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LUND UNIVERSITY

PO Box 117
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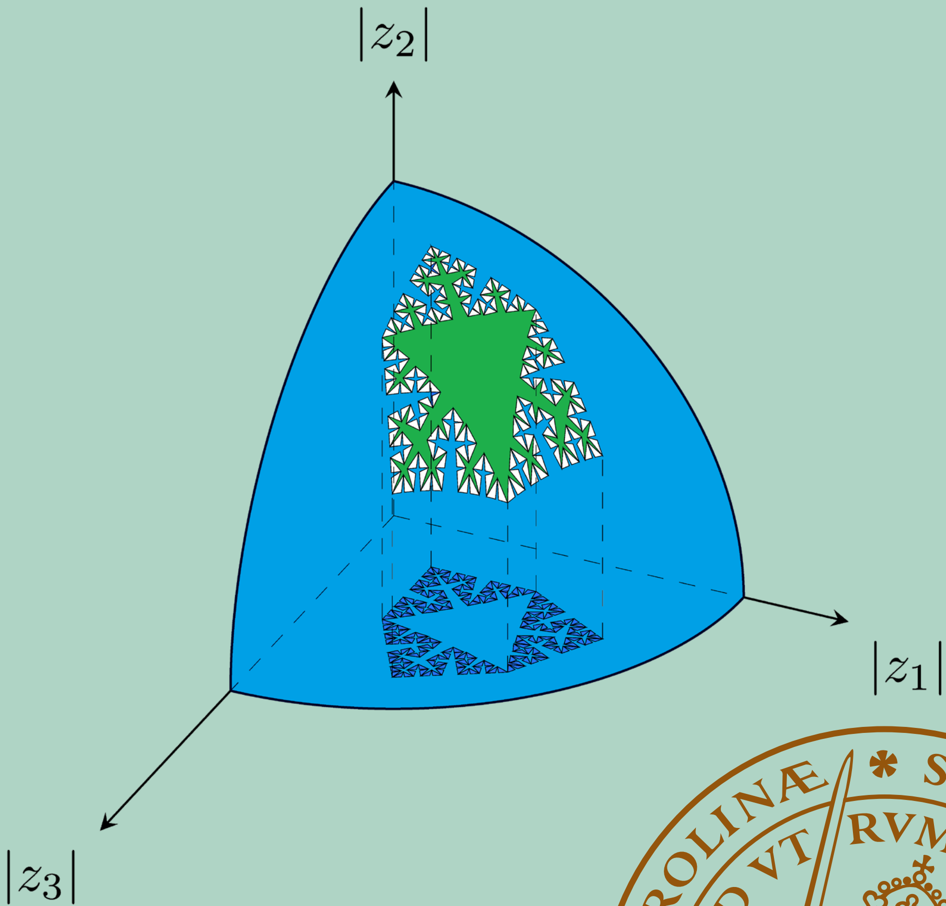
MÅRTEN NILSSON



Lund University
Faculty of Engineering
Centre for Mathematical Sciences
Mathematics

Boundary singularities of plurisubharmonic functions

Boundary singularities of plurisubharmonic functions

Mårten Nilsson

LUND
UNIVERSITY

Doctoral Thesis

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Thesis advisor
Docent Frank Wikström

Co-supervisors
Docent Jacob Stordal Christiansen & Docent Yang Xing

Faculty opponent
Professor Alexander Rashkovskii

Organization
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Centre for Mathematical Sciences
Box 118
SE–221 00 LUND
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Abstract
We study the Perron–Bremermann envelope $P(\mu,\varphi) \coloneqq \sup\{u(z)\ ;\ u \in \mathcal{PSH}(\Omega), (dd^c u)^n \geq \mu, u^* \leq \varphi\}$ on a B-regular domain Ω. Such envelopes occupy a central position within pluripotential theory as they, for suitable μ and φ harmonic and continuous on $\bar{\Omega}$, constitute unique solutions to the Dirichlet problem for the complex Monge–Ampère operator. Much is also known about the measures that guarantee that the solution is continuous, but the corresponding problems for unbounded or discontinuous φ have received very little attention. This is the main theme of this thesis.

In paper I and II, by adapting and expanding Leutwiler and Arsove's theory of quasibounded harmonic functions, we introduce a set of positive plurisubharmonic functions which may be approximated from below by functions in $L^\infty(\Omega) \cap \mathcal{PSH}(\Omega)$ outside a pluripolar set. This approximation property is exploited to show that $P(\mu,\varphi)$ is continuous outside a pluripolar set for a large class of measures, given that φ is bounded from below, is continuous in the extended reals, and have a non-trivial strong majorant, i.e. a plurisuperharmonic majorant whose singularities in a precise sense surpass those of φ. We also show that $P(\mu,\varphi)$ then corresponds to a unique solution to a Dirichlet problem with unbounded boundary data.

In paper III, we show that the Dirichlet problem is uniquely solvable for bounded boundary data with a b-pluripolar discontinuity set, by modifying an extended version of the comparison principle due to Rashkovskii. We also show that the discontinuity set being b-pluripolar is not necessary for the uniqueness. In particular, we construct a class of boundary data for which the Dirichlet problem is uniquely solvable, but where the Lebesgue measure of the set of discontinuities is positive.

In paper IV, we discuss two variations of Edwards' theorem. We prove one version of the theorem for cones not necessarily containing all constant functions, and in particular, we allow the functions in the cone to have a non-empty common zero set. In the other variation, we replace suprema of point evaluations and infima over Jensen measures by suprema of other continuous functionals and infima over a set measures defined through a natural order relation induced by the cone. As applications, we give some results on propagation of discontinuities for Perron–Bremermann envelopes on hyperconvex domains, as well as a characterization of minimal elements in the order relation mentioned above.

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– När ska vi börja?
– Det är inte så enkelt. Först måste du vara redo.
– Jag tror att jag är redo.
– Det här är inget skämt. Du får vänta tills det inte finns några tvivel alls, sedan får du möta honom.
– Måste jag förbereda mig?
– Nej. Du har bara att vänta. Efter ett slag kanske du avstår från hela idén. Du förtröttas lätt. Igår kväll var du färdig att ge upp så snart det blev lite besvärligt.

Carlos Castañeda
Samtalen med don Juan

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should awaken the proper associations. A special mention to the important *Lev najs nu*, providing me with more social activities than manageable.

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Abstract

We study the Perron–Bremermann envelope

$$P(\mu,\varphi) := \sup\{u(z)\ ;\ u \in \mathcal{PSH}(\Omega), (dd^c u)^n \geq \mu, u^* \leq \varphi\}$$

on a B-regular domain Ω. Such envelopes occupy a central position within pluripotential theory as they, for suitable μ and φ harmonic and continuous on $\bar{\Omega}$, constitute unique solutions to the Dirichlet problem for the complex Monge–Ampère operator. Much is also known about the measures that guarantee that the solution is continuous, but the corresponding problems for unbounded or discontinuous φ have received very little attention. This is the main theme of this thesis.

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In paper IV, we discuss two variations of Edwards' theorem. We prove one version of the theorem for cones not necessarily containing all constant functions, and in particular, we allow the functions in the cone to have a non-empty common zero set. In the other variation, we replace suprema of point evaluations and infima over Jensen measures by suprema of other continuous functionals and infima over a set measures defined through a natural order relation induced by the cone. As applications, we give some results on propagation of discontinuities for Perron–Bremermann envelopes on hyperconvex domains, as well as a characterization of minimal elements in the order relation mentioned above.

Populärvetenskaplig sammanfattning på svenska

I denna avhandling studeras en särskild kategori av *subharmoniska* funktioner, det vill säga reellvärda funktioner vars värde i en specifik punkt är mindre än eller lika med funktionens medelvärde på sfärer med centrum i punkten. Sådana funktioner återfinns på flera håll i vetenskapen. Ett exempel är temperaturen i ett bord efter att en värmeslinga slagits på längs bordets rand. I detta fall kommer värmefördelningen vid varje tidpunkt, betraktad som en funktion från ett område i planet till de reella talen, att utgöra en subharmonisk funktion. Ett annat exempel är den elektriska potentialen i ett elektrisk fält som bildas av ett antal positiva laddningar.

Subharmoniska funktioner är också intressanta rent matematiskt, och i synnerhet är teorin för dessa, *potentialteori*, på områden i det komplexa talplanet mycket rik. Funktionerna är då intimt förknippade med komplext deriverbara funktioner, så kallade *holomorfa funktioner*. Till exempel är realdelen, imaginärdelen, beloppet samt logaritmen av beloppet av en holomorf funktion alla subharmoniska. Detta synsätt ger också en teknisk fördel, eftersom subharmoniska funktioner till skillnad från holomorfa funktioner går att modifiera lokalt. I högre dimension, på områden i det komplexa rummet $\mathbb{C}^n$, gäller i hög grad samma förhållanden, om vi begränsar oss till de subharmoniska funktioner som respekterar den komplexa strukturen. I avhandlingen studeras sådana *plurisubharmoniska* funktioner.

Med exemplet med värmeslingan på randen av ett bord i håg kan man fundera över villkor man kan ställa på en plurisubharmonisk funktion som gör den unikt bestämd, i analogi med att en temperaturfördelning i jämvikt enbart bör bero på värmeslingans temperatur och hur värme kan lämna bordet (t ex via luften). Sådana matematiska krav återfinns i *Dirichletproblemet för den komplexa Monge–Ampère-operatorn*,

$$\begin{cases} u \text{ plurisubharmonisk och begränsad,} \\ u = \phi \text{ på randen,} \\ (dd^c u)^n = \mu \end{cases}$$

där $(dd^c u)^n$ är u:s *komplexa Monge–Ampère-mått*, μ är ett fixerat positivt mått, och ϕ är en funktion definierad på områdets rand. Detta problem är mycket välstuderat, och numera vet man att det har en unik lösning, som dessutom är kontinuerlig, om ϕ är kontinuerlig för en stor uppsättning komplexa Monge–Ampère-mått. Här ska det också nämnas att liknande uppställningar, så kallade *komplexa Monge–Ampère-ekvationer*, faktiskt uppkommer i strängfysiken, i frågor som rör hur de extra dimensionerna som man inför där kan se ut rent geometriskt.

Det centrala temat i avhandlingen är att på olika sätt mildra kravet att ϕ ska vara kontinuerlig, och att i dessa fall undersöka huruvida en lösning existerar, om den är unik, samt om och var den är kontinuerlig. Hur diskontinuerlig kan vi låta ϕ vara? Går det att säga någonting om ϕ är obegränsad, ”oändlig” i någon punkt? För att kunna göra detta omformuleras en del resultat i en komplex variabel, där dessa frågor är mer utredda, på ett sådant sätt att de är generaliserbara till högre dimensioner.

List of papers

This thesis is based on the following papers:

I **Quasibounded plurisubharmonic functions**

M. Nilsson, F. Wikström
Internat. J. Math., Vol. 32, No. 9 (2021).

II **Continuity of envelopes of unbounded plurisubharmonic functions**

M. Nilsson
Math. Z. 301 (2022).

III **Plurisubharmonic functions with discontinuous boundary behavior**

M. Nilsson
To appear in Indiana Univ. Math. J.

IV **Variations on a theorem by Edwards**

M. Nilsson, F. Wikström
Preprint (2023).

The papers as they appear in this thesis might differ slightly from the corresponding published versions.

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Introduction

1 Background

This thesis addresses problems within pluripotential theory, which generalizes potential theory in the plane (i.e. on $\mathbb{C}$) to several complex variables. Potential theory is an old branch of mathematics with its roots in classical physics, in particular in electrostatics and in Newton's theory of gravity. At its core, potential theory is the study of *subharmonic functions*, i.e. upper semicontinuous functions u satisfying the local submean value property

$$u(x_0) \leq \frac{1}{\operatorname{Vol} B(x_0, r)} \int_{B(x_0,r)} u \, dV$$

for $r > 0$ small enough. Equivalently, these are the upper semicontinuous functions for which the Laplacian

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

(possibly in a generalized sense) is non-negative. A subharmonic function for which the Laplacian vanishes is called *harmonic*. This turns out to be a quite strong condition, for example implying that the function is smooth. Though the physical scope of this operator is vast, including descriptions of equilibria and waves as a component in both the *heat equation*

$$\Delta T = \frac{\partial T}{\partial t},$$

and the *wave equation*

$$\Delta u = \frac{\partial^2 u}{\partial t^2},$$

it is perhaps most instructive from the viewpoint of potential theory to explain how it arises within electrostatics. In a perfect vacuum, a charge distribution ρ generates an electric flux $\vec{E}$ associating a vector to each point in space and satisfying *Gauss' law* in the form

$$\oint_{\partial K} \vec{E}\, dA = \int_K \rho\, dV,$$

where K is some region of space with boundary ∂K. The physical interpretation of this equation is that the amount of charge within K equals the electric flux that "escapes" through the boundary of K. This equation may be rewritten in differential form as

$$\nabla \cdot \vec{E} = \rho,$$

using the divergence theorem. On the other hand, $\vec{E}$ is a *conservative vector field*; this corresponds to the fact that the amount of work needed to move a charge between two points is independent of the path travelled. A mathematical consequence is that a potential function ϕ for $\vec{E}$ may be defined, in other words a scalar field satisfying

$$\nabla \phi = \vec{E},$$

which implies that

$$\Delta \phi = \nabla \cdot \nabla \phi = \rho.$$

Hence, a subharmonic function corresponds to the electric potential energy generated by a distribution of positive charges. This physical interpretation was a guidance to several important mathematical concepts. A central insight is that given a fixed compact set K, any amount of positive charge will distribute itself as to minimize the total energy. Importantly, this gave rise to the notion of the *equilibrium measure* of K, used to define *polar sets* (a set so small that it only supports charge distributions associated with infinite energy) and the capacity of a set. This viewpoint allows for a quantification of sets complementary to the methods of measure theory.

The adjective "harmonic" originates from the mathematical description of a vibrating string, i.e. a string undergoing harmonic motion. Modeling this situation as a differential equation, the solution may be written as a sum of sines and cosines (sines and cosines hence called "harmonics"), and this sort of representation extends to periodic functions on the real line (which may be seen as functions on a circle) using Fourier analysis. Analogously, considering functions on an n-sphere, one arrives at a representation using the spherical harmonics. These functions have vanishing Laplacian, and over time all functions with vanishing Laplacian came to be known as "harmonic". Subharmonic functions are, as the name suggests, functions that are below the harmonic functions in the following sense: given a harmonic function $h : \Omega \to \mathbb{R}$, we have

$$u \le h \text{ on } \partial U \implies u \le h \text{ on } U$$

for every relatively compact open set $U \subset \Omega$ and any upper semicontinuous function $u : \overline{U} \to \mathbb{R} \cap \{-\infty\}$ which is subharmonic on U. In fact, this maximality property characterizes the harmonic functions among the subharmonic functions.

Since the Laplacian is linear, the space of harmonic functions on a open set is a vector space. The set of subharmonic functions on the other hand, is a *convex cone*, which means that for $a, b \geq 0$,

$$u, v \text{ subharmonic} \quad \Longrightarrow \quad au + bv \text{ subharmonic.}$$

The subharmonic functions also carry a lattice property,

$$u, v \text{ subharmonic} \quad \Longrightarrow \quad \max\{u, v\} \text{ subharmonic,}$$

and any function defined as a decreasing sequence of subharmonic functions is subharmonic. Furthermore, all subharmonic functions on a bounded domain necessarily satisfy the following two properties:

(a) If a subharmonic function attains a global maximum at an interior point, then it is constant.

(b) If a subharmonic function is non-positive on the boundary of the domain, then it is non-positive in the interior as well.

Maximum principles such as these two statements have wide application, and instill intuition for how subharmonic functions may behave. It should also be mentioned that the convex functions also have the above properties. Convex functions are in fact subharmonic as well, which should not come as a surprise as subharmonicity is defined in terms of a local submean value property over balls, where as convexity involves a local submean value property on lines.

Mathematically, the above discussion is valid for $\mathbb{R}^n$, but it turns out that potential theory in the plane stands out as particularly interesting, owing in part to the large set of conformal mappings available compared to other dimensions. In particular, subharmonicity is preserved under biholomorphic mappings, but the connection between potential theory in the plane and complex analysis is even more intimate than immediately discernible from this fact. For example, given a holomorphic function f, the functions $\log|f|$ and $|f|^\alpha$ for $\alpha > 0$ are all subharmonic, and $\mathrm{Re} f$ and $\mathrm{Im} f$ are both harmonic. These connections allow for a cross-pollination between potential theory and complex analysis. In particular, many properties of holomorphic functions are in fact inherited from harmonic functions or even subharmonic ones. Another advantage of potential theory is that subharmonic functions are more flexible to work with as they may be modified locally, allowing for a multitude of applications. Examples include the Riemann mapping theorem (including continuity at the boundary), the Koebe one-quarter theorem, and a sharp quantitative form of Runge's theorem (the Bernstein–Walsh theorem). For more details on this matter, see Ransford (36).

One immediately encounters problems when trying to extend this intimate relation to several complex variables. A first stumbling block is the fact that in general, precomposing a subharmonic function on some open subset of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ with a biholomorphic mapping does *not* produce another subharmonic function. Instead, one only considers *plurisubharmonic functions*, i.e. the subharmonic functions whose compositions with biholomorphic mappings are still subharmonic. Furthermore, we say that a function is *pluriharmonic* if it is both plurisubharmonic and harmonic. These notions extend the relations found in the single variable case to a considerable degree; for example, the real part of a holomorphic function is pluriharmonic, and the logarithm of its modulus is plurisubharmonic. Surprisingly, it is actually enough to consider the subharmonic functions that are subharmonic under all complex linear changes of the coordinates; already such functions are necessarily plurisubharmonic (see (22, Proposition 1.45)). Another defining property of plurisubharmonicity is that the local submean property holds on all complex lines, i.e.

$$z \mapsto u(a + bz)$$

is subharmonic for $z \in \mathbb{C}$, and every $a, b \in \mathbb{C}^n$ such that $a + bz$ lies in the domain of u. Pluriharmonicity is equivalent to all these mappings being harmonic. Here, one should once again make a comparison with the convex functions, functions for which the local submean property holds on all *real* lines. Remarkably, one may alternatively characterize the convex functions as the subharmonic functions that remain subharmonic under all *real* linear changes of the coordinates. Hence, plurisubharmonicity is a natural complex counterpart to convexity.

There are several deep connections between plurisubharmonic functions and holomorphic functions of several variables. Firstly, it should be mentioned that plurisubharmonic functions play an important rôle in a historically central problem in the theory of several complex variables: *The Levi problem*, concerning the classification of domains of holomorphy. This problem was resolved around 1940, when Oka (35) proved that a domain Ω is a domain of holomorphy if and only if it is *pseudoconvex*, i.e. if there exists a continuous plurisubharmonic function φ on Ω such that the set

$$\{z \in \Omega\ ;\ \varphi(z) < c\}$$

is relatively compact for every $c \in \mathbb{R}$. A different proof was provided in the sixties by Hörmander (23), as a consequence of his solution to the $\bar{\partial}$-problem.

Secondly, on domains of holomorphy, the function theories of plurisubharmonic and holomorphic functions are connected in rather striking ways. As noted earlier, a holomorphic function h gives rise to a set of plurisubharmonic functions of the form $c \log |h|$, where $c > 0$. Bremermann (7) provided an elegant argument which shows that any plurisubharmonic function u may be constructed as the upper semicontinuous envelope of the upper limit of a sequence of plurisubharmonic functions of this form. More precisely, u may be written

$$u = \left(\limsup_j \frac{\log |h_j|}{j} \right)^*$$

for some sequence of holomorphic functions h_j. There is also a connection between zero sets of holomorphic functions and singularities of plurisubharmonic functions, due to Bombieri (6) and Hörmander (24): Given a plurisubharmonic function u defined on a domain of holomorphy, the set of points of non-integrability of e^{-u} is an analytic variety. See Kiselman (26) for more details.

One major theme in potential theory is the classical *Dirichlet problem* for the Laplace operator, i.e. the problem of finding a harmonic function with prescribed boundary values, which is always (uniquely) solvable if the domain and the boundary data satisfy some reasonable assumptions. If one naively tries to extend such results to the setting of several complex variables, it becomes painfully evident that pluriharmonic functions do not come in abundance. In particular, Bremmerman (8) showed in the fifties that the *Perron method* does not in general produce a pluriharmonic solution. The procedure does however result in a plurisubharmonic function which carries a certain maximality property akin to harmonic functions, suggesting the existence of an operator for which these *maximal* plurisubharmonic functions indeed do solve a Dirichlet problem. Before giving a more extensive account, we will now, for later comparison, review the classical Dirichlet problem for domains in $\mathbb{C}$.

2 The Dirichlet problem in the complex plane

The material in this section may be found in Ransford (36) and Garnett (19), see also Evans (16). For any (bounded) domain $\Omega \subset \mathbb{C}$, one can consider the Dirichlet problem for the Laplace operator on Ω, namely if it is possible to find a function $h : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} \Delta h = 0 \\ \lim_{w \to z} h(w) = \varphi(z) \qquad \forall z \in \partial\Omega \end{cases}$$

for a given function $\varphi \in C(\partial\Omega)$. That a solution to this problem necessarily is unique is a straightforward consequence of the maximum principle. Indeed, suppose that we have two solutions h_1, h_2. Then $\pm(h_1 - h_2)$ are harmonic as well, extending continuously to zero on the boundary. By the maximum principle,

$$\pm(h_1 - h_2) \leq 0 \text{ on } \Omega,$$

and so $h_1 = h_2$.

There are several ways to establish existence of a solution, given some assumptions on Ω. One way that generalizes to many other similar problems is to consider the envelope of subsolutions, a procedure known as the *Perron method*. In our context, due to the maximality property of harmonic functions, subharmonic functions smaller than φ on the boundary will play the part of subsolutions. Our solution candidate will be the *Perron envelope*

$$P(\varphi)(z) := \sup\{u(z) \; ; \; u \in \mathcal{SH}(\Omega), u^*|_{\partial\Omega} \leq \varphi\}.$$

In fact, one can show that this envelope always is harmonic, and furthermore that it also attains φ as its boundary values given that Ω is *regular*. This means that at all $\zeta_0 \in \partial\Omega$, there exists a neighborhood N and a subharmonic function b defined on $\Omega \cap N$ (a *barrier at* ζ_0) such that

$$b < 0 \text{ on } \Omega \cap N \quad \text{and} \quad \lim_{z \to \zeta_0} b(z) = 0.$$

Regularity of the domain is sufficient and necessary for the Dirichlet problem to be solvable for all $\varphi \in C(\partial\Omega)$.

It is however possible to relax the regularity condition, if one accepts that the boundary values are not attained at every point, but *nearly everywhere*. This means outside a polar set, in other words a set $P \subset \mathbb{C}$ for which there exists a subharmonic function u on $\mathbb{C}$ such that

$$P \subset \{z \in \mathbb{C} \; ; \; u(z) = -\infty\}.$$

Polar sets necessarily have Hausdorff dimension zero, so they are in a sense very small. These notions allow us to formulate a generalized Dirichlet problem which is solvable for any domain $\Omega \subset \mathbb{C}$ with non-polar boundary (under the convention that $\infty \in \partial\Omega$ if Ω is unbounded): For $\varphi : \partial\Omega \to \mathbb{R}$ that is bounded and continuous n.e. (nearly everywhere), there exists a unique bounded harmonic function h such that

$$\lim_{z \to \zeta \in \partial\Omega} h(z) = \varphi(\zeta) \qquad \text{n.e.}$$

On the other hand, if $\partial\Omega$ is polar, then all bounded harmonic functions on Ω are constants; the assumption of having non-polar boundary should therefore not be seen as a great restriction.

For example, all bounded domains have non-polar boundary. There is then a unique solution h_φ corresponding to $\varphi \in C(\partial\Omega)$, which allows us to define

$$\begin{aligned} T_z : C(\partial\Omega) &\to \mathbb{R} \\ \varphi &\mapsto h_\varphi(z). \end{aligned}$$

By the maximum principle, these are positive linear functionals, so by the Riesz representation theorem, there exists a probability measure $\omega(z, \Omega)$ such that

$$h_\varphi(z) = \int_{\partial\Omega} \varphi(y)\, d\omega(z, \Omega)(y).$$

The measure $\omega(z, \Omega)$ is called the *harmonic measure* (of the domain Ω with pole at z). We will now continue by considering a situation where a explicit formula for the harmonic measure is available, allowing us to push the generalization of the Dirichlet problem even further.

2.1 Integral representation in the unit disk

In the unit disk, the harmonic measure is (mutually) absolutely continuous with respect to arc length, and may be written

$$h_\varphi(z) = \int_{\partial\Omega} \varphi(y)\, d\omega(z, \Omega)(y) = \int_0^{2\pi} \varphi(e^{i\theta}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi}.$$

This is the *Poisson integral* of φ, predating the concept of harmonic measure by a century. Notably, this formula continues to produce harmonic functions if we replace the arc length with any probability measure μ on ∂D, and replace φ with any element in $L^1(\partial D, \mu)$. Our previous considerations established a correspondence between continuous functions on the boundary and harmonic functions continuously extendable to the boundary. Is it similarly possible to assign a boundary measure to any harmonic function h?

Given some assumptions on h, the answer is yes! If h is bounded, then the sequence $\varphi_n(e^{i\theta}) = h(r_n e^{i\theta})$ will have a weak star cluster point $\varphi \in L^\infty(\partial D)$ by the Banach–Alaoglu theorem. Since φ_n are bounded from above, the dominated convergence theorem implies that the Poisson integrals of φ_n, viz. $h(r_n e^{i\theta})$, converge to the Poisson integral of φ. Bounded harmonic functions on D are therefore in one-to-one correspondence with bounded Borel functions on ∂D, and we may view h as the unique solution to a Dirichlet problem with φ as boundary data. Also, by Fatou's theorem, the *non-tangential limit* will converge to φ almost everywhere (outside a set of Lebesgue measure zero). Obviously, we are not guaranteed that

$$\lim_{z \to \zeta} h(z) = \varphi(\zeta)$$

for any point $\zeta \in \partial D$ unless φ is continuous at ζ, but assuming that φ is continuous outside a set A_φ of Lebesgue measure zero, there exists a unique bounded harmonic function h such that

$$\lim_{z\to\zeta\in\partial D} h(z) = \varphi(\zeta) \quad \forall \zeta \in \partial D \setminus A_\varphi.$$

Since there are non-polar sets of Lebesgue measure zero, this includes the previous result, which only allowed for a polar discontinuity set. Finding a higher dimensional analogue of this result will be the overarching theme of Section 9–11.

Some classes of unbounded harmonic functions on the unit disk are also represented by integrals. For positive harmonic functions, there is the *Herglotz–Riesz representation theorem*: h is a positive harmonic function on D with $h(0) = K$ if and only if there exists a measure μ on ∂D such that $\mu(\partial D) = K$ and

$$h(z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, d\mu(e^{i\theta}).$$

Implicit in the works of Arsove and Leutwiler (1), there are several equivalent criteria on h which captures when μ is absolutely continuous with respect to arc length, for example that it is possible to approximate h from below with an increasing sequence of bounded harmonic functions. Indeed, let $h_n \nearrow h$ be as sequence of bounded harmonic functions converging to h, and let φ_n be the associated functions in $L^\infty(\partial D)$. By the monotone convergence theorem, we then have $\varphi_n \nearrow \varphi \in L^1(\partial D)$. For the other direction, suppose that h is represented by $\varphi \in L^1(\partial D)$. Letting $\varphi_n := \min\{\varphi, n\} \in L^\infty(\partial D)$, and denoting the corresponding bounded harmonic functions by h_n, it follows from the dominated convergence theorem that $h_n \nearrow h$. Importantly, this show that the Dirichlet problem with positive boundary data in $L^1(\partial D)$ is uniquely solvable in the class of such *quasibounded* harmonic functions.

Several aspects of this setup also generalize to the non-linear theory of higher dimensions, where no integral representation is available. This is the main theme of Section 5 and 6. Leaving the topic of singular boundary values aside for the moment, we will now give a brief introduction to a higher (complex) dimensional analogue of the Laplacian, and an overview of results concerning its associated Dirichlet problem for continuous boundary data.

3 The complex Monge–Ampère operator

In a series of seminal papers published around 1980, Bedford and Taylor (2; 3) introduced the *complex Monge–Ampère operator*, which in many ways plays the rôle of the Laplacian in pluripotential theory. A motivation may be found in the following facts. For a twice continuously differentiable plurisubharmonic function u defined on a domain $\Omega \subset \mathbb{C}^n$, i.e. $u \in C^2(\Omega) \cap \mathcal{PSH}(\Omega)$, the complex Hessian

$$Hu = \begin{bmatrix} \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_1} & \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_2} & \cdots & \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_n} \\ \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_1} & \frac{\partial^2 u}{\partial z_2 \partial \bar{z}_2} & \cdots & \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial z_n \partial \bar{z}_1} & \frac{\partial^2 u}{\partial z_n \partial \bar{z}_2} & \cdots & \frac{\partial^2 u}{\partial z_n \partial \bar{z}_n} \end{bmatrix}$$

is positive semi-definite, where

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$$
$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

are Wirtinger derivatives. The attentive reader should note that this formalism very succinctly conveys that the plurisubharmonic functions constitute the complex analogue to the convex functions. Now suppose that there exists a point $z_0 \in \Omega$ where $\det(Hu(z_0)) > 0$. Then u cannot satisfy the following *maximality property* on Ω:

$$v \in \mathcal{PSH}(G) \cap \mathcal{USC}(\bar{G}) \text{ and } v \le u \text{ on } \partial G \implies v \le u \text{ in } G,$$

for every relatively compact open subset $G \subset \Omega$. To see this, pick a neighborhood U of z_0 on which $\det(Hu) > 0$, and let ϕ be a smooth real valued cutoff function defined on U such that $\phi(z_0) > 0$. For $\varepsilon > 0$ small enough, $v := u + \epsilon\phi$ will be plurisubharmonic, which contradicts the above maximality property.

As the Perron method in pluripotential theory (i.e. constructing the *Perron–Bremermann envelope* of all plurisubharmonic functions with smaller boundary values) in general produces a function satisfying the maximality property, one could ask if it is possible to extend $u \mapsto \det(Hu)$ to functions that are not necessarily in $C^2(\Omega)$. Bedford and Taylor (2) managed to do just that, extending the domain of definition of the operator to all locally bounded plurisubharmonic functions. We will now provide a sketch of their construction. For more details, we refer to the recent textbook by Guedj and Zeriahi (22).

A complex *current* of bidegree (p, q) on $\mathbb{C}^n$ is a differential form

$$T = \sum_{|I|=p, |J|=q} T_{I,J} dz_I \wedge d\bar{z}_J$$

where the coefficients $T_{I,J}$ are distributions. We say that a current of bidegree (p,p) is positive if $T = \overline{T}$ and

$$T \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_{n-p} \wedge \bar{\alpha}_{n-p} = \lambda \beta^n,$$

where λ is a positive distribution,

$$\beta^n = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

and $\{\alpha_i\}_{1 \le i \le n-p}$ are differential test forms of bidegree $(1,0)$. In the spirit of the Riesz representation theorem, the distribution coefficients of a positive current are in fact complex measures. Furthermore, by defining $d^c := i(\bar{\partial} - \partial)$, one may show that $dd^c u = 2i\partial\bar{\partial}u$ extends to a positive current by duality. Note that for smooth u,

$$(dd^c u)^n := \overbrace{dd^c u \wedge \cdots \wedge dd^c u}^{n \text{ times}} = n!\, 4^n \det(Hu)\, \beta^n.$$

Here, we should point out that the precise coefficient preceding the determinant varies in the literature, depending on how one chooses to define d^c. For u locally bounded and T positive and closed, we define

$$dd^c u \wedge T := dd^c(uT),$$

which also turns out to be a closed, positive current. Continuing iteratively, we obtain

$$(dd^c u)^n = \mu,$$

where μ is a positive measure, not necessarily absolutely continuous with respect to the Lebesgue measure. This is the complex Monge–Ampère operator in the sense of Bedford and Taylor. Building on earlier work by Chern, Levine and Nirenberg (12) and Goffman and Serrin (20), they were able to show that this operator is continuous with respect to monotone sequences, and Demailly (13) showed that their construction also encompasses all plurisubharmonic functions with compact singularity set. Important problems in pluripotential theory include investigating the precise domain and range of this operator. See Błocki (5), Cegrell (11) and Kołodziej (30) for more details.

4 Complex Monge–Ampère equations

A very important tool of the Bedford–Taylor theory is *the comparison principle*,

$$\liminf_{z \to \partial\Omega} (u(z) - v(z)) \ge 0 \text{ and } (dd^c u)^n \le (dd^c v)^n \implies v \le u \text{ in } \Omega$$

for $u, v \in L^\infty(\Omega) \cap \mathcal{PSH}(\Omega)$. This principle will to a large extent replace the maximum principle when dealing with the complex Monge–Ampère equation in $\Omega \subset \mathbb{C}^n$ for $n > 1$. In particular, it yields an uniqueness argument for the (possibly inhomogeneous) Dirichlet problem

$$\begin{cases} u \in L^\infty(\Omega) \cap \mathcal{PSH}(\Omega) \\ (dd^c u)^n = \mu \\ \lim_{w \to z} u(w) = \varphi(z) \qquad \forall z \in \partial\Omega, \end{cases}$$

where μ is a positive measure and $\varphi \in C(\partial\Omega)$. The comparison principle also implies that the functions for which the Monge–Ampère measure vanishes are indeed maximal objects, at least in $L^\infty(\Omega) \cap \mathcal{PSH}(\Omega)$. In analogy with ordinary potential theory, this means that the solution must coincide with the Perron–Bremermann envelope

$$P(\varphi, \mu) := \sup\{u(z) \ ; \ u \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega), (dd^c u)^n \geq \mu, u^*|_{\partial\Omega} \leq \varphi\}$$

if it exists. Prior to the works of Bedford and Taylor, Bremermann (8) and Walsh (40) studied envelopes of the form

$$P(\varphi) := \sup\{u(z) \ ; \ u \in \mathcal{PSH}(\Omega), u^* \leq \varphi\}$$

for an extended real-valued function φ, sometimes requiring that $\varphi \in C(\bar{\Omega})$ or that φ is harmonic. If the domain and the dominating function are regular enough, they managed to prove that the envelope indeed attains the sought boundary values and that it is continuous on $\bar{\Omega}$. In this setting, $u^* \leq \varphi$ may be translated into a condition on the boundary as in the definition of $P(\varphi, \mu)$ by assuming that φ is harmonic or identically equal to $+\infty$ on Ω, thereby essentially corresponding to the case $\mu = 0$. Their results showed that whether or not the envelope attains the same boundary values as φ heavily depends on the domain. Over the years, several types of domains have been introduced which to different degrees secure this property.

4.1 Strictly pseudoconvex, B-regular and hyperconvex domains

The most well-behaved class of domains one usually considers in pluripotential theory are the *strictly pseudoconvex* domains. The archetypal example is the unit ball in $\mathbb{C}^n$. Though the precise definition varies slightly, one definition (27) is that

$$\Omega = \{z \in \mathbb{C}^n \ ; \ \rho(z) < 0\},$$

where the defining function ρ is a twice continuously differentiable plurisubharmonic function with its complex Hessian being positive definite on a neighbourhood of $\bar{\Omega}$, such that $d_z\rho \neq 0$ on $\partial\Omega$. These conditions are sufficient to ensure that $P(\varphi)$ attains the same boundary values as φ if $\varphi \in C(\bar{\Omega})$. Many classic results of Bedford–Taylor, Bremermann and Walsh are formulated in terms of strictly pseudoconvex domains, and they find their use in more recent applications as well (29; 30).

However, for any bounded domain Ω with $\varphi \in C(\bar{\Omega})$, Walsh proved that *if* $P(\varphi)$ is continuous at $\partial\Omega$, then necessarily $P(\varphi) \in C(\bar{\Omega})$. Further, he showed that that this holds for all $\varphi \in C(\bar{\Omega})$ if every point $\zeta \in \partial\Omega$ admits a (strong) *plurisubharmonic barrier*. This means that there is a neighborhood N of ζ and a negative plurisubharmonic function u defined on $N \cap \Omega$ which has the limit 0 at ζ, and is bounded away from zero outside any neighborhood of ζ. This suggests the following definition due to Sibony (39). We say that a bounded domain is *B-regular* if every boundary point admits a strong plurisubharmonic barrier. Sibony proved that that this condition is equivalent to requiring that for every $\varphi \in C(\partial\Omega)$, there exists a continuous plurisubharmonic function coinciding with φ on $\partial\Omega$. As these conditions are necessary and sufficient for the Dirichlet problem with $\mu = 0$ to always have a solution for $\varphi \in C(\partial\Omega)$, B-regular domains are often the stage concerning more recent results in pluripotential theory.

Lastly, we mention the *hyperconvex* domains. These are for example considered when one is not too concerned with how the boundary data influence the envelope $P(\varphi, \mu)$, instead focusing on the interplay with the measure μ. This setting is more similar to pluripotential theory on compact Kähler manifolds, where a boundary is altogether absent. A definition of hyperconvexity is that there exists a negative, continuous plurisubharmonic function that vanishes on the boundary, albeit local characterizations are also possible (25; 41). One often restricts the analysis to certain classes of negative plurisubharmonic functions, which allows for partial integration with respect to dd^c, and importantly an extension of the Monge–Ampère operator which in a certain sense is the best possible. See Cegrell (11) for more details. An important example of a hyperconvex domain which is not B-regular is the unit polydisk.

4.2 Existence and continuity of the solution

Bedford and Taylor showed that for positive, continuous f and $\varphi \in C(\partial\Omega)$, the Dirichlet problem

$$\begin{cases} u \in L^\infty(\Omega) \cap \mathcal{PSH}(\Omega) \\ (dd^c u)^n = f\, dV \\ \lim_{w \to z} u(w) = \varphi(z) \qquad \forall z \in \partial\Omega, \end{cases}$$

has a continuous solution on strictly pseudoconvex domains. In addition, they showed that the solution has locally bounded second-order partial derivatives defined almost everywhere if $\varphi \in C^{1,1}(\partial\Omega)$, $f^{1/n} \in C^{1,1}(\bar{\Omega})$. Higher regularity results than this does not seem to be possible if there exists a point where $f = 0$. However, for a smoothly bounded strictly pseudoconvex domain, with φ, f smooth and f strictly positive on $\bar{\Omega}$, Caffarelli, Kohn, Nirenberg and Spruck (9) showed that the unique solution is smooth up to the boundary.

Many have since then investigated the Dirichlet problem for more general measures. Here, we ask two questions: For which measures does there exists a solution, and for which measures is the solution continuous?

In the context of strictly pseudoconvex domains, the deepest results in these directions are due to Kołodziej (29; 30). The question of existence is partially answered by his subsolution theorem: The Dirichlet problem is solvable for a measure μ and $\varphi \in C(\partial\Omega)$ if and only if there exists a bounded plurisubharmonic function u attaining the same boundary values with $(dd^c u)^n \geq \mu$. An important open question is to settle whether the solution inherits continuity if one further assumes that the subsolution is continuous. Kołodziej's results establishing sufficient conditions for a measure to yield a continuous solution are more involved to state, but an important corollary is that the Dirichlet problem always has a continuous solution for $\varphi \in C(\partial\Omega)$ when $\mu = |f|\, dV$, if $f \in L^p(\Omega)$ and $p > 1$.

Note that all the above results assume that the prescribed boundary values constitute a continuous function on $\partial\Omega$. The reasons are twofold: Firstly, the comparison principle a priori only holds for bounded plurisubharmonic functions. It is therefore unclear whether the constructed envelopes correspond to a unique solution to a Dirichlet problem. Secondly, many continuity proofs rely on the original method of Walsh, which breaks down if one introduces singularities on the boundary. In the next chapter, we introduce sufficient conditions on φ allowing us to circumvent some of these difficulties.

Unbounded envelopes of plurisubharmonic functions

5 Quasibounded plurisubharmonic functions

Let Ω be a bounded domain. We say that a plurisubharmonic function u is *quasibounded* if there exists an increasing sequence $\{u_n\}$ of upper bounded, plurisubharmonic functions such that $u_n \nearrow u$. If the convergence only holds outside a pluripolar set, we say that u is *quasibounded quasi-everywhere*. The corresponding notion for harmonic functions is defined analogously, replacing each instance of the word "plurisubharmonic" with the word "harmonic" (of course, we do not have to consider the notion of quasibounded quasi-everywhere in the harmonic case due to the uniqueness theorem for harmonic functions and Harnack's theorem). We will begin this section by providing a characterization of positive, quasibounded harmonic functions due to Leutwiler and Arsove (1). First, they consider the set

$$\mathcal{M}^{\text{A–L}} := \{f\colon \Omega \to [0, +\infty] \ ; \ \exists u \in -\mathcal{SH}(\Omega), f \leq u\},$$

i.e. the set of non-negative functions admitting a superharmonic majorant. On $\mathcal{M}^{\text{A–L}}$, they define a family of operators by

$$(S_\lambda^{\text{A–L}} f)(z) := \Big(\inf\{v(z) \ ; \ v \in -\mathcal{SH}(\Omega), v \geq (f-\lambda)^+\}\Big)_*.$$

We remind the reader that

$$\phi^*(z) := \limsup_{\Omega \ni w \to z} \phi(w) \qquad \phi_*(z) := \liminf_{\Omega \ni w \to z} \phi(w)$$

denote upper and lower regularization, respectively. Since the lower regularization of a family of lower bounded superharmonic functions is superharmonic by the Brelot–Cartan theorem (36), $S_\lambda^{\text{A–L}} : \mathcal{M}^{\text{A–L}} \to -\mathcal{SH}(\Omega)$. Similarly,

$$S^{\text{A–L}}(f) := \big(\lim_{\lambda \to \infty} S_\lambda^{\text{A–L}}(f)\big)_*$$

is a superharmonic function smaller than all these $S_\lambda^{\mathrm{A-L}}(f)$. Their characterization is summarized in the following theorem.

Theorem 5.1. *Let $h \in \mathcal{M}^{A-L}$. Then the following are equivalent:*

1. $S^{A-L}(h) = 0$.
2. *There exists a function $\psi\colon [0,+\infty] \to [0,+\infty]$ such that $\psi(+\infty) = +\infty$,*

$$\lim_{t\to+\infty} \frac{\psi(t)}{t} = +\infty,$$

and $\psi \circ h$ admits a non-trivial superharmonic majorant.

Further, if h is harmonic, then the above statements are if and only if h is quasibounded.

In paper I (31), we translate the methods of Leutwiler and Arsove to the setting of pluripotential theory by defining the corresponding $\mathcal{M}, S_\lambda$ and S in terms of plurisuperharmonic functions instead of superharmonic functions. We show that several properties of the operators are retained, such as the monotonicity of S, i.e.

$$0 \le g \le f \in \mathcal{M} \implies S(g) \le S(f).$$

In fact, it is enough to require that $g \le f$ holds quasi-everywhere. Nevertheless, due to the fact that we cannot use linearity or Harnack's theorem, we are not able to prove that $S(u) = 0$ is equivalent to u being quasibounded quasi-everywhere, even in the case when u is a maximal plurisubharmonic function, although the equivalence between (1) and (2) in Theorem 5.1 remains intact. The arguments of Arsove and Leutwiler are however enough to prove the following theorem, which constitutes the central tool of paper I.

Theorem 5.2. *Let $0 \le u \in \mathcal{M} \cap \mathcal{PSH}(\Omega)$. If $S(u) = 0$, then u is quasibounded quasi-everywhere.*

The corresponding equivalence between (1) and (2) in Theorem 5.1 is expanded upon in paper II (32), and depends on the following definition.

Definition 5.3. *For a given pair $f : \Omega \to [-\infty,+\infty]$ and $v \in -\mathcal{PSH}(\Omega)$ bounded from below, we say that v is a strong majorant to f if*

$$f(z_0) = +\infty \implies v(z_0) = +\infty \quad \textit{and} \quad \frac{v(z)}{f(z)} \to \infty \textit{ as } f(z) \to \infty.$$

We say that a function w is a strong minorant to f if $-w$ is a strong majorant to $-f$.

Remark. Note that the corresponding definition in paper II is slightly different. There, strong majorants are defined for functions on the closure on domain, i.e $f : \bar{\Omega} \to [-\infty, +\infty]$, with $v \in -\mathcal{PSH}(\Omega) \cap \mathcal{LSC}(\bar{\Omega})$. The two definitions are equivalent for f satisfying $f^* = f_*$ on the boundary, if we extend v lower semicontinuously.

Definition 5.3 is central to the developments of paper II, as it will allow for some control over the pluripolar set where the approximating sequence of bounded plurisubharmonic functions (possibly) does not converge to u. Summarizing results of paper I and II, we obtain the following plurisubharmonic counterpart to Theorem 5.1.

Theorem 5.4. *Let $u \in \mathcal{M}$. Then the following are equivalent:*

1. $S(u) = 0$.
2. *There exists a function $\psi\colon [0, +\infty] \to [0, +\infty]$ such that $\psi(+\infty) = +\infty$,*

$$\lim_{t \to +\infty} \frac{\psi(t)}{t} = +\infty,$$

   and $\psi \circ u$ admits a non-trivial plurisuperharmonic majorant.

3. *u has a non-trivial strong majorant.*

If u is plurisubharmonic, then the above statements are if and only if there exists an increasing sequence of upper bounded plurisubharmonic functions $u_n \leq u$ such that $(u_n - u)^*$ is plurisubharmonic and $u_n - u \nearrow 0$ quasi-everywhere.

Here, we should mention that one might add the analogue of the third item to Theorem 5.1, and that the analogy is complete if one instead of considering harmonic functions considers subharmonic functions. The question of whether or not maximal plurisubharmonic functions in some sense occupy a special place in this setting is currently unclear.

Theorem 5.4 has the following straightforward consequence for harmonic functions in the unit disk. We include a proof since it does not appear in paper II.

Theorem 5.5. *A harmonic function h defined on the unit disk has a non-trivial strong majorant and a non-trivial strong minorant if and only if there exists $\varphi \in L^1(\partial D)$ such that*

$$h(z) = \int_0^{2\pi} \varphi(e^{i\theta}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi}.$$

Proof. If h is represented by φ, then the harmonic functions h_1 and h_2 represented by φ^+ and $-\varphi^-$ respectively are quasibounded (see Section 2). By Theorem 5.1, $S(h_1) =$

$S(h_2) = 0$, and by Theorem 5.4, h has a non-trivial strong majorant and non-trivial strong minorant.

Conversely, if h has a non-trivial strong majorant and non-trivial strong minorant, then $S_n(h) = S_n(h^+)$ is a decreasing sequence of positive harmonic functions converging pointwise to zero. As $h_n = S_n(h) - h$ are positive harmonic functions with a common non-trival strong majorant, Theorem 5.4 and Theorem 5.1 implies that h_n is quasibounded. Hence, h_n is represented by elements $-\varphi_n \in L^1(\partial D)$, and by the monotone convergence theorem, h is represented by an element $\varphi \in L^1(\partial D)$ such that $\varphi_n \nearrow \varphi$. □

6 Application to the Dirichlet problem

A consequence of Theorem 5.2 and the monotonicity of S is that the envelopes

$$P(\varphi) := \sup\{u(z) \; ; \; u \in \mathcal{PSH}(\Omega), u^* \leq \varphi\}$$

and

$$\sup\{u(z) \; ; \; u \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega), u^* \leq \varphi\}$$

coincide outside a pluripolar set $\mathcal{P}$ if φ is bounded from below, $\varphi^* = \varphi_*$ on $\bar{\Omega}$ and $S(\varphi^+) = 0$. As $P(\varphi)$ is upper semicontinuous on Ω and the latter coincides with

$$\sup\{u(z) \; ; \; u \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega}), u \leq \varphi\},$$

it follows that $P(\varphi) \in C(\Omega \setminus \mathcal{P})$. If φ is harmonic or plurisuperharmonic, one may further show that $(dd^c P(\varphi))^n = 0$, and that $P(\varphi)$ attains the same boundary values as φ. This means that the Perron–Bremermann envelope is a solution to a Dirichlet problem. Is it the unique solution?

To gain some clarity on the matter, consider a harmonic function h on the unit disk D which is bounded from below and continuous on the boundary in the sense of the extended reals. In other words, the upper and lower limits coincide at all points, but we allow for positive singularities. Assume without loss of generality that h has a singularity at $1 \in \partial D$. Such a function is not uniquely determined by its boundary values, since adding a multiple of the Poisson kernel with pole at 1 yields a larger harmonic function $\tilde{h}$ coinciding with h on ∂D in the sense that

$$\lim_{z \to \zeta \in \partial D} h(z) = \lim_{z \to \zeta \in \partial D} \tilde{h}(z)$$

in the extended reals. Of course, the boundary data differ in the sense of measures, but this difference is undetectable if one solely considers the above limits. There is however a unique quasibounded harmonic function with the prescribed boundary values.

In paper I, we show that this problem persists in the plurisubharmonic setting as well, as illustrated by the two maximal plurisubharmonic functions

$$u_1(z_1, z_2) = \frac{1 - |z_1|^2}{|1 - z_1|^2} - \log|1 - z_1|$$
$$u_2(z_1, z_2) = \frac{|z_2|^2}{|1 - z_1|^2} - \log|1 - z_1|$$

defined on the unit ball in $\mathbb{C}^2$. One way to secure uniqueness is to add the additional requirement that any solution must be majorized by φ, which limits the set of possible solutions to those that are quasibounded quasi-everywhere. In summary, we have the following theorem.

Theorem 6.1. *Let Ω be a B-regular domain, and suppose that φ is a harmonic function bounded from below such that $S(\varphi^+) = 0$ and $\varphi^* = \varphi_*$ on $\bar{\Omega}$. Then $P(\varphi)$ is the unique solution to the Dirichlet problem*

$$\begin{cases} u \in \mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega) \\ (dd^c u)^n = 0 \\ u^* \leq \varphi \\ \lim_{w \to z} u(w) = \varphi(z) \qquad \forall z \in \partial\Omega. \end{cases}$$

Furthermore, $P(\varphi) \in C(\Omega \setminus \mathcal{P})$, where $\mathcal{P}$ is a pluripolar set.

In paper II, we refine this theorem in two ways. Firstly, we show that the discontinuity set $\mathcal{P}$, if non-empty, must be a subset of the set

$$\{z \in \bar{\Omega} \,;\, v(z) = +\infty\},$$

where v is any strong majorant to φ. In particular, this allows us to conclude that $P(\varphi)$ is guaranteed to be continuous at all points where $v^* \neq +\infty$. We also show that it is possible to extend Theorem 6.1 to a class of *inhomogeneous* Dirichlet problems. Specifically, we consider the class of measures for which the Dirichlet problem with regards to continuous boundary values has a bounded solution, which we call *compliant*, and measures for which that solution is always continuous, which we call *continuously compliant*. The existence of compliant measures is equivalent to that the domain is B-regular (see paper III). Also note that for two measures μ, ν such that $\mu \geq \nu$, we have

$$\mu \text{ compliant} \qquad \Longrightarrow \qquad \nu \text{ compliant}$$

by Kołodziej's subsolution theorem. Using this terminology, we have the following statement.

Theorem 6.2. *Let Ω be a B-regular domain, μ be a compliant measure, and let φ satisfy the requirements of Theorem 6.1. Then the Dirichlet problem*

$$\begin{cases} u \in \mathcal{PSH}(\Omega) \cap L^\infty_{loc}(\Omega) \\ (dd^c u)^n = \mu \\ u^* \leq \varphi \\ \lim_{w\to z} u(w) = \varphi(z) \qquad \forall z \in \partial\Omega, \end{cases}$$

has a unique solution. If μ is continuously compliant, then the solution is continuous on the set $\{z \,;\, v^*(z) \neq +\infty\}$, where v is a strong majorant to φ.

We should stress that Theorem 6.2 only guarantees that the solution is continuous on the set $\{z \,;\, v^*(z) \neq +\infty\}$. We have not found any examples where the envelope is not continuous on $\bar{\Omega}$ (in the extended sense).

7 Edwards' theorem

One way to study envelopes is through a dualization argument due to Edwards (15). The main idea is as follows. Let $\mathcal{F}$ be a cone of upper bounded, upper semicontinuous functions on a compact metric space X, and associate to each $x \in X$ a set of positive measures

$$M_x^{\mathcal{F}} := \{\mu \,;\, u(x) \leq \int u \, d\mu \text{ for all } u \in \mathcal{F}\}.$$

Given a Borel function g on X, $x \in X$, we also define

$$S_x(g) := \begin{cases} \sup\{u(x) \,;\, u \in \mathcal{F}, u \leq g\}, & \text{if } \exists\, u \in \mathcal{F} \,;\, u \leq g, \\ -\infty, & \text{otherwise,} \end{cases}$$

$$I_x(g) := \inf\{ \int g \, d\mu \,;\, \mu \in M_x^{\mathcal{F}} \}.$$

Clearly, $S_x(g) \leq I_x(g)$, since

$$u(x) \leq \int u \, d\mu \leq \int g \, d\mu$$

holds for all u in the defining family for $S_x(g)$ and all μ in the defining family for $I_x(g)$. Edwards proved that under certain assumptions, these two operators coincide:

Theorem 7.1. *If $\mathcal{F}$ contains all constants and g is bounded and lower semicontinuous, then $Sg = Ig$ on X.*

By slightly modifying Edwards' original proof, we show in paper II that the requirement that the cone contains all constants is not necessary. Instead, we consider the set of measures

$$C_x^{\mathcal{F}} := \left\{\mu \; ; \; \mu(X) \leq -S_x(-\mathbb{1})\right\} \cap M_x^{\mathcal{F}},$$

where $-\mathbb{1}(x) \equiv -1$. As $C_x^{\mathcal{F}}$ is weak*-compact for all $x \in X$ such that $S_x(-\mathbb{1}) > -\infty$, it is still possible to carry out the compactness arguments of Edwards' proof. Notably, this allows for the following stronger version of Theorem 7.1:

Theorem 7.2. *Let g be lower semicontinuous and bounded from below. For all $x \in X$ such that $S_x(-\mathbb{1}) > -\infty$, we have $S_x(g) = I_x(g)$. It is enough to take the infinum over measures in $C_x^{\mathcal{F}}$.*

Remark. In Section 13, we will discuss versions of Edwards' theorem which encompass Theorem 7.2 as a special case.

We will now briefly explain how Edwards' theorem may be applied in the study of envelopes. Suppose that $\mathcal{F}^1, \mathcal{F}^2$ are two cones of upper bounded, upper semicontinuous functions on a compact metric space X, and that $\mathcal{F}^1 \subset \mathcal{F}^2$. Now consider the equation

$$\sup\{u(x) \; ; \; u \in \mathcal{F}^1, u \leq g\} = \sup\{u(x) \; ; \; u \in \mathcal{F}^2, u \leq g\}.$$

For which $x \in X$ does this hold, and when do the envelopes coincide at all points? In some way, we expect that the answer should depend on how well, if at all, one may approximate elements in $\mathcal{F}^2$ by elements in $\mathcal{F}^1$. It turns out that it is sufficient to require that each element in $\mathcal{F}^2$ may be approximated pointwise by a decreasing sequence of functions in $\mathcal{F}^1$. This is where Edwards' theorem comes into play, as it allows us to translate the equation into

$$I_x^{\mathcal{F}^1}(g) = I_x^{\mathcal{F}^2}(g),$$

which is true at all points where $M_x^{\mathcal{F}^1} = M_x^{\mathcal{F}^2}$. A priori, we know that $M_x^{\mathcal{F}^2} \subset M_x^{\mathcal{F}^1}$, so it is enough to show that $\mu \in M_x^{\mathcal{F}^1}$ implies that $\mu \in M_x^{\mathcal{F}^2}$, or equivalently

$$\int f \, d\mu \geq f(x)$$

for all $f \in \mathcal{F}^2$. Given that there exists $f_n \in \mathcal{F}^1$ such that $f_n \searrow f$, this is an easy consequence of the monotone convergence theorem.

8 Continuity sets of unbounded envelopes

In this section, we continue our study of the continuity set of the Perron–Bremermann envelope

$$P(\varphi) = \sup\{u(z) \; ; \; u \in \mathcal{PSH}(\Omega), u^* \leq \varphi\}.$$

Our starting point is the following theorem due to J.B. Walsh (40).

Theorem 8.1 (J.B. Walsh's theorem). *Let* Ω *be a B-regular domain, and suppose that* $\varphi \in C(\bar{\Omega})$. *Then* $P(\varphi)$ *is continuous on* $\bar{\Omega}$.

The main theorem of paper II is a generalization of the above result, valid for all $\varphi : \bar{\Omega} \to [-\infty, +\infty]$ such that $\varphi^* = \varphi_*$ with non-trivial strong minorant v and non-trivial strong majorant w. There are essentially two techniques involved in the proof, as the positive and the negative singularities separately contribute to the possible discontinuity set. In particular, the positive part of $P(\varphi)$ is quasibounded quasi-everywhere by Theorem 5.4 if w is non-trivial, which allows us to approximate by upper bounded plurisubharmonic functions outside the set

$$\{z \in \bar{\Omega} \,;\, w(z) = +\infty\}.$$

To handle the negative singularities, we employ a variant of the method of Jensen measures (34; 41) made possible by Theorem 7.2. Crucially, this allows us to show that the upper semicontinuous envelope

$$\sup \left\{ u(z) \,;\, u \in \mathcal{PSH}(\Omega), u^* \leq \varphi^- \right\}$$

coincides with the envelope

$$\sup \left\{ u(z) \,;\, u \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega} \setminus \mathcal{P}), u^* \leq \varphi^- \right\},$$

where $\mathcal{P} = \{z \in \bar{\Omega} \,;\, v_*(z) \neq -\infty\}$ and φ^- denote the negative part of φ. As the second envelope is lower semicontinuous on $\bar{\Omega} \setminus \mathcal{P}$, both envelopes must be continuous outside $\mathcal{P}$.

Combining these two observations, we arrive at the following theorem.

Theorem 8.2. *Let* Ω *be a B-regular domain, and let* φ *be a continuous extended real-valued function on* $\bar{\Omega}$ *with a strong minorant* v *and a strong majorant* w. *Then* $P(\varphi)$ *is continuous on*

$$\{z \in \bar{\Omega} \,;\, v_*(z) \neq -\infty\} \cap \{z \in \bar{\Omega} \,;\, w^*(z) \neq +\infty\}.$$

Note that the case $v = w = 0$ corresponds to Theorem 8.1, and that the statement is empty if either $v \equiv -\infty$ or $w \equiv +\infty$.

Plurisubharmonic functions with discontinuous boundary behavior

9 The unit disk revisited

As we saw in the first chapter, an integral representation of the solution permits us to conclude that given a fixed bounded function $\phi : \partial D \to \mathbb{R}$, continuous outside $A_\phi \subset \partial D$, the following two statements are equivalent:

(a) A_ϕ has Lebesgue measure zero.

(b) There exists a unique bounded harmonic function h_ϕ such that

$$\lim_{D \ni \zeta \to z_0} h_\phi(\zeta) = \phi(z_0), \quad \forall z_0 \in \partial D \setminus A_\phi.$$

It possible to list a third property of A_ϕ, equivalent to (a) and (b), namely

(c) A_ϕ is *b-pluripolar.*

Definition 9.1 (Djire and Wiegerinck (14)). *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. We say that a set $F \subset \partial\Omega$ is b-pluripolar if there exists $v \in \mathcal{PSH}(\Omega)$ such that $v \le 0, v \not\equiv -\infty$ and $v^* = -\infty$ on F.*

This follows from a theorem of Fatou (17), which says that we for any compact set $K \subset \partial D$ of Lebesgue measure zero may find a function f such that $f : \partial D \to [-\infty, 0]$ continuously, $f \in L^1(\partial D)$ and

$$f(z) = -\infty \iff z \in K.$$

Since A_φ is a F_σ set, we may write A_φ as a countable union of compact sets K_i. Extending the corresponding f_i to harmonic functions h_i in the interior, we then construct

$$h := \sum_{i=1}^{\infty} c_i h_i,$$

where $c_i > 0$ are chosen such that the sum converges at some point in the interior. By Harnack's theorem, the sum converges everywhere to a harmonic function, which shows that A_φ is b-pluripolar.

Conversely, suppose that A_φ is b-pluripolar, and let v be a negative subharmonic function such that $v^* = -\infty$ on A_φ. Solving the Dirichlet problem for the boundary values $\max\{v, n\}^*$, we get a decreasing sequence of harmonic functions bounded below by v. By Harnack's theorem, this sequence converges to a harmonic function which is represented on the boundary by a function in $L^1(\partial D)$ (see Section 2) with A_φ as a subset of its singularity set. Hence, A_φ must have zero Lebesgue measure.

In the next sections, we will investigate to which degree this relationship remains for the Dirichlet problem for the complex Monge–Ampère operator in higher dimensions.

10 Discontinuous boundary data for B-regular domains in $\mathbb{C}^n$

In the general plurisubharmonic setting, we will replace (b) in the previous section with the statement that the following inhomogeneous Dirichlet problem for the complex Monge–Ampère operator has a unique solution:

$$\begin{cases} u \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega) \\ (dd^c u)^n = \mu \\ \lim_{\Omega \ni \zeta \to z_0} u(\zeta) = \varphi(z_0), \quad \forall z_0 \in \partial\Omega \setminus E_\varphi. \end{cases}$$

Here, we will assume that Ω is B-regular, μ is a compliant measure, and that $\varphi : \partial\Omega \to \mathbb{R}$ is a bounded function, continuous outside E_φ. Note that in one complex dimension, these assumptions reduce the problem to finding a harmonic function with the above boundary behavior, as we may subtract a subharmonic function u with the properties

$$dd^c u = \mu, \quad \lim_{z \to \partial\Omega} u(z) = 0,$$

by the compliance of μ. The main result in paper III (33) is that it is sufficient to assume that the discontinuities form a b-pluripolar set; in other words, the implication (c) $\implies$ (b) remains intact also in higher dimensions, for any B-regular domain.

Theorem 10.1. *Let μ be a compliant measure on a B-regular domain Ω, and let $\varphi : \partial\Omega \to \mathbb{R}$ be a bounded function, continuous outside a b-pluripolar set E_φ. Then the Dirichlet problem*

$$\begin{cases} u \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega) \\ (dd^c u)^n = \mu \\ \lim_{\Omega \ni \zeta \to z_0} u(\zeta) = \varphi(z_0), \quad \forall z_0 \in \partial\Omega \setminus E_\varphi \end{cases}$$

has a unique solution.

The main difficulty of the proof is to show that a solution necessarily is unique. This is due to the fact that the comparison principle (used to establish the uniqueness of the solution when the boundary function is continuous) does not a priori allow for exceptional sets; this is in stark contrast to the Dirichlet problem in the complex plane, where we both have an extended version of maximum principle (allowing for polar exceptional sets) and integral representations at our disposal. Analogously, uniqueness would immediately follow from a corresponding *extended comparison principle*.

A very recent result in this direction is due to Rashkovskii (37), who provided a version of the comparison principle that allows for exceptional sets that are pluripolar. Modifying his proof, we arrive at the following lemma—the essential ingredient in the proof of Theorem 10.1.

Lemma 10.2. *Let $u, v \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega), \Omega \Subset \mathbb{C}^n$ and suppose that*

$$\limsup_{z \to \zeta}(u(z) - v(z)) \le 0 \quad \forall \zeta \in \partial\Omega \setminus F,$$

where $F \subset \partial\Omega$ is b-pluripolar. If $(dd^c v)^n \le (dd^c u)^n$, then $u \le v$ on Ω. In particular, if $\lim_{z \to \zeta}(u(z) - v(z)) = 0$ for all $\zeta \in \partial\Omega \setminus F$ and $(dd^c v)^n = (dd^c u)^n$, then $u = v$.

When μ is continuously compliant, it is possible to estimate at which points the solution $u_{\mu,\varphi}$ might be discontinuous. This estimate is given in terms of the defining family

$$\mathcal{F}_{E_\varphi} := \{u \in \mathcal{PSH}(\Omega) \ ; \ u \not\equiv -\infty, u < 0, u^* |_{E_\varphi} = -\infty\}$$

for the *b-pluripolar hull*

$$\hat{E}_\varphi := \{z \in \bar{\Omega} \ ; \ \forall u \in \mathcal{F}_{E_\varphi}, u^*(z) = -\infty\}$$

of the b-pluripolar set E_φ of discontinuities on the boundary.

Theorem 10.3. *Let μ be a continously compliant measure on a B-regular domain Ω, and let $\varphi : \partial\Omega \to \mathbb{R}$ be a bounded function, continuous outside a b-pluripolar set E_φ. Then $u_{\mu,\varphi}$ is continuous outside the closed set $\cap_{u \in \mathcal{F}_{E_\varphi}} \{u_*(z) = -\infty\}$.*

In general, the set of discontinuities may be nonempty, and may even coincide with $\cap_{u \in \mathcal{F}_{E_\varphi}} \{u_*(z) = -\infty\}$. To see this, consider the plurisubharmonic function

$$\tilde{u}(z_1, z_2) := \max\{\sum_{k=1}^{\infty} 2^{-k} \log |z_1 - 2^{-k}|, -1\}$$

restricted to the unit ball $\mathbb{B} \subset \mathbb{C}^2$. Clearly, $\tilde{u}$ satisfies $(dd^c\tilde{u})^n = 0$, is discontinuous on $\{z_1 = 0\} \cap \mathbb{B}$, and extends continuously to the boundary outside the b-pluripolar set

$$\tilde{E} := E_{\tilde{u}|_{\partial\mathbb{B}}} = \{z_1 = 0\} \cap \partial\mathbb{B}.$$

Since the zero measure is continuously compliant on B-regular domains, $\tilde{u}$ uniquely solves a Dirichlet problem satisfying the conditions of Theorem 10.1 and Theorem 10.3. Precomposing with the analytic disk

$$\begin{aligned} D &\to \mathbb{B} \\ z &\mapsto (0, z), \end{aligned}$$

it is immediate that any element $u \in \mathcal{F}_{\tilde{E}}$ satisfies $u = -\infty$ on $\{z_1 = 0\} \cap \mathbb{B}$. On the other hand, $\log|z_1| \in \mathcal{F}_{\tilde{E}}$, implying that in fact

$$\cap_{u \in \mathcal{F}_{\tilde{E}}} \{u_*(z) = -\infty\} = \{z_1 = 0\} \cap \mathbb{B}.$$

Hence, this example shows that the estimate given in Theorem 10.3 is sharp.

A next natural question is to ask if the reverse implication, (b) $\implies$ (c), also holds in higher dimensions. Before answering this question, we will digress a little bit and consider some results in paper III concerning the continuity of Perron–Bremermann envelopes on Reinhardt domains.

11 Continuity of envelopes on Reinhardt domains

As we saw in Section 7, on a domain $\Omega \subset \mathbb{C}^n$, there is a connection between the continuity of the Perron–Bremermann envelope and the property that plurisubharmonic functions defined on Ω may be approximated pointwise from above by continuous ones. Edwards' theorem provides one way to illuminate this fact; it is also possible to verify this relationship more directly in certain situations.

For example, on B-regular domains, J.B. Walsh's theorem (Theorem 8.1) implies that the envelope $P(\varphi)$ is continuous for any continuous function $\varphi : \bar{\Omega} \to \mathbb{R}$, and since any (upper bounded) plurisubharmonic function u in the defining family extends to a upper semicontinuous function on $\bar{\Omega}$, there is a decreasing sequence of continuous functions φ_n such that $\varphi_n \searrow u$ on $\bar{\Omega}$. Then, as $P(\varphi_n)$ are continuous plurisubharmonic functions such that

$$\varphi_n \geq P(\varphi_n) \geq u,$$

we get a decreasing sequence of continuous *plurisubharmonic* functions converging pointwise to u on $\overline{\Omega}$. Conversely, since $(P(\varphi))^*$ is plurisubharmonic by the Brelot–Cartan theorem and $(P(\varphi))^* \leq \varphi$ for continuous φ, a decreasing sequence $u_n \searrow (P(\varphi))^*$ of continuous plurisubharmonic functions will convergence uniformly since

$$\max\{\varphi, u_n\} \searrow \varphi$$

converges uniformly by Dini's theorem.

Something similar can be achieved on *Reinhardt domains*, i.e. a domain $\Omega \subset \mathbb{C}^n$ with the property that

$$z = (z_1, \dots, z_n) \in \Omega \implies \{w \in \mathbb{C}^n \ ; \ |w_k| = |z_k| \text{ for all } k\} \subset \Omega.$$

For these domains, there is a explicit construction due to Fornæss and Wiegerinck (18) which one can use to approximate *all* plurisubharmonic functions from above by plurisubharmonic functions continuous in the interior. This translates into continuity properties of various Perron–Bremermann envelopes, given some assumptions on the majorant φ. Below, we highlight two such results from paper III. First, to establish the continuity of $P(\varphi)$, it turns out that either of the following two properties are sufficient if Ω is bounded:

(i) φ is uniformly continuous on Ω (in other words, φ extends to an element in $C(\overline{\Omega})$),

(ii) φ is upper semicontinuous, bounded from below and *toric*, i.e.

$$\varphi(z_1, \dots, z_n) = \varphi(|z_1|, \dots, |z_n|)$$

holds for all $z \in \Omega$.

Note that $P(\varphi)$ trivially is continuous (at least outside the coordinate hyperplanes) if φ satisfies the second property, since this implies that $P(\varphi)$ is toric as well, and therefore convex in logarithmic coordinates. It is however not obvious that the first condition is sufficient as well, as we do not require that Ω is B-regular. The following result may hence be viewed as version of J. B. Walsh's theorem valid on bounded Reinhardt domains.

Theorem II.1. *Let Ω be a bounded Reinhardt domain. Then*

$$\varphi \textit{ uniformly continuous on } \Omega \implies P(\varphi) \in C(\Omega).$$

Remark. In contrast to the case when Ω is B-regular, this theorem only guarantees that $P(\varphi)$ is continuous in the interior. There is also a notion of *c-regular domains*, which may be characterized as domains for which the above implication holds, but with continuity up to the boundary. For example, smoothly bounded pseudoconvex Reinhardt domains in $\mathbb{C}^2$ are c-regular. See Göğüş and Şahutoğlu (21) for more details.

Here we should mention that paper III also contain similar results for envelopes where the complex Monge–Ampère measures of the defining family are restricted by a measure of the form $f\,dV$, where $0 \le f \in C(\bar{\Omega})$ and dV denotes the volume measure. If we further assume that Ω is strictly pseudoconvex and that φ is harmonic on Ω, it is possible to weaken the assumptions on f considerably.

Theorem 11.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded, strictly pseudoconvex Reinhardt domain, and assume that φ is toric, harmonic and bounded from below. Then for $0 \le f \in L^p(\Omega)$, $p > 1$,*

$$P(\varphi, f) := \sup\{u(z)\ ;\ u \in \mathcal{PSH}(\Omega), (dd^c u)^n \ge f\,dV, u \le \varphi\}$$

is continuous.

The proof of this theorem depends on the machinery developed by Kołodziej (29), used to prove that the Dirichlet problem for the complex Monge–Ampère operator (on strictly pseudoconvex domains) always has a continuous solution for continuous boundary data when $\mu = |f|\,dV$, if $f \in L^p(\Omega)$ and $p > 1$. In the next section, we will similarly apply Theorem 11.2 to a Dirichlet problem with discontinuous boundary data.

12 A counterexample in the unit ball in $\mathbb{C}^3$

In this section, we will construct a family of examples that allows us to refute the possibility of extending the implication (b) $\implies$ (c) of Section 9 to higher dimensions. The setting will be the unit ball $B \subset \mathbb{C}^n$, and we will consider boundary data of the form

$$\phi_A(z) = \begin{cases} -1, & z \in A \\ 0, & z \in \partial B \setminus A, \end{cases}$$

where $A \subset \partial B$ is an open, toric set such that $\bar{A}$ does not meet the hyperplanes $\{z_j = 0\}$. The relative boundary ∂A will then be the discontinuity set of ϕ_A, and this set is never b-pluripolar. Indeed, for $\zeta \in \partial A$, the set

$$\{z \in \mathbb{C}^n\ ;\ |z_k| = |\zeta_k| \text{ for all } k\} \subset \partial A$$

will be the *Bergman–Shilov boundary* of the polydisk

$$D(0,\zeta) := D(0,|\zeta_1|) \times \cdots \times D(0,|\zeta_n|) \subset B.$$

Therefore, by the maximum principle, for elements in $u \in \mathcal{PSH}(B)$, we have

$$u^* |_{\partial A} = -\infty \implies u |_{D(0,\zeta)} \equiv -\infty \implies u \equiv -\infty.$$

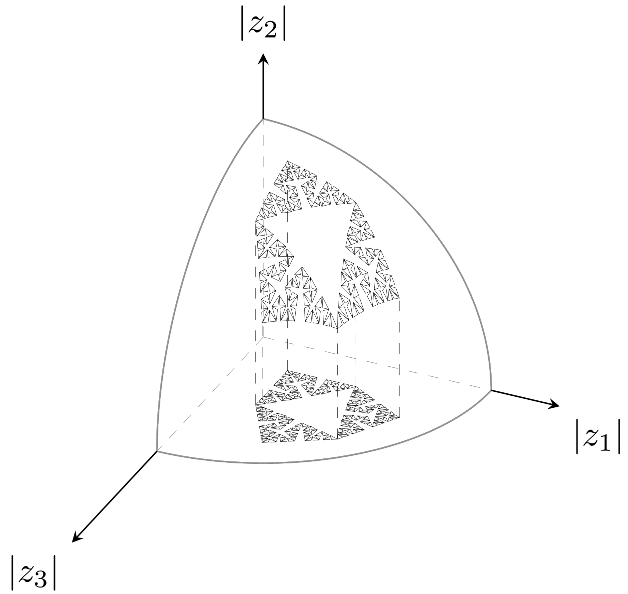
We conclude that ∂A cannot be b-pluripolar.

It is easy to find A such that ∂A has Lebesgue measure zero, which shows that the implication (a) $\implies$ (c) does not hold in general. On the other hand, we are not excluding the possibility that ∂A has positive Lebesgue measure. Although higher dimensional geometry seldomly lends itself to visualization, it is possible to acquire some intuition for this pathological scenario. Consider for example the unit ball in $\mathbb{C}^3$, represented as an object in three real dimensions by identifying points z, w for which

$$(|z_1|, |z_2|, |z_3|) = (|w_1|, |w_2|, |w_3|).$$

Since ∂A is toric, it is faithfully captured in this representation, and may be identified with the boundary of a relatively open subset of an one-eight of the unit sphere.

A pathological set may now be constructed by beginning with a curve that is both an *Osgood curve*, i.e. a non-self-intersecting curve that has positive area, and a *Jordan curve*, dividing the plane into an interior and exterior part. Such curves are guaranteed to exist by the Smith–Volterra–Cantor construction and the Denjoy–Riesz theorem. Alternatively, as illustrated below, one may begin with an arrangement of three triangles and successively cut out smaller and smaller triangles (see Knopp (28)).



The sixth step in the construction of a set ∂A with positive Lebesgue measure.

Alas, the set ∂A may be very complicated, but as a consequence of Djire and Wiegerinck's partial answer (14, Theorem 2.11) to Sadullaev's question concerning when the upper semicontinuous regularization of a variety of boundary extremal functions coincide (38), we can still guarantee that ϕ_A always has the following property.

Lemma 12.1. *Let $A \subset \partial B$ satisfy the requirements above. Then $P(\phi_A)$ is continuous, and may be written as an envelope over uniformly continuous functions.*

This lemma is key to showing that the Dirichlet problem with ϕ_A as boundary data has a unique solution. The rough idea is to replace the step of applying the comparison principle directly to the Perron–Bremermann envelope, instead comparing with elements that extends continuously to the boundary in the defining family. Combining this strategy with Theorem 11.2, we arrive at the following surprising theorem.

Theorem 12.2. *Let $A \subset \partial B$ satisfy the requirements above, and let μ be a compliant measure. Then the Dirichlet problem*

$$\begin{cases} u \in \mathcal{PSH}(\Omega) \cap L^\infty(B) \\ (dd^c u)^n = \mu \\ \lim_{B \ni \zeta \to z_0} u(\zeta) = \phi_A(z_0), \quad \forall z_0 \in \partial B \setminus \partial A \end{cases}$$

has a unique solution. This solution is furthermore continuous on B if μ is a continuously compliant measure of the form $\mu = f\, dV$, where $0 \le f \in L^p(\Omega)$, and $p > 1$.

This result shows that (b) does not imply either (a) or (c) in higher dimensions.

Variations on Edwards' theorem

13 Altering Edwards' proof

In paper IV, we explore how far one may push the original proof of Edwards' theorem. Structurally, the argument may be broken down as follows:

- We begin with the inequality $S_x \le I_x$, by the property of Jensen measures. If one can show that each element in $\varphi \in C(X)$ have a minorant in $\mathcal{F}$, finite at $x \in X$, then $-S_x$ defines a sublinear map $C(X) \to \mathbb{R}$. This is for example guaranteed if $\mathcal{F}$ contains all constants. Importantly, $S_x \le I_x$ is then an inequality between a sublinear operator $-S_x$ and a *family* of positive linear operators.
- On the span of a fixed element $\varphi_0 \in C(X)$, i.e. elements of the form $c\varphi_0, c \in \mathbb{R}$, we can define linear maps $-A_x(c\varphi_0) := -cS_x(\varphi_0)$, which satisfies $-A_x \le -S_x$. By the Hahn–Banach theorem, we can extend $-A_x$ to a linear map $-\tilde{A}_x : C(X) \to \mathbb{R}$ where this inequality holds on the entire space $C(X)$, with equality on φ_0.
- Using the inequality, it follows that $\tilde{A}_x$ is a positive linear operator on $C(X)$, and hence represented by a measure μ_x by the Riesz representation theorem.
- In order to show that μ_x is a Jensen measure, we will employ the following relation between $C(X)$ and $\mathcal{USC}(X)$: For each element $u \in \mathcal{USC}(X)$, there exists a decreasing sequence $g_i \in C(X)$ such that $g_i \searrow u$. In particular, this holds if $u \in \mathcal{F}$. By the monotone convergence theorem, μ_x is a Jensen measure, and since the operators induced by the Jensen measures all majorize S_x, Edwards' theorem is proven true for continuous majorants.
- In order to show that Edwards' theorem also holds for lower semicontinuous majorants, we likewise employ the following relation between $C(X)$ and $\mathcal{LSC}(X)$: For each element $l \in \mathcal{LSC}(X)$, there exists an increasing sequence $g_i \in C(X)$ such that $g_i \nearrow l$. Edwards' theorem follows from another application of the monotone convergence theorem.

There are two settings which suggest natural alterations to this proof.

13.1 Ordering of measures

In the first variation, we replace the Jensen measures by other sets of measures, defined in terms of an ordering of measures. These sets are natural to consider, since any cone $\mathcal{F}$ of real-valued functions on X induces an ordering on the (positive Radon) measures supported on X.

Definition 13.1. *Let μ and ν be positive Radon measures supported on X. We say that $\mu \leq_{\mathcal{F}} \nu$ if*

$$\int u \, d\nu \leq \int u \, d\mu$$

for all $u \in \mathcal{F}$.

In particular, the Jensen measures at $x \in X$ may be defined as all measures smaller than the point mass at x in this order relation. Similarly, one can consider the set of measures smaller than a fixed measure μ with regards to $\mathcal{F}$. Then

$$S_\mu \varphi = \sup \left\{ \int u \, d\mu \; ; \; u \in \mathcal{F}, u \leq \varphi \right\}$$

also become superlinear operators on $C(X)$, coinciding with

$$I_\mu \varphi = \inf \left\{ \int \varphi \, d\nu \; ; \; \nu \leq_{\mathcal{F}} \mu \right\}$$

by slightly modifying Edwards' proof. We will refer to this result as the *first variation*.

13.2 Cones not minorizing the continuous functions

It is apparent that the space $C(X)$ is central to the entire argument, in particular for the measure representation, needed for the monotone convergence. What if there is an element in $C(X)$ that is not minorized by any element in $\mathcal{F}$? This is for example the case if all elements in $\mathcal{F}$ are zero on ∂X. One way to alter Edwards' proof to also encompass such cones is to replace $C(X)$ by a subspace $H \subset C(X)$ where all elements are minorized. For this to work, several modifications to the proof have to be made, and H must satisfy certain criteria (see paper IV for more details). We will refer to this variant of Edwards' theorem as the *second variation.*

14 Application to cones of plurisubharmonic functions

From the viewpoint of pluripotential theory, both variations are motivated by the following ordering of positive measures induced by (negative) plurisubharmonic functions, introduced by Bengtson in (4).

Definition 14.1. *Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain, and let μ and ν be positive Radon measures supported on $\bar{\Omega}$. We say that $\mu \leq_{\mathrm{psh}} \nu$ if*

$$\int u \, \mathrm{d}\mu \geq \int u \, \mathrm{d}\nu$$

for all $u \in \mathcal{E}_0(\Omega)$.

Here, we remind the reader that

$$\mathcal{E}_0(\Omega) = \{u \in \mathcal{PSH}(\Omega) : u < 0, \lim_{\zeta \to p \in \partial\Omega} u(\zeta) = 0, \int_\Omega (dd^c u)^n < \infty\}$$

is the cone of "plurisubharmonic test functions" defined by Cegrell (10). In paper IV, we study several similar orderings induced by plurisubharmonic functions. Among these, for two *finite* positive Radon measures μ, ν supported on $\bar{\Omega}$ we say that $\mu \leq_{\mathrm{c}} \nu$ if

$$\int u \, \mathrm{d}\mu \geq \int u \, \mathrm{d}\nu \quad \text{for all } u \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega}).$$

The advantage of this ordering is that two comparable measures automatically have the same mass, since the generating cone contains all constants. On any hyperconvex domain Ω, the two orderings are related as follows.

Proposition 14.2. *If $\mu \leq_{\mathrm{psh}} \nu$ and $\mu(\Omega) = \nu(\Omega) < \infty$, then $\mu \leq_{\mathrm{c}} \nu$.*

As an application of our first variation of Edwards' theorem, we characterize the minimal elements in the ordering $\leq_{\mathrm{c}}$ on B-regular domains.

Theorem 14.3. *Let $\Omega \subset \mathbb{C}^n$ be a B-regular domain. Then a positive Radon measure μ on $\bar{\Omega}$ has the property that*

$$\nu \leq_{\mathrm{c}} \mu \implies \nu = \mu$$

if and only if $\operatorname{supp} \mu \subset \partial\Omega$.

In Bengtson's ordering, there are no minimal measures in the sense that

$$\nu \leq_{\mathrm{psh}} \mu \implies \nu = \mu,$$

since we always have $\frac{1}{2}\mu \leq_{\mathrm{psh}} \mu$. On the other hand, by combining both variations of Edwards' theorem, we quickly conclude that there are no elements in this ordering (on hyperconvex domains) with the property

$$\nu \leq_{\mathrm{psh}} \mu \quad \Rightarrow \quad \nu = k_\nu \mu$$

for some $k_\nu \in [0, 1]$.

Theorem 14.4. *Let $\Omega \subset \mathbb{C}^n$ be a hyperconvex domain, and suppose that μ is positive Radon measure on Ω satisfying the minimality property above. Then $\mu(\Omega) = 0$.*

The second variation also allows us to study envelopes over cones of negative plurisubharmonic functions defined on the closure of a bounded hyperconvex domain Ω. These results are largely paralleling the case when Ω is B-regular (32; 41). Specifically, we consider the two cones

$$\begin{aligned}&\mathcal{PSH}^0(\bar{\Omega}) := \{u \in \mathcal{USC}(\bar{\Omega}) \cap \mathcal{PSH}^-(\Omega) \; ; \; u|_{\partial\Omega} = 0\},\\ &\mathcal{E}_0(\Omega) \cap C(\bar{\Omega}).\end{aligned}$$

By an approximation theorem of Cegrell (11), the Jensen measures of these two cones coincide. Denoting this set of measures by J_z^-, the second variation implies the following result.

Theorem 14.5. *Let $g : \bar{\Omega} \to \mathbb{R}$ be a lower bounded function such that $g_* \leq g$, with equality outside a pluripolar set P, and suppose that $g_*(\zeta) = 0$ for all $\zeta \in \partial\Omega$. Then*

$$\begin{aligned}\sup\left\{u(x) \; ; \; u \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega}), u \leq g\right\} &= \sup\left\{u(x) \; ; \; u \in \mathcal{PSH}^0(\Omega), u \leq g\right\}\\ &= \inf\left\{ \int g \, d\mu \; ; \; \mu \in J_z^-\right\}\end{aligned}$$

holds outside the pluripolar hull of P. In particular, if g is upper semicontinuous and P is a closed, complete pluripolar set, then the envelopes are continuous outside P.

We also get a version of Theorem 8.2 valid for hyperconvex domains.

Theorem 14.6. *Let Ω be a hyperconvex domain, and let $g : \bar{\Omega} \to [-\infty, 0]$ be such that $g^* = g_*$ on $\bar{\Omega}$, $g = 0$ on $\partial\Omega$, and that g has a non-trivial strong minorant. Then the envelope*

$$\sup\left\{u(z) \; ; \; u \in \mathcal{PSH}^-(\Omega), u^* \leq g\right\}$$

is continuous on $\{z \in \bar{\Omega} \; ; \; v_*(z) \neq -\infty\}$.

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Scientific papers

Author contributions

Paper I: Quasibounded plurisubharmonic functions

Both authors contributed equally to the main ideas of the paper and the construction of examples.

Paper IV: Variations on a theorem by Edwards

I provided the material in Section 2, and contributed to the formulation of Theorem 5.2 and the formulation and proof of Theorem 5.6. Both authors contributed equally to the writing of the paper.

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