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ADAPTIVE DECOUPLING OF MULTIVARIABLE SYSTEMS*

by

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ABSTRACT

For many industrial processes it is of interest to design a decoupling precompensator. The precompensator makes it possible to design the controllers based on single-input-single output models of the process. A model of the process must be known to design the precompensator. This report shows how the precompensator can be designed adaptively using input output measurements.

The precompensator design is first done for the case when the process is known. The adaptive precompensator is then constructed using the certainty equivalence principle. The convergence properties and the implementation of the adaptive decoupler are discussed. It is shown that the adaptive scheme will converge provided the system is persistently excited and that a suitable model structure is used in the estimation.

1. INTRODUCTION

Most control systems for industrial processes are designed from a single-input-single-output (SISO) point of view. This is appropriate only if the couplings between the different loops are weak. There are, however, many multivariable processes where there are strong couplings between the loops. It is then of importance to consider this coupling when designing the control system. Further it may be desirable to control the different loops individually. To do this a decoupler must be designed to separate the different control loops. The design of a decoupler requires good process models since the decoupler critically depends on the internal structure and parameters of the system. The problem of decoupling using linear state feedback has been given large attention. An overview based on geometric concepts is given in Morse and Wonham (1971) and using frequency domain methods in Wolovich (1974). The decoupling problem is also discussed in Pernebo (1981 a, b). The fundamental questions are the priori knowledge that is needed about the system and what closed loop systems that are achievable. These problems are discussed in Desoer and Gündes (1986). They give a parameterization of all achievable stable decoupled systems using dynamic output feedback.

Our approach is aimed at making adaptive decoupling. This implies that the design has to be made based on input-output models. Adaptive controllers for multiple input multiple output (MIMO) are discussed for instance in Borisson (1979), Koivo (1980), Prager and Wellstead (1981), Favier and Hassani (1982), Elliott and Wolovich (1984). The adaptive decoupling problem is solved approximately in McDermott and Mellichamp (1986). Adaptive decoupling and prior knowledge is discussed in Singh and Narendra (1984).

How is decoupling done in the process industry today? The usual approach is to design a precompensator such that the new system essentially becomes decoupled. A survey of different ways to measure the interaction in a system is given in Jensen, Fisher and Shah (1986). Procedures for designing the precompensators are discussed in Wolovich (1974) and Wolovich (1981). The first reference gives a design procedure that can be used if the process has a stable inverse. In this paper we propose a decoupling procedure based on a precompensator. The precompensator is designed such that no unstable pole-zero cancellations occur. The decoupling precompensator will separate the controller design into a number of single input single output design problems. A design procedure along these lines is hinted at in Johansson (1983, p44).

The paper is organized in the following way: Section 2 contains a discussion of the design method for known systems. The class of systems that are considered are MIMO systems with equal number of inputs and outputs. Further the design should be based on input-output models. The adaptive version of the decoupling precompensator is discussed in Section 3. The convergence properties of the algorithm are also discussed. Section 4 gives a discussion of the implementation of the adaptive algorithm and of the computational problems of the proposed scheme. Section 5 contains a summary and conclusions.

Our contributions relative to earlier works are:

a) We derive an adaptive decoupling algorithm for a large class of systems. The decoupling is done without any unstable pole-zero cancellations.

b) The convergence of the design scheme is established under the assumption that the input signals are persistently exciting.

2. A DECOUPLING PRECOMPENSATOR FOR KNOWN SYSTEMS

The Process

The process to be controlled is assumed to be a m-input m-output linear system. The system is described by the sampled data input output model:

$$A(q^{-1})y(t) = B(q^{-1})u(t) \tag{2.1}$$

where $A(q^{-1})$ and $B(q^{-1})$ are polynomial matrices in the backward shift operator. It is assumed that $A(q^{-1})$ is diagonal. This is no loss of generality, but may require that more parameters have to be estimated when the process is unknown. It is assumed that $B(q^{-1})$ is invertable. This implies that det $(B(q^{-1}))$ is not identical equal to zero. The internal couplings in the process are of great importance for the design of multivariable control systems. The interactor matrix is used in Elliott and Wolovich (1984) and Dugard et al (1984). The internal structure matrix has been used in Pernebo (1981 a, b) and Johansson (1983). In many adaptive control schemes it is assumed that the interactor matrix is known or estimated. The problem of estimating the structure matrix is avoided in our case since it is assumed that det $(B) \neq 0$. In this case the B matrix plays a similar role as the structure matrix.

The priori knowledge about the process that is needed for the design of the precompensator is the "delay structure" of the process. This is important for the design and to be able to determine the achievable closed loop system. See Pernebo (1981 a, b) and Desoer and Gündes (1986). The importance of the delay structure is also discussed in Holt and Morari (1985). In this paper it is assumed that the delay between each of the inputs and the outputs are known. For the adaptive implementation it suffices to assume that the shortest delay in each row of the B-matrix is known. To be able to make the estimation of the unknown system it is also necessary to have an upper bound on the orders of the polynomials in the A and B matrices.

The assumptions about the process are summarized as

Assumtion 2.1: The process is described by the m-input m-output system (2.1) with A diagonal and B invertable, i.e. det(B) is not identical zero.

Assumption 2.2: Upper bounds of the orders of the polynomials in $A(q^{-1})$ and $B(q^{-1})$ are known.

Design of the precompensator

Introduce a precompensator of the form

$$u(t) = P(q^{-1})v(t) \tag{2.2}$$

where $P(q^{-1})$ is a causal, stable transfer operator matrix. The precompensator will now be derived using the results in Desoer and Gündes (1986). They give a characterization of all decoupled systems that are achievable by a stabilizing controller.

The purpose of the precompensator is to diagonalize the system i.e. it should be chosen such that

$$A(q^{-1})y(t) = B(q^{-1})u(t) = B(q^{-1})P(q^{-1})v(t)$$

is a diagonal system. The diagonalization should be done without any unstable pole-zero cancellations. Further it must be possible to make a causal implementation of $P(q^{-1})$.

The achievable closed loop systems are determined by the time delays and the unstable zeros of the open loop system. We first extract the common factors of each row of B. We thus define the diagonal matrix

$$B_\ell(q^{-1}) = \text{diag}\,[B_{\ell 1}^{-} q^{-k_1} \;.\;.\;.\; B_{\ell m}^{-} q^{-k_m}] \qquad (2.3)$$

where $B_{\ell i}^{-} q^{-k_i}$ $i = 1,\ldots,m$ are the greatest common factors that are not allowed to be cancelled in each row of $B(q^{-1})$. This implies that $B_{\ell i}^{-}(q^{-1})q^{-k_i}$ are the common nonminimum phase zeros and delays for the i:th row.

<u>Remark 2.1</u>

It is convenient to consider $B(q^{-1})$ as a polynomial matrix in q^{-1}. Both nonminimum phase zeros and time delays can then be treated analogously. The "unstable" region in q^{-1} is the interior of the unit circle. Compare Pernebo (1981 a) and Desoer and Gündes(1986). Since we will not discuss the system theoretic questions in this paper extent we will not introduce any new notation for q^{-1}.

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The matrix $B(q^{-1})$ can now be written as

$$B(q^{-1}) = B_\ell(q^{-1})\tilde{B}(q^{-1}) \qquad (2.4)$$

Since det (B) $\neq 0$ it follows that the inverse of $\tilde{B}$ exists

$$\tilde{B}^{-1}(q^{-1}) = \left[\frac{N_{ij}(q^{-1})}{D_{ij}(q^{-1})} \right] \tag{2.5}$$

where N_{ij}/D_{ij} is the (i,j)th element of $\tilde{B}^{-1}$. Further it is assumed that N_{ij} and D_{ij} are coprime, i.e. all common factors are removed. Without loss of generality we can normalize $\tilde{B}^{-1}$ such that the first non-zero coefficient of $D_{ij}(q^{-1})$ is equal to one.

If the denominator polynomials in $\tilde{B}^{-1}$ don't contain any pure time delays (i.e. that q^{-1} is not a factor of $D_{ij}(q^{-1})$ and no unstable roots we can use $\tilde{B}^{-1}$ as the precompensator. In the general case to make the precompensator causal and stable we first have to extract the unstable parts (and pure time delays) of each column of $\tilde{B}^{-1}$. This gives a stable and causal precompensator which can be written

$$P(q^{-1}) = [\tilde{B}^{-1}(q^{-1})B_r(q^{-1})]_+ \tag{2.6}$$

i.e.

$$B_r(q^{-1}) = \text{diag}[D_j^-(q^{-1})] \tag{2.7}$$

where the notation $[\cdot]_+$ denotes that all common unstable poles and zeros are eliminated in each element.

The polynomials $D_j^-(q^{-1})$ contains pure time delays and all unstable poles in each column of $\tilde{B}^{-1}$. $D_j^-(q^{-1})$ $j = 1, \ldots ,m$ are thus a least common multiple of the j:th column of $\tilde{B}^{-1}$.

Through the construction it follows that $P(q^{-1})$ is stable and causal. Combining (2.1), (2.2), (2.4) and (2.6) gives

$$A(q^{-1})y(t) = B(q^{-1})P(q^{-1})v(t)$$

$$= B_\ell(q^{-1})\tilde{B}(q^{-1})\tilde{B}(q^{-1})^{-1}B_r(q^{-1})v(t)$$

$$= B_\ell(q^{-1})B_r(q^{-1})v(t) \qquad (2.8)$$

where A, B_ℓ and B_r are diagonal polynomial matrices. This implies that the precompensator P decouples the MIMO system into m SISO systems. It is now possible to use SISO design methods to determine the m controllers for (2.8).

Remark 2.2

In Desoer and Gündes (1986, Theorem 3.2) it is shown that the system (2.1) can be stabled decoupled if and only if B_ℓ and B_r, as defined by (2.3) and (2.7), are factors of the decoupled system as in (2.8).

The diagonal matrices B_ℓ and B_r thus defines the time delays and the nonminimum phase zeros that must be present in the decoupled system. The construction of the precompensator shows that it may be necessary to introduce extra time delays and/or nonminimum phase zeros in order to get a decoupled system. The design of the precompensator given above explains the heuristic discussion in Holt and Morari (1985), who claim that "better" MIMO systems can sometimes be constructed by introduction of extra time delays in the regulator.

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Remark 2.3

It is always possible to determine a P such that BP is triangular without introducing any extra time delays or nonminimum phase zeros Pernebo (1986). Both P and P^{-1} are then causal and stable.

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Remark 2.4

Notice that when using the decoupling design given above it is not necessary to make any pairing of inputs and outputs. The precompensator automatically defines a control signal v_i that is used to control the output y_i.

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Remark 2.5

It is also possible to let B_r consist of the least common multiple all the denominators of each column of $\tilde{B}^{-1}$. $P(q^{-1})$ will then be a polynomial matrix i.e. $P(q^{-1})$ is a moving average compensator. This choice will introduce more stable zeros in the decoupled system. This will, however, not change the closed loop system if the design method cancels stable zeros. This is the method that is used in the adaptive version presented in Section 3.

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SISO-pole placement

Each loop in (2.8) is now a single input single output system of the form

$$A_i(q^{-1})y_i(t) = B_i(q^{-1})v_i(t) \qquad (2.9)$$

where $B_i(q^{-1}) = B_{\ell i}(q^{-1})B_{ri}(q^{-1})$

SISO-design methods based for instance on pole-placement can now be used, see e.g. Åström and Wittenmark (1984). Since $B_i(q^{-1})$ will contain nonminimum phase zeros it is necessary to use a design method, which does not cancel any unstable zeros.

The controller is now obtained by solving the Diophantine equation

$$A_iR_i + B_iS_i = A_{oi}A_{mi}B_i^+ \qquad (2.10)$$

for R_i and S_i. A_{oi} is an arbitrary stable polynomial, which can be interpreted as an observer polynomial. A_{mi} is the desired closed loop poles. B_i^+ is a monic polynomial that defines the zeros of B_i that are allowed to be cancelled by the controller i.e. $B_i = B_i^+ B_i^-$. The controller for the i:th loop is

$$R_i v_i = - S_i y_i + T_i y_{ri} \tag{2.11}$$

where y_{ri} is the reference signal for the i:th loop and

$$T_i(q^{-1}) = A_{oi}(q^{-1}) B'_{mi}(q^{-1}) A_{mi}(1) / B_i^-(1) B'_{mi}(1) \tag{2.12}$$

$B'_{mi}(q^{-1})$ is used to introduce new zeros in the closed loop transfer function. Equation (2.12) contains a normalization factor to obtain unit steady state gain from y_{ri} to y_i.

Using (2.12) and (2.11) in (2.9) give the closed loop system

$$y_i(t) = \frac{A_{oi}(q^{-1}) B'_{mi}(q^{-1}) B_i^+(q^{-1}) B_i^-(q^{-1}) A_{mi}(1)}{A_{oi}(q^{-1}) A_{mi}(q^{-1}) B_i^+(q^{-1}) B_i^-(1) B'_{mi}(1)} y_{ri}(t)$$

$$= \frac{A_{mi}(1)}{B_i^-(1) B'_{mi}(1)} \cdot \frac{B'_{mi}(q^{-1}) B_i^-(q^{-1})}{A_{mi}(q^{-1})} y_{ri}(t)$$

It is also easy to introduce integrators in the design. The pole placement design and its robustness properties are extensively discussed in Åström and Wittenmark (1984).

Examples

Two examples will now be given to illustrate the construction of the precompensator. The examples illustrate which decoupled systems that are possible to achieve. The examples also show the computations that are needed for the design.

Example 2.1

Consider the system

$$\begin{bmatrix} 1-q^{-1} & 0 \\ 0 & 1-q^{-1} \end{bmatrix} y(t) = \begin{bmatrix} q^{-1} & -q^{-1} \\ q^{-2}+q^{-3} & q^{-3} \end{bmatrix} u(t)$$

From (2.4) we get

$$B(q^{-1}) = \begin{bmatrix} q^{-1} & 0 \\ 0 & q^{-2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1+q^{-1} & q^{-1} \end{bmatrix} = B_{\ell}(q^{-1})\tilde{B}(q^{-1})$$

$B_{\ell}(q^{-1})$ contains the common time delays in each row and there are no nonminimum phase zeros in B_{ℓ}. $\det(\tilde{B}) = 1 + 2q^{-1}$ implies that the system has a nonminimum phase zero in -2.

$$\tilde{B}^{-1} = \frac{1}{1+2q^{-1}} \begin{bmatrix} q^{-1} & 1 \\ -(1+q^{-1}) & 1 \end{bmatrix}$$

This gives

$$B_r(q^{-1}) = \mathrm{diag}\,[1 + 2q^{-1} \quad 1 + 2q^{-1}]$$

and

$$P(q^{-1}) = \begin{bmatrix} q^{-1} & 1 \\ -(1+q^{-1}) & 1 \end{bmatrix}$$

The decoupled system is thus

$$\begin{bmatrix} 1-q^{-1} & 0 \\ 0 & 1-q^{-1} \end{bmatrix} y(t) = \begin{bmatrix} q^{-1}(1+2q^{-1}) & 0 \\ 0 & q^{-2}(1+2q^{-1}) \end{bmatrix} v(t)$$

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In Example 2.1 the same nonminimum phase zero occurs in both loops. That this is not always the case is shown in the next example.

<u>Example 2.2</u>

Let the system be described by

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} y(t) = \begin{bmatrix} q^{-k_1}B_1 & q^{-k_{12}}B_{12} \\ 0 & q^{-k_2}B_2 \end{bmatrix} u(t) \qquad k_{12} \leq k_1$$

Let

$$B_1 = B_1^+ B_1^-$$

where B_1^+ and B_1^- correspond to the minimum phase and nonminimum phase parts of B_1 respectively. B_{12} and B_2 are factored analogously. It is further assumed that B_1^- and B_{12}^- don't have any common factors. The same assumption is also made for B_2^- and B_{12}^-. We now get

$$B_\ell(q^{-1}) = \text{diag } [q^{-k_{12}} \qquad q^{-k_2}]$$

$$\tilde{B}(q^{-1}) = \begin{bmatrix} q^{-k}B_1 & B_{12} \\ 0 & B_2 \end{bmatrix} \qquad k = k_1-k_{12}$$

$$\tilde{B}(q^{-1})^{-1} = \begin{bmatrix} \dfrac{1}{q^{-k} \; B_1} & -\dfrac{B_{12}}{q^{-k} \; B_1 B_2} \\ 0 & \dfrac{1}{B_2} \end{bmatrix}$$

This gives

$$B_r(q^{-1}) = \text{diag} \left[\; q^{-k} \; B_1^- \qquad q^{-k} \; B_1^- \, B_2^- \; \right]$$

and

$$P(q^{-1}) = \begin{bmatrix} \dfrac{1}{B_1^+} & -\dfrac{B_{12}}{B_1^+ \, B_2^+} \\ 0 & q^{-k} \, \dfrac{B_1^-}{B_2^+} \end{bmatrix}$$

The decoupled system is

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} y(t) = \begin{bmatrix} q^{-k_1} \; B_1^- & 0 \\ 0 & q^{-(k_1+k_2-k_{12})} B_1^- \; B_2^- \end{bmatrix} v(t)$$

The decoupled has a longer time delay in the second loop than the original system. Further extra nonminimum pluse zeros are introduced in the second loop. This is in correspondence with the discussion in Holt and Morari (1985).

If we assume that $k_1 < k_{12}$ then it is easily found that

$$B_\ell(q^{-1}) = \text{diag}\,[q^{-k_1} \quad q^{-k_2}]$$

$$B_r(q^{-1}) = \text{diag}\,[B_1^- \quad B_1^- B_2^-]$$

$$P(q^{-1}) = \begin{bmatrix} \frac{1}{B_1^+} & -\frac{q^{-(k_{12}-k_1)} B_{12}}{B_1^+ B_2^+} \\ 0 & \frac{B_1^-}{B_2^+} \end{bmatrix}$$

and

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} y(t) = \begin{bmatrix} q^{-k_1} B_1^- & 0 \\ 0 & q^{-k_2} B_1^- B_2^- \end{bmatrix} v(t)$$

The original system in this case is triangular with the shortest delay for each row in the diagonal. For such systems it is easy to design controllers using feedforward. The price for decoupling is that extra nonminimum phase zeros are introduced in the second loop.

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Summary of the decoupling design

The design of the decoupling precompensator for a known system consists of the following steps:

1. Find common delays and nonminimum phase zeros of each row of B to determine B_ℓ defined in (2.3).
2. Determine the inverse of $\tilde{B}$ and remove all common factors in each element.

3. Determine a least common multiple of each column of $\tilde{B}^{-1}$ and determine B_r.

4. The precompensator P is defined by (2.6) where all common factors have been removed.

Each step above includes finding the common factors of polynomials. This can be done using Euclidean algorithm. It is then also possible to introduce a bound to detect "near" common factors. The main computational burden is to find the inverse of the polynomial matrix $\tilde{B}$ in the second step. Finally it is necessary to factor polynomials into stable and unstable parts.

3. ADAPTIVE DECOUPLING

Parameter estimation

The process is assumed to be described by (2.1) where $A(q^{-1})$ is diagonal. Assumptions 2.1 and 2.2 gives that each row of (2.1) can be written as

$$A_i(q^{-1})y_i(t) = \sum_{j=1}^{m} B_{ij}(q^{-1})u_j(t) \tag{3.1}$$

where the orders of the polynomials A_i, B_{ij} are assumed to be known (or at least upper bounds are known). Introduce the regressor vectors

$$\varphi_i^T(t) = [-y_i(t-1),\ldots,u_1(t-k_{i1}),\ldots u_m(t-k_{im}),\ldots]$$

and the parameter vectors

$$\theta_i = [a_1^i \ \ldots \ a_{ni}^i \ b_1^{i1} \ \ldots \ b_{n_{i1}}^{i1},\ldots \ b_1^{im} \ \ldots \ b_{n_{im}}^{im} \ ]$$

where n_i and n_{ij} are the orders of the A_i and B_{ij} polynomials respectively. The system (3.1) can now be written as

$$y_i(t) = \varphi_i(t)^T\theta_i \qquad i = 1, \ \ldots \ ,m \tag{3.2}$$

The system is linear in the parameters and the parameter vectors θ_i can be estimated using recursive estimation algorithm. Let the estimation be defined by

$$\hat{\theta}_i(t) = \hat{\theta}_i(t-1) + \frac{P_i(t-1)\varphi_i(t)e_i(t)}{1 + \varphi_i(t)^T \ P_i(t-1)\varphi_i(t)} \tag{3.3}$$

$$P_i(t) = P_i(t-1) - \frac{P_i(t-1)\varphi_i(t)\varphi_i(t)^T P_i(t-1)}{1 + \varphi_i(t)^T P_i(t-1)\varphi_i(t)} + Q_i(t-1) \tag{3.4}$$

where

$$e_i(t) = y_i(t) - \varphi_i(t)^T\hat{\theta}_i(t-1) \tag{3.5}$$

The matrix $Q_i(t) \geq 0$ and should ensure that $P_i(t)$ is uniformly bounded. The estimation algorithm (3.3) - (3.5) is the ordinary least squares algorithm if $Q_i(t) = 0$.

An adaptive decoupler can now be obtained by combining the design procedure in Section 2 with the estimation algorithm given above. A certainty equivalence controller is obtained by using the estimated parameters as if they are equal to the true ones. This implies that the estimated parameters are updated and the design in Section 2 is repeated at each sampling interval.

The Adaptive Decoupling Algorithm

Let the true system be

$$A(q^{-1})y(t) = \hat{B}_\ell(q^{-1})\tilde{B}(q^{-1})u(t) \tag{3.6}$$

It is assumed that (3.6) fulfils Assumption 2.1. We will also introduce the following assumptions.

Assumption 3.1

The zeros of $D = \det(\tilde{B})$ are distinct.

Assumption 3.2

$\hat{B}_\ell(q^{-1})$ is a known diagonal polynomial matrix, such that each element of $\hat{B}_\ell$ is a factor of the corresponding element in B_ℓ defined through (2.3)

Remark 3.1

Assumption 3.1 is made for technical reasons and does not impose any larger restriction since the zeros of $\det(\tilde{B})$ are continuous in the coefficients of $\tilde{B}$.

$\hat{B}_\ell(q^{-1})$ is an estimate of the delay structure. We also allow that known common nonminimum phase zeros of the rows of B are included in $\hat{B}_\ell(q^{-1})$. The structure imposed by $\hat{B}_\ell$ should be used in the estimation in order to decrease the number of estimated parameters. This is done by filtering the inputs in the regressors φ_i by $\hat{B}_{\ell i}$.

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Define

$$C_{ij}(q^{-1}) \quad = \quad \text{The monic greatest common divisor of } D \text{ and } B_{a_{ij}} \tag{3.7}$$

and

$$\nu_{ij} \quad = \quad \deg C_{ij} \tag{3.8}$$

The integers ν_{ij} are the orders of the common factors of $D = \det(\tilde{B})$ and the elements of $B_a = \operatorname{adj}(\tilde{B})$. Finally we introduce three additional assumptions

Assumption 3.3

Any cancellations between $\operatorname{adj}(\tilde{B})$ and $\det(\tilde{B})$ for the true system are exact. All other factors of $\operatorname{adj}(\tilde{B})$ and $\det(\tilde{B})$ are further than ϵ apart, where ϵ is known.

Assumption 3.4

The true system is "diagonally stabilizable", i.e. let

$$B_{ri} \triangleq \frac{D}{C_i} \tag{3.9}$$

where

$$C_i \triangleq \text{h.c.f.}\ \{C_{1i}, C_{2i} \ldots C_{ni}\}$$

(the highest common factor of column i of $[C_{ij}]$)

then A_i and $\hat{B}_{\ell i}B_{ri}$ should not have any common zeros in the unstable region.

Assumption 3.5

The input signal $u_i(t)$ is assumed to be persistently exciting.

The estimated system will rarely have exact cancellations. This implies that extra care needs to be taken in defining the estimates of C_{ij}, $\hat{C}_{ij}$ and of ν_{ij}, $\hat{\nu}_{ij}$. We define $\hat{\nu}_{ij}$ to be the largest degree of a monic polynomial $\hat{C}_{ij}$ such that $\hat{C}_{ij}$ divides $\hat{D}$ and the remainder on dividing $\hat{B}_{a_{ij}}$ by $\hat{C}_{ij}$ is "smaller" than $\epsilon_1 < \epsilon$. The size of a polynomial is defined as the norm of the vector of coefficients. The $\hat{C}_{ij}$ defined above may be non-unique and we finally define $\hat{C}_{ij}$ to be a polynomial in q^{-1} with the properties above, which also gives the least remainder on dividing $\hat{B}_{a_{ij}}$ by $\hat{C}_{ij}$. Formal definitions of $\hat{C}_{ij}$ and $\hat{\nu}_{ij}$ are given in Appendix A.

The adaptive algorithm is now defined through the following steps:

a) Update the estimates using (3.3) - (3.5).

b) Form $\hat{A}$ and $\hat{B}$ and let $\hat{B}_a = \text{adj}(\hat{\tilde{B}})$ and $\hat{D} = \det(\hat{\tilde{B}})$

c) Compute $\hat{C}_{ij}$ and $\hat{\nu}_{ij}$ as defined above and $\hat{C}_i$ as the highest common factor of $\hat{C}_{1i},\ldots,\hat{C}_{n-i}$.

d) Let $\hat{Q}_{ij}$ be such that

$$\hat{B}_{a_{ij}} = \hat{Q}_{ij}\hat{C}_{ij} + \mathbb{R}\,(\hat{B}_{a_{ij}},\hat{C}_{ij})$$

then

$$\hat{P}_{ij} = \hat{Q}_{ij}\frac{\hat{C}_{ij}}{\hat{C}_i}$$

$$\hat{B}_r = \text{diag}\ [\hat{D}/\hat{C}_i]$$

e) Solve the Diophantine equation

$$\hat{A}_i\hat{R}_i + \hat{B}_{\ell i}\hat{B}_{ri}\hat{S}_i = A_{oi}A_{mi}\hat{B}^+_{ri} \qquad i = 1, \ldots , m$$

for $\hat{R}_i$ and $\hat{S}_i$ where A_{oi}, A_{mi} and $\hat{B}^+_{ri}$ are stable.

f) Implement the control law as

$$\hat{R}_i(q^{-1})v_i(t) = -\hat{S}_i(q^{-1})y_i(t) + \hat{T}_i(q^{-1})y_{ri}(t)$$

$$u(t) = \hat{P}(q^{-1})V(t)$$

where

$$\hat{T}_i(q^{-1}) = \frac{A_{oi}(q^{-1})B'_{mi}(q^{-1})A_{mi}(1)}{\hat{B}^-_i(1)\ B'_{mi}(1)}$$

Compare (2.12)

Remark 3.2

If the Diophantine equation in Step e) is non-solvable we can use the previous values of $\hat{R}_i$ and $\hat{S}_i$. Assumption 3.4 and 3.5 guarantee that the equation will be solvable at a future time.

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Convergence and stability analysis

The convergence properties of the algorithm will now be analysed. The formal proof is done in the following steps:

- By assuming persistently excitation it is shown that $\hat{\theta}_i \rightarrow \theta_i$. Sufficient conditions are given in Lemma 3.1.
- "Smoothness" properties of the algorithm is used to show that the estimated precompensator will converge to the desired value of the compensator (Lemma 3.2).
- Theorem 3.1 shows that the output and inputs as well as the states of the adaptive controller are bounded.

Sufficient conditions for parameter convergence is given by the following lemma:

Lemma 3.1

Let A_i be of order n_i, $\hat{B}_{\ell i}$ of order m_i and B_{ij} of order n_{ij} where

$$m_i + n_{ij} \leq n_i \qquad \forall i,j \tag{3.10}$$

Then a sufficient condition to guarantee that $\hat{\theta}_i \rightarrow \theta_i$ is that $v_i(t)$ contains at least $2\bar{n}$ sinewaves at frequencies ω_{ij} $j = 1, \dots, 2\bar{n}$ where

$$0 \leq \omega_{ij} \leq \omega_s/2 = \pi f_s \tag{3.11}$$

$$\omega_{ij} = \omega_{k\ell} \Rightarrow i = k \text{ and } j = \ell \tag{3.12}$$

$$\bar{n} = \max_{i = 1,\dots,m} \{n_i\} \tag{3.13}$$

f_s is the sampling frequency (Hz) and a block processing approach is used.

Proof: (Outline)

The proof is as in for instance Goodwin and Teoh (1985), or Feuer and Heyman (1986) where in this case we treat each estimator separately. Condition (3.11) and (3.12) are necessary only to ensure that the outputs $y_i(t)$ are sufficiently excited i.e. that the input excitations do not cancel each other.

▽▽▽

Lemma 3.2

Provided $\hat{\theta}(t) \to \theta$, i.e. the parameter estimates converge to their true values we have

a) $\hat{D} = \det \hat{\tilde{B}} \to D = \det \tilde{B}$

and

$\hat{B}_a = \operatorname{adj} \hat{\tilde{B}} \to B_a = \operatorname{adj} \tilde{B}$

b) $\hat{\upsilon}_{ij} \to \upsilon_{ij}$ and $\hat{C}_{ij} \to C_{ij} \quad \forall i,j$

c) $\hat{C}_i \to C_i \quad \forall i$

d) $\hat{P} \to P$, $\hat{R} \to R$ $\hat{S} \to S$ and $\hat{T} \to T$

Proof: See Appendix A.

The proof is quite straight forward but lengthy and technical. Lemma 3.2 establishes the necessary smoothness properties and parameter convergence for the adaptive algorithm. The convergence and stability of the algorithm is now given by the following theorem.

Theorem 3.1

Consider the adaptive algorithm defined through steps a) - f) above applied to the system (3.1) subject to Assumptions 2.1-2.2, 3.1 - 3.5, then for any bounded reference signal and any finite initial conditions we have

i) $y(t)$, $v(t)$ and all states of the adaptive controller are bounded.

ii) If $\bar{y}(t)$ is the output in the ideal case, i.e. $\hat{\theta}(t) = \theta \ \forall t$, then $y(t) \rightarrow \bar{y}(t)$.

<u>Proof</u>:

Consider the (hypothetical) system for the case of known parameters:

$$A\bar{y}(t) - \hat{B}_\ell B_r \bar{v}(t) = 0 \tag{3.14}$$

$$S\bar{y}(t) + R\bar{v}(t) = T\, y_r(t) \tag{3.15}$$

or

$$\begin{bmatrix} A & -\hat{B}_\ell B_r \\ S & R \end{bmatrix} \bar{x}(t) = \begin{bmatrix} 0 \\ T \end{bmatrix} y_r(t) \tag{3.16}$$

$$\mathcal{A}\bar{x}(k) = \mathcal{B} y_r(k) \tag{3.17}$$

$\mathcal{A}^{-1}$ is a stable operator since

$$\begin{aligned} \det(\mathcal{A}) &= \det(A)\det(R + SA^{-1}\hat{B}_\ell B_r) \\ &= \det(A\,R + \hat{B}_\ell B_r S) \qquad \text{(since } A, R, \hat{B}_\ell, B_r \text{ and } S \text{ are diagonal)} \\ &= A_o A_m B_r^+ \end{aligned} \tag{3.18}$$

is a stable polynomial. Thus, the ideal response, $\bar{x}(t)$, is bounded, for any bounded $y_r(t)$. Now consider the equations of the adaptive control law:

$$Ay(t) - \hat{B}_\ell B\hat{P}v(t) = 0 \tag{3.19}$$

$$\hat{S}y(t) + \hat{R}\, v(t) = \hat{T}y_r(t) \tag{3.20}$$

or $$\hat{\mathscr{A}}x(t) = \hat{\mathscr{B}}y_r(t) \qquad (3.21)$$

From Lemma 3.2, we have $\lim_{t\to\infty} \bar{\mathscr{A}}(t) = \mathscr{A}$ and so $\lim (\hat{\mathscr{A}}(t+1) - \hat{\mathscr{A}}(t)) = 0$. Since $\mathscr{A}^{-1}$ is an exponentially stable operator, and $\hat{\mathscr{A}}(t)$ is bounded, it then follows that $\hat{\mathscr{A}}^{-1}(t)$ is an exponentially stable operator (See for example Willems(1970) p 113). Then since $\hat{\mathscr{B}}$ is bounded it follows that $x(t)$ is bounded. Since $u(t) = \hat{P}v(t)$ we then have $v(t)$ and $y(t)$ bounded. Let $\Delta x = \bar{x} - x$, $\hat{\mathscr{B}} = \hat{\mathscr{B}} - \mathscr{B}$ and $\tilde{\mathscr{A}} = \hat{\mathscr{A}} - \mathscr{A}$. Then from (3.21) and (3.17) we have

$$\Delta x(t + 1) = \mathscr{A}\Delta x(t) - \tilde{\mathscr{A}}x(t) - \tilde{\mathscr{B}}y_r(t) \qquad (3.22)$$

Then since $\tilde{\mathscr{A}} \to 0$ and $\tilde{\mathscr{B}} \to 0$ (see Lemma 3.2) and $\mathscr{A}$ is exponentially stable, we have

$$\Delta x(t) \to 0$$

and so $y(t) \to \bar{y}(t)$.

▽▽▽

Lemma 3.2 shows that the precompensator will converge to the desired precompensator provided the system is sufficiently excited. Theorem 3.1 then establishes that the signals and the states are bounded and that the output will converge to the output that would be obtained if the system was known.

The need for persistent excitation in the above theory is for two reasons. Firstly, persistant excitation is one method of overcoming the stabilizability problem, present in all indirect adaptive control schemes. Secondly, and more peculiarly to the multivariable decoupling problem, one needs to ensure smoothness (at least asymptotically) of the control law with respect to the estimated parameters. This is difficult in the multivariable problem because of the processes of taking least common multiples and highest common factors. The algorithm presented in this paper has been structured so that these difficulties are overcome if one can ensure $\hat{\theta}(t) \rightarrow \theta$.

4. DISCUSSION OF THE ALGORITHM

The adaptive algorithm defined by the estimator (3.3) - (3.5) and steps a) - f) looks quite complicated at the first glance. It is divided into three parts:

- Estimation
- Design of the precompensator
- Design of SISO regulators

The estimation part is well suited for recursive computations. It is essentially m different estimators one for each output of the process. A stochastic noise part could also be included in the model. i.e., the model (3.1) is replaced by

$$A_i(q^{-1})y_i(t) = \sum_{j=1}^{m} B_{ij}(q^{-1})\, u_j(t) + C_i(q^{-1})\, e_i(t)$$

where e_i is normal distributed white noise. It is no loss of generality to assume that e_i is independent of e_j when $i \neq j$. The design of the precompensator and the design of the regulator will be the same as when (3.1) is used as the model.

The design of the precompensator requires more calculations as discussed in the end of Section 2. The determinant and the adjoint of $\tilde{B}$ have to be determined.

In steps c) and d) of the adaptive algorithm it is necessary to determine the common factor and the remained of two polynomials. This is conveniently done using the Euclidean algorithm. The adaptive algorithm is set up such that it is not necessary to separate stable and unstable parts of polynomials. For the design of the precompensator it is convenient to have good software for manipulation of polynomials.

The controller design is the same as for SISO systems. It is, however, necessary to make the design such that no unstable zeros are cancelled. Pole-placement using the Diophantine equation (2.10) is one such method. The $\hat{B}_i = \hat{B}_\ell \hat{B}_r$ must first be separated into the stable and unstable parts. The solution of (2.10) is essentially the same as solving a set of linear equations. The order of the problem is $\partial(R_i) + \partial(S_i) + 1$.

The computations required for the adaptive decoupling controller is more time consuming (but not much more) than for a SISO system of the same order. With the increasing power of microcomputers we consider it feasable to implement the adaptive decoupler.

In Singh and Narendra (1984) it is shown that necessary and sufficient conditions for the existence of a decoupling precompensator are the knowledge of the relative degrees and that the process is invertable. These conditions are in our case captured in Assumptions 2.1 and 2.2. The diagonalization of the A matrix is here done automatically through the identification while the diagonalization of the B matrix is obtained using the precompensator P in (2.6).

5. CONCLUSIONS

The adaptive decoupling problem has been solved for a general class of systems. It is believed that the design method is the first one that can give complete decoupling. Previously published methods only give approximate decoupling, see for instance McDermott and Mellichamp (1986). The design procedure of the precompensator also gives good insight into the problem of decoupling of MIMO systems. The procedure gives a parameterization of possible closed loop systems. The price for complete decoupling is determined by the diagonal matrices B_ℓ and B_r.

The computational burden for the design procedure is clearly within the capacity of today's microprocessors.

The theoretical analysis of the algorithm shows that convergence and stability is achieved through persistently excitation of the process.

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APPENDIX A

Proof of Lemma 3.2

To prove the lemma we need to formally define $\hat{\nu}_{ij}$ and $\hat{C}_{ij}$.

$$\hat{\nu}_{ij} \triangleq \max_{\nu \in Z^+} \{\partial(\hat{C}_{ij}) = \nu;\ \hat{C}_{ij} | \hat{D},\ \hat{C}_{ij} \text{ monic}$$

$$\text{and } \|\mathbb{R}(\hat{B}_{a_{ij}}, C_{ij})\| \leq \epsilon_1 < \epsilon\}$$

$$\hat{C}_{ij} \triangleq \arg\min_{\substack{\partial(\hat{C}_{ij}) = \hat{\nu}_{ij} \\ \hat{C}_{ij} | \hat{D} \\ \hat{C}_{ij} \text{ monic}}} \{\|\mathbb{R}(\hat{B}_{a_{ij}}, \hat{C}_{ij})\|\}$$

$\mathbb{R}(x,y)$ is the unique remainder satisfying

$$X = QY + \mathbb{R}(X,Y)$$

$$\partial(\mathbb{R}(X,Y)) < \partial(Y))$$

and

$$\|P\| \triangleq \|[p_o\ p_1\ \ldots\ p_n]\|$$

where

$$P(q^{-1}) = p_o + p_1 q^{-1} + \ldots + p_n q^{-n}$$

and $Y|X$ denotes $\mathbb{R}(X,Y) \equiv 0$, i.e. Y divides X.

(a) The assumption in Lemma 3.2 that $\hat{\theta} \to \theta$ is clearly equivalent to assuming $\hat{A} \to A$ and $\hat{B} \to B$. Part (a) of the Lemma follows since both the determinant and the adjoint of a matrix are continuous with respect to the matrix.

(b) Since $\hat{\nu}_{ij}$ and ν_{ij} are integers, we are in essence establishing that there exists a T such that $\forall t > T$, $\hat{\nu}_{ij}(t) = \nu_{ij}$. We first establish that a T exists such that $\forall t > T$, $\hat{\nu}_{ij}(t) \geq \nu_{ij}$.

In view of the definition of C_{ij} we have:

$$\mathbb{R}(B_{a_{ij}}, C_{ij}) \equiv 0 \tag{A.1}$$

Since the remainder is continuous with respect to the divisor and the dividend, it follows from (a) that

$$\lim_{t\to\infty} \mathbb{R}(\hat{B}_{a_{ij}}(t), C_{ij}) \to 0 \tag{A.2}$$

Since the roots of a polynomial are continuous with respect to the coefficients of the polynomial, it follows that there exists a $\hat{C}_{ij}$ of degree $\hat{\nu}_{ij}$ such that

$$\hat{C}_{ij} | \hat{D} \qquad \forall_{i,j} \qquad \forall t \tag{A.3}$$

$$\hat{C}_{ij} \text{ is monic}$$

$$\text{and} \qquad \hat{C}_{ij} \to C_{ij} \tag{A.4}$$

It thus follows that

$$\lim_{t\to\infty} \mathbb{R}(\hat{B}_{a_{ij}}(t), \hat{C}_{ij}(t)) = 0 \tag{A.5}$$

In view of (A.3) - (A.5) and part (c) of the adaptive control algorithm, it is clear that $\exists\, T_1$ such that $\forall t > T_1$,

$$\hat{\nu}_{ij}(t) \geq \nu_{ij} \qquad (A.6)$$

We now show (by contradiction) that $\hat{\nu}_{ij}(t)$ cannot be greater than ν_{ij} more than a finite number of times. Suppose $\hat{\nu}_{ij}(t) > \nu_{ij}$ at the infinite sequence of times,

$$t = \{k_1, k_2, k_3, \ldots\ldots\ldots\} \qquad (A.7)$$

From part (c) of the control algorithm, it follows that

$$\|\mathbb{R}(\hat{B}_{a_{ij}}(k_\ell), \hat{C}_{ij}(k_\ell)\| \leq \epsilon_1 \qquad \forall \ell \qquad (A.8)$$

and

$$\partial(\hat{C}_{ij}(k_\ell)) > \nu_{ij} \qquad \forall \ell \qquad (A.9)$$

Since $\hat{C}_{ij}(t)$ divides $\hat{D}(t)$ for all t and $\hat{D} \rightarrow D$ it then follows that must exist at least one infinite subsequence, $\{\bar{k}_1, \bar{k}_2, \ldots\ldots\}$ of $\{k_1, k_2 \ldots\}$ such that

$$\lim_{\ell \to \infty} \{\hat{C}_{ij}(\bar{k}_\ell)\} \text{ exists.}$$

Let

$$\lim_{\ell \to \infty} \{\hat{C}_{ij}(\bar{k}_\ell)\} = \bar{F}_{ij} \qquad (A.10)$$

Since $\hat{D} \to 0$, it follows that $\overline{F}_{ij} / D$. Also, $\hat{C}_{ij}(\overline{k}_\ell)$ is monic and of degree greater than ν_{ij}, so $\overline{F}_{ij}$ is monic and of degree greater than ν_{ij}. (A.8) then implies that

$$\|\mathbb{R}(B_{a_{ij}}, \overline{F}_{ij})\| \leq \epsilon_1 < \epsilon \tag{A.11}$$

This contradicts Assumption 3.3 regarding the true plant and so it follows that $\exists\ T_2$ such that

$$\hat{\nu}_{ij}(t) \leq \nu_{ij} \qquad \forall t > T_2 \tag{A.12}$$

Using (A.12) and (A.6) we see that for $T = \max\{T_1, T_2\}$,

$$\hat{\nu}_{ij}(t) = \nu_{ij} \qquad \forall t > T \tag{A.13}$$

Without loss of generality, we shall hereafter assume that $\hat{\nu}_{ij} = \nu_{ij}$. We now wish to establish that $\hat{C}_{ij} \to C_{ij}$. Arguing as before, i.e. $\hat{D} \to D$ and $\hat{C}_{ij}(t)$ divides $\hat{D}(t)$ for all t, it can be seen that there exists an infinite subsequence,

$$\{k_1, k_2, k_3 \ldots\ldots\} \subset N \tag{A.14}$$

such that

$$\lim_{\ell \to \infty} \{\hat{C}_{ij}(k_\ell)\} \text{ exists.}$$

Suppose

$$\lim_{\ell \to \infty} \{\hat{C}_{ij}(k_\ell)\} = \tilde{C}_{ij} \neq C_{ij} \tag{A.15}$$

Also, let $\hat{G}_{ij}$ denote the sequence defined by:

$$\lim_{t\to\infty} \{\hat{G}_{ij}(t)\} = C_{ij} \tag{A.16}$$

$\hat{G}_{ij}$ is monic

and
$$\mathbb{R}(\hat{D}, \hat{G}_{ij}) = 0 \tag{A.17}$$

Clearly, since $\hat{D} \to D$ and $\mathbb{R}(D, C_{ij}) = 0$ the above sequence exists. Note also from the adaptive control algorithm, (c),

$$\|\mathbb{R}(\hat{B}_{a_{ij}}, \hat{C}_{ij})\| \leq \|\mathbb{R}(\hat{B}_{a_{ij}}, \hat{G}_{ij})\| \tag{A.18}$$

Now

$$\lim_{t\to\infty} \mathbb{R}(B_{a_{ij}}(t), \hat{G}_{ij}(t)) = \mathbb{R}(\lim_{t\to\infty} \hat{B}_{a_{ij}}(t), \lim_{t\to\infty} \hat{G}_{ij}(t))$$

$$= \mathbb{R}(B_{a_{ij}}, C_{ij})$$

$$= 0 \tag{A.19}$$

From (A.18) and (A.19) we then see that

$$\lim_{t\to\infty} \mathbb{R}(\hat{B}_{a_{ij}}(t), \hat{C}_{ij}(t)) = 0 \tag{A.20}$$

and so

$$\lim_{\ell\to\infty} \mathbb{R}(\hat{B}_{a_{ij}}(k_\ell), \hat{C}_{ij}(k_\ell)) = 0$$

$$= \mathbb{R}(B_{a_{ij}}, \tilde{C}_{ij}) \quad (A.21)$$

(A.21) and (A.15) contradict the Assumption 3.3 regarding the true plant since the highest common multiple of $\hat{C}_{ij}$ and C_{ij} must divide $B_{a_{ij}}$. It thus follows that the only 'terms' which may occur in $\hat{C}_{ij}(t)$ an infinite number of times are the terms $\hat{G}_{ij}(t)$. Since there are only a finite number of possible choices for $\hat{C}_{ij}(t)$ which divide $\hat{D}(t)$, it can be shown that $\exists$ T such that $\forall t > T$, $\hat{C}_{ij}(t) = \hat{G}_{ij}(t)$ and thus

$$\hat{C}_{ij} \rightarrow \hat{G}_{ij} \rightarrow C_{ij} \quad (A.22)$$

(c) Normally, the process of taking the highest common factor (h.c.f.) of several polynomials is a discontinuous operation. In this case, however, we note that

$$\mathbb{R}(\hat{D}, \hat{C}_{ij}) \equiv 0 \qquad \forall t, \forall i, \forall j \quad (A.23)$$

Thus the h.c.f. required reduces to the problem of finding the following set:

$$S_i = \{\hat{z}: C_{1i}(\hat{z}) = 0, C_{2i}(\hat{z}) = 0, \ldots \hat{C}_{ni}(\hat{z}) = 0\} \quad (A.24)$$

$$= \hat{S}_{1i} \cap \hat{S}_{2i} \cap \ldots\ldots \cap \hat{S}_{ni} \quad (A.25)$$

where:

$$\hat{S}_{ji} = \{\hat{z}: \hat{C}_{ji}(\hat{z}) = 0\} \quad (A.26)$$

and the only values of $\hat{z}$ which need to be checked in any of the above sets are $\hat{z}$ such that $\hat{D}(\hat{z}) = 0$. Then, as shown in (b) we have

$$\hat{S}_{ji} \rightarrow S_{ji} \tag{A.27}$$

(where S_{ji} is the set of zeros of C_{ji}). Since the zeros of D (and hence C_{ji}) are distinct and $\hat{D} \rightarrow D$ it follows that the intersection performed in (A.25) is continuous and

$$\bar{S}_i \rightarrow S_i \tag{A.28}$$

i.e. $\bar{C}_i \rightarrow C_i$

(d) It now follows that $\hat{B}_r \rightarrow B_r$ and $\hat{P} \rightarrow P$. By assumption, $B_\ell B_r^-$ and A are coprime and so the solution to the Diophantine equation is (at least for all $t >$ some T) continuous.

▽▽▽