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ABSTRACT

It is shown that the limiting values of the extra zeros introduced through the sampling process are on the negative real axis.

1 Introduction

The zeros of sampled systems are discussed, for instance, in Åström-Hagander-Sternby (1984) and Edmunds (1976). There is, in general, no simple connection between the zeros of the continuous time system and the zeros of the sampled data system. The limiting values of the zeros when the sampling period goes to zero are given in Åström-Hagander-Sternby (1984). In this note some further properties of the limiting zeros are given. These properties are of interest when fast sampling and δ-operators are used, see Middleton and Goodwin (1984) and Goodwin-Leal-Mayne-Middleton (1986).

The contribution of this note is that the limiting extra zeros all are on the negative real axis. An alternative proof can be found in Mårtensson (1982).

The paper is organized in the following way. Section 2 gives the notations and defines the problem. A recursive way to define the limiting extra zeros are derived in Section 3 and a proof of the properties is given. Section 4 gives some implications and conclusions of the result. References are given in Section 5.

2 Problem Formulation

Let the continuous time system have the transfer function

$$G(z) = \frac{B_c(s)}{A_c(s)} \qquad (2.1)$$

where $\deg A_c = n$, $\deg B_c = m$, $r = n-m$. Assume that the pole excess $r > 0$. Sampling the system (2.1) using zero order hold gives the pulse transfer function

$$H(z) = \frac{B_d(z)}{A_d(z)}$$

where $\deg A_d = n$ and in general $\deg B_d = n-1$. This implies that $r-1 = n-m-1$ extra zeros are introduced through the sampling process. In Åström-Hagander-Sternby (1984) it is shown that as the sampling period h goes to zero m zeros of H go to 1 as $\exp(z_i h)$ where z_i are the zeros of $G(s)$. The remaining $n-m-1$ zeros of H go to the zeros defined by the polynomial $B_r(z)$. The polynomial B_r is defined as

$$B_r(z) = b_1^r z^{r-1} + b_2^r z^{r-2} + \ldots + b_r^r \tag{2.2}$$

where

$$b_k^r = \sum_{\ell=1}^{k} (-1)^{k-\ell} \ell^n \binom{n+1}{k-\ell} \qquad k = 1, \ldots, r \tag{2.3}$$

It can be shown that the coefficients of the polynomials B_r can be computed recursively

$$\begin{aligned} b_k^r &= k b_k^{r-1} + (r - k + 1) b_{k-1}^{r-1} \qquad k = 2, \ldots, r-1 \\ b_1^r &= b_r^r = 1 \end{aligned} \tag{2.4}$$

The first five B_r polynomials are

$$B_1(z) = 1$$
$$B_2(z) = z + 1$$
$$B_3(z) = z^2 + 4z + 1$$

$$B_4(z) = z^3 + 11z^2 + 11z + 1$$

$$B_5(z) = z^4 + 26z^3 + 66z^2 + 26z + 1$$

The polynomials B_r have zeros outside or on the unit circle for $r \geq 2$. The unstable zeros for some values of r are listed in Åström-Hagander-Sternby (1984). All numerical calculated zeros are on the negative real axis. The aim of this note is to show that this is true for a polynomial of general order defined by (2.2) and (2.3). In the derivation we need the notion of reciprocal polynomials. Given a polynomial $F(z)$ with $\deg F = \ell$ then the reciprocal polynomial $F^*(z)$ is defined as

$$F^*(z) = z^\ell F(z^{-1})$$

3 Properties of the Limiting Extra Zeros

The properties of the limiting zeros are given by the following theorem :

Theorem

The polynomials defined by (2.2). (2.3) and (2.4) have the following properties :

1. If z_i is a root of B_r then $1/z_i$ is also a root;
2. $B_r^*(z) = B_r(z)$;
3. There is a root in -1 when r even;
4. $B_r(z)$ has all roots on the negative real axis.

□

The proof of the theorem will be done through induction in the order r. The following lemma gives a recursive way of determining $B_r(z)$ given $B_{r-1}(z)$.

Lemma

Given a polynomial $B_{r-1}(z)$ such that

$$B^{*}_{r-1}(z) = B_{r-1}(z)$$

then the recursion (2.4) implies that

$$B_r(z) = D_{r-1}(z) + z\,D^{*}_{r-1}(z) \tag{3.1}$$

where

$$D_{r-1}(z) = \frac{d}{dz}\Big[z\,B_{r-1}(z)\Big] \tag{3.2}$$

Proof of lemma

The recursion formula (2.4) implies that B_r can be written as

$$B_r(z) = B_{r1}(z) + B_{r2}(z)$$

where

$$B_{r1}(z) = (r-1)b_1^{r-1}\,z^{r-2} + (r-2)b_2^{r-1}\,z^{r-3} + \ldots + 2b_{r-2}^{r-1}\,z + b_{r-1}^{r-1}$$

$$B_{r2}(z) = b_1^{r-1}\,z^{r-1} + 2b_1^{r-1}\,z^{r-2} + \ldots + (r-1)b_{n-1}^{n-1}\,z$$

But

$$B_{r1}(z) = \frac{d}{dz}\,(z\,B_{r-1}(z)) = D_{r-1}(z) \qquad \deg D_{r-1} = r-2$$

Since

$$B_{r-1}(z) = B^{*}_{r-1}(z) = b_{r-1}^{r-1}\,z^{r-2} + b_{r-2}^{r-1}\,z^{r-3} + \ldots + b_1^{r-1}$$

it follows that

$$D_{r-1}(z) = \frac{d}{dz}\,(z\,B_{r-1}\,(z)) = (r-1)b_{r-1}^{r-1}\,z^{r-2} + \ldots + b_1^{r-1}$$

and

$$B_{r2}(z) = z^{r-1} D_{r-1}(z^{-1}) = z D^*_{r-1}(z)$$

Finally

$$B_r(z) = D_{r-1}(z) + z D^*_{r-1} (z)$$

□

Example

Start with $B_3(z)$ then

$$D_3(z) = \frac{d}{dz} \left[z^3 + 4z^2 + z\right] = 3z^2 + 8z + 1$$

and

$$B_4(z) = 3z^2 + 8z + 1 + z \cdot z^2\left[3z^{-2} + 8z^{-1} + 1\right]$$

$$= z^3 + 11z^2 + 11z + 1$$

Further

$$D_4 = \frac{d}{dz} \left[z^4 + 11z^3 + 11z^2 + z\right] = 4z^3 + 33z^2 + 22z + 1$$

This gives

$$B_5(z) = 4z^3 + 33z^2 + 22z + 1 + z \cdot z^3\left[4z^{-3} + 33z^{-2} + 22z^{-1} + 1\right]$$

$$= z^4 + 26z^3 + 66z^2 + 26z + 1$$

$B_4(z)$ and $B_5(z)$ are the same as given above.

□

Proof of the theorem

1. Assume that z_i is a root of $B_r(z)$. The definition of $B_r(z)$ implies that $z_i \neq 0$. From (3.1) it follows that

$$B_r\left[z_i^{-1}\right] = D_{r-1}\left[z_i^{-1}\right] + z_i^{-1} D^*_{r-1}\left[z_i^{-1}\right]$$

$$= z_i^{-r+1}\left[z_i z_i^{r-2} D_{r-1}\left[z_i^{-1}\right] + z^{r-2} D^*_{r-1}\left[z_i^{-1}\right]\right]$$

$$= z_i^{-r+1}\left[z_i D^*_{r-1}\left[z_i\right] + D_{r-1}\left[z_i\right]\right] = 0$$

2. $$B^*_r(z) = z^{r-1}\left[D_{r-1}(z^{-1}) + z^{-1} D^*_{r-1}(z^{-1})\right]$$

$$= z D^*_{r-1}(z) + D_{r-1}(z)$$

$$= B_r(z)$$

3. From Property 2 it follows that the coefficients of $B_r(z)$ are "symmetric" i.e. if r is even

$$b_1^r = b_r^r$$
$$b_2^r = b_{r-1}^r$$
$$\vdots$$
$$b_{r/2}^r = b_{r/2+1}^r$$

This implies that $B_r(-1) = 0$ when r is even.

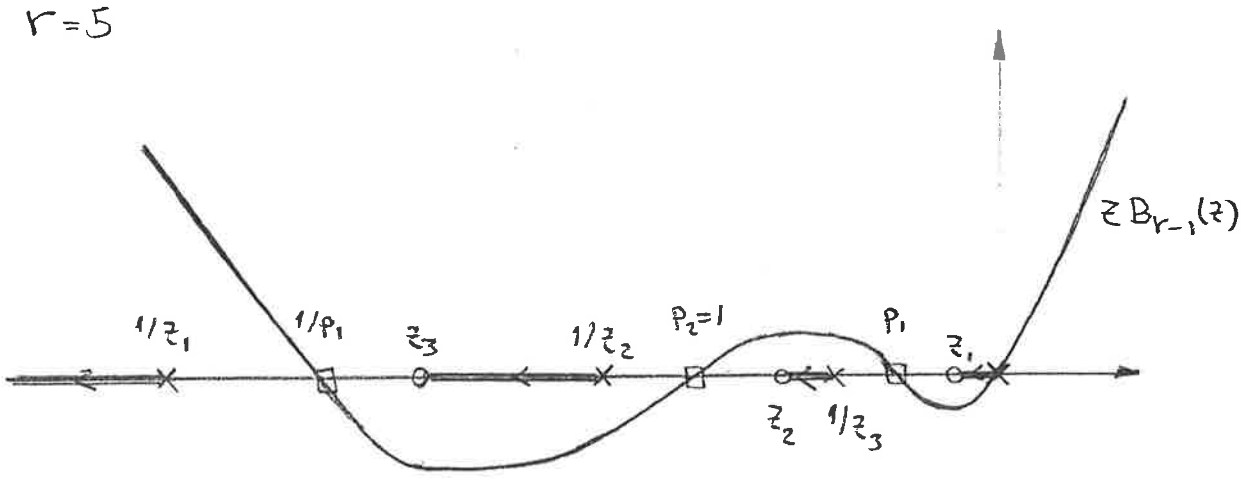
4. Consider a polynomial $B_{r-1}(z) = B^*_{r-1}(z)$ defined by (2.2) and (2.3) and with distinct roots on the negative real axis. We, then, want to show that $B_r(z)$ has the same properties. The $r-1$ roots of $z\, B_{r-1}$ divides the negative real axis into $r-1$ intervals. From the definition, it follows that D_{r-1} will have one root in

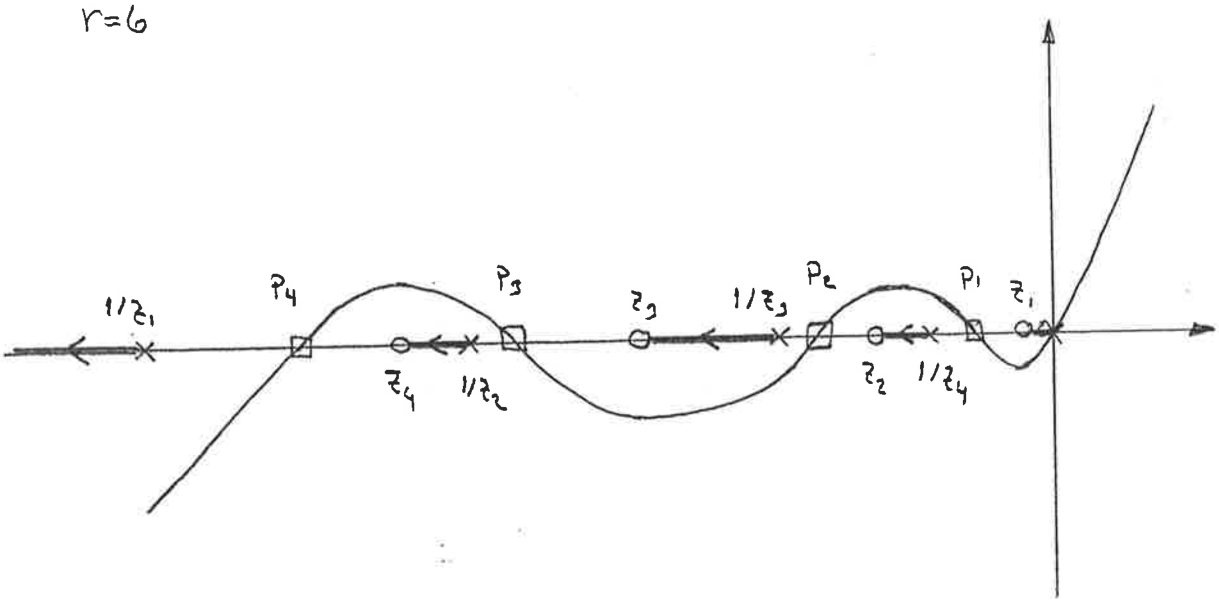
each of the intervals except the one furthest to the left. Further the roots of $D^*_{r-1}(z)$ will also be one in each interval except for the interval closest to the origin.

The patterns of the different zeros are illustrated in Figure 1. This implies that $z\,D^*_{r-1}(z)$ and $D_{r-1}(z)$ each has one root in all intervals except for the leftmost interval where $z\,D^*_{r-1}$ only has a root. Write (3.1) as

$$z\,D^*_{r-1}(z) + K \cdot D_{r-1}(z) = 0 \tag{3.3}$$

This equation can be regarded as a root locus equation and we find that all roots of (3.3) are on the negative real axis independent of the value of K. For our problem we are only interested in K=1.





□ roots of $B_{r-1}(z)$ p_i

× roots of $z\ D^{*}_{r-1}(z)$ $(1/z_i \quad i=1, \ldots, r-2, 0)$

0 roots of $D_{r-1}(z)$ $z_i \quad i=1, \ldots, r-2$

FIGURE 1 Root Patterns for the Different Polynomials Used to define $B_r(z)$. z_i are the roots of $D_{r-1}(z)$.

By induction we can then show that by starting with a B_{r-1} with the desired properties then also B_r will have the same properties and the proof of the theorem is completed.

□

Remark 1

It should be remarked that the ordering of the roots of $z\ D^{*}_{r-1}(z)$ and $D_{r-1}(z)$ is unimportant as long as there is only one root from each polynomial in each interval defined by the roots of $z\ B_{r-1}(z)$.

□

Remark 2

A similar idea with the root distribution in intervals can be found in Fuller (1955). The root locus argument is, however, believed to be new. □

4 Conclusions

The properties of the theorem are of interest when fast sampling is used. The poles and the zeros of the sampled system are continuous in the sampling period, at least for short sampling intervals. Using the result in this paper it is possible to obtain rough estimates of the zeros of the sampled data system. In the design of digital regulators it is important that zeros outside the unit circle are not cancelled. Cancellation of such zeros will lead to an unstable generation of the control signal. The experience from the design of digital controllers indicate that zeros on the negative real axis can be kept when determining the desired closed loop zeros. Further fast sampling is likely to occur if the δ-operator approach is used for the design of digital controllers. Still a further implication of the result is the possibility to incorporate *a priori* knowledge about the continuous time system in adaptive regulators.

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