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CONVERGENCE ANALYSIS OF THE NONOVERLAPPING ROBIN–ROBIN METHOD FOR NONLINEAR ELLIPTIC EQUATIONS*

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Abstract. We prove convergence for the nonoverlapping Robin–Robin method applied to nonlinear elliptic equations with a p-structure, including degenerate diffusion equations governed by the p-Laplacian. This nonoverlapping domain decomposition is commonly encountered when discretizing elliptic equations, as it enables the usage of parallel and distributed hardware. Convergence has been derived in various linear contexts, but little has been proven for nonlinear equations. Hence, we develop a new theory for nonlinear Steklov–Poincaré operators based on the p-structure and the L^p-generalization of the Lions–Magenes spaces. This framework allows the reformulation of the Robin–Robin method into a Peaceman–Rachford splitting on the interfaces of the subdomains, and the convergence analysis then follows by employing elements of the abstract theory for monotone operators. The analysis is performed on Lipschitz domains and without restrictive regularity assumptions on the solutions.

Key words. Robin–Robin method, nonoverlapping domain decomposition, nonlinear elliptic equation, convergence, Steklov–Poincaré operator

AMS subject classifications. 65N55, 65J15, 35J70, 47N20

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1. Introduction.

Approximating the solution of an elliptic partial differential equation (PDE) typically demands large-scale computations requiring the usage of parallel and distributed hardware. In this context, a nonoverlapping domain decomposition method is a suitable choice, as it can be implemented in parallel with local communication. After decomposing the equation's spatial domain into nonoverlapping subdomains, the method consists of an iterative procedure that solves the equation on each subdomain and thereafter communicates the results via the boundaries to the adjacent subdomains. For a general introduction, we refer the reader to [32, 37].

There is a vast amount of methods in the literature, employing different transmission conditions between the subdomains. The standard examples are based on the alternate use of Dirichlet and Neumann boundary conditions, but a competitive alternative is the Robin–Robin method, where the same type of Robin boundary condition is used for all subdomains. The Robin–Robin method was introduced in [26] together with a convergence proof when applied to linear elliptic equations. After applying a finite element discretization, convergence rates of the form $1 - \mathcal{O}(\sqrt{h})$, with h denoting the mesh width, have been derived in various linear contexts [21, 28, 38]; also see [17, 18]. For generalizations and further applications of the Robin–Robin method applied to linear PDEs, we refer the reader to [3, 7, 8, 9, 21] and references therein.

When considering nonlinear elliptic PDEs, the literature is more limited. Convergence studies relating to overlapping Schwarz methods are given in [14, 25, 35, 36]. However, there are hardly any results dealing with nonoverlapping domain

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decomposition schemes. One exception is [5], where the convergence of the Dirichlet–Neumann and Robin–Robin methods is analyzed for a family of one-dimensional elliptic equations. A related study is [34], where the equivalence between a class of nonlinear elliptic equations and the corresponding transmission problems is proven for nonoverlapping decompositions with cross points, but no numerical scheme is considered. Apart from [36], all these nonlinear studies rely on frameworks similar to the linear case, e.g., assuming that the diffusion is uniformly positive. Hence, the aim of this paper is to derive a genuinely nonlinear extension of the linear convergence result given in [26] for the nonoverlapping Robin–Robin method.

We will focus on nonlinear elliptic equations of the form

$$\begin{cases} -\nabla\cdot\alpha(\nabla u)+g(u)=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d=1,2,\dots$, with boundary $\partial\Omega$. The functions α and g are assumed to have a p-structure, defined in section 2. This structure enables a clear-cut convergence analysis for a broad family of degenerate elliptic equations; i.e., $\alpha(\nabla u)$ may vanish for nonzero values of u. The latter typically prevents the existence of a strong solution in $W^{2,p}(\Omega)$. Examples of other numerical studies concerning nonlinear equations with a p-structure include [2, 12, 15, 16].

The archetypical examples of nonlinear elliptic equations with a p-structure are those governed by the p-Laplacian, where $\alpha(z)=|z|^{p-2}z$. Examples include the computation of the nonlinear resolvent

$$-\nabla\cdot(|\nabla u|^{p-2}\nabla u)+\lambda u=f, \tag{1.2}$$

arising in the context of an implicit Euler discretization of the parabolic p-Laplace equation, and the nonlinear reaction-diffusion problem

$$-\nabla\cdot(|\nabla u|^{p-2}\nabla u)+\lambda|u|^{p-2}u=f. \tag{1.3}$$

For sake of simplicity, we decompose the original domain Ω into two nonoverlapping subdomains Ω_i, $i=1,2$, with boundaries denoted by $\partial\Omega_i$ and separated by the interface Γ, i.e.,

$$\overline{\Omega}=\overline{\Omega}_1\cup\overline{\Omega}_2,\quad \Omega_1\cap\Omega_2=\emptyset,\quad \text{and}\quad \Gamma=(\partial\Omega_1\cap\partial\Omega_2)\setminus\partial\Omega.$$

Two examples of such decompositions are illustrated in Figure 1a and 1b, respectively. The analysis presented here can also, in a trivial fashion, be extended to the case when Ω_i is a union of nonadjacent subdomains, i.e., $\Omega_i=\cup_{\ell=1}^N\Omega_{i\ell}$ such that $\overline{\Omega}_{i\ell}\cap\overline{\Omega}_{ik}=\emptyset$ for $\ell\neq k$. An example of this is the stripewise domain decomposition illustrated in Figure 1c.

For a given domain decomposition, we can then restate the nonlinear elliptic equation (1.1) as two equations on Ω_i connected via Γ; i.e., we consider the nonlinear transmission problem

$$\begin{cases} -\nabla\cdot\alpha(\nabla u_i)+g(u_i)=f & \text{in } \Omega_i,\\ u_i=0 & \text{on } \partial\Omega_i\setminus\Gamma \text{ for } i=1,2,\\ u_1=u_2 & \text{on } \Gamma,\\ \alpha(\nabla u_1)\cdot\nu_1=-\alpha(\nabla u_2)\cdot\nu_2 & \text{on } \Gamma, \end{cases} \tag{1.4}$$

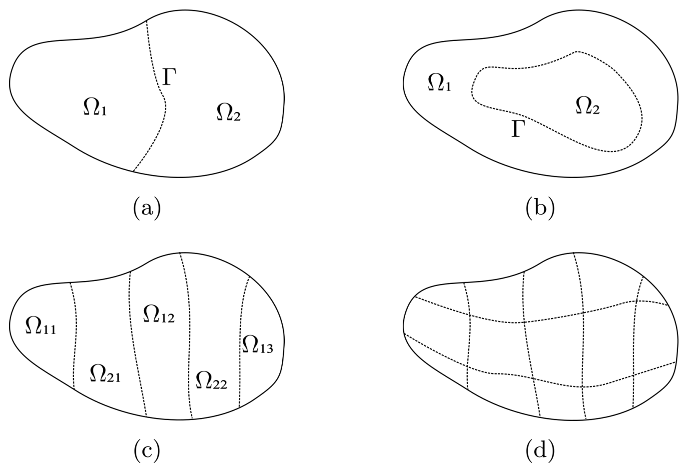

Fig. 1. *Examples of different domain decompositions:* (a) *a domain decomposition with two intersection points between* $\partial\Omega$ *and* Γ*;* (b) *a decomposition without intersection points between* $\partial\Omega$ *and* Γ*;* (c) *a stripewise decomposition without cross points;* (d) *a full decomposition with cross points.*

where ν_i denotes the unit outward normal vector of $\partial\Omega_i$. As $\nu_1 = -\nu_2$ on Γ, the last two equations of (1.4) are equivalent to the Robin conditions

$$\alpha(\nabla u_1) \cdot \nu_i + su_1 = \alpha(\nabla u_2) \cdot \nu_i + su_2 \quad \text{on } \Gamma \text{ for } i = 1, 2,$$

where s is nonzero. Alternating between the subdomains then leads to the Robin–Robin method, which is given by finding (u_1^n, u_2^n) for $n = 1, 2, \ldots$ such that

$$\text{(1.5)} \qquad \begin{cases} -\nabla \cdot \alpha(\nabla u_1^{n+1}) + g(u_1^{n+1}) = f & \text{in } \Omega_1, \\ u_1^{n+1} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma, \\ \alpha(\nabla u_1^{n+1}) \cdot \nu_1 + su_1^{n+1} = \alpha(\nabla u_2^n) \cdot \nu_1 + su_2^n & \text{on } \Gamma, \\ -\nabla \cdot \alpha(\nabla u_2^{n+1}) + g(u_2^{n+1}) = f & \text{in } \Omega_2, \\ u_2^{n+1} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma, \\ \alpha(\nabla u_2^{n+1}) \cdot \nu_2 + su_2^{n+1} = \alpha(\nabla u_1^{n+1}) \cdot \nu_2 + su_1^{n+1} & \text{on } \Gamma, \end{cases}$$

where u_2^0 is a given initial guess and $s > 0$ is referred to as the method parameter. Here, u_i^n and $u_i^n|_\Gamma$ approximate $u_i = u|_{\Omega_i}$ and $u_i|_\Gamma = u|_\Gamma$, respectively. Note that the method in itself is sequential, but the computation of each u_i^n can be implemented in a parallel fashion if Ω_i is a union of nonadjacent subdomains.

The convergence analysis is organized as follows. For linear elliptic equations, i.e., equations with a 2-structure, the analysis relies on the trace operator from $H^1(\Omega_i)$ onto $H^{1/2}(\partial\Omega_i)$ and the Lions–Magenes spaces $H_{00}^{1/2}(\Gamma)$. We therefore start by introducing the generalized p-version of the trace operator, now given from $W^{1,p}(\Omega_i)$ onto $W^{1-1/p,p}(\partial\Omega_i)$, and the corresponding Lions–Magenes spaces Λ_i; see sections 3 and 4. Results related to these spaces are, e.g., discussed in [4, 20]. There is, however, a surprising lack of proofs in the literature dealing with this generalized p-setting. Hence, we will make an effort to give precise definitions and proof references.

With the correct function spaces in place, we prove that the weak forms of the elliptic equation and the transmission problem are equivalent in Theorem 5.2 and introduce the new nonlinear Steklov–Poincaré operators, as maps from Λ_i to Λ_i^* in section 6. The latter yields that the transmission problem can be stated as a problem on Γ, and the Robin–Robin method reduces to the Peaceman–Rachford splitting. The main challenge is then to derive the fundamental properties of the nonlinear Steklov–Poincaré operators from the p-structure, which is achieved in section 7. By interpreting the nonlinear Steklov–Poincaré operators as unbounded, monotone maps on $L^2(\Gamma)$, We finally prove that the Robin–Robin method is well defined on $W^{1,p}(\Omega_1)\times W^{1,p}(\Omega_2)$ (see Corollary 8.6) and convergent in the same space; see Theorem 8.9. The latter relies on the abstract theory of the Peaceman–Rachford splittings [27].

The continuous convergence analysis presented here also holds in the finite-dimensional case obtained after a suitable spatial discretization, e.g., by employing finite elements. However, we will limit ourselves to the continuous case in this paper. Hence, important issues including convergence rates for the finite-dimensional case and the influence of the mesh width on the optimal choice of the method parameter s will be explored elsewhere.

Finally, (c, c_i, C, C_i) will denote generic positive constants that assume different values at different occurrences.

2. Nonlinear elliptic equations with p-structure. Throughout the paper, we will consider the nonlinear elliptic equation (1.1) with $f \in L^2(\Omega)$ and Ω being a bounded Lipschitz domain. The equation is assumed to have a p-structure of the following form.

Assumption 2.1. The parameters (p, r) and the functions $\alpha : \mathbb{R}^d \to \mathbb{R}^d, g : \mathbb{R} \to \mathbb{R}$ satisfy the following properties:

- Let $p \in [2, \infty)$ and $r \in (1, \infty)$. If $p < d$, then $r \leq dp/\big(2(d-p)\big) + 1$.
- The functions α and g are continuous and satisfy the growth conditions

$$|\alpha(z)| \leq C|z|^{p-1} \quad \text{and} \quad |g(x)| \leq C|x|^{r-1} \quad \text{for all } z \in \mathbb{R}^d, x \in \mathbb{R}.$$

- The function α is strictly monotone with the bound

$$\big(\alpha(z) - \alpha(\tilde{z})\big) \cdot (z - \tilde{z}) \geq c|z - \tilde{z}|^p \quad \text{for all } z, \tilde{z} \in \mathbb{R}^d.$$

- The function α is coercive with the bound

$$\alpha(z) \cdot z \geq c|z|^p \quad \text{for all } z \in \mathbb{R}^d.$$

- The function g is strictly monotone and coercive with the bounds

$$\big(g(x) - g(\tilde{x})\big)(x - \tilde{x}) \geq c|x - \tilde{x}|^r \quad \text{and} \quad g(x)x \geq c|x|^r \quad \text{for all } x, \tilde{x} \in \mathbb{R}.$$

Let $V = W_0^{1,p}(\Omega)$, and define the form $a : V \times V \to \mathbb{R}$ by

$$a(u, v) = \int_\Omega \alpha(\nabla u) \cdot \nabla v + g(u)v \,\mathrm{d}x.$$

The weak form of (1.1) is to find $u \in V$ such that

$$a(u, v) = (f, v)_{L^2(\Omega)} \quad \text{for all } v \in V. \tag{2.1}$$

The p-structure implies that there exists a unique weak solution of (2.1); see, e.g., [33, Theorem 2.36]. A central part of the existence proof and of our convergence analysis as well is to observe that the p-structure directly implies that the form a is bounded, strictly monotone, and coercive.

LEMMA 2.2. *If Assumption* 2.1 *holds, then* $a : V \times V \to \mathbb{R}$ *is well defined and satisfies the upper bound*

$$|a(u,v)| \le C_1\big(\|\nabla u\|_{L^p(\Omega)^d}^{p-1}\|\nabla v\|_{L^p(\Omega)^d} + \|u\|_{L^r(\Omega)}^{r-1}\|v\|_{L^r(\Omega)}\big),$$

the strict monotonicity bound

$$a(u,u-v) - a(v,u-v) \ge c_1\big(\|\nabla(u-v)\|_{L^p(\Omega)^d}^p + \|u-v\|_{L^r(\Omega)}^r\big),$$

and the coercivity bound

$$a(u,u) \ge c_2\big(\|\nabla u\|_{L^p(\Omega)^d}^p + \|u\|_{L^r(\Omega)}^r\big)$$

for all $u, v \in V$.

Example 2.3. The equation (1.2) satisfies Assumption 2.1 with $\alpha(z) = |z|^{p-2}z$, $\lambda > 0$, $g(x) = \lambda x$, and $r = 2$. The same holds for equation (1.3) with $g(x) = \lambda|x|^{p-2}x$ and $r = p$.

Remark 2.4. The last assertion of Assumption 2.1 is made in order to ensure that the convergence analysis of the domain decomposition is valid without employing the Poincaré inequality, which allows decompositions where $\partial\Omega\setminus\partial\Omega_i = \emptyset$; see Figure 1b. If the latter setting is excluded, then the analysis is valid for a broader class of functions g, especially $g = 0$.

Remark 2.5. Possible generalizations of Assumption 2.1, which we omit for sake of notational simplicity, include dependence on the spatial variable and first-order terms, e.g., $\alpha(\nabla u) = \alpha(x,u,\nabla u)$, and the parameter choice $p \in (2d/(d+1), 2)$, which requires an additional set of embedding results for trace spaces. One can also extend the analysis to other continuous functions α that give rise to bounded, strictly monotone, and coercive forms a on $V \times V$, e.g.,

$$\alpha(z) = (1+|z|^2)^{(p-2)/2}z \quad \text{and} \quad \alpha(z) = (|z_1|^{p-2}z_1, \dots, |z_d|^{p-2}z_d);$$

see [10, section 4] and [39, section 26.5], respectively, for proofs and further details.

In order to conduct the convergence analysis, we also make the following additional regularity assumption on the weak solution.

Assumption 2.6. The weak solution $u \in V$ of (2.1) satisfies $\alpha(\nabla u) \in C(\overline{\Omega})^d$.

Note that the above regularity assumption does *not* imply that u is a strong solution in $W^{2,p}(\Omega)$. A possible generalization of Assumption 2.6 is discussed in Remark 8.3.

Example 2.7. Consider the equations given by the p-Laplacian in Example 2.3. If $p \ge d$, then the weak solution $u \in V$ is also in $C(\overline{\Omega})$. If in addition $f \in L^\infty(\Omega)$ and the boundary $\partial\Omega$ is $C^{1,\beta}$, then [24, Theorem 1] yields that $u \in C^{1,\beta}(\overline{\Omega})$. The latter implies that Assumption 2.6 is valid in this context.

Finally, we will make frequent use of the fact that, under Assumption 2.1, the standard $W^{1,p}(\Omega)$-norm is equivalent to the norm

$$u \mapsto \|\nabla u\|_{L^p(\Omega)^d} + \|u\|_{L^r(\Omega)}. \tag{2.2}$$

For $r \ge p$, this follows directly by the Sobolev embedding theorem together with the assumed restrictions on (p,r). For $r < p$, the equivalence can, e.g., be proven by Ehrling's lemma and the observation that $W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^p(\Omega) \hookrightarrow L^r(\Omega)$; see [33, Theorem 1.21 and Lemma 7.6].

3. Function spaces and trace operators on $\boldsymbol{\Omega_i}$. We start by considering a manifold $\mathcal{M}$ in $\mathbb{R}^d$, which will play the role of $\partial\Omega_i$ or Γ. The manifold $\mathcal{M}$ is said to be Lipschitz if there exist finitely many open, overlapping sets Θ_m such that

$$\mathcal{M} = \bigcup_{m=1}^{M} \Theta_m,$$

where each Θ_m can be described as the graph of a Lipschitz continuous function b_m. More precisely, there exists $(d-1)$-dimensional cubes θ_m and local charts $\psi_m : \Theta_m \to \theta_m$ that are bijective and Lipschitz continuous. The charts have the structure $\psi_m^{-1} = A_m^{-1} \circ Q_m$, where $A_m : \mathbb{R}^d \to \mathbb{R}^d$ is a coordinate transformation, i.e., $A_m x = \tilde{A}_m x + v_m$, where $\tilde{A}_m$ is an orthonormal matrix with $\det \tilde{A}_m = 1$ and

$$Q_m : \theta_m \to \mathbb{R}^d : x_m \mapsto \big(x_m, b_m(x_m)\big)$$

for the Lipschitz continuous map $b_m : \theta_m \to \mathbb{R}$. A function $\mu : \mathcal{M} \to \mathbb{R}$ now has the local components $\mu \circ \psi_m^{-1}$. We refer the reader to [22, section 6.2] for further details.

On a Lipschitz manifold, we may introduce a measure [29, Chapter 3] and thus define the integral and the space $L^p(\mathcal{M})$; see, e.g., [11]. From [11, Chapters 3.4–3.5], it follows that $L^p(\mathcal{M})$ is a Banach space and that $L^2(\mathcal{M})$ is a Hilbert space with the inner product

$$(\eta, \mu)_{L^2(\mathcal{M})} = \int_{\mathcal{M}} \eta\mu \,\mathrm{d}S.$$

Let $\{\varphi_m\}$ be a partition of unity of $\mathcal{M}$ subordinate to $\{\Theta_m\}$. The integral then satisfies

$$\int_{\mathcal{M}} \mu \,\mathrm{d}S = \sum_{m=1}^{M} \int_{\theta_m} (\mu\varphi_m) \circ \psi_m^{-1} |n_m| \,\mathrm{d}x, \tag{3.1}$$

where $n_m = (\partial_1 b_m, \partial_2 b_m, \ldots, \partial_{d-1} b_m, -1)$; see [29, Theorem 3.9]. Seemingly obvious properties of the integral, including

$$\int_{\mathcal{M}} \mu \,\mathrm{d}S = \int_{\mathcal{M}_0} \mu \,\mathrm{d}S + \int_{\mathcal{M} \setminus \mathcal{M}_0} \mu \,\mathrm{d}S,$$

rely heavily on the observation that the integral is independent of the representation (Θ_m, A_m, b_m) and the choice of partition of unity $\{\varphi_m\}$; see [29, Theorems 3.5 and 3.7] and the comments thereafter.

The equality (3.1) also shows that our integral and L^p-spaces are equivalent to the ones used in [22]. Moreover, by [22, Lemma 6.3.5], the L^p-norm used here is equivalent to the norm

$$\mu \mapsto \left(\sum_{m=1}^{M} \|\mu \circ \psi_m^{-1}\|_{L^p(\theta_m)}^p \right)^{1/p}.$$

Finally, recall that for a Lipschitz manifold $\mathcal{M}$, the unit outward normal vector $\nu = (\nu^1, \ldots, \nu^d)$ is defined almost everywhere; see [22, section 6.10.1]. The normal vector is given locally by $\nu \circ \psi_m^{-1} = n_m / |n_m|$, and the Lipschitz continuity of b_m yields that $\nu^\ell \in L^\infty(\mathcal{M})$ for all $\ell = 1, \ldots, d$.

Assumption 3.1. The boundaries $\partial\Omega_i$ and the interface Γ are all $(d-1)$-dimensional Lipschitz manifolds.

We use the notation $(\Theta_m^i, \theta_m^i, M_i, \psi_m^i, b_m^i, \phi_m^i, \nu_i)$ for the quantities related to the local representations of $\partial\Omega_i$. Next, we define the fractional Sobolev spaces on the $(d-1)$-dimensional cubes θ_m. Let $0 < s < 1$. Then $W^{s,p}(\theta_m)$ is defined as all $u \in L^p(\theta_m)$ such that

$$|u|_{s,\theta_m} = \left(\int_{\theta_m}\int_{\theta_m} \frac{|u(x)-u(y)|^p}{|x-y|^{d-1+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} < \infty.$$

The corresponding norm is given by

$$\|u\|_{W^{s,p}(\theta_m)} = \|u\|_{L^p(\theta_m)} + |u|_{s,\theta_m}.$$

Having defined the fractional Sobolev spaces on θ_m^i, we can also define them on $\partial\Omega_i$. For $0 < s < 1$, introduce

$$W^{s,p}(\partial\Omega_i) = \{\mu \in L^p(\partial\Omega_i) : \mu \circ (\psi_m^i)^{-1} \in W^{s,p}(\theta_m^i) \text{ for } m = 1, \dots, M_i\},$$

equipped with the norm

$$\|\mu\|_{W^{s,p}(\partial\Omega_i)} = \left(\sum_{m=1}^{M_i} \|\mu \circ (\psi_m^i)^{-1}\|^p_{W^{s,p}(\theta_m^i)}\right)^{1/p}.$$

By the definitions of the norms, it follows directly that

$$\|\mu\|_{L^p(\partial\Omega_i)} \le C\|\mu\|_{W^{s,p}(\partial\Omega_i)}.$$

Furthermore, the space $W^{s,p}(\partial\Omega_i)$ is complete and for $p > 1$ reflexive; see [22, Definition 6.8.6] and the comment thereafter. Next, we recapitulate the trace theorem for $W^{1,p}$-functions on Lipschitz domains; see, e.g., [22, Theorems 6.8.13 and 6.9.2].

LEMMA 3.2. *If $p > 1$ and the Assumption 3.1 is valid, then there exists a surjective bounded linear operator $T_{\partial\Omega_i} : W^{1,p}(\Omega_i) \to W^{1-1/p,p}(\partial\Omega_i)$ such that $T_{\partial\Omega_i} u = u|_{\partial\Omega_i}$ when $u \in C^\infty(\overline{\Omega}_i)$. The operator $T_{\partial\Omega_i}$ has a bounded linear right inverse $R_{\partial\Omega_i} : W^{1-1/p,p}(\partial\Omega_i) \to W^{1,p}(\Omega_i)$.*

We can then define the Sobolev spaces on Ω_i required for the domain decomposition, namely,

$$V_i^0 = W_0^{1,p}(\Omega_i) \quad \text{and} \quad V_i = \{v \in W^{1,p}(\Omega_i) : (T_{\partial\Omega_i} v)|_{\partial\Omega_i \setminus \Gamma} = 0\}.$$

The spaces are equipped with the norm

$$\|v\|_{V_i} = \|\nabla v\|_{L^p(\Omega_i)^d} + \|v\|_{L^r(\Omega_i)}.$$

As for (2.2), this norm is equivalent to the standard $W^{1,p}(\Omega_i)$-norm under Assumption 2.1. Furthermore, the spaces V_i^0 and V_i are reflexive Banach spaces.

4. Function spaces and trace operators on Γ. The L^p-form of the Lions–Magenes space can be defined as

$$\Lambda_i = \{\mu \in L^p(\Gamma) : E_i\mu \in W^{1-1/p,p}(\partial\Omega_i)\}, \quad \text{with} \quad \|\mu\|_{\Lambda_i} = \|E_i\mu\|_{W^{1-1/p,p}(\partial\Omega_i)}.$$

Here, $E_i\mu$ denotes the extension by zero of μ to $\partial\Omega_i$. We also define the trace space

$$\Lambda = \{\mu \in L^p(\Gamma) : \mu \in \Lambda_i, \text{ for } i = 1, 2\}, \quad \text{with} \quad \|\mu\|_\Lambda = \|\mu\|_{\Lambda_1} + \|\mu\|_{\Lambda_2}.$$

Lemma 4.1. *If $p > 1$ and Assumption 3.1 holds, then Λ_i and Λ are reflexive Banach spaces.*

Proof. Observe that E_i is a linear isometry from Λ_i onto

$$R(E_i) = \{\mu \in W^{1-1/p,p}(\partial\Omega_i) : \mu|_{\partial\Omega\setminus\Gamma} = 0\}. \tag{4.1}$$

Next, consider a sequence $\{\mu^k\} \subset R(E_i)$ such that $\mu^k \to \mu$ in $W^{1-1/p,p}(\partial\Omega_i)$. Then $\mu^k|_{\Omega_i\setminus\Gamma} = 0$ and $\mu^k \to \mu$ in $L^p(\partial\Omega_i)$, which implies that

$$\int_{\partial\Omega_i\setminus\Gamma} |\mu|^p \,\mathrm{d}S = \int_{\partial\Omega_i\setminus\Gamma} |\mu^k - \mu|^p \,\mathrm{d}S \leq \int_{\partial\Omega_i} |\mu^k - \mu|^p \,\mathrm{d}S \to 0 \quad \text{as } k \to \infty. \tag{4.2}$$

Hence, $\mu \in R(E_i)$, and consequently $R(E_i)$ is a closed subset of $W^{1-1/p,p}(\partial\Omega_i)$. The space Λ_i is therefore isomorphic to a closed subset of the reflexive Banach space $W^{1-1/p,p}(\partial\Omega_i)$; i.e., Λ_i is complete and reflexive [23, Chapter 8, Theorem 15].

To prove that the same holds true for Λ, introduce the reflexive Banach space $X = W^{1-1/p,p}(\partial\Omega_1) \times W^{1-1/p,p}(\partial\Omega_2)$, with the norm

$$\|(\mu_1, \mu_2)\|_X = \|\mu_1\|_{W^{1-1/p,p}(\partial\Omega_1)} + \|\mu_2\|_{W^{1-1/p,p}(\partial\Omega_2)}$$

and the operator $E : \Lambda \to X$ defined by $E\mu = (E_1\mu, E_2\mu)$. As E is a linear isometry from Λ onto

$$R(E) = \{(\mu_1, \mu_2) \in X : \mu_1|_{\partial\Omega_1\setminus\Gamma} = 0,\, \mu_2|_{\partial\Omega_2\setminus\Gamma} = 0,\, \mu_1|_\Gamma = \mu_2|_\Gamma\},$$

it is again sufficient to prove that $R(E)$ is a closed subset of X. Let $\{(\mu_1^k, \mu_2^k)\} \subset R(E)$ be a convergent sequence in X with the limit (μ_1, μ_2). By the same argument as (4.2), we obtain that $\mu_i|_{\Omega_i\setminus\Gamma} = 0$. As $\mu_1^k|_\Gamma = \mu_2^k|_\Gamma$, we also have the limit

$$\int_\Gamma |\mu_1 - \mu_2|^p \,\mathrm{d}S \leq 2^{p-1} \left(\int_\Gamma |\mu_1^k - \mu_1|^p \,\mathrm{d}S + \int_\Gamma |\mu_2^k - \mu_2|^p \,\mathrm{d}S \right) \to 0 \quad \text{as } k \to \infty,$$

i.e., $\mu_1|_\Gamma = \mu_2|_\Gamma$ in $L^p(\Gamma)$, and we obtain that $(\mu_1, \mu_2) \in R(E)$. Thus, $R(E)$ is closed, and Λ is therefore a reflexive Banach space. ∎

Lemma 4.2. *If $p \geq 2$ and Assumption 3.1 holds, then Λ_i and Λ are dense in $L^2(\Gamma)$.*

The proof of the lemma is almost identical to the proof of [22, Theorem 6.6.3] and is therefore left out.

Remark 4.3. We conjecture that $\Lambda_1 = \Lambda_2$. The difficulty in proving this conjecture lies in the fact that the norms $\|\cdot\|_{\Lambda_i}$ are dependent on the whole of $\partial\Omega_i$ instead of just being localized to Γ. However, we will move on to a $L^2(\Gamma)$-framework for which it is not necessary to make this identification.

Together, Lemmas 4.1 and 4.2 yield the Gelfand triplets

$$\Lambda_i \hookrightarrow L^2(\Gamma) \cong L^2(\Gamma)^* \hookrightarrow \Lambda_i^* \quad \text{and} \quad \Lambda \hookrightarrow L^2(\Gamma) \cong L^2(\Gamma)^* \hookrightarrow \Lambda^*.$$

For future use, we introduce the Riesz isomorphism on $L^2(\Gamma)$ given by

$$J : L^2(\Gamma) \to L^2(\Gamma)^* : \mu \mapsto (\mu, \cdot)_{L^2(\Gamma)},$$

which satisfies the relations

$$\langle J\eta, \mu_i\rangle_{\Lambda_i^*\times\Lambda_i} = (\eta, \mu_i)_{L^2(\Gamma)} \quad \text{and} \quad \langle J\eta, \mu\rangle_{\Lambda^*\times\Lambda} = (\eta, \mu)_{L^2(\Gamma)}$$

for all $\eta \in L^2(\Gamma)$, $\mu_i \in \Lambda_i$, and $\mu \in \Lambda$. Here, $\langle\cdot,\cdot\rangle_{X^*\times X}$ denotes the dual pairing between a Banach space X and its dual X^*. In the following, we will drop the subscripts on the dual parings.

In order to relate the spaces V_i and Λ_i, we observe that for $v \in V_i$, one has $T_{\partial\Omega_i}v \in R(E_i)$; see (4.1). Hence, the trace operator

$$T_i : V_i \to \Lambda_i : v \mapsto \big(T_{\partial\Omega_i}v\big)|_\Gamma$$

is well defined. We also introduce the linear operator

$$R_i : \Lambda_i \to V_i : \mu \mapsto R_{\partial\Omega_i}E_i\mu.$$

LEMMA 4.4. *If $p > 1$ and Assumption 3.1 holds, then T_i and R_i are bounded, and R_i is a right inverse of T_i.*

Proof. For $v \in V_i$ and $\mu \in \Lambda_i$ we have, by Lemma 3.2, that

$$\|T_iv\|_{\Lambda_i} = \|E_i\big((T_{\partial\Omega_i}v)|_\Gamma\big)\|_{W^{1-1/p,p}(\partial\Omega_i)} = \|T_{\partial\Omega_i}v\|_{W^{1-1/p,p}(\partial\Omega_i)} \le C_i\|v\|_{V_i}$$

$$\text{and} \quad \|R_i\mu\|_{V_i} = \|R_{\partial\Omega_i}E_i\mu\|_{V_i} \le C_i\|E_i\mu\|_{W^{1-1/p,p}(\partial\Omega_i)} = C_i\|\mu\|_{\Lambda_i}.$$

Hence, the linear operators T_i and R_i are bounded. Furthermore, for every $\mu \in \Lambda_i$, we have

$$T_iR_i\mu = \big(T_{\partial\Omega_i}R_{\partial\Omega_i}E_i\mu\big)|_\Gamma = (E_i\mu)|_\Gamma = \mu;$$

i.e., R_i is a right inverse of T_i. ◻

We continue by deriving a few useful properties related to the operator T_i.

LEMMA 4.5. *If $p > 1$, Assumption 3.1 holds, and $v \in V$, then $\mu = T_1v|_{\Omega_1} = T_2v|_{\Omega_2}$ is an element in Λ.*

Proof. Let $v \in V$. As $C_0^\infty(\Omega)$ is dense in V, there exists a sequence $\{v^k\} \subset C_0^\infty(\Omega)$ such that $v^k \to v$ in V. Set $v_i = v|_{\Omega_i}$ and $v_i^k = v^k|_{\Omega_i}$. Clearly, $T_1v_1^k = T_2v_2^k$. Since $v^k \to v$ in V, we also have that $v_i^k \to v_i$ in V_i. The continuity of T_i then implies that $T_iv_i^k \to T_iv_i$ in Λ_i. Putting this together gives us

$$\Lambda_1 \ni T_1v_1 = \lim_{k\to\infty} T_1v_1^k = \lim_{k\to\infty} T_2v_2^k = T_2v_2 \in \Lambda_2 \quad \text{in } L^p(\Gamma).$$

If we now define $\mu = T_1v_1 = T_2v_2$, then μ is an element in $\Lambda = \Lambda_1 \cap \Lambda_2$. ◻

LEMMA 4.6. *Let $p > 1$ and Assumption 3.1 hold. If two elements $v_1 \in V_1$ and $v_2 \in V_2$ satisfy $T_1v_1 = T_2v_2$, then $v = \{v_1 \text{ on } \Omega_1;\, v_2 \text{ on } \Omega_2\}$ is an element in V.*

Proof. It is clear that $v \in L^p(\Omega)$. For each component $1 \le \ell \le d$, there exists a weak derivative $\partial_\ell v_i \in L^p(\Omega_i)$ of $v_i \in V_i \subset W^{1,p}(\Omega_i)$. If we define

$$z_\ell = \{\partial_\ell v_1 \text{ on } \Omega_1;\, \partial_\ell v_2 \text{ on } \Omega_2\},$$

then $z_\ell \in L^p(\Omega)$. Let $w \in C_0^\infty(\Omega)$, and set $w_i = w|_{\Omega_i} \in C^\infty(\Omega_i)$. The $W^{1,p}(\Omega_i)$-version of Green's formula [30, section 3.1.2] yields that

$$\begin{aligned}\int_\Omega z_\ell w \,\mathrm{d}x &= \sum_{i=1}^2 \int_{\Omega_i} \partial_\ell v_i\, w_i \,\mathrm{d}x = \sum_{i=1}^2 -\int_{\Omega_i} v_i\, \partial_\ell w_i \,\mathrm{d}x + \int_{\partial\Omega_i} (T_{\partial\Omega_i}v_i) w_i \nu_i^\ell \,\mathrm{d}S \\ &= -\int_\Omega v\partial_\ell w \,\mathrm{d}x + \sum_{i=1}^2 \int_\Gamma (T_iv_i) w \nu_i^\ell \,\mathrm{d}S = -\int_\Omega v\partial_\ell w \,\mathrm{d}x;\end{aligned}$$

i.e., z_ℓ is the ℓth weak partial derivative of v. By construction, $T_{\partial\Omega} v = 0$, and v is therefore an element in V. ◻

5. Transmission problem and the Robin–Robin method. The framework given in section 4 enables us to introduce the weak forms of the nonlinear transmission problem and the Robin–Robin method. It also allows us to prove equivalence between the nonlinear elliptic equation and the transmission problem along the same lines as done for linear equations [32, Lemma 1.2.1]. To this end, on each V_i, we define $a_i : V_i \times V_i \to \mathbb{R}$ by

$$a_i(u_i, v_i) = \int_{\Omega_i} \alpha(\nabla u_i) \cdot \nabla v_i + g(u_i) v_i \,\mathrm{d}x.$$

We also define $f_i = f|_{\Omega_i} \in L^2(\Omega_i)$.

LEMMA 5.1. *If Assumptions* 2.1 *and* 3.1 *hold, then* $a_i : V_i \times V_i \to \mathbb{R}$ *is well defined and satisfies the growth, strict monotonicity, and coercivity bounds stated in Lemma* 2.2*, with the terms* (a, V, Ω) *replaced by* (a_i, V_i, Ω_i).

The weak form of the nonlinear transmission problem (1.4) is then to find $(u_1, u_2) \in V_1 \times V_2$ such that

$$\begin{cases} a_i(u_i, v_i) = (f_i, v_i)_{L^2(\Omega_i)} & \text{for all } v_i \in V_i^0, i = 1, 2, \\ T_1 u_1 = T_2 u_2, & \\ \sum_{i=1}^2 a_i(u_i, R_i\mu) - (f_i, R_i\mu)_{L^2(\Omega_i)} = 0 & \text{for all } \mu \in \Lambda. \end{cases} \tag{5.1}$$

THEOREM 5.2. *Let Assumptions* 2.1 *and* 3.1 *hold. If* $u \in V$ *solves* (2.1)*, then* $(u_1, u_2) = (u|_{\Omega_1}, u|_{\Omega_2})$ *solves* (5.1)*. Conversely, if* (u_1, u_2) *solves* (5.1)*, then* $u = \{u_1 \text{ on } \Omega_1;\, u_2 \text{ on } \Omega_2\}$ *solves* (2.1).

Proof. Assume that $u \in V$ solves (2.1), and define $(u_1, u_2) = (u|_{\Omega_1}, u|_{\Omega_2}) \in V_1 \times V_2$. For $v_i \in V_i^0$, we can extend by zero to $w_i \in V$ by using Lemma 4.6. Then

$$a_i(u_i, v_i) = a(u, w_i) = (f, w_i)_{L^2(\Omega)} = (f_i, v_i)_{L^2(\Omega_i)}.$$

Moreover, $T_1 u_1 = T_2 u_2$ follows immediately from Lemma 4.5. For an arbitrary $\mu \in \Lambda$, let $v_i = R_i\mu$. Then, by Lemma 4.6, $v = \{v_1 \text{ on } \Omega_1;\, v_2 \text{ on } \Omega_2\}$ is an element in V, and

$$a_1(u_1, R_1\mu) + a_2(u_2, R_2\mu) = a(u, v) = (f, v)_{L^2(\Omega)} = (f_1, R_1\mu)_{L^2(\Omega_1)} + (f_2, R_2\mu)_{L^2(\Omega_2)}.$$

This proves that (u_1, u_2) solves (5.1). Conversely, let $(u_1, u_2) \in V_1 \times V_2$ be a solution to (5.1), and define $u = \{u_1 \text{ on } \Omega_1;\, u_2 \text{ on } \Omega_2\}$. By Lemma 4.6, we have that $u \in V$. Next, consider $v \in V$, and let $v_i = v|_{\Omega_i} \in V_i$. From Lemma 4.5, we have that $\mu = T_i v_i$ is well defined and $\mu \in \Lambda$. The observation that $v_i - R_i\mu \in V_i^0$ for $i = 1, 2$ implies the equality

$$\begin{aligned} a(u, v) &= \sum_{i=1}^2 a_i(u_i, v_i - R_i\mu) + a_i(u_i, R_i\mu) = \sum_{i=1}^2 (f_i, v_i - R_i\mu)_{L^2(\Omega_i)} + a_i(u_i, R_i\mu) \\ &= \sum_{i=1}^2 (f_i, v_i)_{L^2(\Omega_i)} + a_i(u_i, R_i\mu) - (f_i, R_i\mu)_{L^2(\Omega_i)} = (f, v)_{L^2(\Omega)}. \end{aligned}$$

As v can be chosen arbitrarily, u solves (2.1). ◻

Remark 5.3. As the nonlinear elliptic equation (2.1) has a unique weak solution, Theorem 5.2 implies that the same holds true for the nonlinear transmission problem (5.1).

In order to approximate the weak solution $(u_1^n, u_2^n) \in V_1 \times V_2$ of the transmission problem by the Robin–Robin method, we require the method's weak form. Multiplying by test functions and formally applying Green's formula to the equations (1.5) yields that the weak form of the method is given by finding $(u_1^n, u_2^n) \in V_1 \times V_2$ for $n = 1, 2, \ldots$ such that

$$\tag{5.2} \begin{cases} a_1(u_1^{n+1}, v_1) = (f_1, v_1)_{L^2(\Omega_1)} & \text{for all } v_1 \in V_1^0, \\ a_1(u_1^{n+1}, R_1\mu) - (f_1, R_1\mu)_{L^2(\Omega_1)} + a_2(u_2^n, R_2\mu) & \\ \qquad - (f_2, R_2\mu)_{L^2(\Omega_2)} = s(T_2 u_2^n - T_1 u_1^{n+1}, \mu)_{L^2(\Gamma)} & \text{for all } \mu \in \Lambda, \\ a_2(u_2^{n+1}, v_2) = (f_2, v_2)_{L^2(\Omega_2)} & \text{for all } v_2 \in V_2^0, \\ a_2(u_2^{n+1}, R_2\mu) - (f_2, R_2\mu)_{L^2(\Omega_2)} + a_1(u_1^{n+1}, R_1\mu) & \\ \qquad - (f_1, R_1\mu)_{L^2(\Omega_1)} = s(T_1 u_1^{n+1} - T_2 u_2^{n+1}, \mu)_{L^2(\Gamma)} & \text{for all } \mu \in \Lambda, \end{cases}$$

where $u_2^0 \in V_2$ is an initial guess and $s > 0$ is the given method parameter.

6. Interface formulations. The ambition is now to reformulate the nonlinear transmission problem and the Robin–Robin method, which are all given on the domains Ω_i, into problems and methods only stated on the interface Γ. As a preparation, we observe that nonlinear elliptic equations on Ω_i with inhomogeneous Dirichlet conditions have unique weak solutions.

LEMMA 6.1. *If Assumptions* 2.1 *and* 3.1 *hold, then for each* $\eta \in \Lambda_i$ *there exists a unique* $u_i \in W^{1,p}(\Omega_i)$ *such that*

$$\tag{6.1} a_i(u_i, v_i) = (f_i, v_i)_{L^2(\Omega_i)} \quad \textit{for all } v_i \in V_i^0$$

and $T_{\partial\Omega_i} u_i = E_i \eta$ *in* $W^{1-1/p,p}(\partial\Omega_i)$.

The proof can be found in, e.g., [33, Theorem 2.36]. With the notation of Lemma 6.1, consider the operator

$$F_i : \eta \mapsto u_i,$$

i.e., the map from a given boundary value on Γ to the corresponding weak solution of the nonlinear elliptic problem (6.1) on Ω_i. From the statement of Lemma 6.1, we see that

$$F_i : \Lambda_i \to V_i \quad \text{and} \quad T_i F_i \eta = \eta \text{ for } \eta \in \Lambda_i.$$

In other words, the operator F_i is a nonlinear right inverse of T_i. This property will be frequently used, as it, together with the boundedness and linearity of T_i, gives rise to bounds of the forms

$$\|\eta\|_{\Lambda_i} \leq C_i \|F_i \eta\|_{V_i} \quad \text{and} \quad \|\eta - \mu\|_{\Lambda_i} \leq C_i \|F_i \eta - F_i \mu\|_{V_i}.$$

We can now define the nonlinear Steklov–Poincaré operators S_i and S as

$$\langle S_i \eta, \mu \rangle = a_i(F_i \eta, R_i \mu) - (f_i, R_i \mu)_{L^2(\Omega_i)} \quad \text{for all } \eta, \mu \in \Lambda_i \quad \text{and}$$

$$\langle S\eta, \mu \rangle = \sum_{i=1}^{2} \langle S_i \eta, \mu \rangle = \sum_{i=1}^{2} a_i(F_i \eta, R_i \mu) - (f_i, R_i \mu)_{L^2(\Omega_i)} \quad \text{for all } \eta, \mu \in \Lambda.$$

Thus, we may restate the nonlinear transmission problem (5.1) as the Steklov–Poincaré equation, i.e., finding $\eta \in \Lambda$ such that

$$\langle S\eta, \mu\rangle = 0 \quad \text{for all } \mu \in \Lambda. \tag{6.2}$$

That the reformulation is possible follows directly from the definitions of the operators F_i and S, but we state this as a lemma for future reference.

LEMMA 6.2. *Let Assumptions* 2.1 *and* 3.1 *hold. If* (u_1, u_2) *solves* (5.1)*, then* $\eta = T_1u_1 = T_2u_2$ *solves* (6.2)*. Conversely, if* η *solves* (6.2)*, then* $(u_1, u_2) = (F_1\eta, F_2\eta)$ *solves* (5.1)*.*

Before turning to the Robin–Robin method, consider either the stationary problem $(A_1+A_2)v = 0$ or the initial value problem $\mathrm{d}v/\mathrm{d}t+(A_1+A_2)v = 0$, with $v(0) = v_0$. In both settings, the Peaceman–Rachford splitting

$$v^{n+1} = (sI + A_2)^{-1}(sI - A_1)(sI + A_1)^{-1}(sI - A_2)v^n$$

has been proposed as an efficient approximation scheme; see [27, 31]. In the current context of nonlinear Steklov–Poincaré operators, the weak form of the Robin–Robin method is in fact equivalent to the Peaceman–Rachford splitting on the interface Γ. This observation was made for linear elliptic equations in [1]. The equivalence was also utilized in [13, section 4.4.1] for the linear setting of the Stokes–Darcy coupling.

The weak form of the Peaceman–Rachford splitting is given by finding $(\eta_1^n, \eta_2^n) \in \Lambda_1 \times \Lambda_2$ for $n = 1, 2, \ldots$ such that

$$\begin{cases} \langle (sJ + S_1)\eta_1^{n+1}, \mu\rangle = \langle (sJ - S_2)\eta_2^n, \mu\rangle, \\ \langle (sJ + S_2)\eta_2^{n+1}, \mu\rangle = \langle (sJ - S_1)\eta_1^{n+1}, \mu\rangle \end{cases} \tag{6.3}$$

for all $\mu \in \Lambda$, where $\eta_2^0 \in \Lambda_2$ is an initial guess.

LEMMA 6.3. *Let Assumptions* 2.1 *and* 3.1 *be valid. If* $(u_1^n, u_2^n)_{n\geq 1}$ *is a weak Robin–Robin approximation given by* (5.2)*, then* $(\eta_1^n, \eta_2^n)_{n\geq 1} = (T_1u_1^n, T_2u_2^n)_{n\geq 1}$ *is a weak Peaceman–Rachford approximation given by* (6.3)*, with* $\eta_2^0 = T_2u_2^0$*. Conversely, if* $(\eta_1^n, \eta_2^n)_{n\geq 1}$ *is given by* (6.3)*, then* $(u_1^n, u_2^n)_{n\geq 1} = (F_1\eta_1^n, F_2\eta_2^n)_{n\geq 1}$ *fulfills* (5.2)*, with* $u_2^0 = F_2\eta_2^0$*.*

Proof. First, assume that $(u_1^n, u_2^n)_{n\geq 1} \subset V_1 \times V_2$ is a weak Robin–Robin approximation, and define $\eta_i^n = T_iu_i^n \in \Lambda_i$. This definition, the existence of a unique solution of (6.1), and the first and third assertions of (5.2) yield the identification $u_i^n = F_i\eta_i^n$. Inserting this into the second and fourth assertion of (5.2) gives us

$$\begin{aligned} &s(T_1F_1\eta_1^{n+1}, \mu)_{L^2(\Gamma)} + a_1(F_1\eta_1^{n+1}, R_1\mu) - (f, R_1\mu)_{L^2(\Omega_1)} \\ &\quad = s(T_2F_2\eta_2^n, \mu)_{L^2(\Gamma)} - a_2(F_2\eta_2^n, R_2\mu) + (f, R_2\mu)_{L^2(\Omega_2)} \quad \text{and} \\ &s(T_2F_2\eta_2^{n+1}, \mu)_{L^2(\Gamma)} + a_2(F_2\eta_2^{n+1}, R_2\mu) - (f, R_2\mu)_{L^2(\Omega_2)} \\ &\quad = s(T_1F_1\eta_1^{n+1}, \mu)_{L^2(\Gamma)} - a_1(F_1\eta_1^{n+1}, R_1\mu) + (f, R_1\mu)_{L^2(\Omega_1)} \end{aligned}$$

for all $\mu \in \Lambda$, which is precisely the weak form of the Peaceman–Rachford splitting (6.3), with $\eta_2^0 = T_2u_2^0$. Conversely, suppose that $(\eta_1^n, \eta_2^n)_{n\geq 1} \subset \Lambda_1 \times \Lambda_2$ is a weak Peaceman–Rachford approximation, and define $u_i^n = F_i\eta_i^n \in V_i$. Inserting this into (6.3) directly gives that $(u_1^n, u_2^n)_{n\geq 1}$, with $u_2^0 = F_2\eta_2^0$, is a weak Robin–Robin approximation (5.2). ∎

Remark 6.4. For now, we do not know if the weak Robin–Robin and Peaceman–Rachford approximations actually exist in the sense that there exists a solution to (6.3) at each iteration. We will return to this issue in section 8.

Remark 6.5. The interface formulations of the transmission problem and the Robin–Robin method can easily be generalized to more than two subdomains, without cross points as in Figure 1c. However, for domain decompositions with cross points, such as in Figure 1d, or overlapping subdomains, the situation becomes more delicate. For example, consider a decomposition into disjoint subdomains Ω_i, $i = 1, \dots, N$, and let $\Gamma_{i\ell}$ denote the interface between two adjacent subdomains Ω_i and Ω_ℓ. If cross points are present, one can still prove equivalence between the elliptic equation (2.1) and a transmission problem when $p = 2$. The result is due to [34], and the proof can most likely be generalized to our setting with $p \geq 2$. One then obtains a Steklov–Poincaré equation for each interface $\Gamma_{i\ell}$, and the corresponding operators have the form $\langle S_{i\ell}\eta, \mu\rangle = a_i(F_{i\ell}\eta, R_{i\ell}\mu) - (f_i, R_{i\ell}\mu)_{L^2(\Omega_i)}$, where η and μ are elements in the trial and test spaces

$$\{\eta \in L^p(\cup_\ell \Gamma_{i\ell}) : E_i\eta \in W^{1-1/p,p}(\partial\Omega_i)\} \text{ and } \{\mu \in L^p(\Gamma_{i\ell}) : E_i\mu \in W^{1-1/p,p}(\partial\Omega_i)\},$$

respectively. This possible mismatch between trial and test spaces requires a further extension of our framework, which we will address elsewhere.

Finally, note that an alternative numerical scheme to the Robin–Robin method is proposed in [6]. The method enables the usage of domain decompositions with cross points in the context of the Richards equation, but no convergence analysis is presented.

7. Properties of the nonlinear Steklov–Poincaré operators. We proceed by deriving the central properties of the Steklov–Poincaré operators S_i, S when interpreted as maps from Λ_i, Λ into the corresponding dual spaces.

LEMMA 7.1. *If Assumptions* 2.1 *and* 3.1 *hold, then* $S_i : \Lambda_i \to \Lambda_i^*$ *and* $S : \Lambda \to \Lambda^*$ *are well defined.*

Proof. Let $\eta_i \in \Lambda_i$ and $\eta \in \Lambda$. The linearity of the functionals $S_i\eta_i$ and $S\eta$ follows by definition. As $F_i\eta_i \in V_i$, we have, by Lemma 5.1, that

$$\begin{aligned}
|\langle S_i\eta_i, \mu\rangle| &\leq |a_i(F_i\eta_i, R_i\mu)| + |(f_i, R_i\mu)_{L^2(\Omega_i)}| \\
&\leq c_i\big(\|\nabla F_i\eta_i\|_{L^p(\Omega_i)^d}^{p-1}\|\nabla R_i\mu\|_{L^p(\Omega_i)^d} + \|F_i\eta_i\|_{L^r(\Omega_i)}^{r-1}\|R_i\mu\|_{L^r(\Omega_i)}\big) \\
&\quad + \|f_i\|_{L^2(\Omega_i)}\|R_i\mu\|_{L^2(\Omega_i)} \\
&\leq C_i(\|\nabla F_i\eta_i\|_{V_i}, \|f_i\|_{L^2(\Omega_i)})\|R_i\mu\|_{V_i} \leq C_i\|\mu\|_{\Lambda_i}
\end{aligned}$$

for all $\mu \in \Lambda_i$. Thus, $S_i\eta_i$ is a bounded functional on Λ_i. The boundedness of $S\eta$ follows directly by summing up the bounds for $S_i\eta$. ◻

LEMMA 7.2. *If Assumptions* 2.1 *and* 3.1 *hold, then the operators* $S_i : \Lambda_i \to \Lambda_i^*$ *and* $S : \Lambda \to \Lambda^*$ *are strictly monotone with*

$$\langle S_i\eta - S_i\mu, \eta - \mu\rangle \geq c_i\big(\|\nabla(F_i\eta - F_i\mu)\|_{L^p(\Omega_i)^d}^p + \|F_i\eta - F_i\mu\|_{L^r(\Omega_i)}^r\big)$$

for all $\eta, \mu \in \Lambda_i$ *and*

$$\langle S\eta - S\mu, \eta - \mu\rangle \geq c\sum_{i=1}^{2}\big(\|\nabla(F_i\eta - F_i\mu)\|_{L^p(\Omega_i)^d}^p + \|F_i\eta - F_i\mu\|_{L^r(\Omega_i)}^r\big)$$

for all $\eta, \mu \in \Lambda$, *respectively.*

Proof. Since, $w_i = R_i(\eta - \mu) - (F_i\eta - F_i\mu) \in V_i^0$ for all $\eta, \mu \in \Lambda_i$, we have, according to the definition of F_i, that

$$a_i(F_i\eta, w_i) - a_i(F_i\mu, w_i) = 0. \tag{7.1}$$

By this equality and Lemma 5.1, it follows that

$$\begin{aligned}\langle S_i\eta - S_i\mu, \eta - \mu\rangle &= a_i\big(F_i\eta, R_i(\eta-\mu)\big) - a_i\big(F_i\mu, R_i(\eta-\mu)\big)\\ &= a_i(F_i\eta, w_i) + a_i(F_i\eta, F_i\eta - F_i\mu) - a_i(F_i\mu, w_i) - a_i(F_i\mu, F_i\eta - F_i\mu)\\ &\geq c_i\big(\|\nabla(F_i\eta - F_i\mu)\|^p_{L^p(\Omega_i)^d} + \|F_i\eta - F_i\mu\|^r_{L^r(\Omega_i)}\big)\end{aligned}$$

for all $\eta, \mu \in \Lambda_i$, which proves the monotonicity bound for S_i. The bound for S follows directly by summing up the bounds for S_i. □

LEMMA 7.3. *If Assumptions* 2.1 *and* 3.1 *hold, then the operators $S_i : \Lambda_i \to \Lambda_i^*$ and $S : \Lambda \to \Lambda^*$ are coercive, i.e.,*

$$\lim_{\|\eta\|_{\Lambda_i}\to\infty} \frac{\langle S_i\eta, \eta\rangle}{\|\eta\|_{\Lambda_i}} = \infty \quad \textit{and} \quad \lim_{\|\eta\|_{\Lambda}\to\infty} \frac{\langle S\eta, \eta\rangle}{\|\eta\|_{\Lambda}} = \infty.$$

Proof. As $\|F_i\eta\|_{V_i} = \|\nabla F_i\eta\|_{L^p(\Omega_i)^d} + \|F_i\eta\|_{L^r(\Omega_i)} \geq c_i\|\eta\|_{\Lambda_i}$, we have that

$$P(\|\nabla F_i\eta\|_{L^p(\Omega_i)^d}, \|F_i\eta\|_{L^r(\Omega_i)}) \to \infty \quad \text{as } \|\eta\|_{\Lambda_i} \to \infty,$$

where $P(x, y) = (x^p + y^r)/(x + y)$. In particular, we assume from now on that

$$P(\|\nabla F_i\eta\|_{L^p(\Omega_i)^d}, \|F_i\eta\|_{L^r(\Omega_i)}) \geq \|f_i\|_{L^2(\Omega_i)}.$$

By observing that $R_i\eta - F_i\eta \in V_i^0$, Lemma 5.1 yields the lower bound

$$\begin{aligned}\langle S_i\eta, \eta\rangle &= a_i(F_i\eta, F_i\eta) + a_i(F_i\eta, R_i\eta - F_i\eta) - (f_i, R_i\eta)_{L^2(\Omega_i)}\\ &= a_i(F_i\eta, F_i\eta) + (f_i, R_i\eta - F_i\eta)_{L^2(\Omega_i)} - (f_i, R_i\eta)_{L^2(\Omega_i)}\\ &\geq c_i(\|\nabla F_i\eta\|^p_{L^p(\Omega_i)^d} + \|F_i\eta\|^r_{L^r(\Omega_i)}) - (f_i, F_i\eta)_{L^2(\Omega_i)}\\ &\geq c_iP(\|\nabla F_i\eta\|_{L^p(\Omega_i)^d}, \|F_i\eta\|_{L^r(\Omega_i)})\|F_i\eta\|_{V_i} - \|f_i\|_{L^2(\Omega_i)}\|F_i\eta\|_{V_i}\\ &\geq c_i\big(P(\|\nabla F_i\eta\|_{L^p(\Omega_i)^d}, \|F_i\eta\|_{L^r(\Omega_i)}) - \|f_i\|_{L^2(\Omega_i)}\big)\|\eta\|_{\Lambda_i},\end{aligned}$$

which implies that S_i is coercive. For S, we obtain that

$$\frac{\langle S\eta, \eta\rangle}{\|\eta\|_{\Lambda}} \geq \frac{\sum_{i=1}^2\big(c_i\,P(\|\nabla F_i\eta\|_{L^p(\Omega_i)^d}, \|F_i\eta\|_{L^r(\Omega_i)}) - \|f_i\|_{L^2}\big)\|\eta\|_{\Lambda_i}}{\|\eta\|_{\Lambda_1} + \|\eta\|_{\Lambda_2}},$$

which tends to infinity as $\|\eta\|_{\Lambda}$ tends to infinity. Thus, S is also coercive. □

In order to prove that the operators S_i, S are demicontinuous, i.e., if $\eta_i^k \to \eta_i$ in Λ_i and $\eta^k \to \eta$ in Λ, then

$$\langle S_i\eta_i^k - S_i\eta_i, \mu_i\rangle \to 0 \quad \text{and} \quad \langle S\eta^k - S\eta, \mu\rangle \to 0,$$

for all $\mu_i \in \Lambda_i$ and $\mu \in \Lambda$, we first consider the continuity of the operators F_i.

LEMMA 7.4. *If Assumptions* 2.1 *and* 3.1 *hold, then the nonlinear operators F_i : $\Lambda_i \to V_i$ are continuous.*

Proof. Let η, μ be elements in Λ_i. Using the equality (7.1), together with Lemma 5.1 gives us the bound

$$
\begin{aligned}
&c_i(\|\nabla(F_i\eta - F_i\mu)\|^p_{L^p(\Omega_i)^d} + \|F_i\eta - F_i\mu\|^r_{L^r(\Omega_i)})\\
&\quad\le a_i(F_i\eta, F_i\eta - F_i\mu) - a_i(F_i\mu, F_i\eta - F_i\mu)\\
&\quad= a_i\big(F_i\eta, R_i(\eta-\mu)\big) - a_i(F_i\eta, w_i) - a_i\big(F_i\mu, R_i(\eta-\mu)\big) + a_i(F_i\mu, w_i)\\
&\quad\le C_i\big(\|\nabla F_i\eta\|^{p-1}_{L^p(\Omega_i)^d}\|\nabla R_i(\eta-\mu)\|_{L^p(\Omega_i)^d} + \|F_i\eta\|^{r-1}_{L^r(\Omega_i)}\|R_i(\eta-\mu)\|_{L^r(\Omega_i)}\\
&\qquad+ \|\nabla F_i\mu\|^{p-1}_{L^p(\Omega_i)^d}\|\nabla R_i(\eta-\mu)\|_{L^p(\Omega_i)^d} + \|F_i\mu\|^{r-1}_{L^r(\Omega_i)}\|R_i(\eta-\mu)\|_{L^r(\Omega_i)}\big)\\
&\quad\le C_i\big(\|\nabla F_i\eta\|^{p-1}_{L^p(\Omega_i)^d} + \|F_i\eta\|^{r-1}_{L^r(\Omega_i)}\\
&\qquad+ \|\nabla F_i\mu\|^{p-1}_{L^p(\Omega_i)^d} + \|F_i\mu\|^{r-1}_{L^r(\Omega_i)}\big)\|\eta-\mu\|_{\Lambda_i}.
\end{aligned}
\tag{7.2}
$$

Letting $\mu = 0$ in (7.2) and employing the inequality $|x|^p - 2^{p-1}|y|^p \le 2^{p-1}|x-y|^p$ twice yields that

$$
\begin{aligned}
&c_i(\|\nabla F_i\eta\|^p_{L^p(\Omega_i)^d} + \|F_i\eta\|^r_{L^r(\Omega_i)} - 2^{p-1}\|\nabla F_i0\|^p_{L^p(\Omega_i)^d} - 2^{r-1}\|F_i0\|^r_{L^r(\Omega_i)})\\
&\quad\le c_i(2^{p-1}\|\nabla(F_i\eta - F_i0)\|^p_{L^p(\Omega_i)^d} + 2^{r-1}\|F_i\eta - F_i0\|^r_{L^r(\Omega_i)})\\
&\quad\le C_i\big(\|\nabla F_i\eta\|^{p-1}_{L^p(\Omega_i)^d} + \|F_i\eta\|^{r-1}_{L^r(\Omega_i)} + \|\nabla F_i0\|^{p-1}_{L^p(\Omega_i)^d} + \|F_i0\|^{r-1}_{L^r(\Omega_i)}\big)\|\eta\|_{\Lambda_i}.
\end{aligned}
$$

Thus, we have a bound of the form

$$
\frac{\|\nabla F_i\eta\|^p_{L^p(\Omega_i)^d} + \|F_i\eta\|^r_{L^r(\Omega_i)} - c_1}{\|\nabla F_i\eta\|^{p-1}_{L^p(\Omega_i)^d} + \|F_i\eta\|^{r-1}_{L^r(\Omega_i)} + c_2} \le C_i\|\eta\|_{\Lambda_i}
\tag{7.3}
$$

for every $\eta \in \Lambda_i$, where $c_\ell = c_\ell(\|\nabla F_i 0\|_{L^p(\Omega_i)^d}, \|F_i0\|_{L^r(\Omega_i)}) \ge 0$.

Assume that $\eta^k \to \eta$ in Λ_i. As η^k is bounded in Λ_i, the bound (7.3) implies that $\nabla F_i\eta^k$ and $F_i\eta^k$ are bounded in $L^p(\Omega_i)^d$ and $L^r(\Omega_i)$, respectively. By setting $\mu = \eta^k$ in (7.2), we finally obtain that $\nabla F_i\eta^k \to \nabla F_i\eta$ in $L^p(\Omega_i)^d$ and $F_i\eta^k \to F_i\eta$ in $L^r(\Omega_i)$, i.e., $F_i\eta^k \to F_i\eta$ in V_i. ∎

LEMMA 7.5. *If Assumptions* 2.1 *and* 3.1 *hold, then the operators* $S_i : \Lambda_i \to \Lambda_i^*$ *and* $S : \Lambda \to \Lambda^*$ *are demicontinuous.*

Proof. Assume that $\eta_i^k \to \eta_i$ in Λ_i. Lemma 7.4 then implies that $\nabla F_i\eta^k \to \nabla F_i\eta$ in $L^p(\Omega_i)^d$ and $F_i\eta^k \to F_i\eta$ in $L^r(\Omega_i)$. By the assumed continuity and boundedness of the functions $\alpha : z \mapsto \big(\alpha_1(z), \dots, \alpha_d(z)\big)$ and g, we also have that the corresponding Nemyckii operators $\alpha_\ell : L^p(\Omega_i)^d \to L^{p/(p-1)}(\Omega_i)$ and $g : L^r(\Omega_i) \to L^{r/(r-1)}(\Omega_i)$ are continuous [39, Proposition 26.6]. The demicontinuity of S_i then holds, as

$$
\begin{aligned}
|\langle S_i\eta_i^k - S_i\eta_i, \mu_i\rangle| \le \Bigg(&\sum_{\ell=1}^d \|\alpha_\ell(\nabla F_i\eta_i^k) - \alpha_\ell(\nabla F_i\eta_i)\|_{L^{p/(p-1)}(\Omega_i)}\\
&+ \|g(F_i\eta_i^k) - g(F_i\eta_i)\|_{L^{r/(r-1)}(\Omega_i)}\Bigg)\|R_i\mu_i\|_{V_i}
\end{aligned}
$$

for every $\mu_i \in \Lambda_i$. To prove demicontinuity of S, assume that $\eta^k \to \eta$ in Λ. Then $\eta^k \to \eta$ in Λ_i, and we have the inequality

$$
|\langle S\eta^k - S\eta, \mu\rangle| \le \sum_{i=1}^2 |\langle S_i\eta^k - S_i\eta, \mu\rangle|
$$

for every $\mu \in \Lambda$. The demicontinuity of S_i now implies the same property for S. ◻

THEOREM 7.6. *If Assumptions* 2.1 *and* 3.1 *hold, then the nonlinear Steklov–Poincaré operators* $S_i : \Lambda_i \to \Lambda_i^*$ *and* $S : \Lambda \to \Lambda^*$ *are bijective.*

Proof. The spaces Λ_i and Λ are real, reflexive Banach spaces, and, by Lemmas 7.2, 7.3, and 7.5, the operators $S_i : \Lambda_i \to \Lambda_i^*$ and $S : \Lambda \to \Lambda^*$ are all strictly monotone, coercive, and demicontinuous. With these properties, the Browder–Minty theorem (see, e.g., [39, Theorem 26.A(a,c,f)]) implies that the operators are bijective. ◻

The next corollary follows by the same argumentation as for the bijectivity of S_i.

COROLLARY 7.7. *If Assumptions* 2.1 *and* 3.1 *hold, then the operators* $sJ + S_i : \Lambda_i \to \Lambda_i^*$ *are bijective for every* $s > 0$.

8. Existence and convergence of the Robin–Robin method. There does not seem to be a general convergence analysis in the literature for the weak form of the Peaceman–Rachford splitting (6.3), i.e., with operators mapping reflexive Banach spaces into their duals. However, there are such results for unbounded monotone operators on Hilbert spaces, e.g., in the study [27]. Hence, we will restrict the domains of the operators S_i, S such that the Steklov–Poincaré equation (6.2) and the Peaceman–Rachford splitting can be interpreted on $L^2(\Gamma)$ instead of on the dual spaces Λ_i^*, Λ^*. This comes at the cost of requiring more regularity of the weak solution (see Assumption 2.6 and Remark 8.3) and of the initial guess η_2^0. See Remark 8.7 for how the latter can be handled in practice.

More precisely, we define the operators $\mathcal{S}_i : D(\mathcal{S}_i) \subseteq L^2(\Gamma) \to L^2(\Gamma)$ as

$$D(\mathcal{S}_i) = \{\mu \in \Lambda_i : S_i\mu \in L^2(\Gamma)^*\} \quad \text{and} \quad \mathcal{S}_i\mu = J^{-1}S_i\mu \text{ for } \mu \in D(\mathcal{S}_i).$$

Analogously, we introduce $\mathcal{S} : D(\mathcal{S}) \subseteq L^2(\Gamma) \to L^2(\Gamma)$ given by

$$D(\mathcal{S}) = \{\mu \in \Lambda : S\mu \in L^2(\Gamma)^*\} \quad \text{and} \quad \mathcal{S}\mu = J^{-1}S\mu \text{ for } \mu \in D(\mathcal{S}).$$

As the zero functional obviously is an element in $L^2(\Gamma)^*$, the unique solution $\eta \in \Lambda$ of the Steklov–Poincaré equation is in $D(\mathcal{S})$ and

$$\mathcal{S}\eta = 0.$$

Remark 8.1. By the above construction, one obtains that $D(\mathcal{S}_1) \cap D(\mathcal{S}_2) \subseteq D(\mathcal{S})$ and

$$\mathcal{S}\mu = \mathcal{S}_1\mu + \mathcal{S}_2\mu \quad \text{for all } \mu \in D(\mathcal{S}_1) \cap D(\mathcal{S}_2).$$

However, the definition of the domains does not ensure that $D(\mathcal{S})$ is equal to $D(\mathcal{S}_1) \cap D(\mathcal{S}_2)$, as $(S_1 + S_2)\mu \in L^2(\Gamma)^*$ does not necessarily imply that $S_i\mu \in L^2(\Gamma)^*$.

If the weak solution of the nonlinear elliptic equation (2.1) satisfies the additional regularity property stated in Assumption 2.6, then the corresponding solution of $\mathcal{S}\eta = 0$ is in fact an element in $D(\mathcal{S}_1) \cap D(\mathcal{S}_2)$. This propagation of regularity will be crucial when proving convergence of the Peaceman–Rachford splitting.

LEMMA 8.2. *If Assumptions* 2.1, 2.6, *and* 3.1 *hold and* $\mathcal{S}\eta = 0$, *then* $\eta \in D(\mathcal{S}_1) \cap D(\mathcal{S}_2)$.

Proof. As $u = \{F_1\eta \text{ on } \Omega_1; F_2\eta \text{ on } \Omega_2\}$ is the weak solution of (2.1), we have that

$$\int_\Omega \alpha(\nabla u) \cdot \nabla v \,\mathrm{d}x = -\int_\Omega \big(g(u) - f\big) v \,\mathrm{d}x \quad \text{for all } v \in C_0^\infty(\Omega).$$

The restrictions on r in Assumption 2.1 yield that $u \in W^{1,p}(\Omega) \hookrightarrow L^{2(r-1)}(\Omega)$. This together with the observation $|g(u)|^2 \leq C|u|^{2(r-1)}$ implies that $g(u) - f \in L^2(\Omega)$; i.e., the distributional divergence of $\alpha(\nabla u)$ is in $L^2(\Omega)^d$. By Assumption 2.6 and restricting to Ω_i, we arrive at

$$\alpha(\nabla F_i\eta) \in H(\mathrm{div}, \Omega_i) \cap C(\overline{\Omega}_i)^d, \quad \alpha(\nabla F_i\eta) \cdot \nu_i \in L^\infty(\partial\Omega_i), \quad \text{and}$$
$$\nabla \cdot \alpha(\nabla F_i\eta) = g(F_i\eta) - f_i \in L^2(\Omega_i).$$

The $H(\mathrm{div}, \Omega_i)$-version of Green's formula [19, Chapter 1, Corollary 2.1] then gives us

$$\int_{\Omega_i} \alpha(\nabla F_i\eta) \cdot \nabla v \,\mathrm{d}x = -\int_{\Omega_i} \nabla \cdot \alpha(\nabla F_i\eta) v \,\mathrm{d}x + \int_{\partial\Omega_i} \alpha(\nabla F_i\eta) \cdot \nu_i \, T_{\partial\Omega_i} v \,\mathrm{d}S$$

for all $v \in H^1(\Omega_i)$. Hence,

$$\begin{aligned}\langle S_i\eta, \mu\rangle &= \int_{\Omega_i} \alpha(\nabla F_i\eta) \cdot \nabla R_i\mu \,\mathrm{d}x + \int_{\Omega_i} \big(g(F_i\eta) - f_i\big) R_i\mu \,\mathrm{d}x \\ &= \int_{\partial\Omega_i} \alpha(\nabla F_i\eta) \cdot \nu_i \, T_{\partial\Omega_i} R_i\mu \,\mathrm{d}S = \big(\alpha(\nabla F_i\eta) \cdot \nu_i, \mu\big)_{L^2(\Gamma)} \quad \text{for all } \mu \in \Lambda_i,\end{aligned}$$

which implies that $\eta \in D(\mathcal{S}_i)$ for $i = 1, 2$. ∎

Remark 8.3. From the proof, it is clear that the regularity assumption $\alpha(\nabla u) \in C(\overline{\Omega})^d$ is stricter than necessary and could be replaced by assuming that the normal component of $\alpha(\nabla u)$ on Γ can be interpreted as an element in $L^2(\Gamma)^*$. However, characterizing the spatial regularity of u required to satisfy this weaker assumption demands a more elaborate trace theory than the one considered in section 3.

LEMMA 8.4. *If Assumptions* 2.1 *and* 3.1 *hold, then the operators* $\mathcal{S}_i$ *are monotone, i.e.,*

$$(\mathcal{S}_i\eta - \mathcal{S}_i\mu, \eta - \mu)_{L^2(\Gamma)} \geq 0 \quad \textit{for all } \eta, \mu \in D(\mathcal{S}_i),$$

and the operators $sI + \mathcal{S}_i : D(\mathcal{S}_i) \to L^2(\Gamma)$ *are bijective for any* $s > 0$.

Proof. The monotonicity follows by Lemma 7.2, as

$$\begin{aligned}(\mathcal{S}_i\eta - \mathcal{S}_i\mu, \eta - \mu)_{L^2(\Gamma)} &= (J^{-1}S_i\eta - J^{-1}S_i\mu, \eta - \mu)_{L^2(\Gamma)} \\ &= \langle S_i\eta - S_i\mu, \eta - \mu\rangle \geq 0 \quad \text{for all } \eta, \mu \in D(\mathcal{S}_i) \subseteq \Lambda_i.\end{aligned}$$

For a fixed $s > 0$ and an arbitrary $\mu \in L^2(\Gamma)$, we have, due to Corollary 7.7, that there exists a unique $\eta \in \Lambda_i$ such that $(sJ + S_i)\eta = J\mu$ in Λ_i^*, i.e.,

$$S_i\eta = J(\mu - s\eta) \in L^2(\Gamma)^*.$$

Hence, $\eta \in D(\mathcal{S}_i)$ and $(sI + \mathcal{S}_i)\eta = \mu$ in $L^2(\Gamma)$. The operators $sI + \mathcal{S}_i : D(\mathcal{S}_i) \to L^2(\Gamma)$ are therefore bijective. ∎

The Peaceman–Rachford splitting on $L^2(\Gamma)$ is now given by finding $(\eta_1^n, \eta_2^n) \in D(\mathcal{S}_1) \times D(\mathcal{S}_2)$ for $n = 1, 2, \ldots$ such that

$$\begin{cases}(sI + \mathcal{S}_1)\eta_1^{n+1} = (sI - \mathcal{S}_2)\eta_2^n, \\ (sI + \mathcal{S}_2)\eta_2^{n+1} = (sI - \mathcal{S}_1)\eta_1^{n+1},\end{cases} \tag{8.1}$$

where $\eta_2^0 \in D(\mathcal{S}_2)$ is an initial guess. Lemma 8.4 then directly yields the existence of the approximation; i.e., each iteration of the Peaceman–Rachford approximation has a unique solution η_i^{n+1}.

Corollary 8.5. *If Assumptions* 2.1 *and* 3.1 *hold and* $\eta_2^0 \in D(\mathcal{S}_2)$*, then there exists a unique Peaceman–Rachford approximation* $(\eta_1^n, \eta_2^n)_{n\geq 1} \subset D(\mathcal{S}_1)\times D(\mathcal{S}_2)$ *given by* (8.1) *in* $L^2(\Gamma)$.

Corollary 8.6. *Let Assumptions* 2.1 *and* 3.1 *hold,* $\eta_2^0 \in D(\mathcal{S}_2)$*, and set* $u_2^0 = F_2\eta_2^0$*. The Peaceman–Rachford approximation* $(\eta_1^n, \eta_2^n)_{n\geq 1} \subset D(\mathcal{S}_1) \times D(\mathcal{S}_2)$ *also satisfies the weak formulation* (6.3)*, and* $(u_1^n, u_2^n)_{n\geq 1} = (F_1\eta_1^n, F_2\eta_2^n)_{n\geq 1}$ *is a weak Robin–Robin approximation* (5.2).

Proof. Assume that $(\eta_1^n, \eta_2^n)_{n\geq 1} \subset D(\mathcal{S}_1) \times D(\mathcal{S}_2)$ is a Peaceman–Rachford approximation in $L^2(\Gamma)$. Then

$$\begin{aligned}
\big((sI + \mathcal{S}_1)\eta_1^{n+1}, \mu\big)_{L^2(\Gamma)} &= \big((sI - \mathcal{S}_2)\eta_2^n, \mu\big)_{L^2(\Gamma)} && \text{for all } \mu \in L^2(\Gamma),\\
\big((sI + \mathcal{S}_1)\eta_1^{n+1}, \mu\big)_{L^2(\Gamma)} &= \langle (sJ + S_1)\eta_1^{n+1}, \mu\rangle && \text{for all } \mu \in \Lambda_1, \quad \text{and}\\
\big((sI - \mathcal{S}_2)\eta_2^n, \mu\big)_{L^2(\Gamma)} &= \langle (sJ - S_2)\eta_2^n, \mu\rangle && \text{for all } \mu \in \Lambda_2.
\end{aligned}$$

This implies that

$$\langle (sJ + S_1)\eta_1^{n+1}, \mu\rangle = \langle (sJ - S_2)\eta_2^n, \mu\rangle \quad \text{for all } \mu \in \Lambda = \Lambda_1 \cap \Lambda_2;$$

i.e., the first assertion of (6.3) holds. The same argumentation yields that the second assertion of (6.3) is valid. As $(\eta_1^n, \eta_2^n)_{n\geq 1}$ satisfies (6.3), Lemma 6.3 directly implies that $(u_1^n, u_2^n)_{n\geq 1} = (F_1\eta_1^n, F_2\eta_2^n)_{n\geq 1}$ is a weak Robin–Robin approximation (5.2). ∎

Remark 8.7. At first glance, finding an initial guess satisfying $\eta_2^0 \in D(\mathcal{S}_2)$ might seem limiting, as $D(\mathcal{S}_2)$ is not explicitly given. However, such an initial guess can, e.g., be found by solving $\langle S_2\eta_2^0, \mu\rangle = 0$ for all $\mu \in \Lambda_2$.

With this $L^2(\Gamma)$-framework, the key part of the convergence proof follows by the abstract result [27, Proposition 1]. For sake of completeness, we state a simplified version of the short proof in the current notation.

Lemma 8.8. *Consider the solution of* $\mathcal{S}\eta = 0$ *and the Peaceman–Rachford approximation* $(\eta_1^n, \eta_2^n)_{n\geq 1}$*. If* $\eta_2^0 \in D(\mathcal{S}_2)$ *and Assumptions* 2.1, 2.6, *and* 3.1 *hold, then*

$$(\mathcal{S}_i\eta_i^n - \mathcal{S}_i\eta, \eta_i^n - \eta)_{L^2(\Gamma)} \to 0 \quad \text{as } n \to \infty \tag{8.2}$$

for $i = 1, 2$.

Proof. By the hypotheses and Lemma 8.2, we obtain that $\eta \in D(\mathcal{S}_1) \cap D(\mathcal{S}_2)$ and $\mathcal{S}_1\eta = -\mathcal{S}_2\eta$. Furthermore,

$$\eta_1^{n+1} = (sI+\mathcal{S}_1)^{-1}(sI-\mathcal{S}_2)\eta_2^n \in D(\mathcal{S}_1) \text{ and } \eta_2^{n+1} = (sI+\mathcal{S}_2)^{-1}(sI-\mathcal{S}_1)\eta_1^{n+1} \in D(\mathcal{S}_2).$$

Next, we introduce the notation

$$\mu^n = (sI + \mathcal{S}_2)\eta_2^n, \quad \mu = (sI + \mathcal{S}_2)\eta, \quad \lambda^n = (sI - \mathcal{S}_2)\eta_2^n, \quad \text{and} \quad \lambda = (sI - \mathcal{S}_2)\eta,$$

which yields the representations

$$\begin{aligned}
\eta &= \frac{\mu + \lambda}{2s}, & \mathcal{S}_2\eta &= \frac{\mu - \lambda}{2}, & \mathcal{S}_1\eta &= \frac{\lambda - \mu}{2},\\
\eta_2^n &= \frac{\mu^n + \lambda^n}{2s}, & \mathcal{S}_2\eta_2^n &= \frac{\mu^n - \lambda^n}{2}, & &\\
\eta_1^{n+1} &= \frac{\mu^{n+1} + \lambda^n}{2s}, & \mathcal{S}_1\eta_1^{n+1} &= \frac{\lambda^n - \mu^{n+1}}{2}. & &
\end{aligned}$$

The monotonicity of $\mathcal{S}_i$ then gives the bounds

$$\begin{aligned}0 &\le (\mathcal{S}_2\eta_2^n - \mathcal{S}_2\eta, \eta_2^n - \eta)_{L^2(\Gamma)}\\ &= \frac{1}{4s}\big((\mu^n - \mu) - (\lambda^n - \lambda), (\mu^n - \mu) + (\lambda^n - \lambda)\big)_{L^2(\Gamma)}\\ &= \frac{1}{4s}\big(\|\mu^n - \mu\|^2_{L^2(\Gamma)} - \|\lambda^n - \lambda\|^2_{L^2(\Gamma)}\big)\end{aligned}$$

and

$$\begin{aligned}0 &\le (\mathcal{S}_1\eta_1^{n+1} - \mathcal{S}_1\eta, \eta_1^{n+1} - \eta)_{L^2(\Gamma)}\\ &= \frac{1}{4s}\big((\lambda^n - \lambda) - (\mu^{n+1} - \mu), (\lambda^n - \lambda) + (\mu^{n+1} - \mu)\big)_{L^2(\Gamma)}\\ &= \frac{1}{4s}\big(\|\lambda^n - \lambda\|^2_{L^2(\Gamma)} - \|\mu^{n+1} - \mu\|^2_{L^2(\Gamma)}\big).\end{aligned}$$

Putting this together yields that

$$\|\mu^{n+1} - \mu\|^2_{L^2(\Gamma)} \le \|\lambda^n - \lambda\|^2_{L^2(\Gamma)} \le \|\mu^n - \mu\|^2_{L^2(\Gamma)},$$

and we obtain the telescopic sum

$$0 \le \sum_{n=0}^{N}\big(\|\mu^n - \mu\|^2_{L^2(\Gamma)} - \|\mu^{n+1} - \mu\|^2_{L^2(\Gamma)}\big) \le \|\mu^0 - \mu\|^2_{L^2(\Gamma)} - \|\mu^{N+1} - \mu\|^2_{L^2(\Gamma)},$$

i.e., $\|\mu^n - \mu\|^2_{L^2(\Gamma)} - \|\mu^{n+1} - \mu\|^2_{L^2(\Gamma)} \to 0$ as $n \to \infty$. The latter together with the bounds above imply the limits (8.2). ∎

THEOREM 8.9. *Consider the Peaceman–Rachford approximation* $(\eta_1^n, \eta_2^n)_{n\ge1}$, *given by* (8.1), *of the Steklov–Poincaré equation* $\mathcal{S}\eta = 0$ *in* $L^2(\Gamma)$ *together with the corresponding Robin–Robin approximation* $(u_1^n, u_2^n)_{n\ge1} = (F_1\eta_1^n, F_2\eta_2^n)_{n\ge1}$ *of the weak solution* $u = \{F_1\eta$ *on* $\Omega_1; F_2\eta$ *on* $\Omega_2\}$ *to the nonlinear elliptic equation* (2.1).
If $\eta_2^0 \in D(\mathcal{S}_2)$ *and Assumptions* 2.1, 2.6, *and* 3.1 *hold, then*

$$\|\eta_1^n - \eta\|_{\Lambda_1} + \|\eta_2^n - \eta\|_{\Lambda_2} \to 0, \quad \text{and} \quad \|u_1^n - u\|_{W^{1,p}(\Omega_1)} + \|u_2^n - u\|_{W^{1,p}(\Omega_2)} \to 0 \tag{8.3}$$

as n tends to infinity.

Proof. By the monotonicity bound in Lemma 7.2, the property that $\eta_i^n, \eta \in D(\mathcal{S}_i)$, and Lemma 8.8, we have the limits

$$\begin{aligned}c_i\big(\|\nabla(F_i\eta_i^n - F_i\eta)\|^p_{L^p(\Omega_i)^d} + \|F_i\eta_i^n - F_i\eta\|^r_{L^r(\Omega_i)}\big) &\le \langle S_i\eta_i^n - S_i\eta, \eta_i^n - \eta\rangle\\ &= (\mathcal{S}_i\eta_i^n - \mathcal{S}_i\eta, \eta_i^n - \eta)_{L^2(\Gamma)} \to 0 \quad \text{as } n \to \infty\end{aligned}$$

for $i = 1, 2$. Hence, each of the terms $\|\nabla(F_i\eta_i^n - F_i\eta)\|_{L^p(\Omega_i)^d}$ and $\|F_i\eta_i^n - F_i\eta\|_{L^r(\Omega_i)}$ tends to zero, which yields that

$$\|\eta_i^n - \eta\|_{\Lambda_i} \le C\|F_i\eta_i^n - F_i\eta\|_{V_i} \to 0 \quad \text{as } n \to \infty$$

for $i = 1, 2$. The desired convergence (8.3) is then proven, as $\|\cdot\|_{V_i}$ and $\|\cdot\|_{W^{1,p}(\Omega_i)}$ are equivalent norms. ∎

Remark 8.10. As already stated in section 1, the convergence rate of the Robin–Robin method applied to a spatially discretized linear elliptic equation deteriorates as the spatial parameter tends to zero [17], i.e., when one considers the continuous case. It is therefore unlikely that one could derive a stronger result than the $W^{1,p}$-convergence of Theorem 8.9 for the Robin–Robin method applied to general nonlinear elliptic equations.

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