Critical recurrence in real quadratic and rational dynamics

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Mats Bylund

LUND UNIVERSITY

Doctoral thesis
Thesis advisor: Docent Magnus Aspenberg
Faculty opponent: Professor Jacek Graczyk

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Abstract
In this thesis we study the dynamics of real quadratic functions on the interval, and rational functions on the Riemann sphere. The problems we are considering are concerned with the recurrent nature of the critical orbit(s). In Paper I we investigate the real quadratic family and prove a theorem regarding the rate of recurrence of the critical point to itself, extending a previous result by Avila and Moreira. In Paper II and Paper III we consider rational functions. Here we do not study the rate of recurrence, rather we assume that the critical points approach each other at a slow rate, and investigate some of the consequences. Assuming this slow recurrence condition, we prove in Paper II that certain Collet–Eckmann rational functions can in a strong sense be approximated by hyperbolic ones. In Paper III we observe that within the family of slowly recurrent rational maps, the well-known Collet–Eckmann conditions are all equivalent.

Key words
Quadratic family, rational dynamics, critical points, recurrence, Collet–Eckmann conditions

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Populärvetenskaplig sammanfattning

Studiet av dynamiska system grundar sig i att förstå det långsiktiga beteendet hos ett system som fortskrider i tiden, enligt vissa för systemet specifika regler. Dynamiska system uppkommer naturligt inom olika vetenskapliga discipliner, exempelvis då man vill studera planeternas rörelse, ta fram väderprognoser, eller förstå hur ett virus sprider sig i samhället.

För att studera dessa naturliga system behöver man matematiska modeller. Dessa modeller är naturligt parametriserade och det är därför av intresse att inte enbart studera ett specifikt dynamiskt system, utan en parametriserad familj av dynamiska system. En viktig fråga man kan ställa är hur robusta dessa system är, eller med andra ord, hur dynamiken förändras vid små störningar av parametrarna. Fastän modellerna man tar fram ofta är förenklade, och parameterberoendet väldigt explicit, uppkommer teoretiskt intressanta och mycket icke-triviala problem. Av betydande intresse är interaktionen mellan tamt beteende och kaotiskt beteende. I parameterrummet är dessa två skilda företeelser ofta komplext sammanvävda.

I denna avhandling studeras små störningar av kaotiska system. Dessa system kommer att beskrivas av funktioner på intervallet och på Riemannsfären. Systemen vi studerar har kritiska punkter, det vill säga punkter där funktionens derivata är lika med noll. Hur dynamiken för dessa specifika punkter ter sig visar sig ha stor betydelse för den globala dynamiken. En viktig aspekt är rekurrent beteende: med vilken hastighet återkommer de kritiska punkterna till varandra under iteration? Avhandlingen bygger vidare på tidigare väl etablerade resultat, och det centrala temat är just dessa frågeställningar angående rekurrens och dess konsekvenser.

List of papers

This thesis is based on the following three papers.

Paper I **Critical recurrence in the real quadratic family**

M. Bylund
To appear in Ergodic Theory and Dynamical Systems
Preprint: arXiv:2103.17200

Paper II **Slowly recurrent Collet–Eckmann maps with non-empty Fatou set**

M. Aspenberg, M. Bylund, W. Cui
Submitted
Preprint: arXiv:2207.14046

Paper III **Equivalence of Collet–Eckmann conditions for slowly recurrent rational maps**

M. Bylund
Submitted
Preprint: arXiv:2209.05237

Paper II is the result of an equal contribution from the authors regarding all aspects of the work.

The papers as they appear in this thesis might differ slightly from the corresponding preprint versions.

Contents

Part I

Introduction and summary

Chapter 1

Introduction

This introductory chapter gives a brief overview of the theory and results on which the scientific papers of this thesis are based upon. It is divided into five sections as follows. We begin by introducing what a dynamical system is, and some of the most fundamental notions. The second section is devoted to the real quadratic family, which is the system studied in Paper I, and which is one of the most well studied families of dynamical systems. In Paper II and Paper III we study the dynamics of rational functions on the Riemann sphere, and this topic is briefly introduced in the third section. In the fourth section we discuss the Collet–Eckmann condition and some of its variants. These are conditions of non-hyperbolicity and they play a central role in the thesis. In the final section we give a schematic outline of the Benedicks–Carleson techniques, which are the foundational tools used in Paper I and Paper II.

These sections below are by no means complete in terms of their scope, and many important results and notions are left out. Rather, the goal is to give the minimal information needed to motivate the problems studied in Paper I–III. Relevant references will be given throughout the text, but for the more general theory of interval dynamics and complex (rational) dynamics, we refer to [dMvS93, Dev92] and [CG93, Mil06, Bea91], respectively.

1 Some notions in dynamical systems

In this thesis we are concerned with the study of *discrete dynamical systems*. At its core this constitutes a set X of points and a mapping $f : X \to X$. The set X is usually referred to as the *state space* (or *phase space*), with each $x \in X$ representing a specific state of the system.

The mapping f is the *evolution mapping* which determines the future of the system, taking state x to its future state $f(x)$. One of the main objectives when studying a dynamical system is to understand its long term behaviour: given a state $x \in X$, how does its *orbit*

$$x, f(x), f(f(x)), \dots, f^n(x), \dots$$

distribute in state space? Here and elsewhere, f^n always denotes the nth iterate of f. That is, $f^0 = \mathrm{id}$ and $f^n = f \circ f^{n-1}$, with $n \geq 1$ an integer.

More generally, given one or more parameters λ belonging to some *parameter space*, one can consider a *family* of dynamical systems $f_\lambda : X \to X$. In this setting it is of interest to understand how certain behaviours of the system are affected by small changes of the parameter value.

The above questions are of course too general to answer if no structure on X nor regularity on f are imposed. In this thesis we study the so-called real quadratic family

$$x \mapsto x^2 + a = Q_a(x),$$

acting on the real line, and more general rational functions

$$z \mapsto \frac{a_d z^d + a_{d-1} z^{d-1} + \dots + a_0}{b_d z^d + b_{d-1} z^{d-1} + \dots + b_0} = R(z),$$

acting on the Riemann sphere. The quadratic family can be seen as a 'toy model' for the more general study of rational maps, but has also been used in, for instance, biological modelling [May76]. Nevertheless, already in this analytically simple family of dynamical systems one finds very rich dynamics.

To understand the dynamics of a function f such as above, acting on some appropriate space, it is important to look for points which are left invariant under the action of f, and to study the local behaviour of f near these points. Such points are called *fixed points*, and per definition they solve the equation $f(x) = x$. More generally, one can look for so-called *periodic points*. A point x is a periodic point of f if there exists an integer $k > 0$ such that

$$x \mapsto f(x) \mapsto f^2(x) \mapsto \dots \mapsto f^k(x) = x.$$

Such an above orbit is usually referred to as a *cycle*, and if $k > 0$ is the least integer such that the above holds, then k is called the length of the cycle. A cycle of length k is classified as

- *attracting* if $|(f^k)'(x)| < 1$,
- *repelling* if $|(f^k)'(x)| > 1$,
- *neutral* if $|(f^k)'(x)| = 1$.

These names are very suggestive: nearby points get closer to the cycle under iteration if the cycle is attracting, get further away if the cycle is repelling, and in the neutral case both instances may occur.

Another important notion is that of *critical points*. A point x is a critical point of f if the derivative of f at x vanishes, i.e. critical points are the solutions to the equation $f'(x) = 0$. From now on we denote the set of critical points of f by $\text{Crit}(f)$. It turns out that the behaviour of the critical orbit(s) is of great importance to the global dynamics, and we give some motivation to this claim in the following sections.

The results of this thesis are in one way or another concerned with the notion of *critical recurrence*. In Paper I we investigate the real quadratic family and prove a theorem regarding the rate of recurrence of the critical point to itself. This extends a previous result, and completes the picture of so-called *polynomial recurrence*. In Paper II and Paper III we consider rational functions. Here we do not prove any results regarding the rate of recurrence, rather we investigate some of the consequences when the critical points are allowed to approach each other only at a slow rate.

2 The real quadratic family

A quadratic polynomial acting on the real line is from an analytic point of view the simplest non-linear dynamical system one can study. Let $x \mapsto Ax^2 + Bx + C = p(x)$ be a quadratic polynomial with real coefficients $A \neq 0$, B, and C. Conjugating this polynomial with $x \mapsto Ax$ we get the monic quadratic polynomial $x \mapsto x^2 + Bx + AC$, and further conjugating with $x \mapsto x + B/2$, i.e. translating the critical point to the origin, we end up with the so-called *real quadratic family*

$$x \mapsto x^2 + a = Q_a(x),$$

with $a = B/2 - B^2/4 + AC$ being the *parameter*. Given a real parameter a, going the other way around does not determine a unique quadratic polynomial. Rather, each a corresponds to a *conjugacy class*. In this thesis we are concerned with the recurrent behaviour of the critical orbit. To motivate this study, and also settle some notation, let us first briefly mention some of the major results regarding this family of dynamical systems.

To understand the dynamics of Q_a for different values of a, understanding the behaviour of the critical orbit is of interest, as can be understood from the following result.

Proposition 2.1. *For each parameter a there can exist at most one (finite) attracting cycle for the corresponding quadratic map Q_a. Moreover, if an attracting cycle exists, the orbit of the critical point $x = 0$ will accumulate along this cycle.*

As a first step towards understanding the behaviour of the iterations of the critical point, we allow ourselves to restrict the parameter interval.

Proposition 2.2. *If a does not belong to the interval $[-2, 1/4]$, then $Q_a^n(0)$ tends to infinity as n tends to infinity. On the other hand, if a belongs to $[-2, 1/4]$ then there exists an interval $I_a \subset [-2, 2]$, containing the critical point, such that $Q_a(I_a) \subset I_a$.*

To begin the study of the qualitative behaviour of the real quadratic family, the following proposition can be checked by hand.

Proposition 2.3. *For the quadratic family Q_a:*

(1) For $a = 1/4$, there is a single fixed point that is neutral.

(2) For $-3/4 < a < 1/4$, there is an attracting fixed point.

(3) For $a = -3/4$, the attracting fixed point given in (2) becomes neutral.

(4) For $-5/4 < a < -3/4$, there is an attracting cycle of length two.

Hence, for parameter values in the interval $(-5/4, 1/4]$, the dynamics is rather trivial. In fact, for such a parameter, almost every point of I_a (with respect to Lebesgue measure) will tend to the attracting fixed point, or 2-cycle, under iteration. To calculate attracting cycles by hand soon becomes impractical, and one must rely on more qualitative and sophisticated techniques. The transition from an attracting fixed point to an attracting 2-cycle is an example of a so-called *period-doubling bifurcation*. By plotting the iterations of the critical point for different values of a, this period-doubling bifurcation can be illustrated as in Figure 1.1. Here one sees, going from right to left, the transition from an attracting fixed point to an attracting 2-cycle, from an attracting 2-cycle to an attracting 4-cycle, and so on. At the parameter value $a = -1.401 \ldots$ (the so-called Feigenbaum point), we see a sudden change in the behaviour of the orbit of the critical point. Namely, the orbit does not seem to be attracted to any cycle. This motivates the following definition.

Definition 2.4. A parameter $c \in [-2, 1/4]$ is called a *regular parameter* if $x \mapsto x^2 + c$ has an attracting cycle, and otherwise it is called a *nonregular parameter*. The set of regular parameters is denoted $\mathcal{R}$, and the set of nonregular parameters is denoted $\mathcal{NR}$.

It is customary to call the corresponding function Q_a regular (or nonregular) if the parameter a is regular (or nonregular). Looking at the bifurcation diagram of Figure 1.1, the 'white windows' correspond to regular parameters, while the 'black lines' correspond to nonregular parameters. To understand these two sets of parameters, and how they are intertwined, has been a central topic of study during the last couple of decades.

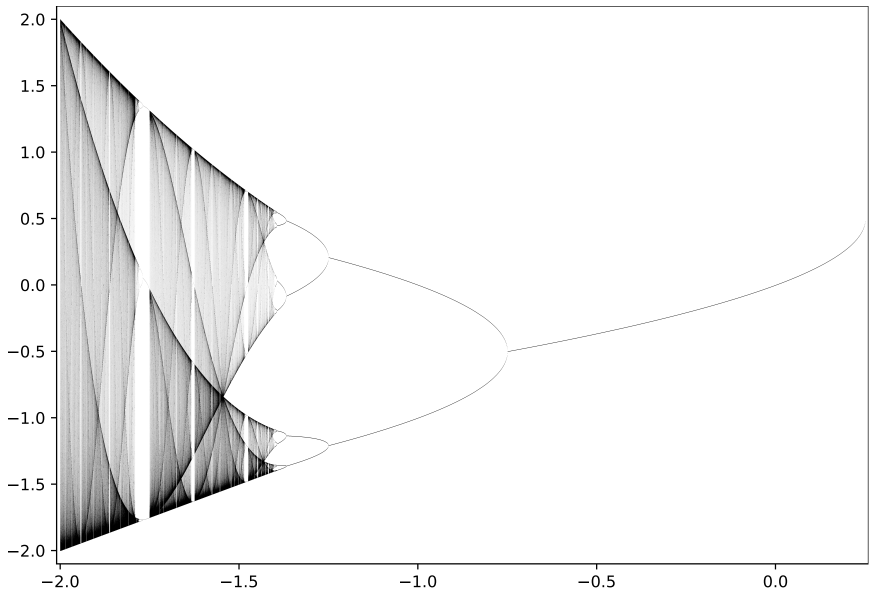

Figure 1.1: Bifurcation diagram for $x \mapsto x^2 + a$, $a \in [-2, 1/4]$.

When studying a parameterised family of dynamical systems, one is often interested in whether some specific property holds on a positive measure set of parameters. In the case of the quadratic family, the natural measure on the parameter interval is the Lebesgue measure (which we from now on denote by Leb). For instance, it is obvious that the set of regular parameters has positive measure since the interval $(-3/4, 1/4)$ is contained in $\mathcal{R}$. Moreover, it is not difficult to show that the set of parameters having neutral cycles constitute only a set of measure zero. More difficult is the question about the measure of the set of nonregular parameters. In 1981, M. Jakobson [Jak81] initiated the study of nonregular parameters by proving that there exists a set Δ_J of positive measure such that for each $a \in \Delta_J$ there exists an absolutely continuous (with respect to Lebesgue) invariant probability measure (acip) for the corresponding quadratic function Q_a. This in turn implies that the Lebesgue measure of $\mathcal{NR}$ is positive, since for a regular parameter any finite invariant measure is necessarily singular with respect to Lebesgue measure, being the sum of point measures along the attracting cycle. We make the following definition for this subset of the nonregular parameters.

Definition 2.5. A parameter $a \in [-2, 1/4]$ is called a *stochastic parameter* if $x \mapsto x^2 + a$ has an absolutely continuous (with respect to Lebesgue) invariant probability measure. The set of stochastic parameters is denoted $\mathcal{S}$.

We recall that the measure μ is acip with respect to the function f if it is a probability measure and if, for every measurable set A, $\mu(f^{-1}(A)) = \mu(A)$ and

$$\mu(A) = \int_A \frac{d\mu}{d\,\mathrm{Leb}} d\,\mathrm{Leb},$$

with $d\mu/d\,\mathrm{Leb}$ denoting the so-called Radon-Nikodým derivative.

Having an acip is one characterisation of being nonregular. Other characterisations can be formulated in terms of the derivative along the critical orbit. Indeed, since for a regular map the critical orbit accumulates on the attracting cycle, the condition

$$\liminf_{n\to\infty} |(Q_a^n)'(a)| > 0$$

clearly implies a being nonregular. However this condition is not necessary: in [Bru94] examples of parameters a are provided such that $x \mapsto x^2 + a$ has no attracting or neutral cycles, but $\liminf_{n\to\infty} |(Q_a^n)'(a)| = 0$. Instead, let us denote by $\chi_-(a)$ the so-called *lower Lyapunov exponent*

$$\chi_-(a) = \liminf_{n\to\infty} \frac{\log |(Q_a^n)'(a)|}{n}.$$

It turns out that the condition $\chi_-(a) \geq 0$ is the correct one to consider, since it is not only sufficient for a to be nonregular, but also necessary [NS98, LPS16].

Focusing on a similar condition as the above, M. Benedicks and L. Carleson [BC85] proved in the early 1980s that there exists a set Δ_{BC} of positive measure such that, for each $a \in \Delta_{BC}$, the derivative along the critical orbit grows at least subexponentially:

$$\liminf_{n\to\infty} \frac{\log |(Q_a^n)'(a)|}{\sqrt{n}} > 0.$$

Moreover, for each $a \in \Delta_{BC}$, the corresponding quadratic map has an acip. In the subsequent paper [BC91], working with the so-called Hénon family, Benedicks and Carleson improved this growth condition and showed that it is in fact exponential. This condition of having exponential growth of the derivative along the critical orbit is called the *Collet–Eckmann condition*, and it was first introduced by P. Collet and J. P. Eckmann [CE83,CE80] where they used this condition to prove the abundance of functions with chaotic dynamics within certain families of dynamical systems. The Collet–Eckmann condition, and some of its variants, are further discussed in Section 4 below. For the quadratic family, we make the following definition.

Definition 2.6. A parameter $a \in [-2, 1/4]$ is called a *Collet–Eckmann parameter* if the corresponding quadratic map satisfies the *Collet–Eckmann condition*

$$\liminf_{n\to\infty} \frac{\log |(Q_a^n)'(a)|}{n} > 0.$$

The set of Collet–Eckmann parameters is denoted $\mathcal{CE}$.

The techniques developed in [BC85,BC91] are of great importance in the field of dynamical systems, and are also central to this thesis. We come back to these in Section 5.

It turns out that both the property of being stochastic, and that of being Collet–Eckmann, are typical within the real quadratic family, namely

$$\operatorname{Leb} \mathcal{NR} = \operatorname{Leb} \mathcal{S} = \operatorname{Leb} \mathcal{CE}.$$

That the stochastic parameters are typical within nonregular parameters was proved by M. Lyubich [Lyu02], following the work in [Lyu00, MN00]. That the Collet–Eckmann parameters are typical within nonregular parameters was proved by A. Avila and C. G. Moreira [AM05]. For this reason, one can consider both of these conditions as good characterisations of being nonregular.

Considering the set of regular parameters, one can with an application of the inverse function theorem show that this set is open, i.e. small changes in the parameter value of a regular map do not alter the existence of an attracting cycle. A much deeper result is that these parameters form a dense set in $[-2, 1/4]$. This result, known as the *real Fatou conjecture*, was proved by J. Graczyk and G. Świątek [GS97, GS98b], and independently by Lyubich [Lyu97]. This genericity result was later extended to the class of real polynomials of arbitrary fixed degree, by O. Kozlovski, W. Shen, and S. van Strien [KSvS07]. With the characterisation of nonregular maps, and the density of regular maps, one can say that from a qualitative point of view, the real quadratic family is well-understood.

Considering the orbit of the critical point, we know from Proposition 2.1 that if a is a regular parameter, then its orbit accumulates on the attracting cycle. If a on the other hand is a nonregular parameter then, by definition, there can be no accumulation on an attracting cycle, and we are left with two possible cases:

$$\text{either} \quad \liminf_{n\to\infty} |Q_a^n(0)| > 0 \quad \text{or} \quad \liminf_{n\to\infty} |Q_a^n(0)| = 0.$$

The first case is known as the *Misiurewicz case*, and it implies that there exists $\delta = \delta(a) > 0$ such that $|Q_a^n(0)| > \delta$ for all $n \geq 1$. It was conjectured by M. Misiurewicz in the early 1980s that these parameters constitute only a set of measure zero, and this conjecture was proved to be true by D. Sands [San98]. Hence for a typical nonregular parameter the second case holds, and we simply call this the *recurrent case*. In this recurrent case, it is natural to ask at what rate the critical point returns to itself or, more precisely, what are the correct conditions on δ_n that guarantee

$$|Q_a^n(0)| < \delta_n \quad \text{for infinitely many } n. \tag{1.1}$$

It was conjectured by Y. Sinai that in the recurrent case, the critical point typically returns with exponent 1. This can be formulated as, for almost every nonregular parameter a,

$$\limsup_{n\to\infty} \frac{-\log|Q_a^n(0)|}{\log n} = 1.$$

This conjecture was indeed proved to be true by Avila and Moreira [AM05]. Another way to phrase this result is as follows: for almost every nonregular parameter a, the set of n such that $|Q_a^n(0)| < 1/n^\theta$ is finite if $\theta > 1$, and infinite if $\theta < 1$. This result motivated Paper I, namely to study the case of the critical exponent $\theta = 1$.

3 Rational dynamics

The study of iterations of rational maps on the Riemann sphere $\widehat{\mathbb{C}}$ was first initiated by P. Fatou [Fat19, Fat20a, Fat20b] and G. Julia [Jul18] around the 1920s. With the emergence of computers with better power of computation, this theory got more popular in the 1980s, much due to the many beautiful pictures. Let us briefly introduce the fundamental notions of rational dynamics.

We consider rational functions of one complex variable z belonging to the *Riemann sphere* $\widehat{\mathbb{C}}$. The Riemann sphere is the complex plane together with the abstract 'point at infinity'. Through stereographic projection, $\widehat{\mathbb{C}}$ is identified with the usual euclidean sphere in $\mathbb{R}^3$, and by pulling back the euclidean metric $|\cdot|$ this provides us with the so-called *chordal metric* σ. For points z and w in the plane the distance between them with respect to the chordal metric is

$$\sigma(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}},$$

and if $w = \infty$ then

$$\sigma(z,\infty) = \lim_{w\to\infty} \sigma(z,w) = \frac{2}{\sqrt{1+|z|^2}}.$$

Instead of the chordal metric one can also consider the equivalent so-called *spherical metric* σ_0, which is defined as

$$\sigma_0(z,w) = \inf_\gamma \int_\gamma \frac{|dt|}{1+|t|^2},$$

where the infimum is taken over all continuous curves γ joining z and w.

Each rational function can be represented as the quotient of two polynomials

$$z \mapsto \frac{a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0}{b_d z^d + b_{d-1} z^{d-1} + \cdots + b_0} = \frac{P(z)}{Q(z)} = R(z),$$

with a_i and b_i belonging to $\mathbb{C}$. We always assume that the P and Q do not share any common factors, and if not both a_d and b_d are equal to 0, we say that the *degree of* R $\deg(R)$ is equal to d. Thus, a rational map of degree d is a d-to-1 covering of the Riemann sphere onto itself. The *spherical derivative* of $z \mapsto R(z)$ is defined as

$$DR(z) = R'(z) \frac{1 + |z|^2}{1 + |R(z)|^2},$$

and we notice that it satisfies the chain rule.

The parameter space of rational maps (of a fixed degree d) is more complicated than that of the interval. We can assume that either $a_d = 1$ or $b_d = 1$, thus the parameter space of rational maps of degree d is a $(2d + 1)$-dimensional complex manifold, and also a subspace of the projective space $\mathbb{CP}^{2d+1}$. On each of the two charts corresponding to $a_d = 1$ and $b_d = 1$, respectively, the Lebesgue measures are mutually absolutely continuous. The Lebesgue measures on each chart are also mutually absolutely continuous to the induced Fubini-Study measure on $\mathbb{CP}^{2d+1}$. In Paper II we use a special normalisation of rational functions of degree d, due to G. Levin [Lev14]. We identify two rational functions of degree d as being equal if they are conjugated by a Möbius transformation. Up to equivalence, we then consider the space of rational functions (of degree d) with exactly p' different critical points $c_1, c_2, \dots, c_{p'}$, with corresponding multiplicities $\overline{p'} = (m_1, m_2, \dots, m_{p'})$. Within this space, which we denote by $\Lambda_{d,\overline{p'}}$, critical points move analytically with respect to the parameter. In particular, if all critical points are simple, i.e. $\overline{p'} = (1, 1, \dots, 1)$, then $\Lambda_{d,\overline{p'}}$ is locally equal to the entire parameter space.

An early and important step in the theory of rational dynamics was made by Fatou and Julia when they described a decomposition of the Riemann sphere into two invariant sets with respect to a rational function, namely the *Fatou set* and its complement, the *Julia set*. The Fatou set of a rational map R is denoted $\mathcal{F}(R)$ and is by definition the *domain of normality*: for each $z \in \mathcal{F}(R)$ there exists a neighbourhood U, containing z, such that the set of consecutive iterates of R restricted to U forms a normal family. That is to say, there exists an increasing sequence n_k such that $f^{n_k}|_U$ converges locally uniformly on compact subsets of U, with respect to the spherical metric. Intuitively, nearby points belonging to the Fatou set share similar limiting behaviour, and for this reason the dynamics on the

Fatou set is considered stable. From the definition, it follows that the Fatou set is open, hence the Julia set is compact. We denote the Julia set by $\mathcal{J}(R)$. Using Montel's theorem, one can prove that the Julia set is equal to the closure of the repelling cycles. Hence, nearby points belonging to the Julia set will repel each other, and one speaks of chaotic dynamics. On the Julia set, it therefore makes sense to talk about Lyapunov exponents, invariant measures, and so on.

The dynamics in the Fatou set for a rational function is well understood, and for completeness we state the following classification result. A component U of the Fatou set $\mathcal{F}(R)$ is called *fixed* if $R(U) = U$, *periodic* if $R^k(U) = U$ for some $k > 0$, and *pre-periodic* if $R^l(U)$ is periodic for some $l > 0$. That these are the only possibilities was proved by D. Sullivan [Sul85]: a component U of the Fatou set of a rational map is either fixed, periodic, or pre-periodic. This result by Sullivan, which is often called *Sullivan's no-wandering-domain theorem*, is a milestone in rational dynamics, and introduced the new idea of using quasiconformal mappings in dynamics.

Assuming U to be a fixed component, the dynamics can be classified as follows.

Proposition 3.1. *Let U be a fixed component of the Fatou set of a rational function. Then one of the following alternatives is true.*

(1) U contains an attracting fixed point for which all points in U converge to under iteration,

(2) ∂U contains a neutral fixed point for which all point in U converge to under iteration,

(3) U is either conformally equivalent to the disk or an annulus, and the dynamics is conjugated to a euclidean rotation.

In case (2) above, the neutral fixed point, say $z = R(z)$, is in fact a so-called *parabolic fixed point*. By definition this means that $DR(z) = e^{ip/q}$, with p and q being integers. If U is of type (3), it is called a *Siegel disk* if it is conformally equivalent to the disk, and a *Herman ring* if it is conformally equivalent to an annulus. The rotation angle is, in either case, irrational. Proposition 3.1 can be naturally generalised to periodic components by considering a suitable iterate of the rational map.

Let us now begin to consider the dynamics on the Julia set. It is illustrative to consider the most simple function, namely a complex quadratic one

$$z \mapsto z^2 + a = P_a(z),$$

with $a \in \mathbb{C}$. The following result tells us that the behaviour of the critical point has direct consequences for the geometry of the Julia set.

Proposition 3.2. *If $P_a^n(0)$ tends to infinity as n tends to infinity, then the Julia set $\mathcal{J}(P_a)$ is totally disconnected. Otherwise it is connected.*

The above result motivates the definition of the so-called *connectedness locus*, which is the set consisting of those parameters a for which $\mathcal{J}(P_a)$ is connected. In the case of the (complex) quadratic family this set is usually called the *Mandelbrot set*, after B. Mandelbrot [Man80] who was the first to obtain high quality pictures of it (see also [BM81]). We denote the Mandelbrot set by $\mathcal{M}$, and from Proposition 2.2 we know that $\mathcal{M}$ intersects the real line in $[-2, 1/4]$. Moreover we have the following result.

Proposition 3.3. *$\mathcal{M}$ is a closed simply connected subset of the disk $\{|a| \leq 2\}$, and consists of precisely those a such that $P_a^n(0) \leq 2$ for all $n \geq 0$.*

Figure 1.2 provides a picture of the Mandelbrot set, and we notice the close connection with the bifurcation diagram of Figure 1.1. Indeed, the parameter values for which period doubling bifurcation occurs are precisely those parameters in the Mandelbrot set lying on the real axis connecting the components.

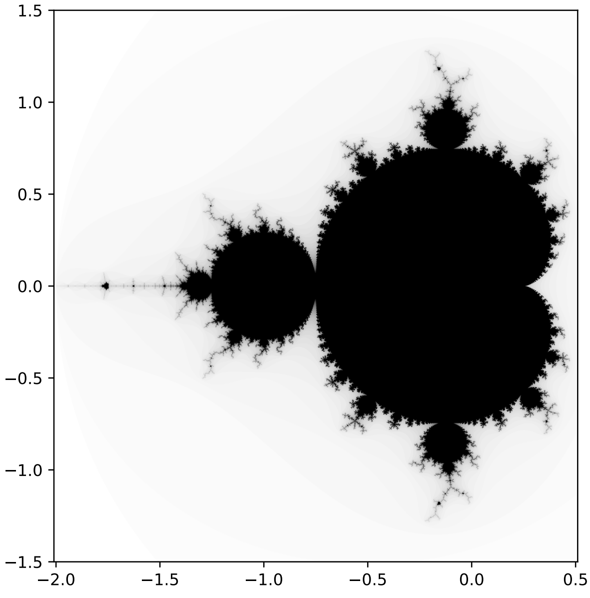

Figure 1.2: Connectedness locus for $z \mapsto z^2 + c$.

In the rational setting there is no analogue of Proposition 3.2, however the behaviour of the critical orbits are equally important for the global dynamics. The following result resembles that of Proposition 2.1.

Proposition 3.4. *For each attracting cycle of a rational function of degree $d \geq 2$ there is at least one critical point whose orbit accumulates on this cycle. The number of critical points (counting multiplicity) is at most $2d - 2$, hence there are at most $2d - 2$ attracting cycles.*

In order to understand the dynamics on the Julia set, the following definition is of central importance. We notice the resemblance with Definition 2.5.

Definition 3.5. A rational function $z \mapsto R(z)$ is called *hyperbolic* if every critical point belongs to the Fatou set $\mathcal{F}(R)$ and is attracted to an attracting cycle. Otherwise it is called *non-hyperbolic*.

Being hyperbolic is equivalent to the existence of a metric, smoothly equivalent to the spherical metric in a neighbourhood of the Julia set, for which R is *expanding*. If we assume that $\infty \notin \mathcal{J}(R)$, then this is equivalent to the existence of $C > 0$ and $\gamma > 0$ such that

$$|(R^n)'(z)| \geq Ce^{\gamma n}$$

for all $z \in \mathcal{J}(R)$ and $n \geq 1$. (This latter notion of expanding on the Julia set is in fact the usual definition of being hyperbolic, and our definition can be proved to be equivalent.)

One of the great open conjectures in the field of rational dynamics is the so-called *Hyperbolicity conjecture*: the set of hyperbolic rational maps form an (open) dense set in parameter space. Even in the case of the quadratic family $z \mapsto z^2 + a$ it is not yet known whether the set of (complex) parameters a forms an open dense set (this is the so-called *Fatou conjecture*).

4 The Collet–Eckmann conditions

As mentioned earlier, the Collet–Eckmann condition was first introduced by Collet and Eckmann [CE83, CE80] in their study of certain real families of dynamical systems, and was used to prove the abundance of acip's.

The Collet–Eckmann condition has proven to be very fruitful to consider also in the rational setting, although things naturally become more complex. We give the following definition.

Definition 4.1. A rational function R without parabolic cycles is said to satisfy the *Collet–Eckmann condition* (CE) if there exist $C > 0$ and $\gamma > 0$ such that, for each critical point c in the Julia set of R,

$$|DR^n(R(c))| \geq Ce^{\gamma n},$$

for all $n \geq 0$.

The requirement of no parabolic cycles is a technical one since, for instance, one usually wants uniform expansion outside a neighbourhood of the critical points in the Julia set. From now on we denote by $\text{Crit}'(R)$ the set of critical points in the Julia set of R, i.e. $\text{Crit}'(R) = \text{Crit}(R) \cap \mathcal{J}(R)$.

The study of rational Collet–Eckmann maps was initiated by F. Przytycki [Prz96, Prz98]. For instance, in [Prz96] it is proved that if $\mathcal{J}(R) \neq \widehat{\mathbb{C}}$, then $\text{Leb}\,\mathcal{J}(R) = 0$, i.e. for a rational Collet–Eckmann map, either the Julia set is the entire sphere, or it has measure zero. Moreover, by assuming an extra condition by M. Tsujii, namely that the average distance of $R^n(\text{Crit}')$ to Crit' is not too small, it was also proved that the Hausdorff dimension of $\mathcal{J}(R)$ is strictly less than 2 (provided $\mathcal{J}(R) \neq \widehat{\mathbb{C}}$, of course). Later, Graczyk and Smirnov [GS98a] proved, among other things, that rational Collet–Eckmann maps can have no rotation domains, and the Fatou components are Hölder domains. (Using a result by P. Jones and N. Makarov [JM95], this latter property implies that, for a rational Collet–Eckmann map with at least one fully invariant Fatou component, the Hausdorff dimension of its Julia set is strictly less than 2.)

That rational Collet–Eckmann maps are interesting from a measure point of view was established by M. Aspenberg [Asp04, Asp13] in his doctoral thesis: the set of Collet–Eckmann maps has positive (Lebesgue) measure in the parameter space of rational functions of any fixed degree $d \geq 2$. Moreover, using the results of Przytycki [Prz96], and Graczyk and Smirnov [GS98a], these maps described by Aspenberg also support acip's. The existence of a positive measure set of rational maps having acip's was first proved by M. Rees [Ree86].

Considering the recurrent nature of rational functions, Aspenberg [Asp09] furthermore proved that the set of rational Misiurewicz functions of any fixed degree $d \geq 2$ constitutes only a set of measure zero in the parameter space. Therefore, analogous to the case of real quadratic functions, the critical points belonging to the Julia set of a typical non-hyperbolic rational function are recurrent. Results regarding the rate of recurrence of the critical points for non-hyperbolic rational functions are more sparse than in the real quadratic setting. In the quadratic (and even unicritical) setting $z \mapsto z^2 + a = P_a(z)$, the Collet–Eckmann parameters are known to constitute only a set of measure zero [ALS11]. However, Graczyk and Świątek [GS00] proved that for a typical parameter with respect to *harmonic measure* on the boundary of the Mandelbrot set, the Collet–Eckmann condition is satisfied (see also [Smi00]). Moreover, they proved in [GS15] that the *Lyapunov exponent* $\chi(a)$ exists: for a typical parameter $a \in \partial\mathcal{M}$ with respect to harmonic measure,

$$\chi(a) = \lim_{n\to\infty} \frac{\log|(P_a)'(a)|}{n} = \log 2.$$

This in turn immediately gives us a recurrence result: for every $\alpha > 0$ there exists a constant

$C = C(\alpha) > 0$ such that

$$|P_a^n(0)| \geq Ce^{-\alpha n},$$

for all $n \geq 1$. For a rational function, we make the following definition.

Definition 4.2. A rational function R of degree $d \geq 2$ is said to satisfy the *slow recurrence condition* (SR) if for every $\alpha > 0$ there exists $C = C(\alpha) > 0$ such that, for every critical point $c \in \mathrm{Crit}'(R)$,

$$\mathrm{dist}(R^n(c), \mathrm{Crit}') \geq Ce^{-\alpha n},$$

for all $n \geq 1$.

Not much is known about the measure of rational functions satisfying the slow recurrence condition, however it is conjectured to be satisfied for almost every rational Collet–Eckmann map. We should also mention that, to the author's knowledge, no results exist regarding the typical rate of recurrence in the rational setting, i.e. for what δ_n do we have, given $c \in \mathrm{Crit}'$,

$$\mathrm{dist}(R^n(c), \mathrm{Crit}') < \delta_n$$

for infinitely many n? We do believe, however, that the techniques of Paper I can be carried over to the rational setting.

Focusing on this slow recurrence condition, Aspenberg [Asp21] recently proved the following consequence. Let R be a rational Collet–Eckmann map of degree $d \geq 2$, satisfying the slow recurrence condition, and such that $\mathcal{J}(R) = \widehat{\mathbb{C}}$. Then R is a Lebesgue density point of rational Collet–Eckmann maps of degree d within the space $\Lambda_{d,\overline{p}'}$. In particular, this generalises the results in [Asp04, Asp13]. Motivated by this result, together with Aspenberg and W. Cui, in Paper II we consider functions as above but with $\mathcal{J}(R) \neq \widehat{\mathbb{C}}$, and prove that these are density points of hyperbolic maps. In particular, assuming that almost every rational Collet–Eckmann map satisfies the slow recurrence condition, then almost every Collet–Eckmann map has its Julia set equal to the Riemann sphere.

Let us finish this section with discussing some other closely related conditions of non-hyperbolicity. Already in [CE83, CE80], a condition now known as the *second (or backward) Collet–Eckmann condition* was considered. The definition in the rational setting is as follows.

Definition 4.3. A rational map R of degree $d \geq 2$ is said to satisfy the *second Collet–Eckmann condition* (CE2) if there exist constants $C_2 > 1$ and $\gamma_2 > 0$ such that, for every $n \geq 1$ and every $w \in R^{-n}(c)$, for $c \in \mathrm{Crit}'(R)$ not in the forward orbit of other critical points,

$$|DR^n(w)| \geq C_2 e^{\gamma_2 n}.$$

Graczyk and Smirnov [GS98a] proved that CE and CE2 are equivalent in the unicritical setting $z \mapsto z^d + a$. In Paper I and Paper II, this condition is utilised to prove strong expansion results outside a neighbourhood of the critical point(s).

In their study of the geometry of Collet–Eckmann Julia sets, Przytycki and S. Rohde [PR98] formulated the following condition.

Definition 4.4. A rational map R of degree $d \geq 2$ is said to satisfy the *topological Collet–Eckmann condition* (TCE) if there exist $M \geq 0$, $P \geq 0$ and $r > 0$ such that for every $z \in \mathcal{J}(R)$ there exists a strictly increasing sequence of positive integers n_j, $j = 1, 2, \ldots$, such that $n_j \leq Pj$ and, for each j,

$$\# \left\{ k : 0 \leq k < n_j, \mathrm{Comp}_{R^k(z)} R^{-(n_j - k)} \left(B(R^{n_j}(z), r) \right) \cap \mathrm{Crit} \neq \emptyset \right\} \leq M.$$

Here in the above definition, Comp_w denotes the connected component containing w. Since the above condition is formulated in topological terms, it is invariant under topological conjugacy. One of the more useful properties of the topological Collet–Eckmann condition is its many equivalent formulations [PRLS03, PRL07, RL10]. In particular, CE and CE2 independently imply TCE.

Much work has been done to understand the relationships between these three characterisations of non-hyperbolicity. Przytycki, Smirnov, and J. Rivera-Letelier [PRLS03] made an extensive study and proved, among other things, that these conditions are equivalent within the family of unicritical functions $z \mapsto z^d + a$. In Paper III, we observe yet another consequence of the slow recurrence condition, namely that within the family of slowly recurrent rational maps of degree $d \geq 2$, all of these conditions are equivalent. Since there are known examples where CE does not imply CE2, CE2 does not imply CE, and TCE does not imply CE or CE2, this shows that the slow recurrence condition is in some sense essential for equivalence to hold.

5 The Benedicks–Carleson techniques

In their seminal papers, Benedicks and Carleson [BC85, BC91] developed techniques to prove the abundance of Collet–Eckmann real quadratic functions, and the existence of acip's. However, this machinery of theirs is far reaching, as can be realised by the many papers utilising it. In fact, it is the foundational tool used in Paper I and Paper II of this thesis. In this section we try to provide a schematic outline of these *parameter exclusion techniques*.

At its core, these techniques constitute a technical induction argument, with the Collet–Eckmann condition being the driving force. For the sake of explanation, let $f = f_0$ be the so-called *unperturbed map*, acting on some space X. We ask of this map to satisfy the Collet–Eckmann condition: there exist constants $C > 0$ and $\gamma > 0$ such that, for all critical points c of f belonging to $\mathcal{J}(f)$,

$$|(f^n)'(f(c))| \geq Ce^{\gamma n},$$

for all $n \geq 0$.

For a in some subset $\Delta = \Delta_0$ of the parameter space, we let f_a denote a perturbation of f. The goal is to show that for a large (or small) set of parameters, the corresponding perturbations f_a share similar properties as the unperturbed map.

To this end, suppose that f only has one critical point $c = c(0)$, and that the corresponding perturbation f_a only has one critical point $c(a)$. In fact, let us assume a normalisation so that $c(a) = 0$ for all $a \in \Delta$. We will iterate the critical point simultaneously for different parameters, and we let $\xi_n : \Delta \mapsto X$ denote the function $a \mapsto \xi_n(a) = f_a^n(0)$.

If Δ is chosen sufficiently small then, up to some large time N, the Collet–Eckmann condition is inherited by all perturbations. In particular, as long as the derivatives of f_a^N and f_b^N, evaluated at their corresponding critical values, are comparable, the Collet–Eckmann condition gives *expansion* of the image. This property of having comparable derivatives is called *distortion*. At some time $m_1 \geq N$, the image of Δ will come very close to, and might even cover, the critical point. At this stage one makes a *partition*: $\Delta = \bigcup_k \Delta_{1,k}$. This partition is made so that on each partition element $\Delta_{1,k}$, we have good distortion control. Each of the partition elements will then be iterated individually until the same situation occurs. That is to say, the partition element $\Delta_1 = \Delta_{1,k}$, for instance, will be iterated until at some time $m_2 \geq m_1$ its image $\xi_{m_2}(\Delta_1)$ gets close to the critical point. At this stage we once again make a partition $\Delta_1 = \bigcup_k \Delta_{2,k}$, and the procedure continuous indefinitely.

At each stage of partitioning, one might have to discard parameters that belong to partition elements that come too close to the critical point. The reason for this is to make sure that not too much derivative is lost, hence ensuring a Collet–Eckmann condition for future iterates. This approach rate condition is usually referred to as the *basic assumption*: for all $a \in \Delta$ we ask that

$$\operatorname{dist}\left(f_a^n(0), 0\right) \geq \delta_n,$$

for all $n \geq 1$, and for some suitable sequence δ_n.

Even though some derivative is lost when returning close to the critical point, much (but not all) of what was lost will be recovered during the so called *bound period*. Indeed, the Collet–Eckmann condition is a standing induction assumption, and for some time after the partition, the future iterates will stay close to the past iterates. Using this fact, one can show that during this bound period, derivative from the past iterates will be inherited by the future iterates.

In order to estimate what is left in parameter space after each partition stage, one needs to be able to compare the parameter derivative of ξ_n with the phase derivative of f_a^{n-1}. This kind of comparison is called *transversality*. Assuming good distortion estimates, and good transversality estimates, the measure of what is left in parameter space after infinitely long time is essentially determined by whether the sequence δ_n in the basic assumption is summable or not.

Chapter 2

Summary of results

Paper I

In this paper we study the real quadratic family

$$x \mapsto x^2 + a = Q_a(x),$$

acting on $X = [-2, 2]$, and with parameter $a \in [-2, 1/4]$. Our goal is to investigate the *typical recurrence rate* of the critical point $x = 0$ to itself, when a is a nonregular parameter, i.e. when a is such that $x \mapsto x^2 + a$ has no attracting cycle. With typical recurrence rate we mean a sequence δ_n such that, for almost every nonregular parameter a,

$$|Q_a^n(0)| < \delta_n$$

holds true for infinitely many n. Without loss of generality we may assume $a \in [-2, -1]$, and for such parameters we instead study the equivalent family

$$x \mapsto 1 - ax^2 = F(x; a),$$

acting on $X = [-1, 1]$, and with parameter $a \in [1, 2]$.

A. Avila and C. G. Moreira [AM05] proved two important results regarding the real quadratic family. The first result states that almost every nonregular parameter satisfies the Collet–Eckmann condition. The second result concerns recurrence, and states that for almost every nonregular parameter a

$$\limsup_{n\to\infty} \frac{-\log|F^n(0; a)|}{\log n} = 1.$$

Introducing the set $\Lambda(\delta_n) = \{a \in \mathcal{NR} : |F^n(0; a)| < \delta_n \text{ for infinitely many } n\}$ the above equality can be rephrased as

$$\operatorname{Leb} \Lambda(n^{-\theta}) = \begin{cases} \operatorname{Leb} \mathcal{NR} & \text{if } \theta < 1, \\ 0 & \text{if } \theta > 1. \end{cases}$$

The above lim sup-result is strong and gives us both a typical recurrence rate, namely $\delta_n = n^{-(1-\epsilon)}$ for any $\epsilon > 0$, but also a *typical approach rate*: for almost every nonregular parameter a and $\epsilon > 0$ there exists a constant $C = C(a, \epsilon)$ such that

$$|F^n(0; a)| \geq \frac{C}{n^{1+\epsilon}} \quad \text{for all} \quad n \geq 1.$$

What the lim sup cannot see, though, is the sharpness of the exponent, i.e. the case of $\epsilon = 0$, and to investigate this is the main concern of Paper I.

Let us call a sequence δ_n *admissible* if there exists a constant $K > 0$ and an exponent $\sigma \geq 0$ such that

$$\delta_n \geq \frac{K}{n^{\sigma}} \quad \text{for all} \quad n \geq 1.$$

In Paper I we prove the following result. There exists $\tau \in (0, 1)$ such that if δ_n is admissible and

$$\sum \frac{\delta_n}{\log n} \tau^{(\log^* n)^3} = \infty,$$

then $\operatorname{Leb} \Lambda(\delta_n) = \operatorname{Leb} \mathcal{NR}$. Here $\log^*$ is the so-called iterated logarithm, and it is defined as

$$\log^* x = \begin{cases} 1 & \text{if } x \leq 1, \\ 1 + \log^* \log x & \text{if } x > 1. \end{cases}$$

In particular, $\log^*$ grows slower than any $\log_j = \log \circ \log_{j-1}$, $j \geq 0$. Therefore as a direct corollary we find that

$$\operatorname{Leb} \Lambda(n^{-1}) = \operatorname{Leb} \mathcal{NR},$$

thus covering the missing case of $\theta = 1$.

The proof utilises the Benedicks–Carleson techniques [BC85, BC91], together with more recent developments [Asp21, Lev14]. The main innovation of this paper is the introduction of unbounded distortion estimates.

Paper II

We consider slowly recurrent rational functions of a fixed degree and whose Julia set is not equal to the entire sphere. By assuming that the critical points approach each other only at a slow rate, i.e. by assuming the so-called slow recurrence condition, we prove that these functions can be approximated in a strong sense by hyperbolic functions.

Let us call two rational functions equivalent if they are conjugated by a Möbius transformation. In the parameter space of rational functions of a fixed degree $d \geq 2$, let $\Lambda_{d,\overline{p'}}$ denote the subspace of rational functions, up to equivalence, with exactly p' critical points $c_1, c_2, \ldots, c_{p'}$, and with corresponding multiplicities $\overline{p'} = (m_1, m_2, \ldots, m_{p'})$. Within this subspace, critical points do not split, and move analytically with the parameter. In this paper, we look at small perturbation of $R = R_0 \in \Lambda_{d,\overline{p'}}$, where R satisfies the Collet–Eckmann condition, and $\mathcal{J}(R) \neq \widehat{\mathbb{C}}$. Moreover, R also satisfies the slow recurrence condition: for any $\alpha > 0$ there exists $C > 0$ such that, for every $c \in \mathrm{Crit}'$,

$$\mathrm{dist}\left(R^n(c), \mathrm{Crit}'\right) \geq Ce^{-\alpha n},$$

for all $n \geq 1$. In Paper II we prove that such a rational function is a Lebesgue density point of hyperbolic functions (within $\Lambda_{d,\overline{p'}}$). Moreover, if all critical points are simple, then such a function is a Lebesgue density point of hyperbolic functions in the entire space of rational functions of degree d.

To prove the above result, we utilise the parameter exclusion techniques developed by Benedicks and Carleson [BC85, BC91], together with its evolvement in the rational setting by Aspenberg [Asp04, Asp13, Asp09, Asp21], and strong transversality results by Levin [Lev14]. In fact, Aspenberg [Asp21] recently proved a contrasting result. Namely, if $R \in \Lambda_{d,\overline{p'}}$ satisfies the Collet–Eckmann condition, if $\mathcal{J}(R) = \widehat{\mathbb{C}}$, and if R satisfies the slow recurrence condition, then it is a Lebesgue density point of Collet–Eckmann functions (within $\Lambda_{d,\overline{p'}}$).

The techniques used in Paper II are similar to those in [Asp21]. We begin with a small parameter square centred at R, and our goal is for this square to reach to so-called *large scale*. Since $\mathcal{J}(R) \neq \widehat{\mathbb{C}}$, the measure of the Julia set $\mathcal{J}(R)$ is equal to zero [Prz96]. Therefore, upon reaching the large scale, a large portion of our square will correspond to parameters whose critical points lie in the Fatou set. We show that the large scale is reached under bounded transversality, and bounded distortion, and the conclusion is that in parameter space, most parameters correspond to hyperbolic maps, hence our density result.

Paper III

In Paper III we consider rational functions acting on the Riemann sphere $\widehat{\mathbb{C}}$, and the relationships between the Collet–Eckmann condition (CE), the second Collet–Eckmann condition (CE2), and the topological Collet–Eckmann condition (TCE). Much work has been made investigating these conditions. In particular it is known that CE or CE2 implies TCE, whereas to any other possible implication there are known counterexamples. In the unicritical, on the other hand, all of these conditions are equivalent. (See [PRLS03] and references therein.)

In this paper we observe that within the family of slowly recurrent rational functions, all of the above conditions are equivalent. Moreover these conditions are invariant under topological conjugation. The proofs in this paper are short, even though the results on which they are based upon require technical machinery. Indeed, the techniques are those of *shrinking neighbourhoods* as developed by Przytycki [Prz98], and used by Graczyk and Smirnov [GS98a].

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