Approximation problems and weights

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LUND UNIVERSITY

DOCTORAL THESIS
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Abstract
This thesis is mainly concerned with investigations of approximation problems on spaces of analytic functions on the unit disc in the complex plane, which naturally arise in connection to spectral problems of certain classes of linear operators acting on the spaces in question. For instance, quasi-nilpotency of certain analytic paraproducts on the Hardy spaces and on the Bergman spaces are investigated. These problems can be interpreted in terms of approximation problems in the corresponding symbol classes that induce bounded paraproducts therein. Another substantial part of the thesis is devoted to studying smooth approximations in the model spaces and in the de Branges-Rovnyak spaces. It turns out that these questions have dual reformulations in terms of Beurling-type theorems for shift operators on certain spaces of analytic functions. In all this work, various notions of weights play a central role, either explicitly or implicitly.

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”Problems worthy of attack, prove their worth by hitting back.” – Piet Hein.

Contents

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Populärvetenskaplig sammanfattning

Inom kvantfysiken modelleras fysikaliska storheter som uppfyller superpositionsprincipen av linjära operatorer på Hilbertrum, där spektrala egenskaper hos operatorerna har tolkningar i termer av mätvärden hos storheterna. Ett klassiskt exempel är Schrödinger operatorn, som beskriver dynamiken hos ett kvantmekaniskt tillstånd, exempelvis tillståndet hos elektroner innanför atomer. Matematiskt är det ofta fördelaktigt att studera linjära operatorer genom att låta de verka på Hilbertrum av funktioner, som ofta besitter särskilda egenskaper förknippade till operatorn i fråga. En viktig och väletablerad operatorteoretisk modell härrör från Sz-Nagy & Foais teorin utvecklad omkring 50-talet, och härleder hur skiftoperatorer på underrum i det klassiska Hardyrummet beskriver spektrala egenskaper hos en generell klass av kontraktiva linjära operatorer på Hilbertrum, med tillämpningar inom bland annat signal behandling och kontrollteori. Förståelse för skiftoperatorer på Hardyrummen anses idag vara god och beror till stor del på en viktig insats av Arne Beurling år 1949, där en funktionsteoretisk beskrivning av de underrum till Hardyrummet som är invarianta under skiftoperatorerna tillkännagavs. Detta resultat är i modern tid är att betraktas som en utav de viktigaste milstolparna inom operatorteorin. Särskilda funktionsrum som uppträder i dessa sammanhang är modellrummen och mer generellt, de Branges-Rovnyak rummen, som båda erhåller mycket subtila funktionsteoretiska egenskaper. En annan viktig klass av operatorer som introducerades av Paul Halmos år 1952, är klassen av subnormala operator, som i många intressanta fall kan modelleras av skiftoperator på polynomtillslutningar av Lebesguerum med mått som har kompakt stöd i det komplexa talplanet.

I denna avhandling behandlas olika frågeställningar som berör tillnärmningsproblem i särskilda funktionsrum på enhetsskivan i det komplexa talplanet, som naturligt uppträder inom spektralteorin för klassiska linjära operatorer, såsom skiftoperatorer och paraprodukter. Exempel på funktionsrum som betraktas är de klassiska Hardyrummen och Bergmanrummen, samt särskilda rum däri, såsom modellrummen. Exempelvis så handlar ett utav projekten om att studera spektrala egenskaper av analytiska paraprodukter, som visar sig ha att göra med tillnärmningsproblem i symbolklassen som inducerar begränsade paraprodukter på diverse funktionsrum. En betydande del utav avhandlingen tillägnas åt att studera reguljära tillnärmningsproblem i extrema de Branges-Rovnyak rum, såsom modellrummen. Det visar sig att dessa frågeställningar har intima kopplingar till osäkerhetsprinciper relaterat till irreducibilitet inom teorin för subnormala operatorer, som är av fundamental betydelse inom både operatorteorin och funktionsteorin i komplex och harmonisk analys.

List of publications

This thesis is based on the following list of publications:

I **Generalized Cesáro operators: geometry of spectra and quasi-nilpotency**

A. Limani & B. Malman
Published in *International Mathematics Research Notices*, 2020, Vol. 2021, Issue 23, Pages 17695-17707.

II **Bloch functions and Bekollé-Bonami weights**

A. Limani & A. Nicolau
Accepted for publication in *Indiana University Mathematics Journal*, 2021.

III **On model spaces and density of functions smooth on the boundary**

A. Limani & B. Malman
Pre-print 2021 (submitted)

IV **An abstract approach to approximations in spaces of pseudocontinuable functions**

A. Limani & B. Malman
Published in *Proceedings of the American Mathematical Society*, 2021, Vol. 150, Number 6, Pages 2509-2519.

V **Inner functions, Invariant subspaces and cyclicity in $\mathcal{P}^t(\mu)$-spaces**

A. Limani & B. Malman
Pre-print 2021 (submitted)

VI **On the problem of smooth approximations in de Branges-Rovnyak spaces and connections to subnormal operators**

A. Limani & B. Malman
Pre-print 2021 (submitted)

VII **Sparse Lerner operators in infinite dimensions**

A. Limani & S. Pott
Pre-print 2021.

Preface

Preface

1 Summary

The main part of my PhD thesis revolves around investigating specific approximation problems in different spaces of analytic functions, which often and naturally appear in connection to spectral problems of certain bounded linear operators therein. The subjects treated here are best captured within the realms of complex analysis, harmonic analysis and operator theory. In this thesis, we are concerned with the treatment of 4 different, but somewhat related research projects, summarized in 7 research articles.

The first project is concerned with studying spectral properties of so-called analytic paraproducts, also known as Cesaró operarors, through the theory of weights, which turns out to be connected to certain approximation problems in the spaces of symbols that induce bounded analytic paraproducts. These results are contained in the following two articles, where the first one is in collaboration with my collaborator Dr. Bartosz Malman, affiliated with the Royal Institute of Technology in Stockholm, while the second article is a collaborative work together with Prof. Artur Nicolau from Universitet Autónoma de Barcelona.

[**Paper I**] A. Limani, B. Malman, Generalized Cesáro operators: geometry of spectra and quasi-nilpotency, published in International Mathematics Research Notices (2020).

[**Paper II**] A. Limani, A. Nicolau, Bloch functions and Bekollé-Bonami weights, accepted for publication in Indiana University Mathematics Journal (2021).

The second project concerns approximation problems in the classical model spaces in the Hardy space H^2 and their H^p-counterparts, usually referred to as the spaces of pseudocontinuable functions. Here, the central questions are to investigate when certain classes of functions with sufficiently regular boundary values form a dense subset in a model space or in a space of pseudocontinuable functions. This work is contained in the following two papers, both in collaboration with Dr. Bartosz Malman.

[**Paper III**] A. Limani, B. Malman, On model spaces and density of functions smooth on the boundary, pre-print 2021 (submitted).

[**Paper IV**] A. Limani, B. Malman, An abstract approach to approximations in spaces of pseudocontinuable functions, published in Proceedings of the American Mathematical Society (2021).

The next project in line was initially intended as an extension of the previous research project on smooth approximations in model spaces, to the setting of de Branges-Rovnyak spaces. However, our investigations revealed that these questions were significantly more involved, as they closely tie to the theory of cyclic subnormal operators, which can be modeled by the multiplication operator $M_z(f)(z) = zf(z)$ on some $\mathcal{P}^2(\mu)$-space, the closure of analytic polynomials in the usual Lebesgue space $L^2(\mu)$, where μ is a compactly supported Borel measure in the complex plane. This work was carried in collaboration with Dr. Bartosz Malman and is contained in the following articles.

[**Paper V**] A. Limani, B. Malman, Inner functions, Invariant subspaces and cyclicity in $\mathcal{P}^t(\mu)$-spaces, pre-print 2021 (submitted).

[**Paper VI**] A. Limani, B. Malman, On the problem of smooth approximations in de Branges-Rovnyak spaces and connections to subnormal operators, pre-print 2021 (submitted).

The final project in this thesis deviates from the previous theme of works and is strictly contained within the area of Harmonic Analysis. In this joint work together with my supervisor Prof. Sandra Pott, we explored sharp bounds for sparse Lerner operators in the infinite dimensional setting of operator-valued weights and investigated sparse domination models for the Bergman projection. The content of our work is contained in the following paper.

[**Paper VII**] A. Limani, S. Pott, Sparse Lerner operators in infinite dimensions, pre-print 2021.

2 Spectral properties of analytic paraproducts from the perspective of weight theory

2.1 Background

In [Paper I], B. Malman and I investigated spectral properties of a certain class of analytic paraproducts on the Hardy spaces and on the Bergman spaces, both defined on the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. Among experts, the analytic paraproducts of our consideration are sometimes referred to as generalized Cesáro operators or generalized Volterra operators. Given an analytic function g on $\mathbb{D}$, the analytic paraproduct T_g with symbol g is defined as the linear operator acting on analytic functions f on $\mathbb{D}$ via the integral formula

$$T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta, \qquad z \in \mathbb{D}.$$

Here the analyticity of f and g on a simply connected domain ensures that the integral is independent of the simple path that joins 0 and z. The label *paraproduct* should be understood in a conceptual manner, as suggested by S. Janson and J. Petree in [23], which is justified by the property $fg = T_g(f) + T_f(g)$. In this framework, we considered the spectral properties of analytic paraproducts on the classical Hardy spaces H^p and on the standard weighted Bergman spaces $L_a^{p,\alpha}$, where we recall that H^p for $p > 0$, is the space of analytic functions f on $\mathbb{D}$ satisfying

$$\sup_{0<r<1} \int_{\partial\mathbb{D}} |f(r\zeta)|^p dm(\zeta) < \infty,$$

while $L_a^{p,\alpha}$ for $p > 0$ and $\alpha > -1$, consists of analytic functions f on $\mathbb{D}$ with

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty$$

As in the equations above, we shall denote by dm the normalized arc-length measure on the unit circle $\partial\mathbb{D}$, and denote by dA the area measure on $\mathbb{D}$. In the setting of the Hardy spaces H^p, it is well-known that $T_g : H^p \to H^p$ defines a bounded linear operator, if and only if g belongs to **BMOA**, that is, g belongs to some Hardy space H^p with $p > 0$, and its boundary extension (also denoted by g) satisfies the BMO-condition on the unit circle $\partial\mathbb{D}$:

$$\|g\|_{BMO} := |g(0)| + \sup_{J \subset \partial\mathbb{D}} \frac{1}{m(J)} \int_J \left| g - \frac{1}{m(J)} \int_J g dm \right| dm < \infty,$$

In the setting of Bergman spaces $L_a^{p,\alpha}$, the right condition for $T_g : L_a^{p,\alpha} \to L_a^{p,\alpha}$ to define a bounded linear operator is that g belongs to the Bloch space $\mathcal{B}$, that is, the Banach space of analytic functions on $\mathbb{D}$ equipped with the norm

$$\|g\|_{\mathcal{B}} := |g(0)| + \sup_{z \in \mathbb{D}} (1-|z|^2)|g'(z)| < \infty.$$

Spectral properties of T_g operators have previously been studied for particular subclasses of symbols, such as for rational functions g in [1] by Albrecht, E., T. L.

Miller, and M. M. Neumann, and in [42] [41] by S. W. Young, in [32] by A-M. Persson, and by others (see references therein). However, the first characterization of the spectrum for general symbols g was only obtained by A. Aleman and O. Constantin in the Bergman space setting [4], and later by A. Aleman and J. Á. Peláez in the Hardy space setting [8]. The resolvent set of T_g on these spaces are described in terms a weight condition related to the symbol g. More precisely, it was proved that $\lambda \neq 0$ does not belong the spectrum of T_g on H^p, denoted by $\sigma(T_g|H^p)$, if and only if the weight $\omega_{\lambda,p} := |\exp(pg/\lambda)|$ satisfies the Muckenhoupt $\mathcal{A}_\infty$-condition (see Theorem 1.1 in [Paper I]). In the Bergman space setting, the role of Muckenhoupt weights is replaced by the class of Békollé-Bonami weights and the result is phrased as follows: $\lambda \notin \sigma(T_g|L_a^{p,\alpha})$ if and only if the adjusted weight $w(z) := (1-|z|)^\alpha \omega_{\lambda,p}(z)$ satisfies a B_∞-type condition (see Theorem 1.2 in [Paper I]). A particularly striking consequence of these descriptions of the spectrum is that, in conjunction with the basis fact that the spectrum of continuous linear operators are compact subsets of $\mathbb{C}$, one can derive deep Gehring-type results on self-improvement properties for a large class of weights, see Corollary D in [8]. In [Paper I] our intention was to reverse this perspective and instead utilize the rich and well-developed theory of weights in Harmonic Analysis, in order to derive geometric properties of the spectrum of T_g operators.

2.2 Geometry of spectra and quasi-nilpotency

To highlight some of our results, we shall in this section let X denote either H^p or $L_a^{p,\alpha}$, where $p > 0$ and $\alpha > -1$ should be regarded as arbitrary but fixed. An important result in our work illustrates that the fact that Muckenhoupt weights and Bekollé-Bonami weights are closed under the formations of log-convex combinations and satisfy self-improving properties, naturally corresponds to certain geometric properties of the spectrum of analytic paraproducts.

Theorem 2.1 (Theorem 2.5 in [Paper 1). *] Let $T_g : X \to X$ with the property that there exists a non-zero $\lambda \in \sigma(T_g|X)$. Then for any $0 < r < 1$, there exists a circular arc $J_{r,\lambda} \subset \{z : |z| = r|\lambda|\}$, centered at the point $r\lambda$, such that the circular sector created by taking the convex hull of 0 and $J_{r,\lambda}$ is contained in $\sigma(T_g|X)$.*

We refer the reader to the figure 1 in [Paper 1] for a clarifying illustration of the theorem above. We denote by $T(X)$ the induced Banach spaces of symbols g, equipped with the norm

$$\|g\|_{T(X)} := \|T_g\|_{X \to X}.$$

As briefly indicated in the previous paragraphs, one can show that $T(H^p) = \mathbf{BMOA}$ and $T(L_a^{p,\alpha}) = \mathcal{B}$, thus the above induces equivalent norms on the spaces of symbols giving rise to continuous T_g-operators. We denote by $M(X)$ the algebra of multipliers on X, which for our spaces X is just the algebra H^∞ of bounded analytic functions, and thus we always have the containment $M(X) \subset T(X)$. In fact, it turns out that the containment remains true for yet another wide range of interesting Banach spaces of analytic functions X, such as the family of weighted sequence spaces (see [12]), where the corresponding multiplier algebras $M(X)$ are known to be strictly smaller than H^∞ and complicated to describe. Our next result reveals a connection between quasi-nilpotency of analytic paraproducts and approximation problems in the spaces associated to the symbol classes.

Theorem 2.2 (Theorem 2.1 and Theorem 2.2 in [Paper 1]). *] Let $T_g, T_h : X \to X$ be bounded. If $\sigma(T_h|X) = \{0\}$, then the spectrum of T_g is stable under perturbation with T_h, that is*

$$\sigma(T_{g+h}|X) = \sigma(T_g|X).$$

Moreover, the symbol g induces a quasi-nilpotent operator $T_g : X \to X$, if

$$\inf_{h \in M(X)} \|g - h\|_{T(X)} = 0.$$

A natural question that appears in this context is if the converse of the latter statement holds true. More precisely, if T_g is quasi-nilpotent on X, does that necessarily imply that g belongs to the closure of $M(X) = H^\infty$ in $T(X)$? For the sake of brevity, we shall denote the later condition by $g \in \text{Clos}(M(X))_{T(X)}$. We remark that in the context of analytic paraproducts on Bergman spaces $X = L_a^{p,\alpha}$, this question suggests a plausible conjecture for the description of $\text{Clos}(H^\infty)_{\mathcal{B}}$, which was initially raised in 1974 by J. Anderson, J. Clunie and C. Pommerenke in [35]. In [Paper I], we answered the above question in the affirmative for the Hardy spaces $X = H^p$, while together with Prof. Artur Nicolau in [Paper II], we answered the above question in the negative in the context of the Bergman spaces $X = L_a^{p,\alpha}$. Our findings are summarized in the following theorems.

Theorem 2.3 (Theorem 2.4 in [Paper I). *]Let $p > 0$ and $g \in BMOA$. Then $\sigma(T_g|H^p) = \{0\}$ if and only if $g \in Clos(H^\infty)_{BMOA}$.*

Theorem 2.4 (Theorem 1.4 in [Paper II). *] There exists an analytic function $g \in \mathcal{B}$ with the property that $\sigma(T_g|L_a^{p,\alpha}) = \{0\}$, for all $p > 0$ and $\alpha > -1$, but $g \notin Clos(H^q \cap \mathcal{B})_{\mathcal{B}}$, for any $0 < q \leq \infty$.*

The proof of the first theorem relies on the classical Helson-Szegö theorem for Muckenhoupt A_2-weights, which is an absent tool in the setting of Békollé-Bonami weights. Meanwhile, the construction of the counterexample proving the second assertion relies on the previous work by A. Nicolau and N. Galán on the characterization of $\mathbf{Clos}(H^q \cap \mathcal{B})_{\mathcal{B}}$, for $0 < q < \infty$, using Lusin square-area functions [16]. Our result above shows that quasi-nilpotency of analytic paraproducts on the Bergman spaces $L^{p,\alpha}_a$ are far from capturing the closure of H^∞ in $\mathcal{B}$. However, the question of describing the symbols g that induce quasi-nilpotent T_g operators on Bergman spaces still remains relevant.

2.3 Control of spectral radius and distance formulas

We introduce the space $BMO(\mathbb{D})$ as the set of complex-valued functions on $\mathbb{D}$, satisfying

$$\|f\|_{BMO(\mathbb{D})} = \sup_D \frac{1}{A(D \cap \mathbb{D})} \int_{D\cap\mathbb{D}} |f(z) - f_{D\cap\mathbb{D}}| dA(z) < \infty$$

where $f_S := \frac{1}{A(S)} \int_S f dA$ denotes the average of f over a Borel set $S \subset \mathbb{D}$ and the supremum is taken over disks D centered at points inside $\mathbb{D}$. An important observation by R. Coifman, R. Rochberg and G. Weiss in [14] was that the closed subspace of $BMO(\mathbb{D})$ consisting of analytic functions on $\mathbb{D}$ coincides with $\mathcal{B}$, and their respective semi-norms are equivalent. Denoting the standard hyperbolic metric on $\mathbb{D}$ by β (See [Paper II] for a precise definition) and the spectral radius of T_g on L^p_a by $\|\sigma_p(g)\|$, our main result in this direction can be phrased as follows.

Theorem 2.5 (Theorem 1.1 and Corollary 4.1 in [Paper II). *] There exists a universal constant $C > 0$, such that for any $g \in \mathcal{B}$, we have*

$$C^{-1} \frac{\|\sigma_p(g)\|}{p} \leq \inf_{h \in L^\infty(\mathbb{D})} \|g - h\|_{BMO(\mathbb{D})} \leq C \frac{\|\sigma_p(g)\|}{p}.$$

Moreover, let $\varepsilon(g)$ denotes the infimum of $\varepsilon > 0$ for which g satisfies the following property: there exists a constant $C(\varepsilon) > 0$, such that

$$|g(z) - g(\lambda)| \leq C(\varepsilon) + \varepsilon\beta(z, \lambda) \qquad z, \lambda \in \mathbb{D}. \tag{1}$$

Then we have that $\qquad 2p \cdot \varepsilon(g) \leq \|\sigma_p(g)\| \leq 4p \cdot \varepsilon(g).$

The first statement of the above theorem implies that T_g is quasi-nilpotent on some (any) Bergman space, if and only if g belongs to the closure of $L^\infty(\mathbb{D})$ in

$BMO(\mathbb{D})$, thus in light of Theorem 1.4 in [Paper II], such bounded approximates to g may generally fail to be analytic on $\mathbb{D}$. The second part of the statement asserts that quasi-nilpotency of T_g is also equivalent to g having arbitrary small hyperbolic Lipschitz characteristic, which essentially says that g behaves like a little Bloch function at points of large hyperbolic distance. In fact, results of similar kind that extend beyond the framework of analytic functions were also obtained (see Theorem 1.1 in [Paper II]). We also mention that appropriate adaptations in conjunction with Theorem 1.3 from [Paper I] also gives a spectral radius formula for T_g on the Hardy spaces H^p, but instead in terms of the distance to H^∞ in **BMOA**.

2.4 Directions for further work

2.4.1 Imposing better boundary conditions

In Theorem 1.4 of [Paper II], we constructed a super lacunary power series $g \in \mathcal{B}$, which does not belong to the closure of $H^p \cap \mathcal{B}$ in $\mathcal{B}$, for any $p > 0$, but belongs to the closure of $L^\infty(\mathbb{D})$ in $BMO(\mathbb{D})$. As lacunary series are notorious for their lack of boundary values on $\partial\mathbb{D}$, it is natural to ask whether such obscure constructions could be prohibited by initially imposing some better boundary behavior on g. For instance, one can phrase the following question:

Problem 2.6. Suppose g belongs to $\bigcup_{p>0} H^p$ or perhaps even $BMOA$, with the additional property that T_g is quasi-nilpotent on some Bergman space. Does that imply that $g \in \mathbf{Clos}(H^\infty)_{\mathcal{B}}$?

Besides the fact that the above assumptions imposed on g exclude any counterexample of the type provided by Theorem 1.4 of [Paper II] and makes the tools of Hardy-space factorization available, it is not clear how any of these assumptions should be implemented in order to answer the phrased question.

2.4.2 Connection to the theory of conformal maps

Another interesting direction of work would be to consider spectral properties of analytic paraproducts on Bergman spaces in relation to the corresponding conformal maps that induce the symbols. For instance, which univalent maps $\psi : \mathbb{D} \to \mathbb{C}$ with normalizations $\psi(0) = 0$ and $\psi'(0) = 1$, induce a Bloch function $\log \psi'$ to be the symbol of a quasi-nilpotent analytic paraproduct $T_{\log \psi'}$ on a Bergman space? Assuming that $\log \psi'$ satisfies condition (1) of Theorem 1.4 for all $\varepsilon > 0$, we can find constants $C(\varepsilon) > 0$ such that

$$\frac{|\psi'(z)|}{|\psi'(\lambda)|} \leq C(\epsilon)(1 - |\phi_\lambda(z)|)^{-\varepsilon} \qquad \lambda, z \in \mathbb{D}.$$

Here $\phi_\lambda(z) = \frac{\lambda - z}{1-\bar{\lambda}z}$ denotes the standard conformal self-map on $\mathbb{D}$. If we set $\lambda = 0$, then the estimate above reduces to a growth condition of ψ' on $\mathbb{D}$, which is equivalent to ψ being $(1-\varepsilon)$-Hölder continuous on $\partial\mathbb{D}$, for any $0 < \varepsilon < 1$. Roughly speaking, this suggests that quasi-nilpotency of $T_{\log \psi'}$ on a Bergman space is related to regularity properties on $\partial\mathbb{D}$ of some family of normalized hyperbolic translates of the conformal map ψ. Perhaps the problem of characterizing the closure of H^∞ in the Bloch space also has a plausible interpretation in terms of conformal maps. For the moment, we can unfortunately neither offer any plausible conjectures nor phrase any concrete questions in order to further pursue any of these problems.

3 Regular approximations in model spaces

3.1 Background

The contents of [Paper III] and [Paper IV] revolve around approximation problems in the classical model spaces, which are the only invariant subspaces for the backward shift operator

$$\mathcal{L}(f)(z) = \frac{f(z) - f(0)}{z}, \qquad z \in \mathbb{D}$$

on the Hardy space H^2. An immediate consequence of Beurling's theorem is that any model space is the orthogonal complement of ΘH^2 in H^2 for some inner function Θ, hence these spaces all take the form $K_\Theta := H^2 \ominus \Theta H^2$. The label model space stems from the operator model theory developed by Sz.-Nagy and Foias, which implies that any completely non-unitary bounded linear operator on a separable Hilbert space is unitary equivalent to the backward shift restricted to some model space (possibly vector-valued). Consequently, any such operator on a separable Hilbert space can be modeled by the shift operator on a model space. The model spaces in the setting of the upper-half plane also appear in connection to completeness problems of solutions to certain classes of symmetric Schrödinger operators, see [30]. Besides their intrinsic operator theoretical nature, these spaces also have very subtle function theoretical properties. A classical theorem that illustrates this phenomena is due to A. B. Aleksandrov, which asserts that the set of functions in a model space which extend continuously to the boundary always form a dense subset therein, despite the fact that in many model spaces, it is very difficult to construct even a single such function [3]. In light of this magnificent result, the following questions naturally arise in this context.

For which model spaces can one find a dense subset of functions with better regularity properties on the boundary than mere continuity? If possible, how can one explicitly construct such functions?

3.2 Smooth approximations in the model spaces

Inspired by the classical theory of partial differential equations, a natural way of measuring the order of regularity is to introduce the analytic Dirichlet-Sobolev spaces of order $\alpha > 0$, denoted by $\mathcal{H}^\alpha$ and consisting of analytic functions f on $\mathbb{D}$ with Fourier coefficients $\{f_n\}_{n\geq 0}$, satisfying

$$\sum_{n=0}^{\infty} (n+1)^\alpha |f_n|^2 < \infty.$$

As the parameter $\alpha > 0$ increases, so does the smoothness properties on $\partial\mathbb{D}$ of functions f in $\mathcal{H}^\alpha$, thus $\mathcal{A}^\infty := \cap_{\alpha>0}\mathcal{H}^\alpha$ consists of analytic functions on $\mathbb{D}$ with C^∞-smooth extensions to $\partial\mathbb{D}$. A striking result by K. Dyakonov and D. Khavinson in [15] asserts that some model spaces may not even contain a single non-trivial function belonging to $\mathcal{H}^\alpha$, for any $\alpha > 0$. In fact, they proved that a model space K_Θ contains a non-trivial function from $\mathcal{H}^\alpha$, precisely when either Θ has a zero inside $\mathbb{D}$ or when the associated singular measure ν_Θ assigns mass to some set $E \subset \partial\mathbb{D}$ of finite Beurling-Carleson entropy (see the Theorem below). In a similar fashion, our main result in [Paper III] provides a complete characterization of the model spaces K_Θ for which the linear manifolds $\mathcal{H}^\alpha \cap K_\Theta$ are dense in K_Θ.

Theorem 3.1 (Theorem 1.1 of [Paper III). *] Let $\Theta = BS_\nu$ be an inner function with Blaschke factor B, and singular inner factor S_ν with associated singular measure ν. Then the following conditions are equivalent:*

(i) *$\mathcal{H}^\alpha \cap K_\Theta$ is dense in K_Θ, for some $\alpha > 0$*

(ii) *$\mathcal{A}\infty \cap K_\Theta$ is dense in K_Θ.*

(iii) *The singular measure ν is concentrated on a countable union of closed sets E with finite Beurling-Carleson entropy:*

$$\sum_n m(I_n) \log m(I_n) > -\infty$$

where $\{I_n\}$ denote the connected components of $\partial\mathbb{D} \setminus E$.

The necessity of condition (iii) follows from a duality argument involving the Korenblum-Roberts theorem on cyclicity of inner functions on the Bergman spaces (see [26], [37]). Conversely, if the associated singular measure ν satisfies condition (iii), then we can actually provide a constructive approximation scheme of functions in $\mathcal{A}^\infty \cap K_\Theta$, using Toeplitz operators to smooth out singularities on sets of finite Beurling-Carleson entropy.

3.3 An abstract approach to approximation in spaces of pseudocontinauble functions

The content of [Paper IV] is essentially concerned with generalizing the results in [Paper III] to spaces of pseudocontinuable functions K^p_Θ, which can be regarded as appropriate H^p-versions of the classical model spaces, for $p > 0$. This time, we actually approach these problems from a more abstract point of view and with the following central question in mind:

For which linear spaces X, consisting of bounded analytic functions in $\mathbb{D}$ with certain degree of regular extensions to $\partial\mathbb{D}$, and inner functions Θ, are the linear manifolds $X \cap K^p_\Theta$ dense in K^p_Θ, for some $p > 0$?

If X is invariant under the backward shift, as practically any reasonable class X containing functions with regular extensions to $\partial\mathbb{D}$ is, then our first important observation is the following extrapolation principle: if $X \cap K^p_\Theta$ is dense in K^p_Θ for some $p > 0$, then automatically $X \cap K^p_\Theta$ ia dense in K^p_Θ, for all $p > 0$ (see Theorem 1.1 in [Paper IV]). This result is attributed to A. B. Aleksandrov, as it generalizes some fundamental ideas of his in [2]. As a consequently, it suffices to investigate the above question in the setting of model spaces $K_\Theta := K^2_\Theta$. To this end, we shall denote the Cauchy dual of X by X' (see section 2.1 in [Paper IV] for a detailed definition) and for an inner function Θ, we set $[\Theta]_{X'}$ to be the weak-star closure of analytic polynomial multiples of Θ in X'. Our main result is that density with a wide range of regularity classes X in K_Θ is equivalent to the preservation of the inner factor Θ in the weak-star topology on the Cauchy dual X' of X.

Theorem 3.2 (Theorem 1.3 in [Paper IV]. *] The linear manifold $X \cap K_\Theta$ forms a dense subset of K_Θ if and only if the following (P)-property of the pair (X, Θ) holds:*

$$[\Theta]_{X'} \cap H^2 \subseteq \Theta H^2.$$

In other words, the inner-factor Θ is preserved under weak-star convergence in X'.

Observe that if X is a reflexive Banach space, then the weak and weak-star topologies on X' are equivalent, thus since $[\Theta]_{X'}$ is a convex set, one can rephrase the (P)-property in terms of the norm-closure of analytic polynomial multiples of Θ in X'. Furthermore, if the Cauchy dual X' can be identified with a Bergman space, then Theorem 1 in [37] actually provides a complete description of the inner functions Θ for which the pair (X, Θ) satisfies the (P)-property. Consequently, this line of reasoning also provides a different proof of Theorem 1.1 in [Paper III]. In our next result, we provide a fairly practical condition for checking when a the linear manifolds $X \cap K_\Theta$ are dense in K_Θ, for all inner functions Θ.

Theorem 3.3 (Theorem 1.4 in [Paper IV). *] Let $X \subset H^\infty$ be a Banach spaces containing the polynomials as a dense subset. If $X' \hookrightarrow H^p$ for some $p > 0$, then the linear manifold $X \cap K_\Theta$ forms a dense subset of K_Θ, for any inner function Θ.*

Note that the theorem above, in conjunction with the well-known fact that the Cauchy dual of the disk algebra $\mathcal{A}$ is contained in $\bigcap_{0<p<1} H^p$, provides a new proof of Aleksandrov's density theorem. In fact, using a deep result by S. Vinogradov [39], one can slightly improve Aleksandrov's density theorem by extending it to the finer class U_a, which consists of analytic functions on $\mathbb{D}$ with uniformly convergent Taylor series on $\overline{\mathbb{D}}$.

Corollary 3.4 (Corollary 1.5 in [Paper IV). *] The linear manifold $U_a \cap K^p_\Theta$ is dense in K^p_Θ, for any inner function Θ and any $0 < p \leq \infty$.*

Here, the case $p = \infty$ should be understood in the sense of the weak-star topology on K^∞_Θ, inherited from $L^\infty(\partial\mathbb{D}, m)$. In light of this Corollary, it is remarkable that if we substitute "uniform" convergent by "absolutely" convergent, then the result above fails drastically. In fact, there exists an inner function Θ, such that $W_a \cap K_\Theta = \{0\}$, where W_a denotes the Wiener algebra consisting of analytic functions on $\mathbb{D}$ with absolutely summable Taylor coefficients (see Corollary 1.5 in [Paper IV]). With additional efforts, it should not come as a surprise if the Corollary above extends to the framework of de Branges-Rovnyak spaces, and perhaps even to certain spaces of analytic functions that are contractive with respect to the backward shift, appearing in [7].

3.4 Direction for further work

3.4.1 Connection to boundary zero sets of Banach algebras

A closed set $E \subset \partial\mathbb{D}$ of Lebesgue measure zero is said to be a boundary zero set for a Banach algebra $X \subseteq \mathcal{A}$, if there exists a non-trivial function $f \in X$ with $f = 0$ on

E. In our work, we have gathered some heuristic evidence suggesting that there is an intimate relationship between density of such Banach algebras X in the model spaces and the support of the associated singular inner functions on boundary zero sets of X. More precisely, given an inner function $\Theta = BS_\nu$ and a Banach algebra $X \subseteq \mathcal{A}$ equipped with pointwise multiplication, our examples suggests that the following phenomena occurs.

Conjecture 1. Let $X \subset \mathcal{A}$ be a Banach algebra and $\Theta = BS_\nu$ be an inner function. Then the linear manifold $X \cap K_\Theta$ forms a dense subset of K_Θ if and only if the singular measure ν is concentrated on a countable union of boundary zero sets of X.

Recall that the boundary zero sets for a wide collection of Banach algebras X of analytic functions on $\mathbb{D}$, ranging from the analytic Hölder spaces to $\mathcal{A}^\infty$, are in fact geometrically characterized in terms of having finite Beurling-Carleson entropy [11], [38]. This result in conjunction with our main Theorem in [Paper III] settles the above statement for all such Banach algebras X. In a similar way, one can also view Aleksandrov's density theorem from this perspective, as any closed set $E \subset \partial\mathbb{D}$ is a boundary zero set for a function in $\mathcal{A}$. The claim stipulated above also provides a plausible hint to why we have not been able to resolve the question as to which model spaces K_Θ contain functions from the Wiener algebra W_a as a dense subset. Indeed, R. Kaufman proved that boundary zero sets of such functions are not even invariant under simple disk automorphisms, which suggests that they can be extremely complicated [24]. In light of Theorem 1.3 of [Paper IV], we also suspect the following connection to shift-invariant subspaces generated by inner functions on the Cauchy-duals X' of X.

Conjecture 2. Let $X \subseteq \mathcal{A}$ be a Banach algebra and $\Theta = BS_\nu$ be an inner function. Then there exists a unique decomposition $\nu = \nu_X + \nu_{X^c}$ consisting of non-negative Borel measures ν_X, ν_{X^c}, where ν_X is concentrated on a countable union of boundary zero sets of X and ν_{X^c} vanishes on all such sets. Moreover, the decomposition of ν gives rise to a unique factorization of $\Theta = BS_{\nu_X} S_{\nu_{X^c}}$, with the following properties

(i) (**Preserving part**): $[BS_{\nu_X}]_{X'} \cap H^2 \subset BS_{\nu_X} H^2$.

(ii) (**Cyclic part**): $S_{\nu_{X^c}}$ is weak-star cyclic in X'.

Note that this statement is precisely the Korenblum-Roberts theorem (see [26], [37]) when the boundary zero sets of X are characterized by the Beurling-Carleson entropy and X' can be identified with a standard Bergman space. Indeed, we recall that for reflexive Banach spaces X, we can equivalently reformulate all the above conditions in terms of norm-convergence in X'. We remark that only the second part of the statement is of genuine interest here, while the first part follows from a simple measure theory argument in conjunction with the algebra assumption of X. With the previous statements taken into account, we suggest that there is a deep connection between approximation with Banach algebras X in model spaces, boundary zero sets of X, and Beurling-type theorems on their Cauchy-duals X'.

3.4.2 Smooth normalized Cauchy transforms of singular measures

Given a positive finite singular Borel measure μ on $\partial\mathbb{D}$, we define the *normalized Cauchy transform* C_μ with respect to μ on $L^2(d\mu)$ by the formula

$$C_\mu(f)(z) := \frac{\int_{\partial\mathbb{D}} \frac{f(\zeta)}{1-\bar{\zeta}z} d\mu(\zeta)}{\int_{\partial\mathbb{D}} \frac{1}{1-\bar{\zeta}z} d\mu(\zeta)} = \frac{\mathcal{K}(f d\mu)(z)}{\mathcal{K}(d\mu)(z)} \qquad z \in \mathbb{D}.$$

Here $\mathcal{K}$ denotes the usual Cauchy transform on $\partial\mathbb{D}$. Recall that for any singular probability measure μ on Borel sets of $\partial\mathbb{D}$, there exists a uniquely associated inner function Θ_μ on $\mathbb{D}$ with $\Theta_\mu(0) = 0$, such that the following Herglotz representation formula holds

$$\frac{1+\Theta_\mu(z)}{1-\Theta_\mu(z)} = \int_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z} d\mu(\zeta), \qquad z \in \mathbb{D}. \tag{2}$$

Conversely, one can also show that for any inner function Θ with $\Theta(0) = 0$ there exists a uniquely associated singular probability measure μ_Θ, which is related to Θ via (2). Consequently, the Herglotz representation formula provides a one-to-one correspondence between the set of singular probability Borel measures on $\partial\mathbb{D}$ and the set of inner functions Θ, normalized by $\Theta(0) = 1$. The measure μ_Θ is usually referred to as the *Aleksandrov-Clark measure* associated to Θ. A compelling consequence of this correspondence between $\Theta \leftrightarrow \mu_\Theta$ was initially established by D. Clark in [13], and says that the normalized Cauchy transform C_{μ_Θ} maps $L^2(d\mu_\Theta)$ unitarily onto the model space K_Θ. In particular, for any $f \in K_\Theta$, we can find a unique $h \in L^2(d\mu_\Theta)$, such that

$$f(z) = C_{\mu_\Theta}(h)(z) = (1-\Theta(z)) \int_{\partial\mathbb{D}} \frac{h(\zeta)}{1-\bar{\zeta}z} d\mu_\Theta(\zeta), \qquad z \in \mathbb{D}.$$

This expression is sometimes referred to as the Aleksandrov-Clark measure representation for the model space K_Θ, and a classical theorem by A. Poltoratski in [34] asserts that $f = C_{\mu_\Theta}(h)$ has non-tangential limit equal to $h \in L^2(d\mu_\Theta)$, at μ_Θ-a.e every point on $\partial\mathbb{D}$. Now a natural problem that arises in this context, is how the construction of smooth functions in a certain model space K_Θ, for which the inner function Θ satisfies the hypothesis of Theorem 1.1 in [Paper III], can be carried out through the Aleksandrov-Clark measure representation. To this end, we shall let Θ be an inner function, for which the singular measure ν_Θ associated to the singular inner factor of Θ is concentrated on a set of finite Beurling-Carleson entropy (perhaps even a countable union of such sets), and denote by μ_Θ the corresponding Aleksandrov-Clark measure. In this context, the following questions would be of considerable interest:

Can we find a reasonable description for the set of functions $h \in L^2(\mu_\Theta)$, for which $C_{\mu_\Theta}(h) \in \mathcal{A}^\infty$?

A modest initial approach to question (i) would be to understand the classical toy-model case when $\Theta(z) = \exp\left(-\frac{1+z}{1-z}\right)$, in which ν_Θ is just the Dirac measure with charge at $z = 1$, and the carrier set of the associated Aleksandrov-Clark measure μ_Θ can be explicitly computed.

4 Models for cyclic subnormal operators and smooth approximation in de Branges-Rovnyak spaces

4.1 Background

A subnormal $\mathcal{T}$ is the restriction of a normal operator T on a Hilbert space $\mathcal{H}$ to an invariant subspace $\mathcal{K}$, thus $\mathcal{T} = T|_{\mathcal{K}}$. This class of operators was introduced by P. Halmos in the early 1950's, with the intent that the considerable success that has characterized the history of normal operators could be extended to these more general objects [18]. The essential difference between these and that of normal operators is that the theory of normal operators hinges on measure theory, while the theory of subnormal operators calls for greater reliance on the theory of analytic functions. Particularly interesting is the class of cyclic subnormal operators, as their spectral properties can be modelled by the multiplication operator $M_z f(z) = zf(z)$ on some $\mathcal{P}^2(\mu)$-space, where $\mathcal{P}^2(\mu)$ denotes the closure of the analytic polynomials in the standard Lebesgue space $L^2(\mathbb{C}, \mu)$, and μ is a compactly supported measure in the complex plane $\mathbb{C}$. If the spectrum of the minimal normal extension of the subnormal operator is the closure of a (bounded) simply connected domain Ω in $\mathbb{C}$, then the measure μ in the corresponding spectral model $(M_z, \mathcal{P}^2(\mu))$ is supported on the closure of Ω, see [40]. For this subclass of cyclic subnormal operators, the corresponding spectral theory roughly branches out in two fundamentally different directions. In the former case, the measure μ induces a genuine Hilbert space of analytic functions $\mathcal{P}^2(\mu)$, that is, the restriction map $f \mapsto f|_\Omega$ is injective on $\mathcal{P}^2(\mu)$ and $f|_\Omega$ is holomorphic, thus we are within the realm of analytic function theory (see [9] and references therein) and we say that $\mathcal{P}^2(\mu)$ is *irreducible*. In the second case, $\mathcal{P}^2(\mu)$ contains non-analytic functions and consequently, these spaces contain objects that are better understood within the framework of measure theory, similar to the theory of normal operators. These sorts of questions are commonly referred to as problems of irreducibility or reducibility/splitting, as we shall discuss later.

The content of [Paper V] is devoted to investigating invariant subspaces for the multiplication operator $M_z(f)(z) = zf(z)$ are generated by bounded analytic functions on a certain class of $\mathcal{P}^t(\mu)$-spaces, which are genuine spaces of analytic functions, for $t \geq 1$. A classical framework occurs when the measure μ is assumed to be supported inside the closed unit disc $\overline{\mathbb{D}}$. To provide examples, if μ is the Lebesgue measure m on $\partial\mathbb{D}$, then $\mathcal{P}^t(\mu)$ coincides with the usual Hardy spaces H^t and Beurling's theorem provides a complete characterization of the invariant subspaces of M_z: outer functions are cyclic, while inner functions generate proper invariant subspaces. If $dA_\alpha(z) = (1 - |z|^2)^\alpha dA(z)$ with $\alpha > -1$ and dA denotes the area measure on $\mathbb{D}$, then $\mathcal{P}^t(dA_\alpha)$ is a standard weighted Bergman space and the invariant subspaces generated by inner functions were completely described in the work of B. Korenblum [26] and J. Roberts [37], independently of each other. Denote by $[\Theta]_{\mathcal{P}^t(\mu)}$ the closure in $\mathcal{P}^t(\mu)$ of analytic polynomial multiples of Θ. The Korenblum-Roberts theorem can be phrased as follows:

Theorem 4.1 (Korenblum-Roberts ([26], [37])). *Let $\Theta = BS_\nu$ be an inner function and decompose $\nu = \nu_\mathcal{C} + \nu_\mathcal{K}$, where $\nu_\mathcal{C}$ is concentrated on a countable union of sets if finite Beurling-Carleson entropy and $\nu_\mathcal{K}$ assigns no mass to such sets. Then for any $t > 0$ and $\alpha > -1$, the following statements hold:*

(i) $BS_{\nu_\mathcal{C}}$ generates a proper M_z-invariant subspace on the Bergman space $\mathcal{P}^t(dA_\alpha)$ with the property: $[BS_{\nu_\mathcal{C}}]_{\mathcal{P}^t(dA_\alpha)} \cap H^2 \subseteq BS_{\nu_\mathcal{C}} H^2$.

(ii) $S_{\nu_\mathcal{K}}$ is a cyclic vector for M_z on the Bergman space $\mathcal{P}^t(dA_\alpha)$.

4.2 A Beurling-type theorem on $\mathcal{P}^t(\mu)$

The purpose of our work in [Paper V] was to carry out similar investigations for spaces of analytic functions which are within the realm of the classical Bergman spaces, but considerably larger than the Hardy spaces. More precisely, we considered the class of $\mathcal{P}^t(\mu)$-spaces for which the measures μ are supported in $\overline{\mathbb{D}}$, and of the form

$$d\mu = (1 - |z|^2)^\alpha dA(z) + \omega dm \tag{3}$$

where ω is a weight on $\partial\mathbb{D}$ and $\alpha > -1$. At this point, it may happen that $\mathcal{P}^t(\mu)$ is not a genuine space of analytic functions. In other word, the mapping of f in $\mathcal{P}^t(\mu)$ onto its analytic restriction $f|_{\mathbb{D}}$ on $\mathbb{D}$, may not be injective. Thus, there may exist a function $f \in \mathcal{P}^t(\mu)$ which entirely lives on some subset of $\partial\mathbb{D}$. Indeed, this happens if either the geometry of the carrier set E of ω is too "weird", or the size of ω is too small on E (see [Paper V] and [Paper VI] for indicatory examples). A complete description of this phenomenon is far from being understood and is related to notoriously difficult problems on irreducibility and splitting, see [27] and references therein. Nevertheless, we found some appropriate conditions on the geometry of the carrier set E of ω, and conditions on the size of ω on E, which guarantee that $\mathcal{P}^t(\mu)$ is a genuine space of analytic functions. In order to phrase these conditions, we shall first recall that a closed set $F \subset \partial\mathbb{D}$ of positive Lebesgue measure is said to have finite Beurling-Carleson entropy, if

$$\sum_n m(I_n) \log m(I_n) > -\infty,$$

where $\{I_n\}_n$ denotes the collection of connected components of $\partial\mathbb{D} \setminus F$. Now the conditions we impose are the following:

(**Geometry**) The carrier set E of ω can be expressed as a countable union of closed sets $\{E_N\}_N$, where each set E_N has finite Beurling-Carleson entropy.

(**Size**) The weight ω satisfies on each E_N:

$$\int_{E_N} \log \omega dm > -\infty.$$

Under these conditions on ω, $\mathcal{P}^t(\mu)$ is a genuine space of analytic functions (see Theorem 1.1 of [Paper V]). Moreover, we have the following complete description of M_z-invariant subspaces generated by inner functions.

Theorem 4.2 (Corollary 1.4 of [Paper V]. *] Let $t \in [1, \infty)$ and μ be defined by* (3), *and ω satisfy the above conditions on geometry and size. Let $\Theta = BS_\nu$ be an inner function and decompose ν according to*

$$\nu = \nu_{\mathcal{C}} + \nu_{\mathcal{K}|E} + \nu_{\mathcal{K}|E^c}$$

where $E^c := \partial\mathbb{D} \setminus E$, and $\nu_{\mathcal{C}}, \nu_{\mathcal{K}}$ are as in the statement of the Korenblum-Roberts Theorem. Then the following statements hold:

(i) $\Theta_0 := BS_{\nu_{\mathcal{C}}} S_{\nu_{\mathcal{K}|E}}$ *generates a proper* M_z*-invariant subspace of* $\mathcal{P}^t(\mu)$ *with the property:* $[\Theta_0]_{\mathcal{P}^t(\mu)} \cap H^2 \subseteq \Theta_0 H^2$.

(ii) $S_{\nu_{\mathcal{K}|E^c}}$ *is a cyclic vector for* M_z *on* $\mathcal{P}^t(\mu)$.

In light of the Korenblum-Roberts theorem, our result above has a flavor which demonstrates the following interplay between classical inner functions on the Hardy spaces and cyclic inner functions on the Bergman spaces. The part of $\nu_{\mathcal{K}}$ that lives inside the carrier set E of ω behaves like an inner function in the Hardy space, while the part of $\nu_{\mathcal{K}}$ outside E acquires the behavior of a cyclic singular inner function a Bergman space. The proof of part (i) of the theorem above essentially relies on the construction of smooth Cauchy transforms of functions supported on Beurling-Carleson sets, which closely relates to the work of S. Khrushchev in [25]. On the other hand, the proof of (ii) intrinsically relies on techniques developed by J. Roberts in [37], but requires noticeable adaptions to our setting, which is mainly attributed to the fact that the measure μ now also lives on $\partial\mathbb{D}$.

4.3 Smooth approximations in de Branges-Rovnyak spaces

The results of [Paper V] have some important applications to smooth approximations in de Branges-Rovnyak spaces on the unit disc. Given a bounded analytic function b in the unit-ball of H^∞, a de Branges-Rovnyak space $\mathcal{H}(b)$ can be realized as a reproducing kernel Hilbert space of analytic functions on the unit disc $\mathbb{D}$, determined by the reproducing kernels

$$\kappa_b(z,\lambda) = \frac{1 - \overline{b(\lambda)}b(z)}{1-\overline{\lambda}z} \qquad z,\lambda \in \mathbb{D}.$$

The classical functional model says that any completely non-isometric contraction T on a separable Hilbert space is unitarily equivalent to the backward shift operator $\mathcal{L}f(z) = (f(z) - f(0))/z$ on some de Branges-Rovnyak space [10] (possibly vector-valued). Roughly speaking, the theory of de Branges-Rovnyak spaces bifurcates into two parts, depending whether or not the symbol b is an extreme point in the unit ball of H^∞. A classical result by D. Sarason says that polynomials form a dense subset in $\mathcal{H}(b)$ if and only if $\log(1-|b|^2)$ is integrable on $\partial\mathbb{D}$, where the integrability condition is equivalent to b not being an extreme point in the unit ball of H^∞. In contrast to this work, A. Aleman and B. Malman proved that in any $\mathcal{H}(b)$-space, functions with continuous extensions to $\partial\mathbb{D}$ always form a dense subset (see [6]), thus extending Aleksandrov's density theorem to the setting of de Branges-Rovnyak spaces. In [Paper VI], we studied density results for extreme de

Branges-Rovnyak spaces with functions of higher order regularity. Again, similarly to [Paper III], the appropriate spaces in this context are the family of analytic Dirichlet-Sobolev spaces of $\mathcal{H}^\alpha$ (see previous section). Given a number $\alpha > 0$ and an extreme point b in the unit ball of H^∞, we introduce the associated measure $\mu = \mu(\alpha, b)$, given by $d\mu = (1-|z|)^{\alpha-1}dA + (1-|b|^2)dm$, where dA is the area-measure on $\mathbb{D}$. Our main result in [Paper VI] reads as follows.

Theorem 4.3 (Theorem 1.1 from [Paper VI]. *] Let $\mu = \mu(\alpha, b)$ denote the measure defined in the previous paragraph, and let $b = b_0\Theta$, where b_0 denotes the outer factor of b and Θ denotes the inner factor of b. Then the linear manifold $\mathcal{H}^\alpha \cap \mathcal{H}(b)$ forms a dense subset in $\mathcal{H}(b)$ if and only if the following two conditions hold:*

- *(i) The multiplication operator $M_z : \mathcal{P}^2(\mu) \to \mathcal{P}^2(\mu)$ is completely non-isometric, that is, $\mathcal{P}^2(\mu)$ is a genuine space of analytic functions (irreducible).*
- *(ii) Any H^2-function f contained in the invariant subspace $[\Theta]_{\mathcal{P}^2(\mu)} \subset \mathcal{P}^2(\mu)$ satisfies $f/\Theta \in H^2$.*

Note that condition (i) is equivalent to $\mathcal{P}^2(\mu)$ being a genuine space of analytic functions, that is, irreducible. Since $\mu(\alpha, b) = \mu(\alpha, b_0)$, condition (i) actually only depends on the outer factor b_0 of b. As previously mentioned in [Paper V], a function theoretical description of the outer factors b_0 that give rise to (i) is far from being completely understood, and seems to depend on an intricate interplay between the geometry of the carrier set $E := \{\zeta \in \partial\mathbb{D} : |b_0(\zeta)| < 1\}$ of the measure $(1-|b_0|^2)dm$, and the *size* of the weight $(1-|b_0|^2)dm$, determined by the integrability of $\log(1-|b_0|^2)$ on E. Meanwhile, condition (ii) is a version of the (P)-property in the setting of $\mathcal{P}^2(\mu)$ (compare with Theorem 1.3 in [Paper IV]), for which our Corollary 1.4 in [Paper V] provides a complete function theoretical description in terms of the inner factor Θ of b.

4.4 Direction for further work

4.4.1 Irreducibility

As we briefly mentioned in the previous paragraphs, a function theoretic description of the functions b for which condition (i) holds is still lacking. For instance, if b is outer and an extreme point in the unit ball H^∞, then condition (ii) is redundant and the density of $\mathcal{H}^\alpha$-functions in $\mathcal{H}(b)$ is equivalent to $\mathcal{P}^2(\mu(\alpha, b))$ being a genuine space of analytic functions. With this observation in mind, one can phrase the following general problem:

Problem 4.4. For $d\mu = dA_\alpha + \omega dm$, describe the non-negative weights $\omega \in L^1(\partial\mathbb{D}, dm)$ for which $\mathcal{P}^2(\mu)$ is irreducible, that is, a genuine space of analytic functions.

Recall that we proved in [Paper V] that if $E = \cup_n E_n$ is a countable union of (closed) sets E_n having finite Beurling-Carleson entropy and $\int_{E_n} \log \omega dm > -\infty$ for each n, then $\mathcal{P}^2(\mu)$ is a genuine space of analytic functions. Our impression is that these conditions are fairly close to being necessary as well, but we are far from being able to establish such results. At this stage, we have only found some strongly indicative examples, contained in the work on splitting by T. Kriete and B. MacCluer in [27]. An important example stems from the work of S. Khrushchev in [25], and asserts that any weight ω concentrated on a "bad" set F, in the sense that F contains no closed subset of positive Lebesgue measure with finite Beurling-Carleson entropy, induces a $\mathcal{P}^2(\mu)$-space which splits, that is

$$\mathcal{P}^2(\mu) = \mathcal{P}^2(dA_\alpha) \oplus L^2(\omega dm), \qquad \alpha > -1.$$

As a consequence, the Hilbert space $\mathcal{P}^2(\mu)$ is certainly not a space of analytic functions, see Theorem 1.2 in [27]. However, even if the carrier set of a weight ω is a very nice set, such as the entire unit circle $\partial\mathbb{D}$, splitting may still occur if ω is too small everywhere on the unit circle. An example of such a weight ω was provided by A. Volberg (see p 82 in [19]), satisfying $0 < \omega < 1$ on $\partial\mathbb{D}$, and $\int_I \log \omega dm = -\infty$ for any arc $I \subset \partial\mathbb{D}$. In fact, this example led to the phrasing of general conjectures on problems of splitting (see Conjecture 1 and 2 in [27]), which still remain open to this date.

4.4.2 Smooth Cauchy transforms

In [25], S. Khrushchev studied the following problem (among others):

For which Borel sets $E \subset \partial\mathbb{D}$ of positive Lebesgue measure does there exist a non-trivial integrable function ϕ supported on E, with the property that its Cauchy transform $\mathcal{K}(\phi)$ belongs to $\mathcal{H}^\alpha$, for some $\alpha > 0$?

Surprisingly, the answer is independent of $\alpha > 0$ and was completely solved by S. Khrushchev, where the necessary and sufficient condition is that E contains a closed set of positive Lebesgue measure with finite Beurling-Carleson entropy. An interesting observation is that the logical negation of the above problem has the following dual reformulation in terms of splitting (see Theorem 1.2 and Theorem 7.5 in [25]):

There exists no non-trivial function $f \in L^2(dm)$ for which $\mathcal{K}(1_E f)$ belongs to some $\mathcal{H}^\alpha$, if and only if the following splitting holds:

$$\mathcal{P}^2(dA_{\alpha-1} + 1_E dm) = \mathcal{P}^2(dA_\alpha) \oplus L^2(1_E dm).$$

As a consequence, the problem of splitting in $\mathcal{P}^2(\mu)$ for measures of the form $d\mu = dA_{\alpha-1} + 1_E dm$ has a complete solution, described in terms of the geometry of the set E. In a similar way, one can also rephrase the problem of irreducibility of the space $\mathcal{P}^2(dA_{\alpha-1} + 1_E dm)$, that is, $\mathcal{P}^2(dA_{\alpha-1} + 1_E dm)$ is a genuine space of analytic functions if and only if the set $\{f \in L^2(1_E dm) : \mathcal{K}(1_E f) \in \mathcal{H}^\alpha\}$ is a dense subset of $L^2(1_E dm)$. In fact, similar to S. Khrushchev's work, one can carry out such reformulations beyond the setting of measures μ of the specific form above (see Theorem 1.2 in [25]). Now an interesting problem would be to modify Khrushchev's problem, but consider a general weight ω on $\partial\mathbb{D}$ instead of 1_E.

For which weights ω on $\partial\mathbb{D}$ can we find a non-trivial function $f \in L^1(\omega dm)$ with the property that $\mathcal{K}(fw) \in \mathcal{H}^\alpha$, for some $\alpha > 0$?

By Theorem 1, the carrier set of ω must necessarily contain a closed set E of finite Beurling-Carleson entropy, but the size of the weight ω on E may still interfere with the "good" set E. However, if we additionally assume that $\log\omega \in L^1(1_E dm)$, then there is a short and beautiful argument by B. Malman, which allows one to construct for any $\alpha > 0$, a function $f \in L^\infty(\partial\mathbb{D})$ with $\mathcal{K}(f\omega) \in \mathcal{H}^\alpha$. To this end, let W denote the outer function whose modulus is equal to ω on E and equal to 1 on $\partial\mathbb{D} \setminus E$. According to Lemma 4.2 in [Paper V], there exists for any integer $N > 0$, an outer function s with the property that the function $t \mapsto e^{it}s(e^{it})W(e^{it})1_{\partial\mathbb{D}\setminus E}(e^{it})$ is N-times continuously differentiable on $\partial\mathbb{D}$. Since Cauchy transforms of co-analytic functions in $L^2(dm)$ vanish, we have $\mathcal{K}(\overline{\zeta s W}) = 0$, and thus

$$\int_E \frac{\overline{\zeta s(\zeta)W(\zeta)}}{1-\bar{\zeta}z} dm(\zeta) = -\int_{\partial\mathbb{D}\setminus E} \frac{\overline{\zeta s(\zeta)W(\zeta)}}{1-\bar{\zeta}z} dm(\zeta) \qquad z \in \mathbb{D}.$$

Now since $e^{it}s(e^{it})W(e^{it})1_{\partial\mathbb{D}\setminus E}(e^{it})$ is sufficiently regular on $\partial\mathbb{D}$, its Cauchy transform belongs to $\mathcal{H}^\alpha$, for some $\alpha > 0$ depending on N, which can be chosen as large as we wish. But this precisely means that the expression on the right hand side of the equation above belongs to $\mathcal{H}^\alpha$, thus so does the left hand side, which is the Cauchy transform of an $L^\infty(\partial\mathbb{D})$-function supported on E.

One is tempted to believe that the sufficient condition that the carrier of ω contains a Beurling-Carleson set E with $\log\omega \in L^1(1_E dm)$ is also necessary. In fact, that particular claim is equivalent to a specific version of the notorious problem on splitting phrased as Conjecture 2 in [27].

5 Sparse operators and dyadic models for the Bergman projection

5.1 Background

Let $\mathcal{D}$ denote the standard grid of dyadic cubes in $\mathbb{R}^d$ and recall that a subcollection $\mathcal{S} \subset \mathcal{D}$ is said to be *sparse*, if for any dyadic cube $Q \in \mathcal{D}$, we have

$$\sum_{Q' \in Ch_{\mathcal{S}}(Q)} |Q'| \leq \frac{1}{2}|Q|,$$

where $Ch_{\mathcal{S}}(Q)$ denotes the collection of maximal cubes in $\mathcal{S}$ that are strictly contained in Q. For any sparse collection $\mathcal{S}$ of dyadic cubes in $\mathbb{R}^d$, we consider the associated family of sparse Lerner operators $L^{\mathcal{S}}_{\varphi,\psi}$, parametrized by

$$L^{\mathcal{S}}_{\varphi,\psi} f(x) = \sum_{Q \in \mathcal{S}} \varphi_Q(x) \fint_Q \psi_Q(y) f(y) dy$$

where each pair of φ_Q, ψ_Q are complex-valued functions supported on Q and belong to the unit-ball of L^{∞}. Versions of these operators have notably appeared in the pioneering works on sparse domination models for Calderón-Zygmund operators by A. Lerner [28], and more relevant to our purposes, in the vectorial setting on Convex body domination of Calderón-Zygmund operators and alike in [31].

5.2 Weighted bounds for sparse Lerner operators

Together with my supervisor Sandra Pott, we found a new and simple proof that sparse Lerner operators are bounded on matrix-weighted L^2-spaces. In fact, given a separable Hilbert space $\mathcal{H}$ and an operator-valued weight $W : \mathbb{R}^d \to \mathcal{B}(\mathcal{H})$, then the family of canonical extensions $\{L^{\mathcal{S}}_{\varphi,\psi} \otimes \mathbb{1}_{\mathcal{H}}\}_{\varphi,\psi}$ are uniformly bounded on $L^2_W(\mathcal{H})$ if and only if W satisfies the Muckenhoupt A_2-condition, restricted to the sparse collection $\mathcal{S}$, given by

$$[W]_{\mathcal{A}_2^{\mathcal{S}}} := \sup_{Q \in \mathcal{S}} \left\| \langle W \rangle_Q^{1/2} \langle W^{-1} \rangle_Q^{1/2} \right\|^2_{\mathcal{B}(\mathcal{H})}.$$

Moreover, there exists a constant $C = C_{\mathcal{S}} > 0$ only depending on $\mathcal{S}$, such that

$$\frac{1}{C}[W]_{\mathcal{A}_2^{\mathcal{S}}}^{1/2} \leq \sup_{\varphi,\psi} \left\| T_{\varphi,\psi}^{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{H}} \right\|_{L_W^2 \to L_W^2} \leq C[W]_{\mathcal{A}_2^{\mathcal{S}}}^{3/2}.$$

In our proof, we do not rely on any self-improving Gehring-type properties of the weight W, such as the reverse Hölder inequality, thus as a consequence the theorem also holds in the context of Bekollé-Bonami weights. Moreover, our techniques extend beyond the setting of finite dimensions, which in light of the standard identification $L^2(\mathbb{R}^d \times \mathbb{R}^s) \cong L^2(\mathbb{R}^d; L^2(\mathbb{R}^s))$, it also has applications to the multiparameter setting. We remark that the upper bound in terms of the $[W]_{\mathcal{A}_2^{\mathcal{S}}}$–constant, which we prove agrees with the sharpest one to this date, even in the finite-dimensional setting. In contrast to weighted inequalities for other classical operators, it is well-known that the matrix-weighted L^2-norm of the Hilbert transform grows at least logarithmically with the dimension [17], while the Bergman projection was proven to have dimensionally independent bounds [5], thus it behaves similarly to our family of sparse Lerner operators. Roughly speaking, our approach relies on decomposing any sparse collection into a union of disjoint family of decaying stopping times, which means that any sparse operator can be expressed as a sum of sparse operators of simpler forms. The decaying stopping times allow us to prove that these simpler operators form an almost orthogonal set, hence making the principle of almost orthogonality by M. Cotlar and E. Stein applicable.

5.3 Sparse domination for the Bergman projection

Our initial motivation for studying these families of sparse Lerner operators was to find an appropriate sparse domination model for the Bergman projection, with the intent to improve the bound obtained in [5]. Using a simple dyadic model developed by S. Pott and M. C. Reguera in [36], in conjunction with our result on sparse Lerner operators, we obtained a convex body domination for the Bergman projection on the upper-half plane $\mathbb{C}_+$:

$$Pf(z) = \int_{\mathrm{Im}(\xi)>0} \frac{f(\xi)}{(\bar{\xi} - z)^2} dA(\xi).$$

As a consequence, an upper bound for the operator norm of Pf on $L_W^2(\mathcal{H})$ was provided in terms of $[W]_{\mathcal{A}_2}^{3/2}$, when $\mathbf{dim}(\mathcal{H}) < \infty$. In fact, this improves the recent $[W]_{\mathcal{A}_2}^2$-bound for matrix-weights, obtained by Z. Huo and B. Wick in [20]. The improvement is attributed to our abstract approach using Cotlar's lemma, which also does not require any self-improving assumptions on our Bekollé-Bonami weights. However, the convex body domination still heavily relies on the dimensionally dependent John-Ellipsoid theorem, via Lemma 2.8 in [31]. Since our upper bound completely relies on the bound obtained for sparse Lerner operators, an improvements of the current $[W]_{\mathcal{A}_2}^{3/2}$-bound for the Bergman project would most likely lead to in substantial progress on the matrix-weighted $\mathcal{A}_2$-conjecture for general Calderón-Zygmund operators.

5.4 Direction for further work

5.4.1 The Bergman projection as an average of simpler operators

An astonishing result by S. Petermichl in [33], later refined and simplified by T. Hytönen in [21], says that the classical Hilbert transform H on $\mathbb{R}$ can be approximated in the strong operator topology by random averages of dyadic shift operators:

$$T^{\omega,\rho} f := \sum_{k \in \mathbb{Z}} \sum_{J \in \rho \mathcal{D}_k^{\omega}} H_J \langle f, h_J \rangle_{L^2}$$

where $\rho\mathcal{D}^{\omega} := \{\rho J = [\rho(a+\omega), \rho(b+\omega)) : [a,b) \in \mathcal{D}\}$ for the parameters $1 \leq \rho < 2$, $\omega = \{\omega_j\}_{j \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ and $\mathcal{D} = \cup_{k \in \mathbb{Z}} \mathcal{D}_k$ denotes the standard dyadic grid on $\mathbb{R}$, while $\{h_J\}$ are the associated Haar functions and $H_J = (h_{J_-} - h_{J_+})$. More precisely, it was shown that

$$H(f)(x) = -\frac{8}{\pi} \int_1^2 \int_{\{0,1\}^{\mathbb{Z}}} T^{\omega,\rho} f(x) d\mathbb{P}(\omega) \frac{d\rho}{\rho}$$

where $\mathbb{P}$ denotes the canonical probability Bernoulli measure on $\{0,1\}^{\mathbb{Z}}$. A surprising result which extends these ideas to general classes of Calderón-Zygmund operators was announced by T. Hytönen in [22], which ultimately resolved the famous A_2-conjecture. In a similar fashion, it would be interesting to find an appropriate family of dyadic models, so that when taking averages over certain scales, converges in the strong operator topology to the Bergman projection on $\mathbb{C}_+$. The main difficulty in this task seems to emerge from the fact that the Bergman projection is conveniently viewed through the lens of hyperbolic geometry and enjoys certain intricate symmetries wrt the group of automorphisms on $\mathbb{C}_+$, which likely needs to be accounted for in a random averaging procedure. A result of this kind is would not only be very interesting in its own right, but in conjunction with our operator-valued weighted bound of sparse Lerner operators, it would provide an essentially sharp bound for the Bergman projection in the setting of operator-valued weights.

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