LUND UNIVERSITY

# Fast reconstruction of harmonic functions from Cauchy data using integral equation techniques

Helsing, Johan; Johansson, B. Tomas

*Published in:*
Inverse Problems in Science and Engineering

*DOI:*
10.1080/17415971003624322

2010

Link to publication

*Citation for published version (APA):*
Helsing, J., & Johansson, B. T. (2010). Fast reconstruction of harmonic functions from Cauchy data using integral equation techniques. *Inverse Problems in Science and Engineering*, *18*(3), 381-399. https://doi.org/10.1080/17415971003624322

*Total number of authors:*
2

**General rights**
Unless other specific re-use rights are stated the following general rights apply:
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.
• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal

Read more about Creative commons licenses: https://creativecommons.org/licenses/

**Take down policy**
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

January 2010

To appear in *Inverse Problems in Science and Engineering*

# Fast reconstruction of harmonic functions from Cauchy data using integral equation techniques

Johan Helsing
Centre for Mathematical Sciences
Lund University
Lund, Box 118, 221 00, Sweden
E-mail: helsing@maths.lth.se

B. Tomas Johansson
School of Mathematics
University of Birmingham
Birmingham B15 2TT, UK
E-mail: b.t.johansson@bham.ac.uk

**Abstract**

We consider the problem of stable determination of a harmonic function from knowledge of the solution and its normal derivative on a part of the boundary of the (bounded) solution domain. The alternating method is a procedure to generate an approximation to the harmonic function from such Cauchy data and we investigate a numerical implementation of this procedure based on Fredholm integral equations and Nyström discretization schemes, which makes it possible to perform a large number of iterations (millions) with minor computational cost (seconds) and high accuracy. Moreover, the original problem is rewritten as a fixed point equation on the boundary, and various other direct regularization techniques are discussed to solve that equation. We also discuss how knowledge of the smoothness of the data can be used to further improve the accuracy. Numerical examples are presented showing that accurate approximations of both the solution and its normal derivative can be obtained with much less computational time than in previous works.

*Mathematics Subject Classification* (2000): 35R25.

**Keywords**. Alternating method, Second kind boundary integral equation, Cauchy problem, Laplace equation.

## 1 Introduction

In many engineering problems for fluid and heat flow, such as in non-destructive testing and tomography, the governing model is the Laplace equation and overspecified data is given on a part of the boundary of the solution domain in the form of the solution and its normal derivative, see, for example, [9, 25]. This is termed a Cauchy problem, and even if a solution exists to it, usually it can not be numerically calculated via classical methods since it is ill-posed, thus highly sensitive to measurement errors in the data.

To formulate our model, let $\Omega$ be a bounded domain in $\mathbf{R}^2$ with Lipschitz boundary $\Gamma$ and let $\Gamma_{\mathrm{C}}$ be a non-empty (open) arc of $\Gamma$. We are interested in finding a harmonic function $u$ such that this function and its normal derivative satisfy given Cauchy conditions on $\Gamma_{\mathrm{C}}$, and this means that $u$ solves

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f_{\mathrm{C}} & \text{on } \Gamma_{\mathrm{C}}, \\ \dfrac{\partial u}{\partial \nu} = g_{\mathrm{C}} & \text{on } \Gamma_{\mathrm{C}}, \end{cases} \tag{1}$$

where $f_C$ and $g_C$ are given functions, and $\nu$ is the outward unit normal to the boundary. Note that uniqueness of the solution $u$ is well established, see for example [6, 7]. We shall assume that data are given such that there exists a solution.

Due to the importance of the model (1), there are many different numerical methods in the literature for this Cauchy problem. In 1989, Kozlov and Maz'ya [18] proposed a procedure denoted as the alternating iterative method for solving Cauchy problems for general strongly elliptic and formally self-adjoint systems in bounded domains. One of the advantages of this method is that the governing partial differential operator is preserved and the regularizing character is achieved by appropriate change of the boundary conditions. In [19], the alternating iterative method is applied and investigated for Cauchy problems for the equations of anisotropic elasticity.

The alternating method gives relatively accurate numerical approximations and is suitable for use in practical applications. This has been verified in numerous papers, see for example [1, 3, 4, 5, 8, 12, 15, 20, 21, 22, 23]. The number of iterations needed for an accurate approximation in this procedure can be large and time-consuming; in [14], it was noted that around 50 000 iterations was needed to get a reasonable approximation in Stokes flow and it took more than one day to perform these computations using the boundary element method. Of course, Stokes flow in itself is a more complex phenomena to numerically simulate than stationary heat flow but also for the Laplace operator the alternating method can be time-consuming. Note that there are various relaxation techniques to improve the rate of convergence, see [16, 17], but the choice of relaxation parameters can be difficult.

However, in each iterative step of the alternating method, mixed problems are solved for the governing differential operator but only the solution or its normal derivative on the boundary of the solution domain are of interest. Thus, it should be possible to considerably speed up the procedure using boundary integral techniques and Nyström discretization schemes, and this possibility will be explored in this work.

Recently, see [13], a highly accurate and efficient method based on Fredholm integral equations of the second kind was proposed to solve mixed boundary value problems for the Laplace equation and the biharmonic equation. We employ this method for the alternating procedure and show that it is easy to do millions of iterations in only few seconds on an ordinary computer. The approximations are at least as accurate as those earlier reported in the literature and which are mainly based on the boundary element method. We then rewrite the Cauchy problem as a fixed point equation on the boundary, and discuss some additional techniques to solve that equation. In particular, to further improve the accuracy, we show how knowledge of the smoothness of the solution and its normal derivative on the boundary can be used to restrict the solution space, decrease the number of degrees of freedom, speed up the computations, and increase stability. With such additional information, we can accurately reconstruct both the solution and its normal derivative on the remaining part of the boundary using little computational cost.

We point out that, in principle, in 2-dimensions one can apply conformal mappings to solve boundary value problems for the Laplace equation. We do not invoke this however, since we are interested in methods that can be generalized to 3-dimensional domains.

The outline of the paper is the following. In Section 2, we introduce some notation and function spaces. In Section 3, we recall the alternating method and some of its properties. Recursive forms of these methods in terms of updating boundary data are given in Section 4, and we also present direct methods for the Cauchy problem. Geometry and parameters for the numerical investigations are presented in Section 5. In Section 6, the numerical implementation of the alternating procedure is discussed, together with the integral equation method used, and the discretisations employed. To further improve the

results and efficiency, construction of suitable initial guesses of the solution on remaining part $\Gamma \setminus \overline{\Gamma}_{\mathrm{C}}$ of the boundary $\Gamma$ are given in Section 7, together with ideas on incorporation projections into the procedure. The final results show that accurate approximations can be obtained both for the solution and its normal derivative on the remaining part of the boundary using very little computational time.

## 2 Notation and function spaces

Let $\Omega$, $\Gamma$ and $\Gamma_{\mathrm{C}}$ be as above and define $\Gamma_{\mathrm{U}} = \Gamma \setminus \overline{\Gamma}_{\mathrm{C}}$, and note that $\Gamma_{\mathrm{U}}$ is the open part of the boundary where the solution and its normal derivative are unknown in (1). As usual, $H^1(\Omega)$ is the Sobolev space of real-valued functions in $\Omega$ with finite norm

$$\|u\|_{H^1(\Omega)} = \left( \int_\Omega u^2 \, dx + \int_\Omega |\nabla u|^2 \, dx \right)^{1/2}, \tag{2}$$

where $\nabla = (\partial_{x_1}, \partial_{x_2})$. By $H_0^1(\Omega)$, we denote the subspace of functions of $H^1(\Omega)$ that vanish on $\Gamma$. The space of functions in $H^1(\Omega)$ vanishing on $\Gamma_{\mathrm{C}}$ ($\Gamma_{\mathrm{U}}$), is denoted by $H^1_{\Gamma_{\mathrm{C}}}(\Omega)$ ($H^1_{\Gamma_{\mathrm{U}}}(\Omega)$).

The space of traces of functions from $H^1(\Omega)$ on $\Gamma$, is denoted by $H^{1/2}(\Gamma)$. This space is equipped with the norm

$$\|u\|_{H^{1/2}(\Gamma)} = \left( \int_\Gamma u(x)^2 \, d\sigma_x + \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^2}{|x-y|^2} \, d\sigma_x \, d\sigma_y \right)^{1/2}, \tag{3}$$

where $d\sigma$ is an element of arc-length. Restrictions of elements in $H^{1/2}(\Gamma)$ to the boundary part $\Gamma_{\mathrm{C}}$ ($\Gamma_{\mathrm{U}}$) constitute the space $H^{1/2}(\Gamma_{\mathrm{C}})$ ($H^{1/2}(\Gamma_{\mathrm{U}})$), where the norm is defined by (3) with $\Gamma$ replaced by $\Gamma_{\mathrm{C}}$ ($\Gamma_{\mathrm{U}}$).

We also use the space $H^{1/2}_{00}(\Gamma_{\mathrm{C}})$ that consists of elements from $H^{1/2}(\Gamma)$ vanishing on $\Gamma_{\mathrm{U}}$. This is then a subspace of $H^{1/2}(\Gamma)$ and one of the equivalent norms in this space is

$$\|u\|_{H^{1/2}_{00}(\Gamma_{\mathrm{C}})} = \left( \int_{\Gamma_{\mathrm{C}}} \frac{u^2(x)}{\mathrm{dist}(x, \Gamma_{\mathrm{U}})} \, d\sigma_x + \int_{\Gamma_{\mathrm{C}}} \int_{\Gamma_{\mathrm{C}}} \frac{|u(x) - u(y)|^2}{|x-y|^2} \, d\sigma_x \, d\sigma_y \right)^{1/2}. \tag{4}$$

In the similar way, $H^{1/2}_{00}(\Gamma_{\mathrm{U}})$ is defined.

## 3 The alternating method and some of its properties

To present the alternating method of [18] for finding a stable approximation to (1), we introduce the following mixed boundary value problems:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g_{\mathrm{U}} & \text{on } \Gamma_{\mathrm{U}}, \\ u = f_{\mathrm{C}} & \text{on } \Gamma_{\mathrm{C}}, \end{cases} \tag{5}$$

and

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f_{\mathrm{U}} & \text{on } \Gamma_{\mathrm{U}}, \\ \dfrac{\partial u}{\partial \nu} = g_{\mathrm{C}} & \text{on } \Gamma_{\mathrm{C}}. \end{cases} \tag{6}$$

The procedure is then:

(i) The first approximation, $u_1$, to $u$ of (1), is obtained by solving (5) with $g_{\mathrm{U}} = g_0$, where $g_0 \in (H^{1/2}_{00}(\Gamma_{\mathrm{U}}))^*$ is an arbitrary initial guess of the normal derivative on $\Gamma_{\mathrm{U}}$.

(ii) Given that $u_{2k-1}$ has been constructed, we find $u_{2k}$ by solving problem (6) with $f_{\mathrm{U}} = f_{k-1/2}$, where $f_{k-1/2} = u_{2k-1}|_{\Gamma_{\mathrm{U}}}$.

(iii) The element $u_{2k+1}$ is obtained by solving (5) with $g_{\mathrm{U}} = g_k$, where

$$g_k = \frac{\partial u_{2k}}{\partial \nu}|_{\Gamma_{\mathrm{U}}}. \tag{7}$$

In case of exact Cauchy data the alternating procedure continues by iterating in the last two steps.

Note that the alternating method puts few restrictions on the governing operator and the solution domain. In fact, it can be applied to strongly elliptic self-adjoint operators in Lipschitz domains.

In the remark below we give another version of the alternating method and, to distinguish between the various sequences obtained, we use the index $k$ for the normal derivative on $\Gamma_{\mathrm{U}}$ and $k + 1/2$ for function values on $\Gamma_{\mathrm{U}}$ in the first method. The opposite notation, $k$ for function values and $k + 1/2$ for derivatives on $\Gamma_{\mathrm{U}}$, is used in the second method.

*Remark* 3.1. We point out that instead of starting the alternating procedure by guessing the normal derivative on $\Gamma_{\mathrm{U}}$ and solving (5), one can instead make a guess of the function itself on $\Gamma_{\mathrm{U}}$, say $f_{\mathrm{U}} = f_0$, and start by solving (6). The scheme is then:

(i) The first approximation, $u_1$, to $u$ of (1), is obtained by solving (6) with $f_{\mathrm{U}} = f_0$, where $f_0 \in H^{1/2}(\Gamma_{\mathrm{U}})$ is an arbitrary initial guess of the solution on $\Gamma_{\mathrm{U}}$.

(ii) Given that $u_{2k-1}$ has been constructed, we find $u_{2k}$ by solving problem (5) with $g_{\mathrm{U}} = g_{k-1/2}$, where $g_{k-1/2} = \frac{\partial u_{2k-1}}{\partial \nu}|_{\Gamma_{\mathrm{U}}}$.

(iii) The element $u_{2k+1}$ is obtained by solving (6) with $f_{\mathrm{U}} = f_k$, where

$$f_k = u_{2k}|_{\Gamma_{\mathrm{U}}}. \tag{8}$$

We will refer to this latter method as the second version of the alternating procedure.

In Section 4, we rewrite these schemes in term of recursions in boundary data on $\Gamma_{\mathrm{U}}$ and then these recursions will be numerically implemented and investigated in Sections 6 and 7.

### 3.1 Convergence

For the convergence of the alternating method, we have the following result [18, 19]:

**Theorem 3.2.** *Let $f_{\mathrm{C}} \in H^{1/2}(\Gamma_{\mathrm{C}})$ and $g_{\mathrm{C}} \in (H^{1/2}_{00}(\Gamma_{\mathrm{C}}))^*$ be chosen such that (1) has a solution $u \in H^1(\Omega)$. Let $u_k$ be the $k$-th approximate solution in the alternating procedure. Then*

$$\lim_{k\to\infty} \|u - u_k\|_{H^1(\Omega)} = 0 \tag{9}$$

*for any initial data element $g_0 \in (H^{1/2}_{00}(\Gamma_{\mathrm{U}}))^*$.*

Note that using trace estimates, we also get convergence of $u_k|_{\Gamma_{\mathrm{U}}}$ to the correct solution. Moreover, as was pointed out in [4], in the interior, we have convergence also of derivatives of higher order.

**Corollary 3.3.** *Let the assumptions of Theorem 3.2 be fulfilled and let $\Omega'$ be a domain with $\overline{\Omega'} \subset \Omega$. Then, for $l = 1, 2, \ldots,$*

$$\lim_{k\to\infty} \|u - u_k\|_{H^{l+1}(\Omega')} = 0 \tag{10}$$

*for any initial data $g_0 \in (H^{1/2}_{00}(\Gamma_{\mathrm{U}}))^*$.*

To see this, note that since $u_k - u$ satisfies system (1), we can use local estimates for the Laplace equation, and get

$$\|u_k - u\|_{H^{l+1}(\Omega')} \leq C\|u_k - u\|_{H^1(\Omega)}. \tag{11}$$

Then the results of the Corollary follow by referring to Theorem 3.2.

Let us briefly recall the ideas in [18, 19] of proving Theorem 3.2, since this gives a reformulation of the Cauchy problem that we shall use for effective numerical implementation.

Let $u_1$ be the solution to (5), for given functions $g_{\mathrm{U}}$ and $f_{\mathrm{C}} = 0$. Then let $u_2$ be the solution to (6) with $g_{\mathrm{C}} = 0$ and $f_{\mathrm{U}} = u_1$ on $\Gamma_{\mathrm{U}}$. The operator $B_{\mathrm{NN}} : (H^{1/2}_{00}(\Gamma_{\mathrm{U}}))^* \to (H^{1/2}_{00}(\Gamma_{\mathrm{U}}))^*$ is defined by

$$B_{\mathrm{NN}} g_{\mathrm{U}} = \frac{\partial u_2}{\partial \nu}|_{\Gamma_{\mathrm{U}}}. \tag{12}$$

This is a well-defined linear operator. In the similar way, let $v_2$ be the element obtained from the second approximation in the alternating procedure, with initial guess $g_{\mathrm{U}} = 0$, and define the element $G_{\mathrm{NN}}(f_{\mathrm{C}}, g_{\mathrm{C}})$ by

$$G_{\mathrm{NN}}(f_{\mathrm{C}}, g_{\mathrm{C}}) = \frac{\partial v_2}{\partial \nu}|_{\Gamma_{\mathrm{U}}}. \tag{13}$$

The Cauchy problem (5) is equivalent with the fixed point equation

$$B_{\mathrm{NN}} g_{\mathrm{U}} + G_{\mathrm{NN}}(f_{\mathrm{C}}, g_{\mathrm{C}}) = g_{\mathrm{U}}. \tag{14}$$

Then in the alternating procedure when $G_{\mathrm{NN}}(f_{\mathrm{C}}, g_{\mathrm{C}}) = 0$, one can show that $g_k = B^k_{\mathrm{NN}} g_0$. Thus, for the convergence one has to investigate properties of the operator $B_{\mathrm{NN}}$. From [18], it can be shown that $B_{\mathrm{NN}}$ is self-adjoint, non-negative, non-expansive, and that the number one is not an eigenvalue. This implies convergence of the procedure (for non-discretized operators) in the Sobolev space stated in the above theorem.

*Remark* 3.4. For the second version of the alternating method, now let $u_1$ be the solution to (6), for given functions $f_{\mathrm{U}}$ and $g_{\mathrm{C}} = 0$ and let $u_2$ be the solution to (5) with $f_{\mathrm{C}} = 0$ and

$$g_{\mathrm{U}} = \frac{\partial u_1}{\partial \nu}|_{\Gamma_{\mathrm{U}}}. \tag{15}$$

The operator $B_{\mathrm{DD}} : H^{1/2}(\Gamma_{\mathrm{U}}) \to H^{1/2}(\Gamma_{\mathrm{U}})$ is defined by

$$B_{\mathrm{DD}} f_{\mathrm{U}} = u_2|_{\Gamma_{\mathrm{U}}}. \tag{16}$$

Then let $v_2$ be the element obtained from the second approximation in this version of the alternating procedure, with initial guess $f_{\mathrm{U}} = 0$, and define the element $G_{\mathrm{DD}}(f_{\mathrm{C}}, g_{\mathrm{C}})$ by

$$G_{\mathrm{DD}}(f_{\mathrm{C}}, g_{\mathrm{C}}) = v_2|_{\Gamma_{\mathrm{U}}}. \tag{17}$$

The Cauchy problem (5) is then equivalent with the fixed point equation

$$B_{\mathrm{DD}} f_{\mathrm{U}} + G_{\mathrm{DD}}(f_{\mathrm{C}}, g_{\mathrm{C}}) = f_{\mathrm{U}}. \tag{18}$$

To prove convergence in this case, let $Sf_{\mathrm{U}}$ be the restriction of the normal derivative to $\Gamma_{\mathrm{U}}$ of the solution to the Laplace equation in $\Omega$ with Dirichlet conditions $u = f_{\mathrm{C}}$ on $\Gamma_{\mathrm{C}}$ and $u = f_{\mathrm{U}}$ on $\Gamma_{\mathrm{U}}$. Starting the alternating method with $g_0 = Sf_0$, the second step and onwards will be precisely like starting the second version of the alternating procedure with initial guess $f_{\mathrm{U}} = f_0$. Thus, from Theorem 3.2, convergence is settled also for the second version of the alternating procedure. Corresponding to (14) and (18), we have the equation

$$B_{\mathrm{NN}} S f_{\mathrm{U}} + G_{\mathrm{NN}}(f_{\mathrm{C}}, g_{\mathrm{C}}) = S f_{\mathrm{U}}. \tag{19}$$

# 4 Recursive schemes of the alternating methods

## 4.1 Recursion in the original alternating method

We can rewrite the alternating procedure in various ways for an effective numerical implementation. In particular, one only needs to construct function values or normal derivatives on the boundary part $\Gamma_{\mathrm{U}}$. To exemplify this, let $B_{\mathrm{NN}}$ and $G_{\mathrm{NN}}$ be defined by (12) and (13), respectively. Then the original alternating method can be written in the form

$$g_{k+1} = B_{\mathrm{NN}} g_k + G_{\mathrm{NN}}(f_{\mathrm{C}}, g_{\mathrm{C}}) \quad \text{for} \quad k = 0, 1, 2, 3, \ldots, \tag{20}$$

where

$$g_k = \frac{\partial u_{2k}}{\partial \nu}|_{\Gamma_{\mathrm{U}}} \tag{21}$$

and $g_0$ is the initial guess. The element $g_0$ is usually taken to be zero; another natural guess is $G_{\mathrm{NN}}(f_{\mathrm{C}}, g_{\mathrm{C}})$. The evaluation of the operator $B_{\mathrm{NN}}$ involves the solution of two mixed boundary value problems corresponding to steps (ii) and (iii) in the alternating procedure. In fact, the operator $B_{\mathrm{NN}}$ can be split into a product of two operators. To show this, let

$$B_{\mathrm{DN}} g = u|_{\Gamma_{\mathrm{U}}}, \tag{22}$$

where $u$ solves (5) with $f_{\mathrm{C}} = 0$ and $g_{\mathrm{U}} = g$. Furthermore, let

$$B_{\mathrm{ND}} f = \frac{\partial u}{\partial \nu}|_{\Gamma_{\mathrm{U}}}, \tag{23}$$

with $u$ solving (6) with $g_{\mathrm{C}} = 0$ and $f_{\mathrm{U}} = f$. Then one can check that

$$B_{\mathrm{NN}} g = B_{\mathrm{ND}}(B_{\mathrm{DN}} g). \tag{24}$$

We discuss how this operator can be efficiently discretized and numerically evaluated in Sections 6 and 7.

Similarly, we introduce

$$G_{\mathrm{DC}} f = v|_{\Gamma_{\mathrm{U}}}, \tag{25}$$

where $v$ solves (5) with $f_{\mathrm{C}} = f$ and $g_{\mathrm{U}} = 0$, and

$$G_{\mathrm{NC}} g = \frac{\partial v}{\partial \nu}|_{\Gamma_{\mathrm{U}}}, \tag{26}$$

with $u$ solving (6) with $g_{\mathrm{C}} = g$ and $f_{\mathrm{U}} = 0$. Note that

$$G_{\mathrm{NN}}(f_{\mathrm{C}}, g_{\mathrm{C}}) = B_{\mathrm{ND}} G_{\mathrm{DC}} f_{\mathrm{C}} + G_{\mathrm{NC}} g_{\mathrm{C}} \tag{27}$$

and

$$G_{\mathrm{DD}}(f_{\mathrm{C}}, g_{\mathrm{C}}) = G_{\mathrm{DC}} f_{\mathrm{C}} + B_{\mathrm{DN}} G_{\mathrm{NC}} g_{\mathrm{C}}. \tag{28}$$

Given the normal derivative $g_k$ on $\Gamma_{\mathrm{U}}$, we can obtain the function value $f_{k+1/2}$ by performing a single-step as

$$f_{k+1/2} = B_{\mathrm{DN}} g_k + G_{\mathrm{DC}} f_{\mathrm{C}}. \tag{29}$$

We can also find a representation for the function values $f_{k+1/2}$ on $\Gamma_{\mathrm{U}}$ corresponding to (20). This formula can be obtained by using the operators for the second version of the alternating method and will be derived in the next section.

We point out that in a direct method instead of iterating we can rewrite (14) as

$$(I - B_{\mathrm{NN}})g = G_{\mathrm{NN}} \tag{30}$$

and solve for $g$. Then, if needed, simply compute $f$ using (29).

### 4.2 Recursion in the second version of the alternating method

Let $B_{\mathrm{DD}}$ and $G_{\mathrm{DD}}$ be defined by (16) and (17), respectively. Then the second version of the alternating method can be written in the form

$$f_{k+1} = B_{\mathrm{DD}} f_k + G_{\mathrm{DD}}(f_{\mathrm{C}}, g_{\mathrm{C}}) \quad \text{for} \quad k = 0, 1, 2, 3, \ldots, \tag{31}$$

where $f_k = u_{2k}|_{\Gamma_{\mathrm{U}}}$ and $f_0$ is the initial guess. The element $f_0$ is usually taken to be zero, another natural guess is $G_{\mathrm{DD}}(f_{\mathrm{C}}, g_{\mathrm{C}})$. One can check, using the operators introduced in the previous section, that

$$B_{\mathrm{DD}} f = B_{\mathrm{DN}}(B_{\mathrm{ND}} f). \tag{32}$$

Given the function value $f_k$ on $\Gamma_{\mathrm{U}}$, we can obtain the normal derivative $g_{k+1/2}$ from

$$g_{k+1/2} = B_{\mathrm{ND}} f_k + G_{\mathrm{NC}} g_{\mathrm{C}}. \tag{33}$$

There is also a formula for the normal derivative using $B_{\mathrm{NN}}$. Simply use $g_{1/2}$ from (33) in (20), with, for example, $f_0 = 0$.

In the same way, there is a formula for function values in the original alternating method using $B_{\mathrm{DD}}$ and it is obtained by using $f_{1/2}$ from (29) in (31) with, for example, initial guess $g_0 = 0$.

We point out also here that in a direct method instead of iterating, in the second version of the alternating method, we can rewrite (18) as

$$(I - B_{\mathrm{DD}})f = G_{\mathrm{DD}} \tag{34}$$

and solve for $f$. Then, if needed, simply compute $g$ using (33).

*Remark* 4.1. Since solving the Cauchy problem (1) is equivalent of finding a fixed point of equation (14), then according to [24, Chapt. 3, Sect. 3], the discrepancy principle can be employed as a stopping rule for the alternating method in the case of noisy data. Thus let the noisy data $f_{\mathrm{C}}^{\delta}$ and $g_{\mathrm{C}}^{\delta}$, where $\delta > 0$, be given such that

$$\|G_{NN}(f_{\mathrm{C}}^{\delta}, g_{\mathrm{C}}^{\delta}) - G_{NN}(f_{\mathrm{C}}, g_{\mathrm{C}})\| \leq \delta. \tag{35}$$

Then if $k = k(\delta)$ is the smallest integer with

$$\left\| \frac{\partial u_{2k+1}^{\delta}}{\partial \nu} - \frac{\partial u_{2k-1}^{\delta}}{\partial \nu} \right\| \leq b\delta \tag{36}$$

for some given $b > 1$, then $u_{k(\delta)}^{\delta}$ converges to the exact solution of (1) when $\delta \to 0$.

*Remark* 4.2. For the second version of the alternating method, we have equation (18) to solve. Thus, one can apply the discrepancy principle also for that method.

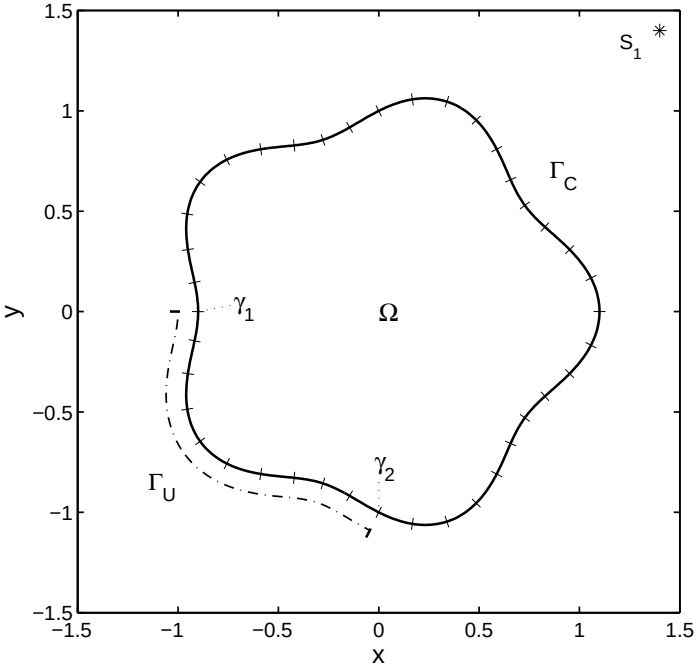


Figure 1: The solution domain $\Omega$ with boundary $\Gamma = \Gamma_{\rm U} \cup \Gamma_{\rm C}$ given by (37) and (38). The closure of the arcs $\Gamma_{\rm U}$ and $\Gamma_{\rm C}$ meet at the two points $\gamma_1$ and $\gamma_2$. A mesh of 40 quadrature panels is constructed on $\Gamma$, 10 of which are located on $\Gamma_{\rm U}$. A source $S_1$, for the generation of Cauchy data via (39), is marked by '$*$'.

## 5 Test geometry, software, and hardware for the numerics

To keep the notation short we make no distinction between points in the real plane $\mathbf{R}^2$ and points in the complex plane $\mathbf{C}$; all points will be denoted $z$ or $\tau$. We mainly use the same geometry throughout the numerical experiments, namely an interior domain with boundary parameterization

$$\tau(t) = (1 + 0.1\cos 5t)e^{{\rm i}t}\,, \quad -\pi < t \le \pi\,, \tag{37}$$

see Fig. 1, but we shall also vary this geometry to see how other geometries influence the numerical results. In fact, for the integral equation method outline in the next section, the main requirement on the solution domain is that its boundary can be parameterized by a piecewise smooth function $\tau(t)$.

The boundary given by (37) is simple to produce, yet not trivial since its curvature is varying. The two arcs $\Gamma_{\rm U}$ and $\Gamma_{\rm C}$ are

$$\tau(t) \in \Gamma_{\rm U}\,, \quad -\pi < t < -\frac{\pi}{2}\,, \quad \text{and} \quad \tau(t) \in \Gamma_{\rm C}\,, \quad -\frac{\pi}{2} < t < \pi\,. \tag{38}$$

The two points where the closure of $\Gamma_{\rm U}$ and $\Gamma_{\rm C}$ meet are $\gamma_1 = \tau(\pi)$ and $\gamma_2 = \tau(-\pi/2)$, and are referred to as *singular boundary points*. The Cauchy data will be chosen compatible with the harmonic function

$$u(z) = \Re\left\{\frac{1}{z - S_1}\right\}\,, \tag{39}$$

where $S_1 = 1.4 + 1.4{\rm i}$, see Fig. 1.

All numerical experiments presented are performed in MATLAB version 7.6 and executed on an ordinary workstation equipped with an Intel Core2 Duo E8400 CPU at 3.00 GHz.

# 6 An integral equation method for the mixed problems

The numerical solution to the mixed problems (5) and (6) can be obtained by a recently developed integral equation based solver [13], which we now briefly review.

## 6.1 Fredholm second kind equations

Fredholm second kind integral equations

$$\left(I + K^{(j)}\right) \rho^{(j)}(z) = h^{(j)}(z)\,, \quad j = 1, 2\,, \quad z \in \Gamma\,, \tag{40}$$

are constructed for (5) and (6), corresponding to equations (22)–(23) in [13]. Here superscript '(1)' refers to (5) and superscript '(2)' refers to (6) and $K^{(j)}$ are integral operators. Due to the singular boundary points, $K^{(j)}$ is not compact, but each $K^{(j)}$ can be decomposed into the sum of a compact and a non-compact operator. The right-hand sides in (40) are

$$h^{(1)}(z) = g_{\mathrm{U}}(z)\,, \quad h^{(2)}(z) = f_{\mathrm{U}}(z)\,, \quad z \in \Gamma_{\mathrm{U}}\,, \tag{41}$$

$$h^{(1)}(z) = f_{\mathrm{C}}(z)\,, \quad h^{(2)}(z) = g_{\mathrm{C}}(z)\,, \quad z \in \Gamma_{\mathrm{C}}\,. \tag{42}$$

The solutions sought, $u(z)$ or $\partial u(z)/\partial\nu$ with $z \in \Gamma_{\mathrm{U}}$, can be obtained from the layer density $\rho^{(j)}(\tau)$ via integral representations that are a mix of single-, double-, and quadruple-layer potentials, see further the equations (21) and (24) in [13].

## 6.2 Discretization and transformation of the integral equations

The integral equations (40) are discretized using a Nyström scheme, relying on composite 16-point Gauss–Legendre quadrature as the basic quadrature tool; for more on Nyström schemes, see [2, Chaper 4]. A number $n_\Lambda = 40$ of quadrature panels, equisized in parameter, are, for this purpose, placed on $\Gamma$, see Fig. 1. This corresponds to $N = 640$ discretization points on $\Gamma$, of which $N_{\mathrm{U}} = 160$ points are located on $\Gamma_{\mathrm{U}}$ and $N_{\mathrm{C}} = 480$ points are located on $\Gamma_{\mathrm{C}}$. This is more than sufficient for full resolution, that means achieving the highest possible accuracy in the solver given the boundary and the boundary conditions of Fig. 1. Hypersingular and logarithmic integral operators are discretized using a mix of two techniques denoted local regularization and panelwise evaluation, respectively, see Section 2 of [13].

Upon discretization, the integral equations (40) are transformed, using a recursive compressed inverse preconditioning technique, to resemble the discretizations of Fredholm second kind integral equations with operators that are everywhere compact. The result can be written as

$$\left(\mathbf{I} + \mathbf{K}^{\circ(j)}\mathbf{R}^{(j)}\right) \tilde{\boldsymbol{\rho}}^{(j)} = \mathbf{h}^{(j)}\,, \quad j = 1, 2\,, \tag{43}$$

where all matrices have size $N \times N$, $\mathbf{I}$ is the identity matrix, $\mathbf{K}^{\circ(j)}$ corresponds to the discretization of a compact integral operator, $\mathbf{R}^{(j)}$ is a compressed weighted inverse, $\tilde{\boldsymbol{\rho}}^{(j)}$ is a discrete transformed layer density, and $\mathbf{h}^{(j)}$ is the discretized right-hand side (function values at the discretization points), see further equation (34) of [13].

### 6.3 Post-processor

Once (43) is solved for $\tilde{\boldsymbol{\rho}}^{(j)}$, discrete Neumann–Dirichlet and Dirichlet–Neumann maps can be computed in post-processors

$$\mathbf{h}^{(2)} = \mathbf{L}^{(1)}\tilde{\boldsymbol{\rho}}^{(1)} \quad \text{and} \quad \mathbf{h}^{(1)} = \mathbf{L}^{(2)}\tilde{\boldsymbol{\rho}}^{(2)}\,, \tag{44}$$

where $\mathbf{L}^{(j)}$ are $N \times N$ matrices. If the right-hand sides $h^{(j)}(z)$ of (40) are piecewise smooth, then the scheme given in [13] can produce highly accurate results irrespective of what the smoothness of the solution is in $\Omega$.

When $N$ is large and one only has to solve mixed problems a small number of times it is appropriate to use an iterative solver for (43) and implement the action of the post-processor (44) without explicitly forming the matrices $\mathbf{L}^{(j)}$. This is the approach taken in equations (39) and (49) of [13]. In the present context, where $N$ is small and we may wish to solve millions of mixed problems, it is effective in the end to construct the $N \times N$ matrices

$$\mathbf{A}^{(j)} = \mathbf{L}^{(j)}\left(\mathbf{I} + \mathbf{K}^{\circ(j)}\mathbf{R}^{(j)}\right)^{-1}, \quad j = 1, 2\,. \tag{45}$$

Let the matrices $\mathbf{B}_{\mathrm{DN}}$, $\mathbf{G}_{\mathrm{DC}}$, $\mathbf{B}_{\mathrm{ND}}$, and $\mathbf{G}_{\mathrm{NC}}$ be discretizations via the above schemes of the operators $B_{\mathrm{DN}}$, $G_{\mathrm{DC}}$, $B_{\mathrm{ND}}$, and $G_{\mathrm{NC}}$ of (22), (25), (23), and (26), respectively. Then these matrices can be extracted as blocks from $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ in the following way:

$$\mathbf{B}_{\mathrm{DN}} = A^{(1)}_{i_{\mathrm{U}},i_{\mathrm{U}}}\,, \quad \mathbf{G}_{\mathrm{DC}} = A^{(1)}_{i_{\mathrm{U}},i_{\mathrm{C}}}\,, \quad \mathbf{B}_{\mathrm{ND}} = A^{(2)}_{i_{\mathrm{U}},i_{\mathrm{U}}}\,, \quad \text{and} \quad \mathbf{G}_{\mathrm{NC}} = A^{(2)}_{i_{\mathrm{U}},i_{\mathrm{C}}}\,, \tag{46}$$

where $i_{\mathrm{U}}$ is the set of indices for points on $\Gamma_{\mathrm{U}}$ and $i_{\mathrm{C}}$ is the set of indices for points on $\Gamma_{\mathrm{C}}$.

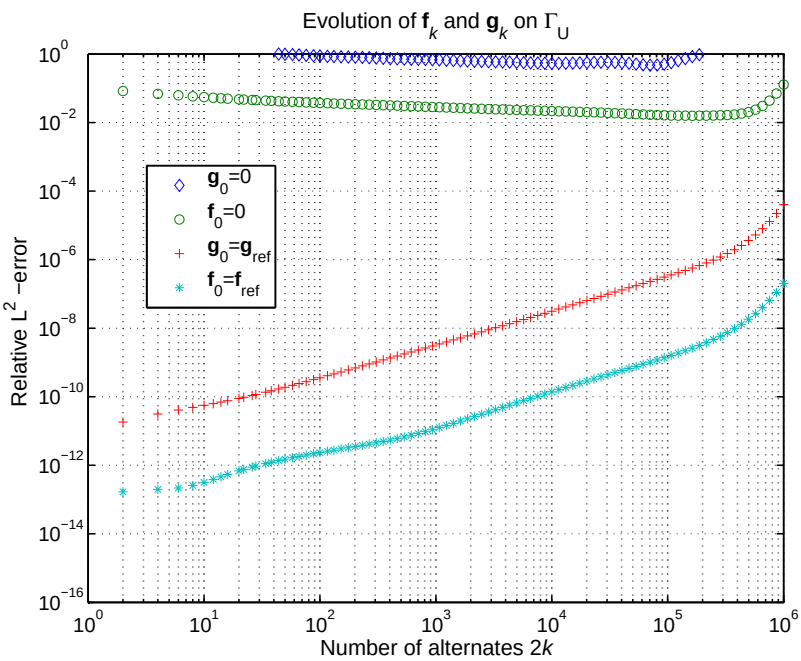


Figure 2: The double-step recursions (31) and (20) are run for $2k = 0, 2, 4, \ldots, 10^6$ with the initial guesses $\mathbf{f}_0 = 0$, $\mathbf{f}_0 = \mathbf{f}_{\mathrm{ref}}$, $\mathbf{g}_0 = 0$, and $\mathbf{g}_0 = \mathbf{g}_{\mathrm{ref}}$. The symbols '$\circ$' and '$*$' refer to $\mathbf{f}_k$, while '$\diamond$' and '$+$' refer to $\mathbf{g}_k$.

The discretizations of the operators $B_{\mathrm{NN}}$, $G_{\mathrm{NN}}$, $B_{\mathrm{DD}}$, and $G_{\mathrm{DD}}$ can be obtained by replacing these operators with their discrete counterparts in (12), (13), (16) and (17). The

discretizations of $f_k$ and $g_k$ are denoted $\mathbf{f}_k$ and $\mathbf{g}_k$. From now on in the text we shall refer to equations and recursions and let the context determine whether we mean discretized or continuous versions. Discretized quantities will always appear in boldface.

### 6.4 Accuracy, stability, and speed

To illustrate the intrinsic stability and speed of our solver we run the double-step recursions (31) and (20) until $2k = 10^6$, starting with initial guesses $\mathbf{f}_0 = \mathbf{f}_{\text{ref}}$ and $\mathbf{g}_0 = \mathbf{g}_{\text{ref}}$, which is the reference solution on $\Gamma_{\text{U}}$ given by (39). The evolution of $\mathbf{f}_k$ and $\mathbf{g}_k$ are shown in Fig. 2. The relative $L^2$-error at step $k = 1$ is only $2 \cdot 10^{-13}$ for $\mathbf{f}_1$ and $2 \cdot 10^{-11}$ for $\mathbf{g}_1$, which confirm the high accuracy claimed for our solver. As $k$ increases, the errors grow, and this is due to the fact that the spectral radii of $\mathbf{B}_{\text{DD}}$ and $\mathbf{B}_{\text{NN}}$ are larger than unity. In fact, the largest eigenvalue of both these matrices is approximately 1.00001. Note that, for example, the continuous operator $B_{\text{DD}}$ acts on $H^{1/2}(\Gamma_U)$, thus properties such as being non-expansive and self-adjoint are inherited for the discretization of this operator with respect to a discretization of the inner product in $H^{1/2}(\Gamma_U)$ and do therefore not necessarily hold in the standard $L^2$-setting. The schemes used for the discretization might then not be fully optimal and to correct for this we shall introduce certain projections in the next section.

With 50 recursion steps in the construction of the $\mathbf{R}^{(j)}$ matrices, see Section 6 in [13] and in particular the discussion after equation (44) in [13], the setup time for the matrices of (46) is around 6 seconds. The time needed to reach $2k = 10^6$ in (20) and (31) is around 5.5 seconds per initial guess.

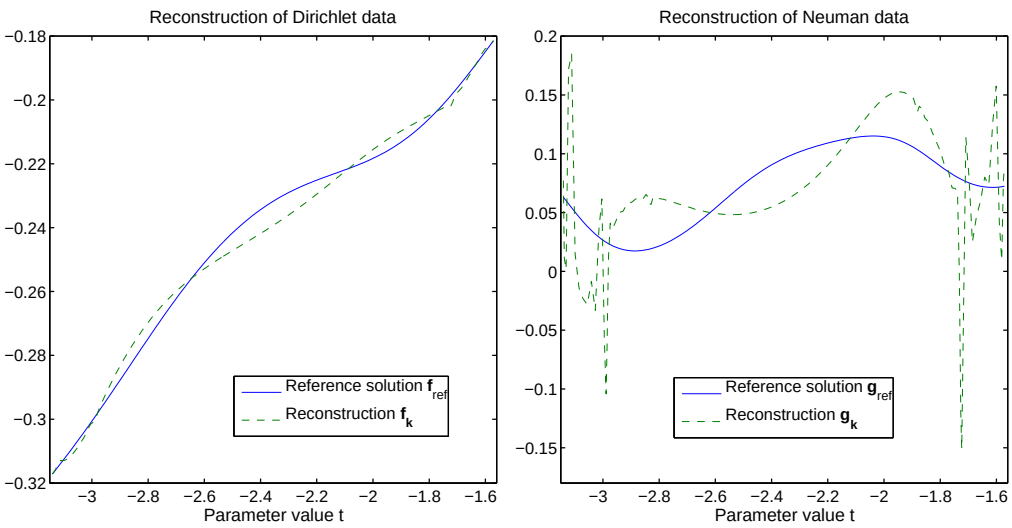


Figure 3: Reconstruction of the Dirichlet data $\mathbf{f}_k$ and Neumann data $\mathbf{g}_k$ with the double-step recursions (31) and (20) at $2k = 93260$. The initial guesses are $\mathbf{f}_0 = 0$ and $\mathbf{g}_0 = 0$, respectively.

For comparison, Fig. 2 also include results for the initial guesses $\mathbf{f}_0 = 0$ and $\mathbf{g}_0 = 0$. Fig. 3 shows $\mathbf{f}_k$ and $\mathbf{g}_k$ at $2k = 93260$, which is close to an optimal number of steps for both methods.

# 7 Improved reconstructions from additional information

Cauchy problems are known to be severely ill-posed and thus discretizations of them give rise to matrices that are notoriously ill-conditioned. Note that the spaces in which the data are taken from contain a large class of functions and not necessarily only continuous ones. The reconstructions of Fig. 3 are far from satisfactory, but may very well be the best one can get under such general assumptions. At least this holds in terms of accuracy. For improved speed one can rewrite (31) as

$$\mathbf{f}_k = \mathbf{B}_{\mathrm{DD}}^k \mathbf{f}_0 + \left(\mathbf{I} - \mathbf{B}_{\mathrm{DD}}^k\right) \left(\mathbf{I} - \mathbf{B}_{\mathrm{DD}}\right)^{-1} \mathbf{c}\,, \tag{47}$$

where

$$\mathbf{c} = \mathbf{G}_{\mathrm{DC}} \mathbf{f}_{\mathrm{C}} + \mathbf{B}_{\mathrm{DN}} \mathbf{G}_{\mathrm{NC}} \mathbf{g}_{\mathrm{C}}\,, \tag{48}$$

and compute $\mathbf{f}_k$, for any $k$, essentially by evaluating the single $N_{\mathrm{U}} \times N_{\mathrm{U}}$ matrix power $\mathbf{B}_{\mathrm{DD}}^k$. The results produced by the closed multi-step expression (47) are surprisingly similar to those produced by the double-step recursion (31). In fact, for the evolution of $\mathbf{f}_k$ in Fig. 2 and for both $\mathbf{f}_0 = 0$ and $\mathbf{f}_0 = \mathbf{f}_{\mathrm{ref}}$, the relative difference between results produced by (31) and (47) is of the order of machine precision for small $k$ and not larger than $10^{-8}$ for large $k$.

If one has some additional information, beside the Cauchy data, to feed into the solver then this can change the situation completely, also in terms of accuracy. In this section we show how one can achieve much improved reconstructions by supplying such extra information.

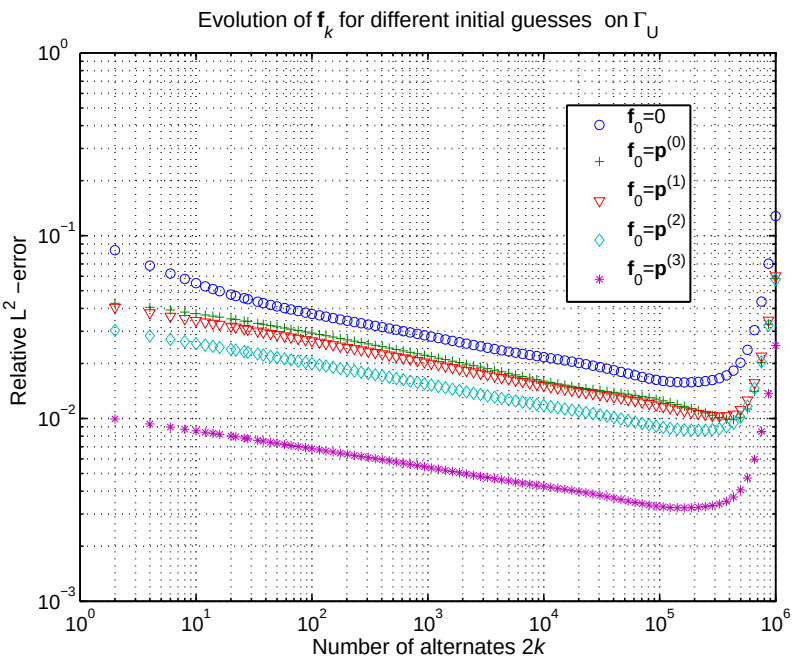


Figure 4: The double-step recursion (31) is run for $2k = 0, 2, 4, \ldots, 10^6$. The initial guess $\mathbf{f}_0$ is either zero or equal to a discretized matching polynomial $\mathbf{p}^{(m)}$ of degree $2m + 1$.

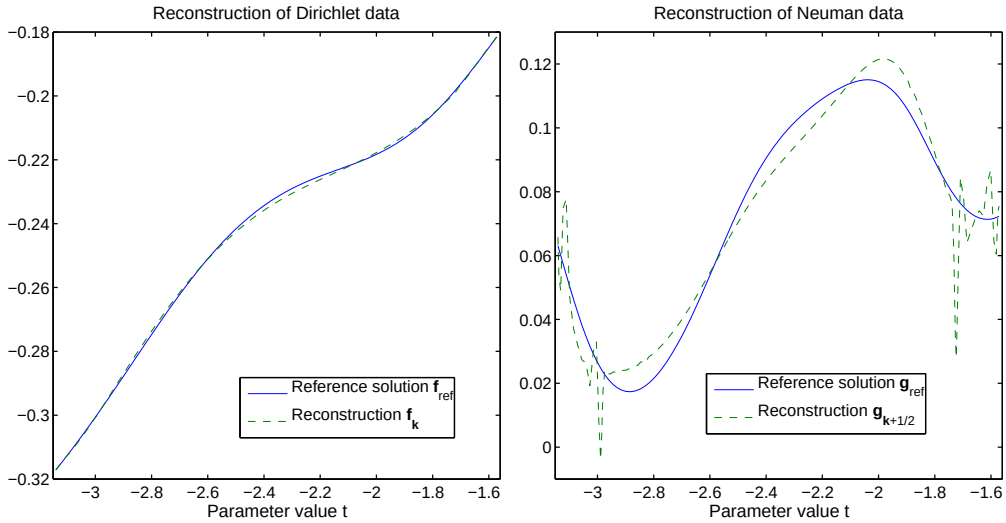


Figure 5: Reconstruction of Dirichlet data $\mathbf{f}_k$ and Neumann data $\mathbf{g}_{k+1/2}$ with the double-step recursion (31) at $2k = 93260$. The initial guess is $\mathbf{f}_0 = \mathbf{p}^{(3)}$. A single recursion step (33) is taken to get from $\mathbf{f}_k$ to $\mathbf{g}_{k+1/2}$.

### 7.1 Initial guesses

Information about the regularity of $u(z)$ and $\partial u(z)/\partial\nu$ across the two singular boundary points can be used to construct efficient initial guesses for the alternating schemes. As an example, assume that $u(z)$ and $m$ of its higher order tangential derivatives along $\Gamma$ are known at the points $\gamma_1$ and $\gamma_2$. Then one can construct a polynomial $p^{(m)}(t)$ of degree $2m+1$ in the parameter $t$ on $\Gamma_{\mathrm{U}}$, which matches these conditions. Upon discretization this polynomial can then be used as an initial guess for $\mathbf{f}_0$. Fig. 4 shows that with $\mathbf{p}^{(3)}$, which corresponds to the discretization of a matching polynomial of degree seven as initial guess, one can get almost an extra digit for $\mathbf{f}_k$ in the double-step recursion (31), compared to using $\mathbf{f}_0 = 0$ as initial guess. The improved reconstruction at $2k = 93260$, as is shown in Fig. 5, also illustrates this, compared with Fig. 3.

### 7.2 Smoothing projections

Information about the the regularity of the solution $u(z)$ and the normal derivative $\partial u(z)/\partial\nu$ on the interior of $\Gamma_{\mathrm{U}}$ is even more useful. It can be used to restrict the solution space, decrease the number of degrees of freedom, speed up the computations, and increase stability by reducing the largest eigenvalue of the $N_{\mathrm{U}} \times N_{\mathrm{U}}$ matrices $\mathbf{B}_{\mathrm{NN}}$ and $\mathbf{B}_{\mathrm{DD}}$. In the example of Fig. 1 we know that $u(z)$ and $\partial u(z)/\partial\nu$ are both smooth and can be well approximated with low degree polynomials on $\Gamma_{\mathrm{U}}$.

To use this smoothness information to improve the approximation, introduce a scaled parameter $s = 4t/\pi + 3$, that is $s \in [-1, 1]$ on $\Gamma_{\mathrm{U}}$, and consider the $n$ monomials $s^j$, $j = 0, \ldots, n-1$. Let $\mathbf{s}^{(j)}$ be the discretization of $s^j$ on $\Gamma_{\mathrm{U}}$, and let $\mathbf{Q}_n$ be a matrix whose columns form an orthonormal basis for the space spanned by the vectors $\mathbf{s}^{(j)}$. Clearly, $\mathbf{Q}_n$ can be obtained by a reduced $QR$-factorization of a Vandermonde matrix $\mathbf{V}_n$ with $\mathbf{s}^{(j)}$ as columns. Now introduce the representation of the vector $\mathbf{f}_k$ of length $N_{\mathrm{U}}$ in terms of the vector $\tilde{\mathbf{f}}_k$ of length $n$ via

$$\mathbf{f}_k = \mathbf{Q}_n \tilde{\mathbf{f}}_k \,. \tag{49}$$

Then, for example, the discrete projected counterpart of (31) reads

$$\tilde{\mathbf{f}}_{k+1} = \tilde{\mathbf{B}}_{\mathrm{DD}}\tilde{\mathbf{f}}_k + \tilde{\mathbf{c}}\,, \quad k = 0, 1, 2, 3, \ldots\,, \tag{50}$$

where

$$\tilde{\mathbf{B}}_{\mathrm{DD}} = \mathbf{Q}_n^{\mathrm{T}}\mathbf{B}_{\mathrm{DD}}\mathbf{Q}_n\,, \tag{51}$$

$$\tilde{\mathbf{c}} = \mathbf{Q}_n^{\mathrm{T}}\mathbf{G}_{\mathrm{DC}}\mathbf{f}_{\mathrm{C}} + \mathbf{Q}_n^{\mathrm{T}}\mathbf{B}_{\mathrm{DN}}\mathbf{G}_{\mathrm{NC}}\mathbf{g}_{\mathrm{C}}\,, \tag{52}$$

and where superscript 'T' denotes the transpose. The spectral radius of the projected $n \times n$ matrix $\tilde{\mathbf{B}}_{\mathrm{DD}}$ appears to grow monotonically with $n$, starting at 0 for $n = 1$. It may be a good idea to choose $n$ as the largest integer which still keeps this spectral radius less than unity. In the example of Fig. 1 this corresponds to $n \approx 15$.

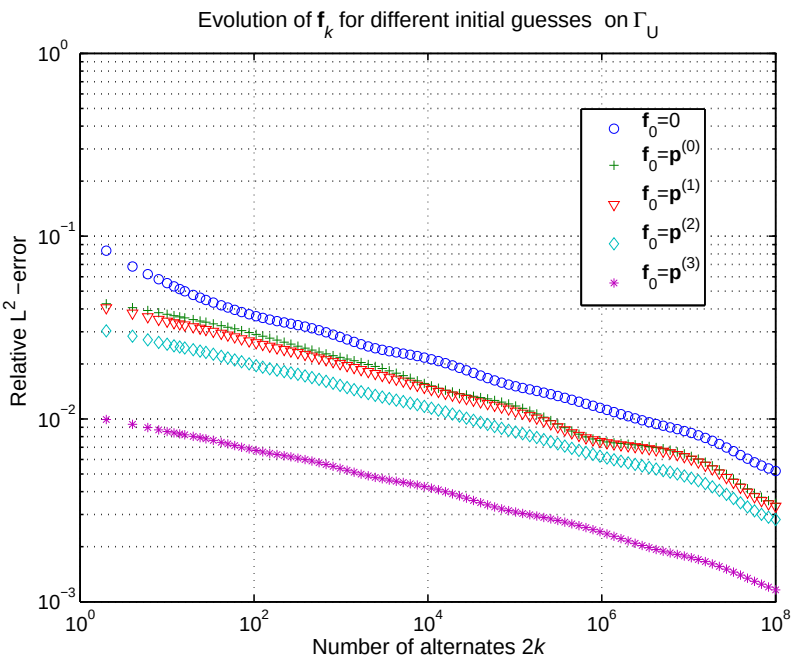


Figure 6: The projected double-step recursion (50) is run with $n = 15$ for $2k = 0, 2, 4, \ldots, 10^8$. The initial guess $\tilde{\mathbf{f}}_0 = \mathbf{Q}_n^{\mathrm{T}}\mathbf{f}_0$ is either zero or equal to a transformed discretized matching polynomial $\mathbf{Q}_n^{\mathrm{T}}\mathbf{p}^{(m)}$ of degree $2m + 1$.

Figure 6 shows how this projection stabilizes the alternating method; since the projected $n \times n$ matrix $\tilde{\mathbf{B}}_{\mathrm{DD}}$ of (50) is much smaller than the original $N_{\mathrm{U}} \times N_{\mathrm{U}}$ matrix $\mathbf{B}_{\mathrm{DD}}$ of (31), we can take more recursion steps in a given period of time. With $n = 15$ the time needed to reach $2k = 10^8$ in the double-step recursion (50) is around 60 seconds per initial guess.

### 7.3 Direct methods

Direct methods may be even more efficient than the alternating schemes in the context of smoothing projections, and especially if $n$ is small. The discrete projected counterpart of (34) reads

$$\left(\mathbf{I} - \tilde{\mathbf{B}}_{\mathrm{DD}}\right)\tilde{\mathbf{f}} = \tilde{\mathbf{c}}\,. \tag{53}$$

This $n \times n$ linear system can easily be solved with Gaussian elimination and partial pivoting (MATLAB's backslash for square system matrices) in less than a tenth of a millisecond. The solutions produced for $\mathbf{f}$ and $\mathbf{g}$ via $\mathbf{f} = \mathbf{Q}_n \tilde{\mathbf{f}}$ and $\mathbf{g} = \mathbf{B}_{\mathrm{ND}}\mathbf{f} + \mathbf{G}_{\mathrm{NC}}\mathbf{g}_{\mathrm{C}}$, are rather accurate. The relative $L^2$-error at $n = 14$, not shown in any figure, is around $2 \cdot 10^{-5}$ for $\mathbf{f}$ and around $5 \cdot 10^{-4}$ for $\mathbf{g}$. That is, they are approximately three digits better than the best results obtained by (31) and (20) with $\mathbf{f}_0 = 0$ and $\mathbf{g}_0 = 0$, see Figs. 2 and 4.

The highest quality in the reconstruction of $\mathbf{f}$ and $\mathbf{g}$ from Cauchy data in our example, however, is obtained by the representation of the vector $\mathbf{f}$ in terms of a vector $\hat{\mathbf{f}}$ and the Vandermonde matrix itself

$$\mathbf{f} = \mathbf{V}_n \hat{\mathbf{f}} \,. \tag{54}$$

The discrete counterpart of (34) now reads

$$\left(\mathbf{V}_n - \hat{\mathbf{B}}_{\mathrm{DD}}\right) \hat{\mathbf{f}} = \mathbf{c} \,, \tag{55}$$

where

$$\hat{\mathbf{B}}_{\mathrm{DD}} = \mathbf{B}_{\mathrm{DD}} \mathbf{V}_n \,, \tag{56}$$

and $\mathbf{c}$ is as in (48). The overdetermined $N_{\mathrm{U}} \times n$ linear system (55) is easily solved in the least squares sense with $QR$-factorization via Householder triangularization (MATLAB's backslash for rectangular system matrices).

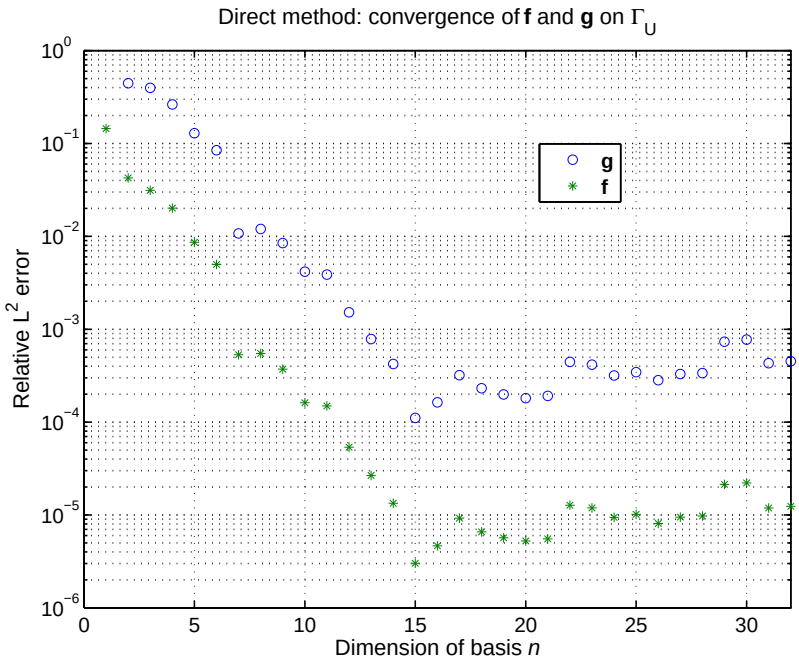


Figure 7: The direct method for $\mathbf{f}$ and $\mathbf{g}$ via (54), (55), and $\mathbf{g} = \mathbf{B}_{\mathrm{ND}}\mathbf{f} + \mathbf{G}_{\mathrm{NC}}\mathbf{g}_{\mathrm{C}}$.

Figure 7 shows the relative $L^2$-errors for the vectors $\mathbf{f} = \mathbf{V}_n \hat{\mathbf{f}}$ and $\mathbf{g} = \mathbf{B}_{\mathrm{ND}}\mathbf{f} + \mathbf{G}_{\mathrm{NC}}\mathbf{g}_{\mathrm{C}}$ for various $n$. The best results are obtained with $n = 15$, which is in agreement with the prediction of the eigenvalue analysis of Section 7.2. Furthermore, the system matrix of (55) becomes rank deficient in finite precision arithmetic when $n = 17$, and that too, can be used as a criterion for how large the number $n$ should be chosen. The relative error for $\mathbf{f}$ at $n = 15$ is around $3 \cdot 10^{-6}$ and for $\mathbf{g}$ around $10^{-4}$; these are almost four digits better than the best value obtained by (31) and (20) with $\mathbf{f}_0 = 0$ and $\mathbf{g}_0 = 0$, see Figs. 2 and 4.

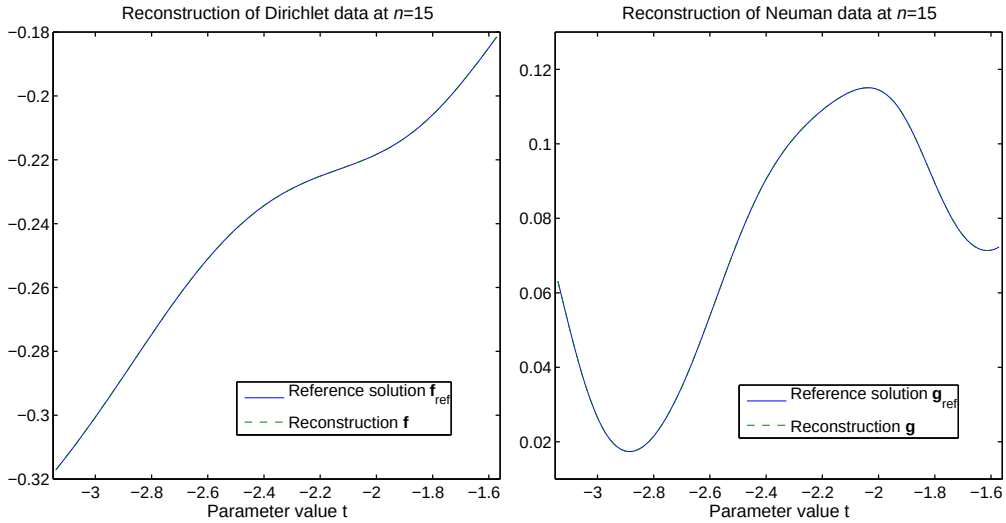


Figure 8: Reconstruction of the Dirichlet data $\mathbf{f}$ and Neumann data $\mathbf{g}$ via (54), (55), and $\mathbf{g} = \mathbf{B}_{\mathrm{ND}}\mathbf{f} + \mathbf{G}_{\mathrm{NC}}\mathbf{g}_{\mathrm{C}}$ at $n = 15$.

Figure 8 shows $\mathbf{f}$ and $\mathbf{g}$ at $n = 15$, and the agreement with the reference solutions is now excellent, compare with Figs. 3 and 5.

Encouraged by the fast and accurate results produced by the direct method (54) and (55), we also performed an experiment with a more challenging geometry, similar to that of Fig. 1, but where the interior domain has the boundary parameterization

$$\tau(t) = (1 + 0.3\cos 5t)e^{\mathrm{i}t}\,, \quad -\pi < t \leq \pi\,. \tag{57}$$

The number of quadrature panels $n_\Lambda$ placed on $\Gamma$ was increased to $n_\Lambda = 84$, to ensure full resolution, see Fig. 1 in [13] for an illustration, but note that the reference solution (39) is still generated by a single source $S_1$.

The results produced by (54) and (55) for the geometry of (57), not shown, are similar to those of Figs. 7 and 8. Accurate results are again obtained for $n \geq 15$, with the smallest relative $L^2$-error for $\mathbf{f}$ of around $3 \cdot 10^{-4}$, and for $\mathbf{g}$ of around $5 \cdot 10^{-3}$ at $n = 24$.

# 8 Conclusion

We have investigated an implementation of the alternating method [18] using a recent technique [13] based on Fredholm integral equations and Nyström discretization schemes to solve mixed problems for the Laplace equation, for fast and stable numerical reconstruction of harmonic functions from Cauchy data. The Cauchy problem is rewritten as a fixed point equation on the boundary and various direct regularizing techniques for solving this equation have also been investigated. Using the method in [13], it was shown that it is possible to perform millions of iterations with the alternating method in only a few seconds on an ordinary computer. It was also demonstrated how additional information about the smoothness of the data can be incorporated into the procedure to obtain very fast and accurate approximations of both the function and its normal derivative. Future work involves employing these ideas to equations of elasticity and to use other function spaces for further improvement of the performance of the methods, and finally to also consider conjugate gradient schemes like those of [10, 11].

**Acknowledgment** The first author was supported by the Swedish Research Council under the contract 621-2007-6234.

## References

[1] Andrieux, S., Baranger, T. N. and Ben Abda, A., Solving Cauchy problems by minimizing an energy-like functional, *Inverse Problems* **22** (2006), 115–133.

[2] Atkinson, K. E., *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, Cambridge, (1997).

[3] Avdonin, S., Kozlov, V., Maxwell, D., and Truffer, M., Iterative methods for solving a nonlinear boundary inverse problem in glaciology, *J. Inverse Ill-Posed Problems* **17** (2009), 239–258.

[4] Bastay, G., Johansson, T., Kozlov, V. and Lesnic, D., An alternating method for the stationary Stokes system, *ZAMM* **86** (2006), 268-280.

[5] Baumeister, J. and Leitão, A., On iterative methods for solving ill-posed problems modeled by partial differential equations, *J. Inv. Ill-Posed Probl.* **9** (2001), 13–29.

[6] Calderón, A.-P., Uniqueness in the Cauchy problem for partial differential equations, *Amer. J. Math.* **80** (1958), 16–36.

[7] Carleman, T., Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes, (French) *Ark. Mat., Astr. Fys.* **26** (1939), 1–9.

[8] Chapko, R. and Johansson, B. T., An alternating boundary integral based method for a Cauchy problem for Laplace equation in semi-infinite domains, *Inverse Probl. Imaging* **3** (2008), 317–333.

[9] Clerc, M. and Kybic, J., Cortical mapping by Laplace Cauchy transmission using a boundary element method, *Inverse Problems* **12** (2007), 2589–2601.

[10] Dinh Nho Hào and Reinhardt, H. J., Gradient methods for inverse heat conduction problems, *Inv. Probl. Engng.* **6** (1998), 177–211.

[11] Dinh Nho Hào and Lesnic, D., The Cauchy problem for Laplace's equation via the conjugate gradient method, *IMA J. Appl. Math.* **65** (2000), 199-217.

[12] Engl, H. W. and Leitão, A., A Mann iterative regularization method for elliptic Cauchy problems, *Numer. Funct. Anal. Optim.* **22** (2001), 861–884.

[13] Helsing, J., Integral equation methods for elliptic problems with boundary conditions of mixed type, *J. Comput. Phys.*, **228** (2009), 8892–8907.

[14] Johansson, B. T. and Lesnic, D., A relaxation of the alternating method for the reconstruction of a stationary flow from incomplete boundary data, *Advances in Boundary Integral Methods - Proceedings of the Seventh UK Conference on Boundary Integral Methods*, (Eds. H. Power, A. La Rocca and S. J. Baxter), University of Nottingham, UK, (2009), 161–169.

[15] Jourhmane, M. and Nachaoui, A., An alternating method for an inverse Cauchy problem, *Numer. Algorithms* **21** (1999), 247–260.

[16] Jourhmane, M. and Nachaoui, A., Convergence of an alternating method to solve the Cauchy problem for Poission's equation, *Appl. Anal.* **81** (2002), 1065–1083.

[17] Jourhmane, M, Lesnic, D., and Mera, N. S., Relaxation procedures for an iterative algorithm for solving the Cauchy problem for the Laplace equation, *Eng. Anal. Boundary Elements* **28** (2004), 655–665.

[18] Kozlov, V. A. and Maz'ya, V. G., On iterative procedures for solving ill-posed boundary value problems that preserve differential equations, *Algebra i Analiz* **1** (1989), 144–170. English transl.: *Leningrad Math. J.* **1** (1990), 1207–1228.

[19] Kozlov, V. A., Maz'ya, V. G. and Fomin, A. V., An iterative method for solving the Cauchy problem for elliptic equations, *Zh. Vychisl. Mat. i Mat. Fiz.* **31** (1991), 64–74. English transl.: *U.S.S.R. Comput. Math. and Math. Phys.* **31** (1991), 45–52.

[20] Lesnic, D., Elliot, L. and Ingham, D. B., An iterative boundary element method for solving numerically the Cauchy problem for the Laplace equation, *Eng. Anal. Bound. Elem.* **20** (1997), 123–133.

[21] Marin, L., Elliott, L., Ingham, D.B., and Lesnic, D., Boundary element method for the Cauchy problem in linear elasticity, *Eng. Anal. Boundary Elem.* **25** (2001), 783-793.

[22] Maxwell, D., Truffer, M., Avdonin, S. and Stuefer, M., Determining glacier velocities and stresses with inverse methods: an iterative scheme, *J. Glaciol.* **54** (2008), 888–898.

[23] Mera, N. S., Elliott, L., Ingham, D. B. and Lesnic, D., The boundary element solution of the Cauchy steady heat conduction problem in an anisotropic medium, *Int. J. Numer. Meth. Engng.* **49** (2000), 481–499.

[24] Vainikko, G. M. and Veretennikov, A. Y, *Iteration Procedures in Ill-Posed Problems*, Nauka Publ., Moscow, 1986 (in Russian).

[25] Yang, X., Choulli, M. and Cheng, J., An iterative BEM for the inverse problem of detecting corrosion in a pipe, *Numer. Math. J. Chin. Univ.* (Engl. Ser.) **14** (2005), 252–266.