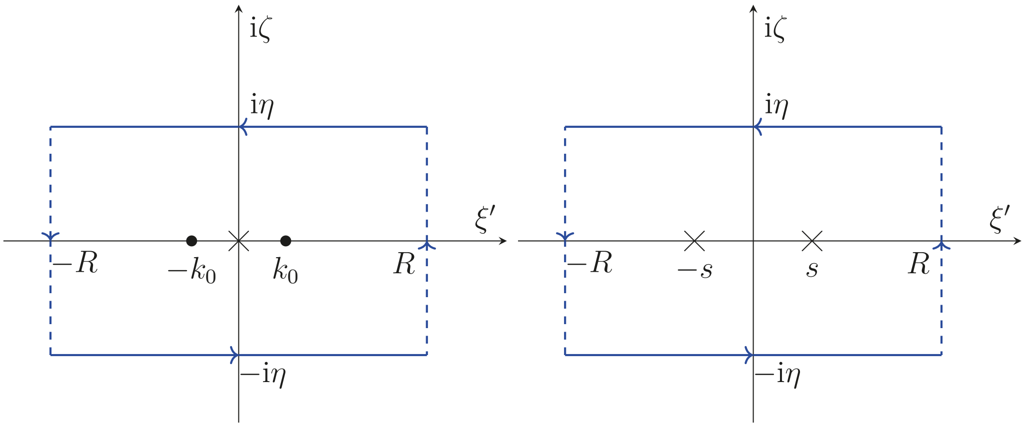

Figure 4: The rectangular contour Γ_R in Theorem 3.3. It consists of the arcs $\{\xi' \pm \mathrm{i}\eta \;:\; |\xi'| \leq R\}$ and $\{\pm R + \mathrm{i}\zeta \;:\; |\zeta| \leq \eta\}$. The left picture illustrates the case (8) and the right illustrates (9). Here, dots and crosses represent zeroes of $1 - c_0\ell(\xi)$ with multiplicity one and two, respectively.

which implies that $|\ell(\xi' \pm \mathrm{i}\eta)| \to 0$ as $|\xi'| \to \infty$. Consequently, $A(0, 1, 0, \pm 1) = 1$. By similar calculations, we find

$$A(x_0, x, \xi_0, \xi) = \begin{cases} 1 - c_0\ell\left(\frac{\xi}{\sqrt{1-\xi^2}} - \mathrm{i}\eta\right), & \text{on } \Gamma_1 \setminus \{(0,1,0,\pm 1)\} \\ 1 - c_0\ell\left(\frac{\xi}{\sqrt{1-\xi^2}} + \mathrm{i}\eta\right), & \text{on } \Gamma_2 \setminus \{(0,-1,0,\pm 1)\} \\ 1, & \text{on } \Gamma_3 \cup \Gamma_4. \end{cases}$$

In view of Lemma 3.2, A is smooth and nowhere vanishing on $\Gamma = \cup_{j=1}^4 \Gamma_j$ and hence Proposition A.1 implies that $\tilde{\mathcal{T}}$ is a Fredholm operator, as desired.

Next, we compute the Fredholm index of the operator $\tilde{\mathcal{T}}$ by computing the winding number of A along Γ transversed in the counter-clockwise direction (as described above). Setting $\xi' = \xi(1-\xi^2)^{-1/2}$ for $|\xi| < 1$ we see that traversing from $(0, 1, 0, -1)$ to $(0, 1, 0, 1)$ along Γ_1 corresponds to considering $1 - c_0\ell(\xi' - \mathrm{i}\eta)$ as ξ' varies from $\xi' = -\infty$ to $\xi' = \infty$, while traversing from $(0, -1, 0, 1)$ to $(0, -1, 0, -1)$ along Γ_2 corresponds to considering $1 - c_0\ell(\xi' + \mathrm{i}\eta)$ as ξ' varies from $\xi' = \infty$ to $\xi' = -\infty$. Further, since A is constant along Γ_3 and Γ_4, traversing along these arcs does not contribute to the winding number of A along Γ. To compute the winding number, choose τ, c_0 to satisfy either (8) or (9). Let $R > 0$ be strictly larger than the corresponding values $k_0 > 0$ or $s > 0$ from Lemma 3.2, and consider the rectangular contour Γ_R with vertices at $(\pm R, \pm \mathrm{i}\eta)$: see Figure 4. The winding number of A along Γ is in fact the limit of the winding number of

$1 - c_0\ell$ along Γ_R as $R \to \infty$. The latter is computed via

$$\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_R}\frac{c_0\ell'(z)}{1-c_0\ell(z)}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\Big(-\int_{-R}^{R}\frac{c_0\ell'(\xi'+\mathrm{i}\eta)}{1-c_0\ell(\xi'+\mathrm{i}\eta)}\,\mathrm{d}\xi' + \int_{-R}^{R}\frac{c_0\ell'(\xi'-\mathrm{i}\eta)}{1-c_0\ell(\xi'-\mathrm{i}\eta)}\,\mathrm{d}\xi'$$
$$+\int_{-\eta}^{\eta}\frac{c_0\ell'(R+\mathrm{i}\zeta)}{1-c_0\ell(R+\mathrm{i}\zeta)}\,\mathrm{d}\zeta - \int_{-\eta}^{\eta}\frac{c_0\ell'(-R+\mathrm{i}\zeta)}{1-c_0\ell(-R+\mathrm{i}\zeta)}\,\mathrm{d}\zeta\Big).$$

Since $1 - c_0\ell$ is analytic, it follows from Lemma 3.2 and the residue theorem that the above integrals sum to exactly four. Noting that the last two integrals vanish as $R \to \infty$ since $|\ell(\pm R + \mathrm{i}\zeta)| \to 0$ and $|\ell'(\pm R + \mathrm{i}\zeta)| \to 0$ as $R \to \infty$ and $|\zeta| \leq \eta$, the winding number of A around Γ is four. Proposition A.1 now gives that the Fredholm index of $\tilde{\mathcal{T}}$, and hence that of $\mathcal{T}$, is indeed four.

Finally, it remains to characterize the kernel of $\mathcal{T}$ acting on $H^5_{-\eta}$ when τ and c_0 satisfy either (8) or (9). Observe here that the equation $\mathcal{T}f = 0$ with $f \in L^2_{-\eta}$ can not be studied directly by the Fourier transform since the Fourier transform of such f is not a tempered distribution. To this end, we argue along the same lines as [33, Proposition 2.9] and consider instead the range of $\mathcal{T}\colon L^2_\eta \to L^2_\eta$, which is the adjoint of $\mathcal{T}\colon L^2_{-\eta} \to L^2_{-\eta}$ under the L^2-pairing. The Fourier transform of the range equation $\mathcal{T}f = g$ in L^2_η is precisely

$$(1 - c_0\ell(\xi))\mathcal{F}f(\xi) = \mathcal{F}g(\xi).$$

In view of Lemma 3.2, it follows that the range of $\mathcal{T}$ on L^2_η consists of functions g whose Fourier transforms vanish on the zero set of $1 - c_0\ell(\xi)$. If τ, c_0 satisfy (8), it follows from Lemma 3.2 that the range of $\mathcal{T}$ acting on L^2_η consists of functions g that satisfy

$$\mathcal{F}g(0) = (\mathcal{F}g)'(0) = \mathcal{F}g(\pm k_0) = 0$$

or, equivalently,

$$\int_{\mathbb{R}} 1\cdot g(x)\,\mathrm{d}x = \int_{\mathbb{R}} x\cdot g(x)\,\mathrm{d}x = \int_{\mathbb{R}} \exp(\mp\mathrm{i}k_0x)g(x)\,\mathrm{d}x = 0.$$

By duality, such τ, c_0 the kernel of $\mathcal{T}$ acting on $L^2_{-\eta}$ is given by (12), which clearly also belongs to $H^5_{-\eta}$.

Similarly, if τ, c_0 satisfy (9) then Lemma 3.2 implies that the kernel of $\mathcal{T}$ acting on L^2_η consists of functions g satisfying

$$\mathcal{F}g(\pm s) = (\mathcal{F}g)'(\pm s) = 0$$

or, equivalently,

$$\int_{\mathbb{R}} \exp(\pm\mathrm{i}sx)g(x)\,\mathrm{d}x = \int_{\mathbb{R}} x\cdot\exp(\pm\mathrm{i}sx)g(x)\,\mathrm{d}x = 0.$$

Again by duality, this shows that the kernel of $\mathcal{T}$ on $L^2_{-\eta}$ is given by (13), which again belongs to $H^5_{-\eta}$. $\square$

We use a nonlocal center manifold theorem, originally introduced in [16] and later adapted in [33] to account for the non-integrability of K'. For completeness, the general result used here is recorded in Appendix B. In this section, we apply this general result to the nonlocal profile equation (10) together with the modified equation

$$\mathcal{T}\varphi + \mathcal{N}^{\delta}(\varphi,\mu) = 0, \tag{14}$$

where

$$\mathcal{N}^{\delta}(\varphi,\mu) = \mathcal{N}(\chi^{\delta}(\varphi),\mu)$$

and $\chi^{\delta} : H^5_{-\eta} \to H^5_{\mathrm{u}}$ is the nonlocal and translationally invariant cutoff operator defined in (38) in Appendix B. In particular, χ^{δ} maps $\varphi \in H^5_{-\eta}$ to a ball of radius $C\delta$ in H^5_{u}, the space of uniform locally H^5 functions, with norm

$$\|\varphi\|_{H^5_{\mathrm{u}}} = \sup_{t\in\mathbb{R}} \|\varphi(\,\cdot\, + t)\|_{H^5([0,1])}.$$

More precisely, there exists a constant $C > 0$ such that

$$\chi^{\delta}(\varphi) = \begin{cases} \varphi & \text{if } \ \|\varphi\|_{H^5_{\mathrm{u}}} \le C\delta \\ 0 & \text{if } \ \|\varphi\|_{H^5_{\mathrm{u}}} \ \text{is sufficiently large} \end{cases}$$

and hence for $\|\varphi\|_{H^5_{\mathrm{u}}} \le C\delta$ we have $\mathcal{N}^{\delta}(\varphi,\mu) = \mathcal{N}(\varphi,\mu)$. Then, such small solutions of (14) are also solutions of the original profile equation (10). Furthermore, note that since H^5_{u} is continuously embedded in $H^5_{-\eta}$ for all $\eta > 0$, the operator χ^{δ} also serves as a cutoff in the $H^5_{-\eta}$ norm as well. For more details, see Appendix B.

A central ingredient of the center manifold reduction is the construction of a bounded projection $\mathcal{Q} : H^5_{-\eta} \to H^5_{-\eta}$ onto $\operatorname{Ker}\mathcal{T}$, which could be any bounded projection having a continuous extension to $H^4_{-\eta}$ and commuting with the inclusion map from $H^5_{-\eta}$ to $H^5_{-\eta'}$ for all $0 < \eta' < \eta$. Since the nonlocal profile equation (10) is invariant under all spatial translations, a specific choice of $\mathcal{Q}$ simplifies the computations significantly. Indeed, from Theorem 3.3 we know $\operatorname{Ker}\mathcal{T}$ has dimension four and hence, keeping generality for the moment, we may take

$$\operatorname{Ker}\mathcal{T} = \operatorname{span}\{e_1, e_2, e_3, e_4\}$$

for appropriately chosen, linearly independent functions e_j. Follow the recommendation in [16], we aim to choose a projection $\mathcal{Q}\colon H^5_{-\eta} \to \operatorname{Ker}\mathcal{T}$

$$\mathcal{Q}\colon \varphi \mapsto Ae_1 + Be_2 + Ce_3 + De_4,$$

which relates the coefficients A, B, C and D to $\varphi(0), \varphi'(0), \varphi''(0)$ and $\varphi'''(0)$ via a transition matrix $\mathscr{T}$

$$\mathscr{T}\colon (\varphi(0), \varphi'(0), \varphi''(0), \varphi'''(0)) \mapsto (A, B, C, D).$$

Using that $\mathcal{Q}\varphi = \mathcal{Q}^2\varphi$, a straightforward computation yields

$$\mathscr{T} = \begin{pmatrix} e_1(0) & e_2(0) & e_3(0) & e_4(0) \\ e_1'(0) & e_2'(0) & e_3'(0) & e_4'(0) \\ e_1''(0) & e_2''(0) & e_3''(0) & e_4''(0) \\ e_1'''(0) & e_2'''(0) & e_3'''(0) & e_4'''(0) \end{pmatrix}^{-1}.$$

When the parameters τ, c_0 satisfy (8), $\operatorname{Ker} \mathcal{T} = \operatorname{span}\{1, x, \cos(k_0 x), \sin(k_0 x)\}$ according to Theorem 3.3 and the transition matrix with respect to these basis functions is

$$\mathscr{T}_1 = \begin{pmatrix} 1 & 0 & k_0^{-2} & 0 \\ 0 & 1 & 0 & k_0^{-2} \\ 0 & 0 & -k_0^{-2} & 0 \\ 0 & 0 & 0 & -k_0^{-3} \end{pmatrix}, \tag{15}$$

which gives the explicit choice

$$\begin{aligned} \mathcal{Q}_1\varphi(x) = \Big(\varphi(0) + k_0^{-2}\varphi''(0)\Big) + \Big(\varphi'(0) + k_0^{-2}\varphi'''(0)\Big)\, x \\ - k_0^{-2}\varphi''(0)\cos(k_0 x) - k_0^{-3}\varphi'''(0)\sin(k_0 x). \end{aligned} \tag{16}$$

Similarly, when τ, c_0 satisfy (9) we have $\operatorname{Ker} \mathcal{T} = \operatorname{span}\{\cos(sx), x\cos(sx), \sin(sx), x\sin(sx)\}$, and the transition matrix with respect to these basis functions is

$$\mathscr{T}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & -(2s^2)^{-1} \\ 0 & 3(2s)^{-1} & 0 & (2s^3)^{-1} \\ s/2 & 0 & (2s)^{-1} & 0 \end{pmatrix}, \tag{17}$$

which gives the explicit choice

$$\begin{aligned} \mathcal{Q}_2\varphi(x) = \varphi(0)\cos(sx) - \left(\frac{1}{2}\varphi'(0) + \frac{1}{2s^2}\varphi'''(0)\right) x\cos(sx) \\ + \left(\frac{3}{2s}\varphi'(0) + \frac{1}{2s^3}\varphi'''(0)\right)\sin(sx) + \left(\frac{s}{2}\varphi(0) + \frac{1}{2s}\varphi''(0)\right) x\sin(sx). \end{aligned} \tag{18}$$

Remark 4.1. Our analysis up until this point holds in any space $H^m_{-\eta}$ with $m \geq 1$ and the choice of space $H^5_{-\eta}$ is made here. The projections $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are required to have a continuous extension to $H^{m-1}_{-\eta}$. Because these involve pointwise evaluation of φ''', we need at least $m - 1 = 4$ which explains the choice $H^5_{-\eta}$.

Lastly, the shift operator $\varphi \mapsto \varphi(\,\cdot\, + t)$ will be denoted by S_t. We are in position to apply the nonlocal center manifold theorem to equation (10).

Theorem 4.2. *There exist a neighborhood $\mathcal{V}$ of $0 \in \mathbb{R}$, a cutoff radius $\delta > 0$, a weight $\eta < \min\{\eta^*, \tilde{\eta}\}$ and a map*

$$\Psi \colon \mathbb{R}^4 \times \mathcal{V} \to \operatorname{Ker} \mathcal{Q} \subset H^5_{-\eta}$$

with the center manifold

$$\mathscr{M}_0^\mu = \{Ae_1 + Be_2 + Ce_3 + De_4 + \Psi(A, B, C, D, \mu) \,|\, (A, B, C, D) \in \mathbb{R}^4\}$$

as its graph for each $\mu \in \mathcal{V}$. Here, $\operatorname{Ker} \mathcal{T} = \operatorname{span}\{e_j\}_{j=1}^4$ and functions e_j are taken to be as in Theorem 3.3 for the given choices of τ, c_0. The following statements hold:

(i) (smoothness) Ψ is $\mathscr{C}^4$;

(ii) (tangency) $\Psi(0,0,0,0,0) = 0$ and $\mathrm{D}_{(A,B,C,D)}\Psi(0,0,0,0,0) = 0$;

(iii) (global reduction) $\mathscr{M}_0^\mu$ consists precisely of solutions $\varphi \in H^5_{-\eta}$ with parameter μ to the modified equation (14)*;*

(iv) (local reduction) any φ solving (10) *with parameter μ and $\|\varphi\|_{H^5_{\mathrm{u}}} \lesssim \delta$ is contained in $\mathscr{M}_0^\mu$;*

(v) (correspondence) $\varphi \in \mathscr{M}_0^\mu$ if and only if it solves the ODE

$$\varphi''''(t) = g(\mathscr{T}(\varphi(t), \varphi'(t), \varphi''(t), \varphi'''(t)), \mu), \tag{19}$$

where

$$g(A, B, C, D, \mu) = \frac{\mathrm{d}^4}{\mathrm{d}x^4}\Big(Ae_1(x) + Be_2(x) + Ce_3(x) + De_4(x) + \Psi(A, B, C, D, \mu)(x)\Big)\Big|_{x=0}$$

and $\mathscr{T}$ is the transition matrix $\mathscr{T}_1$ in (15)*, $\mathscr{T}_2$ in* (17) *for* (8) *and* (9)*, respectively;*

(vi) (reversibility) equation (10) *possesses the translation symmetries S_t and a reflection symmetry $R\varphi(x) \coloneqq \varphi(-x)$, meaning*

$$\mathcal{T}S\varphi = S\mathcal{T}\varphi \quad \text{and} \quad \mathcal{N}(S\varphi, \mu) = S\mathcal{N}(\varphi, \mu)$$

if S is S_t or R. Equation (10) *is thus reversible; so is the modified equation* (14) *and the reduced ODE* (19) *in item (v).*

Proof. We use the nonlocal center manifold theorem from [33], which for completeness is stated in Theorem B.5. Hypothesis B.1 on the linear operator $\mathcal{T}$ has been verified in the previous section and, further, Hypothesis B.3 for the nonlinearity $\varphi \mapsto \varphi^2$ in $H^m_{-\eta}$ for

$m \geq 1$ was verified in[7] [17, 33]. Using that convolution with K is a bounded linear mapping on $H^5_{-\eta}$ it follows that Hypothesis B.3 holds for our nonlocal nonlinearity $\varphi \mapsto K * \varphi^2$ as well. Note that the regularity $k \geq 2$ in Hypothesis B.3 is arbitrary for $\mathcal{N}^\delta$ in (14), possibly at the cost of a smaller cutoff radius δ and $\eta > 0$. Forthcoming computations with the reduced ODEs motivate the choice of $k = 4$, which we now take. Finally, the symmetries in item (vi) above are easily checked, using that equation (10) is steady, that the cutoff χ^δ commutes with both R and S_t, and that K is an even function. It follows that Theorem B.5 applies directly to the present case, giving statements (i)–(iv) and (vi).

It remains to prove the claim in (v) above. To this end, let $\varphi \in \mathscr{M}^\mu_0$ and note that, since Theorem B.5(vi) implies that $\mathscr{M}^\mu_0$ is invariant under translation symmetries, we have $S_t\varphi \in \mathscr{M}^\mu_0$ for all $t \in \mathbb{R}$. Consequently, there exist functions $A(t)$, $B(t)$, $C(t)$ and $D(t)$ defined for all $t \in \mathbb{R}$ such that

$$\begin{aligned} S_t\varphi(x) =& A(t)e_1(x) + B(t)e_2(x) + C(t)e_3(x) + D(t)e_4(x) \\ &+ \Psi(A(t), B(t), C(t), D(t), \mu)(x). \end{aligned} \tag{20}$$

for each $t \in \mathbb{R}$. Noting that the left-hand side of (19) can be rewritten as

$$\varphi''''(t) = \varphi''''(x+t)\Big|_{x=0} = \frac{\mathrm{d}^4}{\mathrm{d}x^4} S_t\varphi\Big|_{x=0},$$

differentiating the identity (20) four times in x and evaluating at $x = 0$ yields precisely the right-hand side in (19) with $A(t)$, $B(t)$, $C(t)$ and $D(t)$. Statement (v) is now proved by using the transition matrix $\mathscr{T}$ to rewrite (19) in terms of $\varphi(t)$, $\varphi'(t)$, $\varphi''(t)$ and $\varphi'''(t)$. □

Remark 4.3. The above proof shows that the projection coefficients $A(t), B(t), C(t)$ and $D(t)$, defined by the shift action S_t on $\varphi \in \mathscr{M}^\mu_0$, are in fact $H^5_{-\eta}$ functions in t because

$$(A(t), B(t), C(t), D(t)) = \mathscr{T}(\varphi(t), \varphi'(t), \varphi''(t), \varphi'''(t)), \qquad t \in \mathbb{R}.$$

Here, $\mathscr{T}$ is the transition matrix $\mathscr{T}_1$ for (8) and $\mathscr{T}_2$ for (9).

In the next sections, we will verify our main results Theorem 1.2 and Theorem 1.3 by studying the reduced ODE equations for (14) for appropriate values of of the parameters τ and c_0.

5 Existence of generalized solitary waves

We now establish Theorem 1.2 by deriving and studying the reduced ODE for equation (14) for τ, c_0 satisfying (8). First, we assume that $\varphi \in \mathscr{M}^\mu_0$ is so small in the H^5_{u} norm

[7]Technically, the hypothesis was verified in [17] in the case $m = 1$, and then later extended to $m \geq 1$ in [33].

that it is a solution of (10). Expanding the reduced function Ψ in A, B, C, D and μ, and then substituting into (10) gives the reduced ODE up to second-order terms. We observe from the linear part of the truncated ODE that we have a reversible $0^{2+}(\mathrm{i}k_0)$ bifurcation and then apply normal form theory for this bifurcation phenomenon. It turns out that equation (19) at leading orders is almost identical to the reduced ODE for the two-dimensional gravity–capillary water wave equations in this parameter region. Theorem 1.2 is then established after a persistence argument.

5.1 The reduced system Recall that Remark 4.3 highlights how the projection coefficients A, B, C and D may be interpreted as differentiable functions and, further, Theorem 4.2(v) suggests working with these rather than the $\varphi(t), \varphi'(t), \varphi''(t)$ and $\varphi'''(t)$ directly. Next, we Taylor expand the function Ψ up to second-order terms to obtain the following truncated system of ODEs.

Proposition 5.1. *Equation* (19) *in terms of* A, B, C *and* D *is*

$$\begin{cases} \dfrac{\mathrm{d}A}{\mathrm{d}t} = B \\ \dfrac{\mathrm{d}B}{\mathrm{d}t} = \dfrac{1}{k_0^2}\Psi(A,B,C,D,\mu)''''(0) \\ \dfrac{\mathrm{d}C}{\mathrm{d}t} = k_0 D \\ \dfrac{\mathrm{d}D}{\mathrm{d}t} = -k_0 C - \dfrac{1}{k_0^3}\Psi(A,B,C,D,\mu)''''(0). \end{cases} \tag{21}$$

Moreover, with $\sigma = \ell''(0)^{-1} = 1/(1/3 - \tau_0)$, *we have*

$$\begin{aligned} \Psi(A,B,C,D,\mu)''''(0) = {}& 2\sigma k_0^2 \mu A - \frac{2k_0^3}{\ell'(k_0)}\mu C - 2\sigma k_0^2 A^2 + \frac{4k_0^3}{\ell'(k_0)} AC \\ & - \left(\frac{3\sigma^{-2} - 4\sigma^{-1} - 4/15}{3}\sigma^2 k_0^2 - 4\sigma \right) B^2 \\ & + \left(2k_0^3 \frac{\ell''(k_0) - 2\ell'(k_0)^2}{\ell'(k_0)^2} - \frac{10k_0^2}{\ell'(k_0)} \right) BD \\ & + \left(\frac{8\ell(2k_0)k_0^4}{\ell(2k_0) - 1} - \sigma k_0^2 \right) C^2 - \left(\frac{8\ell(2k_0)k_0^4}{\ell(2k_0) - 1} + \sigma k_0^2 \right) D^2 \\ & + \mathcal{O}\left(|(A,B,C,D)| \left(\mu^2 + |A|^2 + |B|^2 + |C|^2 + |D|^2 \right) \right). \end{aligned} \tag{22}$$

Proof. Deriving equation (21) from (19) using the transition matrix $\mathscr{T}_1$ is straightforward.

Indeed, simply note that (19) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \mathscr{T}_1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathscr{T}_1^{-1} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ g(A,B,C,D,\mu) \end{pmatrix}.$$

Noting that, in this case,

$$g(A,B,C,D,\mu) = k_0^4 C + \Psi(A,B,C,D,\mu)''''(0),$$

a direct calculation shows that the above is precisely (21).

It remains to compute the asymptotic expansion (22). Specifically, we focus on computing the function $\Psi(A,B,C,D,\mu)''''$ evaluated at $x=0$ up to order two in A,B,C,D and μ. According to Theorem 4.2(i), Ψ is $\mathscr{C}^4$ in (A,B,C,D,μ). Together with item (ii) in Theorem 4.2 and the fact that $\varphi \equiv 0$ is a solution to (10) for all $\mu \in \mathbb{R}$, it follows that the Taylor expansion of Ψ must be of the form

$$\Psi(A,B,C,D,\mu)(x) = \sum_{\substack{2 \le p+q+l+m+n \le 3 \\ n \ge 1}} \Psi_{pqlmn}(x) \cdot A^p B^q C^l D^m \mu^n + \cdots$$

where each $\Psi_{pqlmn} \colon \mathbb{R} \to \mathbb{R}$ belongs to $\operatorname{Ker} \mathcal{Q}_1 \subset H^5_{-\eta}$. It thus remains to compute $\Psi''''_{pqlmn}(0)$ for $p+q+l+m+n = 2$ and $n \ge 1$. To this end, let $\varphi \in \mathscr{M}_0^\mu$ and $\|\varphi\|_{H^5_\mathrm{u}} \lesssim \delta$ and note by Theorem 4.2(iv) that φ solves (10). To conveniently group the terms, we rewrite the left-hand side of equation (10) to have

$$\mathcal{T}\varphi + (\mathrm{Id} - \mathcal{T})(\varphi^2 - \mu\varphi) = 0.$$

Since φ belongs to $\mathscr{M}_0^\mu$, we know that $\varphi(x) = \mathcal{Q}_1\varphi(x) + \Psi(A,B,C,D,\mu)(x)$, and plugging this into the above equation gives

$$\mathcal{T}(\mathcal{Q}_1\varphi + \Psi) + (\mathrm{Id} - \mathcal{T})\Big((\mathcal{Q}_1\varphi + \Psi)^2 - \mu(\mathcal{Q}_1\varphi + \Psi)\Big) = 0.$$

Using that $\mathcal{T}\mathcal{Q}_1\varphi = 0$ by definition, the above can be rearranged as

$$\mathcal{T}\Psi + (\mathcal{Q}_1\varphi)^2 - \mu\mathcal{Q}_1\varphi - \mathcal{T}(\mathcal{Q}_1\varphi)^2 = -\left(\mathrm{Id} - \mathcal{T}\right)\Big(2\left(\mathcal{Q}_1\varphi\right)\Psi + \Psi^2 - \mu\Psi\Big),$$

where we note the right-hand side above consists of all terms that are at least cubic in (A,B,C,D,μ). Linear equations for the functions Ψ_{pqlmn} can now be read off easily, and are recorded in Appendix C.1. Note that by the condition $\mathcal{Q}_1\Psi_{pqlmn} = 0$, these coefficient functions Ψ_{pqlmn} are uniquely determined. Indeed, as seen in Appendix C.1, if there are two solutions Ψ_{pqlmn} and $\tilde{\Psi}_{pqlmn}$, then their difference must belong to $\operatorname{Ker} \mathcal{T} \cap \operatorname{Ker} \mathcal{Q}_1$, and hence must be zero. Further, we observe that symmetries can be used to greatly simplify the necessary computations. Indeed, note that the basis functions $1, x, \cos(k_0 x)$ and $\sin(k_0 x)$ are either even or odd and that the operators $\mathcal{T}$, $\mathrm{Id} - \mathcal{T}$ and $\mathcal{Q}_1$ map even to even and odd to odd functions. Consequently, as seen in Appendix C.1 the linear equations for Ψ_{pqlmn} involve either only even or odd functions and hence the solutions Ψ_{pqlmn} are also necessarily either even or odd functions. Since only even functions Ψ_{pqlmn} contribute to $\Psi(A, B, C, D, \mu)''''$ evaluated at 0, equations for odd Ψ_{pqlmn} may be disregarded. The computations for Ψ_{pqlmn} are detailed in Appendix C.1 and these give equation (22). $\square$

5.2 Normal form reduction

We use normal form theory to study the reduced system (21), which can be written as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \mathbf{L}U + \mathbf{R}(U, \mu), \tag{23}$$

where $U = (A, B, C, D)$, and $\mathbf{L}$ is precisely the linearization of (21) about the origin, that is,

$$\mathbf{L} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_0 \\ 0 & 0 & -k_0 & 0 \end{pmatrix},$$

and $\mathbf{R}$ is $\mathscr{C}^4$ in a neighborhood of $(0, 0) \in \mathbb{R}^4 \times \mathbb{R}$, satisfying $\mathbf{R}(0, 0) = 0$ and $\mathrm{D}_U \mathbf{R}(0, 0) = 0$. The spectrum of $\mathbf{L}$ consists of the algebraically double and geometrically simple eigenvalue 0, as well as the pair of simple purely imaginary eigenvalues $\pm \mathrm{i} k_0$. Further, we note that the reflection symmetry $R\varphi(x) = \varphi(-x)$ on $\operatorname{Ker} \mathcal{T}$ with respect to the basis functions $1, x, \cos(k_0 x)$ and $\sin(k_0 x)$ is

$$A + B(-x) + C\cos(-k_0 x) + D\sin(-k_0 x) = A - Bx + C\cos(k_0 x) - D\sin(k_0 x).$$

This shows that R restricted to $\operatorname{Ker} \mathcal{T}$ is a linear mapping on $\mathbb{R}^4$, given by

$$R\colon (A, B, C, D) \mapsto (A, -B, C, -D)$$

and, clearly, $R^2 = \mathrm{Id}$. Further, by Theorem 4.2(vi), R anticommutes with $\mathbf{L}$ and $\mathbf{R}$, that is, $R\mathbf{L}U = -\mathbf{L}RU$ and $R\mathbf{R}(U, \mu) = -\mathbf{R}(RU, \mu)$. Taken together, it follows that it is natural to expect that the origin undergoes a reversible $0^{2+}(\mathrm{i}k_0)$ bifurcation for parameters μ sufficiently small. In this section, we use the corresponding normal form theory for such bifurcations from [19, Chapter 4.3.1] to study (23) near the origin for μ sufficiently small.

To begin, we note that the eigenvectors and generalized eigenvectors of $\mathbf{L}$ are given by

$$\xi_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \xi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and } \zeta = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \mathrm{i} \end{pmatrix},$$

which are readily seen to satisfy

$$\begin{aligned} \mathbf{L}\xi_0 &= 0, & \mathbf{L}\xi_1 &= \xi_0, & \mathbf{L}\zeta &= \mathrm{i}k_0\zeta, \\ R\xi_0 &= \xi_0, & R\xi_1 &= -\xi_1, & R\zeta &= \overline{\zeta}. \end{aligned} \tag{24}$$

Based on the structure of $\mathbf{L}$, throughout the remainder of this section $\mathbb{R}^4$ will be identified with $\mathbb{R}^2 \times \widetilde{\mathbb{R}^2}$ where $\widetilde{\mathbb{R}^2} := \{(C, \overline{C}) : C \in \mathbb{C}\}$. We are now in the position to directly apply the normal form result [19, Lemma 3.5]. This result implies that there exist neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of $0 \in \mathbb{R}^2 \times \widetilde{\mathbb{R}^2}$ and $0 \in \mathbb{R}$, respectively, and a polynomial change of variables

$$U = \mathbf{A}\xi_0 + \mathbf{B}\xi_1 + \mathbf{C}\zeta + \overline{\mathbf{C}\zeta} + \Phi(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu) \tag{25}$$

defined in $\mathcal{V}_1$ and $\mathcal{V}_2$, which transforms the reduced system (21) into the normal form

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \mathbf{B} \\ \dfrac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = P(\mathbf{A}, |\mathbf{C}|^2, \mu) + \rho_{\mathbf{B}}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu) \\ \dfrac{\mathrm{d}\mathbf{C}}{\mathrm{d}t} = \mathrm{i}k_0\mathbf{C} + \mathrm{i}\mathbf{C}Q(\mathbf{A}, |\mathbf{C}|^2, \mu) + \rho_{\mathbf{C}}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu), \end{cases} \tag{26}$$

where P and Q are polynomials of degree two and one in $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}})$, respectively. Here, the function Φ is $\mathscr{C}^4$, satisfying

$$\Phi(0,0,0,0,0) = 0, \qquad \partial_{(\mathbf{A},\mathbf{B},\mathbf{C},\overline{\mathbf{C}})}\Phi(0,0,0,0,0) = 0$$
$$\Phi(\mathbf{A}, -\mathbf{B}, \overline{\mathbf{C}}, \mathbf{C}, \mu) = R\Phi(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu)$$

while the remainders $\rho_{\mathbf{B}}$ and $\rho_{\mathbf{C}}$ are $\mathscr{C}^4$ with

$$|\rho_{\mathbf{B}}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu)| + |\rho_{\mathbf{C}}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu)| = o((|\mathbf{A}| + |\mathbf{B}| + |\mathbf{C}|)^2).$$

For proof and more details, see [19, Chapter 4.3.1].

Let

$$\begin{aligned} P(\mathbf{A}, |\mathbf{C}|^2, \mu) &= p_0\mu + p_1\mu\mathbf{A} + p_2\mathbf{A}^2 + p_3|\mathbf{C}|^2 \\ Q(\mathbf{A}, |\mathbf{C}|^2, \mu) &= q_0\mu + q_1\mathbf{A}. \end{aligned}$$

The scalar coefficients p_0, p_1, p_2, p_3, q_0 and q_1 are computed in Appendix D.1. Setting $\sigma = (1/3 - \tau)^{-1}$ these calculations yield the normal form of (21) as

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \mathbf{B} \\ \dfrac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = 2\sigma\mu\mathbf{A} - 2\sigma\mathbf{A}^2 - 4\sigma|\mathbf{C}|^2 + \mathcal{O}(|\mu|^2 + (|\mu| + |\mathbf{A}| + |\mathbf{C}|)^2)) \\ \dfrac{\mathrm{d}\mathbf{C}}{\mathrm{d}t} = \mathrm{i}k_0\mathbf{C} + \dfrac{\mathrm{i}}{\ell'(k_0)}\mu\mathbf{C} - \dfrac{2\mathrm{i}}{\ell'(k_0)}\mathbf{A}\mathbf{C} + \mathcal{O}(|\mathbf{C}|(|\mu| + |\mathbf{A}| + |\mathbf{C}|^2)^2). \end{cases} \tag{27}$$

5.3 **Generalized solitary waves** Next, we consider the normal form system (27) truncated at second-order terms, i.e.

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \mathbf{B} \\ \dfrac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = 2\sigma\mu\mathbf{A} - 2\sigma\mathbf{A}^2 - 4\sigma|\mathbf{C}|^2 \\ \dfrac{\mathrm{d}\mathbf{C}}{\mathrm{d}t} = \mathrm{i}k_0\mathbf{C} - \dfrac{\mathrm{i}}{\ell'(k_0)}\mu\mathbf{C} + \dfrac{2\mathrm{i}}{\ell'(k_0)}\mathbf{A}\mathbf{C}. \end{cases} \tag{28}$$

The change of variables

$$\begin{gathered} t = \frac{1}{\sqrt{2}}w, \qquad \mathbf{A}(t) = -\frac{3}{2}\widetilde{\mathbf{A}}(w), \qquad \mathbf{B}(t) = -\frac{3}{\sqrt{2}}\widetilde{\mathbf{B}}(w), \\ \mathbf{C}(t) = |\mu|k^{1/2}\exp(\mathrm{i}\Theta(t)) \end{gathered} \tag{29}$$

transforms (28) into the system (3.14) studied by Iooss & Kirchgässner in [24] in their search for generalized solitary waves in the context of the full gravity–capillary water wave problem. The only difference between our rescaled system and that studied in [24] is the coefficients of terms involving **C**, which is inconsequential. Note in [24] that the small parameter used is $\frac{1}{c^2} - 1$, which corresponds to $-\mu$ in our case. Here, k is fixed but arbitrary. We observe that

$$\Theta'(t) = k_0 - \frac{\mu}{\ell'(k_0)} + \frac{2}{\ell'(k_0)}\mathbf{A}(t).$$

Equations (3.17)–(3.19) in [24] provide us with a one-parameter family of explicit solutions, parametrized by k, of the rescaled truncated system given by

$$\begin{aligned} \widetilde{\mathbf{A}}(w) &= -\frac{\mu}{3}(1 - \mathrm{sgn}(\mu)\rho^{1/2}) - |\mu|\rho^{1/2}\mathrm{sech}^2\left(\frac{\rho^{1/4}|\mu|^{1/2}\sigma^{1/2}w}{2}\right), \\ \widetilde{\mathbf{B}}(w) &= \widetilde{\mathbf{A}}'(w). \end{aligned} \tag{30}$$

Then, substituting $\mathbf{A}(t) = -3\widetilde{\mathbf{A}}(w)/2$ into the differential equation for $\Theta(t)$ gives

$$\begin{aligned} \Theta(t) = \Theta_* &+ \left(k_0 - \frac{\mu}{\ell'(k_0)} + \frac{2\mu}{\ell'(k_0)}\left(1 - \mathrm{sgn}(\mu)\rho^{1/2}\right)\right)t \\ &+ \frac{3\sqrt{2}\,\rho^{1/4}|\mu|^{1/2}}{\sigma^{1/2}\ell'(k_0)}\tanh\left(\frac{\rho^{1/4}|\mu|^{1/2}\sigma^{1/2}t}{\sqrt{2}}\right), \end{aligned}$$

where $\Theta_* \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ is arbitrary and $\rho = 1 + 24k$.

It remains to see if the above family of solutions of the rescaled truncated normal-form system persist as solutions of the full rescaled normal-form system. Luckily, the persistence of (30) under reversible perturbations has received considerable treatment (see, for example, the work of Iooss & Kirchgässner in [24]). In particular, these persistence results are summarized for $\mathscr{C}^m$ vector fields in [19, Theorem 3.10] and, in the present context, this work guarantees that the family of explicit solutions (30) persists provided that

$$r = |\mu| k^{1/2} > r_*(\mu) = \mathcal{O}(|\mu|^{1/2}). \tag{31}$$

In particular, note that since μ is small the persistence condition (31) is effectively a lower bound on the frequency k, corresponding to high-frequency oscillation in $\tilde{\mathbf{C}}$.

Finally, we undo the above variable changes to return to the original unknown function φ. Undoing (29) in (5.3) yields

$$\begin{aligned}
\mathbf{A}(t) &= \frac{\mu}{2}(1 - \operatorname{sgn}(\mu)\rho^{1/2}) + \frac{3}{2}|\mu|\rho^{1/2}\operatorname{sech}^2\left(\frac{\rho^{1/4}|\mu|^{1/2}\sigma^{1/2}t}{\sqrt{2}}\right), \\
\mathbf{B}(t) &= \mathbf{A}'(t), \\
\mathbf{C}(t) &= |\mu|k^{1/2}\exp\left(\mathrm{i}(k_0 + \mathcal{O}(\mu))t + \mathrm{i}\Theta_* + \mathcal{O}(\mu)\right),
\end{aligned}$$

while undoing the polynomial change of variables (25) in the above normal form analysis yields

$$\begin{aligned}
A(t) = \mathbf{A}(t) + \mathcal{O}(\mu^2\rho^{1/2}), \quad B(t) = \mathbf{B}(t) + \mathcal{O}(\mu^2\rho^{1/2}), \\
C(t) = \frac{1}{2}(\mathbf{C} + \overline{\mathbf{C}}) + \mathcal{O}(\mu^2\rho^{1/2}).
\end{aligned} \tag{32}$$

Recalling now that (16) implies $A(t) = \varphi(t) + k_0^{-2}\varphi''(t)$, $C(t) = -k_0^{-2}\varphi''(t)$ and switching back to the original variable x, it follows that

$$\begin{aligned}
\varphi(x) &= A(x) + C(x) \\
&= \frac{3}{2}|\mu|\rho^{1/2}\operatorname{sech}^2\left(\frac{\rho^{1/4}|\mu|^{1/2}\sigma^{1/2}x}{\sqrt{2}}\right) + \frac{\mu}{2}(1 - \operatorname{sgn}(\mu)\rho^{1/2}) \\
&\qquad + |\mu|k^{1/2}\cos\Big((k_0 + \mathcal{O}(\mu))x + \Theta_* + \mathcal{O}(\mu)\Big) + \mathcal{O}(\mu^2\rho^{1/2}).
\end{aligned}$$

Here, $\Theta_* \in \mathbb{R}/2\pi\mathbb{Z}$ is an arbitrary integration constant. Due to the hyperbolic tangent in Θ, there is an asymptotic phase shift in the cosinus term between $x = -\infty$ and $x = \infty$ of order $\mathcal{O}(\rho^{1/4}|\mu|^{1/2})$.

Provided the persistence condition (31) holds, the function φ above solves the modified profile equation (14). For φ to be a solution to the original profile equation (10) with parameter μ, it must additionally satisfy the smallness assumption $\|\varphi\|_{H^5_{\mathrm{u}}} \lesssim \delta$. This can be achieved by setting, for example,

$$k = k'|\mu|^{-1-2\kappa}, \qquad \text{for some } \kappa \in [0, 1/2) \text{ and some constant } k' > 0.$$

Indeed, under this condition the persistence condition (31) is clearly met and the functions $\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}$ have amplitude $\mathcal{O}(|\mu|^{1/2-\kappa})$ which, in turn, implies that $\varphi, \varphi', \varphi'', \varphi'''$ are also $\mathcal{O}(|\mu|^{1/2-\kappa})$ via (32) and (16). This bound is carried over to the fourth and fifth derivatives by differentiating (19) twice (see [33, Theorem 3.3]). It follows from choosing μ sufficiently small that the H^5_{u} norm of φ is small, and hence that φ is a solution to (10) with parameter μ. This establishes Theorem 1.2.

Remark 5.2. When $\mu > 0$, (28) features an orbit which is homoclinic to the saddle equilibrium $(\mathbf{A}, \mathbf{B}) = (0, 0)$ once projected onto the $(\mathbf{A}, \mathbf{B})$-plane. When $\mu < 0$, it is homoclinic to $(\mathbf{A}, \mathbf{B}) = (\frac{\mu}{2}(1 + \rho^{1/2}), 0)$, which is close to $(0, 0)$. In the latter case, we point out that this solution has supercritical wave speed. Indeed, equation (1) is invariant under a Galilean change of variable

$$\varphi \mapsto \varphi + v, \quad c \mapsto c - 2v, \quad (1-c)^2 b \mapsto (1-c)^2 b + (1-c)v + v^2,$$

where b is an integration constant which doesn't affect the critical wavespeed: see [22]. Putting $v = \mu/2(1 + \rho^{1/2})$, the new wave speed is

$$c - 2v = 1 + \mu - 2 \cdot \frac{\mu}{2}(1 + \rho^{1/2}) = 1 + |\mu|\rho^{1/2} > 1.$$

To summarize, all generalized solitary-wave solutions in Theorem 1.2 have supercritical wave speed $c > 1$.

6 Existence of modulated solitary waves

The aim of this section is to prove existence of modulated solitary waves in (9). As for the classical two-dimensional gravity–capillary water wave equations, the signs of two terms in the normal form are to be determined – one of those terms will be of cubic order. Instead of deriving the full reduced ODE as in Section 5, we only determine it roughly using the symmetries. We then perform a normal form reduction and determine linear equations for the relevant normal form coefficients. From these, it will be clear which center manifold coefficients are necessary. Throughout this section, we assume that the parameters τ and c_0 satisfy (9).

6.1 Normal form reduction As in Section 5, we will work with projection coefficients A, B, C, and D rather than $\varphi, \varphi', \varphi''$ and φ'''. Using the transition matrix $\mathscr{T}_2$ from Section 4 and proceeding along the same lines as the proof of Proposition 5.1, we find that

(19) in this case is equivalent to the system

$$
\begin{cases}
\dfrac{\mathrm{d}A}{\mathrm{d}t} = B + sC \\
\dfrac{\mathrm{d}B}{\mathrm{d}t} = sD - \dfrac{1}{2s^2}\Psi(A,B,C,D,\mu)''''(0) \\
\dfrac{\mathrm{d}C}{\mathrm{d}t} = -sA + D + \dfrac{1}{2s^3}\Psi(A,B,C,D,\mu)''''(0) \\
\dfrac{\mathrm{d}D}{\mathrm{d}t} = -sB.
\end{cases} \tag{33}
$$

Letting $U = (A, B, C, D)$, (33) can be rewritten as

$$
\frac{\mathrm{d}U}{\mathrm{d}t} = \mathbf{L}U + \mathbf{R}(U,\mu), \tag{34}
$$

where here

$$
\mathbf{L} = \begin{pmatrix} 0 & 1 & s & 0 \\ 0 & 0 & 0 & s \\ -s & 0 & 0 & 1 \\ 0 & -s & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{R}(U,\mu) = \frac{1}{2s^3}\begin{pmatrix} 0 \\ -s\Psi(U,\mu)''''(0) \\ \Psi(U,\mu)''''(0) \\ 0 \end{pmatrix}.
$$

In particular, in view of Theorem 4.2(ii) the matrix $\mathbf{L}$ is precisely the linearization of (33) about the trivial solution $0 \in \mathbb{R}^4$. The spectrum of $\mathbf{L}$ is readily seen to consist of a pair of algebraically double and geometrically simple eigenvalues $\mathrm{i}s$ and $-\mathrm{i}s$ with corresponding eigenvectors and generalized eigenvectors

$$
\zeta_0 = \begin{pmatrix} 1 \\ 0 \\ \mathrm{i} \\ 0 \end{pmatrix}, \quad \overline{\zeta_0} = \begin{pmatrix} 1 \\ 0 \\ -\mathrm{i} \\ 0 \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \mathrm{i} \end{pmatrix} \quad \text{and} \quad \overline{\zeta_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\mathrm{i} \end{pmatrix},
$$

that satisfy

$$
\begin{aligned}
(\mathbf{L} - \mathrm{i}s)\zeta_0 &= 0, \qquad (\mathbf{L} - \mathrm{i}s)\zeta_1 = \zeta_0, \\
(\mathbf{L} + \mathrm{i}s)\overline{\zeta_0} &= 0, \qquad (\mathbf{L} + \mathrm{i}s)\overline{\zeta_1} = \overline{\zeta_0}.
\end{aligned}
$$

As such, it is natural to expect that the system undergoes an $(\mathrm{i}s)^2$ bifurcation.

To analyze this bifurcation, observe that the set $\{\zeta_0, \zeta_1, \overline{\zeta_0}, \overline{\zeta_1}\}$ spans $\widetilde{\mathbb{R}^2} \times \widetilde{\mathbb{R}^2} \sim \mathbb{R}^4$, and that the reversible symmetry $R\varphi(x) = \varphi(-x)$ restricted on $\operatorname{Ker} \mathcal{T}$ takes the form

$$
R\colon (A, B, C, D) \mapsto (A, -B, -C, D)
$$

with respect to the basis $\{\cos(sx), x\cos(sx), \sin(sx), x\sin(sx)\}$. Furthermore, the vectors ζ_0 and ζ_1 also satisfy

$$R\zeta_0 = \overline{\zeta_0} \quad \text{and} \quad R\zeta_1 = -\overline{\zeta_1}.$$

Normal form theory for $(\mathrm{i}s)^2$ bifurcations now asserts that there exists a polynomial change of variable

$$U = \mathbf{A}\zeta_0 + \mathbf{B}\zeta_1 + \overline{\mathbf{A}\zeta_0} + \overline{\mathbf{B}\zeta_1} + \Phi(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, \mu),$$

where Φ is a polynomial in $(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}})$ of degree 3, that transforms (34) into the normal form

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \mathrm{i}s\mathbf{A} + \mathbf{B} + \mathrm{i}\mathbf{A}P\left(|\mathbf{A}|^2, \dfrac{\mathrm{i}}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B})\right) + \rho_{\mathbf{A}}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, \mu) \\ \dfrac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = \mathrm{i}s\mathbf{B} + \mathrm{i}\mathbf{B}P\left(|\mathbf{A}|^2, \dfrac{\mathrm{i}}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B})\right) \\ \qquad + \mathbf{A}Q\left(|\mathbf{A}|^2, \dfrac{\mathrm{i}}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B})\right) + \rho_{\mathbf{B}}(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, \mu). \end{cases} \tag{35}$$

Here, the polynomials P and Q have degree 2 in $(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}})$. For details, see [19, Section 4.3.3] and, specifically, Lemma 3.17 in that reference.

6.2 Modulated solitary waves We now aim to consider the normal form system (35) truncated at second-order terms. Let

$$\begin{aligned} P\left(|\mathbf{A}|^2, \frac{\mathrm{i}}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B})\right) &= p_0\mu + p_1|\mathbf{A}|^2 + \frac{\mathrm{i}p_2}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B}), \\ Q\left(|\mathbf{A}|^2, \frac{\mathrm{i}}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B})\right) &= q_0\mu + q_1|\mathbf{A}|^2 + \frac{\mathrm{i}q_2}{2}(\mathbf{A}\overline{\mathbf{B}} - \overline{\mathbf{A}}\mathbf{B}). \end{aligned} \tag{36}$$

The coefficients q_0 and q_1 are computed in Appendices C.2 and D.2,

$$q_0 = \frac{2}{c_0^2\ell''(s)} \quad \text{and} \quad q_1 = \frac{4(-c_0 + \ell(2s)^{-1})^{-1} + 8(1 - c_0)^{-1}}{c_0^2\ell''(s)}.$$

One can check that this agrees with the formulas given in Theorem 1.3 by using that $m(s) = \ell(s)^{-1}$, $\ell(s) = c_0^{-1}$ and $\ell'(s) = 0$. Moreover, q_0 and q_1 are both negative because $c_0 < 1$, $c_0\ell(2s) < 1$ while $\ell''(s) < 0$ for each $s > 0$ as illustrated in Figure 3. Recalling that $\mu < 0$ in this case, the above puts (35) into the subcritical case considered by Iooss & Pèrouéme in [25, Section IV3]. Through the change of variables

$$\mathbf{A}(t) = r_0(t)\exp(\mathrm{i}(st + \Theta_0(t))), \qquad \mathbf{B}(t) = r_1(t)\exp(\mathrm{i}(st + \Theta_1(t))),$$

the normal form truncated at third order terms has explicit homoclinic solutions

$$
\begin{aligned}
r_0(t) &= \sqrt{\frac{-2q_0\mu}{q_1}}\operatorname{sech}(\sqrt{q_0\mu}\,t),\\
r_1(t) &= |r_0'|,\\
\Theta_0(t) &= p_0\mu t - \frac{2p_1\sqrt{q_0\mu}}{q_1}\tanh(\sqrt{q_0\mu}\,t) + \Theta_*,\\
\Theta_1 - \Theta_0 &\in \{0,\pi\},
\end{aligned}
$$

(see [19, pp.217–223]). Here, p_0, p_1 are as in (36) and $\Theta_* \in \mathbb{R}$ is an arbitrary integration constant, resulting in a full circle of homoclinic solutions. However, only two distinct homoclinic solutions persist under reversible perturbation, when $\Theta_* = 0$ and $\Theta_* = \pi$. Tracing back to the original unknown φ and variable x, we get

$$
\varphi(x) = \sqrt{\frac{-8q_0\mu}{q_1}}\operatorname{sech}(\sqrt{q_0\mu}\,x)\cos\left(sx + \mathcal{O}\left(|\mu|^{1/2}\right)\right) + \mathcal{O}(\mu^2),
$$

and

$$
\varphi(x) = -\sqrt{\frac{-8q_0\mu}{q_1}}\operatorname{sech}(\sqrt{q_0\mu}\,x)\cos\left(sx + \mathcal{O}\left(|\mu|^{1/2}\right)\right) + \mathcal{O}(\mu^2).
$$

The first solution φ is often referred to as a modulated solitary wave of elevation and the latter is a modulated solitary wave of depression. We illustrate the elevation case in Figure 1. Due to the hyperbolic tangent, there is an asymptotic phase shift of order $\mathcal{O}(|\mu|^{1/2})$ between $x = -\infty$ and $x = \infty$. Lastly, it can be shown that $\varphi, \varphi', \varphi''$ and φ''' are of order $\mathcal{O}(|\mu|^{1/2})$ by arguing as in the previous section. The uniform locally Sobolev norm $\|\varphi\|_{H^5_{\mathrm{u}}}$ can thus be made arbitrarily small, qualifying these as solutions to (10) with parameters (9). This establishes Theorem 1.3.

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A Fredholm theory for pseudodifferential operators

In this appendix, we review a Fredholm theory for pseudodifferential operators developed by Grushin in [18]. This theory is applied in Section 3 to determine the Fredholm properties of the linear operator $\mathcal{T}$.

Let $x^* = (x_0, x) \in \mathbb{R} \times \mathbb{R}^n$ and

$$X^* = \{x^* \in \mathbb{R}^{n+1} \mid x_0 \geq 0, x^* \neq 0\}.$$

Similarly, let $\xi^* = (\xi_0, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and

$$E^* = \{\xi^* \in \mathbb{R}^{n+1} \mid \xi_0 \geq 0, \xi^* \neq 0\}.$$

Let $\mathcal{A}$ be the class of functions $A(x^*, \xi^*) \in C^\infty(X^* \times E^*)$ such that A is positive-homogeneous of degree 0 in x^* and ξ^*, that is,

$$A(\lambda x^*, \xi^*) = A(x^*, \lambda \xi^*) = A(x^*, \xi^*), \qquad \lambda > 0.$$

Let $\mathbb{S}^n_+$ denote the hemisphere $|x^*| = 1$ and $x_0 > 0$, or $|\xi^*| = 1$ and $\xi_0 > 0$. Let $\overline{\mathbb{S}^n_+}$ denote the relative closure of $\mathbb{S}^n_+$ in X^*, or in E^*, that is, $\overline{\mathbb{S}^n_+}$ is the hemisphere $|x^*| = 1$ (or $|\xi^*| = 1$), $x_0 \geq 0$ (or $\xi_0 \geq 0$). Clearly, each $A \in \mathcal{A}$ is uniquely determined by its values on $\mathbb{S}^n_+ \times \mathbb{S}^n_+$. Conversely, each function $\tilde{A} \in C^\infty(\overline{\mathbb{S}^n_+} \times \overline{\mathbb{S}^n_+})$ can be uniquely homogeneously extended to $X^* \times E^*$. So, $\mathcal{A} \cong C^\infty(\overline{\mathbb{S}^n_+} \times \overline{\mathbb{S}^n_+})$. By $S^0_{\mathcal{A}}$, we denote the set of symbols $p_A(x, \xi)$ which are given by

$$p_A(x, \xi) = A(1, x,\ 1, \xi),$$

for some $A \in \mathcal{A}$. For $p_A \in S^0_{\mathcal{A}}$, we have the following result, which combines Theorems 4.1 and 4.2 in Grushin [18].

Theorem A.1. *If* $p_A(x, \xi) \in S^0_{\mathcal{A}}$ *and* $\det A(x^*, \xi^*) \neq 0$ *on* Γ, *then*

$$p_A(x, D) \colon H^s \to H^s$$

is Fredholm and the index is

$$\operatorname{ind} p_A(x, D) = \frac{1}{2\pi}\Big(\arg\det\ A(x^*, \xi^*)\Big|_\Gamma \Big),$$

where Γ *is the boundary of* $\overline{\mathbb{S}^n_+} \times \overline{\mathbb{S}^n_+}$, *and* $\arg\det\ A(x^*, \xi^*)|_\Gamma$ *is the increase in the argument of* $\det\ A(x^*, \xi^*)$ *around* Γ *oriented counterclockwise.*

B A nonlocal center manifold theorem

In this section, we record a version due to Truong, Wahlén & Wheeler [33] of the nonlocal center manifold theorem originally developed by Faye & Scheel [16, 17]. This result is the main analytical tool used throughout Section 5 and Section 6.

We consider nonlocal nonlinear parameter-dependent problems of the form

$$\mathcal{T} v + \mathcal{N}(v, \mu) = 0, \tag{37}$$

where

$$\mathcal{T}v = v + \mathcal{K} * v,$$

in the weighted Sobolev spaces $H^m_{-\eta}$ for some $\eta > 0$ and positive integer m. $\mathcal{T}$ is referred to as the linear part and $\mathcal{N}$ as the nonlinear part of (37). Before introducing the modified equation, we define a cutoff operator χ which is invariant under all translations and reversible symmetries. The translation map by $t \in \mathbb{R}$, that is $\varphi \mapsto \varphi(\,\cdot\, + t)$, is denoted by S_t. First, let $\underline{\chi}\colon \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function satisfying $\underline{\chi} = 1$ for $|x| < 1$, 0 for $|x| > 2$ and $\sup_{x\in\mathbb{R}} |\underline{\chi}'(x)| \leq 2$. Secondly, let $\theta\colon \mathbb{R} \to \mathbb{R}$ be an even and smooth function with

$$\sum_{j\in\mathbb{Z}} \theta(x-j) = 1, \quad \operatorname{supp}\theta \subset [-1,1], \quad \theta\left(\left[0,\tfrac{1}{2}\right]\right) \subset \left[\tfrac{1}{2},1\right],$$

for all $x \in \mathbb{R}$. Define

$$\chi\colon v \mapsto \int_{\mathbb{R}} \underline{\chi}(\|S_y\theta\cdot v\|_{H^m})\theta(x-y)v(x)\,\mathrm{d}y \quad \text{and} \quad \chi^\delta\colon v \mapsto \delta\cdot\chi\left(\frac{v}{\delta}\right), \quad \delta > 0. \tag{38}$$

It has been shown in [17] that $\chi : H^m_{-\eta} \to H^m_{\mathrm{u}}$ is well-defined, Lipschitz continuous, invariant under all S_t and reversible symmetries on $\mathbb{R}$, and its image is contained in a ball in H^m_{u}. As a consequence, the scaled cutoff χ^δ inherits all these properties except for its image, which will be contained in a ball of radius δ in H^m_{u}. The modified equation is

$$\mathcal{T}v + \mathcal{N}^\delta(v,\mu) = 0 \quad \text{where} \quad \mathcal{N}^\delta(v,\mu) := \mathcal{N}(\chi^\delta(v),\mu). \tag{39}$$

Also, let $\mathcal{Q}\colon H^m_{-\eta} \to H^m_{-\eta}$ be a bounded projection on the nullspace $\operatorname{Ker}\mathcal{T}$ of $\mathcal{T}$ with a continuous extension to $H^{m-1}_{-\eta}$, such that $\mathcal{Q}$ commutes with the inclusion map from $H^m_{-\eta}$ to $H^m_{-\eta'}$, for all $0 < \eta' < \eta$.

Hypothesis B.1 (The linear part $\mathcal{T}$).

(i) There exists $\eta_0 > 0$ such that $\mathcal{K} \in L^1_{\eta_0}$.

(ii) The operator

$$\mathcal{T}\colon v \mapsto v + \mathcal{K} * v, \qquad H^m_{-\eta} \to H^m_{-\eta}$$

is Fredholm for $\eta \in (0,\eta_0)$, its nullspace $\operatorname{Ker}\mathcal{T}$ is finite-dimensional and $\mathcal{T}$ is onto.

Remark B.2. A straightforward application of Young's inequality shows that Hypothesis B.1(i) implies that the operator $\mathcal{T}\colon H^m_{-\eta} \to H^m_{-\eta}$ is bounded for each $\eta \in (0,\eta_0)$ for each choice of c_0. In the works [16, 17] the authors additionally assumed that $K' \in L^1_{\eta_0}$ for some $\eta_0 > 0$ which, as seen from Proposition 3.1, does not hold for the current case. This assumption, however, is used to guarantee Hypothesis B.1(ii) which, here, we instead require directly.

Hypothesis B.3 (The nonlinear part $\mathcal{N}$). *There exist $k \geq 2$, a neighborhood $\mathcal{U}$ of $0 \in H^m_{-\eta}$ and $\mathcal{V}$ of $0 \in \mathbb{R}$, such that for all sufficiently small $\delta > 0$, we have*

(a) $\mathcal{N}^\delta \colon H^m_{-\eta} \times \mathcal{V} \to H^m_{-\eta}$ *is* $\mathscr{C}^k$. *Moreover, for all non-negative pairs (ζ, η) such that $0 < k\zeta < \eta < \eta_0$, $\mathrm{D}^l_v \mathcal{N}^\delta(\,\cdot\,, \mu) \colon (H^j_{-\zeta})^l \to H^j_{-\eta}$ is bounded for all $0 < l\zeta \leq \eta < \eta_0$ and $0 \leq l \leq k$, and is Lipschitz in v for $1 \leq l \leq k-1$ uniformly in $\mu \in \mathcal{V}$.*

(b) $\mathcal{N}^\delta$ *commutes with translations of v,*

$$\mathcal{N}^\delta(S_t v, \mu) = S_t \mathcal{N}(v, \mu), \qquad \textit{for all } t \in \mathbb{R}.$$

(c) $\mathcal{N}^\delta(0,0) = 0$, $\mathrm{D}_v \mathcal{N}^\delta(0,0) = 0$ *and as $\delta \to 0$, the Lipschitz constant*

$$Lip_{H^m_{-\eta} \times \mathcal{V}} \mathcal{N}^\delta = \mathcal{O}(\delta + |\mu|).$$

Let $v \colon \mathbb{R} \to \mathbb{R}$ be a function. A symmetry is a triple $(\rho, S_t, \kappa) \in \mathbf{O}(1) \times (\mathbb{R} \times \mathbf{O}(1))$, acting on v in the following way: the orthogonal linear transformation $\rho \in \mathbf{O}(1)$ acts on the value $v(x) \in \mathbb{R}$, while S_t and κ act on the variable $x \in \mathbb{R}$. A symmetry (ρ, S_t, κ) is equivariant if $\kappa = \mathrm{Id}$, and reversible otherwise.

Hypothesis B.4 (Symmetries). *There exists a symmetry group S, under which the equation is invariant, that is*

$$\gamma(\mathcal{T} v) = \mathcal{T}(\gamma v), \qquad \mathcal{N}(\gamma v, \mu) = \gamma \mathcal{N}(v, \mu), \qquad \textit{for all } \gamma \in S,$$

such that S contains all translations on the real line.

Theorem B.5. *Assume that Hypotheses B.1, B.3 and B.4 are met for the* (37). *Then, by possibly shrinking the neighborhood $\mathcal{V}$ of $0 \in \mathbb{R}$, there exists a cutoff radius $\delta > 0$, a weight $\eta > 0$ and a map*

$$\Psi \colon \operatorname{Ker} \mathcal{T} \times \mathcal{V} \subset H^m_{-\eta} \times \mathbb{R} \to \operatorname{Ker} \mathcal{Q} \subset H^m_{-\eta}$$

with the center manifold

$$\mathcal{M}^\mu_0 = \{ v_0 + \Psi(v_0, \mu) \,|\, v_0 \in \operatorname{Ker} \mathcal{T}, \mu \in \mathcal{V} \} \subset H^m_{-\eta},$$

as its graph for each μ. The following statements hold:

(i) (smoothness) $\Psi \in \mathscr{C}^k$, *where k is as in Hypothesis B.3;*

(ii) (tangency) $\Psi(0,0) = 0$ *and* $\mathrm{D}_{v_0} \Psi(0,0)$;

(iii) (global reduction) $\mathcal{M}^\mu_0$ consists precisely of functions v such that $v \in H^m_{-\eta}$ is a solution of the modified equation (39) *with parameter μ;*

(iv) *(local reduction) any function v solving (37) with $\|v\|_{H^m_u} \lesssim \delta$ is contained in $\mathcal{M}^\mu_0$;*

(v) *(translation invariance) the shift S_t, $t \in \mathbb{R}$ acting on $\mathcal{M}^\mu_0$ induces a μ-dependent flow*

$$\Phi_t \colon \operatorname{Ker} \mathcal{T} \to \operatorname{Ker} \mathcal{T}$$

through $\Phi_t = \mathcal{Q} \circ S_t \circ (\mathrm{Id} + \Psi)$;

(vi) *(reduced vector field) the reduced flow $\Phi_t(v_0, \mu)$ is of class $\mathscr{C}^k$ in v_0, μ, t and is generated by a reduced parameter dependent vector field f of class $\mathscr{C}^{k-1}$ on the finite-dimensional $\operatorname{Ker} \mathcal{T}$;*

(vii) *(correspondence) any element $v = v_0 + \Psi(v_0, \mu)$ of $\mathcal{M}^\mu_0$ corresponds one-to-one to a solution of*

$$\frac{\mathrm{d}v_0}{\mathrm{d}t} = f(v_0) := \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{Q}(S_t v)\Big|_{t=0};$$

(viii) *(equivariance) $\operatorname{Ker} \mathcal{T}$ is invariant under Γ and $\mathcal{Q}$ can be chosen to commute with all $\gamma \in S$. Consequently, Ψ commutes with $\gamma \in S$ and $\mathcal{M}^\mu_0$ is invariant under Γ. Finally, the reduced vector field f in item (vi) commutes with all translations S_t and anticommutes with reversible symmetries in S.*

C Coefficients in center manifold reduction

In this appendix, we compute the center manifold coefficients $\psi_{pqlmn} := \Psi''''_{pqlmn}(0)$ up to second-order terms in Proposition 5.1 as well as the coefficients

$$\psi_{10001}, \quad \psi_{20000}, \quad \psi_{00200}, \quad \psi_{10010}, \quad \psi_{01100}, \quad \psi_{10200}, \quad \psi_{30000}$$

from Section 6. The proof of Proposition 5.1 observes that $\mathcal{T}$, $\mathrm{Id} - \mathcal{T}$ and $\mathcal{Q}$ map even to even, and odd to odd functions. Also, the basis functions of $\operatorname{Ker} \mathcal{T}$ are either even or odd functions. Using these, vanishing ψ_{rslmn} are identified and excluded. Then, linear equations for non-vanishing ψ_{rslmn} are written down. To compute ψ_{rslmn}, we will extensively use

$$\begin{aligned}
\mathbf{m}(\mathrm{D}) \colon x^{2k} \cos(yx) \mapsto & \sum_{j=0}^{k} \binom{2k}{2j} (-1)^j \mathbf{m}^{(2j)}(y) \cdot x^{2(k-j)} \cos(yx) \\
& + \sum_{j=0}^{k-1} \binom{2k}{2j+1} (-1)^j \mathbf{m}^{(2j+1)}(y) \cdot x^{2(k-j)-1} \sin(yx) \\
\mathbf{m}(\mathrm{D}) \colon x^{2k+1} \sin(yx) \mapsto & \sum_{j=0}^{k} \binom{2k+1}{2j} (-1)^j \mathbf{m}^{(2j)}(y) \cdot x^{2(k-j)+1} \sin(yx) \\
& + \sum_{j=0}^{k} \binom{2k+1}{2j+1} (-1)^{j+1} \mathbf{m}^{(2j+1)}(y) \cdot x^{2(k-j)} \cos(yx),
\end{aligned} \tag{40}$$

where $\mathbf{m}\colon \mathbb{R} \to \mathbb{R}$ is an even multiplier and $y \in \mathbb{R}$. Finally, we observe that if f satisfies $\mathcal{T} f = g$, then $h \coloneqq f - \mathcal{Q} f$ satisfies $\mathcal{T} h = g$ and $\mathcal{Q} h = 0$.

C.1 For generalized solitary waves Here, let τ, c_0 satisfy (8) and note, specifically, that $c_0 = 1$ here. Equation (10) with $\varphi = \mathcal{Q}_1\varphi + \Psi$ is

$$\mathcal{T}\Psi - \mu\mathcal{Q}_1\varphi + (\mathrm{Id} - \mathcal{T})(\mathcal{Q}_1\varphi + \Psi)^2 = 0.$$

By noting that second-order μ-inhomogeneous terms come from $\mathcal{T}\Psi$ and $(\mathrm{Id} - \mathcal{T})(\mathcal{Q}_1\varphi)^2$, the linear equations from grouping A^2, B^2, C^2, D^2, AB, AC, AD, BC, BD and CD terms are given by

$$\begin{aligned}
&\mathcal{T}\Psi_{20000} + (\mathrm{Id} - \mathcal{T})1 = 0, && \mathcal{T}\Psi_{00200} + (\mathrm{Id} - \mathcal{T})\cos^2(k_0x) = 0,\\
&\mathcal{T}\Psi_{02000} + (\mathrm{Id} - \mathcal{T})\,x^2 = 0, && \mathcal{T}\Psi_{00020} + (\mathrm{Id} - \mathcal{T})\sin^2(k_0x) = 0,\\
&\mathcal{T}\Psi_{11000} + 2(\mathrm{Id} - \mathcal{T})\,x = 0, && \mathcal{T}\Psi_{10100} + 2(\mathrm{Id} - \mathcal{T})\cos(k_0x) = 0,\\
&\mathcal{T}\Psi_{10010} + 2(\mathrm{Id} - \mathcal{T})\sin(k_0x) = 0, && \mathcal{T}\Psi_{01100} + 2(\mathrm{Id} - \mathcal{T})\,x\cos(k_0x) = 0,\\
&\mathcal{T}\Psi_{00110} + (\mathrm{Id} - \mathcal{T})\sin(2k_0x) = 0, && \mathcal{T}\Psi_{01010} + 2(\mathrm{Id} - \mathcal{T})\,x\sin(k_0x) = 0,
\end{aligned}$$

and, by noting that the μ-homogeneous terms come from $\mathcal{T}\Psi$ and $-\mu\mathcal{Q}_1\varphi$,

$$\begin{aligned}
&\mathcal{T}\Psi_{10001} - 1 = 0, && \mathcal{T}\Psi_{00101} - \cos(k_0x) = 0,\\
&\mathcal{T}\Psi_{01001} - x = 0, && \mathcal{T}\Psi_{00011} - \sin(k_0x) = 0.
\end{aligned}$$

Note that equations arising from grouping $AB, AD, BC, CD, \mu B$ and μD terms are excluded here since they involve only odd functions. Using (40) with $\mathbf{m} = \ell$ and $y = 0, k_0$ or $2k_0$, we arrive at

$$\begin{aligned}
&\mathcal{T}\Psi_{10001} = 1, && \mathcal{T}\Psi_{00101} = \cos(k_0x),\\
&\mathcal{T}\Psi_{20000} = -1, && \mathcal{T}\Psi_{10100} = -2\cos(k_0x),\\
&\mathcal{T}\Psi_{02000} = -x^2 + \ell''(0) && \mathcal{T}\Psi_{00200} = -\tfrac{1}{2} - \tfrac{1}{2}\ell(2k_0)\cos(2k_0x),\\
&\mathcal{T}\Psi_{01010} = -2x\sin(k_0x) + 2\ell'(k_0)\cos(k_0x), && \mathcal{T}\Psi_{00020} = -\tfrac{1}{2} + \tfrac{1}{2}\ell(2k_0)\sin(2k_0x),
\end{aligned}$$

all subjected to the condition $\mathcal{Q}_1\Psi_{pqlmn} = 0$, which ensures uniqueness.

Let $\sigma = \ell''(0)^{-1} = (1/3 - \tau)^{-1}$. Lengthy but straightforward calculations employing (40) with $\mathbf{m} = 1 - \ell$ now yield

$$\psi_{10001} = -\psi_{20000} = 2\sigma k_0^2, \qquad \psi_{00101} = -\tfrac{1}{2}\psi_{10100} = -\frac{2k_0^3}{\ell'(k_0)},$$

$$\psi_{02000} = -\frac{\ell''''(0) - 6\sigma^{-2}}{3}\sigma^2 k_0^2 - 4\sigma, \quad \psi_{01010} = 2\frac{\ell''(k_0) - 2\ell'(k_0)^2}{\ell'(k_0)^2}k_0^3 - \frac{10}{\ell'(k_0)}k_0^2,$$

$$\psi_{00200} = \frac{8\ell(2k_0)}{\ell(2k_0) - 1}k_0^4 - \sigma k_0^2, \qquad \psi_{00020} = -\frac{8\ell(2k_0)}{\ell(2k_0) - 1}k_0^4 - \sigma k_0^2.$$

C.2 For modulated solitary waves Now, let τ, c_0 satisfy (9) and note in this case that both c_0 and τ_0 are parametrized by $s \in (0, \infty)$. The linear equations for the center manifold coefficients are

$$\begin{aligned}
\mathcal{T}\Psi_{10001} &= \frac{1}{c_0}(\mathrm{Id} - \mathcal{T})\cos(sx),\\
\mathcal{T}\Psi_{20000} &= -\frac{1}{c_0}(\mathrm{Id} - \mathcal{T})\cos^2(sx),\\
\mathcal{T}\Psi_{10100} &= -\frac{1}{c_0}(\mathrm{Id} - \mathcal{T})\sin(2sx),\\
\mathcal{T}\Psi_{00200} &= -\frac{1}{c_0}(\mathrm{Id} - \mathcal{T})\sin^2(sx),\\
\mathcal{T}\Psi_{10010} = \mathcal{T}\Psi_{01100} &= -\frac{1}{c_0}(\mathrm{Id} - \mathcal{T})x\sin(2sx),\\
\mathcal{T}\Psi_{30000} &= -\frac{2}{c_0}(\mathrm{Id} - \mathcal{T})\cos(sx)\Psi_{20000},\\
\mathcal{T}\Psi_{10200} &= -\frac{2}{c_0}(\mathrm{Id} - \mathcal{T})(\cos(sx)\Psi_{00200} + \sin(sx)\Psi_{10100}),
\end{aligned}$$

where all Ψ_{pqlmn} are subject to $\mathcal{Q}_2\Psi_{pqlmn} = 0$ and $\mathcal{Q}_2$ is given in (18). Using (40) with $\mathbf{m} = 1 - c_0\ell, y = 0, s, 2s$ or $3s$ again, then evaluating $\Psi''''_{pqlmn}(0) = \psi_{pqlmn}$ gives

$$\begin{aligned}
\psi_{10001} &= -8s^2 e,\\
\psi_{20000} &= s^4(a + 9b),\\
\psi_{00200} &= s^4(a - 9b),\\
\psi_{10010} = \psi_{01100} &= 9s^4 c - 48s^3 b,\\
\psi_{30000} &= \Big(\big(-2a(a+b) - 18b(a+b) + 128bd - \frac{9s}{2}(a - 3b)c\big)s^2\\
&\qquad + 24s^2 b(a - 3b) + 8(2a + b)e\Big)s^2,
\end{aligned}$$

and

$$\psi_{10200} = \Big(\big(-2a(a-b) - 18b(a-b) - 128bd - \frac{9s}{2}(a+3b)c\big)s^2 \\ + 24s^2 b(a+3b) + 8(2a-b)e\Big)s^2 \\ + \Big(\big(-54sbc + 4ab - 36b^2 - 256bs\big)s^2 + 288s^2b^2 + 16be\Big)s^2,$$

with

$$a = \frac{1}{2c_0}\left(1 - \frac{1}{1-c_0}\right), \quad b = \frac{1}{2c_0}\left(1 - \frac{1}{1-c_0\ell(2s)}\right),$$
$$c = \frac{\ell'(2s)}{(1-c_0\ell(2s))^2}, \qquad d = \frac{1}{2c_0}\left(1 - \frac{1}{1-c_0\ell(3s)}\right),$$

and finally

$$e = \frac{1}{c_0^2\ell''(s)}.$$

D Coefficients in normal form reduction

D.1 For generalized solitary waves Our goal here is to compute coefficients p_0, p_1, p_2, p_3, q_0 and q_1 in Section 5.2. The expansion of Φ in $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}})$ and μ up to second-order terms is

$$\Phi(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}}, \mu) \\ = \phi_{00001}\mu + \phi_{10001}\mu\mathbf{A} + \phi_{01001}\mu\mathbf{B} + \phi_{00101}\mu\mathbf{C} + \phi_{00011}\mu\overline{\mathbf{C}} + \phi_{20000}\mathbf{A}^2 \\ + \phi_{11000}\mathbf{AB} + \phi_{10100}\mathbf{AC} + \phi_{10010}\mathbf{A}\overline{\mathbf{C}} + \phi_{02000}\mathbf{B}^2 + \phi_{01100}\mathbf{BC} + \phi_{01010}\mathbf{B}\overline{\mathbf{C}} \\ + \phi_{00200}\mathbf{C}^2 + \phi_{00110}|\mathbf{C}|^2 + \phi_{00020}\overline{\mathbf{C}}^2 + \mathcal{O}(|\mu^2| + (|\mu| + |(\mathbf{A}, \mathbf{B}, \mathbf{C}, \overline{\mathbf{C}})|)^3)).$$

Denote the coefficients in front of $A^pB^qC^lD^m\mu^n$ in (21) by ψ_{pqlmn}. We also Taylor expand the nonlinear term

$$\mathbf{R}(U,\mu) = \mu\mathbf{R}_{11}(U) + \mathbf{R}_{20}(U,U) + \mathcal{O}(|\mu|^2 + |(U,\mu)|^3),$$

where

$$\mathbf{R}_{11}(x,y,z,w) = \frac{1}{k_0^3}\begin{pmatrix} 0 \\ k_0\psi_{10001}x + k_0\psi_{00101}z \\ 0 \\ -\psi_{10001}x - \psi_{00101}z \end{pmatrix}, \ \mathbf{R}_{20}(U,\tilde{U}) = \frac{1}{k_0^3}\begin{pmatrix} 0 \\ k_0H(U,\tilde{U}) \\ 0 \\ -H(U,\tilde{U}) \end{pmatrix}$$

with

$$
\begin{aligned}
H(U,\tilde{U}) &= H((x,y,z,w),(\tilde{x},\tilde{y},\tilde{z},\tilde{w})) \\
&= \psi_{20000}x\tilde{x} + \frac{\psi_{10100}}{2}(x\tilde{z}+z\tilde{x}) + \psi_{02000}y\tilde{y} \\
&\quad + \frac{\psi_{01010}}{2}(y\tilde{w}+w\tilde{y}) + \psi_{00200}z\tilde{z} + \psi_{00020}w\tilde{w}.
\end{aligned}
$$

Plugging $U = \mathbf{A}\xi_0 + \mathbf{B}\xi_1 + \mathbf{C}\zeta + \overline{\mathbf{C}\zeta} + \Phi$ into (23), relevant linear equations are identified

$$
\begin{aligned}
\mathcal{O}(\mu):&\quad p_0\xi_1 = \mathbf{L}\phi_{00001} \\
\mathcal{O}(\mu\mathbf{A}):&\quad p_1\xi_1 + p_0\phi_{11000} = \mathbf{L}\phi_{10001} + \mathbf{R}_{11}(\xi_0) \\
\mathcal{O}(\mathbf{A}^2):&\quad p_2\xi_1 = \mathbf{L}\phi_{20000} + \mathbf{R}_{20}(\xi_0,\xi_0) \\
\mathcal{O}(|\mathbf{C}|^2):&\quad p_3\xi_1 = \mathbf{L}\phi_{00110} + 2\mathbf{R}_{20}(\zeta,\overline{\zeta}) \\
\mathcal{O}(\mu\mathbf{C}):&\quad \mathrm{i}q_0\zeta + p_0\phi_{01100} + \mathrm{i}k_0\phi_{00101} = \mathbf{L}\phi_{00101} + \mathbf{R}_{11}(\zeta) \\
\mathcal{O}(\mathbf{AC}):&\quad \mathrm{i}q_1\zeta + \mathrm{i}k_0\phi_{10100} = \mathbf{L}\phi_{10100} + 2\mathbf{R}_{20}(\xi_0,\zeta).
\end{aligned}
$$

Since ξ_1 is not in the range of $\mathbf{L}$, $p_0 = 0$. Similarly, the equation from $\mathcal{O}(\mu\mathbf{A})$ terms

$$
\mathbf{L}\phi_{10001} = p_1\xi_1 - \mathbf{R}_{11}(\xi_0) = \frac{1}{k_0^3}\begin{pmatrix} 0 \\ k_0^3 p_1 - k_0\psi_{10001} \\ 0 \\ \psi_{10001} \end{pmatrix}
$$

is solvable if and only if $k_0^3 p_1 - k_0\psi_{10001} = 0$. Equations for p_2 and p_3 are handled in the same fashion. To solve for q_0, we note that

$$
(\mathbf{L} - \mathrm{i}k_0)\phi_{00101} = \mathrm{i}q_0\zeta - \mathbf{R}_{11}(\zeta)
$$

Writing $\phi_{00101} = x_0\xi_0 + x_1\xi_1 + x_3\zeta + \overline{x_3\zeta}$, equation (24) can be used to show that $x_3\zeta = 0$ on the left-hand side. The right-hand side in the basis $\{\xi_0, \xi_1, \zeta, \overline{\zeta}\}$ is

$$
\mathrm{i}q_0\zeta - \mathbf{R}_{11}(\zeta) = -\frac{1}{k_0^2}\psi_{00101}\xi_1 + \left(\mathrm{i}q_0 - \frac{\mathrm{i}}{2k_0^3}\psi_{00101}\right)\zeta + \frac{\mathrm{i}}{2k_0^3}\overline{\zeta},
$$

which gives $q_0 = (2k_0^3)^{-1}\psi_{00101}$. Using results from Appendix C.1 and writing $\sigma = (1/3-\tau)^{-1}$, we get

$$
p_0 = 0,\ \ p_1 = 2\sigma = -p_2,\ \ p_3 = -4\sigma,\ \ q_0 = -\frac{1}{\ell'(k_0)}\ \text{ and }\ q_1 = \frac{2}{\ell'(k_0)}.
$$

D.2 For modulated solitary waves As before, the Taylor expansion of Φ is

$$\Phi(\mathbf{A}, \mathbf{B}, \overline{\mathbf{A}}, \overline{\mathbf{B}}, \mu) = \sum_{\substack{2 \le p+q+l+m+n \le 3 \\ n \ge 1}} \phi_{pqlmn} \mathbf{A}^p \mathbf{B}^q \overline{\mathbf{A}}^l \overline{\mathbf{B}}^m \mu^n + \cdots .$$

Our goal here is to determine the Taylor expansion of $\mathbf{R}(U, \mu)$ from (34) up to order three. Using the symmetries as in the proof of Proposition 5.1, contributing terms are

$$\begin{aligned} &\mu A, \mu D, \mu^2 A, \mu^2 D, \\ &A^2, B^2, C^2, D^2, AD, BC, \mu A^2, \mu B^2, \mu C^2, \mu D^2, \mu AD, \mu BC, \\ &A^3, D^3, A^2 D, AB^2, AC^2, AD^2, ABC, B^2 D, BCD, C^2 D. \end{aligned}$$

In short, if a multiplication between a pair or a triple of $\cos(sx)$, $x\cos(sx)$, $\sin(sx)$, $x\sin(sx)$ is an even function in x, the product of their coefficients A, B, C, D will contribute to $\Psi(A, B, C, D, \mu)''''(0)$. Denote the coefficients of $A^p B^q C^l D^m \mu^n$ by ψ_{pqlmn}. The Taylor expansion of $\mathbf{R}(U, \mu)$ is

$$\begin{aligned} \mathbf{R}(U, \mu) = \mathbf{R}_{00} + \mathbf{R}_{01}\mu + \mathbf{R}_{10}U + \mathbf{R}_{11}\mu U + \mathbf{R}_{20}(U, U) \\ + \mu \mathbf{R}_{21}(U, U) + \mu^2 \mathbf{R}_{12} U + \mathbf{R}_{30}(U, U, U), \end{aligned}$$

where relevant terms for us are

$$\mathbf{R}_{01} = \frac{1}{2s^3} \begin{pmatrix} 0 \\ -sH_0 \\ H_0 \\ 0 \end{pmatrix}, \qquad \mathbf{R}_{11}(U) = \frac{1}{2s^3} \begin{pmatrix} 0 \\ -sH_1(U) \\ H_1(U) \\ 0 \end{pmatrix},$$

$$\mathbf{R}_{20}(U, \tilde{U}) = \frac{1}{2s^3} \begin{pmatrix} 0 \\ -sH_2(U, \tilde{U}) \\ H_2(U, \tilde{U}) \\ 0 \end{pmatrix}, \quad \mathbf{R}_{30}(U, \tilde{U}, \hat{U}) = \frac{1}{2s^3} \begin{pmatrix} 0 \\ -sH_3(U, \tilde{U}, \hat{U}) \\ H_3(U, \tilde{U}, \hat{U}) \\ 0 \end{pmatrix},$$

with $U = (x, y, z, w), \tilde{U} = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}), \hat{U} = (\hat{x}, \hat{y}, \hat{z}, \hat{w})$,

$$\begin{aligned} &H_0 = 0, \\ &H_1(x, y, z, w) \\ &\quad = \psi_{10001} x + \psi_{00011} w, \\ &H_2((x, y, z, w), (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w})) \\ &\quad = \psi_{20000} x\tilde{x} + \psi_{02000} y\tilde{y} + \psi_{00200} z\tilde{z} + \psi_{00020} w\tilde{w} \\ &\qquad + \frac{\psi_{10010}}{2}(x\tilde{w} + \tilde{x}w) + \frac{\psi_{01100}}{2}(y\tilde{z} + \tilde{y}z), \end{aligned}$$

and

$$\begin{aligned}
&H_3((x,y,z,w),(\tilde{x},\tilde{y},\tilde{z},\tilde{w}),(\hat{x},\hat{y},\hat{z},\hat{w}))\\
&\quad=\psi_{30000}x\tilde{x}\hat{x}+\psi_{00030}w\tilde{w}\hat{w}\\
&\qquad+\frac{\psi_{20010}}{3}(x\tilde{x}\hat{w}+x\hat{x}\tilde{w}+\tilde{x}\hat{x}w)+\frac{\psi_{12000}}{3}(x\tilde{y}\hat{y}+\tilde{x}y\hat{y}+\hat{x}y\tilde{y})\\
&\qquad+\frac{\psi_{10200}}{3}(x\tilde{z}\hat{z}+\tilde{x}z\hat{z}+\hat{x}z\tilde{z})+\frac{\psi_{10020}}{3}(x\tilde{w}\hat{w}+\tilde{x}w\hat{w}+\hat{x}w\tilde{w})\\
&\qquad+\frac{\psi_{11100}}{6}(x\tilde{y}\hat{z}+x\hat{y}\tilde{z}+\tilde{x}y\hat{z}+\tilde{x}\hat{y}z+\hat{x}y\tilde{z}+\hat{x}\tilde{y}z)\\
&\qquad+\frac{\psi_{01110}}{6}(y\tilde{z}\hat{w}+y\hat{z}\tilde{w}+\tilde{y}z\hat{w}+\tilde{y}\hat{z}w+\hat{y}z\tilde{w}+\hat{y}\tilde{z}w)\\
&\qquad+\frac{\psi_{02010}}{3}(y\tilde{y}\hat{w}+y\hat{y}\tilde{w}+\tilde{y}\hat{y}w)+\frac{\psi_{00210}}{3}(z\tilde{z}\hat{w}+z\hat{z}\tilde{w}+\tilde{z}\hat{z}w).
\end{aligned}$$

Let $\zeta_1^*=\frac{1}{2}(0,1,0,-\mathrm{i})^{\mathrm{T}}$. It is a vector orthogonal to the range of $\mathbf{L}-\mathrm{i}s$ and satisfies

$$\langle\zeta_0,\zeta_1^*\rangle=0,\quad\langle\overline{\zeta_0},\zeta_1^*\rangle=0,\quad\langle\zeta_1,\zeta_1^*\rangle=0,\quad\langle\overline{\zeta_1},\zeta_1^*\rangle=0,$$

and $R\zeta_1^*=-\overline{\zeta_1^*}$. Equations (D.45) and (D.47), [19, Appendix D.2], give

$$\begin{aligned}
q_0&=\langle\mathbf{R}_{11}\zeta_0+2\mathbf{R}_{20}(\zeta_0,\phi_{00001}),\zeta_1^*\rangle,\\
q_1&=\langle 2\mathbf{R}_{20}(\zeta_0,\phi_{10100})+2\mathbf{R}_{20}(\overline{\zeta_0},\phi_{20000})+3\mathbf{R}_{30}(\zeta_0,\zeta_0,\overline{\zeta_0}),\zeta_1^*\rangle,
\end{aligned}$$

respectively. Here, $\phi_{00001},\phi_{10100},\phi_{20000}$ satisfy

$$\begin{aligned}
&\mathbf{L}\phi_{00001}+\mathbf{R}_{01}=0,\\
&\mathbf{L}\phi_{10100}+2\mathbf{R}_{20}(\zeta_0,\overline{\zeta_0})=0,\\
&(\mathbf{L}-2\mathrm{i}s)\phi_{20000}+\mathbf{R}_{20}(\zeta_0,\zeta_0)=0.
\end{aligned}$$

A computation gives

$$\phi_{00001}=0,\ \phi_{10100}=\frac{\psi_{20000}+\psi_{02000}}{s^3}\begin{pmatrix}2/s\\0\\0\\1\end{pmatrix},\ \phi_{20000}=\frac{\psi_{20000}-\psi_{02000}}{9s^4}\begin{pmatrix}1\\3\mathrm{i}s\\-\mathrm{i}\\-3s/2\end{pmatrix},$$

which in turn yields

$$\begin{aligned}
q_0&=-\frac{1}{4s^2}\psi_{10001},\\
q_1&=-\frac{\psi_{20000}+\psi_{00200}}{2s^5}\left(\frac{2}{s}\psi_{20000}+\frac{\psi_{10010}}{2}\right)\\
&\quad-\frac{1}{18s^6}(\psi_{20000}-\psi_{00200})\left(\psi_{20000}-\psi_{00200}+\frac{3s}{2}\psi_{01100}-\frac{3s}{4}\psi_{01100}\right)\\
&\quad-\frac{3}{4s^2}\left(\psi_{30000}+\frac{\psi_{10200}}{3}\right).
\end{aligned}$$

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Paper III

Transverse dynamics of two-dimensional traveling periodic gravity–capillary water waves

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Abstract

We study the transverse dynamics of two-dimensional traveling periodic waves for the gravity–capillary water-wave problem. The governing equations are the Euler equations for the irrotational flow of an inviscid fluid layer with free surface under the forces of gravity and surface tension. We focus on two open sets of dimensionless parameters (α, β), where α and β are the inverse square of the Froude number and the Weber number, respectively. For each arbitrary but fixed pair (α, β) in one of these sets, two-dimensional traveling periodic waves bifurcate from the trivial constant flow. In one open set we find a one-parameter family of periodic waves, whereas in the other open set we find two geometrically distinct one-parameter families of periodic waves. Starting from a transverse spatial dynamics formulation of the governing equations, we investigate the transverse linear instability of these periodic waves and the induced dimension-breaking bifurcation. The two results share a common analysis of the purely imaginary spectrum of the linearization at a periodic wave. We apply a simple general criterion for the transverse linear instability problem and a Lyapunov center theorem for the dimension-breaking bifurcation. For parameters (α, β) in the open set where there is only one family of periodic waves, we prove that these waves are linearly transversely unstable. For the other open set, we show that the waves with larger wavenumber are transversely linearly unstable. We identify an open subset of parameters for which both families of periodic waves are tranversely linearly unstable. For each of these transversely linearly unstable periodic waves, a dimension-breaking bifurcation occurs in which three-dimensional doubly periodic waves bifurcate from the two-dimensional periodic wave.

Keywords: Gravity–capillary water waves, periodic waves, transverse linear stability, dimension-breaking bifurcation.

1 Introduction

We consider a three-dimensional inviscid fluid with constant density ρ occupying a region

$$D_\eta = \{(X, Y, z) \in \mathbb{R}^3 \,:\, 0 < Y < h + \eta(X, z, t)\},$$

where (X, Y, z) are Cartesian coordinates, h is the mean fluid depth, and $\eta > -h$ is the unknown free surface of the fluid depending on the horizontal spatial variables X, z and the time variable t. The fluid is under the influence of the gravitational force with acceleration constant g and surface tension with coefficient T. We assume that the flow is irrotational and denote by ϕ an Eulerian velocity potential. Choosing a coordinate frame moving from left to right along the X-axis with constant velocity $c > 0$, the fluid motion is described by Laplace's equation

$$\phi_{XX} + \phi_{YY} + \phi_{zz} = 0 \quad \text{for} \quad 0 < Y < 1 + \eta, \tag{1}$$

with boundary conditions

$$\begin{aligned} &\phi_Y = 0 && \text{on } Y = 0, \\ &\phi_Y = \eta_t - c\eta_X + \eta_X\phi_X + \eta_z\phi_z && \text{on } Y = 1 + \eta, \\ &\phi_t - c\phi_X + \frac{1}{2}\left(\phi_X^2 + \phi_Y^2 + \phi_z^2\right) + \alpha\eta - \beta\mathcal{K} = 0 && \text{on } Y = 1 + \eta. \end{aligned} \tag{2}$$

Here, we have used dimensionless variables by taking the characteristic length scale h and characteristic time scale h/c. The dimensionless parameters

$$\alpha = \frac{gh}{c^2} \quad \text{and} \quad \beta = \frac{T}{\rho h c^2}$$

are the inverse square of the Froude number and the Weber number, respectively, and the quantity $\mathcal{K}$ is twice the mean curvature of the free surface η, given by

$$\mathcal{K} = \left[\frac{\eta_X}{\sqrt{1 + \eta_X^2 + \eta_z^2}}\right]_X + \left[\frac{\eta_z}{\sqrt{1 + \eta_X^2 + \eta_z^2}}\right]_z.$$

The set of equations (1)–(2) are the Euler equations for gravity–capillary waves on water of finite depth. The case $\beta = 0$, that we do not consider in this work, corresponds to gravity water waves.

We are interested in the transverse dynamics of two-dimensional traveling periodic waves. In the above formulation these are steady solutions which are periodic in X and do not depend on the second horizontal coordinate z and on the time t. Their existence is well-recorded in the literature; e.g., see [7, 12, 13, 22] and the references therein. Many of these results are obtained using methods from bifurcation theory. Bifurcations of two-dimensional periodic waves are determined by the positive roots of the linear dispersion relation

$$\mathcal{D}(k) := (\alpha + \beta k^2)\sinh|k| - |k|\cosh(k) = 0, \tag{3}$$

obtained by looking for nontrivial solutions to the steady system (1)–(2) linearized at 0 of the form $(\eta(X), \phi(X, Y)) = (\eta_k, \phi_k(Y))\exp(\mathrm{i}kX)$. Associated to any positive root

k of the linear dispersion relation, one finds a one parameter family of periodic waves $\{(\eta_\varepsilon(X), \phi_\varepsilon(X,Y))\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ with wavenumbers close to k, for sufficiently small $\varepsilon_0 > 0$. These periodic waves bifurcate from the trivial solution $(\eta_0(X), \phi_0(X,Y)) = (0,0)$.

Depending on the values of the two parameters α and β, the linear dispersion relation (3) possesses positive roots in the following cases:

1. one positive simple root $k_* > 0$ if $\alpha \in (0,1)$ and $\beta > 0$; we refer to this set of parameters as Region I.

2. two positive simple roots $0 < k_{*,1} < k_{*,2}$ if $\alpha > 1$ and $0 < \beta < \beta(\alpha)$, where $(\alpha, \beta(\alpha))$ belongs to the curve Γ with parametric equations

$$\alpha = \frac{s^2}{2\sinh^2(s)} + \frac{s}{2\tanh(s)}, \quad \beta = -\frac{1}{2\sinh^2(s)} + \frac{1}{2s\tanh(s)}, \quad s \in (0,\infty); \tag{4}$$

   we refer to this set of parameters as Region II;

3. one positive simple root $k_* > 0$ if $\alpha = 1$ and $\beta < 1/3$;

4. one positive double root k_* if (α, β) belongs to the curve Γ given in (4).

The linear dispersion relation being even in k, together with any positive root k we also find the negative root $-k$. We illustrate these properties in the left panel of Figure 1.

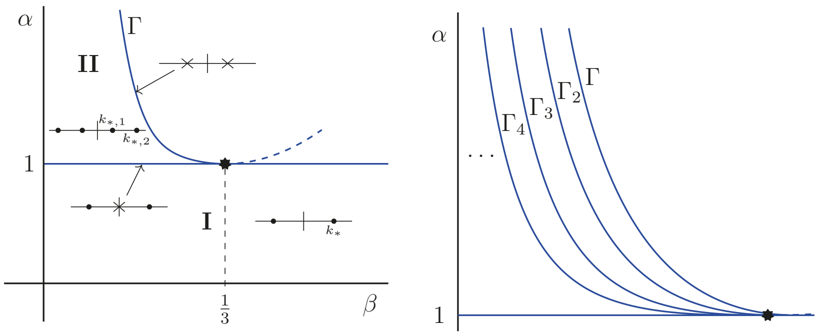

Figure 1: Left: In the (β, α)-plane, sketch of the nonzero roots of the linear dispersion relation (3). We use dots to indicate simple roots and crosses to indicate double roots. Right: In Region II, plot of the curves Γ_m for $m = 2, 3, 4$ which are excluded from our analysis.

Here, we focus on the periodic waves which bifurcate in the two open parameter regions I and II. For simplicity, in Region II we assume that $k_{*,1}$ and $k_{*,2}$ satisfy the non-resonance condition $k_{*,2}/k_{*,1} \notin \mathbb{N}$. This assumption means that (α,β) does not belong to any of the curves Γ_m for $m \in \mathbb{N}$, $m \geq 2$, with parametric equations

$$\begin{aligned} \alpha &= -\frac{m^2 s}{(1-m^2)\tanh(s)} + \frac{ms}{(1-m^2)\tanh(ms)} \\ \beta &= \frac{1}{(1-m^2)s\tanh(s)} - \frac{m}{(1-m^2)s\tanh(ms)} \end{aligned}, \quad s \in (0,\infty);$$

see the right panel in Figure 1. Then, for any fixed (α,β) in Region I, there is a one parameter family of two-dimensional periodic waves $\{(\eta_\varepsilon(X), \phi_\varepsilon(X,Y))\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ with wavenumbers close to k_*, whereas for (α,β) in Region II, there are two geometrically distinct families of periodic waves

$$\{(\eta_{\varepsilon,1}(X), \phi_{\varepsilon,1}(X,Y))\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)} \text{ and } \{(\eta_{\varepsilon,2}(X), \phi_{\varepsilon,2}(X,Y))\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$$

with wavenumbers close to $k_{*,1}$ and $k_{*,2}$, respectively.

The purpose of our transverse dynamics analysis is twofold: to identify the periodic waves in regions I and II which are transversely linearly unstable and to discuss the induced dimension-breaking bifurcations. Roughly speaking, a two-dimensional wave is transversely linearly unstable if the Euler equations (1)–(2) linearized at the wave possess solutions which are bounded in the horizontal coordinates (X,z) and exponentially growing in time t. The dimension-breaking bifurcation is the bifurcation of three-dimensional solutions emerging from the two-dimensional transversely unstable wave. Typically, these three-dimensional solutions are periodic in the transverse horizontal coordinate z; see Figure 2 for an illustration in the case of a two-dimensional periodic wave. Though of different type, these two questions share a common spectral analysis of the linear operator at the two-dimensional wave. This is the key, and most challenging, part of our analysis.

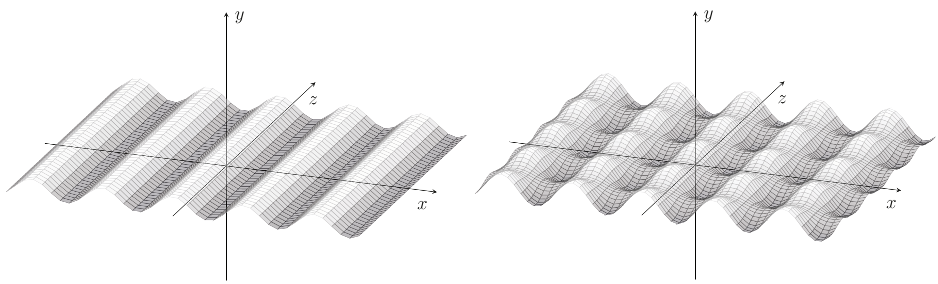

Figure 2: Illustration of a dimension-breaking bifurcation. Left: plot of a two-dimensional periodic wave. Right: plot of a bifurcating three-dimensional doubly periodic wave.

The transverse stability of periodic waves was mostly studied for simpler model equations obtained from the Euler equations (1)–(2) in different parameter regimes: the Kadomtsev–Petviashvili-I equation for the regime of large surface tension ($\alpha \sim 1,\ \beta > 1/3$) was considered in [19, 30, 18], the Davey–Stewartson system for the regime of weak surface tension ((α, β) close to the curve Γ) in [10], and a fifth order KP equation for the regime of critical surface tension ($\alpha \sim 1,\ \beta \sim 1/3$) in [24]; see also the recent review paper [21]. All these results predict that gravity–capillary periodic waves are linearly transversely unstable. We point out that pure gravity periodic, or solitary, water waves ($\beta = 0$) are expected to be linearly transversely stable [1, 23].

For the Euler equations, previous works on transverse instability mostly treat the case of solitary waves; see [14, 40, 41] for the large surface regime and the more recent work [17] for the weak surface tension regime close to the curve Γ. In both regimes, the dimension-breaking bifurcation has been studied in [15] (large surface tension) and [17] (weak surface tension). For periodic waves, the transverse instability predicted in the regime of large surface tension ($\alpha \sim 1,\ \beta > 1/3$) has been confirmed in [20]. In addition, the dimension-breaking bifurcation was studied showing the bifurcation of a one parameter family of three-dimensional doubly periodic waves, as illustrated in Figure 2. In the present work, we treat these two questions for the periodic waves bifurcating in the open parameter regions I and II.

For completeness, we mention that there are other stability/instability results for these periodic waves. When the perturbations are constant in z, the references [34, 8, 9] through formal expansions have provided a characterization for the Benjamin–Feir instability[1], the work [6] demonstrates numerically that periodic waves are sometimes spectrally unstable even when the Benjamin–Feir instability is not present, references [37, 38, 39, 31, 32, 42] indicate through both numerical and theoretical investigations that harmonic resonances feature even more intriguing instability phenomena, e.g. nested instabilities or multiple high-frequency instability bubbles. The works [27, 28, 29] approach these questions through their own proposals of fully dispersive model equations. In particular, [28, 29] find qualitatively the same instability characterization for periodic waves in their models as [34, 8]. Instability under three-dimensional perturbations has been considered numerically, experimentally and using various model equations, such as the Davey–Stewartson equation [5, 8, 25, 26, 44]. In particular, the instability criterion that we arrive at here can be formally obtained by taking $l = 0$ in equation (3.9) in [5], and using the formulas for the coefficients for gravity–capillary waves in [8]. Note that in contrast to the previous studies, we restrict our attention to perturbations which have the same wavelength as the periodic wave in the X-direction.

Our approach to transverse dynamics follows the ideas developed for solitary waves in [14, 15, 17]. The starting point of the analysis is a spatial dynamics formulation of the

[1]This is linear instability with respect to sideband perturbations, which has a different period than that of the main periodic wave. It is first discovered for gravity waves in deep fluids by Benjamin & Feir [4], Benjamin [3] and independently by Whitham [43].

three-dimensional, time-dependent equations (1)–(2) in which the horizontal coordinate z, transverse to the direction of propagation, plays the role of time.

For the transverse linear instability problem, we consider the linearization of this dynamical system at a two-dimensional periodic wave and apply a simple general instability criterion [10] adapted to the Euler equations in [17]. In Region I, we show that the periodic waves $\{(\eta_\varepsilon(X), \phi_\varepsilon(X,Y))\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ are transversely linearly unstable, provided ε_0 is sufficiently small. In Region II, we obtain transverse linear instability for the periodic waves $\{(\eta_{\varepsilon,2}(X), \phi_{\varepsilon,2}(X,Y))\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ with wavenumbers close to the largest root $k_{*,2}$ of the linear dispersion relation. For the second family of periodic waves, $\{(\eta_{\varepsilon,1}(X), \phi_{\varepsilon,1}(X,Y))\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ with wavenumbers close to $k_{*,1}$, we can only conclude on transverse instability for parameter values (α,β) situated in the open region between the curves Γ and Γ_2; see the right panel in Figure 1. The dimension-breaking bifurcation is studied for the transversely linearly unstable periodic waves. Here, we use the time-independent, but nonlinear, version of the dynamical system above. Applying a Lyapunov center theorem, we prove that from each unstable periodic wave bifurcates a family of doubly periodic waves.

The common part of the proofs of these two results is the analysis of the purely imaginary spectrum of the linearized operator at the two-dimensional periodic wave. This analysis is the major part of our work. Our main result shows that this linear operator possesses precisely one pair of simple nonzero purely imaginary eigenvalues. Though it relies upon standard perturbation arguments for linear operators, the proof is rather long because of the complicated formulas for the linear operator. This spectral result is the key property allowing to apply both the transverse instability criterion and the Lyapunov center theorem.

In the following theorem, we summarize the results obtained for Region I.

Theorem 1.1 (Region I) *Fix (α,β) in Region I and let $k_* > 0$ be the unique positive root of the linear dispersion relation* (3).

(i) (Existence) There exist $\varepsilon_0 > 0$ and a one-parameter family of two-dimensional steady solutions $\{(\eta_\varepsilon(X), \phi_\varepsilon(X,Y))\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ to equations (1)–(2)*, such that $(\eta_0,\phi_0) = (0,0)$ and $(\eta_\varepsilon,\phi_\varepsilon)$ are periodic in X with wavenumber $k_\varepsilon = k_* + \mathcal{O}(\varepsilon^2)$.*

(ii) (Transverse instability) There exists $\varepsilon_1 > 0$ such that for each $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ the periodic solution $(\eta_\varepsilon(X), \phi_\varepsilon(X,Y))$ is transversely linearly unstable.

(iii) (Dimension-breaking bifurcation) There exists $\varepsilon_2 > 0$, such that for each $\varepsilon \in (-\varepsilon_2, \varepsilon_2)$ there exist $\delta_\varepsilon > 0$, $\ell_\varepsilon^* > 0$, and a one-parameter family of three-dimensional doubly periodic waves $\{(\eta_\varepsilon^\delta(X,z), \phi_\varepsilon^\delta(X,Y,z))\}_{\delta\in(-\delta_\varepsilon,\delta_\varepsilon)}$, with wavenumber k_ε in X and wavenumber $\ell_\delta = \ell_\varepsilon^* + \mathcal{O}(\delta^2)$ in z, bifurcating from the periodic solution $(\eta_\varepsilon(X), \phi_\varepsilon(X,Y))$.

We point out that $\pm i\ell_\varepsilon^*$ where $\ell_\varepsilon^* > 0$ is given in Theorem 1.1(iii) are the two nonzero purely imaginary eigenvalues of the linearization at the periodic wave. The results found for Region II are summarized in Theorem 5.1 from Section 5.

In our presentation we focus on Region I, the arguments being, up to some computations, the same for Region II. In Section 2 we recall the spatial dynamics formulation of the three-dimensional time-dependent Euler equations (1)–(2) from [17] and the existence result for two-dimensional periodic waves given in Theorem 1.1(i). We also give some explicit expansions of these solutions which are computed in Appendix B. In Section 3 we prove the results for the linear operator. Some of the long computations needed here are given in Appendices C and D. In Section 4 we present the transverse dynamics results and in Section 5 we discuss the results for Region III. Finally, in Appendix A we recall an infinite-dimensional version of the Lyapunov center theorem.

2 Preliminaries

In this section, we recall the spatial dynamics formulation from [17] and the result on existence of two-dimensional steady periodic solutions.

2.1 Spatial dynamics formulation

Following [17], we make the change of variables

$$Y = y(1+\eta(X,z,t)), \quad \phi(X,Y,z,t) = \Phi(X,y,z,t),$$

in (1)–(2) to flatten the free surface. Since we consider periodic solutions, in addition, we set $X = kx$ with k the wavenumber in x. We introduce two new variables,

$$\begin{aligned} \omega &= -\int_0^1 \left(\Phi_z - \frac{y\eta_z\Phi_y}{1+\eta}\right) y\Phi_y \,\mathrm{d}y + \frac{\beta\eta_z}{(1+k^2\eta_x^2+\eta_z^2)^{1/2}}, \\ \xi &= (1+\eta)\left(\Phi_z - \frac{y\eta_z\Phi_y}{1+\eta}\right). \end{aligned}$$

and set $U = (\eta,\omega,\Phi,\xi)^{\mathrm{T}}$. Then, the equations (1)–(2) can be written as dynamical system of the form

$$\frac{\mathrm{d}U}{\mathrm{d}z} = DU_t + F(U), \tag{5}$$

with boundary conditions

$$\Phi_y = y\eta_t + B(U) \quad \text{on} \quad y = 0,1. \tag{6}$$

Here, D is the linear operator defined by

$$DU = (0, \Phi|_{y=1}, 0, 0)^{\mathrm{T}},$$

F is the nonlinear mapping $F(U) = (F_1(U), F_2(U), F_3(U), F_4(U))^{\mathrm{T}}$ given by

$$
\begin{aligned}
F_1(U) &= W\left(\frac{1+k^2\eta_x^2}{\beta^2 - W^2}\right)^{1/2}, \\
F_2(U) &= \frac{F_1(U)}{(1+\eta)^2}\int_0^1 y\Phi_y\xi\,\mathrm{d}y - k\left[k\eta_x\frac{W}{F_1(U)}\right]_x + \alpha\eta - k\Phi_x|_{y=1} \\
&\quad + \int_0^1 \left\{\frac{\xi^2-\Phi_y^2}{2(1+\eta)^2} + \frac{k^2}{2}\left(\Phi_x - \frac{y\eta_x\Phi_y}{1+\eta}\right)^2 + k^2\left[\left(\Phi_x - \frac{y\eta_x\Phi_y}{1+\eta}\right)y\Phi_y\right]_x \right. \\
&\quad \left. + k^2\left(\Phi_x - \frac{y\eta_x\Phi_y}{1+\eta}\right)\frac{y\eta_x\Phi_y}{1+\eta}\right\}\mathrm{d}y, \\
F_3(U) &= \frac{\xi}{1+\eta} + \frac{y\Phi_y}{1+\eta}F_1(U),
\end{aligned}
$$

and

$$
\begin{aligned}
F_4(U) &= -\frac{\Phi_{yy}}{1+\eta} - k^2\left[(1+\eta)\left(\Phi_x - \frac{y\eta_x\Phi_y}{1+\eta}\right)\right]_x \\
&\quad + k^2\left[\left(\Phi_x - \frac{y\eta_x\Phi_y}{1+\eta}\right)y\eta_x\right]_y + \frac{(y\xi)_y}{1+\eta}F_1(U),
\end{aligned}
$$

where

$$W = \omega + \frac{1}{1+\eta}\int_0^1 y\Phi_y\xi\,\mathrm{d}y,$$

and B is the nonlinear mapping defined by

$$B(U) = -ky\eta_x + k^2y\eta_x\Phi_x + \frac{\eta\Phi_y}{1+\eta} - \frac{k^2y^2\eta_x^2\Phi_y}{1+\eta} + \frac{y\xi}{1+\eta}F_1(U).$$

The choice of the function spaces is made precise later in Sections 3 and 4.

The system (5)–(6) inherits the symmetries of the Euler equations (1)–(2). As a consequence of the horizontal spatial reflection $z \mapsto -z$, the system (5)–(6) is reversible with reversibility symmetry R acting by

$$R\begin{pmatrix}\eta\\ \omega\\ \Phi\\ \xi\end{pmatrix}(x,y,z,t) = \begin{pmatrix}\eta\\ -\omega\\ \Phi\\ -\xi\end{pmatrix}(x,y,-z,t), \tag{7}$$

which anti-commutes with D and F and commutes with B. The second horizontal spatial reflection $x \mapsto -x$, implies that the system (5)–(6) possesses a reflection symmetry

$$S\begin{pmatrix}\eta\\ \omega\\ \Phi\\ \xi\end{pmatrix}(x,y,z,t) = \begin{pmatrix}\eta\\ \omega\\ -\Phi\\ -\xi\end{pmatrix}(-x,y,z,t), \tag{8}$$

which commutes with D, F, and B. There are in addition two continuous symmetries, which are the horizontal spatial translations in x and z.

2.2 Two-dimensional steady periodic waves Spatial dynamics also provides an efficient method for the study of the existence of two-dimensional steady waves of the Euler equations (1)–(2). This idea, which goes back to the work by Kirchgässner [35], consists in writing the two-dimensional steady Euler equations as a dynamical system of the form

$$\frac{\mathrm{d}U}{\mathrm{d}x} = LU + R(U), \tag{9}$$

in which x is now the timelike variable, and L and R denote linear and nonlinear parts, respectively. A phase space $\mathcal{X}$ consisting of y-dependent functions is chosen such that the linear operator L is closed with densely and compactly embedded domain $\mathcal{Y} \subset \mathcal{X}$. The first two boundary conditions in (2) are part of the definition of the domain $\mathcal{Y}$. There are several different such formulations of two-dimensional steady problem; see, for instance, [36, 22] for two different formulations as a reversible dynamical system, and [12] for a formulation as a Hamiltonian system. Two-dimensional steady water waves are bounded solutions of the dynamical system (9) and can be found using tools from the theory of dynamical systems and bifurcation theory.

In particular, periodic waves can be obtained by a direct application of the Lyapunov center theorem; see Theorem A.1. The key observation is that the purely imaginary eigenvalues of the operator L are given by the real roots of the linear dispersion relation (3). As shown, for instance in [36], for any pair of parameters (α, β) in Region I, the linear operator L possesses precisely one pair of simple purely imaginary eigenvalues $\pm \mathrm{i}k_*$, with k_* the unique positive root of the linear dispersion relation (3). Similarly, in Region II there are two pairs of simple purely imaginary eigenvalue $\pm \mathrm{i}k_{*,1}$ and $\pm \mathrm{i}k_{*,2}$. Then, the reversibility of the dynamical system (9) together with a direct check of the resolvent estimates (33) allow to apply Theorem A.1 and prove the result in Theorem 1.1(i) for Region I and the result in Theorem 5.1(i) for Region II.

In addition, for our purposes we need to compute the first two terms of the expansion in ε of the two-dimensional periodic solutions. For (α, β) in Region I, we write

$$X = k_\varepsilon x, \quad \eta_\varepsilon(X) = \widetilde{\eta}_\varepsilon(x), \quad \Phi_\varepsilon(X, y) = \widetilde{\Phi}_\varepsilon(x, y),$$

so that $\widetilde{\eta}_\varepsilon$ and $\widetilde{\Phi}_\varepsilon$ are 2π-periodic in x, and consider the expansions

$$\begin{aligned} k_\varepsilon &= k_* + \varepsilon^2 k_2 + \mathcal{O}(\varepsilon^3), \\ \widetilde{\eta}_\varepsilon(x) &= \varepsilon\eta_1(x) + \varepsilon^2\eta_2(x) + \mathcal{O}(\varepsilon^3), \\ \widetilde{\Phi}_\varepsilon(x, y) &= \varepsilon\Phi_1(x, y) + \varepsilon^2\Phi_2(x, y) + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{10}$$

where k_* is the positive root of the linear dispersion relation (3). Substituting these expansions into the Euler equations (1)–(2), we obtain in Appendix B the following explicit formulas:

$$k_2 = \frac{k_*^3}{d(k_*)}\Big((9\alpha\beta + 16)\, k_* - 12\alpha\beta k_* \cosh(2k_*) + 3\alpha\beta k_* \cosh(4k_*) \\ - 8\alpha(2c(k_*) - 1)\sinh(2k_*) - 4\alpha(c(k_*) + 2)\sinh(4k_*)\Big), \tag{11}$$

and

$$\begin{aligned} \eta_1(x) &= \sinh(k_*)\cos(x), \quad \Phi_1(x,y) = \cosh(k_* y)\sin(x), \\ \eta_2(x) &= \frac{k_*}{4}\,(c(k_*) + 1)\sinh(2k_*)\cos(2x) - \frac{k_*^2}{4\alpha}, \\ \Phi_2(x,y) &= \frac{k_*}{4}\,(c(k_*)\cosh(2k_* y) + 2\sinh(k_*) y \sinh(k_* y))\sin(2x), \end{aligned} \tag{12}$$

where

$$\begin{aligned} c(k_*) &= -1 - \frac{k_*(\cosh(2k_*) + 2)}{\mathcal{D}(2k_*)}, \\ d(k_*) &= 32\alpha\left(2\beta k_*(\cosh(2k_*) - 1) + 2k_* - \sinh(2k_*)\right), \end{aligned} \tag{13}$$

and $\mathcal{D}(k)$ is the linear dispersion relation (3). In addition, the function $\widetilde{\eta}_\varepsilon$ is even in x, whereas $\widetilde{\Phi}_\varepsilon$ is an odd function.

For each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the solution $(\eta_\varepsilon, \Phi_\varepsilon)$ of the Euler equations (1)–(2) provides a solution

$$U_\varepsilon(x,y) = (\widetilde{\eta}_\varepsilon(x), 0, \widetilde{\Phi}_\varepsilon(x,y), 0)^{\mathrm{T}} \tag{14}$$

of the dynamical system (5)–(6) for $k = k_\varepsilon$, hence satisfying

$$\begin{cases} F(U_\varepsilon) = 0, \\ \widetilde{\Phi}_{\varepsilon y} = B(U_\varepsilon), \quad \text{on } y = 0, 1. \end{cases} \tag{15}$$

In addition, the above parity properties of $\widetilde{\eta}_\varepsilon$ and $\widetilde{\Phi}_\varepsilon$ imply that $SU_\varepsilon = U_\varepsilon$ where S is the reflection symmetry given in (8).

3 Analysis of the linear operator

For fixed (α, β) in Region I, we denote by L_ε the linear operator which appears in the linearization of the dynamical sytem (5)–(6) at the periodic wave U_ε for $k = k_\varepsilon$. We prove the properties of L_ε needed for the transverse dynamics analysis in Section 4.

For notational simplicity we remove the tilde from (15) and write from now on η_ε and Φ_ε instead of $\widetilde{\eta}_\varepsilon$ and $\widetilde{\Phi}_\varepsilon$, respectively.

3.1 The linear operator L_ε A direct computation of the differential of F at the periodic wave U_ε gives the following explicit formulas for $L_\varepsilon U := \mathrm{d}F[U_\varepsilon]U$,

$$L_\varepsilon U = \begin{pmatrix} \omega/\beta + H_1(\omega,\xi) \\ \alpha\eta - \beta k_\varepsilon^2 \eta_{xx} - k_\varepsilon \Phi_x|_{y=1} + H_2(\eta,\Phi) \\ \xi + H_3(\omega,\xi) \\ -k_\varepsilon^2\Phi_{xx} - \Phi_{yy} + H_4(\eta,\Phi) \end{pmatrix}, \quad U = \begin{pmatrix} \eta \\ \omega \\ \Phi \\ \xi \end{pmatrix}, \tag{16}$$

where

$$\begin{aligned}
H_1(\omega,\xi) &= \frac{(1+k_\varepsilon^2\eta_{\varepsilon x}^2)^{1/2}}{\beta}\left(\omega + \frac{1}{1+\eta_\varepsilon}\int_0^1 y\Phi_{\varepsilon y}\xi\,\mathrm{d}y\right) - \frac{\omega}{\beta},\\
H_2(\eta,\Phi) &= \beta k_\varepsilon^2\eta_{xx} - \beta k_\varepsilon^2\left[\frac{\eta_x}{(1+k_\varepsilon^2\eta_{\varepsilon x}^2)^{3/2}}\right]_x\\
&\quad + \int_0^1 \Bigg\{ k_\varepsilon^2 \Phi_{\varepsilon x}\Phi_x - \frac{\Phi_{\varepsilon y}\Phi_y}{(1+\eta_\varepsilon)^2} + \frac{\Phi_{\varepsilon y}^2\eta}{(1+\eta_\varepsilon)^3} - k_\varepsilon^2\frac{y^2\eta_{\varepsilon x}^2\Phi_{\varepsilon y}\Phi_y}{(1+\eta_\varepsilon)^2}\\
&\qquad - k_\varepsilon^2\frac{y^2\eta_{\varepsilon x}\Phi_{\varepsilon y}^2\eta_x}{(1+\eta_\varepsilon)^2} + k_\varepsilon\frac{y^2\eta_{\varepsilon x}\Phi_{\varepsilon y}^2\eta}{(1+\eta_\varepsilon)^3}\\
&\qquad + k_\varepsilon^2\left[y\Phi_{\varepsilon y}\Phi_x + y\Phi_{\varepsilon x}\Phi_y - \frac{2y^2\eta_{\varepsilon x}\Phi_{\varepsilon y}\Phi_y}{1+\eta_\varepsilon} - \frac{y^2\Phi_{\varepsilon y}^2\eta_x}{1+\eta_\varepsilon} + \frac{y^2\Phi_{\varepsilon y}^2\eta_{\varepsilon x}\eta}{(1+\eta_\varepsilon)^2}\right]_x\Bigg\}\,\mathrm{d}y,
\end{aligned}$$

$$\begin{aligned}
H_3(\omega,\xi) &= -\frac{\eta_\varepsilon\xi}{1+\eta_\varepsilon} + \frac{1}{1+\eta_\varepsilon}\left(H_1(\omega,\xi) + \frac{\omega}{\beta}\right)y\Phi_{\varepsilon y},\\
H_4(\eta,\Phi) &= k_\varepsilon^2\Big[-\eta_\varepsilon\Phi_x - \Phi_{\varepsilon x}\eta + y\Phi_{\varepsilon y}\eta_x + y\eta_{\varepsilon x}\Phi_y\Big]_x + \Big[k_\varepsilon y\eta_x + B_{l\varepsilon}(\eta,\Phi)\Big]_y.
\end{aligned}$$

To this expression of $L_\varepsilon U$ we add the linear boundary conditions obtained by taking the differential of B at U_ε,

$$\Phi_y = B_{l\varepsilon}(U) := \mathrm{d}B[U_\varepsilon]U = 0 \quad \text{on } y = 0,1, \tag{17}$$

where

$$\begin{aligned}
B_{l\varepsilon}(U) &= k_\varepsilon y(-\eta_x + k_\varepsilon\eta_{\varepsilon x}\Phi_x + k_\varepsilon\Phi_{\varepsilon x}\eta_x)\\
&\qquad + \frac{\eta_\varepsilon\Phi_y}{1+\eta_\varepsilon} + \frac{\Phi_{\varepsilon y}\eta}{(1+\eta_\varepsilon)^2} + k_\varepsilon^2\frac{y^2\eta_{\varepsilon x}^2\Phi_{\varepsilon y}\eta}{(1+\eta_\varepsilon)^2} - k_\varepsilon^2\frac{y^2\eta_{\varepsilon x}^2\Phi_y}{1+\eta_\varepsilon} - 2k_\varepsilon^2\frac{y^2\eta_{\varepsilon x}\Phi_{\varepsilon y}\eta_x}{1+\eta_\varepsilon}.
\end{aligned}$$

Notice that $B_{l\varepsilon}(U)$ only depends on the components η and Φ of U. We sometimes write $B_{l\varepsilon}(\eta,\Phi)$ for convenience.

For $s \geq 0$, we define the Hilbert space

$$\mathcal{X}^s = H^{s+1}_{\mathrm{per}}(\mathbb{S}) \times H^s_{\mathrm{per}}(\mathbb{S}) \times H^{s+1}_{\mathrm{per}}(\Sigma) \times H^s_{\mathrm{per}}(\Sigma), \tag{18}$$

where $\mathbb{S} = (0, 2\pi)$, $\Sigma = \mathbb{S} \times (0, 1)$, and

$$
\begin{aligned}
H^s_{\text{per}}(\mathbb{S}) &= \{u \in H^s_{\text{loc}}(\mathbb{R}) \,:\, u(x + 2\pi) = u(x),\ x \in \mathbb{R}\},\\
H^s_{\text{per}}(\Sigma) &= \{u \in H^s_{\text{loc}}(\mathbb{R} \times (0, 1)) \,:\, u(x + 2\pi, y) = u(x, y),\ y \in (0, 1),\ x \in \mathbb{R}\}.
\end{aligned}
$$

The action of the operator L_ε is taken in $\mathcal{X}^0$ with domain of definition

$$
\mathcal{Y}^1_\varepsilon = \{U = (\eta, \omega, \Phi, \xi)^{\mathrm{T}} \in \mathcal{X}^1 \,:\, \Phi_y = B_{l\varepsilon}(\eta, \Phi) \text{ on } y = 0, 1\},
$$

chosen to include the boundary conditions. Then L_ε is well-defined and closed in $\mathcal{X}^0$, and its domain $\mathcal{Y}^1_\varepsilon$ is compactly embedded in $\mathcal{X}^0$. The latter property implies that the operator L_ε has pure point spectrum consisting of isolated eigenvalues with finite algebraic multiplicity. As a consequence of the reflection symmetry S given in (8), which commutes with F and leaves invariant U_ε, the subspaces

$$
\mathcal{X}^0_+ = \{U \in \mathcal{X}^0 \,:\, SU = U\}, \quad \mathcal{X}^0_- = \{U \in \mathcal{X}^0 \,:\, SU = -U\}, \tag{19}
$$

are invariant under the action of L_ε.

One inconvenience of this functional-analytic setting is that the domain of definition $\mathcal{Y}^1_\varepsilon$ of the linear operator L_ε depends on ε. This difficulty is well-known and can be handled using an appropriate change of variables first introduced for the three-dimensional steady nonlinear Euler equations in [16]. Here, we proceed as in [14] and replace Φ by $\Upsilon = \Phi + \chi_y$, where χ is the unique solution of the elliptic problem

$$
\begin{aligned}
-k_\varepsilon^2 \chi_{xx} - \chi_{yy} &= B_{l\varepsilon}(U) && \text{in } \Sigma,\\
\chi &= 0 && \text{on } y = 0, 1.
\end{aligned}
$$

so that Υ satisfies the boundary conditions $\Upsilon_y = 0$ on $y = 0, 1$ which do not depend on ε. The linear mapping defined by $G_\varepsilon(\eta, \omega, \Phi, \xi)^{\mathrm{T}} = (\eta, \omega, \Upsilon, \xi)^{\mathrm{T}}$, is a linear isomorphism in both $\mathcal{X}^0$ and $\mathcal{X}^1$, it depends smoothly on ε and the same is true for its inverse G_ε^{-1}. Setting $\widetilde{L}_\varepsilon = G_\varepsilon L_\varepsilon G_\varepsilon^{-1}$ the operator $\widetilde{L}_\varepsilon$ acts in $\mathcal{X}^0$ with domain of definition

$$
\mathcal{Y}^1 = \{U = (\eta, \omega, \Upsilon, \xi)^{\mathrm{T}} \in \mathcal{X}^1 \,:\, \Upsilon_y = 0 \text{ on } y = 0, 1\},
$$

which does not depend on ε anymore. While $\widetilde{L}_\varepsilon$ allows us to rigorously apply general results for linear operators, it is more convenient to use L_ε for explicit computations.

3.2 Spectral properties of L_0 The unperturbed operator L_0 obtained for $\varepsilon = 0$ is a differential operator with constant coefficients. Therefore, eigenvalues, eigenfunctions, and generalized eigenfunctions can be explicitly computed using Fourier series in the variable x. In particular, for purely imaginary values $\mathrm{i}\ell$ with $\ell \in \mathbb{R}$ the eigenvalue problem $(L_0 - \mathrm{i}\ell\,\mathbb{I})U = 0$ possesses nontrivial solutions in the n^{th} Fourier mode if and only if

$$
(\alpha + \beta\sigma^2)\sigma \sinh\sigma - n^2 k_*^2 \cosh\sigma = 0 \quad \text{with} \quad \sigma^2 = n^2 k_*^2 + \ell^2.
$$

For fixed (α, β) in Region I, this equality holds if and only if $\ell = 0$ and $n \in \{0, \pm 1\}$; see also [11]. Consequently, 0 is the only purely imaginary eigenvalue of L_0 and it has geometric multiplicity three. The associated eigenvectors are given by the explicit formulas:

$$\zeta_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \zeta_- = \begin{pmatrix} -\sinh(k_*)\sin(x) \\ 0 \\ \cosh(k_* y)\cos(x) \\ 0 \end{pmatrix}, \quad \zeta_+ = \begin{pmatrix} \sinh(k_*)\cos(x) \\ 0 \\ \cosh(k_* y)\sin(x) \\ 0 \end{pmatrix}. \tag{20}$$

Associated to each eigenvector there is a Jordan chain of length two, so that the algebraic multiplicity of the eigenvalue 0 is six. The generalized eigenvectors associated to ζ_0, ζ_- and ζ_+ are given by, respectively,

$$\psi_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ -\beta\sinh(k_*)\sin(x) \\ 0 \\ \cosh(k_* y)\cos(x) \end{pmatrix}, \quad \psi_+ = \begin{pmatrix} 0 \\ \beta\sinh(k_*)\cos(x) \\ 0 \\ \cosh(k_* y)\sin(x) \end{pmatrix}. \tag{21}$$

Notice that the reflection symmetry S given in (8) acts on these eigenvectors as follows:

$$\begin{aligned} S\zeta_0 &= -\zeta_0, \qquad & S\zeta_- &= -\zeta_-, \qquad & S\zeta_+ &= \zeta_+, \\ S\psi_0 &= -\psi_0, \qquad & S\psi_- &= -\psi_-, \qquad & S\psi_+ &= \psi_+. \end{aligned}$$

These formulas are consistent with the ones already found in [11]. The remaining eigenvalues of L_0 are bounded away from the imaginary axis.

3.3 Main result We summarize in the next theorem the properties of the linear operator L_ε needed for our transverse dynamics analysis. The same properties hold for the operator $\widetilde{L}_\varepsilon$.

Theorem 3.1 (Linear operator) *There exist positive constants ε_1, C_1, and ℓ_1, such that for each $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ the following properties hold.*

(i) The linear operator L_ε acting in $\mathcal{X}^0$ with domain $\mathcal{Y}^1_\varepsilon$ has an eigenvalue 0 with algebraic multiplicity four, and two simple purely imaginary eigenvalues $\pm \mathrm{i}\ell_\varepsilon$ with $\ell_\varepsilon > 0$ and $\ell_0 = 0$. Any other purely imaginary value $\mathrm{i}\ell \in \mathrm{i}\mathbb{R} \setminus \{0, \pm \mathrm{i}\ell_\varepsilon\}$ belongs to the resolvent set of L_ε.

(ii) The restriction of L_ε to the invariant subspace $\mathcal{X}^0_+$ has the two simple purely imaginary eigenvalues $\pm \mathrm{i}\ell_\varepsilon$ and any other value $\mathrm{i}\ell \in \mathrm{i}\mathbb{R} \setminus \{\pm \mathrm{i}\ell_\varepsilon\}$ belongs to the resolvent set.

(iii) The inequality

$$\left\| (L_\varepsilon - \mathrm{i}\ell\,\mathbb{I})^{-1} \right\|_{\mathcal{L}(X^0)} \leq \frac{C_1}{|\ell|},$$

holds for each real number ℓ with $|\ell| > \ell_1$.

Proof. We rely on the properties of the operator L_0 and perturbation arguments for ε sufficiently small. The operators $\widetilde{L}_\varepsilon$ and $\widetilde{L}_0$ having the same domain of definition $\mathcal{Y}^1$, standard perturbation arguments show that $\widetilde{L}_\varepsilon$ is a small relatively bounded perturbation of $\widetilde{L}_0$ for ε sufficiently small. The result in item (iii) is an immediate consequence of this property. Indeed, for $\varepsilon = 0$ the inequality from (iii) is given in [11], which implies that a similar inequality holds for $\widetilde{L}_0$, with possibly different values C_1 and ℓ_1. The operator $\widetilde{L}_\varepsilon$ being a relatively bounded perturbation of $\widetilde{L}_0$ for sufficiently small ε, from the inequality for $\widetilde{L}_0$ we obtain that item (iii) holds for $\widetilde{L}_\varepsilon$, and then for L_ε. It remains to prove items (i) and (ii). This is the main part of the proof of the theorem.

Spectral decomposition. The results in Section 3.2 show that the spectrum $\sigma(L_0)$ of the linear operator L_0 satisfies

$$\sigma(L_0) = \{0\} \cup \sigma_1(L_0), \quad \sigma_1(L_0) \subset \{\lambda \in \mathbb{C} \,:\, |\mathrm{Re}\,\lambda| > d_1\},$$

for some $d_1 > 0$, where 0 is an eigenvalue with algebraic multiplicity six and geometric multiplicity three, and the same is true for the linear operator $\widetilde{L}_0$. The six-dimensional spectral subspace $\mathcal{E}_0$ associated to the eigenvalue 0 of L_0 is spanned by the eigenvectors $\zeta_0,\ \zeta_\pm$ given in (20) and generalized eigenvectors $\psi_0,\ \psi_\pm$ given in (21). For $\varepsilon \neq 0$ sufficiently small, $\widetilde{L}_\varepsilon$ is a small relatively bounded perturbation of $\widetilde{L}_0$. Consequently, there exists a neighborhood $V_0 \subset \mathbb{C}$ of the origin such that

$$V_0 \subset \{\lambda \in \mathbb{C} \,:\, |\mathrm{Re}\,\lambda| < d_1/4\}$$

and

$$\sigma(\widetilde{L}_\varepsilon) = \sigma_0(\widetilde{L}_\varepsilon) \cup \sigma_1(\widetilde{L}_\varepsilon), \quad \sigma_0(\widetilde{L}_\varepsilon) \subset V_0, \quad \sigma_1(\widetilde{L}_\varepsilon) \subset \{\lambda \in \mathbb{C} \,:\, |\mathrm{Re}\,\lambda| > d_1/2\},$$

for sufficiently small ε, where the spectral subspace associated to $\sigma_0(\widetilde{L}_\varepsilon)$ is six-dimensional, and the same is true for L_ε. Moreover, for the operator L_ε, there exists a basis $\{\zeta_0(\varepsilon), \zeta_\pm(\varepsilon), \psi_0(\varepsilon), \psi_\pm(\varepsilon)\}$ of the six-dimensional spectral subspace $\mathcal{E}_\varepsilon$ associated to $\sigma_0(L_\varepsilon)$ which is the smooth continuation, for sufficiently small ε, of the basis $\{\zeta_0,\ \zeta_\pm,\ \psi_0,\ \psi_\pm\}$ of the six-dimensional spectral subspace $\mathcal{E}_0$ associated to the eigenvalue 0 of L_0. The two bases share the symmetry properties,

$$\begin{aligned} S\zeta_0(\varepsilon) &= -\zeta_0(\varepsilon), & S\zeta_-(\varepsilon) &= -\zeta_-(\varepsilon), & S\zeta_+(\varepsilon) &= \zeta_+(\varepsilon), \\ S\psi_0(\varepsilon) &= -\psi_0(\varepsilon), & S\psi_-(\varepsilon) &= -\psi_-(\varepsilon), & S\psi_+(\varepsilon) &= \psi_+(\varepsilon). \end{aligned}$$

Thus, we have the decomposition $\mathcal{E}_\varepsilon = \mathcal{E}_{\varepsilon,+} \oplus \mathcal{E}_{\varepsilon,-}$ with

$$\begin{aligned} \mathcal{E}_{\varepsilon,+} &= \{U \in \mathcal{E}_\varepsilon \,:\, RU = U\} &&= \mathrm{span}\{\zeta_+(\varepsilon), \psi_+(\varepsilon)\}, \\ \mathcal{E}_{\varepsilon,-} &= \{U \in \mathcal{E}_\varepsilon \,:\, RU = -U\} &&= \mathrm{span}\{\zeta_0(\varepsilon), \zeta_-(\varepsilon), \psi_0(\varepsilon), \psi_-(\varepsilon)\}. \end{aligned}$$

These spaces $\mathcal{E}_{\varepsilon,\pm}$ are invariant under the action of L_ε.

Purely imaginary eigenvalues of L_ε necessarily belong to the neighborhood V_0 of 0. Therefore, they are detemined by the action of L_ε on the spectral subspace $\mathcal{E}_\varepsilon$. This action is represented by a 6×6 matrix. The decomposition $\mathcal{E}_\varepsilon = \mathcal{E}_{\varepsilon,+} \oplus \mathcal{E}_{\varepsilon,-}$ above, implies that we can further decompose the action of L_ε by restricting to the invariant subspaces $\mathcal{E}_{\varepsilon,\pm}$. In other words, the 6×6 matrix is a block matrix with a 2×2 block representing the action of L_ε on $\mathcal{E}_{\varepsilon,+}$ and a 4×4 block representing the action of L_ε on $\mathcal{E}_{\varepsilon,-}$. Our task is to determine the eigenvalues of these two matrices. This will prove the result in part (i) of the theorem. For the restriction of the linear operator L_ε to the invariant subspace $\mathcal{X}^0_+$ in part (ii) of the theorem, it is enough to consider the eigenvalues of the 2×2 matrix.

Eigenvalues of the 4×4 matrix. It turns out that a basis of the subspace $\mathcal{E}_{\varepsilon,-}$ can be explicitly obtained using the symmetries of the Euler equations. First, the Euler equations (1)–(2) are invariant under the transformation $\phi \mapsto \phi + C$ for any real constant C. This implies that the dynamical system (5)–(6) is invariant under the transformation $U \mapsto U + \zeta_0$ where $\zeta_0 = (0,0,1,0)^{\mathrm{T}}$. Consequently, ζ_0 belongs to the kernel of L_ε and since $S\zeta_0 = -\zeta_0$ it belongs to $\mathcal{E}_-$. We choose $\zeta_0(\varepsilon) = \zeta_0$ and then a direct computation gives the generalized eigenvector

$$\psi_0(\varepsilon) = \begin{pmatrix} 0 \\ -\int_0^1 y\Phi_{\varepsilon y}\,\mathrm{d}y \\ 0 \\ 1+\eta_\varepsilon \end{pmatrix},$$

satisfying $L_\varepsilon \psi_0(\varepsilon) = \zeta_0$ and $S\psi_0(\varepsilon) = -\psi_0(\varepsilon)$.

Next, the invariance of the Euler equations under horizontal spatial translations in x implies that the derivative $U_{\varepsilon x} = (\eta_{\varepsilon x}, 0, \Phi_{\varepsilon x}, 0)^{\mathrm{T}}$ of the periodic wave belongs to the kernel of L_ε. Since $SU_{\varepsilon x} = -U_{\varepsilon x}$, the vector $U_{\varepsilon x}$ belongs $\mathcal{E}_{\varepsilon,-}$. From the expansions (10), we find that $U_{\varepsilon x} = \varepsilon\zeta_- + \mathcal{O}(\varepsilon^2)$. This gives a second vector $\zeta_-(\varepsilon) = \varepsilon^{-1}U_{\varepsilon x}$ which belongs to the kernel of L_ε, and also to the invariant subspace $\mathcal{E}_-$, with the property that $\zeta_-(\varepsilon) \to \zeta_-$ as $\varepsilon \to 0$. The corresponding generalized eigenvector is given by

$$\psi_-(\varepsilon) = \frac{1}{\varepsilon}\begin{pmatrix} 0 \\ \dfrac{\eta_{\varepsilon x}\beta}{(1+k_\varepsilon^2\eta_{\varepsilon x}^2)^{1/2}} - \int_0^1 y\Phi_{\varepsilon y}\left(\Phi_{\varepsilon x} - \dfrac{\eta_{\varepsilon x} y \Phi_{\varepsilon y}}{1+\eta_\varepsilon}\right)\mathrm{d}y \\ 0 \\ (1+\eta_\varepsilon)\left(\Phi_{\varepsilon x} - \dfrac{\eta_{\varepsilon x} y\Phi_{\varepsilon y}}{1+\eta_\varepsilon}\right) \end{pmatrix}.$$

The above shows that there is a basis $\{\zeta_0(\varepsilon), \psi_0(\varepsilon), \zeta_-(\varepsilon), \psi_-(\varepsilon)\}$ for $\mathcal{E}_{\varepsilon,-}$ satisfying

$$L_\varepsilon\zeta_0(\varepsilon) = 0, \quad L_\varepsilon\psi_0(\varepsilon) = \zeta_0(\varepsilon), \quad L_\varepsilon\zeta_-(\varepsilon) = 0, \quad L_\varepsilon\psi_-(\varepsilon) = \zeta_-(\varepsilon).$$

Thus, 0 is the only eigenvalue of the 4×4 matrix representing the action of L_ε onto $\mathcal{E}_{\varepsilon,-}$ and it has geometric multiplicity two and algebraic multiplicity four.

Eigenvalues of the 2×2 matrix. We consider a basis $\{\zeta_+(\varepsilon), \psi_+(\varepsilon)\}$ of the subspace $\mathcal{E}_{\varepsilon,+}$ which is the smooth continuation of the basis $\{\zeta_+, \psi_+\}$ of $\mathcal{E}_{0,+}$, and denote by $\mathcal{M}(\varepsilon)$ the 2×2 matrix representing the action of L_ε on this basis. At $\varepsilon = 0$, we have that $L_0\zeta_+ = 0$ and $L_0\psi_+ = \zeta_+$, which implies that

$$\mathcal{M}(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For $\varepsilon \neq 0$, we write

$$\mathcal{M}(\varepsilon) = \begin{pmatrix} m_{11}(\varepsilon) & 1 + m_{12}(\varepsilon) \\ m_{21}(\varepsilon) & m_{22}(\varepsilon) \end{pmatrix}.$$

The invariance of the Euler equations under horizontal spatial translations in x, implies that the periodic waves translated by a half-period π are also periodic solutions. Comparing their expansions in ε with the ones of $(\eta_\varepsilon, \Phi_\varepsilon)$ we conclude that

$$\eta_\varepsilon(x) = \eta_{-\varepsilon}(x + \pi), \quad \Phi_\varepsilon(x, y) = \Phi_{-\varepsilon}(x + \pi, y).$$

Since the 2×2 matrices corresponding to these solutions are the same this implies that $\mathcal{M}(\varepsilon) = \mathcal{M}(-\varepsilon)$, and as a consequence, we have the expansion $m_{ij}(\varepsilon) = m_{ij}^{(2)}\varepsilon^2 + \mathcal{O}(\varepsilon^4)$, for ε sufficiently small.

Next, the reversibility of L_ε implies that the spectrum of L_ε is symmetric with respect to the origin in the complex plane. Moreover, because L_ε is a real operator, its spectrum is also symmetric with respect to the real line. These observations combined imply that the two eigenvalues of $\mathcal{M}(\varepsilon)$ are either both real or both purely imaginary, and their sum is equal to 0. Consequently, $m_{11}(\varepsilon) = -m_{22}(\varepsilon)$. Further, the product of these two eigenvalues is equal to the determinant of $\mathcal{M}(\varepsilon)$. Therefore, the eigenvalues are both real if $\det \mathcal{M}(\varepsilon) < 0$ and both purely imaginary if $\det \mathcal{M}(\varepsilon) > 0$. We have

$$\det \mathcal{M}(\varepsilon) = -m_{11}^2(\varepsilon) - m_{21}(\varepsilon)(1 + m_{12}(\varepsilon)) = -m_{21}^{(2)}\varepsilon^2 + \mathcal{O}(\varepsilon^4),$$

so the result in theorem holds provided $m_{21}^{(2)} < 0$.

The final step is the computation of the sign of $m_{21}^{(2)}$. We prove in Appendix C that

$$m_{21}^{(2)} = -4k_* \cdot \frac{4\beta k_* \sinh^2(k_*) + 2k_* - \sinh(2k_*)}{4\beta k_* \sinh^2(k_*) + 2k_* + \sinh(2k_*)} \cdot k_2, \tag{22}$$

where k_2 is the coefficient in the expansion of the wavenumber k_ε given in (11). Replacing the formula for k_2 we obtain

$$m_{21}^{(2)} = \frac{k_*^4}{8\alpha} \cdot \frac{1}{4\beta k_* \sinh^2(k_*) + 2k_* + \sinh(2k_*)} \cdot \widetilde{m}_{21}^{(2)},$$

where

$$
\begin{aligned}
\widetilde{m}_{21}^{(2)} = &-(9\alpha\beta+16)\,k_* + 12\alpha\beta k_*\cosh(2k_*) - 3\alpha\beta k_*\cosh(4k_*) \\
&+ 8\alpha(2c(k_*)-1)\sinh(2k_*) + 4\alpha(c(k_*)+2)\sinh(4k_*),
\end{aligned}
\tag{23}
$$

with $c(k_*) < -1$ given in (13). Clearly, $m_{21}^{(2)}$ and $\widetilde{m}_{21}^{(2)}$ have the same sign. Proposition D.1 in Appendix D shows that $\widetilde{m}_{21}^{(2)} < 0$ for (α,β) in Region I. This completes the proof of the theorem. □

4 Transverse dynamics

We show that the two-dimensional periodic waves in Theorem 1.1(i) are linearly transversely unstable for ε sufficiently small, and then discuss the induced dimension-breaking bifurcation. These two results prove the parts (ii) and (iii) of Theorem 1.1.

Throughout this section, we consider a two-dimensional periodic wave U_ε such that the associated linearized of operator L_ε studied in Section 3 possesses two simple purely imaginary eigenvalues $\pm \mathrm{i}\ell_\varepsilon$ as in Theorem 3.1, hence by fixing $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$.

4.1 Transverse linear instability Linearizing the system (5)–(6) at U_ε we obtain the linear system

$$
\frac{\mathrm{d}U}{\mathrm{d}z} = DU_t + \mathrm{d}F[U_\varepsilon]U, \tag{24}
$$

with boundary conditions

$$
\Phi_y = y\eta_t + B_{l\varepsilon}(U) \quad \text{on } y = 0, 1. \tag{25}
$$

The periodic wave U_ε is transversely linearly unstable if the linear equation has a solution of the form $\exp(\sigma t)U_\sigma(z)$ with $\operatorname{Re}\sigma > 0$ and $U_\sigma \in C^1_{\mathrm{b}}(\mathbb{R}, \mathcal{X}^0) \cap C_{\mathrm{b}}(\mathbb{R}, \mathcal{X}^1)$. For the construction of such a function, we closely follow the approach developed in [17] where the authors studied the transverse instability of solitary waves for the Euler equations. The only difference is that the functions were localized in $x \in \mathbb{R}$ in [17], whereas here they are periodic in x. We use the following general result from [17], which we have slightly modified; see Remark 4.2.

Theorem 4.1 (Theorem 1.3 [17]) *Consider real Banach spaces $\mathcal{X}$, $\mathcal{Z}$, $\mathcal{Z}_i$, $i = 1, 2$, and a partial differential equation of the form*

$$
\frac{\mathrm{d}U}{\mathrm{d}z} = D_1U_t + D_2U_{tt} + LU. \tag{26}
$$

Assume that the following properties hold:

(i) $\mathcal{Z} \subset \mathcal{Z}_i \subset \mathcal{X}$, $i = 1, 2$, with continuous and dense embeddings;

(ii) *L, D_1, and D_2 are closed linear operators in $\mathcal{X}$ with domains $\mathcal{Z}$, $\mathcal{Z}_1$, and $\mathcal{Z}_2$, respectively;*

(iii) *the spectrum of L contains a pair of isolated purely imaginary eigenvalues $\pm \mathrm{i}\ell_*$ with odd multiplicity;*

(iv) *there exists an involution $R \in \mathcal{L}(\mathcal{X})$ which anticommutes with L and D_i, $i = 1, 2$, i.e., the equation (26) is reversible.*

Then, for each sufficiently small $\sigma > 0$, equation (26) has a solution of the form $\exp(\sigma t)U_\sigma(z)$ with $\operatorname{Re}\sigma > 0$ and $U_\sigma \in C^1(\mathbb{R}, \mathcal{X}) \cap C(\mathbb{R}, \mathcal{Z})$ a periodic function.

Remark 4.2. Theorem 1.3 [17] assumes that the linear operators L, D_1, and D_2 have the same domain of definition, while we in Theorem 4.1 allow for different domains, just like in Theorem 2.1 of [10]. This is needed since the operators D_1 and D_2 in our application (and in fact also in [17]) have different domains than L. Note in particular that the hypotheses imply that D_1 and D_2 are relatively bounded perturbations of L by Remarks 1.4 and 1.5, Chapter 4.1.1 [33].

This general result does not directly apply to the system (24)–(25) because the boundary condition (25) contains the extra term $y\eta_t$ which involves a derivative with respect to t. We proceed as in [17] and eliminate this term by an appropriate change of variables, similar to the one used for L_ε in Section 3.

We replace the variable Φ in U by a new variable $\Theta = \Phi + \theta_{yt}$ where θ is the unique solution of the elliptic boundary value problem

$$\begin{aligned} -k_\varepsilon^2\theta_{xx} - \theta_{yy} + B_{l\varepsilon}(0, \theta_y) = y\eta \quad & \text{in } \Sigma, \\ \theta = 0 \quad & \text{on } y = 0, 1, \end{aligned}$$

where we set $B_{l\varepsilon}(\eta, \Phi) = B_{l\varepsilon}(U)$ because $B_{l\varepsilon}(U)$ only depends on η and Φ. In the boundary value problem for θ, we regard t as a parameter and assume analytic dependence on t. Then the mapping defined by $Q(\eta, \omega, \Phi, \xi)^{\mathrm{T}} = (\eta, \omega, \Theta, \xi)^{\mathrm{T}}$ is a linear isomorphism on both $\mathcal{X}^0$ and $\mathcal{X}^1$. The transformed linearized problem (24)–(25) for $V = (\eta, \omega, \Theta, \xi)^{\mathrm{T}}$ is of the form

$$\frac{\mathrm{d}V}{\mathrm{d}z} = D_1 V_t + D_2 V_{tt} + L_\varepsilon V, \tag{27}$$

with boundary conditions

$$\Theta_y = B_{l\varepsilon}(V) \quad \text{on } y = 0, 1. \tag{28}$$

The two linear operators D_1 and D_2 are bounded in $\mathcal{X}^0$ and defined by

$$D_1 \begin{pmatrix} \eta \\ \omega \\ \Theta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ \Theta|_{y=1} + k_\varepsilon \theta_{xy}|_{y=1} - H_2(0, \theta_y) \\ \widehat{\theta}_y \\ -\eta - k_\varepsilon^2(-\eta_\varepsilon \theta_{xy} + y\eta_{\varepsilon x}\theta_{yy})_x \end{pmatrix}, \quad D_2 \begin{pmatrix} \eta \\ \omega \\ \Theta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ -\theta_y|_{y=1} \\ 0 \\ 0 \end{pmatrix},$$

where $\widehat{\theta}$ is the unique solution of elliptic boundary value problem

$$\begin{aligned} -k_\varepsilon^2\widehat{\theta}_{xx} - \widehat{\theta}_{yy} + B_{l\varepsilon}(0,\widehat{\theta}_y) &= y\left(\frac{\omega}{\beta} + H_1(\omega,\xi)\right) &&\text{in } \Sigma,\\ \hat{\theta} &= 0 &&\text{on } y = 0,1. \end{aligned}$$

We use the system (27)–(28) and the result in Theorem 4.1 to prove the transverse linear instability of the periodic wave U_ε.

Proof of Theorem 1.1(ii). We apply Theorem 4.1 to the equation (27) with Hilbert spaces $\mathcal{X} = \mathcal{Z}_i = \mathcal{X}^0$, $\mathcal{Z} = \mathcal{Y}_\varepsilon^1$, $i = 1,2$, operators D_1, D_2 defined as above, and L_ε. Since D_1 and D_2 are bounded on $\mathcal{X}^0$, they are closed operators in $\mathcal{X}^0$. The first two hypotheses *(i)* and *(ii)* are satisfied. The spectral condition *(iii)* is verified by Theorem 3.1(i). The reverser is R defined in Section 2.1 and its anti-commutativity with D_1, D_2 and L is preserved by the change of variables Q. Thus, equation (27) is reversible. Theorem 4.1 now gives the statement of Theorem 1.1(ii). □

4.2 Dimension-breaking bifurcation

We look for three-dimensional steady solutions of the system (5)–(6) which bifurcate from the transversely unstable periodic wave U_ε. Taking

$$U(x,y,z) = U_\varepsilon(x,y) + \widetilde{U}(x,y,z), \quad \widetilde{U} = (\eta,\omega,\Phi,\xi)^{\mathrm{T}},$$

in (5)–(6) we obtain the equation

$$\frac{\mathrm{d}\widetilde{U}}{\mathrm{d}z} = F(U_\varepsilon + \widetilde{U}), \tag{29}$$

together with the boundary conditions

$$\Phi_y = B(U_\varepsilon + \widetilde{U}) - B(U_\varepsilon) \quad \text{on } y = 0,1. \tag{30}$$

The mappings F and B are defined on an open neighborhood M of $0 \in \mathcal{X}^1$ which is contained in the set

$$\{(\eta,\omega,\Phi,\xi)^{\mathrm{T}} \in \mathcal{X}^1 \,:\, |W(x)| < \beta, \eta(x) > -1 \text{ for all } x \in \mathbb{R}\},$$

and are analytic. The periodic wave U_ε belongs to M, for sufficiently small ε, and we look for bounded solutions $\widetilde{U}$ such that $U_\varepsilon + \widetilde{U}(z) \in M$, for all $z \in \mathbb{R}$.

General bifurcation results cannot be directly applied to this system because the boundary condition (30) is nonlinear. We make a nonlinear change of variables which transforms these nonlinear boundary conditions into linear boundary conditions. Similarly to our previous changes of variables from Section 3.1 and Section 4.1, we replace Φ by a new variable $\Theta = \Phi + \theta_y$ where θ is the unique solution of the elliptic boundary value problem

$$\begin{aligned} -k_\varepsilon^2\theta_{xx} - \theta_{yy} + B_{l\varepsilon}(0,\theta_y) &= B(U) &&\text{in } \Sigma,\\ \theta &= 0 &&\text{on } y = 0,1. \end{aligned}$$

and define $Q(\eta, \omega, \Phi, \xi)^{\mathrm{T}} = (\eta, \omega, \Theta, \xi)^{\mathrm{T}}$. Using the method from [14] (see also [17]) one can show that Q is a near-identity analytic diffeomorphism from a neighborhood M_1 of $0 \in \mathcal{X}^1$ onto possibly a different neighborhood M_2 of $0 \in \mathcal{X}^1$ and that for each $U \in M_1$, the linear operator $\mathrm{d}Q[U] \colon \mathcal{X}^1 \to \mathcal{X}^1$ extends to an isomorphism $\widehat{\mathrm{d}Q}[U] \colon \mathcal{X}^0 \to \mathcal{X}^0$ which depends analytically on U and the same holds for the inverse $\widehat{\mathrm{d}Q}[U]^{-1}$. Then the equation (29) is transformed into

$$\frac{\mathrm{d}V}{\mathrm{d}z} = L_\varepsilon V + N(V), \tag{31}$$

where

$$N := \widetilde{F} - L_\varepsilon, \quad \widetilde{F}(V) = \widehat{\mathrm{d}Q}[Q^{-1}(V)](F(U_\varepsilon + Q^{-1}(V))),$$

and the boundary condition (30) becomes linear,

$$\Theta_y = B_{l\varepsilon}(V) \quad \text{on } y = 0, 1.$$

In particular, we recover the linear operator L_ε studied in Section 3, and we can apply the Lyapunov center theorem to conclude.

Proof of Theorem 5.1(iii). The equation (31) is a dynamical system in the phase space $\mathcal{X}^0$ with vector field defined in a neighborhood of 0 in $\mathcal{Y}^1_\varepsilon$. Because the change of variables Q preserves reversibility and reflection symmetries, the vector field in (31) anti-commutes with the reverser R and commutes with the reflection S. Consequently, the system (31) is reversible with reverser R and the reflection symmetry S implies that the subspace $\mathcal{X}^0_+$ given in (19) is invariant. Taking $\mathcal{X} = \mathcal{X}^0_+$ and $\mathcal{Y} = \mathcal{Y}^1_\varepsilon \cap \mathcal{X}^0_+$ the results in Theorem 3.1 imply that the hypotheses of Theorem A.1 hold, for ε sufficiently small. This proves Theorem 1.1(iii). □

5 Parameter Region II

The analysis done for (α, β) in Region I can be easily transferred to the parameter Region II. However, the final result is different because the linear dispersion relation (3) possesses two positive roots for (α, β) in this parameter region. We point out the differences and then state the main result for this parameter region.

Denote by $k_{*,1}$ and $k_{*,2}$ the two positive roots of the dispersion relation. Take $k_{*,1} < k_{*,2}$ and assume that $k_{*,2}/k_{*,1} \notin \mathbb{Z}$.

First, the existence of two-dimensional periodic waves is proved in the same way, with the difference that we now find two geometrically distinct families of two-dimensional periodic waves $\{(\eta_{\varepsilon,1}(X), \phi_{\varepsilon,1}(X,Y))\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$ and $\{(\eta_{\varepsilon,2}(X), \phi_{\varepsilon,2}(X,Y))\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$ with wavenumbers $k_{\varepsilon,1} = k_{*,1} + \mathcal{O}(\varepsilon^2)$ and $k_{\varepsilon,2} = k_{*,2} + \mathcal{O}(\varepsilon^2)$, respectively. The expansions (10) remain valid with k_* replaced by $k_{*,1}$ for the first family and by $k_{*,2}$ for the second family, and this is also the case for all other symbolic computations.

Next, the analysis of the linear operator L_ε given in Section 3 stays the same until the last step of the proof of Theorem 3.1 which consists in showing that $m_{21}^{(2)}$ is negative. The formula for $m_{21}^{(2)}$ is the same, but the result is different for the first family of periodic waves. The analysis in Appendix D gives the conclusion that $m_{21}^{(2)}$ is negative for the second family of periodic waves, whereas for the first family of periodic waves it is negative only when $2k_{*,1} > k_{*,2}$. This condition is satisfied if and only if (α, β) belongs to the open region between Γ_2 and Γ in Figure 1.

Consequently, the two-dimensional periodic waves $(\eta_{\varepsilon,2}(X), \phi_{\varepsilon,2}(X, Y))$ are transversely linearly unstable, whereas the periodic waves $(\eta_{\varepsilon,1}(X), \phi_{\varepsilon,1}(X, Y))$ are unstable if (α, β) lies between Γ_2 and Γ. Notice that our approach does not allow us to conclude on stability because the general criterion in Theorem 4.1 only provides sufficient conditions for instability. Finally, the dimension-breaking result holds for all linearly transversely waves.

We summarize these results in the following theorem.

Theorem 5.1 (Region II) *Fix (α, β) in Region II and let $k_{*,1}, k_{*,2}$ be the two positive roots of the dispersion relation* (3). *Assume that $k_{*,1} < k_{*,2}$ and $k_{*,2}/k_{*,1} \notin \mathbb{Z}$. Denote by Γ_2 the (α, β)-parameter curve for which $2k_{*,1} = k_{*,2}$.*

(i) (Existence) There exist $\varepsilon_0 > 0$ and two geometrically distinct families of two-dimensional steady periodic waves

$$\{(\eta_{\varepsilon,1}(X), \phi_{\varepsilon,1}(X, Y))\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)} \quad \text{and} \quad \{(\eta_{\varepsilon,2}(X), \phi_{\varepsilon,2}(X, Y))\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$$

to the equations (1)–(2)*, such that $(\eta_{0,i}, \phi_{0,i}) = (0, 0)$ and $(\eta_{\varepsilon,i}, \phi_{\varepsilon,i})$ are periodic in X with wavenumbers $k_{\varepsilon,i} = k_{*,i} + \mathcal{O}(\varepsilon^2)$ for $i = 1, 2$.*

(ii) (Transverse instability) There exists $\varepsilon_1 > 0$ such that for each $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ the periodic solution $(\eta_{\varepsilon,2}, \phi_{\varepsilon,2})$ is transversely linearly unstable. The solution $(\eta_{\varepsilon,1}, \phi_{\varepsilon,1})$ is transversely linearly unstable if $2k_{*,1} > k_{*,2}$, which occurs for (α, β) in the open region between the curves Γ_2 and Γ.

(iii) (Dimension-breaking bifurcation) There exists $\varepsilon_2 > 0$ such that for each transversely linearly unstable wave $(\eta_{\varepsilon,i}, \phi_{\varepsilon,i})$ with $\varepsilon \in (-\varepsilon_2, \varepsilon_2)$, $i = 1, 2$, there exist $\delta_\varepsilon > 0$, $\ell^*_{\varepsilon,i} > 0$, and a family of three-dimensional doubly periodic waves $\{(\eta^\delta_{\varepsilon,i}(X, z), \phi^\delta_{\varepsilon,i}(X, Y, z))\}_{\delta\in(-\delta_{\varepsilon,i},\delta_{\varepsilon,i})}$, with wavenumber $k_{*,i}$ in X and wavenumber $\ell_\delta = \ell^*_{\varepsilon,i} + \mathcal{O}(\delta^2)$ in z, bifurcating from the periodic solution $(\eta_{\varepsilon,i}, \phi_{\varepsilon,i})$.

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A Lyapunov center theorem

We state a non-resonant version of the Lyapunov center theorem for reversible systems which is a particular case of the more general version from [2].

Theorem A.1 *Let $\mathcal{X}$ and $\mathcal{Y}$ be real Banach spaces such that $\mathcal{Y}$ is continuously embedded in $\mathcal{X}$. Consider the evolutionary equation*

$$\frac{\mathrm{d}U}{\mathrm{d}t} = F(U), \tag{32}$$

where $F \in \mathscr{C}^6(\mathcal{U}, \mathcal{X})$ with $\mathcal{U} \subset \mathcal{Y}$ a neighborhood of 0. Assume that $F(0) = 0$ and that the following properties hold:

(i) there exists an involution $R \in \mathcal{L}(\mathcal{X}) \cap \mathcal{L}(\mathcal{Y})$ which anticommutes with F, i.e., the equation (32) is reversible;

(ii) the linear operator $L := \mathrm{d}F[0]$ possesses a pair of simple eigenvalues $\pm\mathrm{i}\omega_0$ with $\omega_0 > 0$;

(iii) for each $n \in \mathbb{Z} \setminus \{-1, 1\}$, $\mathrm{i}n\omega_0$ belongs to the resolvent set of L;

(iv) there exists a positive constant C such that

$$\|(L - \mathrm{i}n\omega_0\mathbb{I})^{-1}\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \leq \frac{C}{|n|}, \quad \|(L - \mathrm{i}n\omega_0\mathbb{I})^{-1}\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \leq C, \tag{33}$$

as $n \to \infty$.

Then, there exists a neighborhood $E \subset \mathbb{R}$ of 0 and a $\mathscr{C}^4$-curve $\{U(\varepsilon), \omega(\varepsilon)\}_{\varepsilon \in E}$ where $U(\varepsilon)$ is a real periodic solution to (32) with period $2\pi/\omega(\varepsilon)$. Furthermore, $(U(0), \omega(0)) = (0, \omega_0)$.

In the version of the above theorem from [2] the curve $\{U(\varepsilon), \omega(\varepsilon)\}_{\varepsilon \in E}$ was only continuously differentiable and the vector field F was of class $\mathscr{C}^3$. For our purposes we need at least a $\mathscr{C}^4$-dependence on ε and we therefore assume that F is of class $\mathscr{C}^6$.

B Expansion of the two-dimensional periodic waves

For this computation it is more convenient to use the original system (1)–(2) instead of the dynamical system (5). Restricting to two-dimensional steady solutions, we make the change of variables

$$X = kx, \quad Y = y(1 + \eta(X)), \quad \eta(X) = \widetilde{\eta}(x), \quad \phi(X, Y) = \widetilde{\Phi}(x, y).$$

Dropping the tildes we obtain the equations

$$\begin{aligned} &k^2\Phi_{xx} + \frac{1}{(1+\eta)^2}\Phi_{yy} - 2k^2\frac{y\eta_x}{1+\eta}\Phi_{xy} \\ &\qquad + k^2\left(\frac{y\eta_x}{1+\eta}\right)^2\Phi_{yy} + k^2\left(\frac{2y\eta_x^2}{(1+\eta)^2} - \frac{y\eta_{xx}}{1+\eta}\right)\Phi_y = 0, \end{aligned} \tag{34}$$

for $0 < y < 1$ with boundary conditions

$$
\begin{aligned}
&\Phi_y = 0 && \text{on } y = 0,\\
&\Phi_y = (1+\eta)(-k\eta_x + k^2\eta_x\Phi_x) - k^2\eta_x^2\Phi_y && \text{on } y = 1,\\
&\alpha\eta - k\left(\Phi_x - \frac{\eta_x}{1+\eta}\Phi_y\right) - \beta k^2\left(\frac{\eta_x}{(1+k^2\eta_x^2)^{1/2}}\right)_x && \\
&\qquad + \frac{1}{2}\left(k^2\left(\Phi_x - \frac{\eta_x}{1+\eta}\Phi_y\right)^2 + \frac{\Phi_y^2}{(1+\eta)^2}\right) = 0 && \text{on } y = 1.
\end{aligned}
\tag{35}
$$

The scaled periodic wave $(\widetilde{\eta}_\varepsilon(x), \widetilde{\Phi}_\varepsilon(x,y))$ satisfies these equations for $k = k_\varepsilon$. We insert the expansions (10) into equations (34)–(35) and expand the resulting equations in ε. We restrict to solutions with $\widetilde{\eta}_\varepsilon$ even in x and $\widetilde{\Phi}_\varepsilon$ odd in x.

At order $\mathcal{O}(\varepsilon)$ we find the following equations for η_1 and Φ_1:

$$
\begin{aligned}
&k_*^2\Phi_{1xx} + \Phi_{1yy} = 0 && \text{for } 0 < y < 1,\\
&\Phi_{1y} = 0 && \text{on } y = 0,\\
&\Phi_{1y} = -k_*\eta_{1x} && \text{on } y = 1,\\
&\alpha\eta_1 - k_*\Phi_{1x}|_{y=1} - \beta k_*^2\eta_{1xx} = 0 && \text{on } y = 1.
\end{aligned}
\tag{36}
$$

Taking Fourier series in x,

$$
\eta_1(x) = \sum_{n=0}^{\infty}\eta_{1n}\cos(nx), \quad \Phi_1(x,y) = \sum_{n=1}^{\infty}\phi_{1n}(y)\sin(nx),
$$

we obtain the solvability condition $\mathcal{D}(nk_*) = 0$ where $\mathcal{D}$ is the linear dispersion relation in (3). Consequently, $n = 0$ which gives solutions which are constant in x, only, and $n = \pm 1$ which gives the formulas for η_1 and Φ_1 in (12).

Next, at order $\mathcal{O}(\varepsilon^2)$ we obtain the equations for η_2 and Φ_2,

$$
\begin{aligned}
&k_*^2\Phi_{2xx} + \Phi_{2yy} = 2\eta_1\Phi_{1yy} + 2k_*^2 y\eta_{1x}\Phi_{1xy} + k_*^2 y\eta_{1xx}\Phi_{1y} && \text{for } 0 < y < 1,\\
&\Phi_{2y} = 0 && \text{on } y = 0,\\
&\Phi_{2y} + k_*\eta_{2x} = k_*^2\eta_{1x}\Phi_{1x} - k_*\eta_1\eta_{1x} && \text{on } y = 1,\\
&\alpha\eta_2 - \beta k_*^2\eta_{2xx} - k_*\Phi_{2x} = -k_*\eta_{1x}\Phi_{1y} - \tfrac{1}{2}(k_*^2\Phi_{1x}^2 + \Phi_{1y}^2) && \text{on } y = 1.
\end{aligned}
\tag{37}
$$

Inserting the explicit formulas for η_1 and Φ_1 we obtain

$$
\begin{aligned}
&k_*^2\Phi_{2xx} + \Phi_{2yy} = k_*^2\sinh(k_*)\sin(2x)\left(\cosh(k_*y) - \tfrac{3}{2}k_*y\sinh(k_*y)\right) && \text{for } 0 < y < 1,\\
&\Phi_{2y} = 0 && \text{on } y = 0,\\
&\Phi_{2y} + k_*\eta_{2x} = \left(\tfrac{k_*}{2}\sinh^2(k_*) - \tfrac{k_*^2}{2}\sinh(k_*)\cosh(k_*)\right)\sin(2x) && \text{on } y = 1,\\
&\alpha\eta_2 - \beta k_*^2\eta_{2xx} - k_*\Phi_{2x} = -\tfrac{k_*^2}{4}\cosh(2k_*)\cos(2x) - \tfrac{k_*^2}{4} && \text{on } y = 1.
\end{aligned}
\tag{38}
$$

Observing that the right-hand side only involves the second Fourier mode, we find the formulas for η_2 and Φ_2 in (12).

Finally, the coefficient k_2 is determined from the expansion at order $\mathcal{O}(\varepsilon^3)$,

$$\begin{aligned}
k_*^2\Phi_{3xx} + \Phi_{3yy} = {} & -2k_*k_2\Phi_{1xx} - 3\eta_1^2\Phi_{1yy} + 2\eta_2\Phi_{1yy} + 2\eta_1\Phi_{2yy} \\
& + 2k_*^2 y(\eta_{2x}\Phi_{1xy} + \eta_{1x}\Phi_{2xy} - \eta_{1x}\eta_1\Phi_{1xy} - \eta_{1x}^2\Phi_{1y}) \\
& - k_*^2 y^2\eta_{1x}^2\Phi_{1yy} + yk_*^2(\eta_{1xx}\Phi_{2y} + \eta_{2xx}\Phi_{1y} - \eta_{1xx}\eta_1\Phi_{1y})
\end{aligned}$$

for $0 < y < 1$, and

$$\begin{aligned}
&\Phi_{3y} = 0 && \text{on } y = 0,\\
&\Phi_{3y} + k_*\eta_{3x} = -k_*(\eta_1\eta_{2x} + \eta_{1x}\eta_2) - k_2\eta_{1x} \\
&\qquad + k_*^2(\eta_{1x}\Phi_{2x} + \eta_{2x}\Phi_{1x} + \eta_1\eta_{1x}\Phi_{1x}) - k_*^2\eta_{1x}^2\Phi_{1y} && \text{on } y = 1,\\
&\alpha\eta_3 - \beta k_*^2\eta_{3xx} - k_*\Phi_{3x} = k_2\Phi_{1x} - k_*(\eta_{1x}\Phi_{2y} + \eta_{2x}\Phi_{1y} - \eta_{1x}\eta_1\Phi_{1y}) \\
&\qquad + 2\beta k_* k_2\eta_{1xx} - \frac{3}{2}\beta k_*^4\eta_{1xx}\eta_{1x}^2 \\
&\qquad - k_*^2(\Phi_{1x}\Phi_{2x} - \eta_{1x}\Phi_{1x}\Phi_{1y}) - \Phi_{1y}\Phi_{2y} + \eta_1\Phi_{1y}^2 && \text{on } y = 1.
\end{aligned}$$

The right-hand sides involving only Fourier modes 1 and 3, we write

$$\eta_3(x) = \eta_{31}\cos(x) + \eta_{33}\cos(3x), \quad \Phi_3(x,y) = \phi_{31}(y)\sin(x) + \phi_{33}(y)\sin(3x),$$

and the resulting systems for (η_{31}, ϕ_{31}) and (η_{33}, ϕ_{33}) are decoupled. The coefficient k_2 only appears in the terms with Fourier mode 1 so that it is enough to solve the equations for (η_{31}, ϕ_{31}). These equations are of the form

$$\begin{aligned}
-k_*^2\phi_{31}(y) + \phi_{31}''(y) &= F_3 \qquad \text{for } 0 < y < 1,\\
(\alpha + \beta k_*^2)\eta_{31} - k_*\phi_{31}(1) &= g_3,\\
\phi_{31}'(0) &= 0,\\
\phi_{31}'(1) - k_*\eta_{31} &= f_3,
\end{aligned} \tag{39}$$

where, after computations, we find the explicit formulas

$$\begin{aligned}
F_3(y) = {} & \left(\frac{k_*^2\sinh^2(k_*)}{4} - \frac{k_*^4}{2\alpha} - \frac{k_*^3}{4}\left(c(k_*) + 1\right)\sinh(2k_*) + 2k_*k_2\right)\cosh(k_*y) \\
& + \frac{k_*^3\sinh^2(k_*)}{2} y\sinh(k_*y) + k_*^3 c(k_*)\sinh(k_*)\cosh(2k_*y) \\
& + \frac{3k_*^4 c(k_*)\sinh(k_*)}{4} y\sinh(2k_*y),
\end{aligned}$$

and

$$
\begin{aligned}
g_3 &= \left(\frac{k_*^3}{16} - \frac{k_*^3 c(k_*)}{4} + k_2\right)\cosh(k_*) + \left(-\frac{9}{32}\beta k_*^4 + 2\beta k_* k_2\right)\sinh(k_*)\\
&\quad - \frac{k_*^3}{16}\cosh(3k_*) + \frac{3}{32}\beta k_*^4 \sinh(3k_*),\\
f_3 &= -\frac{k_*^2 c(k_*)}{16}\cosh(k_*) + \left(-\frac{k_*^3 c(k_*)}{4} + \frac{k_*^3}{16} - \frac{k_*^3}{4\alpha} + k_2\right)\sinh(k_*)\\
&\quad + \frac{k_*^2 c(k_*)}{16}\cosh(3k_*) + \frac{3}{16}k_*^3 \sinh(3k_*).
\end{aligned}
$$

Observe that the system (39) is equivalent to the linear nonhomogeneous equation

$$
L_0 \begin{pmatrix} \eta_{31}\cos(x) \\ 0 \\ \phi_{31}(y)\sin(x) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ g_3\cos(x) \\ 0 \\ -F_3(y)\sin(x) \end{pmatrix}. \tag{40}
$$

where L_0 is the linear operator from Section 3, together with the linear nonhomogeneous boundary conditions

$$
\phi_{31}'(0) = 0 \quad \text{and} \quad \phi_{31}'(1) - k_*\eta_{31} = f_3. \tag{41}
$$

Consider the dual vector

$$
\zeta_+^* = \begin{pmatrix} 0 \\ \sinh(k_*)\cos(x) \\ 0 \\ \cosh(k_* y)\sin(x) \end{pmatrix}, \tag{42}
$$

which belongs to the kernel of the adjoint operator L_0^*. Taking into account the nonhomogeneous boundary conditions (41), a direct computation of the scalar product of (40) with ζ_+^* leads to the solvability condition

$$
f_3\cosh(k_*) + g_3\sinh(k_*) = \int_0^1 F_3(y)\cosh(k_* y)\,\mathrm{d}y. \tag{43}
$$

Indeed, integrating twice by parts we find

$$
\begin{aligned}
&\int_0^1 F_3(y)\cosh(k_* y)\,\mathrm{d}y - g_3\sinh(k_*)\\
&= \int_0^1 \left(k_*^2\phi_{31}(y) - \phi_{31}''(y)\right)\cosh(k_* y)\,\mathrm{d}y - g_3\sinh(k_*)\\
&= \phi_{31}'(1)\cosh(k_*) - k_*\phi_{31}(1)\sinh(k_*) - g_3\sinh(k_*)\\
&= (f_3 + k_*\eta_{31})\cosh(k_*) - (\alpha + \beta k_*^2)\eta_{31}\sinh(k_*)\\
&= f_3\cosh(k_*),
\end{aligned}
$$

where we have also used the linear dispersion relation $\mathcal{D}(k_*) = 0$. Replacing the explicit formulas for F_3, f_3, and g_3 into the solvability condition (43) and solving for k_2 we obtain the formula (11).

C Computation of the coefficient $m_{21}^{(2)}$

We prove the equality (22) which connects the coefficient $m_{21}^{(2)}$ with the coefficient k_2 in the expansion of the wavenumber k_ε of the periodic wave.

We emphasize the dependence on k of the vector field F in (5) by writing $F(U,k)$ and similarly for B in the boundary conditions (6) we write $B(U,k)$. Setting

$$\widetilde{B}(U,k) = B(U,k) - \Phi_y,$$

the two-dimensional periodic wave U_ε given in (15) satisfies

$$\begin{cases} F(U_\varepsilon, k_\varepsilon) = 0, & y \in (0,1) \\ \widetilde{B}(U_\varepsilon, k_\varepsilon) = 0, & y = 0, 1. \end{cases} \tag{44}$$

The linear operator L_ε is equal to $\mathrm{D}_U F[U_\varepsilon, k_\varepsilon]$ with boundary conditions $\mathrm{D}_U \widetilde{B}[U_\varepsilon, k_\varepsilon] = 0$. To determine $m_{21}^{(2)}$, we study the problem

$$L_\varepsilon \zeta_+(\varepsilon) = m_{11}(\varepsilon)\zeta_+(\varepsilon) + m_{21}(\varepsilon)\psi_+(\varepsilon), \tag{45}$$

with the boundary conditions

$$\mathrm{D}_U \widetilde{B}[U_\varepsilon, k_\varepsilon]\zeta_+(\varepsilon) = 0. \tag{46}$$

We will make a connection between (44) and (45)–(46) using the expansions in ε of F, $\widetilde{B}$, U_ε, k_ε, $\zeta_+(\varepsilon)$, $\psi_+(\varepsilon)$, $m_{11}(\varepsilon)$ and $m_{21}(\varepsilon)$.

For the system (44), we write

$$\begin{aligned} U_\varepsilon &= \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \mathcal{O}(\varepsilon^4), \\ k_\varepsilon &= k_* + \varepsilon^2 k_2 + \mathcal{O}(\varepsilon^4), \end{aligned}$$

and take the Taylor expansion of $F(U,k)$ at $(0,k_*)$,

$$\begin{aligned} F(U,k) = \mathrm{D}_U F[0,k_*]U &+ \frac{1}{2}\mathrm{D}^2_{UU} F[0,k_*](U,U) \\ &+ \mathrm{D}^2_{Uk} F[0,k_*](U, k-k_*) + \frac{1}{6}\mathrm{D}^3_{UUU} F[0,k_*](U,U,U) \\ &+ \mathcal{O}\left(|k-k_*|^2 \|U\|_{\mathcal{X}^1} + |k-k_*| \|U\|^2_{\mathcal{X}^1} + \|U\|^4_{\mathcal{X}^1}\right), \end{aligned}$$

where all derivatives $\mathrm{D}_k^{(q)}F[0,k_*] = 0$ because $F(0,k) = 0$ for all k. A similar Taylor expansion can be written for the nonlinear boundary condition $\widetilde{B}$. In particular $L_0 = \mathrm{D}_U F[0,k_*]$ with boundary conditions $\widetilde{B}_0 U = \mathrm{D}_U \widetilde{B}[0,k_*]U = 0$ on $y = 0,1$. Inserting these expansions into (44), at order $\mathcal{O}(\varepsilon)$, we find

$$\begin{cases} L_0 U_1 = 0, & y \in (0,1), \\ \widetilde{B}_0 U_1 = 0, & y = 0,1. \end{cases} \tag{47}$$

Thus U_1 belongs to the kernel of L_0 and we choose $U_1 = \zeta_+$ which is in agreement with the expansion of U_ε given by (10). At order $\mathcal{O}(\varepsilon^2)$, we obtain the system

$$\begin{cases} L_0 U_2 = -\frac{1}{2}\mathrm{D}^2_{UU}F[0,k_*](U_1,U_1), & y \in (0,1), \\ \widetilde{B}_0 U_2 = -\frac{1}{2}\mathrm{D}^2_{UU}\widetilde{B}[0,k_*](U_1,U_1), & y = 0,1. \end{cases} \tag{48}$$

and at order $\mathcal{O}(\varepsilon^3)$ we find

$$\begin{cases} L_0 U_3 = -\mathrm{D}^2_{UU}F[0,k_*](U_1,U_2) - \mathrm{D}^2_{Uk}F[0,k_*](U_1,k_2) \\ \qquad\quad -\frac{1}{6}\mathrm{D}^3_{UUU}F[0,k_*](U_1,U_1,U_1), & y \in (0,1), \\ \widetilde{B}_0 U_3 = -\mathrm{D}^2_{UU}\widetilde{B}[0,k_*](U_1,U_2) - \mathrm{D}^2_{Uk}\widetilde{B}[0,k_*](U_1,k_2) \\ \qquad\quad -\frac{1}{6}\mathrm{D}^3_{UUU}\widetilde{B}[0,k_*](U_1,U_1,U_1), & y = 0,1. \end{cases} \tag{49}$$

The nonhomogeneous systems (48) and (49) have each a unique solution U_2 and U_3, respectively, up to an element in the kernel of L_0. Furthermore, $U_2 = (\eta_2, 0, \Phi_2, 0)^{\mathrm{T}}$ in agreement with the expansion of U_ε given by (10).

Similarly, for the spectral problem (45)–(46) we take

$$\begin{aligned} \zeta_+(\varepsilon) &= \zeta_+ + \varepsilon\zeta_{+,1} + \varepsilon^2\zeta_{+,2} + \mathcal{O}(\varepsilon^3), \\ \psi_+(\varepsilon) &= \psi_+ + \varepsilon\psi_{+,1} + \varepsilon^2\psi_{+,2} + \mathcal{O}(\varepsilon^3), \\ m_{11}(\varepsilon) &= \varepsilon^2 m_{11}^{(2)} + \mathcal{O}(\varepsilon^4), \\ m_{21}(\varepsilon) &= \varepsilon^2 m_{21}^{(2)} + \mathcal{O}(\varepsilon^4), \end{aligned}$$

the Taylor expansion

$$\begin{aligned} L_\varepsilon &= \mathrm{D}_U F[U_\varepsilon, k_\varepsilon](\,\cdot\,) \\ &= \mathrm{D}_U F[0,k_*](\,\cdot\,) + \mathrm{D}^2_{UU}F[0,k_*](U_\varepsilon,\,\cdot\,) + \mathrm{D}^2_{Uk}F[0,k_*](\,\cdot\,, k_\varepsilon - k_*) \\ &\quad + \frac{1}{2}\mathrm{D}^3_{UUU}F[0,k_*](U_\varepsilon,U_\varepsilon,\,\cdot\,) + \mathcal{O}\Big(|k_\varepsilon - k_*|^2 + |k_\varepsilon - k_*|\|U_\varepsilon\|_{\mathcal{X}^1} + \|U_\varepsilon\|^3_{\mathcal{X}^1}\Big), \end{aligned}$$

and a similar Taylor expansion can be written for $\mathrm{D}_U\widetilde{B}[U_\varepsilon, k_\varepsilon]$. Inserting these expansions in equations (45)–(46), we find at order $\mathcal{O}(1)$ the system

$$\begin{cases} L_0 \zeta_+ = 0, & y \in (0,1), \\ \widetilde{B}_0 \zeta_+ = 0, & y = 0,1. \end{cases}$$

This is precisely the system (47) and we may choose $U_1 = \zeta_+$. Next, at order $\mathcal{O}(\varepsilon)$, we find

$$\begin{cases} L_0\zeta_{+,1} = -\mathrm{D}^2_{UU}F[0,k_*](U_1,U_1), & y\in(0,1),\\ \widetilde{B}_0\zeta_{+,1} = -\mathrm{D}^2_{UU}\widetilde{B}[0,k_*](U_1,U_1), & y=0,1.\end{cases}$$

Comparing to (48), we choose $\zeta_{+,1} = 2U_2$. Finally, at order $\mathcal{O}(\varepsilon^2)$, we have the system

$$\begin{cases} L_0\zeta_{+,2} = -3\mathrm{D}^2_{UU}F[0,k_*](U_1,U_2) - \mathrm{D}^2_{Uk}F[0,k_*](U_1,k_2) \\ \qquad\qquad - \frac{1}{2}\mathrm{D}^3_{UUU}F[0,k_*](U_1,U_1,U_1) + m^{(2)}_{11}\zeta_+ + m^{(2)}_{21}\psi_+, & y\in(0,1),\\ \widetilde{B}_0\zeta_{+,2} = -3\mathrm{D}^2_{UU}\widetilde{B}[0,k_*](U_1,U_2) - \mathrm{D}^2_{Uk}\widetilde{B}[0,k_*](U_1,k_2) \\ \qquad\qquad - \frac{1}{2}\mathrm{D}^3_{UUU}\widetilde{B}[0,k_*](U_1,U_1,U_1), & y=0,1.\end{cases}$$

Substracting the system (49) from the one above we obtain the system

$$\begin{cases} L_0(\zeta_{+,2}-3U_3) = 2\mathrm{D}^2_{Uk}F[0,k_*](U_1,k_2) + m^{(2)}_{11}\zeta_+ + m^{(2)}_{21}\psi_+, & y\in(0,1),\\ \widetilde{B}_0(\zeta_{+,2}-3U_3) = 2\mathrm{D}^2_{Uk}\widetilde{B}[0,k_*](U_1,k_2), & y=0,1,\end{cases} \tag{50}$$

in which k_2 and $m^{(2)}_{21}$ appear in the right-hand sides of the two equalities.

We obtain the connection between k_2 and $m^{(2)}_{21}$ by taking the scalar product of the first equation in (50) with the dual vector ζ_+^* given by (42) which belongs to the kernel of the adjoint operator L_0^*. This leads to the equality

$$m^{(2)}_{21} = \frac{\langle L_0(\zeta_{+,2}-3U_3),\zeta_+^*\rangle}{\langle\psi_+,\zeta_+^*\rangle} - 2\frac{\langle \mathrm{D}^2_{Uk}F[0,k_*](U_1,k_2),\zeta_+^*\rangle}{\langle\psi_+,\zeta_+^*\rangle}. \tag{51}$$

It remains to explicitly compute the right-hand side of this equality.

First, we compute the denominator

$$\begin{aligned}\langle\psi_+,\zeta_+^*\rangle &= \int_0^{2\pi}\int_0^1 \cosh^2(k_*y)\sin^2(x)\,\mathrm{d}y\,\mathrm{d}x + \int_0^{2\pi}\beta\sinh^2(k_*)\cos^2(x)\,\mathrm{d}x\\ &= \pi\left(\frac{1}{2} + \frac{\cosh(k_*)\sinh(k_*)}{2k_*} + \beta\sinh^2(k_*)\right).\end{aligned}$$

Next, observe that $\zeta_{+,2} = (\eta_{+,2},0,\Phi_{+,2},0)^{\mathrm{T}}$, and similarly the second and fourth components of U_3 vanish. Setting

$$\zeta_{+,2} - 3U_3 = (\eta,0,\Phi,0)^{\mathrm{T}},$$

and taking into account the boundary conditions from (50), we find

$$\begin{aligned}\langle L_0(\zeta_{+,2} - 3U_3), \zeta_+^* \rangle &= \int_0^1 \int_0^{2\pi} \left(-k_*^2 \Phi_{xx} - \Phi_{yy}\right) \cosh(k_* y) \sin(x) \, \mathrm{d}x \, \mathrm{d}y \\ &\quad + \int_0^{2\pi} \left(\alpha\eta - \beta k_*^2 \eta_{xx} - k_* \Phi_x\big|_{y=1}\right) \sinh(k_*) \cos(x) \, \mathrm{d}x \\ &= -2 \int_0^{2\pi} k_2 \eta_{1x} \cosh(k_*) \sin(x) \, \mathrm{d}x \\ &= 2k_2 \int_0^{2\pi} \sinh(k_*) \cosh(k_*) \sin^2(x) \, \mathrm{d}x \\ &= \pi k_2 \sinh(2k_*).\end{aligned}$$

where we have integrated twice by parts and used the linear dispersion relation $\mathcal{D}(k_*) = 0$. Finally,

$$\mathrm{D}^2_{Uk} F[0, k_*](U_1, k_2) = k_2 \begin{pmatrix} 0 \\ -2\beta k_* \eta_{1xx} - \Phi_{1x}|_{y=1} \\ 0 \\ -2k_* \Phi_{1xx} \end{pmatrix},$$

which gives

$$\langle \mathrm{D}^2_{Uk} F[0, k_*](U_1, k_2), \zeta_+^* \rangle = \pi k_2 \left(2\beta k_* \sinh^2(k_*) + k_*\right).$$

Replacing these explicit formulas into (51) gives the equality (22).

D Sign of the coefficient $m_{21}^{(2)}$

Consider $\widetilde{m}_{21}^{(2)}$ given by (23) and assume $k_* > 0$ is such that $\mathcal{D}(k_*) = 0$. From the linear dispersion relation, we find that

$$\beta = \frac{1}{k_* \tanh(k_*)} - \frac{\alpha}{k_*^2}.$$

Since $\beta > 0$, we have $0 < \alpha < k_*/\tanh(k_*)$. Setting

$$\alpha = \frac{k_* \cosh(k_*)}{\sinh(k_*)} \cdot w,$$

we obtain

$$\widetilde{m}_{21}^{(2)} = -16k_*^2 \cdot \frac{n_3 w^3 - n_2 w^2 + n_1 w + n_0}{\mathcal{D}(2k_*)},$$

in which $w \in [0,1]$, and

$$\begin{aligned}
n_3 &= 9\cosh^4(k_*)\left(\cosh^2(k_*) - 1\right),\\
n_2 &= 3\cosh^2(k_*)\left(\cosh^4(k_*) + 4\cosh^2(k_*) - 1\right),\\
n_1 &= \cosh^4(k_*)(2\cosh^2(k_*) + 13),\\
n_0 &= 4\cosh^2(k_*) + 2.
\end{aligned}$$

We will now show that the numerator $n_3w^3 - n_2w^2 + n_1w + n_0$ is positive. To this end, note that $n_0 > 0$ for all $k_* > 0$. This gives

$$n_3w^3 - n_2w^2 + n_1w + n_0 > w(n_3w^2 - n_2w + n_1).$$

Observe that the coefficients n_3, n_2 and n_1 have a common factor $\cosh^2(k_*)$. Factoring this out, it suffices to show that the quadratic polynomial in w

$$P(w,k_*) \coloneqq \frac{1}{\cosh^2(k_*)}\left(n_3(k_*)w^2 - n_2(k_*)w + n_1(k_*)\right)$$

is positive for all $k_* > 0$ and $w \in [0,1]$. A simple calculation shows that $P(w,0) > 0$ for $w \in [0,1]$. Differentiating $P(w,k_*)$ with respect to k_* yields

$$\begin{aligned}
\mathrm{D}_{k_*}P(w,k_*) = \sinh(2k_*)\Big(&9(2\cosh^2(k_*) - 1)w^2\\
&- 6(\cosh^2(k_*) + 2)w + 4\cosh^2(k_*) + 13\Big).
\end{aligned}$$

Set $S = \sinh(2k_*)$ and $K = \cosh^2(k_*) \geq 1$. Completing the square in w gives

$$\mathrm{D}_{k_*}P(w,k_*) = 9S(2K-1)\left[\left(w - \frac{K+2}{3(2K-1)}\right)^2 + \frac{7K^2 + 18K - 17}{9(2K-1)^2}\right],$$

where $7K^2 + 18K - 17 > 0$ for all $K \geq 1$. This shows that $\mathrm{D}_{k_*}P(w,k_*) > 0$ for all $k_* > 0$ and $w \in \mathbb{R}$, which together with $P(w,0) > 0$ establishes our claim on the positivity of the numerator for $w \in [0,1]$.

As a consequence, the sign of $\widetilde{m}_{21}^{(2)}$ is completely determined by the sign of $\mathcal{D}(2k_*)$. In Region I, the linear dispersion relation $\mathcal{D}(k) = 0$ has a unique positive simple root $k = k_*$. It follows that the smooth function $k \mapsto \mathcal{D}(2k)$ changes signs exactly once at $k = k_*/2$. Since $\mathcal{D}(0) = 0$ and $\mathcal{D}'(0) < 0$, we conclude that $\mathcal{D}(2k)$ is negative for $k \in (0, k_*/2)$, and positive for $k \in (k_*/2, \infty)$. In particular, we have $\mathcal{D}(2k_*) > 0$. In Region II, the linear dispersion relation $\mathcal{D}(k) = 0$ has two positive simple roots $k = k_{*,1}$ and $k = k_{*,2}$, $0 < k_{*,1} < k_{*,2}$. This means that the smooth function $\mathcal{D}(2k)$ changes signs exactly twice, first at $k = k_{*,1}/2$ and then at $k = k_{*,2}/2$. Since $\mathcal{D}(0) = 0$ and $\mathcal{D}'(0) > 0$, we conclude that $\mathcal{D}(2k)$ is positive on $(0, k_{*,1}/2)$, negative on $(k_{*,1}/2, k_{*,1}/2)$ and positive on $(k_{*,2}/2, \infty)$. This implies that $\mathcal{D}(2k_{*,2}) > 0$, whereas

$$\begin{aligned} \mathcal{D}(2k_{*,1}) > 0 \quad &\text{if} \quad k_{*,1} > k_{*,2}/2, \\ \mathcal{D}(2k_{*,1}) < 0 \quad &\text{if} \quad k_{*,1} < k_{*,2}/2. \end{aligned}$$

The parameter region in which we have $\mathcal{D}(2k_{*,1}) > 0$ is precisely the open region between the curve Γ and Γ_2. We summarize our findings below.

Proposition D.1 *If (α, β) belongs to Region I, the coefficient $m_{21}^{(2)}$ is negative. If (α, β) belongs to Region II, then*

- *$m_{21}^{(2)}$ is negative for both wavenumbers $k_{*,1}$ and $k_{*,2}$ if (α, β) lies in the open region between Γ and Γ_2,*
- *$m_{21}^{(2)}$ is positive for $k_{*,1}$ and negative for $k_{*,2}$ if (α, β) lies to the left of Γ_2.*

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