Direct and inverse resonance problems for seismic surface waves

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Direct and inverse resonance problems for seismic surface waves

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DOCTORAL THESIS
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Abstract

This thesis concerns the investigation of direct and inverse resonance results for the Love and the Rayleigh operators, which arise from decoupling the three-dimensional elastic wave equation in the semiclassical limit in the half-space. Direct resonance results in terms of the asymptotic distribution of resonances in the complex plane are obtained for the Love and the Rayleigh operators. An inverse resonance characterization problem is obtained for the Love operator using a class of Jost function and also through a suitable class of Weyl-Tichtmarsh function.

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MADE IN SWEDEN

Dedicated to my parents, my brother and my grandparents
Patrizia – Orazio – Matteo – Matteo – Rosetta – Giovanna

Contents

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Popular summary in English

Resonant waves are waves whose amplitude decays over time (or space), which means they are present only for a rather short time (or in a rather short region of space). As with any kind of wave, they are determined by their frequency and wave number, which encode their lifetime and effective spatial range, respectively.

The thesis investigates the direct and inverse resonance results regarding seismic waves travelling close to the Earth's surface. First, we want to obtain information about the possible wave numbers for the resonant seismic waves, as well as information about the number of them. This information depends on the elastic properties of the material, in which those waves travel, and it is called direct result. We cannot obtain exact, but only asymptotic values for them, that is, values for large frequencies or large modulus of the wave number.

The typical example to explain what an inverse problem is in mathematics is the example of the drum. In that case, the inverse problem consists in determining uniquely the shape of a drum from the knowledge of the frequencies of its normal modes. Likewise here, supposing we know the values of the wave numbers of those resonant waves, we want to reconstruct the properties of the medium from them. This is what is called an inverse problem.

There are many applications of inverse problems in real life, for instance, in medicine with X-ray tomography measurements, in oil exploration measurements, where one can determine the presence of oil in a specific area of the Earth's inner layers from frequencies or wave numbers measured on the surface, or, in a similar way, in the study of the Mars' inner structure.

Preface

Preface

This thesis investigates the direct and inverse resonance problems for surface waves in seismology. The direct resonance problem aims to find information about resonances from a differential operator with certain properties. Resonances describe the oscillation and the decay of waves on non-compact domains and, likewise for the eigenvalues, they can be computed explicitly only in very few cases, such as the Eckart barrier potential. In general, it is only possible to determine the distribution of the resonances asymptotically, similar to the Weyl law for the eigenvalues. Resonances can be considered an analogue of eigenvalues (see [54]), as well as poles of the resolvent, for a class of operators with continuous spectrum. Unlike eigenvalues, the solutions of the differential equation at the resonances are not L^2. Therefore, to make things work, we need to introduce a cut-off function that allows us to extend the resolvent meromorphically to the whole complex plane.

The theory of inverse problems started to arise in 1966 after the paper of Kac "Can one hear the shape of a drum" (see [27]). In particular, the author posed the question of whether it were possible to draw the shape of a drum from the knowledge of the frequencies of its normal modes. Mathematically, this questions whether it is possible to reconstruct uniquely the domain of the Laplacian from the knowledge of its spectrum. This query remained unsolved for over 26 years, when finally in 1992 Gordon, Webb and Wolpert proved that the answer is negative in the general setting of a compact Riemannian manifold (see [24]). The answer would be positive if the domain were convex and with analytic boundary, as proved by Zelditch later on. Unlike this first example of inverse problem, in our case, the domain is fixed and we want to reconstruct the parameters that determine the differential operator from eigenvalues plus some other quantities, such as, in general, scattering or spectral data. In relation to seismology, this means reconstructing the parameters that determine the elasticity of the medium in the interior of the Earth from measurements performed on the boundary of the Earth's surface, which are, for example, the frequencies or the wave numbers of surface waves (eigenvalues and resonances). The Earth is a compact domain, but, for simplification, we consider it as a flat half space $\mathbb{R}^2 \times (-\infty, 0]$ prescribed with some boundary conditions rendering the problem an exterior boundary value problem.

It is important to stress that we study an inverse resonance problem, where the set of data is limited only to eigenvalues and resonances. The importance that inverse resonance problems carry is two-fold. On the one hand, it leads to data (eigenvalues and resonances), which are easily obtained in the laboratory from scattering cross sections as opposed to other data like scattering functions and normalizing constants, which cannot be obtained directly from laboratory measurements (see [9]). On the other hand, this is a relatively unexplored area of mathematics. In fact, there are only a few examples of complete characterizations of inverse resonance problems, for instance by Korotyaev (see [30]), who solved it on the half-line for compactly supported potentials with Dirichlet boundary condition, or Christiansen, who solved it on the whole line for step-like potentials (see [13]), using some results from an earlier paper of Cohen–Kappeler ([15]). Some other examples are [6, 25, 26, 32]. It is important to stress that what we call resonances in this thesis differs from what physicists usually describe as resonances. In particular, we study the mathematical resonances in terms of the wave number ξ and not the frequency ω, where the latter would lead to a wave with amplitude decreasing in time with an exponential rate given by the imaginary part of the frequency. Hence, these would be spatial resonances, where the amplitude exponentially decays or increases in space with rate involving the imaginary part of the wave number. These resonances would precisely be the so-called Regge poles, which are resonances with respect to the angular variables, in the case of a spherical domain (see [45]).

In the first chapter of the thesis, starting from the elastic wave equation and viewing the effective Hamiltonian as a semi-classical pseudo-differential operator, we recover a system of ordinary differential equations (see [36]) prescribed with some boundary conditions. The system can be decoupled into a scalar exterior Neumann boundary value problem, called the Love problem, and a system of two coupled differential equations with boundary conditions, called the Rayleigh problem.

In the second chapter of the thesis we will study the Love problem. We employ a diffeomorphism to recover a Schrödinger-type differential operator with Robin boundary condition. Then we describe the framework of scattering theory and introduce the Jost solution, the Jost function, and their properties. As direct results, we obtain information on the spectrum and the resonances. In particular, we obtain new results on the asymptotics of the resonance counting function and resonance-free regions for a Schrödinger operator with Robin boundary condition. For the inverse resonance problem, we follow the ideas of Korotyaev [30] (see also [8, 42]) and we adapt them to our formalism and our different boundary conditions. We obtain a new result of characterization of a class $\mathbb{V}_{x_I}$ of potentials by a class W_{x_I} of Jost functions, where the latter can be reconstructed by eigenvalues and resonances (and the known asymptotic behaviour of the Jost function on the physical sheet). Moreover, following a procedure similar to [43, 3], which were done for the Rayleigh problem in the matrix case, and adapted to our Love scalar problem we obtain a new equation for the resolution of the inverse spectral problem, which is similar to the Gelfand–Levitan equation (see [34, 11]). In this way, we can prove a new inverse result from a suitable class $\mathbb{M}_{x_I}$ of Weyl–Titchmarsh functions, that can be recovered by eigenvalues and resonances, and a class $\mathbb{V}^1_{x_I}$ of potentials, which describe the medium in the Love problem.

In the third chapter, we study the Rayleigh problem. First, we introduce the homogeneous case, that was studied by Secher and Colin de Verdiere (see [46, 20]), and show how the resonances, which are present also in the homogeneous case, behave. In the Rayleigh case the setting becomes slightly more difficult as we need to define a four-sheeted Riemann surface for the wave number ξ such that the quasi-momenta for the P and S waves are single-valued and holomorphic. Then we recover some Gauge symmetries of the fundamental solution of the Rayleigh system of equations (Jost solution) that arise from the symmetries of the Hamiltonian operator itself. Those symmetries simplify the reflection matrix and the reflection coefficients, and reduce the number of degrees of freedom. Afterwards, we make a change of variables, following the papers of Pekeris and Markushevich (see [40]) and following the notation as in [19], that makes the Rayleigh operator become Schrödinger-type with eigenvalue $-\xi^2$, but with Robin boundary condition. The Schrödinger-type form of the equation, on the one hand, helps us prove analytic properties of the fundamental solutions, but on the other hand causes the potential to be no longer self-adjoint. Unlike the Love problem, we cannot recover the Jost function from the resonances, as the Jost function is not entire in the complex plane and the Hadamard factorization theorem cannot be applied. Hence, we need to define a function $F(\xi)$, constructed by factors of determinants of the Jost function evaluated in each of the four sheets of the Riemann surface. We prove a new result on this function $F(\xi)$ being entire. Moreover, from the estimates of the Jost function, we prove a new result on the F function to be of exponential type and Cartwright class (see [33, 29, 10, 53]) and consequently obtain new direct results of the resonance counting function and resonance-free region. This thesis arises from a collaboration of more than three years with my former supervisor Alexei Iantchenko, and some of the results in the first part of chapter 3, especially sections 3.3, 3.6, and 3.7 of the thesis, can also be found in the working document [17, 18]. However, various parts of the presentation and results are different, as well as the Riemann surface.

Chapter 1

Chapter 1

Physical background and assumptions

1.1 Physical framework

In this chapter, we follow the same mathematical setting as in [36]. When an earthquake occurs and seismic energy is released, a part of the energy propagates through the body as seismic body waves and another part propagates along the surface as seismic surface waves, which occur when the medium is stratified. Body waves move towards the Earth's surface and divide into two types: the P waves and the S waves.

P waves stand for primary waves because they are faster than the other seismic waves and reach the seismograph station first. They are also pressure waves because they are longitudinal, which means the particles oscillate along the direction of propagation of the wave. The other type of body waves is the S waves, which are secondary waves because they are slower than the P-waves and are shear transverse waves. That means the particles oscillate in an ellipse that lies on a plane perpendicular to the direction of propagation of the wave. Both P and S waves have a spherical wave-front, which means that they propagate radially in every direction from the source of the earthquake and, by the Huygens principle, each point of the wave-front serves as a secondary source. Whenever a ray meets an interface between two layers of different refraction indices, it turns into a refracted wave and a reflected wave. The first law of reflection tells us that the incident wave, the reflected wave and refracted wave must lie in the same plane (incident plane). The reflection angle, that is, the angle between the normal to the interface and the reflected wave, and the refraction angle, the angle between the normal to the interface and the refracted wave, are regulated by the Snell's laws and in particular, depend on the acoustic impedance index of the two layers Z_1 and Z_2. For example, if a ray passes from a layer with velocity α_1 to a layer with higher velocity α_2, the refracted angle will be larger than the incident angle, following

$$\frac{\sin i}{\alpha_1} = \frac{\sin r}{\alpha_2},$$

where i is the incident angle and r is the refracted angle. After a seismic phenomenon, many events of wave reflections and refractions follow and give rise to many kinds of waves. Surface waves are the ones that travel between the ground and a fixed layer with a lower velocity than the body waves. Their amplitude decays exponentially in terms of the depth and with a decay rate depending on the wavelength of the wave, which means the shorter the wavelength, the faster the decay. Surface waves can be divided into Love and Rayleigh waves. Love waves are generated by the constructive interference between horizontally polarized S-waves, and Rayleigh waves are generated by the constructive interference of P waves and vertically polarized S waves. Whenever a wave meets an interface, a part of the energy of the wave is lost because of the refraction, with modulus depending on the incident angle. In particular, for incident angles large enough there could be total reflection. When a seismic body P-wave meets another layer we not only have a reflected P-wave but also a reflected S-wave, as we see in Figure 1.1. In particular for values of the incident angle lower than a certain value, there is neither a transmitted nor a reflected S-wave starting from an incident P-wave. The energy in those processes is broken down following the impedance ratio $\gamma = \frac{Z_1}{Z_2}$, where $Z_i = \rho_i v_i$ is the impedance value of the medium i which depends on the mass density ρ_i and the velocity v_i. In particular for $\gamma = 1$ the energy is completely transmitted and for values of γ close to 0 or ∞ the energy is completely reflected. This is the case of the interface between the Earth and the atmosphere. Usually, when passing from a medium with a certain mass density to another with a lower mass density, the velocity of the acoustic wave decreases.

1.2 The elastic wave equation

The equations that describe the surface waves can be obtained starting from the elastic wave equation in the seismological framework of the half-space with domain $\mathbb{R}^2 \times (-\infty, 0]$. From Newton's second law $F = ma$, applied to an infinitesimal volume of an elastic solid, and taking into account only the linear perturbations of the medium, we recover the *linear solid elastic wave* equation in $\mathbb{R}^3$

$$\operatorname{div} \sigma(u) = \rho \frac{\partial^2 u}{\partial t^2}, \tag{1.1}$$

where $u(x, t)$ is the displacement vector, $\rho(x)$ is the density of mass, and $\sigma(u)$ is the symmetric third-rank stress-tensor which satisfies Hooke's law (in the unidimensional case $F = kx$)

$$\sigma(u) = c\epsilon(u),$$

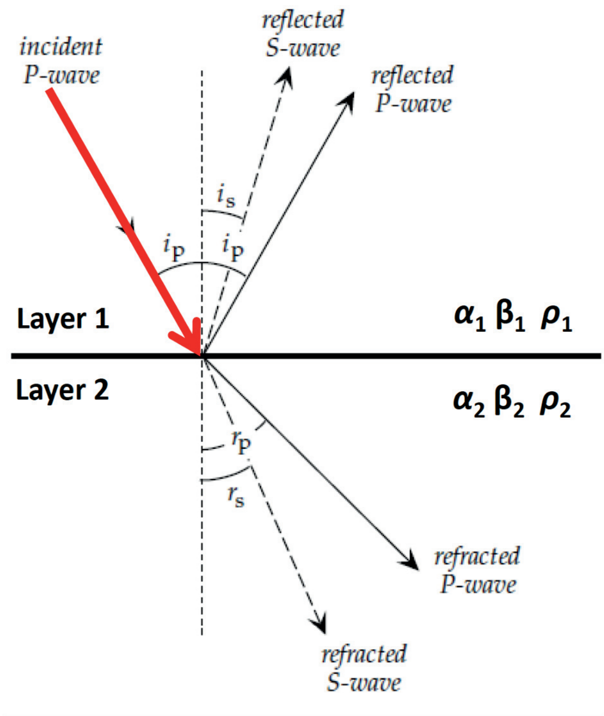

Figure 1.1: Scattering of a P-wave between two different layers [35].

that component-wise becomes

$$\sigma_{ij}(u) = \sum_{k,l=1}^{3} c_{ijkl}\epsilon_{kl}(u), \qquad \epsilon_{kl}(u) = \frac{1}{2}\left(\partial_k u_l + \partial_l u_k\right).$$

The index i of the element σ_{ij} of the stress tensor indicates that the stress is applied on a plane perpendicular to the direction i, while the index j indicates the direction in which the stress is applied. The tensor $\epsilon(u)$ is called infinitesimal strain tensor and, for example, its component ϵ_{kl} informs us about the elongation that the particles along the l axis undergo in the k direction, plus the elongation that the particles along the k axis undergo along the l direction. The tensor c is the fourth-order stiffness tensor that characterizes the elasticity of the medium, that is how much it gets stretched or shrunk under stresses. We consider the solutions of the elastic wave equations in the half-solid space $X = \mathbb{R}^2_{x_1,x_2} \times (-\infty, 0]_z$ augmented with boundary conditions at $\partial X = \{z = 0\}$,

$$\partial_t^2 u_i + M_{il}u_l = 0,$$

$$
\begin{aligned}
&u(t=0,x,z)=0,\\
&\partial_t u(t=0,x_1,x_2,z)=h(x_1,x_2,z),\\
&\frac{c_{i3kl}}{\rho}\partial_k u_l(t,x_1,x_2,z=0)=0,
\end{aligned}
$$

where

$$
\begin{aligned}
M_{il}=&-\frac{\partial}{\partial z}\frac{c_{i33l}(x_1,x_2,z)}{\rho(x_1,x_2,z)}\frac{\partial}{\partial z}-\sum_{j,k=1}^{2}\frac{c_{ijkl}(x_1,x_2,z)}{\rho(x_1,x_2,z)}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\\
&-\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\frac{c_{ij3l}(x_1,x_2,z)}{\rho(x_1,x_2,z)}\frac{\partial}{\partial z}-\sum_{k=1}^{2}\frac{c_{i3kl}(x_1,x_2,z)}{\rho(x_1,x_2,z)}\frac{\partial}{\partial z}\frac{\partial}{\partial x_k}\\
&-\sum_{k=1}^{2}\left(\frac{\partial}{\partial z}\frac{c_{i3kl}(x_1,x_2,z)}{\rho(x_1,x_2,z)}\right)\frac{\partial}{\partial x_k}-\sum_{j,k=1}^{2}\left(\frac{\partial}{\partial x_j}\frac{c_{ijkl}(x_1,x_2,z)}{\rho(x_1,x_2,z)}\right)\frac{\partial}{\partial x_k}.
\end{aligned}
$$

Let x denote the pair of variables $(x_1,\ x_2)$. Since the medium is elastic, the stiffness tensor satisfies the following symmetry condition

$$
c_{ijkl}=c_{jikl}=c_{klij}\qquad\text{for any } i,j,k,l.
$$

Moreover, it satisfies the strong convexity condition (or ellipticity condition)

$$
\sum_{i,j,k,l=1}^{3}\frac{c_{ijkl}}{\rho}\epsilon_{ij}\epsilon_{kl}\geq\delta\sum_{i,j=1}^{3}\epsilon_{ij}^2.
$$

which physically means that the medium will undergo a strict elongation in all directions regardless from which direction we apply the stress, and mathematically means that the displacement vector is strongly convex, that implies the smallest eigenvalue to be positive.

We assume that the medium is stratified: the elastic properties change much more rapidly in the vertical direction than in the horizontal. More specifically (see [36, Section 2]), we make the following assumptions.

Assumption 1.2.1. *We assume that the stiffness tensor and density to satisfy the scaling property*

$$
\frac{c_{ijkl}}{\rho}(x,z)=:C_{ijkl}\left(x,\frac{z}{\epsilon}\right),
$$

where $\epsilon\in(0,\epsilon_0]$ *is the semi-classical parameter.*

Assumption 1.2.2 (Homogeneity). *We assume that below a certain depth* Z_I*, where* $Z=\frac{z}{\epsilon}$*, the medium is homogeneous, so the stiffness tensor is constant*

$$
C_{ijkl}\left(x,Z\right)=C_{ijkl}\left(x,Z_I\right)\qquad\text{for } Z\leq Z_I. \tag{1.2}
$$

Assumption 1.2.3 (Limiting velocity). *We assume that for any wave number ξ the following condition holds*

$$\inf_{Z\leq 0} v_L(x,\xi,Z) < v_L(x,\xi,Z_I). \tag{1.3}$$

Condition (1.3) implies the existence of surface waves. It is called the *limiting velocity condition* and it is the lowest velocity such that the matrix made of the coefficients of the second order derivative and the matrix of the terms without derivative are singular (see [36, Section 3]).

In the case of an *elastically isotropic* material, the stress produces the same elongation regardless of the direction from which the stress is applied. In this case the 81 components of the stiffness tensor c_{ijkl} reduce their number to only 2 independent values μ and λ, called *Lamé parameters*.

Assumption 1.2.4 (Elastically isotropic medium). *We assume the medium to be elastically isotropic, hence the stiffness tensor can be simplified in the following way*

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{1.4}$$

In particular, μ is called shear modulus and it is zero in liquid or gases, and λ is called Lamé first parameter and it depends on how thick the layer is. We set $\hat{\mu} := \frac{\mu}{\rho}$ and $\hat{\lambda} := \frac{\lambda}{\rho}$ that are the density normalized Lamé parameters. After those assumptions, the equation (1.1) becomes

$$\left[\epsilon^2\partial_t^2 + \widehat{H}\right]v = 0,$$

where $v(t,x,Z) := u(t,x,z)$. We view $\widehat{H}$ as a *semi-classical pseudo-differential operator* in the standard quantization on $\mathbb{R}^2$ (see the appendix). Thus,

$$\epsilon^2\partial_t^2 v + Op_\epsilon\left(H(x,\xi)\right)v \sim 0,$$

where $Op_\epsilon\left(H(x,\xi)\right)$ is the semi-classical pseudo-differential operator with operator-valued symbol $H(x,\xi)$. The symbol $H(x,\xi)$ is obtained by the Fourier inversion formula, using a partial Fourier transform in the variables $x = (x_1,x_2)$. With this trick, we get rid of the partial derivatives with respect to x_1 and x_2 and the result is an ordinary differential operator in Z depending on the wave vector components ξ_1 and ξ_2, which are the dual variables of x_1 and x_2. The symbol of $\widehat{H}$ is separated into two different orders through the semi-classical parameter ϵ

$$H(x,\xi) = H_0(x,\xi) + \epsilon H_1(x,\xi),$$

where

$$H_{0,il}(x,\xi) = -\frac{\partial}{\partial Z}C_{i33l}(x,Z)\frac{\partial}{\partial Z} - i\sum_{j=1}^{2}C_{ij3l}(x,Z)\xi_j\frac{\partial}{\partial Z}$$

$$
\begin{aligned}
&- i\sum_{k=1}^{2} C_{i3kl}(x,Z)\frac{\partial}{\partial Z}\xi_k - i\sum_{k=1}^{2}\left(\frac{\partial}{\partial Z}C_{i3kl}(x,Z)\right)\xi_k \\
&+ \sum_{j,k=1}^{2} C_{ijkl}(x,Z)\xi_j\xi_k. \qquad (1.5)
\end{aligned}
$$

and

$$
H_{1,il}(x,\xi) = -\sum_{j=1}^{2}\left(\frac{\partial}{\partial x_j}C_{ij3l}(x,Z)\right)\frac{\partial}{\partial Z} - i\sum_{j,k=1}^{2}\left(\frac{\partial}{\partial x_j}C_{ijkl}(x,Z)\right)\xi_k. \quad (1.6)
$$

For fixed (x,ξ), the operator-valued symbols $H_0(x,\xi)$ and $H_1(x,\xi)$ are ordinary differential operators in the Z variable with domain

$$
\mathcal{D} = \left\{ v \in H^2(\mathbb{R}^-) \,\middle|\, \sum_{l=1}^{3}\left(C_{i33l}(x,0)\frac{\partial v_l}{\partial Z}(0) + i\sum_{k=1}^{2} C_{i3kl}\xi_k v_l(0)\right) = 0 \right\}.
$$

In the case of an *isotropic medium*, we can decouple the effective Hamiltonian H_0 as

$$
H_0(x,\xi) = H_0^L(x,\xi) \bigoplus H_0^R(x,\xi),
$$

where H_0^L is the scalar Love operator with eigenfunctions corresponding to the *surface Love waves*, while H_0^R is the matrix-valued Rayleigh operator with eigenfunctions corresponding to *surface Rayleigh waves*. The spectrum of $H_0(x,\xi)$ consists of a discrete spectrum with elements inside the interval $\left(0, \hat{\mu}(Z_I)\,|\xi|^2\right)$ and an essential spectrum $\left[\hat{\mu}(Z_I)\,|\xi|^2, \infty\right)$ (see [36]).

1.3 Decoupling into Love and Rayleigh operators

In physical experiments, we see that surface waves decouple into Love waves, which are transverse waves where the oscillation of the particles is perpendicular to the direction of propagation of the wave, and Rayleigh waves, which are a composition of a longitudinal wave with oscillation of the particles parallel to the direction of the wave propagation and a transverse wave. The result of these two waves, longitudinal and transverse, makes the single particle move on an ellipse, or more precisely, on a helix since the area of the ellipse will change with time due to the damping of the wave. Mathematically, we would expect to be able to decouple this 3 by 3 matrix-valued differential operator into the direct sum of a scalar differential operator, whose eigenvalues would be the energies of the Love waves, and a 2 by 2 differential operator, whose eigenvalues would be the energies of the Rayleigh waves.

The velocity of the P-wave and S-wave are

$$c_P = \sqrt{\hat{\lambda} + 2\hat{\mu}}, \qquad c_S = \sqrt{\hat{\mu}}. \tag{1.7}$$

We make the substitution $v(x,Z) = P(\xi)\varphi(x,Z)$, where $\varphi(x,Z)$ is a vector $\varphi(x,Z) = (\varphi_1(x,Z), \varphi_2(x,Z), \varphi_3(x,Z))$ and

$$P(\xi) = \begin{pmatrix} |\xi|^{-1}\xi_1 & |\xi|^{-1}\xi_2 & 0 \\ |\xi|^{-1}\xi_2 & -|\xi|^{-1}\xi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is a rotation around the Z-axis. After the substitution, the differential operator becomes

$$P(\xi)^T H_0(x,\xi) P(\xi)$$
$$= \begin{pmatrix} -\frac{\partial}{\partial Z}\hat{\mu}\frac{\partial}{\partial Z} + (\hat{\lambda}+2\hat{\mu})|\xi|^2 & 0 & -i|\xi|\left(\frac{\partial}{\partial Z}\hat{\mu} + \hat{\lambda}\frac{\partial}{\partial Z}\right) \\ 0 & -\frac{\partial}{\partial Z}\hat{\mu}\frac{\partial}{\partial Z} + \hat{\mu}|\xi|^2 & 0 \\ -i|\xi|\left(\frac{\partial}{\partial Z}\hat{\lambda} + \hat{\mu}\frac{\partial}{\partial Z}\right) & 0 & -\frac{\partial}{\partial Z}(\hat{\lambda}+2\hat{\mu})\frac{\partial}{\partial Z} + \hat{\mu}|\xi|^2 \end{pmatrix}$$

and the decoupling is obtained. For the Love waves, we have the following Neumann boundary value problem for the eigenfunctions

$$-\frac{\partial}{\partial Z}\hat{\mu}\frac{\partial \varphi_2}{\partial Z} + \hat{\mu}\,|\xi|^2\varphi_2 = \Lambda\varphi_2, \tag{1.8}$$

$$\frac{\partial \varphi_2}{\partial Z}(0) = 0. \tag{1.9}$$

For the Rayleigh waves, we get the following boundary value problem for the eigenfunctions

$$-\frac{\partial}{\partial Z}\hat{\mu}\frac{\partial \varphi_1}{\partial Z} - i|\xi|\left(\frac{\partial}{\partial Z}(\hat{\mu}\varphi_3) + \hat{\lambda}\frac{\partial}{\partial Z}\varphi_3\right) + (\hat{\lambda}+2\hat{\mu})|\xi|^2\varphi_1 = \Lambda\varphi_1, \tag{1.10}$$

$$-\frac{\partial}{\partial Z}(\hat{\lambda}+2\hat{\mu})\frac{\partial \varphi_3}{\partial Z} - i|\xi|\left(\frac{\partial}{\partial Z}(\hat{\lambda}\varphi_1) + \hat{\mu}\frac{\partial}{\partial Z}\varphi_1\right) + \hat{\mu}|\xi|^2\varphi_3 = \Lambda\varphi_3, \tag{1.11}$$

$$i|\xi|\varphi_3(0) + \frac{\partial \varphi_1}{\partial Z}(0) = 0, \tag{1.12}$$

$$i\hat{\lambda}|\xi|\frac{\partial \varphi_1}{\partial Z}(0) + (\hat{\lambda}+2\hat{\mu})\frac{\partial \varphi_3}{\partial Z}(0) = 0. \tag{1.13}$$

The solutions of (1.8) and (1.10)–(1.11) are the two types of surface modes, Love and Rayleigh waves, that are decoupled in the principal part of the semi-classical differential operator in an isotropic medium. We can construct the lower order term of the solution by a perturbative method, that means expanding both the eigenfunction and the eigenvalue in powers of the semi-classical parameter ϵ and then solve recursively the respective equation at each order of ϵ, as is shown in Theorem A.1.4 in the Appendix.

We suppose that $H_0(x,\xi)$ has $\mathfrak{M}$ eigenvalues

$$\Lambda_1 < \ldots < \Lambda_\alpha < \ldots < \Lambda_{\mathfrak{M}}$$

with eigenfunction $\Phi_{\alpha,0}(Z,x,\xi)$. Then, those eigenfunctions satisfy

$$H_0\Phi_{\alpha,0} = \Lambda_\alpha\Phi_{\alpha,0} + O(\epsilon)$$

and, as in Theorem A.1.4, we can construct $\Phi_{\alpha,\epsilon} \sim \Phi_{\alpha,0}+\epsilon\Phi_{\alpha,1}+\ldots$, $\Lambda_{\alpha,\epsilon} \sim \Lambda_{\alpha,0}+\ldots$ and each order satisfies

$$H \circ \Phi_{\alpha,\epsilon} = \Phi_{\alpha,\epsilon} \circ \Lambda_{\alpha,\epsilon} + O(\epsilon^\infty),$$

where $\circ$ denotes the composition of symbols. Multiplying $\Phi_{\alpha,0}(Z,x,\xi)$ times the factor $\frac{1}{\sqrt{\epsilon}}$ and defining

$$J_{\alpha,\epsilon}\left(Z,x,\xi\right) = \frac{1}{\sqrt{\epsilon}}\Phi_{\alpha,0}(Z,x,\xi)$$

we obtain that $J_{\alpha,\epsilon}\left(Z,x,\xi\right)$ is micro-locally unitary. Then we call $W_{\alpha,\epsilon}\left(t,x,Z\right)$ the solution to the initial value problem

$$[\epsilon^2\partial_t^2 + \Lambda_\alpha(x,D_x)]W_{\alpha,\epsilon}(t,x,Z) = 0, \tag{1.14}$$

$$W_{\alpha,\epsilon}(0,x,Z) = 0, \quad \partial_t W_{\alpha,\epsilon}(0,x,Z) = J_{\alpha,\epsilon}W_\alpha(x,Z), \tag{1.15}$$

with $\alpha = 1,\ldots,\mathfrak{M}$. Then, after finding the solution to the previous initial value problem, thanks to the perturbative method (see Theorem A.1.4 in the Appendix), we can construct the solution to the initial problem (1.1) which is

$$u(t,x,\epsilon Z) = \sum_{\alpha=1}^{\mathfrak{M}} J_{\alpha,\epsilon}\left(Z,x,\epsilon D_x\right)W_{\alpha,\epsilon}\left(t,x,Z\right).$$

Chapter 11

Chapter 2

The Love problem

2.1 Introduction

In this chapter, we will introduce the Love problem and focus on its direct and inverse results. Starting from equations (1.8)–(1.9) we can apply a change of variables so that the resulting boundary value problem assumes a Schrödinger-type form. Classical ways to transform the Love problem are the *calibration transform* and the *Liouville transform*. By those transforms, we get a Schrödinger-type problem with Robin boundary condition with energy k^2, that is related to the usual energy ω^2 by $k^2 = \frac{\omega^2}{\hat{\mu}_I} - \xi^2$. The difference between the two types of transforms is that in the former we obtain a potential depending on the shear modulus μ and the wave number ξ, whereas in the latter we obtain a potential depending on μ and the frequency ω. For our purposes, we will use the calibration transform as we want ξ to be our spectral parameter to be aligned with the Rayleigh problem in Chapter 3, where ξ is the spectral parameter. Once we have performed the calibration transform, we need to solve an inverse resonance Schrödinger problem with Robin boundary condition, where the resonances are the poles of the resolvent with respect to the parameter k (or ξ) as in Definition 2.5.4. This is a new result and is obtained by following the result of Korotyaev for Dirichlet boundary condition (see [30]). The novelty of my work, compared to Korotyaev's work is to change the definitions of some quantities and adapt all the results to our different setting. Moreover, all the proofs in my work are more detailed.

The main goal of this chapter is to retrieve the shear modulus $\hat{\mu}$ ($\hat{\mu} = \mu/\rho$, with ρ being the density (see Chapter 1) as we do in Theorem 2.5.46, which is a new result. Theorem 2.5.46 is an application of a characterization (see Theorem 2.5.44) between a class W_{x_I} of Jost functions (see Definition 2.5.16) and a class $\mathbb{V}_{x_I}$ of potentials (see Definition 2.5.2), which is a new result based on the paper of Korotyaev ([30]) who solved for Dirichlet boundary condition. We also show classical direct results for the spectrum, for example, that all eigenvalues are purely imaginary in k (see Theorem 2.5.20), that there is finite number of them (see Theorem 2.5.24), the proof that $k = 0$ is not in the spectrum (see Theorem 2.5.27) and the simplicity of all the eigenvalues (see Theorem 2.5.28). We obtain new direct results for the resonances consisting in the asymptotics of the counting function (Theorem 2.5.37) and the estimates of the resonances and their forbidden domain (Corollary 2.5.36), which are similar to the results in the Dirichlet case ([30]). In Section 2.6, we present a new alternative method of recovering the Gelfand–Levitan–Marchenko equation (see Proposition 2.6.22), which is based on the papers [3] who did it for the Rayleigh system. In this alternative method the Weyl–Titchmarsh function is defined as in [22], following an approach that can be extended for not self-adjoint problem. Furthermore, we prove a new result of spectral inverse problem in terms of a class $\mathbb{M}_{x_I}$ of Weyl functions (see Definition 2.6.27) as in Theorem 2.6.28.

2.2 The calibration transform

In Chapter 1, after decoupling the system, we obtained the following Love boundary value problem

$$-\frac{\partial}{\partial Z}\hat{\mu}\frac{\partial \varphi_2}{\partial Z} + \hat{\mu}\,|\xi|^2\varphi_2 = \Lambda\varphi_2,$$

$$\frac{\partial \varphi_2}{\partial Z}(0) = 0.$$

where Z ranges in the depth axis, that is, the half-line $]-\infty, 0]$, with $Z = 0$ coinciding with the Earth's surface, $-\infty$ coinciding with the centre of the Earth, $\Lambda = \omega^2$ is the eigenvalue, ω is the frequency, $\hat{\mu}$ is one of the two Lamé parameters and ξ represents the wave vector. We make the calibration substitution

$$\varphi_2 = \frac{1}{\sqrt{\hat{\mu}}}u, \quad \frac{d}{dZ}\left(\hat{\mu}\frac{d}{dZ}\varphi_2\right) = \frac{1}{4}\hat{\mu}^{-\frac{3}{2}}(\hat{\mu}')^2u - \frac{1}{2}\hat{\mu}^{-\frac{1}{2}}\hat{\mu}''u + \hat{\mu}^{\frac{1}{2}}u''$$

and we get

$$u'' - |\xi|^2u = \left[\frac{1}{2}\frac{\hat{\mu}''}{\hat{\mu}} - \frac{1}{4}\left(\frac{\hat{\mu}'}{\hat{\mu}}\right)^2 - \frac{1}{\hat{\mu}}\omega^2\right]u.$$

We set the quasi momentum $k := \sqrt{\frac{\omega^2}{\hat{\mu}_I} - |\xi|^2}$ and

$$V = \frac{1}{2}\frac{\hat{\mu}''}{\hat{\mu}} - \frac{1}{4}\left(\frac{\hat{\mu}'}{\hat{\mu}}\right)^2 - \frac{1}{\hat{\mu}}\omega^2 + \frac{1}{\hat{\mu}_I}\omega^2 = \frac{(\sqrt{\hat{\mu}})''}{\sqrt{\hat{\mu}}} - \frac{1}{\hat{\mu}}\omega^2 + \frac{1}{\hat{\mu}_I}\omega^2, \tag{2.1}$$

where $\hat{\mu}_I := \hat{\mu}(Z_I)$ is the value of the shear modulus at the depth Z_I, below which the medium is homogeneous. By Assumption 1.2.2, $\hat{\mu}(Z) = \hat{\mu}_I$ constant for $Z \le Z_I$, hence also the derivatives $\hat{\mu}'$ and $\hat{\mu}''$ vanish for $Z \leq Z_I$. This implies that the potential V has compact support and depends only on Z as we fixed ω and let our spectral parameter ξ vary. In this way the potential $V = V_\omega$ can be parametrized by ω and the resonances are considered in terms of ξ.

Remark 2.2.1. *In Section 2.5 we will assume that the potential $V \in \mathbb{V}_{x_I}$ (Definition 2.5.2), that implies the Lamé parameter $\hat{\mu}$ to be constant below the depth Z_I and to be different than $\hat{\mu}_I$ in an interval of type $(Z_I, a + Z_I)$ for $a > 0$.*

The Love scalar equation takes the following form:

$$-u'' + Vu = \lambda u, \qquad \lambda = k^2, \tag{2.2}$$

with corresponding boundary condition that becomes of Robin type after the transformation

$$u'(0) + hu(0) = 0, \qquad h = -\frac{1}{2}\frac{\hat{\mu}'(0)}{\hat{\mu}(0)}. \tag{2.3}$$

To resemble the classical formulation, we make the substitution $Z = -x$, which leads the domain to become $[0, +\infty)$ and we study the problem in terms of k. In our case, the potential of the Schrödinger operator is real because we are considering an elastic medium. In the case of an inelastic medium, we would have a complex potential that implies the loss of part of the energy which is converted into heat. We make a self-adjoint realization in $L^2(\mathbb{R}_+)$ of the operator in (2.2) due to the boundary condition (see [11]). Then the operator appearing on the left hand side of (2.2) prescribed with the domain

$$D = \{u \in H^2\,[0, +\infty) : u'(0) + hu(0) = 0\} \tag{2.4}$$

and the L^2 inner product is self-adjoint.

Remark 2.2.2. *We stress that the λ appearing in* (2.2) *is the eigenvalue of the Schrödinger equation and has nothing to do with the λ appearing in* (1.4)*, that is one of the Lamé parameters, and which will always appear later as $\hat{\lambda} := \frac{\lambda}{\rho}$ in the normalized form, thus avoiding any type of confusion between them.*

2.3 The Riemann surface

The presence of a square root in the definition of k suggests that we should define a Riemann surface for ξ in order for k to be a single-valued holomorphic function there. We make an analytic continuation of the real positive variable $|\xi|$ to the whole complex plane and we define the new complex variable as ξ. Let $k_\omega(\xi) = i\sqrt{\xi^2 - \frac{\omega^2}{\hat{\mu}_I}}$ and define the Riemann surface Ω of $k_\omega(\xi)$ by taking two sheets of the complex plane with cuts along $i\mathbb{R} \cup \left[-\frac{\omega}{\sqrt{\hat{\mu}_I}}, \frac{\omega}{\sqrt{\hat{\mu}_I}}\right]$, Ω_+ called physical sheet and Ω_- called unphysical sheet, and gluing them in a crosswise way. On the one hand, Ω_+ is called physical sheet as all the ξ on this sheet correspond to k with positive imaginary part, which lead to a physical L^2 solution. On the other hand, Ω_- is called unphysical sheet as all the ξ on there correspond to k with negative imaginary part, which give rise to a non L^2 solution.

We choose a determination of $k_\omega(\xi)$ by picking the branch of the square root so that $k_\omega(\xi) \in i\mathbb{R}_+$, when $\xi \in \Omega_+$. The function $k_\omega(\xi)$ becomes single-valued holomorphic on the Riemann surface Ω and with non-zero derivative everywhere, hence it is a *conformal mapping*. The quasi momentum $k_\omega(\xi)$ satisfies the following properties

$$k_\omega\left(\left[0 - i0, \frac{\omega}{\sqrt{\hat{\mu}_I}} - i0\right)\right) = \left(0, \frac{\omega}{\sqrt{\hat{\mu}_I}}\right],$$
$$k_\omega\left(\left[0 + i0, \frac{\omega}{\sqrt{\hat{\mu}_I}} + i0\right)\right) = \left[-\frac{\omega}{\sqrt{\hat{\mu}_I}}, 0\right),$$

$$k_\omega(\pm\xi) = i\xi + O(|\xi|^{-1}) \qquad \xi \in \Omega_+, \operatorname{Re}\xi \geq 0,$$
$$k_\omega(\pm\xi) = -i\xi + O(|\xi|^{-1}) \qquad \xi \in \Omega_-, \operatorname{Re}\xi \geq 0,$$

and also

$$k_\omega(\xi) = -\overline{k_\omega(\overline{\xi})} = k_\omega(-\xi) \qquad \text{for } \xi \in \Omega_\pm. \tag{2.5}$$

In (2.5) the conjugation is made through paths non intersecting the cuts. The reflection is made by paths that cross the cuts as in Figure 2.1. Hence, when we pass the first cut on the imaginary axis we get to the sheet Ω_- and when we pass through the cut $\left(-\frac{\omega}{\sqrt{\hat{\mu}_I}}, \frac{\omega}{\sqrt{\hat{\mu}_I}}\right)$ we come back to the original sheet. From (2.5), we see that $k_\omega(\xi)$ is an even function on each single sheet.

2.4 Cartwright class functions

In this section we give some definitions and results from complex analysis that will be useful later on (see [29, Chapter 3] and [33, Chapter 1]).

Definition 2.4.1 (Exponential type function). *An entire function $f(z)$ is said to be of exponential type if there are real-valued constants α, o and A such that*

$$|f(z)| \leq Ae^{\alpha|z|^o} \tag{2.6}$$

for $z \to \infty$ in the complex plane. The infimum of the o and α such that (2.6) is satisfied are called respectively order and type of the exponential type.

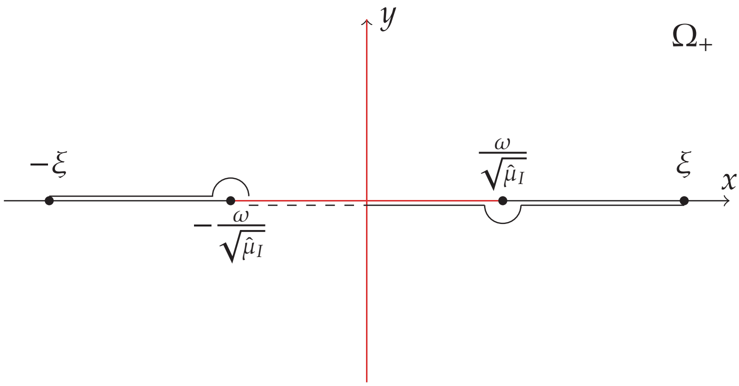

Figure 2.1: Reflection from ξ to $-\xi$ in the physical sheet Ω_+. The dashed line represents a path in the unphysical sheet Ω_-. The red lines represent the cuts of the Riemann sheets.

In the following we present the Hadamard factorization theorem from [16, page 279], which we will be fundamental for our analysis.

Theorem 2.4.2 (Hadamard factorization). *Let $f(z)$ be entire of finite order ρ and denote by a_n the sequence of its zeros $\neq 0$ (with multiplicity counted by repetition), so arranged that*

$$0 < |a_1| \leq |a_2| \leq |a_3| \leq \dots .$$

Then

$$f(z) = z^m e^{g(z)} \prod_n E_p\left(\frac{z}{a_n}\right) \tag{2.7}$$

where $g(z)$ is a polynomial of degree q, $q \leq \rho$, m is the multiplicity of $z = 0$ as a zero of f and

$$E_p(z) = \begin{cases} (1-z) & p = 0 \\ (1-z)e^{\frac{z}{1}+\frac{z^2}{2}+\dots+\frac{z^p}{p}} & p \neq 0 \end{cases}$$

with $p = [\rho]$ being the integer part of ρ. The product (2.7) *is uniformly convergent on compact subsets of $\mathbb{C}$.*

Next, we present a theorem from Lindelöf from [29, page 20], which proves the absolute convergence of the harmonic series of zeros of an entire function of exponential type.

Theorem 2.4.3. *Let $f(z)$ be entire, of exponential type, with $f(0) \neq 0$ and denote by $\{a_n\}$ the sequence of zeros of $f(z)$ with each zero repeated according to its multiplicity. Put*

$$S(r) = \sum_{|a_n| \leq r} \frac{1}{a_n}.$$

Then $|S(r)|$ is bounded as $r \to \infty$.

The next theorem is also from Lindelöf from [29, page 21] and is, basically, the converse of the previous theorem.

Theorem 2.4.4 (Lindelöf). *Let*

$$0 < |a_1| \leq |a_2| \leq |a_3| \leq \dots,$$

denote by $n(r)$ the number of a_k having modulus $\leq r$ (taking account of multiplicities, as usual), and suppose that $n(r) \leq Kr$ for some constant $K \geq 0$. Moreover, suppose that the sums

$$\sum_{|a_n| \leq r} \frac{1}{a_n}$$

remain bounded as $r \to \infty$. Then the product

$$H(z) = \prod_n \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}}$$

is equal to an entire function of exponential type.

The next theorem is from [29, page 22].

Theorem 2.4.5. *Let $f(z)$ and $g(z)$ be entire and of exponential type. If the ratio $\frac{f(z)}{g(z)}$ is also entire, then it is of exponential type.*

Definition 2.4.6 (Cartwright class). *A function f is said to be in the Cartwright class with indices $\rho_+ = A$ and $\rho_- = B$, if $f(z)$ is entire, of exponential type, and the following conditions are fulfilled:*

$$\int_{\mathbb{R}} \frac{\log^+ |f(x)| dx}{1 + x^2} < \infty, \quad \rho_+(f) = A, \quad \rho_-(f) = B \tag{2.8}$$

where $\rho_\pm(f) \equiv \limsup_{y \to \infty} \frac{\log |f(\pm iy)|}{y}$ and $\log^+(x) = \max\{\log x, 0\}$.

Basically, for a function to be of Cartwright class means that it is of exponential order 1, of type A in the upper half-plane and B in the lower half-plane and with positive part of the absolute value of its logarithm in $L^1(\mathbb{R}, \Pi)$, where Π is the Poisson measure (see [37])

$$d\Pi(t) = \frac{dt}{1 + t^2}. \tag{2.9}$$

For these functions, the Hadamard formula (2.7) can be simplified. Cartwright class functions are very useful in view of a version[1] of the Paley-Wiener theorem because they can be written as the Fourier transform of a compactly supported function (see Lemma 2.5.34). Another useful application of the Cartwright class property is the Levinson theorem (see [29, page 69]), which is the counterpart of the Weyl law for the resonances. We denote by $\mathcal{N}_+(r, f)$ the number of zeros of an entire function f with positive imaginary part with modulus $\leq r$, and by $\mathcal{N}_-(r, f)$ the number of zeros with negative imaginary part having modulus $\leq r$. Moreover, $\mathcal{N}_\pm(f) := \lim_{r\to\infty} \mathcal{N}_\pm(r, f)$. The total number of zeros with modulus smaller than r is $\mathcal{N}(r, f) := \mathcal{N}_+(r, f) + \mathcal{N}_-(r, f)$.

Theorem 2.4.7 (Levinson). *Let the function f be in the Cartwright class with $\rho_+ = \rho_- = A$ for some $A > 0$. Then*

$$\mathcal{N}_\pm(r, f) = \frac{Ar}{\pi}(1 + o(1)) \qquad \text{for } r \to \infty.$$

Given $\delta > 0$, the number of zeros of f with modulus $\leq r$ lying outside both of the two sectors $|\arg z| < \delta$, $|\arg z - \pi| < \delta$ is $o(r)$ for large r.

2.5 The scattering problem

By a *direct problem* (or forward problem) we mean the problem of finding the scattering or spectral data associated with a differential operator in a certain class and all their properties. As we can see from (1.8) and (1.9), the boundary value problem is determined by V, for fixed h. Hence, we want to define a suitable class for this pair, such that we can find a mapping from this pair to the scattering data of the problem.

In this section, we introduce the Jost solution and the Jost function because they are the key ingredients we need to be able to obtain information about the scattering data. Below we give some definitions that are essential for our next results.

Definition 2.5.1 (Bargmann-Jost-Kohn). *We define the Bargmann-Jost-Kohn class of potentials[2], and we denote it by $L_{1,1}$, as all the real functions $V(x)$ such that the potential and its first momentum are integrable*

$$\int_0^\infty |(1 + x)\, V(x)| dx < \infty. \tag{2.10}$$

[1]The standard version of the Paley-Wiener theorem states that the Fourier transform of Hardy space functions on a real line (in the upper half plane) are functions in $L^2(\mathbb{R}_+)$. It can be both stated for the upper half-plane and for the unit disc, which can be mapped into each other through a Möbius transformation.

[2]For this class of potentials it is possible to write the Jost solution in terms of a transformation operator, as we will see. Potentials in this class are short-range potentials.

Definition 2.5.2 (Class of potentials). *We denote by* $\mathbb{V}_{x_I}$ *the class of real potentials* V *such that* $V \in L^1(\mathbb{R}_+)$, $\operatorname{supp} V \subset [0, x_I]$ *for some* $x_I > 0$ *and for each* $\epsilon > 0$ *the set* $(x_I - \epsilon, x_I) \cap \operatorname{supp} V$ *has positive Lebesgue measure.*

Remark 2.5.3. *We give these two definitions of classes of potentials* $L_{1,1}$ *and* $\mathbb{V}_{x_I}$ *to point out that we could solve the inverse problem with either class of potentials. If we consider* $V \in \mathbb{V}_{x_I}$ *then we can reconstruct the potential from only eigenvalues and resonances as data. Otherwise, if* $V \in L_{1,1}$ *we can reconstruct the potential from the scattering data, such as the scattering function, the eigenvalues and the* L^2 *norm of the eigenfunctions.*

Associated to (2.2), the resolvent operator $R(k) = \left(-\frac{d^2}{dx^2} + V - k^2\right)^{-1}$ is bounded from $L^2(\mathbb{R})$ to $H^2(\mathbb{R})$ for all but a finite number of k for $\operatorname{Im} k > 0$. The existence of the resolvent for $\operatorname{Im} k > 0$ follows from the spectral theorem. The resolvent operator can be extended from $\operatorname{Im} k > 0$ through the continuous spectrum ($\operatorname{Im} k = 0$) to a meromorphic operator-valued function

$$R(k) : L^2_c(\mathbb{R}_+) \to H^2_{loc}(\mathbb{R}_+), \qquad k \in \mathbb{C}$$

on the complex plane. The proof of the existence of the meromorphic continuation can be found in [21, Theorem 2.2]. This definition has to be interpreted in the sense that for every cut-off function $\chi \in C_c^\infty(\mathbb{R}_+)$ with $\chi = 1$ on $\operatorname{supp} V$, the cut-off resolvent $\chi R \chi(k)$ is bounded for every $k \in \mathbb{C}$, except for a finite number of points which are eigenvalues ($\operatorname{Im} k > 0$) and resonances ($\operatorname{Im} k \leq 0$) (see [21, Section 2.2]).

For an elliptic self-adjoint operator in $L^2(\Sigma)$ of a bounded set Σ, the eigenvalues must be real, countable and accumulate at infinity (see [47, Theorem 8.3, Chapter 1]). The exact values of these can be obtained only in specific cases, such as for the sphere or the disc where the spherical symmetry allows us to write the Dirichlet Laplacian in spherical coordinates, or in the case of a square where by separation of variables we can reduce the problem to two one-dimensional ones. In all the other cases for the Laplace-Beltrami operator, we can usually find an asymptotic formula for the eigenvalues or an asymptotic formula for the number of them, that is, the Weyl law.

On the half-line, for the Laplacian plus a real compact perturbation, the L^2 spectrum consists of an essential spectrum $[0, +\infty)$ and a possible set of negative discrete eigenvalues. In terms of k, where $\lambda = k^2$, the L^2 spectrum, then, consists of a discrete set of pure imaginary eigenvalues and an essential spectrum $(-\infty, +\infty)$. Moreover, some other k arise for which $\left(-\frac{d^2}{dx^2} + V - k^2\right) u = 0$ has a non-trivial solution, but such solutions are not in L^2. These are called the resonances and we define them below.

Definition 2.5.4 (Resonance). *We define a resonance of the differential operator* $P = -\frac{d^2}{dx^2} + V$ *to be a pole of the meromorphic continuation of the resolvent* $R(k)$ *in the unphysical sheet* $\operatorname{Im} k \leq 0$ *with* $k^2 = \lambda$. *We jointly enumerate the eigenvalues and resonances as* $(\lambda_j)_{j \in \mathbb{N}}$, *where* $\lambda_1, \ldots, \lambda_N$ *are the eigenvalues.*

Remark 2.5.5. *In the definition of resonances, we consider the number of eigenvalues N to be finite. Indeed, we will prove this fact later in Theorem 2.5.24.*

Remark 2.5.6. *With a slight abuse of terminology, we might call the k_j, such that $\lambda_j = k_j^2$ is an eigenvalue or a resonance, also an eigenvalue or resonance for the sake of simplicity.*

Remark 2.5.7. *The self-adjointness of the problem implies that if k is a resonance, $-\overline{k}$ is also and with the same multiplicity [21, Section 2.2].*

Eigenvalues and resonances are the scattering data we use to solve the inverse resonance problem. They can be seen as zeros of the Fredholm determinant (see [49]) or of the Jost function. The Jost function is defined in terms of the Jost solution, introduced below (see Definition 2.5.11 for the Jost function).

Definition 2.5.8 (Jost solution). *The Jost solutions $f^{\pm}$ are the unique solutions to the differential equation* (2.2) *that satisfy the following condition*

$$f^{\pm}(x,k) = e^{\pm ikx} \qquad \textit{for } \; x > x_I. \tag{2.11}$$

The radiation condition (2.11) tells us that the solution of the differential equation must behave like a plane wave (eigenfunction of the Helmholtz operator) far from the scattering area (for $x > x_I$) and implies uniqueness. Since $V \in L_{1,1}$ then the Jost solution can be rewritten as:

$$f^{\pm}(x,k) = e^{\pm ikx} + \int_x^{\infty} A(x,t)\, e^{\pm ikt}\, dt, \tag{2.12}$$

where $A(x,t)$ is the kernel of the scattering transformation operator (see [34, Section 4.2] or [22, Theorem 2.1.3]). The transformation operator is a continuous function for $0 \leq x \leq t < \infty$, and satisfies

$$A(x,x) = \frac{1}{2}\int_x^{\infty} V(t)dt, \tag{2.13}$$

$$|A(x,t)| \leq \frac{1}{2} Q_0\left(\frac{x+t}{2}\right) e^{Q_1(x) - Q_1\left(\frac{x+t}{2}\right)}, \tag{2.14}$$

where

$$Q_0(x) := \int_x^{\infty} |V(t)|dt, \tag{2.15}$$

$$Q_1(x) := \int_x^{\infty} Q_0(t)dt = \int_x^{\infty} (t-x)|V(t)|dt. \tag{2.16}$$

The self-adjointness of the differential operator implies that, for real k, $f^-(x,k) = \overline{f^+(x,k)} = f^+(x,-k)$. These properties suggest that we remove the superscript $+$

and – and set $f^+(x,k) =: f(x,k)$. Accordingly, we will refer to f^- as the conjugate of f. By solving (2.2) with the variation of constants method, we can get a Volterra-type equation for the Jost solution $f(x,k)$ (see [34, Section 4.2])

$$f(x,k) = e^{ikx} - \int_x^\infty \frac{\sin[k(x-t)]}{k} V(t) f(t,k) dt. \tag{2.17}$$

In this form, the Jost solution can be naturally expanded as a power series of the potential, by Volterra iteration. Below we state some properties of the Jost solution.

Proposition 2.5.9. *The Wronskian* $W(f,\overline{f})(x) = f(x,k)\overline{f'(x,k)} - f'(x,k)\overline{f(x,k)}$ *is independent of* x.

Proof. This follows from the fact that in our differential equation the term of order one in the derivative is missing. Then the Wronskian between two eigenfunctions for the same (real) eigenvalue is constant in x, for example

$$\frac{d}{dx}W(f,\overline{f})(x) = f'(x,k)\overline{f'(x,k)} + f(x,k)\overline{f''(x,k)} - f'(x,k)\overline{f'(x,k)}$$
$$- f''(x,k)\overline{f(x,k)} = \left(-V + k^2\right)\left(f(x,k)\overline{f(x,k)} - f(x,k)\overline{f(x,k)}\right) = 0. \quad \square$$

Proposition 2.5.10. *The Jost solutions* $f(x,k)$ *and* $\overline{f(x,k)}$ *are linearly independent for real* $k \neq 0$.

Proof. Since the Wronskian is independent of x, we can compute it for $x \to \infty$, as the Jost solution is known by the boundary condition at infinity intrinsic in its definition. Hence, we have

$$W(f,\overline{f}) = \lim_{x\to\infty} W(f,\overline{f})(x) = \lim_{x\to\infty}\left(f\overline{f}' - f'\overline{f}\right)(x) = -2ik. \tag{2.18}$$

This ends the proof. $\square$

The two Jost solutions f and $\overline{f}$ are two linearly independent solutions of the differential equation (2.2), hence they form a basis of the space of solutions and we can write the general solution to the boundary value problem (2.2)–(2.3) as a linear combination of them

$$\varphi(x,k) = a(k)f(x,k) + b(k)\overline{f(x,k)}. \tag{2.19}$$

Imposing the boundary condition (2.3) on φ we get:

$$a(k)\left[f(0,k)h + f'(0,k)\right] + b(k)\left[\overline{f(0,k)}h + \overline{f'(0,k)}\right] = 0$$

which gives us a one-parameter family of solutions

$$\varphi(x,k) = c(k)\left\{\left[\overline{f(0,k)}h + \overline{f'(0,k)}\right] f(x,k) - \left[f(0,k)h + f'(0,k)\right] \overline{f(x,k)}\right\}. \tag{2.20}$$

We let $\varphi(x,k)$ satisfy the condition

$$\varphi(0,k) = 1, \tag{2.21}$$

and we determine the constant $c(k)$

$$c(k) = \frac{1}{W(f,\overline{f})} = -\frac{1}{2ik}.$$

Definition 2.5.11 (Jost function). *We define the Jost function $f_h(k)$ of the Schrödinger operator $-\frac{d^2}{dx^2} + V$ in* (2.2) *with Robin boundary condition* (2.3) *as the quantity*

$$f_h(k) = f(0,k)h + f'(0,k) \tag{2.22}$$

where $f(0,k)$ is the Jost solution evaluated at $x = 0$.

We enumerate the zeros of f_h as $(k_j)_{j\in\mathbb{N}}$ in the same way as in Definition 2.5.4.

Remark 2.5.12. *It is important to point out that if k_j is a zero of $f_h(k)$ then the Jost solution $f(x,k_j)$ satisfies the boundary value problem* (2.2)–(2.3).

The Jost function contains information on the Jost solution at the boundary. It is a fundamental quantity for the problem because its zeros correspond to our data (eigenvalues and resonances). The goal is to find enough analytic properties so that we are able to reconstruct the Jost function from its zeros. Below we define the regular solution (see [11, Section 1.2]).

Definition 2.5.13 (Regular solution). *We define the regular solution φ of the Cauchy problem* (2.2) *with Robin boundary condition* (2.3) *as the quantity*

$$\varphi(x,k) = -\frac{1}{2ik}\left[\overline{f_h(k)}f(x,k) - f_h(k)\overline{f(x,k)}\right]. \tag{2.23}$$

The Jost function corresponds to the Wronskian between the regular solution $\varphi(x,k)$ to the problem (2.2) and the Jost solution $f(x,k)$ evaluated at $x = 0$. Indeed

$$\begin{aligned} W(\varphi, f)(x=0,k) &= \varphi(0,k)f'(0,k) - \varphi'(0,k)f(0,k) \\ &= 1 \cdot f'(0,k) - (-h)f(0,k) = f_h(k). \end{aligned} \tag{2.24}$$

For real $k \neq 0$ we can easily obtain the property

$$\overline{f_h(k)} = f_h(-k), \tag{2.25}$$

since h and the kernel $A(x,t)$ are real. In order to keep the property (2.25) valid also for complex k, we need to define the complex conjugation operation as $f_h^*(k) = \overline{f_h(\overline{k})}$ and we can prove that, similarly, $f_h^*(k) = \overline{f_h(\overline{k})} = f_h(-k)$ holds for $k \in \mathbb{C}$. From (2.25)

$$\begin{aligned}
\varphi(x,k) &= \frac{-1}{2ik}\left\{\left[f(0,-k)h + f'(0,-k)\right] f(x,k) - \left[f(0,k)h + f'(0,k)\right] f(x,-k)\right\} \\
&= \frac{-1}{2ik}\left[f_h(-k)f(x,k) - f_h(k)f(x,-k)\right] \\
&= \frac{-1}{2ik}\left[f_h(-k)f(x,k) - \overline{f_h(-k)}\,\overline{f(x,k)}\right] = \frac{-1}{2ik}\left[2i\,\mathrm{Im}\left\{f_h(-k)f(x,k)\right\}\right],
\end{aligned}$$

so, we obtain

$$\varphi(x,k) = -\frac{1}{k}\,\mathrm{Im}\left\{f_h(-k)f(x,k)\right\}. \tag{2.26}$$

We can write the Jost function in a polar form

$$f_h(k) = |f_h(k)|e^{-i\delta(k)} \tag{2.27}$$

where $\delta(k)$ is called the *scattering phase*. From (2.25) we can recover an interesting property of the scattering phase, indeed

$$\overline{f_h(k)} = |f_h(k)|e^{i\delta(k)} = |f_h(k)|e^{-i\delta(-k)} = f_h(-k)$$

then $\delta(k) = -\delta(-k)$, so the scattering phase is an odd function. From (2.26), (2.27) and (2.12) we can write the general solution in the following form

$$\varphi(x,k) = -\frac{|f_h(k)|}{k}\left\{\sin\left(kx + \delta(k)\right) + \int_x^{\infty} A(x,t;k)\sin\left(kt + \delta(k)\right)dt\right\}.$$

Definition 2.5.14 (Scattering function). *We define the scattering function $S(k)$ of the problem as the negative of the ratio between the Jost function $f_h(k)$ and the reflected Jost function $f_h(-k)$*

$$S(k) = -\frac{f_h(-k)}{f_h(k)} = -e^{2i\delta(k)}. \tag{2.28}$$

Remark 2.5.15. *In the case of Dirichlet boundary condition it is usually defined as $\frac{f(0,-k)}{f(0,k)}$, where $f(0,k)$ is the Jost function in the Dirichlet case (see [30] or [34, Section 4.2]). In the case of Robin boundary condition, it is usually defined as in* (2.28) *(see [51]).*

2.5.1 Properties of the Jost function

In this subsection we will obtain some properties of the Jost function, that will help us with the direct and inverse results. For the following, we recall the definition of $\mathcal{N}_+(f)$. In the following and throughout we define the (complex) Fourier transform

$$\hat{g}(k) := \int_I g(x)e^{2ixk}dx,$$

$k \in \mathbb{C}$, for $g \in L^1(I)$ with bounded support, where I is an interval. Moreover, throughout the thesis we denote by $\mathbb{C}_\pm$ the upper and lower half, respectively, of the complex plane $\mathbb{C}$.

Definition 2.5.16 (Class of Jost function). *We define the class W_{x_I} of Jost functions as the class of all entire functions f such that:*

I $f(k) \neq 0$ for all $k \in \mathbb{R}$ and for some $F \in \mathbb{V}_{x_I}$ the function f is given by

$$f(k) = ik\left[1 - \frac{1}{2ik}\left(\hat{F}(0) - \hat{F}(k)\right)\right], \qquad \hat{F}(k) = \int_0^{x_I} F(x)e^{2ixk}dx. \tag{2.29}$$

II All zeros $k_1, \dots, k_N$ of the function f in $\mathbb{C}_+$ are simple, belong to $i\mathbb{R}_+$ and satisfy for $n = 1, \dots, N = \mathcal{N}_+(f)$:

$$|k_1| > |k_2| > \dots > |k_N| > 0 \quad \textit{and} \quad f_h(-k_n)(-1)^n < 0. \tag{2.30}$$

It will be clear later why we call W_{x_I} class of Jost functions. In particular, the goal is to prove a bijection between $\mathbb{V}_{x_I}$ and W_{x_I} (see Theorem 2.5.44). In order to do so, we first have to prove that the Jost function is entire and that it satisfies the two conditions of the definition of the class.

In the next theorem we prove that the Jost solution and the Jost function are entire in k (see also [38, Lemma 3.1.4.] for the Dirichlet case and $V \in L_{1,1}$ potential).

Theorem 2.5.17. *For each fixed $x \geq 0$, the Jost solution $f(x,k)$ and the Jost function $f_h(k)$ are entire in k.*

Proof. Recall from (2.17) the following Volterra-type equation

$$f(x,k) = e^{ikx} - \int_x^\infty \frac{\sin[k(x-t)]}{k}V(t)f(t,k)dt.$$

Multiplying (2.17) by e^{-ikx} and defining the Faddeev solution as

$$\chi(x,k) = f(x,k)e^{-ikx}$$

we get

$$\chi(x,k) = 1 - \int_x^\infty \frac{1 - e^{-2ik(x-t)}}{2ik} V(t)\chi(t,k)dt. \tag{2.31}$$

Iterating (2.31) gives the series

$$\chi(x,k) = \sum_{l=0}^{\infty} \chi^{(l)}(x,k) \tag{2.32}$$

where

$$\chi^{(0)}(x,k) = 1$$

and

$$\chi^{(l)}(x,k) = (-1)^l \int_x^\infty \int_{t_1}^\infty \cdots \int_{t_{l-1}}^\infty \prod_{j=1}^{l} \frac{1 - e^{-2ik(t_{j-1}-t_j)}}{2ik} V(t_j) dt_l \cdots dt_1$$

with the convention that $t_0 = x$. Moreover, we have the estimate

$$\begin{aligned}
|\chi^{(l)}(x,k)| &\leq \int_x^\infty \int_{t_1}^\infty \cdots \int_{t_{l-1}}^\infty \frac{e^{(|\operatorname{Im} k| - \operatorname{Im} k)[(t_1-x)+(t_2-t_1)+\cdots(t_l-t_{l-1})]}}{(\max(1,|k|))^l} \\
&\qquad |V(t_1)|\cdots|V(t_l)|dt_1\cdots dt_l \\
&= \frac{e^{(|\operatorname{Im} k| - \operatorname{Im} k)(x_I - x)}}{(\max(1,|k|))^l} \int_x^{x_I} \int_t^{x_I} \cdots \int_{t_{l-1}}^{x_I} |V(t_1)|\cdots|V(t_l)|dt_1\cdots dt_l \\
&= \frac{e^{(|\operatorname{Im} k| - \operatorname{Im} k)(x_I - x)}}{(\max(1,|k|))^l} \frac{1}{l!} \left(\int_x^{x_I} |V(t)|dt \right)^l
\end{aligned} \tag{2.33}$$

where we have used the fact that the potential has compact support, $V(x) = 0$ for $x > x_I$, and in the last passage, we have used that those l-integrals with respect to different variables are equal to l times the product of the integral of the potential divided by $l!$. Each term of the power series is bounded by the term appearing in (2.33), which leads to a uniformly convergent series on every compact set of k. By Weierstrass M-test, also the original series (2.32) converges uniformly and absolutely on every compact set, hence, $\chi(x,k)$ is entire in k. Then, also $f(x,k) = \chi(x,k)e^{ikx}$ is entire in k.

For $f'(x,k)$ we have

$$\begin{aligned}
&f'(x,k)e^{-ikx} \\
&= ik - \int_x^\infty \frac{1 + e^{-2ik(x-t_1)}}{2} V(t_1)dt_1
\end{aligned}$$

$$+\int_x^\infty\int_{t_1}^\infty \frac{1+e^{-2ik(x-t_1)}}{2}\frac{1-e^{2ik(t_2-t_1)}}{2ik}V(t_1)V(t_2)\chi(t_2,k)dt_1dt_2$$

that for the same argument lead to $f'(x,k)$ being entire in k. Therefore $f_h(k)$ is also entire. □

From the proof of Theorem 2.5.17 we get

$$|\chi(x,k)| \leq 1+\left(\sum_{l=1}^\infty \frac{1}{l!}\left(\frac{\int_x^{x_I}|V(t)|dt}{\max(1,|k|)}\right)^l\right)e^{(|\operatorname{Im}k|-\operatorname{Im}k)(x_I-x)}$$

$$= 1+\left(\sum_{l=0}^\infty \frac{1}{l!}\left(\frac{\int_x^{x_I}|V(t)|dt}{\max(1,|k|)}\right)^l - 1\right)e^{(|\operatorname{Im}k|-\operatorname{Im}k)(x_I-x)}, \tag{2.34}$$

that becomes

$$|\chi(x,k)| \leq e^{\frac{\int_x^{x_I}|V(t)|dt}{\max(1,|k|)}}e^{(|\operatorname{Im}k|-\operatorname{Im}k)(x_I-x)},$$

since $1 \leq e^{(|\operatorname{Im}k|-\operatorname{Im}k)(x_I-x)}$.

In the physical sheet, $\operatorname{Im}k \geq 0$, (2.34) becomes

$$|\chi(x,k)| \leq e^{\frac{\int_x^{x_I}|V(t)|dt}{\max(1,|k|)}}. \tag{2.35}$$

Subtracting $\chi^{(0)}(x,k)$ in both members of (2.32), and repeating the same steps as in the proof of Theorem 2.5.17 we get

$$|\chi(x,k)-1| \leq e^{\frac{\int_x^{x_I}|V(t)|dt}{\max(1,|k|)}} - 1 \tag{2.36}$$

for $\operatorname{Im}k > 0$.

In the following, we state some properties of the Jost function (Proposition 2.5.18, Proposition 2.5.19 and Theorem 2.5.28) as in [34].

Proposition 2.5.18. *$f_h(k)$ cannot vanish for real $k \neq 0$.*

Proof. We have

$$\det\begin{pmatrix} f_h(k) & f_h(-k) \\ f(0,k) & f(0,-k)\end{pmatrix} = f_h(k)f(0,-k) - f_h(-k)f(0,k)$$
$$= \big(f'(0,k)+hf(0,k)\big)f(0,-k) - \big(f'(0,-k)+hf(0,-k)\big)f(0,k)$$
$$= f'(0,k)f(0,-k) - f'(0,-k)f(0,k) = -W(f,\overline{f})(0),$$

for k real. Using (2.18) we obtain

$$\det \begin{pmatrix} f_h(k) & f_h(-k) \\ f(0,k) & f(0,-k) \end{pmatrix} = 2ik. \tag{2.37}$$

By Proposition 2.5.10 the two Jost solutions are linearly independent for real $k \neq 0$, then it follows that $f_h(\pm k)$ cannot vanish, as $\overline{f_h(k)} = f_h(-k)$. □

The arguments in the proof of Proposition 2.5.18 do not work for complex k in general. Indeed, for complex k, $f(x,-k) \neq \overline{f(x,k)}$ but $\overline{f(x,\overline{k})} = f(x,-k)$ instead. Proposition 2.5.18 is equivalent to saying that there are no eigenvalues on $\mathbb{R}\backslash\{0\}$.

Proposition 2.5.19. *$f_h(k)$ and $\frac{d}{dk} f_h(k)$ cannot vanish simultaneously.*

Proof. We can prove this by differentiating (2.37) with respect to k. □

In the next theorem we show that for our self-adjoint operator, the eigenvalues and eigenfunctions are real.

Theorem 2.5.20. *The zeros k_j of the Jost function $f_h(k)$ in the upper half-plane are purely imaginary and the eigenfunctions $\varphi(x,k_j)$ and $f(x,k_j)$ are real.*

Proof. We adapt the proof of Theorem 2.3.2 in [22]. From (2.24) we have that if k_j is a zero of $f_h(k)$, then

$$f(x,k_j) = C_j \varphi(x,k_j), \qquad C_j \neq 0.$$

For $x = 0$, we have $C_j = f(0,k_j)$ because of condition (2.21), hence

$$f(x,k_j) = f(0,k_j)\varphi(x,k_j), \qquad f(0,k_j) \neq 0.$$

Suppose λ_n and λ_m with $\lambda_n \neq \lambda_m$ are eigenvalues with eigenfunctions $f(x,k_n)$ and $f(x,k_m)$ ($k_n^2 = \lambda_n$), then

$$\begin{aligned} &\int_0^\infty \left(-\frac{d^2}{dx^2} + V(x)\right) f(x,k_n) f(x,k_m) dx \\ &= \int_0^\infty f(x,k_n) \left(-\frac{d^2}{dx^2} + V(x)\right) f(x,k_m) dx \end{aligned}$$

using integration by parts and the self-adjointness of the problem (2.2)–(2.3). Hence

$$\lambda_n \int_0^\infty f(x,k_n) f(x,k_m) dx = \lambda_m \int_0^\infty f(x,k_n) f(x,k_m) dx,$$

which implies

$$\int_0^\infty f(x,k_n)f(x,k_m)dx = 0.$$

Assume that $\lambda_j = a + ib$ is a non real eigenvalue ($b \neq 0$) with eigenfunction $f(x,k_j) \neq 0$. Since V and h are real then $\overline{\lambda_j}$ is also an eigenvalue with eigenfunction $\overline{f(x,k_j)}$, since $\lambda_j \neq \overline{\lambda_j}$, then we get

$$\int_0^\infty f(x,k_j)\overline{f(x,k_j)}dx = 0,$$

which is impossible as $f(0,k_j) \neq 0$. Then all the eigenvalues λ_j are real, k_j are purely imaginary by Proposition 2.5.18, and the eigenfunctions $\varphi(x,k_j)$ and $f(x,k_j)$ are real. □

Remark 2.5.21. *In the case of* $V = 0$ *with Robin boundary condition, there is only one eigenvalue* $-h^2$ *plus the continuous spectrum* $(0,+\infty)$*. If we add a bounded perturbation* V *to* $-\frac{d^2}{dx^2}$ *the number of eigenvalues can increase, but the set of eigenvalues will still be bounded as we can see below.*

The next lemma shows another classical result for some self-adjoint operators, which are lowerly semi-bounded for our class of potentials.

Lemma 2.5.22. *Assume* $V \in \mathbb{V}_{x_I}$*. The self-adjoint realization of* $-\frac{d^2}{dx^2}+V$ *in* $L^2(0,+\infty)$ *with domain* (2.4) *is bounded from below.*

Proof. From [7, Lemma 3, page 221], we know that

$$\int_0^\infty V(x)|f(x)|^2dx \geq -C\left(t^{-1}||f||_{L^2}^2 + t||f'||_{L^2}^2\right) \tag{2.38}$$

for $0 < t \leq 1$. Then the quadratic form defined as

$$q[f] := \left(\left(-\frac{d^2}{dx^2}+V\right)f, f\right)_{L^2}$$

for any $f \in H^2[0,+\infty)$ satisfying the boundary condition (2.3) becomes

$$q[f] = \int_0^\infty \left(|f'(x)|^2 + V(x)|f(x)|^2\right)dx - h|f(0)|^2$$

which using (2.38) yields

$$q[f] \geq (1-Ct)||f'||_{L^2}^2 - Ct^{-1}||f||_{L^2}^2 - h|f(0)|^2. \tag{2.39}$$

Let χ_t be the standard mollifier in $C_c^\infty[0,1)$ such that $\chi_t = 1$ in a neighborhood of zero. Then we can write

$$f(0) = -\int_0^1 (\chi_t f)'\,dx = -\int_0^1 \left(\chi_t' f + \chi_t f'\right) dx,$$

that implies

$$|f(0)| \le ||\chi_t'||_{L^2}||f||_{L^2} + ||\chi_t||_{L^2}||f'||_{L^2}$$

and

$$|f(0)|^2 \le 2||\chi_t'||_{L^2}^2||f||_{L^2}^2 + 2||\chi_t||_{L^2}^2||f'||_{L^2}^2.$$

For any $\epsilon > 0$ there exists a χ_t such that

$$||\chi_t||_{L^2} < \epsilon$$

and we can write (2.39) as

$$q[f] > (1 - Ct - 2|h|\epsilon^2)||f'||_{L^2}^2 - \left(Ct^{-1} + 2|h|||\chi_t'||_{L^2}^2\right)||f||_{L^2}^2, \tag{2.40}$$

so, we take ϵ and t such that

$$\epsilon^2 < \frac{1}{6|h|} \qquad \text{and} \qquad t < \frac{1}{3C}$$

then the coefficient of the first term of (2.40) is positive and there exists a $\tilde{C} > 0$ such that

$$q[f] > -\tilde{C}||f||_{L^2}^2.$$

□

Remark 2.5.23. *In light of Theorem 2.5.20 and Lemma 2.5.22, we conclude that the set of eigenvalues is bounded.*

The next theorem shows a classical result on the finite number of eigenvalues for self-adjoint Schrödinger operators with L^1 potential and first momentum.

Theorem 2.5.24. *Let $V \in \mathbb{V}_{x_I}$, then the number of zeros of the Jost function $f_h(k)$ in the upper half-plane is finite.*

Proof. We partly follow [22, Theorem 2.3.4]. From (2.36) we know that for $k = i\tau$ with $\tau > 0$ we have

$$|f(x, i\tau)e^{\tau x} - 1| \le e^{\frac{\int_x^{x_I} |V(t)|dt}{\max(1,|k|)}} - 1. \tag{2.41}$$

As x approaches x_I, the exponential tends to 1. Then, there exists an $a > 0$ such that

$$e^{\frac{\int_x^{x_I} |V(t)|dt}{\max(1,|\tau|)}} - 1 \leq \frac{1}{2} \qquad x \geq a, \ \tau \geq 0,$$

and for the same $a > 0$ we have

$$f(x, i\tau)e^{\tau x} \geq \frac{1}{2} \qquad x \geq a, \ \tau \geq 0. \tag{2.42}$$

We assume towards a contradiction that the set of eigenvalues $\Lambda' = \{\lambda_j\}$ is infinite. Since Λ' is bounded (see Lemma 2.5.22), then $k_j = i\tau_j \to 0$ for $\tau > 0$. We have

$$\begin{aligned} \int_a^\infty f(x, k_j)f(x, k_n)dx &= \int_a^\infty \left(f(x, k_j)e^{\tau_j x}\right)\left(f(x, k_n)e^{\tau_n x}\right) e^{-(\tau_j + \tau_n)x} dx \\ \geq \frac{1}{4}\int_a^\infty e^{-(\tau_j+\tau_n)x} dx &\geq \frac{e^{-2aT}}{8T} \end{aligned} \tag{2.43}$$

after using (2.42) and that $T := \max_j \tau_j$. The operator described by (2.2)–(2.3) with domain (2.4) is self-adjoint, so the eigenfunctions $f(x, k_j)$ and $f(x, k_n)$ are orthogonal in $L^2(0, \infty)$

$$\begin{aligned} 0 = \int_0^\infty f(x, k_j)f(x, k_n)dx &= \int_a^\infty f(x, k_j)f(x, k_n)dx \\ + \int_0^a \left(f(x, k_j)f(x, k_n) + f^2(x, k_j) - f^2(x, k_j)\right) dx &= \int_a^\infty f(x, k_j)f(x, k_n)dx \\ + \int_0^a f^2(x, k_j)dx + \int_0^a f(x, k_j)\left(f(x, k_n) - f(x, k_j)\right) dx. \end{aligned} \tag{2.44}$$

The first and the second term of (2.44) are, due to (2.43),

$$\int_a^\infty f(x, k_j)f(x, k_n)dx \geq C_a > 0, \qquad \int_0^a f^2(x, k_j)dx \geq 0.$$

In order to get a contradiction, we aim to show that

$$\int_0^a f(x, k_j)\left(f(x, k_n) - f(x, k_j)\right) dx \to 0 \qquad \text{as } j, n \to \infty. \tag{2.45}$$

By (2.35), it holds for $x \geq 0$

$$|f(x, k_j)| \leq e^{\frac{\int_0^{x_I} |V(t)|dt}{\max(1,|k_j|)}} e^{-k_j x} \leq e^{\frac{\int_0^{x_I} |V(t)|dt}{\max(1,|k_j|)}}.$$

Using (2.13), we can write

$$\left|\int_0^a f(x,k_j)\left(f(x,k_n)-f(x,k_j)\right)dx\right| \leq e^{\frac{\int_0^{x} I\ |V(t)|dt}{\max(1,|k_j|)}} \int_0^a \left|e^{-\tau_n x}-e^{-\tau_j x}\right| dx$$
$$+\, e^{\frac{\int_0^{x} I\ |V(t)|dt}{\max(1,|k_j|)}} \int_0^a \left(\int_x^\infty |A(x,t)\left(e^{-\tau_n t}-e^{-\tau_j t}\right)|dt\right) dx, \qquad (2.46)$$

then the first term

$$\int_0^a \left|e^{-\tau_n x}-e^{-\tau_j x}\right| dx \to 0 \quad \text{as } n,j\to\infty$$

by the dominated convergence theorem. We use (2.14) in

$$|A(x,t)| \leq \frac{1}{2}Q_0\left(\frac{x+t}{2}\right)e^{Q_1(x)} \leq \frac{1}{2}Q_0\left(\frac{t}{2}\right)e^{Q_1(0)}$$

since Q_0 and Q_1 are decreasing, the last term of (2.46) becomes

$$\int_0^a \left(\int_x^\infty |A(x,t)\left(e^{-\tau_n t}-e^{-\tau_j t}\right)|dt\right) dx$$
$$\leq e^{Q_1(0)} \int_0^a \int_0^\infty Q_0\left(\frac{t}{2}\right)\left|e^{-\tau_n t}-e^{-\tau_j t}\right| dt\, dx$$
$$\leq C\int_0^\infty Q_0\left(\frac{t}{2}\right)\left|e^{-\tau_n t}-e^{-\tau_j t}\right| dt \leq C\int_0^\infty Q_0\left(\frac{t}{2}\right)\left|e^{-\tau_n t}-e^{-\tau_j t}\right| dt \to 0$$

as $n,j\to\infty$ by the dominated convergence theorem, since the integrand is dominated by $2Q_0\left(\frac{t}{2}\right)$ which is integrable because

$$\int_0^\infty 2Q_0\left(\frac{t}{2}\right) dt = 2Q_1(0) < \infty.$$

Then (2.45) is satisfied and we get a contradiction, thus, Λ' is a finite set. □

Remark 2.5.25. *In Theorem 2.5.24 it is enough to assume the potential to be in $L_{1,1}$, but we would need to use different estimates than* (2.41). *In particular, we could use the estimate $|f(x,i\tau)e^{\tau x}-1| \leq Q_1(x)e^{Q_1(x)}$ with $Q_1(x)$ as in* (2.16) *(see [22, Theorem 2.3.4.]).*

Remark 2.5.26. *The finiteness of the number of eigenvalues and their sign (negative for λ_j with $j=1,\dots,N$) does not surprise us. Indeed, the spectrum for the Laplacian, $-\Delta$, for Dirichlet and Neumann boundary conditions is equal to the essential spectrum $[0,+\infty)$, while in the case of Robin boundary condition there is also the eigenvalue $-h^2$. Since $-\Delta$ is* *self-adjoint and if we add a symmetric and relatively compact perturbation*[3] V *the operator* $-\Delta+V$ *will be self-adjoint (Kato-Rellich Theorem) and the essential spectrum is conserved, hence it will be* $\sigma_{ess}(-\Delta+V)=[0,+\infty)$ *for any dimension. So, the self-adjointness and the essential spectrum are stable under a symmetric and relatively compact perturbation*[4] *(see [28, Theorem 5.35, Chapter IV] and [28, Theorem 4.3, Chapter V]). The number of eigenvalues, instead, can increase. In fact, for the Robin Laplacian, we will have instead of a negative eigenvalue* $-h^2$ *a finite set of real negative eigenvalues* $\lambda_1,\ldots,\lambda_N$ *(hence pure imaginary in k). The fact that the number of eigenvalues remains finite depends on the potential being* $V\in L_{1,1}$ *(see [43, Chapter 1.1, page 12]).*

The next theorem is a classical result for Schrodinger self-adjoint operator with $L_{1,1}$ potentials (see [22, Theorem 2.3.6.]).

Theorem 2.5.27. *Let* $V\in L_{1,1}$ *be real. Then* $k=0$ *is not an eigenvalue of* (2.2).

Proof. We adapt the proof from Theorem 2.3.6. in [22]. The Jost solution $f(x,0)$ is a solution of (2.2) for $k=0$, but the asymptotic condition (2.11) implies

$$\lim_{x\to\infty} f(x,0)=1. \tag{2.47}$$

We take $a>0$ such that

$$f(x,0)>\frac{1}{2} \qquad \text{for } x\geq a,$$

and we define

$$z(x):=f(x,0)\int_a^x \frac{dt}{(f(t,0))^2}.$$

We can see that $z(x)$ is also a solution of (2.2) for $k=0$, because it satisfies

$$z''(x)=V(x)z(x),$$

since $f''(x,0)=V(x)f(x,0)$. Moreover, it satisfies:

$$f(x,0)z'(x)-f'(x,0)z(x)=1,$$

[3]We say that V is relatively compact if for any sequence $x_n\in L^2$, if Δx_n is bounded in L^2 then there exists a convergent (in L^2) subsequence of Vx_n.

[4]For the self-adjointness it is enough the potential to be symmetric and relatively bounded. We say that an operator V is T-bounded if $D(T)\subset D(V)$ and if there exists $a>0$ and $b>0$ so that $\|Vf\|_{L^2}^2\leq a^2\|f\|_{L^2}^2+b^2\|Tf\|_{L^2}^2$ for any $f\in D(V)$.

which means that $f(x,0)$ and $z(x)$ are linearly independent. Using that $f(x,0)$ is positive in (a,x) and (2.47) it follows that

$$\lim_{x\to\infty} z(x) = +\infty. \tag{2.48}$$

We assume towards a contradiction that $y(x)$ is an eigenfunction in L^2 corresponding to the eigenvalue $k = 0$. Since $f(x,0)$ and $z(x)$ satisfy (2.2) for $k = 0$ and since they are linearly independent, we can then write

$$y(x) = C_1 f(x,0) + C_2 z(x).$$

By (2.47) and (2.48), then, it must be that $C_1 = C_2 = 0$. Then $y(x)$ is the trivial solution $y(x) = 0$, so $k = 0$ is not an eigenvalue. □

The following theorem is a classical result on the simplicity of the eigenvalues for self-adjoint Schrödinger operators with $L_{1,1}$ potentials. We follow a proof similar to [34, Chapter 4, page 79].

Theorem 2.5.28. *Let $V \in \mathbb{V}_{x_I}$, then the zeros of the function $f_h(k)$ in the upper half-plane are all simple.*

Proof. We assume towards a contradiction that there exists a root k_0 which is not simple: $f_h(k_0) = 0$ and $\dot{f}_h(k_0) = 0$, where the dot indicates the derivative with respect to k. These two conditions mean that:

$$\begin{aligned} f'(0,k_0) &= -f(0,k_0)h \\ \dot{f}'(0,k_0) &= -\dot{f}(0,k_0)h \end{aligned} \tag{2.49}$$

We consider the differential equation satisfied by the Jost solution $f(x,k)$, and its derivative with respect to k

$$\begin{aligned} -f''(x,k) + V(x)f(x,k) &= k^2 f(x,k) \\ -\dot{f}''(x,k) + V(x)\dot{f}(x,k) &= 2kf(x,k) + k^2\dot{f}(x,k). \end{aligned}$$

We multiply the first equation by $\dot{f}(x,k)$ and the second by $f(x,k)$ and subtract the second from the first

$$-\dot{f}(x,k)f''(x,k) + f(x,k)\dot{f}''(x,k) = -2kf^2(x,k).$$

Then, integrating by parts and considering that

$$\lim_{x\to\infty}\Big(f(x,k)\dot{f}'(x,k) - f'(x,k)\dot{f}(x,k)\Big) = \lim_{x\to\infty}\Big(e^{ikx}(ix)(ik)e^{ikx}$$

$$-(ix)e^{ikx}(ik)e^{ikx}\Big) = 0$$

we get

$$f'(0,k)\dot{f}(0,k) - \dot{f}'(0,k)f(0,k) + \int_0^\infty f'(x,k)\dot{f}'(x,k)dx$$
$$- \int_0^\infty \dot{f}'(x,k)f'(x,k)dx = -2k\int_0^\infty f^2(x,k)dx$$

which becomes

$$f'(0,k)\dot{f}(0,k) - \dot{f}'(0,k)f(0,k) = -2k\int_0^\infty f^2(x,k)dx. \tag{2.50}$$

Evaluating this equation at k_0 and using the condition of a multiple root (2.49) we get a contradiction, in fact, the left-hand side is zero while the right-hand side is not. □

Remark 2.5.29. *For the Robin Laplacian in the half-line the eigenvalue $-h^2$ is simple. In the one-dimensional case, we have seen that the simplicity of the eigenvalue is stable under the addition of a small compact perturbation. However, for dimensions $d \geq 2$, this is not true in general. Indeed, if we consider $-\Delta + V$ in $\mathbb{R}^3$ with V being the Coulomb potential, the eigenvalues are all simple and accumulating at zero. However, if we add a potential coming from an external electric field, we will observe a splitting of the eigenvalues which become multiple (Stark effect). The same happens if we add a potential coming from a magnetic field (Zeeman effect).*

Those properties of the eigenvalues and resonances proved in Theorem 2.5.20 and Theorem 2.5.28 depend on the self-adjointness of the differential operator (see [22]). In the following lemma we obtain some estimates on the Jost function, which will be useful both for the direct and the inverse results.

Lemma 2.5.30 (Uniform bounds on Jost function). *Let $V \in \mathbb{V}_{x_I}$. Then the Jost function is of exponential type and satisfies the following estimates:*

$$|f_h(k) - ik| \leq \|V\|\, e^{(|\operatorname{Im} k| - \operatorname{Im} k)x_I} e^a \tag{2.51}$$

$$\left| f_h(k) - ik - h + \frac{\hat{V}(0) + \hat{V}(k)}{2} \right| \leq \left[|h| + \frac{\|V\|}{2} \right] a e^{(|\operatorname{Im} k| - \operatorname{Im} k)x_I} e^a \tag{2.52}$$

where $\hat{V}(k) = \int_0^{x_I} e^{2ikt}V(t)dt$ is the Fourier transform of the potential[5] V and $a = \frac{||V||}{\max(1,|k|)}$, with $||V|| := \int_{\mathbb{R}} |V(x)|dx$.

[5]In physics the Fourier transform of the potential would be the scattering amplitude.

Proof. These formulas result from the bound on the Jost solution and Jost function and from the definition of the Jost solution iterating the Neumann series up to the first order. We first compute the estimate of the Jost solution (step 1), then we do the same for the derivative of the Jost solution (step 2) and finally, we collect those results to get an estimate of the Jost function (step 3).

- *Step 1.* We start from (2.32) and write it as

$$\chi(x,k) = \sum_{l=0}^{N} \chi^{(l)}(x,k) + R_N(x,k)$$

where $R_N(x,k)$ is the remainder defined as

$$R_N(x,k) = (-1)^{N+1} \int_x^\infty \int_{t_1}^\infty \cdots \int_{t_N}^\infty \prod_{j=1}^{N+1} \left(\frac{1 - e^{-2ik(t_{j-1}-t_j)}}{2ik} V(t_j) \right) \cdot \chi(t_{N+1},k) dt_{N+1} \cdots dt_1$$

with $t_0 = x$ and with

$$R_0(x,k) = -\int_x^\infty \frac{1 - e^{2ik(t_1 - x)}}{2ik} V(t_1)\chi(t_1,k)dt_1.$$

Using (2.34) we get $|\chi(x,k)| \leq e^a e^{(|\operatorname{Im} k| - \operatorname{Im} k)x_I}$. From the definition of the Faddeev solution it follows that $\chi(0,k) = f(0,k)$. The second iterate $\chi^{(1)}(0,k)$ of (2.32) can be written as

$$\chi^{(1)}(0,k) = -\frac{1}{2ik} \int_0^{x_I} V(t)dt + \frac{1}{2ik} \int_0^{x_I} e^{2ikt} V(t)dt = -\frac{\hat{V}(0) - \hat{V}(k)}{2ik}.$$

Then

$$f(0,k) - 1 + \frac{\hat{V}(0) - \hat{V}(k)}{2ik} = \sum_{l=2}^{\infty} \chi^{(l)}(x,k),$$

so

$$\left| f(0,k) - 1 + \frac{\hat{V}(0) - \hat{V}(k)}{2ik} \right| \leq \sum_{l=2}^{\infty} \left| \chi^{(l)}(x,k) \right| \leq e^{(|\operatorname{Im} k| - \operatorname{Im} k)(x_I - x)} \left(e^{a(x)} - 1 \right).$$

From (2.33) we can get an estimate (Gronwall inequality) of the Jost solution. The estimates of the Jost solution after respectively zero and one iteration of the Neumann series are

$$|f(0,k)-1| = |R_0(0,k)| \le e^{(|\operatorname{Im} k|-\operatorname{Im} k)x_I} a e^a \tag{2.53}$$

and

$$\begin{aligned}\left| f(0,k) - 1 + \frac{\hat{V}(0)-\hat{V}(k)}{2ik} \right| &= |R_1(0,k)| \le e^{(|\operatorname{Im} k|-\operatorname{Im} k)x_I} \left(e^a - 1 - a\right) \\ &\le \frac{a^2}{2} e^{(|\operatorname{Im} k|-\operatorname{Im} k)x_I} e^a. \end{aligned} \tag{2.54}$$

- *Step 2.* For $f'(x,k)$ we have

$$f'(x,k) = ike^{ikx} - \int_x^\infty \cos\left[k\left(x-t_1\right)\right] V(t_1) f(t_1,k) dt_1$$

which can be written as

$$\begin{aligned} & f'(x,k)e^{-ikx} \\ & = ik - \int_x^\infty \frac{1+e^{-2ik(x-t_1)}}{2} V(t_1)\chi(t_1,k)dt_1 \\ & = ik - \int_x^\infty \frac{1+e^{-2ik(x-t_1)}}{2} V(t_1)dt_1 \\ & \quad + \int_x^\infty \int_{t_1}^\infty \frac{1+e^{-2ik(x-t_1)}}{2} \frac{1-e^{2ik(t_2-t_1)}}{2ik} V(t_1)V(t_2)\chi(t_2,k)dt_1dt_2 \end{aligned}$$

Evaluating it at $x=0$ and resolving the first integral, we get

$$\begin{aligned} f'(0,k) &= ik - \frac{\hat{V}(0)+\hat{V}(k)}{2} + \\ &+ \int_0^\infty \int_{t_1}^\infty \frac{1+e^{2ikt_1}}{2} \frac{1-e^{2ik(t_2-t_1)}}{2ik} V(t_1)V(t_2)\chi(t_2,k)dt_1dt_2 \end{aligned}$$

from which we derive the estimate

$$\left| f'(0,k) - ik + \frac{\hat{V}(0)+\hat{V}(k)}{2} \right| \le \max(1,|k|) \frac{a^2}{2} e^{(|\operatorname{Im} k|-\operatorname{Im} k)x_I} e^a. \tag{2.55}$$

- *Step 3.* The estimate of the Jost function at the first order in k can be obtained only considering (2.55) truncated at the zeroth order and it becomes

$$|f_h(k) - ik| \leq \|V\| \, e^{(|\operatorname{Im} k| - \operatorname{Im} k)x_I} e^a.$$

Instead, using (2.53) truncated at the zeroth order multiplied by h and using (2.55) we obtain

$$\left| f'(0,k) + hf(0,k) - ik + \frac{\hat{V}(0) + \hat{V}(k)}{2} - h \right| \leq \left[|h| + \frac{\|V\|}{2} \right] a e^{(|\operatorname{Im} k| - \operatorname{Im} k)x_I} e^a. \qquad \square$$

As a corollary of Lemma 2.5.30, we show the form of the previous estimates on the physical sheet.

Corollary 2.5.31. *Let $V \in \mathbb{V}_{x_I}$, then the Jost function f_h is of exponential type and satisfies the following asymptotic expansion in the physical sheet* ($\operatorname{Im} k > 0$)

$$f_h(k) = ik + O(1) \qquad \textit{for } |k| > 1. \tag{2.56}$$

Proof. In the physical sheet $\operatorname{Im} k > 0$, $|\operatorname{Im} k| - \operatorname{Im} k = 0$. Then (2.51) becomes

$$|f_h(k) - ik| \leq \|V\| \, e^a$$

which for $k > 1$ implies

$$|f_h(k) - ik| \leq ||V|| e^{\frac{||V||}{|k|}}.$$

This yields (2.56). $\square$

We shall use a version of Lemma 2.1 from Korotyaev [30] adapted to our setting.

Lemma 2.5.32. *If $V \in \mathbb{V}_{x_I}$ and if $k_1, \dots, k_N \in i\mathbb{R}_+$ are the zeros of the Jost function $f_h(k)$ such that $|k_1| > \dots > |k_N| > 0$, then the normalizing constants m_j, defined as*

$$m_j = \int_0^\infty f^2(x, k_j) dx \tag{2.57}$$

satisfy

$$m_j = -i \left[\frac{\dot{f}_h(k)}{f_h(-k)} \right]_{k=k_j} > 0, \qquad \textit{for } j = 1, \dots, N, \tag{2.58}$$

and the following inequalities hold

$$i(-1)^j \dot{f}_h(k_j) > 0, \quad \textit{and} \quad (-1)^j f_h(-k_j) < 0, \qquad \textit{for } j = 1, \dots, N, \tag{2.59}$$

where the dot denotes the derivative with respect to k.

Proof. Since k_j is an eigenvalue, then $f_h(k_j) = 0$, which means

$$f'(0, k_j) = -f(0, k_j)h. \tag{2.60}$$

Plugging this formula into the Wronskian (2.18) between $f(x, k)$ and $f(x, -k)$ we get

$$-2ik_j = f(0, k_j)\left[f'(0, -k_j) + hf(0, -k_j)\right]$$

and together with formula (2.50) and (2.60) we get

$$\begin{aligned}\int_0^\infty f^2(x, k_j)dx &= \frac{f(0, k_j)\left[-h\dot{f}(0, k_j) - \dot{f}'(0, k_j)\right]}{-2k_j} \\ = \frac{f(0, k_j)\left[-h\dot{f}(0, k_j) - \dot{f}'(0, k_j)\right]}{-if(0, k_j)\left[hf(0, -k_j) + f'(0, -k_j)\right]} &= -i\left[\frac{\frac{d}{dk}f_h(k)}{f_h(-k)}\right]_{k=k_j}.\end{aligned}$$

We can see that for $k \in i\mathbb{R}_+$, as $|k| \to \infty$, the Jost function $f_h(k)$ tends to $-\infty$. So the first zero of the Jost function, k_1, has negative derivative $i\dot{f}_h(k_1) < 0$, consequently the next zero has positive derivative $i\dot{f}_h(k_2) > 0$ and so on. This implies the second inequality in (2.59), since the ratio must be positive. □

Remark 2.5.33. *From proposition 2.5.19 we can see that $\dot{f}_h(k_j)$ is different from zero at the $k_j \in i\mathbb{R}_+$, zero of the Jost function $f_h(k)$. Then* (2.58) *makes sense.*

The following lemma makes a connection between the Jost function and the potential (compare with Lemma 2.2. in Korotyaev [30]), which makes use of the Paley-Wiener theorem for functions in the Cartwright class.

Lemma 2.5.34.

(i) Let f be entire, of exponential type, and let $\rho_+(f) = 0$ and $\rho_-(f) \leq 2x_I$. If the following asymptotic holds

$$f(k) = ik\left[1 - \frac{1}{2ik}\left(C - \hat{g}(k) + O(k^{-1})\right)\right]; \qquad k \to \pm\infty,$$

for some $g \in L^1(0, x_I)$ and some constant C, then there exists $F \in L^1(0, x_I)$ such that

$$f(k) = ik\left[1 - \frac{1}{2ik}(\hat{F}(0) - \hat{F}(k))\right], \qquad k \in \mathbb{C}, \tag{2.61}$$

where $\rho_-(f) = 2\sup\,[\mathrm{supp}\, F]$.

(ii) For each $V \in \mathbb{V}_{x_I}$ there exists $p \in L^1(0, x_I)$ such that

$$f_h(k) = ik(1 + \hat{\zeta}(k)) = ik\left[1 - \frac{1}{2ik}(\hat{p}(0) - \hat{p}(k))\right], \quad \zeta(t) := \int_t^{x_I} p(x)dx. \tag{2.62}$$

Proof. (i) is proved in the same way as [30, Lemma 2.2] with slight modifications.

(ii) From Lemma 2.5.30 we can write

$$f_h(k) = ik\left[1 - \frac{1}{2ik}\left(\hat{V}(0) - 2h + \hat{V}(k) + O(k^{-1})\right)\right].$$

Then the Jost function can be written as

$$f_h(k) = ik\left[1 - \frac{1}{2ik}(C - \hat{g}(k) + u(k))\right]$$

where $C = \hat{V}(0) - 2h$, $\hat{g}(k) = \hat{V}(k)$ and $u(k) = O(k^{-1})$, when $k \to \pm\infty$.
Using the Paley-Wiener theorem, since the function $u(k)$ is entire, of exponential type and square integrable over horizontal lines, there exist a $v \in L^2_c(0, x_I)$ which is the Fourier transform of this function, so $u(k) = \hat{v}(k)$, where $v \in L^2_c(0, x_I) \subset L^1_c(0, x_I)$. Using (i) with $p = g + v$ and $\hat{p}(0) = \hat{g}(0) + \hat{v}(0) = \hat{V}(0) - 2h$ we get (2.62) and integrating by parts we obtain $f_h(k) = ik(1 + \hat{\zeta}(k))$ (see [30]). □

Lemma 2.5.34 tells us that if we have a function in the Cartwright class with $\rho_- = 2x_I$, then by Definition 2.5.16 of the class W_{x_I} property I is satisfied, but this is not enough to have a bijection (we already proved that if f is in W_{x_I} then is in the Cartwright class) because we also need property II to be satisfied.

2.5.2 Direct results

In this subsection, we state the direct resonance results for the Love problem in terms of the parameter k and ξ. We use the property of the Jost function being in the Cartwright class in order to use the Levinson Theorem 2.4.7. In the following lemma, from estimates of the Jost function obtained in the previous subsection we recover estimates on the resonances, which tell us where they are located in the complex plane.

Lemma 2.5.35 (Resonance-free regions). *For any zero k_n, $n \geq 1$, of $f_h(k)$, $V \in \mathbb{V}_{x_I}$, the following estimates are fulfilled:*

$$|k_n| \leq C_0 e^{2|\operatorname{Im} k_n| x_I}, \qquad C_0 = \|V\| \, e^{\|V\|}. \tag{2.63}$$

Additionally, if $V' \in L^1$, then

$$|k_n|^2 \leq C_1 e^{2|\operatorname{Im} k_n| x_I} \qquad C_1 = \left[\|V\|^2 + 2|h| \, \|V\| + \frac{1}{4}\left(|V(0)| + \|V'\|\right)\right] e^{\|V\|}.$$

Proof. Here we adapt the proof of Corollary 2.3 of [30] to our case. Estimate (2.51) evaluated at a zero k_n of $f_h(\cdot, V)$, with $|k_n|$ large, implies

$$|f_h(k_n) - ik_n| \leq \|V\| \, e^{2|\operatorname{Im} k_n| x_I} e^{\|V\|}$$

and hence

$$|k_n| \leq \|V\| \, e^{\|V\|} e^{2|\operatorname{Im} k_n| x_I},$$

which gives (2.63). If moreover $V' \in L^1(0, \infty)$, then (2.52) evaluated at k_n, with $|k_n| > 1$, implies

$$|f_h(k_n) - ik_n| \leq \left| h + \frac{\hat{V}(0) - \hat{V}(k_n)}{2} \right| + \left(\frac{\|V\|^2}{2|k_n|} + \frac{|h| \, \|V\|}{|k_n|} \right) e^{\|V\|} e^{2|\operatorname{Im} k_n| x_I}$$

and hence

$$\left|-ik_n^2\right| \leq \left| k_n \left(-h + \frac{\|V\|}{2} \right) \right| + \left| \frac{\hat{V}(k_n) k_n}{2} \right| + \left(\frac{\|V\|^2}{2} + |h| \, \|V\| \right) e^{\|V\|} e^{2|\operatorname{Im} k_n| x_I}. \tag{2.64}$$

Integrating by parts yields

$$\begin{aligned} \hat{V}(k) &= \int_0^{x_I} e^{2ikx} V(x) dx = \left[\frac{1}{2ik} e^{2ikx} V(x) \right]_0^{x_I} - \int_0^{x_I} \frac{1}{2ik} e^{2ikx} V'(x) dx \\ &= -\frac{1}{2ik} \left(V(0) + \int_0^{x_I} V'(x) e^{2ikx} dx \right), \end{aligned}$$

thus

$$|\hat{V}(k_n)| \leq \frac{1}{2|k_n|} \left[|V(0)| + e^{2|\operatorname{Im} k_n| x_I} \left\| V' \right\| \right].$$

Therefore, (2.64) implies

$$\begin{aligned} |k_n|^2 \leq & \left(\frac{\|V\|^2}{2} + |h| \, \|V\| + \frac{1}{4} \left(|V(0)| + \left\| V' \right\| \right) \right) e^{2|\operatorname{Im} k_n| x_I + \|V\|} \\ & + |k_n| \left(|h| + \frac{\|V\|}{2} \right) \leq C_1 e^{2|\operatorname{Im} k_n| x_I + \|V\|} \end{aligned}$$

where we have used (2.63) in the last passage. □

As a corollary of Lemma 2.5.35, we infer the resonance free-regions (forbidden domain) in terms of the wave number ξ, with $k^2 = \frac{\omega^2}{\hat{\mu}} - \xi^2$.

Corollary 2.5.36. *For any* $\xi_n = \sqrt{\frac{\omega^2}{\hat{\mu}} - k_n^2}$, $n \geq 1$, *where* k_n *are the zeros of* $f_h(k, V)$ *with* $V \in \mathbb{V}_{x_I}$ *the following estimates are fulfilled:*

$$|\xi_n| \leq C_0 e^{2|\operatorname{Re}\xi_n|x_I}, \qquad C_0 = \|V\|\, e^{\|V\|}. \tag{2.65}$$

Additionally, if $V' \in L^1(0, \infty)$, *then*

$$|\xi_n|^2 \leq C_1 e^{2|\operatorname{Re}\xi_n|x_I}, \qquad C_1 = \frac{3}{2}\|V\|^2 + 2|h|\,\|V\| + \frac{1}{4}\left(|V(0)| + \|V'\|\right) e^{\|V\|}.$$

Proof. The proof follows from Lemma2.5.35 after substituting the resonances satisfying $k_n = -i\xi_n + O(1)$ for large ξ_n. □

In the following corollary to Theorem 2.4.7, we assume that $\rho_-(f_h) = 2x_I$, which we will prove later in Lemma 2.5.41.

Corollary 2.5.37 (Number of resonances)**.** *Let* $V \in \mathbb{V}_{x_I}$. *Then*

$$\mathcal{N}(r, f_h) = \frac{2x_I r}{\pi}(1 + o(1)) \qquad \text{for } r \to \infty$$

For each $\delta > 0$ *the number of zeros of the Jost function with real part with modulus* $\leq r$ *lying outside both of the two sectors* $|\arg\xi - \frac{\pi}{2}| < \delta$, $|\arg\xi - \frac{3\pi}{2}| < \delta$ *is* $o(r)$ *for large* r.

Proof. The result follows from Theorem 2.4.7 and the fact that $\rho_-(f_h) = 2x_I$. □

2.5.3 The inverse problem

The goal of the inverse problem is to reconstruct the potential from given data. In the inverse scattering problem, these data can be, for example, the scattering function in addition to eigenvalues and normalizing constant. In the inverse spectral problem, the data are the spectral data, which could be the Weyl function. In our inverse problem, we want to reconstruct the potential starting from eigenvalues and resonances.

In this subsection we present the first inverse resonance result, where from eigenvalues and resonances we can retrieve V after proving a bijection between the class W_{x_I} and $\mathbb{V}_{x_I}$ (see Theorem 2.5.44), following the result of Korotyaev (see [30]). The characterization is made by adapting the Marchenko theorem (see [38, Chapter 3]) to our case with Robin boundary condition, which we state below.

Definition 2.5.38. *For* $N \in \mathbb{N}$, *we define* $\mathcal{S}_N$ *to be the set of functions* $S(k)$ *such that*

1. $S(k)$ *is continuous and satisfies the identities* $S(k) = \overline{S(-k)} = S(-k)^{-1}$ *for each* $k \in \mathbb{R}$.

2. *$S(k)-1=o(1)$, for $|k|\to\infty$, and the function $G(x)=1/2\pi\int_{\mathbb{R}}(S(k)-1)e^{ikx}dk$ satisfies*

$$G'\in L^1(\mathbb{R}_+,(x+1)dx), \qquad G=G_1+G_2,$$
$$G_1\in L^1(\mathbb{R}_+), \qquad G_2\in L^2(\mathbb{R}_+)\cap L^\infty(\mathbb{R}_+).$$

3. *The increment of $S(k)$ and N are related by the following formula*

$$N+\frac{S(0)+1}{4}=\frac{1}{2\pi i}\left[\log\left(-S(+0)\right)-\log\left(-S(+\infty)\right)\right].$$

For $N\in\mathbb{N}$, let

$$\Gamma_N:=\left\{(k_1,\dots,k_N)\in i\mathbb{R}_+^N \quad :|k_1|>\dots>|k_N|>0\right\}.$$

The following theorem is the Marchenko Theorem in [38], which is stated for Dirichlet boundary condition, adapted to our setting with Robin boundary condition.

Theorem 2.5.39 (Marchenko theorem). *Consider the mapping*

$$\Sigma: L_{1,1}(\mathbb{R}_+)\to\bigcup_N S_N\times\mathbb{R}_+^N\times\Gamma_N$$

defined by $\Sigma(V):=\Big(S(k),(m_n)_{1,\dots,N},(k_n)_{1,\dots,N}\Big)$ *where*

i) *$S(k)$ denotes the scattering function defined in* (2.28),

ii) *k_n denote the zeros of the Jost function defined in* (2.22),

iii) *$f(x,k_n)\in L^2(\mathbb{R}_+)$ are the eigenfunctions of $-\frac{d^2}{dx^2}+V$, with $f(x,k)$ being the solution of* (2.2) *with condition* (2.11),

iv)
$$m_n=\int_0^\infty|f(x,k_n)|^2dx, \qquad k_n\in\mathbb{C}_+.$$

Then the mapping Σ is one-to-one and onto. Moreover, for[6] *(see [51, 38])*

$$G(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(S(k)-1)e^{ikx}dk, \qquad G_0(x)=G(x)+\sum_{k_n\in\mathbb{C}_+}m_n^{-1}e^{-x|k_n|}, \tag{2.66}$$

[6]In the case with Dirichlet boundary condition $G(x)$ is defined with a minus sign (see [38, 30]).

we can define the inverse mapping from $\Big(S(k), (m_n)_{1,\dots,N}, (k_n)_{1,\dots,N}\Big)$ *to* $V \in L_{1,1}$ *through*

$$V(x) = -2\frac{d}{dx}A(x,x), \tag{2.67}$$

where $A(x,t)$ *is the unique solution (see [38]) for each* $x > 0$ *of the Marchenko equation*

$$A(x,t) = -G_0(x+t) - \int_x^\infty G_0(t+s)A(x,s)ds, \qquad t \geq x. \tag{2.68}$$

Remark 2.5.40. *Observe that the pair* $(S(k), m_n)$ *does not depend on* $f_h(0)$ *since* $S(k) = -\frac{f_h(-k)}{f_h(k)}$ *and* $m_n = -i\frac{\dot{f}_h(k_n)}{f_h(-k_n)}$, *and in the ratios the constant* $f_h(0)$ *gets cancelled.*

In the next lemma we show that the Jost function of our self-adjoint Schrödinger problem with Robin boundary conditions and $V \in \mathbb{V}_{x_I}$ is of Cartwright class with $\rho_+(f_h) = 0$ and $\rho_-(f_h) = 2x_I$.

Lemma 2.5.41. *If* $V \in \mathbb{V}_{x_I}$ *then the Jost function* (2.22) *is entire and of exponential type. In particular,* $f_h(k)$ *satisfies the following conditions:*

$$\int_{\mathbb{R}} \frac{\log^+|f_h(k)|dk}{1+k^2} < \infty, \quad \rho_+(f_h) = 0,\ \rho_-(f_h) = 2x_I.$$

In other words, f_h *is of Cartwright class (see Definition 2.4.6).*

Proof. In Theorem 2.5.17 we proved that $f_h(k)$ is entire. Let $x := \operatorname{Re} k$ and $y := \operatorname{Im} k$. Then by Corollary 2.5.31, we can write the integral condition as

$$\int_{\mathbb{R}} \frac{\log^+|f_h(x)|dx}{1+x^2} = \int_{\mathbb{R}} \frac{\log|ix + O(1)|}{1+x^2}dx < \infty.$$

Using Corollary 2.5.31 we have

$$\rho_+(f_h) = \limsup_{y\to\infty} \frac{\log|-y+O(1)|}{y} = 0,$$

while using Lemma 2.5.30 we get

$$\rho_-(f_h) \leq \limsup_{y\to-\infty} \frac{2x_I y}{y} = 2x_I. \tag{2.69}$$

In order to prove the equality in (2.69), we recall the definitions (2.66), where if $G_0(x) = 0$ for $x > 2dx_I$ for some $d \geq 1$, then $V = 0$ for $x > x_I$. If $\rho_-(f_h(k)) = 2dx_I < 2x_I$, then, thanks to the Jordan lemma, (2.28) and the residue theorem, taking a contour on the upper half plane, we obtain

$$
\begin{aligned}
G(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}(S(k)-1)e^{ikx}dk = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{f_h(k)+f_h(-k)}{f_h(k)}\right)e^{ikx}dk \\
&= -i\sum_{j=1}^{N}\lim_{k\to k_j}\left((k-k_j)\frac{f_h(k)+f_h(-k)}{f_h(k)-f_h(k_j)}\right)e^{ikx} = -i\sum_{j=1}^{N}\frac{f_h(-k_j)}{\dot{f}_h(k_j)}e^{ik_jx} \\
&= -\sum_{j=1}^{N}\frac{1}{m_j}e^{ik_jx}, \qquad \text{if } x > 2dx_I. \qquad (2.70)
\end{aligned}
$$

Hence, $G_0(x) = G(x) + \sum_{j=1}^{N}\frac{1}{m_j}e^{ik_jx}$ is zero for $x > 2dx_I$, so $V(x) = 0$ for $x > x_I$, thus, $\rho_-(f_h(k)) = 2x_I$ (see also [30]). □

Remark 2.5.42. *The condition $\rho_-(f_h) = 2x_I$ in the definition of the Cartwright class tells us that the resonances distribute in the unphysical sheet following a logarithmic curve. This is important because it implies that $\sum_n \frac{1}{k_n}$ is convergent and that, by Theorem 2.4.3 and Theorem 2.4.4, $\prod_n\left(1-\frac{k}{k_n}\right)$ converges to an entire function of exponential type.*

Lemma 2.5.30 allows us to use the Hadamard factorization (Theorem 2.4.2), where $m = 0$ by Theorem 2.5.27, $E_P(z) = (1-z)e^z$ since the function is of exponential order one, $g(z) = az+b$ with $e^b = f_h(0)$ and $e^{(a+\sum_n \frac{1}{k_n})z} = e^{iz}$ by Corollary 2.5.31. Hence, in our case, formula (2.7) becomes

$$
f_h(k) = f_h(0)e^{ik}\lim_{R\to\infty}\prod_{|k_n|\le R}\left(1-\frac{k}{k_n}\right), \qquad (2.71)
$$

where k_n are the zeros of $f_h(k)$ counted with multiplicity.

Remark 2.5.43. *In the formula (2.71) the constant $f_h(0)$ is uniquely determined by the resonances. It is possible to obtain $f_h(0)$ from the asymptotics obtained in Lemma 2.5.30, because the Jost function must satisfy $f_h(k) = ik + O(1)$ for large k and changing $f_h(0)$ will change the asymptotics. Also Korotyaev in the Dirichlet case (see [30, page 224 at the end of the proof of Theorem 1.1]) claims that the Jost function can be uniquely determined from the resonances.*

$f_h(k)$ is therefore determined uniquely by the resonances k_n as explained in Remark 2.5.43. Since $\sum_{n=1}^{\infty}\frac{1}{k_n}$ is absolutely convergent by Theorem 2.4.3 and since k_j is a resonance if and only if $-\overline{k}_j$ is a resonance, then also $\sum_{n=1}^{\infty}\frac{1}{\overline{k}_n}$ is absolutely convergent.

Furthermore, we have

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{i}{k_n}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{-i}{\overline{k_n}}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{i(\overline{k}_n-k_n)}{|k_n|^2}=\sum_{n=1}^{\infty}\frac{\operatorname{Im}k_n}{|k_n|^2}$$

so the *Blaschke condition* is fulfilled, that is, $\sum_{n=1}^{\infty}\frac{\operatorname{Im}k_n}{|k_n|^2}$ is absolutely convergent. From (2.71), differentiating $f_h(k)$ with respect to k, we obtain

$$\frac{d}{dk}(\log(f_h(k)))=\frac{\dot{f}_h(k)}{f_h(k)}=i+\lim_{R\to\infty}\sum_{|k_n|\le R}\frac{1}{k-k_n}. \tag{2.72}$$

uniformly on compact subsets of $\mathbb{C}\setminus(\{0\}\cup\bigcup\{k_n\})$.

Below we state the main theorem of this section, giving a complete characterization of the class of potentials $\mathbb{V}_{x_I}$.

Theorem 2.5.44 (Characterization). *The map $J_h:\mathbb{V}_{x_I}\to W_{x_I}$ defined by $J_h(V):=f_h$ is well-defined and bijective.*

Proof. We adapt the proof in [30, Theorem 1.1] to our case. First, for fixed $h\in\mathbb{R}$ we prove that the map J_h is well-defined.

Let $V\in\mathbb{V}_{x_I}$, then we need to prove that $f_h(k)\in W_{x_I}$. Using Lemma 2.5.30, 2.5.32 and 2.5.34 we can see that $f_h(k)$ is real on $i\mathbb{R}$ and satisfies (2.62). From Theorem 2.5.17 and Lemma 2.5.41 we know that f_h is in the Cartwright class with $\rho_+=0$ and $\rho_-=2x_I$. Then we can use Lemma 2.5.34 and by (2.61), we get the form of (2.29) which satisfies Condition I of W_{x_I}. Condition II of W_{x_I} is fulfilled by Lemma 2.5.32, hence $f_h(k)\in W_{x_I}$ and J_h is well-defined.

Consequently, $V\in\mathbb{V}_{x_I}$ uniquely determines $f_h(k)\in W_{x_I}$, which uniquely determines $\left(S(k),(m_j,k_j)_{j=1,\dots,N}\right)$, which in turn uniquely determines $V\in L_{1,1}$ through the map Σ of Marchenko theorem. Suppose now f_h is the Jost function of $V_1\in\mathbb{V}_{x_I}$ and $V_2\in\mathbb{V}_{x_I}$, then $\left(S_1(k),(m_j,k_j)_{j=1,\dots,N_1}\right)=\left(S_2(k),(m_j,k_j)_{j=1,\dots,N_2}\right)$, and then using the map Σ we deduce $V_1=V_2$ from Theorem 2.5.39. Hence the map J is injective.

We are left to prove that J is surjective. Fix the scattering data $f_h(k)\in W_{x_I}$. We want to construct $V\in\mathbb{V}_{x_I}$ such that $J_h(V)=f_h$. We show this by proving that from $f_h(k)$ we can construct the scattering data $\left(S(k),(m_j,k_j)_{j=1,\dots,N}\right)$ from (2.58) and (2.28) and they satisfy the conditions of the Marchenko theorem. We show that the scattering function satisfies the conditions (1), (2) and (3) of the Definition 2.5.38.

(1) The scattering matrix $S(k)=-f_h(-k)/f_h(k)$ is continuous for $k\in\mathbb{R}$ and it is analytic everywhere excepts at the k_n, the zeros of $f_h(k)$. From the properties of the

scattering phase, we can easily check that it holds that $S(k) = \overline{S(-k)} = S^{-1}(-k)$ and, if N is even, $S(0) = -(-1)^{\mathcal{N}_0(f_h)}$, where $\mathcal{N}_0(f_h)$ is the multiplicity of 0 as a zero of f_h.

(2) From (2.51) we have that

$$\begin{aligned} &|f_h(k) - ik| \le C, && k \in \overline{\mathbb{C}}_+, \\ &|f_h(-k) + ik| \le Ce^{2x_I|\operatorname{Im} k|}, && k \in \overline{\mathbb{C}}_+, \end{aligned}$$

which imply

$$\begin{aligned} |S(k) - 1| &= \left| \frac{f_h(k) + f_h(-k)}{f_h(k)} \right| \le \frac{C}{|k|} \left(|f_h(k) - ik| + |f_h(-k) + ik| \right) \\ &\le \frac{C_1 e^{2x_I \operatorname{Im} k} + C_2}{|k|} \le \frac{Ce^{2x_I \operatorname{Im} k}}{|k|}, \qquad k \in \overline{\mathbb{C}}_+. \end{aligned} \tag{2.73}$$

Here and throughout C denotes a positive constant that can change from line to line. Using the Jordan lemma as in (2.70), we get that $G_0(x) = G(x) + \sum_{j=1}^{N} \frac{1}{m_j} e^{ik_j x}$ is zero for $x > 2x_I$. From (I) we have

$$\begin{aligned} &-\frac{f_h(-k)}{f_h(k)} - 1 = -1 - \frac{-ik\left[1 + \frac{1}{2ik}\left(\hat{F}(0) - \hat{F}(-k)\right)\right]}{ik\left[1 - \frac{1}{2ik}\left(\hat{F}(0) - \hat{F}(k)\right)\right]} \\ &= -\frac{-\frac{1}{2ik}\left(2\hat{F}(0) - \hat{F}(k) - \hat{F}(-k)\right)}{1 - \frac{1}{2ik}\left(\hat{F}(0) - \hat{F}(k)\right)} = \frac{1}{2ik}\left(2\hat{F}(0) - \hat{F}(k) - \hat{F}(-k)\right) + O(k^{-2}) \end{aligned}$$

where $\hat{F}$ is continuous and bounded as $F \in \mathbb{V}_{x_I}$. We define the following functions

$$g_1 + g_2 = S(k) - 1, \quad g_1 = \frac{2\hat{F}(0) - \hat{F}(k) - \hat{F}(-k)}{2ik}, \quad G_p := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixk} g_p(k)dk$$

for $p = 1, 2$. Then $G_2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ since $g_2(k) = O(k^{-2})$. The function G_1 is odd in $x \in \mathbb{R}$ since g_1 is odd in k:

$$\begin{aligned} G_1(-x) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixk}\frac{\hat{F}(k) + \hat{F}(-k) - 2\hat{F}(0)}{2ik}dk \\ &= \frac{1}{2\pi}\int_{+\infty}^{-\infty} e^{-ixk}\frac{\hat{F}(-k) + \hat{F}(k) - 2\hat{F}(0)}{-2ik}dk = -G_1(x) \end{aligned}$$

and $G_1 \in L^2(\mathbb{R})$ because it is the Fourier transform of g_1 which is in $L^2(\mathbb{R})$. Since $|S - 1| = |g_1 + g_2|$ satisfies (2.73) and g_2 is bounded,

$$|g_1| \le C \le \frac{Ce^{2x_I \operatorname{Im} k}}{|k|}, \qquad k \in \overline{\mathbb{C}}_+,\ |k| \gg 1.$$

Using the Jordan lemma, $G_1(x) = 0$ for $x > 2x_I$, so since G_1 is odd, $G_1(x) = 0$ for $|x| < 2x_I$. By G_1 being in L^2 and with compact support we get $G_1 \in L^1(\mathbb{R})$. For $G'(x)$ we get

$$\begin{aligned} G'(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixk}(ik)\,(S(k)-1)\,dk \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixk}\left(\frac{\hat{F}(k)+\hat{F}(-k)-2\hat{F}(0)}{2} + O(k^{-1})\right)dk \end{aligned} \tag{2.74}$$

and using

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixk}\hat{F}(k)dk &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixk}\left(\int_0^{x_I} e^{2iky}F(y)dy\right)dk \\ &= \frac{1}{2\pi}\int_0^{x_I}\int_{-\infty}^{\infty} e^{2ik(y+\frac{x}{2})}F(y)dkdy = \frac{F\left(-\frac{x}{2}\right)}{2}, \end{aligned}$$

by the Fourier inversion formula, (2.74) becomes

$$G'(x) = -\frac{F\left(\frac{x}{2}\right)+F\left(-\frac{x}{2}\right)-2F(0)}{4} + r(x) = -\frac{F\left(\frac{x}{2}\right)+F\left(-\frac{x}{2}\right)}{4} + r(x)$$

where $r(x) \in L^2(\mathbb{R}_+)$ is obtained through the Paley-Wiener theorem. Then $G'(x) \in L^1(\mathbb{R}_+, (1+x)dx)$, because $\operatorname{supp} G_0 \subset [-2x_I, 2x_I]$.

(3) Since $f_h(k)$ is entire in the upper half plane,

$$\begin{aligned} \int_\gamma \frac{f_h'(z)}{f_h(z)}dz &= \int_\gamma d(\log(f_h(z))) = \lim_{R\to\infty}\left(\int_{\gamma_R} d(\log(f_h(z)))\right. \\ &+ \left.\lim_{r\to\infty}\int_{\gamma_r} d(\log(f_h(z))) + \lim_{\epsilon\to 0}\int_{-R}^{-\epsilon} d(\log(f_h(z))) + \lim_{\epsilon\to 0}\int_{\epsilon}^{R} d(\log(f_h(z)))\right) \end{aligned} \tag{2.75}$$

where γ is a closed curve in the upper half plane made by a part that goes from $-R$ to R passing around $z = 0$ through a semi-circle γ_r and an arc γ_R in the upper-half plane. The integral over the big arc goes to zero because of the Jordan lemma, while the integral over the little arc gives a term $-\pi i \mathcal{N}_0(f_h)$. Hence we have

$$\lim_{\epsilon\to 0}\int_{-\infty}^{-\epsilon} d(\log(f_h(z))) + \lim_{\epsilon\to 0}\int_{\epsilon}^{\infty} d(\log(f_h(z))) = 2\pi i\left(\mathcal{N}(f_h) + \mathcal{N}_0(f_h)/2\right). \tag{2.76}$$

We know that $S(0) = -(-1)^{\mathcal{N}_0(f_h)}$ and we have seen in Theorem 2.5.27 that zero is not an eigenvalue, so $\mathcal{N}_0(f_h) = 0$. So, we can write

$$\frac{S(0)+1}{4} = \mathcal{N}_0(f_h)/2 = 0. \tag{2.77}$$

Computing the first two integrals of (2.76) we have

$$\lim_{\epsilon\to 0, R\to\infty}\int_{-R}^{-\epsilon} d(\log(f_h(z))) + \lim_{\epsilon\to 0}\int_{\epsilon}^{R} d(\log(f_h(z))) = \lim_{\epsilon\to 0, R\to\infty}(\log(f_h(+R)) - \log(f_h(+\epsilon)) - \log(f_h(-R)) + \log(f_h(-\epsilon))) \tag{2.78}$$

and since

$$-\log(-S(z)) = \log(f_h(z)) - \log(f_h(-z)) = \arg f_h(z) - \arg f_h(-z),$$

we finally obtain

$$\begin{aligned}&\lim_{\epsilon\to 0, R\to\infty}\int_{-R}^{-\epsilon} d(\log(f_h(z))) + \lim_{\epsilon\to 0}\int_{\epsilon}^{R} d(\log(f_h(z)))\\ &= \log(-S(+0)) - \log(-S(+\infty)).\end{aligned} \tag{2.79}$$

Inserting (2.79) and (2.77) into (2.76), we obtain

$$\frac{1}{2\pi i}(\log(-S(+0)) - \log(-S(+\infty))) = N + \frac{S(0)+1}{4}.$$

It follows that, all of the conditions of Definition 2.5.38 are satisfied. The other conditions on $\{m_j, k_j\}_{j=1,\ldots,N}$ are implied by Condition II of the class W_{x_I}, hence the Marchenko theorem holds and there exists a unique $V \in L_{1,1}$ corresponding to the Jost function. We proved in (2) that $\operatorname{supp} G_0 \subset [-2x_I, 2x_I]$, which implies $V = 0$ for $x > x_I$.

If $\rho_-(f_h) = 2t$, where $t := x_I - \epsilon_0$ with

$$\epsilon_0 := \inf\{\epsilon > 0 : |(x_I - \epsilon, x_I) \cap \operatorname{supp} V| > 0\},$$

then $V \in \mathbb{V}_t$, but since $\rho_-(f_h) = 2x_I$ as explained in the proof of Lemma 2.5.34, then $V \in \mathbb{V}_{x_I}$ is the unique potential corresponding to the Jost function $f_h \in W_{x_I}$. □

Theorem 2.5.44 suggests an algorithm that enables us to reconstruct the unique potential from a set of resonances.

Algorithm 2.5.45. *Starting from a set of eigenvalues and resonances* $\{k_j\}_1^\infty$ *we can retrieve the potential* $V_\omega(x)$ *using the following algorithm:*

- *Construct the Jost function from* (2.71) *as*

$$f_h(k) = f_h(0)e^{ik}\lim_{R\to\infty}\prod_{|k_n|\le R}\left(1 - \frac{k}{k_n}\right), \tag{2.80}$$

  where $f_h(0)$ *is determined so that* $f_h(k) = ik + O(1)$ *as* $k \to \infty$.

- *Use* $\{k_j\}_1^\infty$ *and* $f_h(k)$ *to construct the scattering data* $\Big(S(k), \{m_j, k_j\}_{j=1,\dots,N}\Big)$ *from equations* (2.28) *and* (2.58) *as*

$$S(k) = -e^{-2ik} \prod_{n\geq 1}^{\infty} \left(\frac{k_n + k}{k_n - k}\right),$$
$$m_j = \frac{e^{-2|k_j|}}{2|k_j|} \prod_{n\geq 1, n\neq j} \left(\frac{k_n - k_j}{k_n + k_j}\right), \qquad j = 1, \dots, N.$$

- *Use the scattering data* $\Big(S(k), \{m_j, k_j\}_{j=1,\dots,N}\Big)$ *to construct* $G_0(x)$ *in* (2.66).
- *Solve* (2.68) *for* $A(x,t)$.
- *Obtain the potential from* (2.67).

After the recovery of the potential $V_\omega(x)$, where we added the subscript ω because it is found for every fixed value of ω, we need to recover the Lamé coefficient $\mu(x)$, which, physically, is more interesting. This can be done from the knowledge of the potential at two different values ω_1 and ω_2, with $\omega_1 \neq \omega_2$, as we present in the following theorem.

Theorem 2.5.46. *Let* $V_{\omega_1}(x)$ *and* $V_{\omega_2}(x)$ *be the potential at the frequencies* ω_1 *and* ω_2, *with* $\omega_1 \neq \omega_2$, *then the Lamé parameter can be retrieved by the following formula*

$$\hat{\mu}(x) = \frac{\hat{\mu}_I \left(\omega_1^2 - \omega_2^2\right)}{\omega_1^2 - \omega_2^2 - \hat{\mu}_I \left(V_{\omega_1}(x) - V_{\omega_2}(x)\right)}. \tag{2.81}$$

Proof. In (2.1) we defined the potential as

$$V_\omega = \frac{(\sqrt{\hat{\mu}})''}{\sqrt{\mu}} - \frac{1}{\hat{\mu}}\omega^2 + \frac{1}{\hat{\mu}_I}\omega^2.$$

Then the potential difference at two different frequencies, respectively ω_1 and ω_2 is

$$V_{\omega_1}(x) - V_{\omega_2}(x) = \left(\frac{1}{\hat{\mu}_I} - \frac{1}{\hat{\mu}(x)}\right)(\omega_1^2 - \omega_2^2)$$

which leads to (2.81). □

Remark 2.5.47. *In the thesis, we do not treat the stability of the resonances. For more details, we refer the reader to [31, 42] who treat the Dirichlet case.*

2.6 The spectral problem

In this section, we introduce the Weyl function formalism and we recover a Gelfand–Levitan type equation (see Proposition 2.6.22) following a similar procedure as in [43, Chapter 1, Section 1] and [3] adapted to our Love scalar boundary value problem. Then, we establish a bijection (see Theorem 2.6.28) between a class $\mathbb{M}_{x_I}$ of Weyl function (see Definition 2.6.27) and the class $\mathbb{V}^1_{x_I}$ (see Definition 2.6.1). We do not follow the usual approach in which the Weyl–Titchmarsh function is defined to be a Herglotz–Nevanlinna function and, from which by using its integral representation and the Stieltjes inversion formula, one can obtain the spectral measure (see [50, Theorem 9.17]). Instead, we follow the approach of [22, Chapter 2] and define the Weyl function in a different way (see Definition 2.6.3), which is more suitable for non self-adjoint problems. Our goal is to use this approach for the inverse Rayleigh problem, where the operator is not self-adjoint and the spectral measure cannot be recovered in the usual way as mentioned above. However, this approach is not extended to the Rayleigh case in this thesis.

2.6.1 Estimates of the regular solution

We want to obtain an estimate for the regular solution φ in the limit $k \to \infty$. We start from the Volterra-type expression for the regular solution φ

$$\varphi(x,k) = \cos kx - h\frac{\sin kx}{k} + \int_0^x \frac{\sin\left[k(x-t)\right]}{k} V(t)\varphi(t,k)dt. \tag{2.82}$$

We can easily see that this function satisfies the differential equation and the boundary condition. Indeed

$$\varphi'(x,k) = -k\sin kx - h\cos kx + \int_0^x \cos\left[k(x-t)\right] V(t)\varphi(t,k)dt \tag{2.83}$$

and

$$\begin{aligned}\varphi''(x,k) &= -k^2\cos kx + hk\sin kx + V(x)\varphi(x,k)\\ &\quad - \int_0^x k\sin\left[k(x-t)\right] V(t)\varphi(t,k)dt;\end{aligned}$$

thus $-\varphi''(x,k) + V(x)\varphi(x,k) = k^2\varphi(x,k)$. Moreover,

$$\varphi'(0,k) = -h, \qquad \varphi(0,k) = 1,$$

so also the boundary condition $\varphi'(0,k)+h\varphi(0,k) = 0$ is satisfied. Taking the absolute value of (2.82) and since $|\sin kx| \le \exp(|\eta|x)$ and $|\cos kx| \le \exp(|\eta|x)$, where $\eta = \operatorname{Im} k$, we get

$$|\varphi(x,k)| \le \exp(|\eta|x) + \frac{\exp(|\eta|x)}{|k|} + \int_0^x \frac{\exp(|\eta|(x-t))}{|k|}|V(t)||f(t,k)|dt.$$

We define $\beta_T(k) = \max_{0\le x\le T}(|\varphi(x,k)|)\exp(-|\eta|x)$ and we have then for $|k| > 1$

$$\beta_T(k) \le C_1 + \frac{1}{|k|}\beta_T(k)\int_0^T |V(t)|dt \le C_1 + \frac{1}{|k|}\beta_T(k)\int_0^\infty |V(t)|dt$$

which for $|k| \to \infty$ implies $\beta_T(k) = O(1)$, hence $\varphi(x,k) = O(\exp(|\eta|x))$. Substituting this estimate on (2.82) we get $|\varphi(x,k)| \le C\exp(|\eta|x)$. Doing the same for the derivative of $\varphi(x,k)$, as in (2.83), we get

$$|\varphi^{(\nu)}(x,k)| \le C|k|^\nu \exp(|\eta|x), \qquad \nu = 0,1, \qquad |k| \gg 1 \tag{2.84}$$

uniformly in x.

2.6.2 Properties of Weyl function

In this section, we will define the Weyl solution and the Weyl function and present their properties. These quantities enable another approach to solve the inverse problem (see [22, Section 2.2]) and will also enable us to recover the Gelfand–Levitan equation in an alternative way (see Subsection 2.6.3), as the Gelfand–Levitan equation is usually recovered from the spectral measure.

First, we define the class $\mathbb{V}^1_{x_I}$, to which the potential belongs throughout the whole section.

Definition 2.6.1. *We denote by* $\mathbb{V}^1_{x_I}$ *the class of real potentials* V *such that* $V, V' \in L^1(\mathbb{R}_+)$, $\operatorname{supp} V \subset [0, x_I]$ *for some* $x_I > 0$ *and for each* $\epsilon > 0$ *the set* $(x_I - \epsilon, x_I) \cap \operatorname{supp} V$ *has positive Lebesgue measure.*

In this section λ and k are always related via $\lambda = k^2$ defined initially for $\operatorname{Im} k > 0$. Below we give a definition of Weyl solution that uses those of the Jost solution (Definition 2.5.8) and the Jost function (Definition 2.5.11) given in the previous section.

Definition 2.6.2 (Weyl solution). *We define the Weyl solution* $\phi(x,\lambda)$ *as the function*

$$\phi(x,\lambda) = \frac{f(x,k)}{f_h(k)}, \qquad \operatorname{Im} k > 0. \tag{2.85}$$

This function satisfies the differential equation $-\phi'' + V\phi = \lambda\phi$ because the Jost solution does, but does not satisfy the Robin Boundary condition. In particular:

$$\phi'(0,\lambda) + h\phi(0,\lambda) = 1 \tag{2.86}$$

$$\phi(x,\lambda) = O(e^{ikx}) \qquad x \to \infty, \quad k \in \Sigma, \tag{2.87}$$

where we define the set $\Sigma := \{k \in \mathbb{C} : \operatorname{Im} k \geq 0, k \neq 0\}$. From (2.53) and (2.52) in Lemma 2.5.30 we get the asymptotics on the Weyl solution for large k

$$\phi^{(\nu)}(x, \lambda) = (ik)^{\nu-1} \exp(ikx) \left(1 + o\left(\frac{1}{k}\right)\right), \qquad \nu = 0, 1, \quad |k| \to \infty. \tag{2.88}$$

The Weyl solution is uniquely determined, modulo a multiplicative constant, by the differential equation $-\phi'' + V\phi = \lambda\phi$ (see (2.2)) and the boundary condition (2.86).

Definition 2.6.3 (Weyl function). *We define the Weyl function $M(\lambda)$ (or Weyl–Titchmarsh function) as the function*

$$M(\lambda) := \phi(0, \lambda) = \frac{f(0,k)}{f_h(k)}, \qquad \lambda = k^2, \operatorname{Im} k > 0.$$

Remark 2.6.4. *In some other textbooks, the Weyl–Titchmarsh function for Robin boundary condition is defined as $M(\lambda) = \frac{hf'(0,k)-f(0,k)}{f_h(k)}$ (see [50, formula 9.52, Chapter 9]). This choice entails that $M(\lambda)$ is a Herglotz–Nevanlinna function and using its properties, it is possible to obtain the spectral measure (see [50, Theorem 9.17]). In our treatment, we follow the approach and the definition of [22, Definition 2.1.69].*

Remark 2.6.5. *The zeros of the Jost function (Definition 2.5.11) correspond to the poles of the Weyl function (see Theorem 2.6.16 below). Indeed, at zeros $k = k_j$ of the Jost function, $f(0, k_j) = -\frac{1}{h} f'(0, k_j) \neq 0$.*

Remark 2.6.6. *The Weyl function M maps $f_h(k)$ to $f(0, k)$, so M is the Robin-to-Dirichlet map, since the Jost function in the Dirichlet boundary value problem ($h = \infty$) is precisely $f(0, k)$ (see [30]). In the case of Dirichlet boundary condition, the Weyl function is usually defined as $\frac{f'(0,k)}{f(0,k)}$ (see [50, formula 9.52, Chapter 9]) which is Herglotz–Nevanlinna and can be reconstructed by the Dirichlet and Neumann eigenvalues and resonances.*

From the asymptotics of the Jost solution and Jost function we obtain the asymptotics of the Weyl function, as described in the following lemma.

Lemma 2.6.7. *Let $V \in \mathbb{V}^1_{x_I}$, then the Weyl function (see Definition 2.6.3) has the asymptotic expansion*

$$M(\lambda) = \frac{1}{ik}\left[1 - \frac{h}{ik} + \frac{\hat{V}(k)}{ik} + o(k^{-1})\right], \qquad |k| \to +\infty. \tag{2.89}$$

Proof. From (2.52) and (2.53) in Lemma 2.5.30, we can get the asymptotics of the Weyl function

$$M(\lambda) = \frac{1}{ik}\left(1 + O\left(\frac{1}{k}\right)\right), \qquad |k| \to +\infty.$$

Using (2.52) and (2.54) we can get higher order terms of the expansion of the Weyl function in terms of k

$$
\begin{aligned}
M(\lambda) &= \left(1 - \frac{\hat{V}(0) - \hat{V}(k)}{2ik} + o(k^{-1})\right)\left(\frac{1}{ik + h - \frac{\hat{V}(0)+\hat{V}(k)}{2} + o(1)}\right) \\
&= \frac{1}{ik}\left(1 - \frac{\hat{V}(0) - \hat{V}(k)}{2ik} + o(k^{-1})\right)\left(\frac{1}{1 + \frac{h}{ik} - \frac{\hat{V}(0)+\hat{V}(k)}{2ik} + o(k^{-1})}\right) \\
&= \frac{1}{ik}\left(1 - \frac{\hat{V}(0) - \hat{V}(k)}{2ik} + o(k^{-1})\right)\left(1 - \frac{h}{ik} + \frac{\hat{V}(0) + \hat{V}(k)}{2ik} + o(k^{-1})\right) \\
&= \frac{1}{ik}\left[1 - \frac{h}{ik} - \frac{\hat{V}(0) - \hat{V}(k)}{2ik} + \frac{\hat{V}(0) + \hat{V}(k)}{2ik} + o(k^{-1})\right] \\
&= \frac{1}{ik}\left[1 - \frac{h}{ik} + \frac{\hat{V}(k)}{ik} + o(k^{-1})\right]. \qquad \square
\end{aligned}
$$

Note that we can write the Weyl solution as

$$\phi(x,\lambda) = \theta(x,k) + M(\lambda)\varphi(x,k) \tag{2.90}$$

where $\varphi(x,k)$ and $\theta(x,k)$ are solution of (2.2) satisfying

$$
\begin{aligned}
\theta(0,k) &= 0 \qquad & \theta'(0,k) &= 1 \\
\varphi(0,k) &= 1 \qquad & \varphi'(0,k) &= -h
\end{aligned}
$$

and $\varphi(x,k)$ is the regular solution as in Definition 2.5.13. We can see that

$$W(\varphi(x,k),\phi(x,\lambda)) = W(\varphi(x,k),\theta(x,k)) = 1. \tag{2.91}$$

We denote

$$\Lambda = \left\{\lambda = k^2 : k \in \Sigma,\ f_h(k) = 0\right\}$$

and

$$\Lambda' = \left\{\lambda = k^2 : \operatorname{Im} k > 0,\ f_h(k) = 0\right\}.$$

The set Λ' consists of all the eigenvalues of the differential equation $-f'' + Vf = \lambda f$ (see (2.2)). By Lemma 2.6.7, the Weyl function at the second order can be also written as

$$M(\lambda) = \frac{1}{ik}\left(1 - \frac{h}{ik} + \frac{1}{ik}\int_0^\infty V(t)e^{2ikt}dt + o\left(\frac{1}{k}\right)\right), \quad |k| \to +\infty, k \in \Sigma. \tag{2.92}$$

The following definition of the domain of λ comes from [22, Chapter 2].

Definition 2.6.8. *We define* Π *as the* λ*-plane with the cut* $\lambda \geq 0$*, and* $\Pi_1 = \overline{\Pi} \setminus \{0\}$. Π *and* Π_1 *must be considered as a subset of the Riemann surface of the square-root function.*

In Definition 2.6.8 we stated that Π and Π_1 must be considered as a subset of the Riemann surface of the square root because λ as a square of k ($k = \sqrt{\lambda}$) lays in two copies of the complex plane with cuts on the positive real axis and glued together. Hence, Π and Π_1 live in the first (physical sheet) of these two sheets. Since the cut is placed in the real positive axis of the Π λ-plane, the Weyl function has a jump between above and below the cut. This motivates the following definition.

Definition 2.6.9 (Jumps of Weyl function). *We define*

$$T(\lambda) = \frac{1}{2\pi i}\left(M^{-}(\lambda) - M^{+}(\lambda)\right), \qquad \lambda > 0, \tag{2.93}$$

to be the jumps of the Weyl function $M(\lambda)$ *(see Definition 2.6.3), where*

$$M^{\pm}(\lambda) = \lim_{z \to 0, \operatorname{Re} z > 0} M(\lambda \pm iz).$$

From Definition 2.93, we can see that $T(\lambda)$ represents the jumps (discontinuity points of the first kind) of the Weyl function. Thanks to (2.92) and (2.93) we get the following expansion for $T(\lambda)$:

$$\begin{aligned} T(\lambda) &= \frac{1}{2i\pi k}\left[-\frac{1}{ik}\left(2 + \frac{1}{ik}\int_0^\infty V(t)\left(e^{2ikt} - e^{-2ikt}\right)dt + o\left(\frac{1}{k}\right)\right)\right] \\ &= \frac{1}{\pi k}\left(1 + \frac{1}{k}\int_0^\infty V(t)\sin 2kt\,dt + o\left(\frac{1}{k}\right)\right), \qquad k \to +\infty. \end{aligned}$$

For $a > 0$, we consider the points $\lambda = a \pm i0$ in Π_1. For $\lambda = k^2$, the point $\lambda = a + i0 \in \Pi_1$ corresponds to $k = \sqrt{a + i0} > 0$ in the positive real axis of the k complex plane, while $\lambda = a - i0 \in \Pi_1$ corresponds to the point $k = \sqrt{a - i0} < 0$ situated on the negative real axis for k.

Definition 2.6.10 (Spectral normalizing constant). *We define the spectral normalizing constant* α_j *to be the complex numbers*

$$\alpha_j := \operatorname{Res}_{\lambda = \lambda_j} M(\lambda), \qquad j = 1, \dots, N$$

where $\{\lambda_j\}_{j=1}^N = \Lambda'$.

In the following proposition we connect the jump $T(\lambda)$ of the Weyl function to the Jost function.

Proposition 2.6.11. *Let $T(\lambda)$ be the jumps of the Weyl function as in Definition 2.6.9, then*

$$T(\lambda) = \frac{k}{\pi |f_h(k)|^2}, \qquad k > 0. \tag{2.94}$$

Proof. We follow the argument in [22, page 134]). Identity (2.94) holds if the following identities are true

$$W(f(x,k), \overline{f(x,k)}) = -2ik \tag{2.95}$$

$$\overline{f(x,k)} = f(x,-k), \qquad \overline{f_h(k)} = f_h(-k). \tag{2.96}$$

Those identities hold as the problem (2.2)–(2.3) with domain (2.4) is self-adjoint. Indeed, $k^2 - iz$ is a complex number with real part $\mathrm{Re}(k^2) + \mathrm{Im}\, z$ and imaginary part equal to $\mathrm{Im}(k^2) - \mathrm{Re}\, z$. This complex number in the k complex plane corresponds to the roots $|k_z|e^{i\theta_z}$ and $|k_z|e^{i(\theta_z+\pi)}$, where

$$\begin{aligned} |k_z| &= \left((\mathrm{Re}(k^2) + \mathrm{Im}\, z)^2 + (\mathrm{Im}(k^2) - \mathrm{Re}\, z)^2\right)^{1/4} \\ \theta_z &= \arctan\left(\frac{\mathrm{Im}(k^2) - \mathrm{Re}\, z}{2(\mathrm{Re}(k^2) + \mathrm{Im}\, z)}\right). \end{aligned}$$

In the limit $z \to 0$ along $z > 0$, these two solutions converge to k and $-k$ respectively. Hence, we have $M^-(\lambda) = \frac{f(0,-k)}{f_h(-k)}$ and

$$\begin{aligned} T(\lambda) &= \frac{1}{2\pi i}\left(\frac{f(0,-k)}{f_h(-k)} - \frac{f(0,k)}{f_h(k)}\right) = \frac{1}{2\pi i}\left(\frac{\overline{f(0,k)}}{\overline{f_h(k)}} - \frac{f(0,k)}{f_h(k)}\right) \\ &= \frac{1}{2\pi i}\left(\frac{\overline{f(0,k)}(f'(0,k) + hf(0,k)) - f(0,k)(\overline{f'(0,k)} + h\overline{f(0,k)})}{|f_h(k)|^2}\right) \\ &= \frac{1}{2\pi i}\left(\frac{W(\overline{f}, f)}{|f_h(k)|^2}\right) = \frac{k}{\pi |f_h(k)|^2}, \end{aligned}$$

where in the second step we used (2.96) and in the last we used (2.95). □

From the previous proposition, we can see that we can recover the jump function from the Jost function, but not the converse. The eigenvalues $\{k_n\}_{n=1}^N$, the spectral norming constants $\{\alpha_n\}_{n=1}^N$ and the jump function $T(\lambda)$ are usually considered in the literature as the data for the inverse spectral problem (see [22, Definition 2.3.1]).

The following results are useful for the inverse result at the end of this section.

Lemma 2.6.12. *The following holds*

$$\frac{k}{f_h(k)} = O(1), \qquad k \to 0, \ \mathrm{Im}\, k \geq 0. \tag{2.97}$$

Proof. We follow the proof of [22, Theorem 2.3.5]. Since $W(f(x,k), f(x,-k)) = -2ik$ and $f_h(k) = f'(x,k) + hf(x,k)$, we have that

$$\begin{aligned}
&-2ik = f(0,k)f'(0,-k) - f'(0,k)f(0,-k) = f(0,k)(f_h(-k) - hf(0,k)) \\
&-(f_h(k) + hf(0,k)f(0,-k) = f(0,k)f_h(-k) - f_h(k)f(0,-k).
\end{aligned}$$

We set

$$g(k) = \frac{2ik}{f_h(k)}$$

so, for real $k \neq 0$, we have

$$g(k) = f(0,-k) + S(k)f(0,k)$$

where $S(k) = -f_h(-k)/f_h(k)$ is the scattering function. Because of the property $\overline{f_h(k)} = f_h(-k)$, we know that $f_h(k)$ and $f_h(-k)$ have the same modulus, so $|S(k)| = 1$ for real $k \neq 0$. Let $\lambda_j = k_j^2$, $k_j = i\tau_j$, $0 < \tau_1 < ... < \tau_m$ and denote Σ_{τ^*} as

$$\Sigma_{\tau^*} = \{k : \operatorname{Im} k > 0, |k| < \tau^*\}$$

where $\tau^* = \tau_1/2$, considering the values k_j corresponding to the eigenvalues λ_j ordered from the smallest to the largest. The function $g(k)$ is analytic in Σ_{τ^*} and continuous in $\bar{\Sigma}_{\tau^*} \setminus \{0\}$ and from the estimates on the Jost solution we can say that

$$|g(k)| \leq C \qquad \text{for real} \quad k \neq 0.$$

With this last estimate, we see that $g(k)$ has a removable singularity in the origin, and consequently $g(k)$ is continuous in Σ_{τ^*} and (2.97) is satisfied. □

Proposition 2.6.13. *The spectral normalizing constants α_j from Definition 2.6.10 are strictly positive and are given by*

$$\alpha_j = 4k_j^2 \left[\frac{-i}{f_h(-k_j)\dot{f}_h(k_j)} \right] > 0. \tag{2.98}$$

Proof. We recall the regular solution

$$\varphi(x,k) = -\frac{1}{2ik}\left[f_h(-k)f(x,k) - f_h(k)f(x,-k)\right]$$

that, when k_j is a zero of the Jost function, becomes the eigenfunction

$$\varphi(x,k_j) = -\frac{1}{2ik_j}\left[f_h(-k_j)f(x,k_j)\right].$$

We know that $\varphi(x,k)$ satisfies $\varphi(0,k)=1$ (see (2.21)), hence

$$-2ik_j = f_h(-k_j)f(0,k_j). \tag{2.99}$$

From the definition of α_j we can write

$$\alpha_j = \mathrm{Res}_{\lambda=\lambda_j} M(\lambda) = \lim_{\lambda\to\lambda_j} \frac{(\lambda-\lambda_j)f(0,k)}{(k-k_j)\frac{d}{dk}f_h(k)} = \frac{2k_j f(0,k_j)}{\frac{d}{dk}f_h(k)|_{k=k_j}}. \tag{2.100}$$

Plugging in (2.99), we get

$$\alpha_j = \frac{2k_j(-2ik_j)}{f_h(-k_j)\frac{d}{dk}f_h(k)|_{k=k_j}} = -\frac{4ik_j^2}{f_h(-k_j)\dot{f}_h(k_j)} = 4k_j^2\left[\frac{-i}{f_h(-k_j)\dot{f}_h(k_j)}\right] > 0$$

where the last inequality follows from (2.59) in Lemma 2.5.32 and k_j^2 being negative (Theorem 2.5.20). □

The following theorem shows a representation formula for the Weyl function $M(\lambda)$, which can be reconstructed from the jumps $T(\lambda)$, the spectral normalizing constants α_j and the eigenvalues λ_j, as in [22, Lemma 2.3.1].

Theorem 2.6.14. *The Weyl function is uniquely determined by the specification of the spectral data* $(T(\lambda), \{\lambda_k,\alpha_k\}_{k=1}^N)$ *via the formula*

$$M(\lambda) = \int_0^\infty \frac{T(\mu)}{\lambda-\mu}d\mu + \sum_{k=1}^N \frac{\alpha_k}{\lambda-\lambda_k}, \qquad \lambda\in\Pi\backslash\Lambda'. \tag{2.101}$$

Proof. We follow the proof in [22, Lemma 2.3.1]. We consider the function

$$I_R(\lambda) := \frac{1}{2\pi i}\int_{|\mu|=R} \frac{M(\mu)}{\lambda-\mu}d\mu.$$

Since $M(\lambda)=O(k^{-1})$ for $k\to\infty$, then $\lim_{R\to\infty} I_R(\lambda)=0$. Now, we deform the contour to avoid the singularity at $\mu=\lambda$ with the little circle $\gamma_r(\lambda)$ and to avoid the cut $]0,+\infty]$. Hence,

$$\lim_{R\to 0} I_R(\lambda) = \lim_{r\to 0}\frac{1}{2\pi i}\int_{\gamma_r(\lambda)}\frac{M(\mu)}{\lambda-\mu}d\mu + \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{+\infty-i\epsilon}^{0-i\epsilon}\frac{M(\mu)}{\lambda-\mu}d\mu$$
$$+\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{0+i\epsilon}^{+\infty+i\epsilon}\frac{M(\mu)}{\lambda-\mu}d\mu - \frac{1}{2\pi i}(2\pi i)\sum_{k=1}^m \mathrm{Res}\left(\frac{M(\mu)}{\lambda-\mu}\right),$$

where the last term is the sum of the residues of $\frac{M(\mu)}{\lambda-\mu}$ viewed as a function of μ. In the first term we apply the residue theorem noticing that the little circle is run through in anti-clockwise direction; in the second term we make the substitution $\eta = \mu + i\epsilon$; in the third term we make the substitution $\eta = \mu - i\epsilon$, and in the last term we replace the residue of the Weyl function with α_k (see Definition 2.6.10):

$$0 = \frac{1}{2\pi i}(-2\pi i) \lim_{\mu\to\lambda} (\mu - \lambda)\frac{M(\mu)}{\lambda - \mu} + \lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{+\infty}^{0} \frac{M(\eta - i\epsilon)}{\lambda - \eta + i\epsilon} d\eta$$
$$+ \lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{0}^{+\infty} \frac{M(\eta + i\epsilon)}{\lambda - \eta - i\epsilon} d\eta - \sum_{k=1}^{m} \frac{\alpha_k}{\lambda - \lambda_k}.$$

Since $T(\eta) = \lim_{z\to 0, \operatorname{Re} z>0} \frac{1}{2\pi i} (M(\eta - iz) - M(\eta + iz))$, we can write:

$$0 = M(\lambda) + \int_{+\infty}^{0} \frac{T(\eta)}{\lambda - \eta} d\eta - \sum_{k=1}^{m} \frac{\alpha_k}{\lambda - \lambda_k},$$

which is (2.101). □

We can write the Weyl function $M(\lambda)$ in terms of the jump function $T(\lambda)$ and the normalizing constants α_k through the formula

$$M(\lambda) = \int_{0}^{\infty} \frac{T(\mu)}{\lambda - \mu} d\mu + \sum_{k=1}^{N} \frac{\alpha_k}{\lambda - \lambda_k}, \qquad \lambda \in \Pi\backslash\Lambda'$$

as we can see in Theorem 2.6.14. In order to reconstruct the Weyl function, we need to know the jump function, the eigenvalues and the normalizing constants.

Remark 2.6.15. *As we can see from Proposition 2.6.11 and Proposition 2.6.13, we can retrieve uniquely the Weyl function from the Jost function* f_h *and the eigenvalues* $\{k_j\}_{1,\dots,N}$.

We state below a theorem from [22, Theorem 2.1.5], which proves the analyticity of $M(\lambda)$ in a certain region of the complex plane, which is related through Definition 2.6.3 to the analyticity of the Jost function and the Jost solution (see Theorem 2.5.17)

Theorem 2.6.16. *The Weyl function* $M(\lambda)$ *is analytic in* $\Pi\backslash\Lambda'$ *and continuous in* $\Pi_1\backslash\Lambda$. *The set of singularities of* $M(\lambda)$ *(as an analytic function) coincides with the set* $\Lambda_0 = \{\lambda : \lambda \geq 0\} \cup \Lambda$.

Next, we prove a uniqueness result for the Weyl function (see [22, Theorem 2.2.1]).

Theorem 2.6.17 (Uniqueness). *Let* V *and* $\tilde{V}$ *be in* $\mathbb{V}^1_{x_I}$ *with Weyl functions* M *and* $\tilde{M}$ *respectively. If* $M = \tilde{M}$, *then* $V = \tilde{V}$.

Proof. We very closely follow the proof given in [22]. We define the matrix $P(x,\lambda) = [P_{j,k=1,2}]$ by the formula

$$P(x,\lambda)\begin{bmatrix}\tilde{\varphi}(x,\lambda) & \tilde{\phi}(x,\lambda)\\ \tilde{\varphi}'(x,\lambda) & \tilde{\phi}'(x,\lambda)\end{bmatrix} = \begin{bmatrix}\varphi(x,\lambda) & \phi(x,\lambda)\\ \varphi'(x,\lambda) & \phi'(x,\lambda)\end{bmatrix}. \tag{2.102}$$

Multiplying both sides of the equation by the inverse of the matrix of the left-hand side we get

$$P(x,\lambda) = \begin{bmatrix}\varphi(x,\lambda) & \phi(x,\lambda)\\ \varphi'(x,\lambda) & \phi'(x,\lambda)\end{bmatrix}\frac{1}{W(\tilde{\varphi},\tilde{\phi})}\begin{bmatrix}\tilde{\phi}'(x,\lambda) & -\tilde{\phi}(x,\lambda)\\ -\tilde{\varphi}'(x,\lambda) & \tilde{\varphi}(x,\lambda)\end{bmatrix}.$$

Since the Wronskian $W(\tilde{\varphi},\tilde{\phi}) = 1$ because of (2.91), we can multiply the two matrices and recover the components of the matrix $P(x,\lambda)$

$$\begin{aligned}P_{j1}(x,\lambda) &= \varphi^{(j-1)}(x,\lambda)\tilde{\phi}'(x,\lambda) - \phi^{(j-1)}(x,\lambda)\tilde{\varphi}'(x,\lambda)\\ P_{j2}(x,\lambda) &= \phi^{(j-1)}(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi^{(j-1)}(x,\lambda)\tilde{\phi}(x,\lambda).\end{aligned} \tag{2.103}$$

Solving (2.102) with respect to ϕ and φ we get

$$\begin{aligned}\varphi(x,\lambda) &= P_{11}(x,\lambda)\tilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda)\\ \phi(x,\lambda) &= P_{11}(x,\lambda)\tilde{\phi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\phi}'(x,\lambda).\end{aligned} \tag{2.104}$$

From (2.88) and (2.84), for $|\lambda| \to \infty$ we get

$$|P_{11}(x,\lambda) - 1| \le \frac{C}{|k|}, \qquad |P_{12}(x,\lambda)| \le \frac{C}{|k|}, \quad |k| \to \infty. \tag{2.105}$$

From (2.103), plugging the definition of the Weyl solution $\phi(x,\lambda)$ and $\tilde{\phi}(x,\lambda)$, as in (2.90), we get

$$\begin{aligned}P_{11} &= \varphi(x,\lambda)\tilde{\theta}'(x,\lambda) - \theta(x,\lambda)\tilde{\varphi}'(x,\lambda) + (\tilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\tilde{\varphi}'(x,\lambda)\\ P_{12} &= \theta(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\tilde{\theta}(x,\lambda) + (\tilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\tilde{\varphi}(x,\lambda).\end{aligned}$$

So, if $M(\lambda) \equiv \tilde{M}(\lambda)$, then for each fixed x, the functions $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire in λ. The estimates (2.105) yield $P_{11}(x,\lambda) \equiv 1$ and $P_{12}(x,\lambda) \equiv 0$. Substituting this into (2.104) we get $\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda)$ and $\phi(x,\lambda) \equiv \tilde{\phi}(x,\lambda)$ for all x and λ, then $V = \tilde{V}$. □

Remark 2.6.18. *Theorem 2.6.17 can be found in the literature in the case of Dirichlet boundary condition under the name of Borg-Marchenko uniqueness theorem (see [5]). The converse of it was proved in a local version by Barry Simon (see [48] but also [2, 23, 4]) employing the Phragmen-Lindelöf theorem and Liouville theorem, under the assumption that if two Weyl functions asymptotically agree modulo an exponentially small function, then the two potential agree in a certain interval.*

2.6.3 The main equation of the inverse spectral problem

In this section we show an alternative way (similar to [3] for the Rayleigh case) to recover the Gelfand–Levitan equation, that is an integral equation from which we can reconstruct the potential V and the boundary coefficient h of a Schrödinger boundary value problem. The ordinary way to obtain the Gelfand–Levitan equation is from the spectral measure. Here we obtain it through a function ψ, that depends on the Weyl function and is discontinuous on the real line with jumps proportional to the jumps of the Weyl function. We are motivated by the fact that in the Rayleigh problem we are not able to recover the Gelfand–Levitan equation through the spectral measure as the operator is not self-adjoint, even though we will not extend the following to the Rayleigh problem.

We recall Definition 2.6.2 and Definition 2.6.3. We define $\phi_{\pm}(x,\lambda)$ as

$$\phi_{\pm}(x,\lambda) = \frac{f(x,\pm k)}{f_h(\pm k)}, \qquad \operatorname{Im} k > 0.$$

Note that for $\lambda > 0$ we have (see also Definition 2.93)

$$M^{\pm}(\lambda) = \lim_{z\to 0, \operatorname{Re} z>0} \frac{f(0,\sqrt{\lambda \pm iz})}{f_h(\sqrt{\lambda \pm iz})} = \frac{f(0,\pm k)}{f_h(\pm k)}, \qquad \lambda = k^2, \quad k > 0.$$

We extend the definition of $M^{\pm}(\lambda)$ to $\lambda \in \mathbb{C}$ and note that $M^{\pm}(\lambda) = M(\lambda)$ for $\lambda \notin [0,\infty)$. In particular, it is easy to check that

$$M^{\pm}(\lambda) = \frac{f(0,\pm k)}{f_h(\pm k)}, \qquad \operatorname{Im} k > 0.$$

From (2.82) we can find the asymptotics, as $|k| \to \infty$, in the upper half plane $\operatorname{Im} k > 0$ for the regular solution $\varphi(x,k)$

$$\begin{aligned}
\varphi(x,k) &= \cos kx - h\frac{\sin kx}{k} + \int_0^x \sin\left[k(x-t)\right] V(t)\varphi(t,k)dt \\
&= \left(\frac{e^{ikx}+e^{-ikx}}{2}\right) - h\left(\frac{e^{ikx}-e^{-ikx}}{2ik}\right) + \int_0^x \sin\left(k(x-t)\right)\cos kt V(t)dt \\
&- h\int_0^x \frac{\sin\left(k(x-t)\right)}{k} V(t)\sin kt\, dt \\
&= e^{-ikx}\left(\frac{1}{2} + \int_0^x \frac{e^{2ik(x-t)}-1}{4i} V(t)dt + O\left(\frac{1}{ik}\right)\right).
\end{aligned}$$

We can do the same for the Jost solution $f(x,k)$ and find its asymptotics as $|k| \to \infty$ in the physical sheet, starting from (2.17)

$$f(x,k) = e^{ikx} - \int_x^\infty \frac{e^{ik(x-t)}-e^{-ik(x-t)}}{2ik} V(t)e^{ikt}dt + o\left(\frac{1}{k}\right)$$

$$= e^{ikx}\left(1 - \int_x^\infty \frac{V(t)}{2ik}dt + o\left(\frac{1}{k}\right)\right)$$

and hence

$$f(x,-k) = e^{-ikx}\left(1 + \int_x^\infty \frac{V(t)}{2ik}dt + o\left(\frac{1}{k}\right)\right).$$

From Lemma 2.6.7, we have the following asymptotic expansion of $M(\lambda)$

$$M(\lambda) = \frac{1}{ik} + \frac{1}{k^2}\left[h - \hat{V}(k)\right] + o(k^{-2}), \qquad |k| \to +\infty. \tag{2.106}$$

If $V' \in L^1(0,\infty)$, then we can integrate by parts the Fourier transform of V and get

$$\hat{V}(k) = -\frac{V(0)}{2ik} - \int_0^{x_I} \frac{V'(t)}{2ik}e^{2ikt}dt,$$

and (2.106) becomes

$$M(\lambda) = \frac{1}{ik} + \frac{1}{k^2}\left[h - V(0)\right] + o(k^{-2}), \qquad |k| \to +\infty.$$

The difference between $M+(\lambda)$ and $M^-(\lambda)$ is

$$\frac{2}{ik} = M^+(\lambda) - M^-(\lambda) + o(k^{-2}). \tag{2.107}$$

Definition 2.6.19. *We define the function $\psi(x,k)$ discontinuous in the real line as*

$$\psi(x,k) = \begin{cases} -ike^{ikx}\left(\phi_+(x,\lambda) + \frac{2i}{k}\varphi(x,k)\right) & \operatorname{Im} k > 0 \\ -ike^{ikx}\phi_-(x,\lambda) & \operatorname{Im} k < 0 \end{cases}$$

and let $\psi_+(x,k)$ denote the restriction of $\psi(x,k)$ to the upper-half plane, and $\psi_-(x,k)$ the restriction of $\psi(x,k)$ to the lower-half plane.

One can compare Definition 2.6.19 with [3, Formula 3.8] for the Rayleigh case. We can see that the function ψ is bounded on $\mathbb{C}$. We can also write the general solution $\varphi(x,k)$ in terms of ψ_+ and ψ_- as

$$2\varphi(x,k) = e^{-ikx}\psi_+(x,k) + e^{ikx}\psi_-(x,-k).$$

Since φ is an even function of k, we also have

$$2\varphi(x,k) = e^{ikx}\psi_+(x,-k) + e^{-ikx}\psi_-(x,k) \tag{2.108}$$

and adding these last two we get

$$4\varphi(x,k) = e^{ikx}\left(\psi_+(x,-k) - \psi_-(x,-k)\right) + e^{-ikx}\left(\psi_+(x,k) - \psi_-(x,k)\right).$$

From (2.90), we see that

$$\begin{aligned}
&\phi_+(x,\lambda) - \phi_-(x,\lambda) = \theta(x,k) + M^+(\lambda)\varphi(x,k) - \theta(x,k) - M^-(\lambda)\varphi(x,k) \\
&= \varphi(x,k)\left(M^+(\lambda) - M^-(\lambda)\right). \qquad (2.109)
\end{aligned}$$

Using (2.109) and (2.107) we can calculate

$$\begin{aligned}
\psi_+(x,k) - \psi_-(x,k) &= e^{ikx}\left(\phi_+(x,\lambda) - \phi_-(x,\lambda) - \frac{2}{ik}\varphi(x,k)\right)(-ik) \\
&= e^{ikx}\varphi(x,k)\left(M^+(\lambda) - M^-(\lambda) - \frac{2}{ik}\right)(-ik) \\
&= -ike^{ikx}\varphi(x,k)\left(j(k) - j(-k)\right), \qquad (2.110)
\end{aligned}$$

where $j(k)$ is of order $O(k^{-2})$, as is the coefficient of the second leading order of $M(\lambda)$, and it is defined as

$$j(\pm k) := M^{\pm}(\lambda) \mp \frac{1}{ik}, \qquad \lambda = k^2, \quad \operatorname{Im} k > 0,$$

and

$$j(\pm k) := M^{\pm}(\lambda) \mp \frac{1}{ik}, \qquad \lambda = k^2, \quad k > 0.$$

The difference $j(k) - j(-k)$ is

$$j(k) - j(-k) = \begin{cases} -\frac{2}{ik} & \operatorname{Im} k > 0, \\ M^+(\lambda) - M^-(\lambda) - \frac{2}{ik} & k > 0, \end{cases}$$

and it is of order $o(k^{-2})$. Now, we want to calculate the asymptotics for ψ_-:

$$\begin{aligned}
&\psi_-(x,k) = -ike^{ikx}\phi_-(x,\lambda) = (-ik)e^{ikx}\frac{f(x,-k)}{f(0,-k)}M^-(\lambda) \\
&= (-ik)e^{ikx}\,\frac{e^{-ikx}\left(1 + \int_x^{x_I}\frac{V(t)}{2ik}dt + o(k^{-1})\right)}{1 + \int_0^{x_I}\frac{V(t)}{2ik}dt + o(k^{-1})}\frac{1}{-ik}\left(1 + \frac{h}{ik} - \frac{\hat{V}(-k)}{ik}\right. \\
&\left.+ o(k^{-1})\right) = \left(1 + \int_x^{x_I}\frac{V(t)}{2ik}dt + o(k^{-1})\right)\left(1 - \int_0^{x_I}\frac{V(t)}{2ik}dt + o(k^{-1})\right) \\
&\left(1 + \frac{h}{ik} - \frac{\hat{V}(-k)}{ik} + o(k^{-1})\right) = 1 + \int_x^{x_I}\frac{V(t)}{2ik}dt + \frac{h}{ik} - \int_0^{x_I}\frac{V(t)}{2ik}dt
\end{aligned}$$

$$-\frac{\hat{V}(-k)}{ik}+o(k^{-1})=1-\int_0^x\frac{V(t)}{2ik}dt+\frac{h}{ik}+o(k^{-1}) \tag{2.111}$$

where we used that

$$\frac{f(x,k)}{f(0,k)}=e^{ikx}\frac{\left(1-\int_x^\infty\frac{V(t)}{2ik}+o(k^{-1})\right)}{\left(1-\int_0^\infty\frac{V(t)}{2ik}+o(k^{-1})\right)}.$$

In the following proposition (see [3, Proposition 4] for the Rayleigh case) we represent the function $\psi(x,k)$ in terms of some coefficients of the asymptotcs of the Weyl function $M(\lambda)$ and its residues.

Proposition 2.6.20. *The function* $\psi(x,k)$ *satisfies*

$$\psi(x,k)=1-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{k'e^{ik'x}\varphi(x,k')\left(j(k')-j(-k')\right)}{k'-k}dk'$$
$$+\sum_{j=1}^{N}\frac{\alpha_j}{2i(k_j+k)}e^{-ik_jx}\varphi(x,k_j)+\sum_{j=1}^{N}\frac{\alpha_j}{2i(k-k_j)}e^{ik_jx}\varphi(x,k_j). \tag{2.112}$$

Moreover, the limit value $\psi_\pm(x,k)=\psi(x,k\pm i0)$ *determines* φ *in* (2.90) *by*

$$2\varphi(x,k)=e^{ikx}\psi_+(x,-k)+e^{-ikx}\psi_-(x,k) \tag{2.113}$$

Proof. We consider

$$\frac{1}{2\pi i}\int_{-R}^{R}\frac{-\psi_+(k')+\psi_-(k')}{k'-k}dk'$$

which can be written as

$$-\frac{1}{2\pi i}\int_{-R}^{R}\frac{\psi_+(k')-1}{k'-k}dk'+\frac{1}{2\pi i}\int_{-R}^{R}\frac{\psi_-(k')-1}{k'-k}dk'$$
$$=\lim_{\epsilon\to 0^+}\left(-\frac{1}{2\pi i}\int_{-R+i\epsilon}^{R+i\epsilon}\frac{\psi_+(k')-1}{k'-k}dk'-\frac{1}{2\pi i}\int_{R-i\epsilon}^{-R-i\epsilon}\frac{\psi_-(k')-1}{k'-k}dk'\right).$$

Then we can write the integral over the interval $(-R+i\epsilon,R+i\epsilon)$ as an integral over the contour $\gamma^+(R,\epsilon)$, which consists of the arc on the upper half plane subtended by the segment $(-R+i\epsilon,R+i\epsilon)$ plus the segment itself; an integral over the arc mentioned before with opposite verse $\Gamma^+(R,\epsilon)$. We do something similar with the integral over the interval $(R-i\epsilon,-R-i\epsilon)$ that we write as integral over the contour $\gamma^-(R,\epsilon)$, which consists of the arc on the lower half plane subtended by the segment $(R-i\epsilon,-R-i\epsilon)$ plus the segment itself; an integral over the arc mentioned above

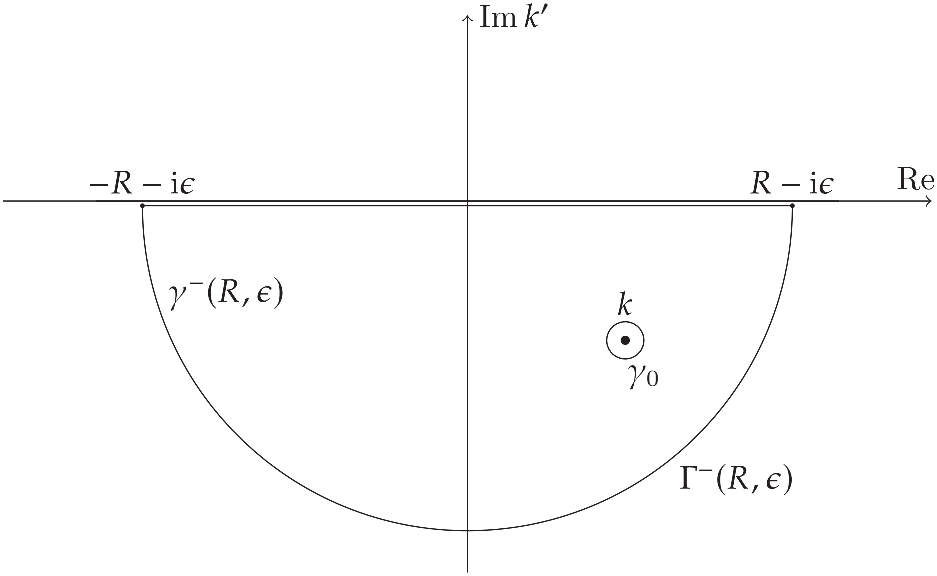

Figure 2.2: The closed contour $\gamma^-(R,\epsilon)$ is made by the segment from $-R - i\epsilon$ to $R - i\epsilon$ plus the arc between them in anti-clockwise way. The arc $\Gamma^-(R,\epsilon)$ is an arc from $-R - i\epsilon$ to $R - i\epsilon$ in a clockwise way. The circle γ_0 is a contour around the pole $k' = k$.

with opposite verse $\Gamma^-(R,\epsilon)$ and an integral over the positive oriented small circle γ_0 around the pole $k' = k$, that we consider lying in $\operatorname{Im} k < 0$, see Figure 2.2.

Then

$$
\begin{aligned}
&-\frac{1}{2\pi i}\int_{-R+i\epsilon}^{R+i\epsilon}\frac{\psi_+(k')-1}{k'-k}dk' - \frac{1}{2\pi i}\int_{R-i\epsilon}^{-R-i\epsilon}\frac{\psi_-(k')-1}{k'-k}dk'\\
&-\frac{1}{2\pi i}\int_{\gamma^+(R,\epsilon)}\frac{\psi_+(k')-1}{k'-k}dk' - \frac{1}{2\pi i}\int_{\gamma^-(R,\epsilon)}\frac{\psi_-(k')-1}{k'-k}dk'\\
&+\frac{1}{2\pi i}\int_{\Gamma^+(R,\epsilon)}\frac{\psi_+(k')-1}{k'-k}dk' + \frac{1}{2\pi i}\int_{\Gamma^-(R,\epsilon)}\frac{\psi_-(k')-1}{k'-k}dk'\\
&-\frac{1}{2\pi i}\int_{\gamma_0}\frac{\psi_-(k')-1}{k'-k}dk'
\end{aligned}
$$

becomes

$$
\begin{aligned}
&-\psi_-(x,k) + 1 - \sum_{j=1}^{N}\operatorname{Res}_{k'=k_j}\frac{\psi_+(k')}{k_j-k} + \sum_{j=1}^{N}\operatorname{Res}_{k'=-k_j}\frac{\psi_-(k')}{k_j+k}\\
&+\frac{1}{2\pi i}\int_{\Gamma^+(R,\epsilon)}\frac{\psi_+(k')-1}{k'-k}dk' + \frac{1}{2\pi i}\int_{\Gamma^-(R,\epsilon)}\frac{\psi_-(k')-1}{k'-k}dk'. \qquad (2.114)
\end{aligned}
$$

We have

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\left(\frac{1}{2\pi i}\int_{\Gamma^+(R,\epsilon)}\frac{\psi_+(k')-1}{k'-k}dk' + \frac{1}{2\pi i}\int_{\Gamma^-(R,\epsilon)}\frac{\psi_-(k')-1}{k'-k}dk'\right) = 0.$$

Indeed, $\psi_\pm - 1$ is of order $1/k'$, and taking into account the term $1/k'$ from the denominator, everything goes to zero by the Jordan Lemma. Thus (2.114) becomes

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi_+(k')}{k'-k}dk' + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi_-(k')}{k'-k}dk' = -\psi_-(x,k)+1$$
$$+\sum_{j=1}^{N}\operatorname{Res}_{k'=-k_j}\frac{\psi_-(k')}{k_j+k} - \sum_{j=1}^{N}\operatorname{Res}_{k'=k_j}\frac{\psi_+(k')}{k_j-k}$$

so

$$\psi(x,k)-1 = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{-\psi_+(k')+\psi_-(k')}{k'-k}dk' + \sum_{j=1}^{N}\operatorname{Res}_{k'=-k_j}\frac{\psi_-(k')}{k_j+k}$$
$$-\sum_{j=1}^{N}\operatorname{Res}_{k'=k_j}\frac{\psi_+(k')}{k_j-k} = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{-\psi_+(k')+\psi_-(k')}{k'-k}dk'$$
$$+\sum_{j=1}^{N}\frac{ik_j}{k_j+k}e^{-ik_jx}\varphi(x,-k_j)\operatorname{Res}_{k'=-k_j}M^-(\lambda)$$
$$+\sum_{j=1}^{N}\frac{ik_j}{k_j-k}e^{ik_jx}\varphi(x,k_j)\operatorname{Res}_{k'=k_j}M^+(\lambda). \tag{2.115}$$

Since $\varphi(x,k)$ is even in k we have $\varphi(x,-k)=\varphi(x,k)$. On the left-hand side, we have a meromorphic function minus its singular terms.

Recalling (2.100), we have that

$$\alpha_j = \operatorname{Res}_{\lambda'=\lambda_j}M(\lambda') = 2k_j\operatorname{Res}_{k'=k_j}M^+(\lambda) = -2k_j\operatorname{Res}_{k'=-k_j}M^-(\lambda).$$

Plugging (2.110) in (2.115) we get

$$\psi(x,k) = 1 - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{k'e^{ik'x}\varphi(x,k')\left(j(k')-j(-k')\right)}{k'-k}dk' +$$
$$-\sum_{j=1}^{N}\frac{i\alpha_j}{2(k_j+k)}e^{-ik_jx}\varphi(x,k_j) - \sum_{j=1}^{N}\frac{i\alpha_j}{2(k-k_j)}e^{ik_jx}\varphi(x,k_j).$$

Formula (2.113) was obtained before in (2.108). □

In the next corollary (see [3, Corollary, page 6701]) we show the connection between the potential V and the eigenvalues k_j, the functions $j(k)$ and the normalizing constants α_j.

Corollary 2.6.21. *The potential V satisfies the identity*

$$\int_0^x V(t)dt - 2h = -\frac{2i}{\pi}\int_{-\infty}^{\infty} k'\varphi(x,k')\cos(k'x)\, j(k')dk'$$
$$-2\sum_{j=1}^{N}\alpha_j\varphi(x,k_j)\cos(k_jx) \tag{2.116}$$

and the function $\varphi(x,k)$ satisfies

$$2\varphi(x,k) = 2\cos(kx) - \frac{i}{\pi}\int_{-\infty}^{+\infty} k'\varphi(x,k')j(k')\left[\frac{\sin(k'-k)x}{k'-k} + \frac{\sin(k'+k)x}{k'+k}\right]$$
$$-\sum_{j=1}^{N}\alpha_j\varphi(x,k_j)\left[\frac{\sin(k_j-k)x}{k_j-k} + \frac{\sin(k_j+k)x}{k_j+k}\right]. \tag{2.117}$$

Proof. We start first with the proof of equation (2.116) (*Step 1*) and then we recover (2.117) (*Step 2*).

- *Step 1.* We already know the asymptotic expansion of ψ from (2.111):

$$\psi(x,k) - 1 = -\frac{1}{2ik}\int_0^x V(t)dt + \frac{h}{ik} + o(k^{-1}).$$

Multiplying $\psi - 1$ by ik and taking the limit as $k \to \infty$, we get

$$\lim_{k\to\infty} ik\left(\psi(x,k) - 1\right) = -\frac{1}{2}\int_0^x V(t)dt + h. \tag{2.118}$$

In (2.112), we multiply by ik and take the limit as $k \to \infty$

$$\lim_{k\to\infty} ik\left(\psi(x,k) - 1\right) = \frac{i}{2\pi}\int_{-\infty}^{\infty} k'e^{ik'x}\varphi(x,k')(j(k') - j(-k'))dk'$$
$$+\frac{1}{2}\sum_{j=1}^{N}\alpha_j e^{-ik_jx}\varphi(x,k_j) + \frac{1}{2}\sum_{j=1}^{N}\alpha_j e^{ik_jx}\varphi(x,k_j) = \frac{i}{2\pi}\int_{-\infty}^{\infty} k'e^{ik'x}$$
$$\varphi(x,k')(j(k') - j(-k'))dk' + \sum_{j=1}^{N}\alpha_j\varphi(x,k_j)\cos(k_jx). \tag{2.119}$$

The first term can be written as

$$\int_{-\infty}^{\infty} k' e^{ik'x}\varphi(x,k')(j(k')-j(-k'))dk' = \int_{-\infty}^{\infty} k'\varphi(x,k')e^{ik'x}j(k')dk'$$
$$- \int_{-\infty}^{\infty} k'\varphi(x,k')e^{ik'x}j(-k')dk' = \int_{-\infty}^{\infty} k'\varphi(x,k')e^{ik'x}j(k')dk'$$
$$- \int_{+\infty}^{-\infty} (-k')\varphi(x,k')e^{-ik'x}j(k')(-dk') = \int_{-\infty}^{\infty} k'\varphi(x,k')e^{ik'x}j(k')dk'$$
$$+ \int_{-\infty}^{\infty} k'\varphi(x,k')e^{-ik'x}j(k')dk' = \int_{-\infty}^{\infty} 2k'\varphi(x,k')\cos(k_j x)\, j(k')dk'.$$

Plugging this result in (2.119) we get

$$\lim_{k\to\infty} ik\,(\psi(x,k)-1) = \frac{i}{2\pi}\int_{-\infty}^{\infty} 2k'\varphi(x,k')\cos(k'x)\, j(k')dk'$$
$$+ \sum_{j=1}^{N} \alpha_j \varphi(x,k_j)\cos(k_j x) \tag{2.120}$$

and comparing (2.118) and (2.120) we get

$$\int_0^x V(t)dt - 2h = -\frac{2i}{\pi}\int_{-\infty}^{\infty} k'\varphi(x,k')\cos(k'x)\, j(k')dk'$$
$$- 2\sum_{j=1}^{N} \alpha_j \varphi(x,k_j)\cos(k_j x).$$

- *Step 2.* We know that

$$2\varphi(x,k) = e^{ikx}\psi_+(x,-k) + e^{-ikx}\psi_-(x,k).$$

Both the function $\psi_+(x,-k)$ and $\psi_-(x,k)$ have poles in the lower half plane, so we can consider k in the lower half plane and use the formula (2.112) for $e^{ikx}\psi_+(x,-k)$:

$$e^{ikx}\psi_+(x,-k) = e^{ikx} - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{k' e^{i(k+k')x}\varphi(x,k')\,(j(k')-j(-k'))}{k'+k}dk'$$
$$+ \sum_{j=1}^{N} \frac{\alpha_j e^{-i(k_j-k)x}}{2i(k_j-k)}\varphi(x,k_j) - \sum_{j=1}^{N} \frac{\alpha_j e^{i(k_j+k)x}}{2i(k+k_j)}\varphi(x,k_j).$$

The first integral can be rewritten as

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{i(k+k')x}\varphi(x,k')\left(j(k')-j(-k')\right)}{k'+k}dk' =$$
$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{i(k+k')x}\varphi(x,k')j(k')}{k'+k}dk'$$
$$-\frac{1}{2\pi}\int_{+\infty}^{-\infty}\frac{(-k')e^{-i(k'-k)x}\varphi(x,k')(-j(k'))}{k-k'}(-dk')$$

where the second integral after the change of variable from k' to $-k'$ becomes

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{-i(k'-k)x}\varphi(x,k')j(k')}{k'-k}dk'.$$

So, in the end we have

$$e^{ikx}\psi_+(x,-k) = e^{ikx} - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{i(k+k')x}\varphi(x,k')j(k')}{k'+k}dk'$$
$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{-i(k'-k)x}\varphi(x,k')j(k')}{k'-k}dk' + \sum_{j=1}^{N}\frac{\alpha_j e^{-i(k_j-k)x}}{2i(k_j-k)}\varphi(x,k_j)$$
$$-\sum_{j=1}^{N}\frac{\alpha_j e^{i(k_j+k)x}}{2i(k+k_j)}\varphi(x,k_j). \tag{2.121}$$

Similarly, for $e^{-ikx}\psi_-(x,k)$ we get

$$e^{-ikx}\psi_-(x,k) = e^{-ikx} - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{i(k'-k)x}\varphi(x,k')j(k')}{k'-k}dk'$$
$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{k'e^{-i(k'+k)x}\varphi(x,k')j(k')}{k'+k}dk' - \sum_{j=1}^{N}\frac{\alpha_j e^{i(k_j-k)x}}{2i(k_j-k)}\varphi(x,k_j)$$
$$+\sum_{j=1}^{N}\frac{\alpha_j e^{-i(k_j+k)x}}{2i(k+k_j)}\varphi(x,k_j). \tag{2.122}$$

Summing (2.122) and (2.121) we get

$$e^{ikx}\psi_+(x,-k) + e^{-ikx}\psi_-(x,k) = 2\cos(kx)+$$
$$-\frac{i}{\pi}\int_{-\infty}^{+\infty}k'\varphi(x,k')j(k')\left[\frac{\sin(k'-k)x}{k'-k}+\frac{\sin(k'+k)x}{k'+k}\right]+$$
$$-\sum_{j=1}^{N}\alpha_j\varphi(x,k_j)\left[\frac{\sin(k_j-k)x}{k_j-k}+\frac{\sin(k_j+k)x}{k_j+k}\right]. \qquad \square$$

Equation (2.116) motivates introducing

$$K(x,y)=\frac{i}{\pi}\int_{-\infty}^{+\infty}k'\varphi(x,k')j(k')\cos(k'y)dk'+\sum_{j=1}^{N}\varphi(x,k_j)\alpha_j\cos(k_jy). \tag{2.123}$$

Then equation (2.117) can be written as

$$\varphi(x,k)=\cos(kx)-\frac{1}{2}\int_{-x}^{x}K(x,t)\cos(kt)dt=\cos(kx)-\int_{0}^{x}K(x,t)\cos(kt)dt.$$

The next proposition shows the Gelfand–Levitan equation and the algorithm one can use to recover the potential. One can compare the following proposition with [3, Proposition 5].

Proposition 2.6.22. *The potential can be reconstructed from the Weyl function through the formula*

$$V(x)=-2\frac{d}{dx}K(x,x), \tag{2.124}$$

where $K(x,y)$ *satisfies the Gelfand–Levitan equation*

$$K(x,y)-g(x,y)+\frac{1}{2}\int_{-x}^{x}K(x,s)g(s,y)ds=0, \tag{2.125}$$

with

$$g(x,y)= \\ =\begin{cases}\frac{i}{\pi}\int_{-\infty}^{+\infty}k'\cos(k'x)j(k')\cos(k'y)dk'+\sum_{j=1}^{N}\cos(k_jx)\alpha_j\cos(k_jy) & x\geq y\\ 0 & x<y\end{cases}. \tag{2.126}$$

Proof. We consider

$$\int_{-x}^{x}K(x,y)\cos(ky)dy=\int_{-x}^{x}\left[\frac{i}{\pi}\int_{-\infty}^{+\infty}k'\varphi(x,k')j(k')\cos(k'y)dk'\right.$$
$$\left.+\sum_{j=1}^{N}\varphi(x,k_j)\alpha_j\cos(k_jy)\right]\cos(ky)dy=\frac{i}{\pi}\int_{-\infty}^{+\infty}k'\varphi(x,k')j(k')$$
$$\int_{-x}^{x}\cos(k'y)\cos(ky)dydk'+\sum_{j=1}^{N}\varphi(x,k_j)\alpha_j\int_{-x}^{x}\cos(k_jy)\cos(ky)dy \tag{2.127}$$

and we can calculate

$$\int_{-x}^{x} \cos(\alpha y)\cos(\beta y)dy = 2\int_{0}^{x} \cos(\alpha y)\cos(\beta y)dy = 2\int_{0}^{x} \frac{1}{2}\left[\cos\left((\alpha+\beta)y\right)\right.$$

$$\left.+\cos\left((\alpha-\beta)y\right)\right]dy = \left[\frac{\sin\left((\alpha+\beta)y\right)}{\alpha+\beta} + \frac{\sin\left((\alpha-\beta)y\right)}{\alpha-\beta}\right]_{0}^{x}$$

$$= \frac{\sin\left((\alpha+\beta)x\right)}{\alpha+\beta} + \frac{\sin\left((\alpha-\beta)x\right)}{\alpha-\beta}.$$

Plugging this in (2.127) we get

$$\int_{-x}^{x} K(x,y)\cos(ky)dy = \frac{i}{\pi}\int_{-\infty}^{+\infty} k'\varphi(x,k')j(k')\left[\frac{\sin\left((k'+k)x\right)}{k'+k}\right.$$

$$\left.+\frac{\sin\left((k'-k)x\right)}{k'-k}\right]dk' + \sum_{j=1}^{N}\varphi(x,k_j)\alpha_j\left[\frac{\sin\left((k_j+k)x\right)}{k_j+k} + \frac{\sin\left((k_j-k)x\right)}{k_j-k}\right].$$

Then, comparing with (2.117), it follows that

$$2\varphi(x,k) - 2\cos(kx) = -\int_{-x}^{x} K(x,y)\cos(ky)dy. \tag{2.128}$$

Taking into account (2.126) and (2.123) we can calculate the difference

$$2g(x,y) - 2K(x,y) = \frac{2i}{\pi}\int_{-\infty}^{+\infty} k'\cos(k'x)j(k')\cos(k'y)dk'$$

$$+2\sum_{j=1}^{N}\cos(k_jx)\alpha_j\cos(k_jy) - \frac{2i}{\pi}\int_{-\infty}^{+\infty} k'\varphi(x,k')j(k')\cos(k'y)dk'$$

$$-2\sum_{j=1}^{N}\varphi(x,k_j)\alpha_j\cos(k_jy) = 2\sum_{j=1}^{N}\cos(k_jy)\alpha_j\left[\cos(k_jx)-\varphi(x,k_j)\right]$$

$$+\frac{2i}{\pi}\int_{-\infty}^{+\infty} k'j(k')\cos(k'y)\left[\cos(k'x)-\varphi(x,k')\right]dk'$$

and, using formula (2.128), we get that

$$\cos(k'x) - \varphi(x,k') = \frac{1}{2}\int_{-x}^{x} K(x,s)\cos(ks)ds.$$

Plugging this result into the previous calculations, we get

$$2g(x,y) - 2K(x,y) = \frac{i}{\pi}\int_{-\infty}^{+\infty}\int_{-x}^{x} k'j(k')\cos(k'y)K(x,s)\cos(ks)ds$$

$$
+\sum_{j=1}^{N}\cos(k_j y)\alpha_j \int_{-x}^{x} K(x,s)\cos(ks)ds = \int_{-x}^{x} K(x,s)
$$

$$
\left[\frac{i}{\pi}\int_{-\infty}^{+\infty} k'\cos(k'y)j(k')\cos(k's)dk' + \sum_{j=1}^{N}\alpha_j\cos(k_j y)\cos(k_j s)\right] ds
$$

$$
= \int_{-x}^{x} K(x,s)g(s,y)ds \qquad \text{for } y \le s \le x.
$$

Hence, the kernel $K(x,y)$ satisfies

$$
K(x,y) - g(x,y) + \frac{1}{2}\int_{-x}^{x} K(x,s)g(s,y)ds = 0. \qquad \square
$$

Remark 2.6.23 (Uniqueness). *Let V and $\tilde{V}$ be in $\mathbb{V}^1_{x_I}$ with Weyl functions M and $\tilde{M}$ respectively. If $M = \tilde{M}$, then $V = \tilde{V}$. Indeed, from equation* (2.123) *we see that*

$$
K(x,x) = \frac{i}{\pi}\int_{-\infty}^{+\infty} k'\varphi(x,k')\left(M(\lambda) - \frac{1}{ik'}\right)\cos(k'x)dk'
$$

$$
+\sum_{j=1}^{N}\varphi(x,k_j)\cos(k_j x)2k_j \operatorname{Res}_{k'=k_j} M(\lambda)
$$

and

$$
\tilde{K}(x,x) = \frac{i}{\pi}\int_{-\infty}^{+\infty} k'\tilde{\varphi}(x,k')\left(\tilde{M}(\lambda) - \frac{1}{ik'}\right)\cos(k'x)dk'
$$

$$
+\sum_{j=1}^{N}\tilde{\varphi}(x,k_j)\cos(k_j x)2k_j \operatorname{Res}_{k'=k_j} \tilde{M}(\lambda).
$$

If $M(\lambda) = \tilde{M}(\lambda)$ then $\varphi(x,k') = \tilde{\varphi}(x,k')$, which also implies $K(x,x) = \tilde{K}(x,x)$, which leads to $V(x) = \tilde{V}(x)$.

Remark 2.6.24. *Since $K(x,s)$ and $g(s,y)$ are even, namely $K(x,-s) = K(x,s)$ and $g(-s,y) = g(s,y)$, we can write* (2.125) *also as*

$$
K(x,y) - g(x,y) + \int_{0}^{x} K(x,s)g(s,y)ds = 0.
$$

The next theorem shows for which condition the Gelfand–Levitan equation (2.125) has a unique solution (see [3, Remark (ii), page 6708]).

Theorem 2.6.25. *The Gelfand–Levitan equation* (2.125) *has a unique solution, for fixed* $x > 0$*, if*

$$\int_0^x \sup_{0 \le s \le t} |g(t,s)| \, dt < \infty \tag{2.129}$$

holds.

Proof. The equation (2.125) is an inhomogeneous Volterra equation, where the inhomogeneous term is $-g(x, y)$. In order to have unique solvability of (2.125), we require that the homogeneous equation

$$K(x,y) + \int_0^x K(x,s)g(s,y)ds = 0$$

only admits the trivial solution $K(x, s) = 0$. One can find the solution of (2.125) from the resolvent $R(s, t)$, which is obtained by iterating the kernel $g(x, y)$

$$R(s,t) = \sum_{k=0}^{\infty} (-1)^k g_{k+1}(s,t)$$

where $g_{k+1}(s, t)$ represents the $k + 1$ iterate of the Volterra kernel. The solution is then

$$K(x,y) = g(x,y) - \int_0^x R(x,t)g(t,y)dt.$$

We consider the second iterate of the kernel $g(x, y)$

$$g_2(x,y) = \int_0^x \int_0^t g(t,s)g(s,y)dtds, \qquad 0 \le y \le s \le t \le x.$$

Since $g(t, s) = 0$ for $s > t$, we have

$$\begin{aligned} |g_2(x,y)| &= \left| \int_0^x \int_0^t g(t,s)g(s,y)dtds \right| \\ &\le \int_0^x \int_0^t \sup_{0 \le s \le t} |g(t,s)| \sup_{0 \le y \le s} |g(s,y)| \, dtds. \end{aligned}$$

We define $d(t) := \sup_{0 \le s \le t} |g(t, s)|$, so we get

$$|g_2(x,y)| \le \int_0^x \int_0^t d(s)d(t)dtds = \frac{1}{2} \left(\int_0^x d(s)ds \right)^2.$$

Similarly,

$$|g_k(x,y)| \le \frac{1}{k!} \left(\int_0^x d(s)ds \right)^k$$

which implies that the homogeneous equation

$$K(x,y) = -\int_0^x K(x,s)g(s,y)ds$$

admits only the trivial solution $K(x,y) = 0$ as long as

$$\int_0^x \sup_{0\le s\le t} |g(t,s)|\,dt < \infty. \qquad \square$$

Condition (2.129) is required to have unique solvability of the Gelfand–Levitan equation. We have

$$\begin{aligned}
&\int_0^x \left| \int_{-\infty}^{\infty} k\cos(kt)\cos(ks)j(k)dk + \sum_{j=1}^{N} \cos(k_j t)\cos(k_j s)\ \alpha_j \right| dt \le \\
&\int_0^x \int_0^{\infty} |k|\,|\cos kt|\ |\cos ks|\ |j(k)-j(-k)|\,dkdt + c_1 x \\
&\le \int_0^x \int_0^{\infty} \frac{dkdt}{|k|^2} + c_1 x \le c_2 x, \qquad (2.130)
\end{aligned}$$

and we see that the condition is satisfied for the Weyl function of our problem, since $x > 0$ is fixed in the Gelfand–Levitan equation.

The resolvent operator for our problem is (see [30, page 215])

$$R(x,y,\lambda) = \frac{\varphi(x,k)f(y,k)}{f_h(k)}.$$

We can recover the density function from it as shown in the next proposition.

Proposition 2.6.26. *The density function for the problem is*

$$\rho_\varphi(x,y,\lambda) = \varphi(x,k)\varphi(y,k)T(\lambda).$$

Proof. The Stieltjes inversion formula tells us that

$$\rho_\varphi(x,y,\lambda) = \lim_{\epsilon\to 0} \frac{R(x,y,\lambda-i\epsilon) - R(x,y,\lambda+i\epsilon)}{2i\pi}.$$

We can notice that the resolvent can be rewritten as

$$\begin{aligned}
R(x,y,\lambda) = \frac{\varphi(x,k)f(y,k)}{f_h(k)} &= \varphi(x,k)\phi(y,k) = \varphi(x,k)\left(\theta(y,k)\right. \\
&\left. +M^+(\lambda)\varphi(y,k)\right) = \varphi(x,k)\theta(x,k) + \varphi(x,k)M^+(\lambda)\varphi(y,k),
\end{aligned}$$

hence

$$R(x,y,\lambda-i\epsilon) - R(x,y,\lambda+i\epsilon) = \varphi(x,k)\varphi(y,k)\left(M(\lambda-i\epsilon) - M(\lambda+i\epsilon)\right)$$

and considering the definition of the jump function $T(\lambda)$ (2.93), we get

$$\rho_\varphi(x,y,\lambda) = \varphi(x,k)\varphi(y,k)T(\lambda). \qquad \square$$

2.6.4 The inverse problem

In this subsection we present an inverse result starting from a class $\mathbb{M}_{x_I}$ of Weyl functions to the class $\mathbb{V}^1_{x_I}$. This motivates the following definition.

Definition 2.6.27 (Class of Weyl function). *For fixed $h \in \mathbb{R}$, we denote by $\mathbb{M}_{x_I}$ the class of functions $M(\lambda)$ satisfying the following properties:*

I) *$M(\lambda)$ is analytic in Π with finite number N of simple poles $\lambda_j < 0$ and residues $\alpha_j = \operatorname{Res}_{\lambda=\lambda_j} M(\lambda) > 0$.*

II) *$M(\lambda)$ is continuous in $\Pi_1 \setminus \{\lambda_1, \dots, \lambda_N, 0\}$ satisfying $kM(\lambda) = O(1)$ as $k \to 0$, $\operatorname{Im} k > 0$.*

III) *Let $M^{\pm}(\lambda) = \lim_{\epsilon \to 0, \operatorname{Re}\epsilon > 0} M(\lambda \pm i\epsilon)$. Then*

$$T(\lambda) := \frac{1}{2\pi i}(M^-(\lambda) - M^+(\lambda)) > 0, \qquad \lambda > 0.$$

IV) *$M(\lambda) = \frac{1}{ik} + \frac{h}{k^2} + \frac{V(0)}{k^2} + o(k^{-2})$, as $|k| \to +\infty$.*

V) *The Gelfand–Levitan equation*

$$g(x,y) + K(x,y) + \int_0^x K(x,s)g(s,y)ds = 0$$

with

$$g(x,y) = \\ = \begin{cases} \frac{i}{\pi}\int_{-\infty}^{+\infty} k' \cos(k'x) j(k') \cos(k'y) dk' + \sum_{j=1}^N \cos(k_j x)\alpha_j \cos(k_j y), & x \geq y, \\ 0, & x < y, \end{cases}$$

and $j(k) = M(\lambda) - \frac{1}{ik}$, for any fixed $x > 0$, has a unique solution $K(x,y)$ with $K(x,x)$ real, absolutely continuous and $\frac{d}{dx}K(x,x) = 0$ for $x > x_I$ and non-zero in a set of non-zero Lebesgue measure $(x_I - \epsilon, x_I)$.

In the following theorem we characterize the class $\mathbb{V}^1_{x_I}$ by the just defined class of Weyl functions $\mathbb{M}_{x_I}$.

Theorem 2.6.28. *The map $\mathcal{J}_h : \mathbb{V}^1_{x_I} \to \mathbb{M}_{x_I}$ defined by $\mathcal{J}_h(V) := M$ is well-defined and bijective.*

Proof. We shall prove that, for fixed $h \in \mathbb{R}$ the map $\mathcal{J}_h$ is well-defined, that is, $\mathcal{J}_h(V) = M \in \mathbb{M}_{x_I}$ for any $V \in \mathbb{V}^1_{x_I}$. In Theorem 2.5.20 and Theorem 2.5.28 we proved that the Jost function $f_h(k)$ has a finite number of zeros in $\mathbb{C}_+$ and that they are all simple and pure imaginary. In Theorem 2.5.17 we proved that the Jost solution $f(x,k)$ and the Jost function $f_h(k)$ are entire in k, hence analytic for $\operatorname{Im} k > 0$ and continuous for $\operatorname{Im} k \geq 0$. Then, by definition of the Weyl function $M(\lambda)$, we can conclude that the Weyl function is analytic in Π, continuous in Π_1 except at the points where the denominator vanishes (see also Theorem 2.6.16), which are the simple and pure imaginary zeros of the Jost function (see Theorem 2.6.16 and Remark 2.6.5). In Proposition 2.6.13, we proved that $\alpha_j > 0$, hence Condition I of the definition of the class of Weyl function is satisfied.

In Lemma 2.6.12 we showed that $\frac{k}{f_h(k)} = O(1)$. Since $f(0,k) = 1 + O(1/k)$, then $kM(\lambda) = O(1)$, which implies Condition II.

Condition III is proved by Proposition 2.6.11 since for $\lambda > 0$ we get $T(\lambda) = \frac{\sqrt{\lambda}}{\pi |f_h(k)|^2} > 0$. By Lemma 2.6.7 we can see that Condition IV is satisfied.

From Proposition 2.6.22, we see that Condition V is satisfied, hence $M \in \mathbb{M}_{x_I}$ and $\mathcal{J}_h$ is well-defined.

The injectivity of the map $\mathcal{J}_h$ is given by Theorem 2.6.17.

To prove surjectivity, we fix $M(\lambda) \in \mathbb{M}_{x_I}$ and we want to prove that there exists a $V \in \mathbb{V}^1_{x_I}$ such that $\mathcal{J}_h(V) = M(\lambda)$. Condition I-IV allow us to define a function $g(x,y)$[7] as in (2.126) and $K(x,y)$ which satisfies the Gelfand–Levitan equation (see Proposition 2.6.22). From $K(x,y)$, solution of (2.125), we can construct (as in (2.128))

$$\varphi(x,k) = \cos kx - \int_0^x K(x,y)\cos(ky)dy$$

that is a solution to the boundary value problem (2.2)–(2.3) with $V(x) = -2\frac{d}{dx}K(x,x)$ and $h = K(0,0)$ given.

From Condition V we know that the Gelfand–Levitan equation (2.125) has a unique solution $K(x,y)$, such that $V = -2\frac{d}{dx}K(x,x)$ is in the class $\mathbb{V}^1_{x_I}$. □

The reader can compare Definition 2.6.27 and Theorem 2.6.28 with the definition of the class **W** and Theorem 2.2.5 in [22], which are obtained for a different class of potentials and through a different Gelfand–Levitan equation.

Algorithm 2.6.29. *Starting from a set of eigenvalues and resonances* $\{k_j\}_1^\infty$ *we can retrieve the potential* $V_\omega(x)$ *using the following algorithm:*

[7] Condition III is needed because $T(\lambda)$ is the spectral measure and it must be non negative.

- *Construct the Jost function from* (2.71) *according to*

$$f_h(k) = f_h(0)e^{ik} \lim_{R\to\infty} \prod_{|k_n|\le R} \left(1 - \frac{k}{k_n}\right),$$

where $f_h(0)$ *is determined so that* $f_h(k) = ik + O(1)$ *as* $k \to \infty$.

- *From* $\{k_j\}_1^\infty$ *and* $f_h(k)$ *we construct the jump function* $T(\lambda)$ *and the normalizing constant* α_k *through formulas* (2.94) *and* (2.98)*:*

$$T(\lambda) = \frac{k}{\pi |f_h(k)|^2},$$
$$\alpha_j = 4k_j^2 \left[\frac{-i}{f_h(-k_j)\dot{f}_h(k_j)}\right].$$

- *Use the spectral data* $\left(T(\lambda), \{\alpha_j, \lambda_j\}_{j=1,\dots,N}\right)$ *to construct the Weyl function via formula* (2.101)

$$M(\lambda) = \int_0^\infty \frac{T(\mu)}{\lambda - \mu} d\mu + \sum_{k=1}^N \frac{\alpha_k}{\lambda - \lambda_k}, \quad \lambda \in \Pi \backslash \Lambda'.$$

- *Then construct* $g(x, y)$ *in* (2.126) *as in*

$$g(x, y) = \\ = \begin{cases} \frac{i}{\pi} \int_{-\infty}^{+\infty} k' \cos(k'x) j(k') \cos(k'y) dk' + \sum_{j=1}^N \cos(k_j x) \alpha_j \cos(k_j y), & x \ge y \\ 0, & x < y \end{cases}.$$

where $j(k) := M(\lambda) - \frac{1}{ik}$.

- *Solve the Gelfand–Levitan equation* (2.125) *with respect to* $K(x, y)$*,*

$$K(x, y) - g(x, y) + \frac{1}{2} \int_{-x}^{x} K(x, s) g(s, y) ds = 0.$$

- *Obtain the potential from* (2.124)*:*

$$V_\omega(x) = -2 \frac{d}{dx} K(x, x).$$

Remark 2.6.30. *After the retrieval of the potential* $V_\omega(x)$*, we can apply Theorem 2.5.46 in order to recover the Lamé parameter* $\hat{\mu}$.

Chapter III

Chapter 3

The Rayleigh problem

3.1 Introduction

In this chapter, we want to study the Rayleigh boundary value problem that we obtained from decoupling the Hamiltonian in Chapter 1. The idea is to follow the same strategy as in Chapter 2, for the scalar case, to solve the inverse resonance problem, although this is not solved in this thesis. In this case, we do not have a Schrödinger-type form of the boundary value problem and we cannot define the Jost function and follow the procedure as in the Love case. Instead, we perform a Pekeris-Markushevic transform (see [40]) that gives us a Schrödinger-type form with eigenvalues $-\xi^2$ and Robin boundary condition depending on the spectral parameter ξ.

This transformed problem is no longer self-adjoint, so we lack some of the properties we had in the Love case. Moreover, the Jost function can no longer be reconstructed by the resonances because it is not entire in the complex plane. Hence, we need to define a function $F(\xi)$ consisting of the product of the Rayleigh determinants of the four different sheets of a Riemann surface (see Section 3.4) defined from the quasi-momenta q_P and q_S. We obtain new results by proving that this function is entire (see Theorem 3.6.12), of exponential-type (see Theorem 3.8.22) and of Cartwright class with indices $\rho_\pm(F) \leq 8H$ (see Theorem 3.9.1). As an application of these results, we also obtain new direct results on the number of resonances (see Corollary 3.9.3) and the forbidden domain for the resonances (see Theorem 3.9.4) in Section 3.9. Even though the setting is made in order to prove the inverse resonance problem, this is not done in this thesis. One of the biggest challenge in this chapter is to be able to define the Riemann surface, and the reflection and conjugation on each sheet in a smart way. Once we have achieved this, we are able to obtain symmetry properties of the Jost solutions and use the mappings $w_\bullet$, with $\bullet = P, S, PS$, to pass from one sheet to another one of the Riemann surface (see [13]). The choice of the Riemann surface is also crucial to obtain the right estimates of the determinants of the Jost function in Section 3.8.5, which is consistent with having divergent exponentials in the unphysical sheets, as it was for the Jost function in Chapter 2.

3.2 Main equations of the Rayleigh problem

In this chapter, we want to study the Rayleigh boundary problem that we obtained after decoupling the Hamiltonian in Chapter 1 (1.10)–(1.13). Let

$$\begin{aligned} H\Phi =& H\begin{pmatrix} \varphi_1 \\ \varphi_3 \end{pmatrix} \\ :=& \begin{pmatrix} -\frac{\partial}{\partial Z}\left(\mu\frac{\partial}{\partial Z}\cdot\right)+(\lambda+2\mu)|\xi|^2 & -i|\xi|\left[\frac{\partial}{\partial Z}(\mu\cdot)+\lambda\left(\frac{\partial}{\partial Z}\cdot\right)\right] \\ -i|\xi|\left[\frac{\partial}{\partial Z}(\lambda\cdot)+\mu(\frac{\partial}{\partial Z}\cdot)\right] & -\frac{\partial}{\partial Z}\left((\lambda+2\mu)\frac{\partial}{\partial Z}\cdot\right)+\mu|\xi|^2 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_3 \end{pmatrix}, \end{aligned} \tag{3.1}$$

with free boundary conditions

$$\begin{aligned} a_-(\Phi) &= i\hat{\lambda}|\xi|\varphi_1(0^-)+(\hat{\lambda}+2\hat{\mu})\frac{\partial\varphi_3}{\partial Z}(0^-)=0 \\ b_-(\Phi) &= i|\xi|\hat{\mu}\varphi_3(0^-)+\hat{\mu}\frac{\partial\varphi_1}{\partial Z}(0^-)=0. \end{aligned} \tag{3.2}$$

We have $\hat{\mu}(Z)=\hat{\mu}_I$, $\hat{\lambda}(Z)=\hat{\lambda}_I$ constant for $Z<Z_I$. Below, we give the definition of the Jost solutions and the unperturbed Jost solutions, namely the Jost solution in the case of the homogeneous medium ($\hat{\mu}(Z)$ and $\hat{\lambda}(Z)$ constant).

Definition 3.2.1 (Unperturbed Jost solution). *If $\hat{\mu}(Z)=\hat{\mu}_I$, $\hat{\lambda}(Z)=\hat{\lambda}_I$ are constants, we define the unperturbed Jost solutions*

$$\begin{aligned} f^\pm_{P,0} &= \begin{pmatrix} |\xi| \\ \pm q_P \end{pmatrix} e^{\pm iZq_P}, \qquad q_P := \sqrt{\frac{\omega^2}{\hat{\lambda}_I+2\hat{\mu}_I}-|\xi|^2}, \\ f^\pm_{S,0} &= \begin{pmatrix} \pm q_S \\ -|\xi| \end{pmatrix} e^{\pm iZq_S}, \qquad q_S := \sqrt{\frac{\omega^2}{\hat{\mu}_I}-|\xi|^2}, \end{aligned} \tag{3.3}$$

for $Z<0$. That is, they are fundamental solutions to $H\Phi=\omega^2\Phi$

Definition 3.2.2 (Jost solution). *We define the Jost solutions $f^\pm_P$, $f^\pm_S$ for $Z<0$ as the fundamental solutions to $H\Phi=\omega^2\Phi$ satisfying the conditions*

$$f^\pm_P=f^\pm_{P,0}, \quad f^\pm_S=f^\pm_{S,0} \qquad \text{for } Z<Z_I,$$

where $f^\pm_{P,0}$ and $f^\pm_{S,0}$ are the solutions in the homogeneous case as in Definition 3.2.1.

The functions $f_P^\pm$, $f_S^\pm$, are

$$f_P^\pm = \begin{pmatrix} \varphi_1^\pm \\ \varphi_3^\pm \end{pmatrix} = \begin{pmatrix} |\xi| \\ \pm q_P \end{pmatrix} e^{\pm iZq_P}, \quad q_P := \sqrt{\frac{\omega^2}{\hat\lambda_I + 2\hat\mu_I} - |\xi|^2}, \qquad Z < Z_I,$$

$$f_S^\pm = \begin{pmatrix} \psi_1^\pm \\ \psi_3^\pm \end{pmatrix} = \begin{pmatrix} \pm q_S \\ -|\xi| \end{pmatrix} e^{\pm iZq_S}, \quad q_S := \sqrt{\frac{\omega^2}{\hat\mu_I} - |\xi|^2}, \qquad Z < Z_I,$$

so they coincide with the homogeneous solutions outside the support ($Z < Z_I$). We make an even extension[1] of the quantities $\hat\lambda$ and $\hat\mu$ to the half-line $Z > 0$ and we drop all the subscripts " $-$ " and denote the differential equation as $Hf = \omega^2 f$, where $f(Z, \xi)$ is the even solution in the whole real Z line.

3.3 Unperturbed problem: constant coefficients.

The Rayleigh problem is easily solvable in the homogeneous isotropic space where $\hat\mu = \hat\mu_I$, $\hat\lambda = \hat\lambda_I$ are constant everywhere. Then the differential operator becomes

$$H_0(x,\xi) := \begin{pmatrix} -\hat\mu_I \frac{\partial^2}{\partial Z^2} + (\hat\lambda_I + 2\hat\mu_I)|\xi|^2 & -i|\xi|(\hat\lambda_I + \hat\mu_I)\frac{\partial}{\partial Z} \\ -i|\xi|(\hat\lambda_I + \hat\mu_I)\frac{\partial}{\partial Z} & -(\hat\lambda_I + 2\hat\mu_I)\frac{\partial^2}{\partial Z^2} + \hat\mu_I|\xi|^2 \end{pmatrix},$$

and we denote

$$a_-(\Phi) := i\hat\lambda_I|\xi|\varphi_1(0^-) + (\hat\lambda_I + 2\hat\mu_I)\frac{\partial\varphi_3}{\partial Z}(0^-),$$

$$b_-(\Phi) := i|\xi|\hat\mu_I\varphi_3(0^-) + \hat\mu_I\frac{\partial\varphi_3}{\partial Z}(0^-),$$

the boundary values of Φ. Then the solutions of the equation $H_0(x,\xi)\Phi = \omega^2\Phi$ are $f_{P,0}^\pm$ and $f_{S,0}^\pm$ (see Definition 3.2.1) with the following boundary values

$$a_-(f_{P,0}^-) = i\hat\mu_I\left(\frac{\omega^2}{\hat\mu_I} - 2|\xi|^2\right), \quad b_-(f_{P,0}^-) = -2i\hat\mu_I|\xi|q_P,$$

$$a_-(f_{S,0}^-) = i2\hat\mu_I|\xi|q_S, \qquad b_-(f_{S,0}^-) = i\hat\mu_I\left(\frac{\omega^2}{\hat\mu_I} - 2|\xi|^2\right),$$

$$a_-(f_{P,0}^+) = i\hat\mu_I\left(\frac{\omega^2}{\hat\mu_I} - 2|\xi|^2\right), \quad b_-(f_{P,0}^+) = 2i\hat\mu_I|\xi|q_P,$$

$$a_-(f_{S,0}^+) = -i2\hat\mu_I|\xi|q_S, \qquad b_-(f_{S,0}^+) = i\hat\mu_I\left(\frac{\omega^2}{\hat\mu_I} - 2|\xi|^2\right).$$

[1]Note that this extension does not need to be smooth or continuous at $Z = 0$.

The determinant

$$\det\begin{vmatrix} a_-(f^-_{P,0}) & a_-(f^-_{S,0}) \\ b_-(f^-_{P,0}) & b_-(f^-_{S,0}) \end{vmatrix} = -\hat{\mu}_I^2\left(\left(\frac{\omega^2}{\hat{\mu}_I} - 2|\xi|^2\right)^2 + 4|\xi|^2 q_P q_S\right) = -\hat{\mu}_I^2 \Delta_{\mathrm{R}}$$

is proportional to the Rayleigh determinant defined as

$$\Delta_{\mathrm{R}} := \left(\left(\frac{\omega^2}{\hat{\mu}_I} - 2|\xi|^2\right)^2 + 4|\xi|^2 q_P q_S\right).$$

The Rayleigh operator $H_0(x,\xi)$ on $\mathbb{R}_-$ is a self-adjoint operator with domain

$$\mathcal{D}_- = \{\Phi \in H^2(\mathbb{R}_-;\mathbb{C}^2), \qquad a_-(\Phi) = b_-(\Phi) = 0\},$$

and it has continuous spectrum $[\hat{\mu}_I|\xi|^2, +\infty)$ (see [46], [20]): continuous spectrum with multiplicity 1 in the interval $[\hat{\mu}_I|\xi|^2, (\hat{\lambda}_I + 2\hat{\mu}_I)|\xi|^2]$ corresponding to pure reflection of S waves with a reflection coefficient of modulus 1; continuous spectrum with multiplicity 2 in the interval $[(\hat{\lambda}_I + 2\hat{\mu}_I)|\xi|^2, \infty)$ corresponding to incident S or P waves which are reflected as a linear combination of both kinds of waves. The equation $\Delta_{\mathrm{R}}(\omega^2) = 0$ for $\omega^2 < \hat{\mu}_I|\xi|^2$ using

$$q_P = i\sqrt{|\xi|^2 - \frac{\omega^2}{\hat{\lambda}_I + 2\hat{\mu}_I}}, \qquad q_S = i\sqrt{|\xi|^2 - \frac{\omega^2}{\hat{\mu}_I}}$$

reads

$$\left(2 - \frac{\omega^2}{|\xi|^2\hat{\mu}_I}\right)^2 - 4\sqrt{1 - \frac{\omega^2}{|\xi|^2(\hat{\lambda}_I + 2\hat{\mu}_I)}}\sqrt{1 - \frac{\omega^2}{1 - |\xi|^2\hat{\mu}_I}} = 0,$$

which we can write extending $|\xi|$ to $\xi \in \mathbb{C}$ and in terms of the parameters $\alpha := \frac{\hat{\lambda}_I + 2\hat{\mu}_I}{\omega^2}$ and $\beta := \frac{\hat{\mu}_I}{\omega^2}$ as in [12]

$$\left(\frac{1}{|\xi|^2\beta}\right)^3 - 8\left(\frac{1}{|\xi|^2\beta}\right)^2 + 8\left(3 - 2\frac{\beta}{\alpha}\right)\left(\frac{1}{|\xi|^2\beta}\right) - 16\left(1 - \frac{\beta}{\alpha}\right) = 0. \quad (3.4)$$

We fix $\alpha = (\hat{\lambda} + 2\hat{\mu})/(\omega^2) = 1$ and let $\beta = \hat{\mu}/(\omega^2)$ vary. Since we are in the constant case, the ratio between the two Lamé parameters, which are constant, is fixed, and by varying β, the frequency ω of the seismic (incident) wave varies. We show the plot of the roots of (3.4) in Figure 3.1 for the intervals $0.1 < \beta < 0.32$, where we have one real root and two complex conjugate roots. In Figure 3.2, we consider the interval $0.5 < \beta < 0.8$, where we have three complex roots:

- for $0.1 < \beta < 0.32$ (which means lower frequencies) we get one real value of ξ^2 (larger than the branching point at $\omega/\sqrt{\hat{\mu}}$) and two complex conjugate values of ξ^2 as in Figure 3.1. The real root of ξ^2 corresponds to real quasi-momenta q_P and q_S in the physical sheet, hence the solution to the problem (eigenfunction) corresponds to a wave with wave number ξ. The two complex conjugate roots of ξ^2 correspond to complex values of the quasi-momenta q_P and q_S. The corresponding solutions are resonant waves.

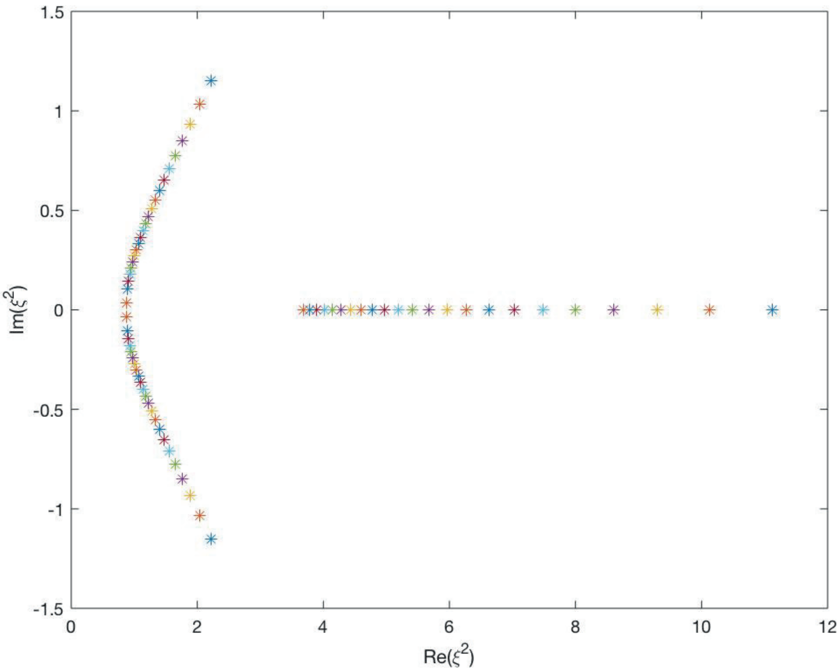

Figure 3.1: The figure shows the roots of (3.4). For each fixed value of β in the range $(0.1, 0.32)$ the corresponding roots of (3.4) are denoted with a star of the same colour.

- for $0.5 < \beta < 0.8$ we get three complex solutions ξ^2 as in Figure 3.2. Hence q_P and q_S are imaginary and the corresponding solutions are resonant waves.

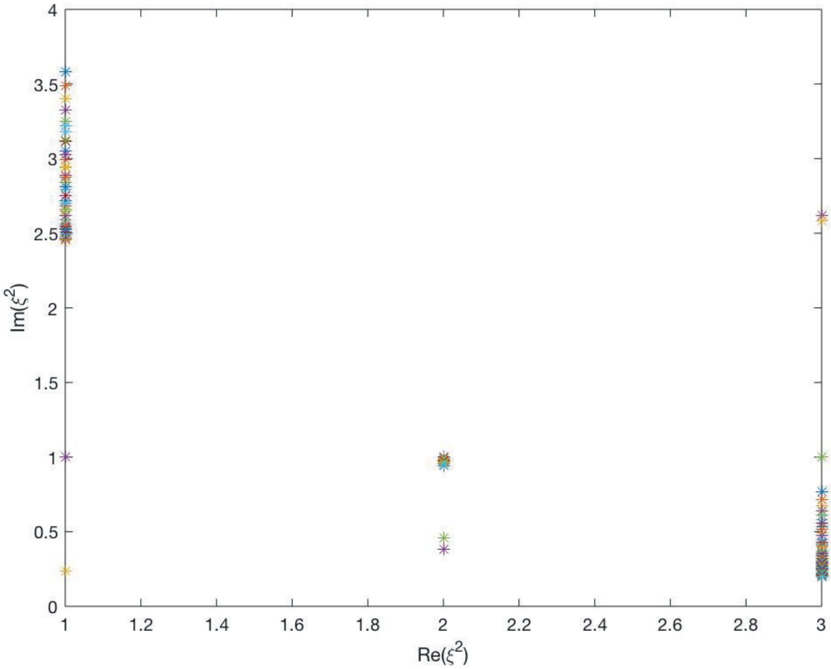

Figure 3.2: The figure shows the roots of (3.4) for each fixed value of β in the range $(0.5, 0.8)$ which are denoted with a star point of the same colour.

3.4 Riemann surface and mappings

We make an analytic continuation of $|\xi|$ to the whole complex plane $\mathbb{C}$ and denote it by ξ. Let the quasi-momenta q_P and q_S be defined as

$$q_P := i\sqrt{\xi^2 - \frac{\omega^2}{\hat{\lambda}_I + 2\hat{\mu}_I}}, \qquad q_S := i\sqrt{\xi^2 - \frac{\omega^2}{\hat{\mu}_I}}.$$

We introduce the branch cuts for q_P along $\left[-\frac{\omega}{\sqrt{\hat{\lambda}_I+2\hat{\mu}_I}}, \frac{\omega}{\sqrt{\hat{\lambda}_I+2\hat{\mu}_I}}\right] \cup i\mathbb{R}$ and for q_S along $\left[-\frac{\omega}{\sqrt{\hat{\mu}_I}}, \frac{\omega}{\sqrt{\hat{\mu}_I}}\right] \cup i\mathbb{R}$ as in Figure 3.3 and we choose the branches of the square root such that $q_P(\xi), q_S(\xi) \in i\mathbb{R}_+$ when $\xi > \frac{\omega}{\hat{\mu}_I}$ is real. Let $\bullet = P, S$, then $q_\bullet(\xi) \in \mathbb{C}_+$ when ξ belongs to the first sheet of the Riemann surface of $q_\bullet(\xi)$ and $q_\bullet(\xi) \in \mathbb{C}_-$ when ξ belongs to the second sheet of the Riemann surface of $q_\bullet(\xi)$. We then consider q_P, q_S defined on the joint four-sheeted Riemann surface Ξ, defined so that q_P and q_S are single-valued and holomorphic. The Riemann surface Ξ is obtained by gluing together

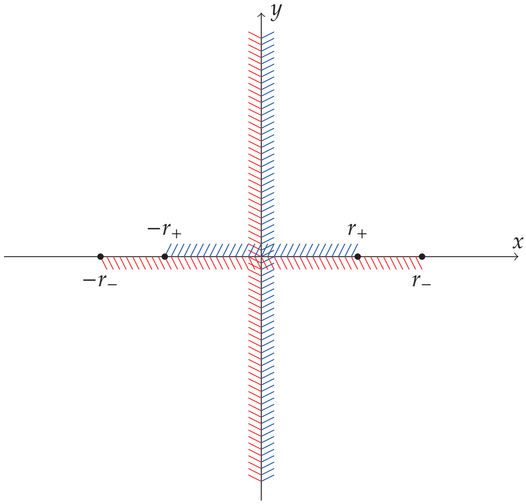

Figure 3.3: We show the branch cuts in a single sheet of Ξ. In the figure $r_- := \frac{\omega}{\sqrt{\hat{\mu}_I}}$ and $r_+ := \frac{\omega}{\sqrt{\hat{\lambda}_I+2\hat{\mu}_I}}$. Blue indicates the branch cuts for the quasi-momentum q_P, while red indicates the branch cuts for the quasi-momentum q_S.

the following sheets:

$$\Xi_{\pm,\pm} := \{\xi : \pm \operatorname{Im} q_P(\xi) > 0, \pm \operatorname{Im} q_S(\xi) > 0\}.$$

We denote by $-\xi$ the point in Ξ belonging to the same sheet as ξ obtained by reflecting ξ with respect to the origin, as in Figure 3.4. Without loss of generality, we start from a point ξ on the sheet $\Xi_{+,+}$ such that $q_\bullet(\xi) \in \mathbb{R}_+$ for $\bullet = P, S$. We approach the imaginary line avoiding the points $r_- := \frac{\omega}{\sqrt{\hat{\mu}_I}}$ and $r_+ := \frac{\omega}{\sqrt{\hat{\lambda}_I+2\hat{\mu}_I}}$, where the quasi-momenta q_P and q_S are not holomorphic. At points $x - i0$, $x \in \left(-\frac{\omega}{\sqrt{\hat{\lambda}_I+2\hat{\mu}_I}}, \frac{\omega}{\sqrt{\hat{\lambda}_I+2\hat{\mu}_I}}\right)$ we have $q_\bullet(x-i0) \in \mathbb{R}_+$. When we pass through the imaginary line we end up on the sheet $\Xi_{-,-}$ (dashed line in Figure3.4) and then we return to the sheet $\Xi_{+,+}$ after passing through the cut $\left[-\frac{\omega}{\sqrt{\hat{\lambda}_I+2\hat{\mu}_I}}, \frac{\omega}{\sqrt{\hat{\lambda}_I+2\hat{\mu}_I}}\right]$. Finally, we reach the point $-\xi$ avoiding again $-r_-$ and $-r_+$, as shown in Figure 3.4.

Since the rotations around $r_\pm$ and $-r_\mp$ are in opposite directions we see that $q_\bullet(-\xi) = q_\bullet(\xi)$, $\bullet = P, S$. Hence, it suffices to study q_P and q_S for $\xi \in \Xi$ such that $\operatorname{Re}\xi \geq 0$. For large ξ with $\operatorname{Re}\xi \geq 0$ we can write

$$\begin{aligned}
q_P(\pm\xi) &= i\xi + O(|\xi|^{-1}), && \xi \in \Xi_{+,\pm}, \operatorname{Re}\xi \geq 0,\\
q_P(\pm\xi) &= -i\xi + O(|\xi|^{-1}), && \xi \in \Xi_{-,\pm}, \operatorname{Re}\xi \geq 0,\\
q_S(\pm\xi) &= i\xi + O(|\xi|^{-1}), && \xi \in \Xi_{\pm,+}, \operatorname{Re}\xi \geq 0,
\end{aligned}$$

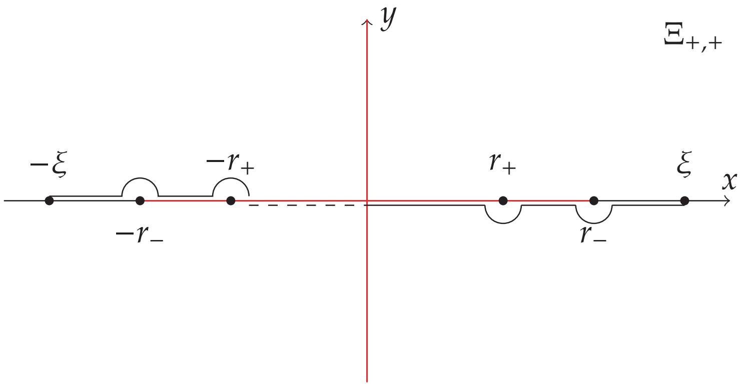

Figure 3.4: Reflection from ξ to $-\xi$ in the physical sheet $\Xi_{+,+}$. The red lines represent the cuts of the Riemann sheets.

$$q_S(\pm\xi) = -i\xi + O(|\xi|^{-1}), \qquad \xi \in \Xi_{\pm,-}, \mathrm{Re}\,\xi \geq 0. \tag{3.5}$$

For example if $\xi \in \Xi_{+,+}$, but $\mathrm{Re}\,\xi < 0$, then we have $q_P(\pm\xi) = -i\xi + O(|\xi|^{-1})$ instead, and equivalently for q_S.

We can define the mappings $w_P, w_S, w_{SP} : \Xi \to \Xi$, where each mapping applied to ξ can change the sign of either one or both the imaginary parts of the quasi-momenta, hence it maps points from one to another fold of the Riemann surface (see [13]). These mappings operate according to the rules

$$q_S(w_S(\xi)) = -q_S(\xi), \qquad q_P(w_S(\xi)) = q_P(\xi); \tag{3.6}$$
$$q_S(w_P(\xi)) = q_S(\xi), \qquad q_P(w_P(\xi)) = -q_P(\xi); \tag{3.7}$$
$$q_S(w_{SP}(\xi)) = -q_S(\xi), \qquad q_P(w_{SP}(\xi)) = -q_P(\xi). \tag{3.8}$$

The function $q_S(\xi)$ maps the horizontal cut $\left[-\frac{\omega}{\sqrt{\hat{\mu}_I}}, \frac{\omega}{\sqrt{\hat{\mu}_I}}\right]$ onto the horizontal slit $\left[-\frac{\omega}{\sqrt{\hat{\mu}_I}}, \frac{\omega}{\sqrt{\hat{\mu}_I}}\right]$ and the imaginary line $i\mathbb{R}$ onto the complementary part of that slit, for example

$$q_S\left(\left[0 - i0, \frac{\omega}{\sqrt{\hat{\mu}_I}} - i0\right)\right) = \left(0, \frac{\omega}{\sqrt{\hat{\mu}_I}}\right],$$
$$q_S\left(\left[0 + i0, \frac{\omega}{\sqrt{\hat{\mu}_I}} + i0\right)\right) = \left[-\frac{\omega}{\sqrt{\hat{\mu}_I}}, 0\right)$$

and equivalently for q_P after adjusting the branching point accordingly. For any value of ξ it holds that

$$q_\bullet(\xi) = -\overline{q_\bullet(\bar{\xi})}, \tag{3.9}$$

where the conjugation of ξ is defined as a normal conjugation in a single sheet of Ξ by contours not passing through the cuts.

Lemma 3.4.1. *We have*

1. $\operatorname{Im}(q_P + q_S) > 0$ *and* $\operatorname{Im}(q_P - q_S) > 0$ *in* $\Xi_{+,\pm}$,

2. $\operatorname{Im}(q_P + q_S) < 0$ *and* $\operatorname{Im}(q_P - q_S) < 0$ *in* $\Xi_{-,\pm}$.

Proof. We know that

$$\operatorname{Im}(q_P + q_S) = \operatorname{Im}\left[\frac{(q_P + q_S)(q_P - q_S)}{q_P - q_S}\right] = (q_P^2 - q_S^2)\operatorname{Im}\left(\frac{1}{q_P - q_S}\right)$$
$$= -\frac{\omega^2(\hat{\lambda} + \hat{\mu})}{\hat{\mu}(\hat{\lambda} + 2\hat{\mu})}\operatorname{Im}\left(\frac{1}{q_P - q_S}\right).$$

Since $\frac{\omega^2(\hat{\lambda}+\hat{\mu})}{\hat{\mu}(\hat{\lambda}+2\hat{\mu})} > 0$ and $\operatorname{sgn}(\operatorname{Im}(1/z)) = -\operatorname{sgn}(\operatorname{Im}(z))$, then $\operatorname{sgn}(\operatorname{Im}(q_P + q_S)) = \operatorname{sgn}(\operatorname{Im}(q_P - q_S))$. Since $\operatorname{Im}(q_P + q_S) > 0$ in $\Xi_{+,+}$ and $\operatorname{Im}(q_P + q_S) < 0$ in $\Xi_{-,-}$, while $\operatorname{Im}(q_P - q_S) > 0$ in $\Xi_{+,-}$ and $\operatorname{Im}(q_P - q_S) < 0$ in $\Xi_{-,+}$, the lemma follows. □

3.5 Parity properties

We define the differential operators $a = a(Z, D_Z, \xi)$ and $b = b(Z, D_Z, \xi)$ such that if we apply them to the function Φ, which belongs in the space of the solutions of $(H - \omega^2)\Phi = 0$, it gives

$$a(\Phi)(Z,\xi) := i\xi\hat{\lambda}(Z)\varphi_1(Z) + \left(\hat{\lambda}(Z) + 2\hat{\mu}(Z)\right)\frac{\partial \varphi_3}{\partial Z}(Z),$$
$$b(\Phi)(Z,\xi) = i\xi\hat{\mu}(Z)\varphi_3(Z) + \hat{\mu}(Z)\frac{\partial \varphi_1}{\partial Z}(Z)$$

so $a_-(\Phi) = a(\Phi)(0^-, \xi)$ and $b_-(\Phi) = b(\Phi)(0^-, \xi)$ with boundary conditions $a(\Phi)$ and $b(\Phi)$ given by (3.2). Since the differential operator H is invariant under the transformation $(Z, \xi) \to (-Z, -\xi)$, if $f_P^{\pm}(Z, \xi)$ (or $f_S^{\pm}(Z, \xi)$) is a solution of the differential equation, then also $f_P^{\pm}(-Z, -\xi)$ (or $f_S^{\pm}(-Z, -\xi)$) is a solution of the equation. The unperturbed solutions $f_{P,0}^{\pm}(Z, \xi)$, $f_{S,0}^{\pm}(Z, \xi)$ defined as

$$f_{P,0}^{\pm} = \begin{pmatrix} |\xi| \\ \pm q_P \end{pmatrix} e^{\pm iZq_P}, \qquad q_P := i\sqrt{\xi^2 - \frac{\omega^2}{\hat{\lambda}_I + 2\hat{\mu}_I}},$$
$$f_{S,0}^{\pm} = \begin{pmatrix} \pm q_S \\ -|\xi| \end{pmatrix} e^{\pm iZq_S}, \qquad q_S := i\sqrt{\xi^2 - \frac{\omega^2}{\hat{\mu}_I}},$$

coincide with the Jost solutions of the problem $f_P^{\pm}(Z,\xi)$, $f_S^{\pm}(Z,\xi)$ for $Z \le Z_I$, by definition of the Jost solution. For the unperturbed solutions for $\xi \in \Xi$, it holds that

$$\begin{aligned} f_{P,0}^{\pm}(-Z,-\xi) &= -f_{P,0}^{\mp}(Z,\xi), \qquad & f_{S,0}^{\pm}(-Z,-\xi) &= -f_{S,0}^{\mp}(Z,\xi), \\ \overline{f_{P,0}^{\pm}(Z,-\overline{\xi})} &= -f_{P,0}^{\pm}(Z,\xi), \qquad & \overline{f_{S,0}^{\pm}(Z,-\overline{\xi})} &= -f_{S,0}^{\pm}(Z,\xi) \end{aligned}$$

where we used (3.9). When $\xi \in \left[-\frac{\omega}{\sqrt{\hat{\lambda}_I+2\hat{\mu}_I}}, \frac{\omega}{\sqrt{\hat{\lambda}_I+2\hat{\mu}_I}}\right]$ with ω real, then q_S, q_P and ξ are real, and we have

$$\overline{f_{P,0}^{\pm}(-Z,\xi)} = f_{P,0}^{\pm}(Z,\xi), \qquad \overline{f_{S,0}^{\pm}(-Z,\xi)} = f_{S,0}^{\pm}(Z,\xi).$$

While if $\xi \in \left[-\frac{\omega}{\sqrt{\hat{\mu}_I}}, \frac{\omega}{\sqrt{\hat{\mu}_I}}\right] \setminus \left[-\frac{\omega}{\sqrt{\hat{\lambda}_I+2\hat{\mu}_I}}, \frac{\omega}{\sqrt{\hat{\lambda}_I+2\hat{\mu}_I}}\right]$, ξ and q_S are real, and q_P is pure imaginary, so the previous properties hold only for $f_{S,0}^{\pm}$:

$$\overline{f_{S,0}^{\pm}(-Z,\xi)} = f_{S,0}^{\pm}(Z,\xi).$$

In the following lemma we show the symmetries of the Jost solution. The idea of looking for symmetry was inspired by [15], where they find symmetries of the reflection coefficients instead.

Lemma 3.5.1. *On the Riemann surface Ξ the following properties hold:*

$$f_P^{\pm}(-Z,-\xi) = -f_P^{\mp}(Z,\xi), \quad f_S^{\pm}(-Z,-\xi) = -f_S^{\mp}(Z,\xi), \qquad \textit{for } \xi \in \Xi; \quad (3.10)$$

$$\overline{f_P^{\pm}(Z,-\overline{\xi})} = -f_P^{\pm}(Z,\xi), \qquad \overline{f_S^{\pm}(Z,-\overline{\xi})} = -f_S^{\pm}(Z,\xi), \qquad \textit{for } \xi \in \Xi. \quad (3.11)$$

Using the projection mappings to the sheets of Ξ we get

$$\begin{aligned} & f_P^{\pm}(Z,w_P(\xi)) = f_P^{\pm}(Z,w_{PS}(\xi)) = f_P^{\mp}(Z,\xi), \\ & f_S^{\pm}(Z,w_S(\xi)) = f_S^{\pm}(Z,w_{PS}(\xi)) = f_S^{\mp}(Z,\xi), \\ & f_P^{\pm}(Z,w_S(\xi)) = f_P^{\pm}(Z,\xi) \\ & f_S^{\pm}(Z,w_P(\xi)) = f_S^{\pm}(Z,\xi). \end{aligned} \qquad (3.12)$$

Proof. We first prove the property (3.10). The operator H is invariant under the transformation $Z \mapsto -Z$ and $\xi \mapsto -\xi$, that means

$$H(-Z, D_{-Z}, D^2_{-Z}, -\xi) = H(Z, D_Z, D^2_Z, \xi).$$

From now on, we will denote $H(Z, D_Z, D_Z^2, \xi)$ as $H(Z, \xi)$. We know that f_P or f_S satisfy the equation $H(Z,\xi) f_{P,S}(Z,\xi) = 0$, and

$$\begin{aligned} H(Z,\xi) f_{P,S}(Z,\xi) = 0 \quad &\Longleftrightarrow \quad H(-Z,-\xi) f_{P,S}(-Z,-\xi) = 0 \\ \Longleftrightarrow \quad & H(Z,\xi) f_{P,S}(-Z,-\xi) = 0, \end{aligned}$$

which means that $f_{P,S}(-Z,-\xi)$ is also a solution of the differential equation. Since H is an ODE operator and since the solutions $f_{P,S}(Z,\xi)$ and $f_{P,S}(-Z,-\xi)$ coincide for $]-\infty, -Z_I] \cup [Z_I, \infty[$, they must coincide for all $Z \in \mathbb{R}$. We can also prove the invariance of H under the transformation $H(Z,\xi)$ to $\overline{H(Z,-\overline{\xi})}$ (following from the property $\left(\overline{\xi}\right)^2 = \overline{(\xi)^2}$) and the invariance under all the other transformations:

$$\begin{aligned} H(Z,\xi) &= H(-Z,-\xi) = \overline{H(Z,-\overline{\xi})} && \text{for } \xi \in \Xi \\ H(Z,\xi) &= \overline{H(-Z,\xi)} && \text{for } \xi \in \Xi. \end{aligned}$$

The invariance of the differential operator under these transformations allows us, following the previous reasoning, to extend those properties of the Jost solution from $]-\infty, -Z_I] \cup [Z_I, \infty[$ to the whole real line.

We can prove (3.12) in the same way, because $H(Z,\xi)$ is invariant under the projection mappings w_S, w_P and $w_{P,S}$ as it is independent on the quasi-momenta q_S and q_P, while the unperturbed solutions $f_{\bullet,0}$, $\bullet = P, S$, satisfy those properties. □

Looking at the boundary conditions (3.2), we define

$$B(Z,\xi) := B(Z, D_Z, \xi) = \begin{pmatrix} i\hat{\lambda}\xi & \left(\hat{\lambda} + 2\hat{\mu}\right)\frac{\partial}{\partial Z} \\ \hat{\mu}\frac{\partial}{\partial Z} & i\hat{\mu}\xi \end{pmatrix}. \tag{3.13}$$

Then (3.2) is equivalent to

$$B(Z,\xi) \left(\begin{array}{c} \varphi_1(Z,\xi) \\ \varphi_3(Z,\xi) \end{array} \right)\Bigg|_{Z=0} = 0.$$

Thanks to the fact that the Lamé parameters are even, we can obtain the following properties of $B(Z,\xi)$:

$$\begin{aligned} B(-Z,-\xi) &= -B(Z,\xi), && \text{for} \quad \xi \in \Xi, \\ \overline{B(Z,-\overline{\xi})} &= B(Z,\xi), && \text{for} \quad \xi \in \Xi. \end{aligned}$$

In the next proposition we show that if the operators $H(Z,\xi)$ and $B(Z,\xi)$ have a certain symmetry, then the solutions to the boundary value problem 3.1–(3.2) also have the same symmetry.

Proposition 3.5.2. *Suppose $\varphi(Z,\xi)$ is a solution of $(H(Z,\xi)-\omega^2)\varphi(Z,\xi) = 0$ that satisfies the boundary conditions* (3.2). *Then $\varphi(Z,\xi)$ must be a linear combination of the Jost solutions $f_{P,S}(Z,\xi)$ and if there exists a transformation T of the variable Z and ξ such that*

$$\begin{aligned} &B(T(Z,\xi)) = \pm B(Z,\xi) \\ &H(T(Z,\xi)) - \omega^2 = \pm\left(H(Z,\xi)-\omega^2\right), \end{aligned}$$

then $\varphi(T(Z,\xi))$ is also a solution to the boundary value problem.

Proof. Let $\varphi(Z,\xi)$ be a solution to

$$\begin{aligned} (H(Z,\xi)-\omega^2)\varphi(Z,\xi) &= 0, \\ B(Z,\xi)\varphi(Z,\xi)|_{Z=0} &= 0. \end{aligned}$$

Then

$$\begin{aligned} 0 &= \pm(H(Z,\xi)-\omega^2)\varphi(T(Z,\xi)) = (H(T(Z,\xi))-\omega^2)\varphi(T(Z,\xi)), \\ 0 &= \pm\, B(Z,\xi)\varphi(T(Z,\xi))|_{Z=0} = B(T(Z,\xi))\varphi(T(Z,\xi))|_{Z=0}\,, \end{aligned}$$

so $\varphi(T(Z,\xi))$ is also a solution to the boundary value problem. □

The previous proposition and Lemma 3.5.1 motivates the following lemma, that shows the symmetries of the boundary conditions applied to the Jost solutions.

Lemma 3.5.3. *For $\xi \in \Xi$, we have*

$$\begin{aligned} a(f_P^{\pm})(-\xi) &= a(f_P^{\mp})(\xi), \qquad & b(f_P^{\pm})(-\xi) &= b(f_P^{\mp})(\xi), \\ a(f_S^{\pm})(-\xi) &= a(f_S^{\mp})(\xi), \qquad & b(f_S^{\pm})(-\xi) &= b(f_S^{\mp})(\xi), \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \overline{a(f_P^{\pm})(-\bar{\xi})} &= -a(f_P^{\pm})(\xi), \qquad & \overline{b(f_P^{\pm})(-\bar{\xi})} &= -b(f_P^{\pm})(\xi), \\ \overline{a(f_S^{\pm})(-\bar{\xi})} &= -a(f_S^{\pm})(\xi), \qquad & \overline{b(f_S^{\pm})(-\bar{\xi})} &= -b(f_S^{\pm})(\xi). \end{aligned} \tag{3.15}$$

Using the projection mappings to the sheets of Ξ we get

$$\begin{aligned} a(f_S^{\pm})(w_S(\xi)) &= a(f_S^{\pm})(w_{PS}(\xi)) = a(f_S^{\mp})(\xi), \\ b(f_S^{\pm})(w_S(\xi)) &= b(f_S^{\pm})(w_{PS}(\xi)) = b(f_S^{\mp})(\xi), \\ a(f_P^{\pm})(w_P(\xi)) &= a(f_P^{\pm})(w_{PS}(\xi)) = a(f_P^{\mp})(\xi), \\ b(f_P^{\pm})(w_P(\xi)) &= b(f_P^{\pm})(w_{PS}(\xi)) = b(f_P^{\mp})(\xi). \end{aligned} \tag{3.16}$$

Proof. We divide the proof of the lemma into the individual proofs of each parity property as follows.

- In order to prove (3.14) consider $\xi \in \Xi$ and calculate

$$a(f_{P,S}^{\pm})(-Z,-\xi) = i\hat{\lambda}(-\xi) f_{P,S}^{\pm,(1)}(-Z,-\xi) + \left(\hat{\lambda} + 2\hat{\mu}\right)\left[-\frac{\partial}{\partial Z} f_{P,S}^{\pm,(2)}(-Z,-\xi)\right]$$
$$= i\hat{\lambda}\xi f_{P,S}^{\mp,(1)}(Z,\xi) + \left(\hat{\lambda} + 2\hat{\mu}\right)\left[\frac{\partial}{\partial Z} f_{P,S}^{\mp,(2)}(Z,\xi)\right] = a(f_{P,S}^{\mp})(Z,\xi)$$

  using (3.10) and where $f_{P,S}^{\pm,(1)}$ and $f_{P,S}^{\pm,(2)}$ are respectively the first and the second component of the vector $f_{P,S}^{\pm}$. Hence

$$a(f_{P,S}^{\pm})(-\xi) = a(f_{P,S}^{\pm})(-Z,-\xi)\Big|_{Z=0} = a(f_{P,S}^{\mp})(Z,\xi)\Big|_{Z=0} = a(f_{P,S}^{\mp})(\xi).$$

  In order to prove the other two properties of (3.14), we start from

$$b(f_{P,S}^{\pm})(-Z,-\xi) = i\hat{\mu}(-\xi) f_{P,S}^{\pm,(2)}(-Z,-\xi) + \hat{\mu}\left[-\frac{\partial}{\partial Z} f_{P,S}^{\pm,(1)}(-Z,-\xi)\right]$$
$$= i\hat{\mu}\xi f_{P,S}^{\mp,(2)}(Z,\xi) + \hat{\mu}\left[\frac{\partial}{\partial Z} f_{P,S}^{\mp,(1)}(Z,\xi)\right] = b(f_{P,S}^{\mp})(Z,\xi)$$

  and again using (3.10) we arrive at

$$b(f_{P,S}^{\pm})(-\xi) = b(f_{P,S}^{\pm})(-Z,-\xi)\Big|_{Z=0} = b(f_{P,S}^{\mp})(Z,\xi)\Big|_{Z=0} = b(f_{P,S}^{\mp})(\xi).$$

- In order to prove (3.15) we consider

$$\overline{a(f_{P,S}^{\pm})(Z,-\overline{\xi})} = -i\hat{\lambda}(-\xi)\overline{f_{P,S}^{\pm,(1)}(Z,-\overline{\xi})} + \left(\hat{\lambda} + 2\hat{\mu}\right)\left[\frac{\partial}{\partial Z}\overline{f_{P,S}^{\pm,(2)}(Z,-\overline{\xi})}\right]$$
$$= -i\hat{\lambda}\xi f_{P,S}^{\pm,(1)}(Z,\xi) - \left(\hat{\lambda} - 2\hat{\mu}\right)\left[\frac{\partial}{\partial Z} f_{P,S}^{\pm,(2)}(Z,\xi)\right] = -a(f_{P,S}^{\pm})(Z,\xi)$$

  where we have used (3.11). Hence

$$\overline{a(f_{P,S}^{\pm})(-\overline{\xi})} = \overline{a(f_{P,S}^{\pm})(Z,-\overline{\xi})}\Big|_{Z=0} = -\,a(f_{P,S}^{\pm})(Z,\xi)\Big|_{Z=0} = -a(f_{P,S}^{\pm})(\xi).$$

  For the other boundary condition we have

$$\overline{b(f_{P,S}^{\pm})(Z,-\overline{\xi})} = -i\hat{\mu}(-\xi)\overline{f_{P,S}^{\pm,(2)}(Z,-\overline{\xi})} + \hat{\mu}\left[\frac{\partial}{\partial Z}\overline{f_{P,S}^{\pm,(1)}(Z,-\overline{\xi})}\right]$$

$$= -i\hat{\mu}\xi f_{P,S}^{\pm,(2)}(Z,\xi) - \hat{\mu}\left[\frac{\partial}{\partial Z} f_{P,S}^{\pm,(1)}(Z,\xi)\right] = -b(f_{P,S}^{\pm})(Z,\xi),$$

and hence

$$\overline{b(f_{P,S}^{\pm})(-\overline{\xi})} = \overline{b(f_{P,S}^{\pm})(Z,-\overline{\xi})}\Big|_{Z=0} = -b(f_{P,S}^{\pm})(Z,\xi)\Big|_{Z=0} = -b(f_{P,S}^{\pm})(\xi).$$

- The proof of (3.16) is straightforward from (3.12) as the coefficients in the boundary conditions do not depend on the quasi-momenta q_P and q_S. $\square$

3.6 Representation of the Jost solution

In the homogeneous case $\hat{\lambda}(Z) = \hat{\lambda}_I$, $\hat{\mu}(Z) = \hat{\mu}_I$, we define:

$$\theta_{P,0} = \frac{1}{2}(f_{P,0}^{+} + f_{P,0}^{-}), \qquad \varphi_{P,0} = \frac{1}{2q_P}(f_{P,0}^{+} - f_{P,0}^{-}),$$
$$\theta_{S,0} = \frac{1}{2}(f_{S,0}^{+} + f_{S,0}^{-}), \qquad \varphi_{S,0} = \frac{1}{2q_S}(f_{S,0}^{+} - f_{S,0}^{-}),$$

and in particular we get

$$\theta_{P,0} = \begin{pmatrix} \xi\cos(q_P Z) \\ iq_P\sin(q_P Z) \end{pmatrix}, \qquad \varphi_{P,0} = \begin{pmatrix} \frac{i\xi}{q_P}\sin(q_P Z) \\ \cos(q_P Z) \end{pmatrix}, \tag{3.17}$$

and

$$\theta_{S,0} = \begin{pmatrix} iq_S\sin(q_S Z) \\ -\xi\cos(q_S Z) \end{pmatrix}, \qquad \varphi_{S,0} = \begin{pmatrix} \cos(q_S Z) \\ -\frac{i\xi}{q_S}\sin(q_S Z) \end{pmatrix}, \tag{3.18}$$

which satisfy the following boundary conditions

$$a(\theta_{P,0}) = i\hat{\mu}_I\left(\frac{\omega^2}{\hat{\mu}_I} - 2\xi^2\right), \qquad b(\theta_{P,0}) = 0,$$
$$a(\theta_{S,0}) = 0, \qquad b(\theta_{S,0}) = i\hat{\mu}_I\left(\frac{\omega^2}{\hat{\mu}_I} - 2\xi^2\right),$$
$$a(\varphi_{P,0}) = 0, \qquad b(\varphi_{P,0}) = 2i\hat{\mu}_I\xi,$$
$$a(\varphi_{S,0}) = -2i\hat{\mu}_I\xi, \qquad b(\varphi_{S,0}) = 0.$$

By expanding $\sin(q_{S,P}Z)$ and $\cos(q_{S,P}Z)$ in series, we see that the functions $\theta_{P,0}^{\pm}$, $\theta_{S,0}^{\pm}$, $\varphi_{P,0}^{\pm}$, and $\varphi_{S,0}^{\pm}$ are entire for $\xi \in \mathbb{C}$ as

$$\theta_{P,0} = \begin{pmatrix} \xi\sum_{n=0}^{\infty}\frac{(-1)^n(q_P Z)^{2n}}{2n!} \\ i\sum_{n=0}^{\infty}\frac{(-1)^n(q_P)^{2n+2}(Z)^{2n+1}}{(2n+1)!} \end{pmatrix}, \; \varphi_{P,0} = \begin{pmatrix} i\xi\sum_{n=0}^{\infty}\frac{(-1)^n(q_P)^{2n}Z^{2n+1}}{(2n+1)!} \\ \sum_{n=0}^{\infty}\frac{(-1)^n(q_P Z)^{2n}}{2n!} \end{pmatrix}$$

and

$$\theta_{S,0} = \begin{pmatrix} i\sum_{n=0}^{\infty} \frac{(-1)^n (q_S)^{2n+2} (Z)^{2n+1}}{(2n+1)!} \\ -\xi \sum_{n=0}^{\infty} \frac{(-1)^n (q_S Z)^{2n}}{2n!} \end{pmatrix}, \ \varphi_{S,0} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n (q_S Z)^{2n}}{2n!} \\ -i\xi \sum_{n=0}^{\infty} \frac{(-1)^n (q_S)^{2n} Z^{2n+1}}{(2n+1)!} \end{pmatrix}.$$

Indeed, we see that ξ never appears inside a square root, but only as powers with natural exponents. Since we can expand those auxiliary functions as a uniformly convergent power series on compact sets, they are analytic in ξ and even[2] in q_P and q_S. We can express the Jost solution $f^{\pm}_{P,0}$ and $f^{\pm}_{S,0}$ in terms of the functions $\theta^{\pm}_{P,0}$, $\theta^{\pm}_{S,0}$, $\varphi^{\pm}_{P,0}$, and $\varphi^{\pm}_{S,0}$:

$$f^{\pm}_{P,0} = \theta_{P,0} \pm q_P \varphi_{P,0}, \qquad f^{\pm}_{S,0} = \theta_{S,0} \pm q_S \varphi_{S,0}. \tag{3.19}$$

Similarly we can extend (3.19) to the inhomogeneous case, by uniqueness of the solutions $\theta_{P,0}$, $\theta_{S,0}$, $\varphi_{P,0}$, $\varphi_{P,0}$, as

$$f^{\pm}_{P} = \theta_{P} \pm q_P \varphi_{P}, \qquad f^{\pm}_{S} = \theta_{S} \pm q_S \varphi_{S},$$

and

$$\theta_P = \frac{1}{2}(f^+_P + f^-_P), \qquad \varphi_P = \frac{1}{2q_P}(f^+_P - f^-_P), \tag{3.20}$$

$$\theta_S = \frac{1}{2}(f^+_S + f^-_S), \qquad \varphi_S = \frac{1}{2q_S}(f^+_S - f^-_S), \tag{3.21}$$

where the function $\theta_{P,S}$ and $\varphi_{P,S}$ satisfy the conditions

$$\begin{aligned} \theta_P &= \theta_{P,0}, \qquad \varphi_P = \varphi_{P,0}, \qquad Z \le Z_I \\ \theta_S &= \theta_{S,0}, \qquad \varphi_S = \varphi_{S,0}, \qquad Z \le Z_I. \end{aligned} \tag{3.22}$$

This motivates the following definition.

Definition 3.6.1 (Auxiliary functions). *We define the auxiliary functions $\theta_\bullet$ and $\varphi_\bullet$, with $\bullet = P, S$, the unique solutions of* (1.10)–(1.11) *satisfying the conditions*

$$\begin{aligned} \theta_P &= \theta_{P,0}, \qquad \varphi_P = \varphi_{P,0}, \qquad Z \le Z_I \\ \theta_S &= \theta_{S,0}, \qquad \varphi_S = \varphi_{S,0}, \qquad Z \le Z_I. \end{aligned} \tag{3.23}$$

with $\theta_{\bullet,0}$ and $\varphi_{\bullet,0}$, $\bullet = P, S$, given by (3.17) *and* (3.18).

This implies that the Jost solutions $f^{\pm}_P$ and $f^{\pm}_S$ are uniquely determined by those auxiliary functions $\theta_{P,0}$, $\theta_{S,0}$, $\varphi_{P,0}$, $\varphi_{P,0}$. For those auxiliary functions we have the following results.

---

[2] They are even in q_P and q_S as there appear only even powers of q_P and q_S in the expansion.

Lemma 3.6.2. *The function θ_P, θ_S, φ_P and φ_S are entire and invariant under the mappings $w_P(\xi)$, $w_S(\xi)$, $w_{PS}(\xi)$:*

$$\begin{aligned}
\theta_P(\xi) &= \theta_P(w_P(\xi)) = \theta_P(w_S(\xi)) = \theta_P(w_{PS}(\xi)),\\
\theta_S(\xi) &= \theta_S(w_P(\xi)) = \theta_S(w_S(\xi)) = \theta_S(w_{PS}(\xi)),\\
\varphi_P(\xi) &= \varphi_P(w_P(\xi)) = \varphi_P(w_S(\xi)) = \varphi_P(w_{PS}(\xi)),\\
\varphi_S(\xi) &= \varphi_S(w_P(\xi)) = \varphi_S(w_S(\xi)) = \varphi_S(w_{PS}(\xi)).
\end{aligned}$$

Proof. The functions θ_P, θ_S, φ_P and φ_S are the unique solutions of an ordinary differential equation (1.10)–(1.11) with analytic dependence on the parameter ξ and satisfying the boundary conditions (3.23), where $\theta_{P,0}$, $\theta_{S,0}$, $\varphi_{P,0}$ and $\varphi_{S,0}$ are entire in ξ. Then θ_P, θ_S, φ_P and φ_S are also entire (see [52, Theorem 2.5.1] or [14, Theorem 8.4, Chapter 1]).

The invariance under the mapping $w_\bullet$, with $\bullet = P, S, PS$, is a consequence of the fact that we can express those auxiliary functions as even powers of the quasi-momenta q_P and q_S, knowing that the mapping w_P, w_S and w_{PS} only change the sign of the quasi-momenta. □

Since the auxiliary functions are written in terms of the Jost solutions (see Definition 3.6.1), we can translate Lemma 3.5.3 in terms of the auxiliary functions as in the following lemmas.

Lemma 3.6.3. *The functions $a(\theta_P), b(\theta_P), a(\theta_S), b(\theta_S)$ are entire and even in ξ in the sense that*

$$\begin{aligned}
a(\theta_{P,S})(-\xi) &= a(\theta_{P,S})(\xi),\\
b(\theta_{P,S})(-\xi) &= b(\theta_{P,S})(\xi),
\end{aligned}$$

while the functions $a(\varphi_P), b(\varphi_P), a(\varphi_S), b(\varphi_S)$ are entire but odd in ξ, namely

$$\begin{aligned}
a(\varphi_{P,S})(-\xi) &= -a(\varphi_{P,S})(\xi)\\
b(\varphi_{P,S})(-\xi) &= -b(\varphi_{P,S})(\xi).
\end{aligned}$$

Proof. By Lemma 3.6.2 and since the boundary conditions (1.12)–(1.13) have analytic dependence on the parameter ξ, then $a(\theta_\bullet), b(\theta_\bullet), a(\varphi_\bullet), b(\varphi_\bullet)$, with $\bullet = P, S$, are also entire.

The oddness and evenness respectively are obtained by the Jost solutions $f_P^\pm$ and $f_S^\pm$ satisfying the following properties

$$\begin{aligned}
a(f_{P,S}^\pm)(Z,-\xi)|_{Z=0} &= a(f_{P,S}^\mp)(Z,\xi)|_{Z=0}\\
b(f_{P,S}^\pm)(Z,-\xi)|_{Z=0} &= b(f_{P,S}^\mp)(Z,\xi)|_{Z=0}
\end{aligned}$$

and since $q_P(-\xi) = q_P(\xi)$ and $q_S(-\xi) = q_S(\xi)$. The claim follows from the definition (3.21) of the auxiliary functions $\theta_{P,S}$ and $\varphi_{P,S}$. $\square$

Lemma 3.6.4. *The functions $a(\theta_\bullet)(\xi)$, $b(\theta_\bullet)(\xi)$, with $\bullet = P, S$, satisfy the conjugation properties*

$$\overline{a(\theta_{P,S})(\bar{\xi})} = -a(\theta_{P,S})(-\xi)$$
$$\overline{b(\theta_{P,S})(\bar{\xi})} = -b(\theta_{P,S})(-\xi),$$

while the functions $a(\varphi_\bullet)(\xi)$, $b(\varphi_\bullet)(\xi)$, with $\bullet = P, S$, satisfy the conjugation properties

$$\overline{a(\varphi_{P,S})(\bar{\xi})} = a(\varphi_{P,S})(-\xi)$$
$$\overline{b(\varphi_{P,S})(\bar{\xi})} = b(\varphi_{P,S})(-\xi).$$

Proof. This follows from (3.20) and the parity properties for the Jost solution $f^{\pm}_{P,S}$. $\square$

The matrix $B(Z, \xi)$ applied to a matrix, whose columns are respectively f_P^- and f_S^-, is

$$B(Z,\xi) \begin{pmatrix} (f_P^-)_1 & (f_S^-)_1 \\ (f_P^-)_2 & (f_S^-)_2 \end{pmatrix} \Bigg|_{Z=0} = \begin{pmatrix} a(f_P^-) & a(f_S^-) \\ b(f_P^-) & b(f_S^-) \end{pmatrix},$$

this motivates the definition of the boundary matrix $B(\xi)$ as it follows.

Definition 3.6.5 (Boundary matrix). *We define the boundary matrix $B(\xi)$ as the quantity*

$$B(\xi) := \begin{pmatrix} a(f_P^-) & a(f_S^-) \\ \\ b(f_P^-) & b(f_S^-) \end{pmatrix}. \tag{3.24}$$

As we have seen in Section 3.3, the eigenvalues and resonances correspond to the zeros of the determinant of the boundary matrix, which we call the Rayleigh determinant.

Definition 3.6.6. *We define the Rayleigh determinant as the determinant of the boundary matrix*

$$\Delta(\xi) := \det \begin{pmatrix} a(f_P^-) & a(f_S^-) \\ \\ b(f_P^-) & b(f_S^-) \end{pmatrix} = a(f_P^-)b(f_S^-) - a(f_S^-)b(f_P^-).$$

As in Chapter 2 for the Love problem, we can distinguish between eigenvalues and resonances whether they are zeros of the Rayleigh determinant located in the physical or unphysical sheet. The physical sheet and the unphysical sheets are determined by imaginary part of the quasi-momenta q_S and q_P being positive or negative and leading to L^2 or not L^2 solutions respectively.

Definition 3.6.7. *We introduce the following discrete sets on the Riemann surface* Ξ

$$\begin{aligned}
\Lambda_{+,+} &:= \{\xi \in \Xi; \quad \Delta(\xi) = 0, \quad \operatorname{Im} q_P(\xi) > 0, \quad \operatorname{Im} q_S(\xi) > 0\}, \\
\Lambda_{+,-} &:= \{\xi \in \Xi; \quad \Delta(\xi) = 0, \quad \operatorname{Im} q_P(\xi) > 0, \quad \operatorname{Im} q_S(\xi) < 0\}, \\
\Lambda_{-,+} &:= \{\xi \in \Xi; \quad \Delta(\xi) = 0, \quad \operatorname{Im} q_P(\xi) < 0, \quad \operatorname{Im} q_S(\xi) > 0\}, \\
\Lambda_{-,-} &:= \{\xi \in \Xi; \quad \Delta(\xi) = 0, \quad \operatorname{Im} q_P(\xi) < 0, \quad \operatorname{Im} q_S(\xi) < 0\},
\end{aligned}$$

and the union of them denoted as Λ

$$\Lambda := \bigcup_{a,b=\pm} \Lambda_{a,b}.$$

It follows from Definition 3.6.7 that the eigenvalues correspond to the set $\Lambda_{+,+}$, zeros of $\Delta(\xi)$ in the physical sheet, and the resonances correspond to the union of the three remaining sets, which are the zeros of $\Delta(\xi)$ in the unphysical sheet.

The Rayleigh determinant can be written in terms of the auxiliary functions θ and φ in the following way:

$$\begin{aligned}
\Delta(\xi) = &[a(\theta_P) - q_P a(\varphi_P)]\,[b(\theta_S) - q_S b(\varphi_S)] - [a(\theta_S) - q_S a(\varphi_S)] \\
&\cdot [b(\theta_P) - q_P b(\varphi_P)] = d_1 + q_P d_2 + q_S d_3 + q_S q_P d_4
\end{aligned} \tag{3.25}$$

where the coefficients d_1, d_2, d_3, d_4 are entire in ξ and are defined as

$$\begin{aligned}
d_1 &:= a(\theta_P)b(\theta_S) - a(\theta_S)b(\theta_P) \\
d_2 &:= -\left[a(\varphi_P)b(\theta_S) - a(\theta_S)b(\varphi_P)\right] \\
d_3 &:= -\left[a(\theta_P)b(\varphi_S) - a(\varphi_S)b(\theta_P)\right] \\
d_4 &:= a(\varphi_P)b(\varphi_S) - a(\varphi_S)b(\varphi_P).
\end{aligned} \tag{3.26}$$

The definition of the functions $d_i(\xi)$, for $i = 1, \ldots, 4$, is made purely to describe $\Delta(\xi)$ in terms of entire functions in a simpler form. The formula (3.25) shows where dependence from the quasi-momenta q_P and q_S is located and make it easier to apply the mappings $w_\bullet$, with $\bullet = P, S$.

In the following we present the symmetries of the functions $d_i(\xi)$, for $i = 1, \ldots, 4$, which are inherited from the symmetries of the auxiliary functions.

Lemma 3.6.8. *The functions* $d_1(\xi)$ *and* $d_4(\xi)$*, defined in* (3.26)*, are entire and satisfy*

$$\begin{aligned}
d_1(\xi) &= d_1(-\xi) = \overline{d_1(\overline{\xi})}, \\
d_4(\xi) &= d_4(-\xi) = \overline{d_4(\overline{\xi})},
\end{aligned}$$

while the functions $d_1(\xi)$ *and* $d_4(\xi)$*, defined in* (3.26)*, are entire and satisfy*

$$d_2(\xi) = -d_2(-\xi) = \overline{d_2(\overline{\xi})},$$
$$d_3(\xi) = -d_3(-\xi) = \overline{d_3(\overline{\xi})}.$$

Proof. The functions d_i, with $i = 1, \dots, 4$, are entire as product of entire functions (see Lemma 3.6.3).

From the definition of $d_1(\xi)$, $d_2(\xi)$, $d_3(\xi)$, $d_4(\xi)$ in (3.26) and Lemma 3.6.3 the evenness of d_1 and d_4 and the oddness of d_2 and d_3 easily follows.

The conjugation properties follow from Lemma 3.6.4. □

Remark 3.6.9. *From Lemma 3.6.8 we can see that* d_1 *and* d_4 *are even in* ξ*, while* d_2 *and* d_3 *are odd in* ξ*.*

Applying the projection maps w_P, w_S and w_{PS} to the Rayleigh determinant we get

$$\begin{aligned}
\Delta(\xi) &= d_1 + q_P d_2 + q_S d_3 + q_S q_P d_4,\\
\Delta(w_P(\xi)) &= d_1 - q_P d_2 + q_S d_3 - q_S q_P d_4,\\
\Delta(w_S(\xi)) &= d_1 + q_P d_2 - q_S d_3 - q_S q_P d_4,\\
\Delta(w_{PS}(\xi)) &= d_1 - q_P d_2 - q_S d_3 + q_S q_P d_4.
\end{aligned} \tag{3.27}$$

From these, we can recover the following identity

$$\Delta + \Delta(w_P) + \Delta(w_S) + \Delta(w_{PS}) = 4d_1,$$

hence the sum of the projections of the Rayleigh determinant onto the four different sheets is an entire function.

Lemma 3.6.10. *The Rayleigh determinant satisfies*

$$\Delta(-\xi) = \Delta(w_{PS}(\xi)),$$
$$\Delta(w_P(-\xi)) = \Delta(w_S(\xi)).$$

Proof. From Lemma 3.6.8 and since q_P and q_S are even function of ξ, we have

$$\begin{aligned}
\Delta(-\xi) &= d_1(-\xi) + q_P(-\xi)d_2(-\xi) + q_S(-\xi)d_3(-\xi) + q_S(-\xi)q_P(-\xi)d_4(-\xi)\\
&= d_1(\xi) + q_P(\xi)(-d_2(\xi)) + q_S(\xi)(-d_3(\xi)) + q_S(\xi)q_P(\xi)d_4(\xi) = \Delta(w_{PS}(\xi))
\end{aligned}$$

and

$$\begin{aligned}
\Delta(w_P(-\xi)) &= d_1(-\xi) - q_P d_2(-\xi) + q_S d_3(-\xi) - q_S q_P d_4(-\xi) =\\
&= d_1 + q_P d_2 - q_S d_3 - q_S q_P d_4 = \Delta(w_S(\xi)).
\end{aligned}$$
□

Lemma 3.6.11. *For the Rayleigh determinant the following properties hold:*

$$\Delta(-\xi) = \overline{\Delta(\overline{\xi})},$$
$$\Delta(w_P(-\xi)) = \overline{\Delta(w_P(\overline{\xi}))}.$$

Proof. Those properties follow from Lemma 3.6.8. □

3.6.1 The entire function $F(\xi)$

We want to consider a function $F(\xi)$ that is entire on the complex plane and whose zeros correspond to the eigenvalues and the resonances of the operator associated to (3.1). From estimates of this function we can obtain estimates of the resonances, which tells us in which areas they are localized.

Theorem 3.6.12. *The function*

$$F(\xi) = \Delta(\xi)\Delta(w_P(\xi))\Delta(w_S(\xi))\Delta(w_{PS}(\xi)), \tag{3.28}$$

is entire.

Proof. Using the definitions of the Rayleigh determinants as in (3.27) we can evaluate the function $F(\xi)$

$$\begin{aligned}
F(\xi) =& (d_1 + q_P d_2 + q_S d_3 + q_S q_P d_4)(d_1 - q_P d_2 - q_S d_3 + q_S q_P d_4) \\
& \cdot (d_1 - q_P d_2 + q_S d_3 - q_S q_P d_4)(d_1 + q_P d_2 - q_S d_3 - q_S q_P d_4) \\
=& \left[(d_1 + q_P q_S d_4)^2 - (q_P d_2 + q_S d_3)^2\right]\left[(d_1 - q_S q_P d_4)^2 - (q_P d_2 - q_S d_3)^2\right] \\
=& (d_1^2 - q_S^2 q_P^2 d_4^2)^2 + (q_P^2 d_2^4 - q_S^2 d_3^2)^2 - (d_1 + q_P q_S d_4)^2 (q_P d_2 - q_S d_3)^2 \\
& - (d_1 - q_S q_P d_4)^2 (q_P d_2 + q_S d_3)^2.
\end{aligned}$$

The first two terms are equal to

$$\begin{aligned}
(d_1^2 - q_S^2 q_P^2 d_4^2)^2 + (q_P^2 d_2^4 - q_S^2 d_3^2)^2 =& d_1^4 + q_S^4 q_P^4 d_4^4 - 2q_S^2 q_P^2 d_1^2 d_4^2 + q_P^4 d_2^4 \\
& + q_S^4 d_3^4 - 2q_P^2 q_S^2 d_2^2 d_3^2,
\end{aligned}$$

and the last two terms are equal to

$$\begin{aligned}
& - (d_1 + q_P q_S d_4)^2 (q_P d_2 - q_S d_3)^2 - (d_1 - q_S q_P d_4)^2 (q_P d_2 + q_S d_3)^2 \\
&= -2q_P^2 d_1^2 d_2^2 - 2q_S^2 d_1^2 d_3^2 - 2q_P^4 q_S^2 d_2^2 d_4^2 - 2q_S^4 q_P^2 d_3^2 d_4^2 + 8q_P^2 q_S^2 d_1 d_2 d_3 d_4.
\end{aligned}$$

Finally, $F(\xi)$ assumes the following form:

$$F(\xi) = d_1^4 + q_P^4 d_2^4 + q_S^4 d_3^4 + q_S^4 q_P^4 d_4^4 - 2q_P^2 d_1^2 d_2^2 - 2q_S^2 d_1^2 d_3^2 - 2q_S^2 q_P^2 d_1^2 d_4^2$$

$$- 2q_P^2 q_S^2 d_2^2 d_3^2 - 2q_P^4 q_S^2 d_2^2 d_4^2 - 2q_S^4 q_P^2 d_3^2 d_4^2 + 8q_P^2 q_S^2 d_1 d_2 d_3 d_4. \tag{3.29}$$

Since d_1, d_2, d_3, d_4 are entire functions in ξ and since ξ is present inside the quasi-momenta, which always have even power, $F(\xi)$ is an entire function for $\xi \in \mathbb{C}$, whose zeros correspond to the set Λ. □

3.7 The Reflection matrix

The two general solutions of (3.1) are

$$\begin{aligned} w_P &= f_P^+ + R_2 f_P^- - q_P R_1 f_S^-, \\ w_S &= f_S^+ + q_S \widetilde{R}_1 f_P^- + \widetilde{R}_2 f_S^-, \end{aligned} \tag{3.30}$$

for $f_S^\pm$ and $f_P^\pm$ that are the Jost solutions (see Definition 3.2.2) and for some coefficients R_j, $\widetilde{R}_j$, $j = 1, 2$. Those solutions must satisfy the boundary conditions, that is, $a(w_P) = b(w_P) = 0$ and $a(w_S) = b(w_S) = 0$, which correspond to the linear systems of equations

$$\begin{cases} R_2 a(f_P^-) - q_P R_1 a(f_S^-) = -a(f_P^+), \\ R_2 b(f_P^-) - q_P R_1 b(f_S^-) = -b(f_P^+), \end{cases} \tag{3.31}$$

and

$$\begin{cases} q_S \widetilde{R}_1 a(f_P^-) + \widetilde{R}_2 a(f_S^-) = -a(f_S^+), \\ q_S \widetilde{R}_1 b(f_P^-) + \widetilde{R}_2 b(f_S^-) = -b(f_S^+). \end{cases} \tag{3.32}$$

We can solve these systems of equations and we find the reflection coefficients

$$\begin{aligned} R_2 &= \frac{1}{\Delta(\xi)} \det \begin{pmatrix} -a(f_P^+) & a(f_S^-) \\ -b(f_P^+) & b(f_S^-) \end{pmatrix}, \quad q_P R_1 = \frac{1}{\Delta(\xi)} \det \begin{pmatrix} a(f_P^-) & a(f_P^+) \\ b(f_P^-) & b(f_P^+) \end{pmatrix} \\ \widetilde{R}_2 &= \frac{1}{\Delta(\xi)} \det \begin{pmatrix} a(f_P^-) & -a(f_S^+) \\ b(f_P^-) & -b(f_S^+) \end{pmatrix}, \quad q_S \widetilde{R}_1 = \frac{1}{\Delta(\xi)} \det \begin{pmatrix} -a(f_S^+) & a(f_S^-) \\ -b(f_S^+) & b(f_S^-) \end{pmatrix} \end{aligned} \tag{3.33}$$

where $\Delta(\xi)$ is as in Definition 3.6.6.

Definition 3.7.1. *We define the reflection matrix R as the quantity*

$$R = \begin{pmatrix} R_2 & q_S \widetilde{R}_1 \\ -q_P R_1 & \widetilde{R}_2 \end{pmatrix} \tag{3.34}$$

with R_j, $\widetilde{R}_j$ as in (3.33).

By Definition 3.7.1 and (3.33) it is clear that the poles of the reflection matrix correspond to the zeros of the Rayleigh determinant $\Delta(\xi)$, hence eigenvalues and resonances.

Remark 3.7.2. *The reflection matrix* (3.34) *is the counterpart on the half-line of the scattering matrix of the whole line case.*

From the systems of equations (3.31)–3.32, we can also represent the reflection matrix R as

$$R = -\begin{pmatrix} a(f_P^-) & a(f_S^-) \\ b(f_P^-) & b(f_S^-) \end{pmatrix}^{-1} \begin{pmatrix} a(f_P^+) & a(f_S^+) \\ b(f_P^+) & b(f_S^+) \end{pmatrix}. \tag{3.35}$$

The next proposition shows that we can recover the reflection matrix from the boundary matrix. This is interesting for the inverse problem because we can also retrieve the reflection matrix by recovering the boundary matrix.

Proposition 3.7.3. *Let R be the reflection matrix defined in Definition 3.7.1 and $B(\xi)$ be the boundary matrix as in Definition 3.13, then the following identity holds*

$$R = -\left[B(\xi)\right]^{-1}\left[B(-\xi)\right].$$

Proof. We have seen that the reflection matrix can be expressed as in (3.35), then using (3.14) we get

$$\begin{pmatrix} a(f_P^+)(\xi) & a(f_S^+)(\xi) \\ b(f_P^+)(\xi) & b(f_S^+)(\xi) \end{pmatrix} = \begin{pmatrix} a(f_P^-)(-\xi) & a(f_S^-)(-\xi) \\ b(f_P^-)(-\xi) & b(f_S^-)(-\xi) \end{pmatrix} = B(-\xi). \qquad \square$$

We know that

$$\Delta(w_S(\xi)) = \det\begin{pmatrix} a(f_P^-) & a(f_S^+) \\ b(f_P^-) & b(f_S^+) \end{pmatrix} = a(f_P^-)b(f_S^+) - a(f_S^+)b(f_P^-) \tag{3.36}$$

and

$$\Delta(w_P(\xi)) = \det\begin{pmatrix} a(f_P^+) & a(f_S^-) \\ b(f_P^+) & b(f_S^-) \end{pmatrix} = a(f_P^+)b(f_S^-) - a(f_S^-)b(f_P^+). \tag{3.37}$$

From the definitions of the reflection coefficients (3.33) and using (3.37) we see that

$$R_2 = \frac{a(f_S^-)b(f_P^+) - a(f_P^+)b(f_S^-)}{\Delta(\xi)} = -\frac{\Delta(w_P(\xi))}{\Delta(\xi)},$$

and it follows that

$$
\begin{aligned}
&R_2(w_P(\xi)) = -\frac{\Delta(\xi)}{\Delta(w_P(\xi))}, \quad R_2(w_S(\xi)) = -\frac{\Delta(w_{PS}(\xi))}{\Delta(w_S(\xi))}, \\
&R_2(w_{PS}(\xi)) = -\frac{\Delta(w_S(\xi))}{\Delta(w_{PS}(\xi))}.
\end{aligned}
$$

From these we get the two identities

$$
\begin{aligned}
&R_2(\xi)\, R_2(w_P(\xi)) = 1, \\
&R_2(w_S(\xi)) R_2(w_{PS}(\xi)) = 1.
\end{aligned}
$$

Using (3.36) we get

$$
\widetilde{R}_2 = \frac{a(f_P^-)b(f_S^+) - a(f_S^+)b(f_P^-)}{\Delta(\xi)} = -\frac{\Delta(w_S(\xi))}{\Delta(\xi)}
$$

and after using the projection maps

$$
\begin{aligned}
&\widetilde{R}_2(w_P(\xi)) = -\frac{\Delta(w_{PS}(\xi))}{\Delta(w_P(\xi))}, \quad \widetilde{R}_2(w_S(\xi)) = -\frac{\Delta(\xi)}{\Delta(w_S(\xi))}, \\
&\widetilde{R}_2(w_{PS}(\xi)) = -\frac{\Delta(w_P(\xi))}{\Delta(w_{PS}(\xi))},
\end{aligned}
$$

we infer that

$$
\begin{aligned}
&\widetilde{R}_2(\xi)\, \widetilde{R}_2(w_S(\xi)) = 1, \\
&\widetilde{R}_2(w_P(\xi)) \widetilde{R}_2(w_{PS}(\xi)) = 1.
\end{aligned}
$$

For the other reflection coefficients, we have

$$
\begin{aligned}
R_1 &= \frac{a(f_P^-)b(f_P^+) - a(f_P^+)b(f_P^-)}{q_P \Delta(\xi)} = \ldots = \frac{2q_P\,(a(\theta_P)b(\varphi_P) - a(\varphi_P)b(\theta_P))}{q_P \Delta(\xi)} \\
&= \frac{\det \mathcal{P}(\xi)}{\Delta(\xi)}; \\
\widetilde{R}_1 &= \frac{a(f_S^+)b(f_S^-) - a(f_S^-)b(f_S^+)}{q_S \Delta(\xi)} = \ldots = \frac{2q_S\,(a(\theta_S)b(\varphi_S) - a(\varphi_S)b(\theta_S))}{q_S \Delta(\xi)} \\
&= \frac{\det \mathcal{S}(\xi)}{\Delta(\xi)},
\end{aligned}
$$

where

$$
\det \mathcal{P}(\xi) := 2\,(a(\theta_P)b(\varphi_P) - a(\varphi_P)b(\theta_P)) = \frac{a(f_P^-)b(f_P^+) - a(f_P^+)b(f_P^-)}{q_P}
$$

$$\det \mathcal{S}(\xi) := 2\left(a(\theta_S)b(\varphi_S) - a(\varphi_S)b(\theta_S)\right) = \frac{a(f_S^+)b(f_S^-) - a(f_S^-)b(f_S^+)}{q_S}$$

and

$$\mathcal{P}(\xi) := \frac{1}{\sqrt{q_P}} \begin{pmatrix} a(f_P^-) & a(f_P^+) \\ b(f_P^-) & b(f_P^+) \end{pmatrix} \tag{3.38}$$

$$\mathcal{S}(\xi) := \frac{1}{\sqrt{q_S}} \begin{pmatrix} a(f_S^+) & a(f_S^-) \\ b(f_S^+) & b(f_S^-) \end{pmatrix}. \tag{3.39}$$

Lemma 3.7.4. *The functions* $\det \mathcal{P}(\xi)$ *and* $\det \mathcal{S}(\xi)$ *are entire and odd functions in* ξ.

Proof. By Lemma 3.6.3, $\det \mathcal{P}(\xi)$ and $\det \mathcal{S}(\xi)$ are entire because the product of entire functions. The oddness and evenness of the functions $a(\theta_{P,S})$, $a(\varphi_{P,S})$, $b(\theta_{P,S})$, $b(\varphi_{P,S})$, formulated in Lemma 3.6.3, implies the oddness of $\det \mathcal{P}(\xi)$ and $\det \mathcal{S}(\xi)$. □

Proposition 3.7.5. *The zeros of* $\det \mathcal{P}(\xi)$ *correspond to the zeros of* $R_1(\xi)$ *and the zeros of* $\det \mathcal{S}(\xi)$ *correspond to the zeros of* $\widetilde{R}_1(\xi)$.

Proof. We know that the reflection coefficients $R_1(\xi)$ and $\widetilde{R}_1(\xi)$ are defined as

$$R_1(\xi) = \frac{\det \mathcal{P}(\xi)}{\Delta(\xi)} \qquad \widetilde{R}_1(\xi) = \frac{\det \mathcal{S}(\xi)}{\Delta(\xi)}.$$

Moreover, the general solutions to the boundary value problems are

$$\begin{aligned} w_P^{\pm}(Z,\xi) &= f_P^{\pm}(Z,\xi) + R_2(\xi) f_P^{\mp}(Z,\xi) - q_P(\xi) R_1(\xi) f_S^{\mp}(Z,\xi), \\ w_S^{\pm}(Z,\xi) &= f_S^{\pm}(Z,\xi) + \tilde{R}_2(\xi) f_S^{\mp}(Z,\xi) + q_S(\xi) \tilde{R}_1(\xi) f_P^{\mp}(Z,\xi). \end{aligned}$$

If we consider the solution w_P^+ and multiply it by $\Delta(\xi)$, we get

$$\Delta(\xi) w_P^+(Z,\xi) = \Delta(\xi) f_P^+(Z,\xi) - \Delta(w_P(\xi)) f_P^-(Z,\xi) - q_P(\xi) \det \mathcal{P}(\xi) f_S^-(Z,\xi). \tag{3.40}$$

Suppose that there exists a ξ which is a zero of both $\det \mathcal{P}(\xi)$ and $\Delta(\xi)$. Substituting it in (3.40) will bring to the contradiction $\Delta(w_P(\xi)) = 0$. Hence, if ξ_j are such that $\det \mathcal{P}(\xi_j) = 0$, then necessarily $\Delta(\xi_j) \neq 0$, so the zeros of $\det \mathcal{P}(\xi)$ correspond to the zeros of $R_1(\xi)$. Repeating the same procedure for w_S^+ we get that the zeros of $\det \mathcal{S}(\xi)$ correspond to the zeros of $\widetilde{R}_1(\xi)$. □

Remark 3.7.6. *Since* $\det \mathcal{S}(\xi)$ *and* $\det \mathcal{P}(\xi)$ *are entire functions, they can be retrieved from their zeros, which correspond to the zeros of* $R_1(\xi)$ *(see Proposition 3.7.5) and* $\widetilde{R}_1(\xi)$ *and are the wave numbers* ξ *such that mode conversion*[3] *does not occur.*

3.8 The Pekeris-Markushevich transform

In this section we recall some known facts of [40] and we use the same notation as in [19, 1]. We make some substitutions in the differential equations such that we can pass from a self-adjoint differential operator to a not self-adjoint Sturm-Liouville problem where the spectral parameter ξ is also present in the boundary condition and the frequency ω is present in the potential and in the boundary condition.

Basically, we lose self-adjointness of the problem but we gain a Schrödinger-type differential equation which can help us in computing the Jost solution through a Volterra-type integral equation. The adjoint problem has transposed potential and boundary condition. As we saw from (3.1)–(3.2) the Rayleigh system of equations is

$$-\frac{\partial}{\partial Z}\left(\hat{\mu}\frac{\partial \varphi_1}{\partial Z}\right) - i\xi\left(\frac{\partial}{\partial Z}(\hat{\mu}\varphi_3) + \hat{\lambda}\frac{\partial}{\partial Z}\varphi_3\right) + (\hat{\lambda}+2\hat{\mu})\xi^2\varphi_1 = \omega^2\varphi_1$$
$$-\frac{\partial}{\partial Z}\left((\hat{\lambda}+2\hat{\mu})\frac{\partial \varphi_3}{\partial Z}\right) - i\xi\left(\frac{\partial}{\partial Z}(\hat{\lambda}\varphi_1) + \hat{\mu}\frac{\partial}{\partial Z}\varphi_1\right) + \hat{\mu}\xi^2\varphi_3 = \omega^2\varphi_3,$$

with boundary conditions

$$a_-(\Phi) = i\hat{\lambda}\xi\varphi_1(0^-) + (\hat{\lambda}+2\hat{\mu})\frac{\partial \varphi_3}{\partial Z}(0^-) = 0,$$
$$b_-(\Phi) = i\xi\hat{\mu}\varphi_3(0^-) + \hat{\mu}\frac{\partial \varphi_1}{\partial Z}(0^-) = 0.$$

Then, as in [40], we define $w_1 := -i\varphi_1$, $w_2 := \varphi_3$ and $\frac{d\Phi(x)}{dx} = -\frac{d\Phi(Z)}{dZ}$ with $x = -Z$ that yields

$$\frac{d}{dx}\Big(\hat{\mu}\frac{dw_1}{dx} - \xi\hat{\mu}w_2\Big) - \xi\hat{\lambda}\frac{dw_2}{dx} + \big(\omega^2 - \xi^2(\hat{\lambda}+2\hat{\mu})\big)w_1 = 0, \tag{3.41}$$

$$\frac{d}{dx}\Big((\hat{\lambda}+2\hat{\mu})\frac{dw_2}{dx} + \xi\hat{\lambda}w_1\Big) + \xi\hat{\mu}\frac{dw_1}{dx} + (\omega^2 - \xi^2\hat{\mu})w_2 = 0, \tag{3.42}$$

with the boundary conditions

$$(\hat{\lambda}+2\hat{\mu})\frac{dw_2}{dx} + \xi\hat{\lambda}w_1\Big|_{x=0} = \psi_2(\xi) := a_-(w) \tag{3.43}$$

[3]Mode conversion refers to when a P (or S) incident wave after hitting the boundary gets reflected as a converted S-wave (or P-wave). When $R_1(\xi) = 0$ or $\widetilde{R}_1(\xi) = 0$ there is no mode conversion, as we can see from (3.30).

$$\hat{\mu}\frac{dw_1}{dx} - \xi\hat{\mu}w_2\Big|_{x=0} = \psi_1(\xi) := b_-(w). \tag{3.44}$$

After some change of variables the boundary value problem (3.41)–(3.44) with $\psi_1 = \psi_2 = 0$ can be reduced to two matrix Sturm-Liouville problems with mutually transposed potentials and boundary conditions (see [44]), as also shown by Markushevich in [39, 40, 41]. Let G be the solution of the Cauchy problem

$$G' = \frac{1}{2}LG, \quad G(0) = I, \tag{3.45}$$

where I is the identity matrix, and

$$L = \begin{pmatrix} 0 & -d \\ -c & 0 \end{pmatrix}, \quad c = \frac{1}{\hat{\mu}_I}\frac{\hat{\mu}(\hat{\lambda}+\hat{\mu})}{(\hat{\lambda}+2\hat{\mu})}, \quad d = -2\hat{\mu}_I\frac{d^2}{dx^2}\left(\frac{1}{\hat{\mu}}\right). \tag{3.46}$$

Following the notation in [41] and in [19], by the substitution

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathfrak{M}^{-1}(F) := \begin{pmatrix} \frac{d}{dx} & 1 \\ -\xi & 0 \end{pmatrix} \begin{pmatrix} \frac{\hat{\mu}_I}{\hat{\mu}} & 0 \\ 0 & \frac{\hat{\mu}}{\hat{\lambda}+2\hat{\mu}} \end{pmatrix} \left(G^{\mathrm{T}}\right)^{-1}F, \tag{3.47}$$

the problem (3.41)–(3.44) reduces to the matrix Sturm-Liouville form

$$F'' - \xi^2 F = QF, \quad x \in (0, +\infty), \tag{3.48}$$

$$F' + \Theta F = \left(D^a\right)^{-1}\Psi, \quad x = 0.$$

Here, $\Psi = (\psi_1, \psi_2)^{\mathrm{T}}$ is the vector-column of the right-hand sides of (3.43) and (3.44), T is transpose operation, $\Theta = \left(D^a\right)^{-1}C^a$ with D^a and C^a being the matrices

$$D^a(\xi) = \begin{pmatrix} -2\hat{\mu}_I\frac{\hat{\mu}'}{\hat{\mu}} & \hat{\mu} \\ -2\hat{\mu}_I\xi & 0 \end{pmatrix}, \quad C^a(\xi) = \begin{pmatrix} \hat{\mu}_I\left(2\xi^2 - \frac{\omega^2}{\hat{\mu}} + \frac{\hat{\mu}''}{\hat{\mu}}\right) & -\frac{\hat{\mu}'\hat{\mu}}{\hat{\lambda}+2\hat{\mu}} \\ 2\hat{\mu}_I\xi\frac{\hat{\mu}'}{\hat{\mu}} & -\xi\frac{\hat{\mu}^2}{\hat{\lambda}+2\hat{\mu}} \end{pmatrix}$$

with $D^a(\xi)$ having inverse:

$$\left(D^a(\xi)\right)^{-1} = \frac{1}{2\hat{\mu}_I\hat{\mu}\xi}\begin{pmatrix} 0 & -\hat{\mu} \\ 2\hat{\mu}_I\xi & -2\hat{\mu}_I\frac{\hat{\mu}'}{\hat{\mu}} \end{pmatrix}.$$

Therefore

$$\Theta(\xi) = (D^a(\xi))^{-1} C^a(\xi) =$$
$$= \begin{pmatrix} -\dfrac{\hat\mu'(0)}{\hat\mu(0)} & \dfrac{1}{2\hat\mu_I}\dfrac{\hat\mu^2(0)}{(\hat\lambda(0)+2\hat\mu(0))} \\ \dfrac{\hat\mu_I}{\hat\mu(0)}\left(2\xi^2 - \dfrac{\omega^2}{\hat\mu(0)} - \hat\mu(0)\dfrac{d^2}{dx^2}\left(\dfrac{1}{\hat\mu}\right)(0)\right) & 0 \end{pmatrix}$$
$$=: \begin{pmatrix} -\theta_3 & \theta_2 \\ 2\frac{\hat\mu_I}{\hat\mu}\xi^2 - \theta_1 & 0 \end{pmatrix}. \tag{3.49}$$

Moreover, the matrix-valued potential Q is defined by

$$Q = \left(G^{-1}BG\right)^{\mathrm{T}}, \quad B = B_1 + \omega^2 B_2, \tag{3.50}$$

$$B_1 =$$
$$\begin{pmatrix} -\frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{\hat\mu}\right)\frac{\hat\mu(\hat\lambda+\hat\mu)}{\hat\lambda+2\hat\mu} + \frac{\hat\mu''}{\hat\mu} & \hat\mu_I\left(2\frac{\hat\mu'}{\hat\mu}\frac{d^2}{dx^2}\left(\frac{1}{\hat\mu}\right) + \frac{d^3}{dx^3}\left(\frac{1}{\hat\mu}\right)\right) \\ \frac{1}{\hat\mu_I}\left(\frac{\hat\lambda'\hat\mu^2+\hat\mu'\hat\lambda(\hat\lambda+\hat\mu)}{(\hat\lambda+2\hat\mu)^2} - \frac{1}{2}\frac{d}{dx}\left(\overline{\frac{\hat\mu(\hat\lambda+\hat\mu)}{\hat\lambda+2\hat\mu}}\right)\right) & \frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{\hat\mu}\right)\frac{(\hat\lambda-\hat\mu)}{\hat\lambda+2\hat\mu} \end{pmatrix} \tag{3.51}$$

$$B_2 = \begin{pmatrix} -\dfrac{1}{\hat\mu} & \hat\mu_I\dfrac{d}{dx}\left(\overline{\dfrac{1}{\hat\mu^2}}\right) \\ 0 & -\dfrac{1}{\hat\lambda+2\hat\mu} \end{pmatrix}. \tag{3.52}$$

We rewrite (3.45) in the form

$$G_{11}' = -\frac{d}{2}G_{21}, \quad G_{12}' = -\frac{d}{2}G_{22}, \quad G_{21}' = -\frac{c}{2}G_{11}, \quad G_{22}' = -\frac{c}{2}G_{12},$$

where $\det G(x) = 1$ (see [39]) and by (3.46) the coefficient d vanishes if $\hat\mu$ is constant. We adopt the notation

$$G_{jk}^H = G_{jk}(H), \qquad j,k = 1,2.$$

Then, the matrix function $G(x)$ in the homogeneous region $x \geq x_I =: H$ solves

$$G' = -\frac{c_I}{2}\begin{pmatrix} 0 & 0 \\ G_{11} & G_{12} \end{pmatrix}, \quad G(H) = \begin{pmatrix} G_{11}^H & G_{12}^H \\ G_{21}^H & G_{22}^H \end{pmatrix}, \tag{3.53}$$

where

$$c_I = \frac{\hat{\lambda}_I + \hat{\mu}_I}{\hat{\lambda}_I + 2\hat{\mu}_I}.$$

By integrating (3.53), we obtain

$$\begin{aligned} G_{11}(x) &= G_{11}^H, & G_{12}(x) &= G_{12}^H, \\ G_{21}(x) &= -\frac{c_I}{2} G_{11}^H (x - H) + G_{21}^H, & G_{22}(x) &= -\frac{c_I}{2} G_{12}^H (x - H) + G_{22}^H. \end{aligned} \quad (3.54)$$

Further, since $\det G(x) = 1$, the inverse matrix is equal to

$$G^{-1} = \begin{pmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{pmatrix}. \quad (3.55)$$

Then plugging (3.51), (3.52), (3.54) and (3.55) into (3.50) we obtain the potential $Q(x)$. We denote the restriction of $Q(x)$ to the region $x \geq H$ by $Q_0(x)$. Then we can write

$$Q_0(x) = \omega^2 \begin{pmatrix} -\frac{1}{\hat{\mu}_I} & 0 \\ 0 & -\frac{1}{\hat{\lambda}_I + 2\hat{\mu}_I} \end{pmatrix} + \omega^2 \frac{\hat{\lambda}_I + \hat{\mu}_I}{\hat{\mu}_I(\hat{\lambda}_I + 2\hat{\mu}_I)} \begin{pmatrix} -G_{12}^H G_{21}(x) & G_{21}(x) G_{11}^H \\ -G_{12}^H G_{22}(x) & G_{12}^H G_{21}(x) \end{pmatrix} \quad (3.56)$$

$Q_0(x)$ extended to $[0, \infty)$ is called background potential. We introduce the perturbed potential $V(x) := Q(x) - Q_0(x)$ which satisfies $V(x) = 0$ for $x \geq H$.

3.8.1 Jost solutions and Jost function

In this subsection, we define the Jost solution and the Jost function following the notation of [19]. The Sturm-Liouville boundary value problem is written in the form

$$-F'' + Q_0 F + VF = -\xi^2 F, \quad (3.57)$$

$$F' + \Theta F = 0, \quad x = 0, \quad (3.58)$$

where Θ is given by (3.49). We construct solutions to the equation

$$-F'' + Q_0 F = -\xi^2 F$$

of the form

$$F_{P,0}^{\pm} = \begin{pmatrix} (F_{P,0}^{\pm})_1 \\ (F_{P,0}^{\pm})_2 \end{pmatrix} e^{\pm i x q_P}, \qquad q_P = \sqrt{\frac{\omega^2}{\hat{\lambda}_I + 2\hat{\mu}_I} - \xi^2},$$

$$F_{S,0}^{\pm} = \begin{pmatrix} (F_{S,0}^{\pm})_1 \\ (F_{S,0}^{\pm})_2 \end{pmatrix} e^{\pm i x q_S}, \qquad q_S = \sqrt{\frac{\omega^2}{\hat{\mu}_I} - \xi^2}.$$

Let $w = (w_1, w_2)^{\mathrm{T}}$ be the solution to the original Rayleigh system,

$$w^{\pm}_{P,0} = \hat{\mu}_I \frac{\xi^2}{\omega^2} \begin{pmatrix} 1 \\ \mp \frac{i}{\xi} q_P \end{pmatrix} e^{\pm ixq_P}, \tag{3.59}$$

$$w^{\pm}_{S,0} = \hat{\mu}_I \frac{\xi^2}{\omega^2} \begin{pmatrix} \mp \frac{i}{\xi} q_S \\ 1 \end{pmatrix} e^{\pm ixq_S}. \tag{3.60}$$

Then the Jost solutions after the Pekeris-Markushevich transform are (see [19])

$$\begin{aligned}\mathfrak{M}(w^{\pm}_{P,0}) := F^{\pm}_{P,0} &= \begin{pmatrix} G_{21}(x) \pm iq_P \frac{\hat{\mu}_I}{\omega^2} G^H_{11} \\ \\ G_{22}(x) \pm iq_P \frac{\hat{\mu}_I}{\omega^2} G^H_{12} \end{pmatrix} e^{\pm ixq_P} \\ &= \begin{pmatrix} \left(-\frac{c_I}{2} G^H_{11}(x-H) + G^H_{21}\right) \pm iq_P \frac{\hat{\mu}_I}{\omega^2} G^H_{11} \\ \\ \left(-\frac{c_I}{2} G^H_{12}(x-H) + G^H_{22}\right) \pm iq_P \frac{\hat{\mu}_I}{\omega^2} G^H_{12} \end{pmatrix} e^{\pm ixq_P},\end{aligned} \tag{3.61}$$

$$\mathfrak{M}(w^{\pm}_{S,0}) := F^{\pm}_{S,0} = -\hat{\mu}_I \frac{\xi}{\omega^2} \begin{pmatrix} G^H_{11} \\ \\ G^H_{12} \end{pmatrix} e^{\pm ixq_S}, \tag{3.62}$$

where we have used that

$$G_{21}(x) = -\frac{c_I}{2} G^H_{11}(x-H) + G^H_{21}, \quad G_{22}(x) = -\frac{c_I}{2} G^H_{12}(x-H) + G^H_{22}.$$

The Jost solutions of (3.57) are given by the conditions

$$F^{\pm}_P = F^{\pm}_{P,0}. \qquad F^{\pm}_S = F^{\pm}_{S,0} \qquad \text{for} \qquad x > H.$$

We define the matrix Jost solution

$$\mathcal{F}(x,\xi) = \begin{pmatrix} (F^+_P)_1 & (F^+_S)_1 \\ \\ (F^+_P)_2 & (F^+_S)_2 \end{pmatrix},$$

and the unperturbed matrix Jost solution

$$\mathcal{F}_0(x,\xi) = \begin{pmatrix} \left(F^+_{P,0}\right)_1 & \left(F^+_{S,0}\right)_1 \\ \\ \left(F^+_{P,0}\right)_2 & \left(F^+_{S,0}\right)_2 \end{pmatrix}$$

$$= \begin{pmatrix} \left(G_{21}(x) + iq_P \frac{\hat{\mu}_I}{\omega^2} G_{11}^H\right) e^{-iq_P x} & -\frac{\hat{\mu}_I \xi}{\omega^2} G_{11}^H e^{-iq_S x} \\ \left(G_{22}(x) + iq_P \frac{\hat{\mu}_I}{\omega^2} G_{12}^H\right) e^{-iq_P x} & -\frac{\hat{\mu}_I \xi}{\omega^2} G_{12}^H e^{-iq_S x} \end{pmatrix}$$

where $\left(F_P^+\right)_i$ denotes the i component of the vector F_P^+ and similarly for F_S^+. The matrix Jost solution satisfies the Volterra-type integral equation

$$\mathcal{F}(x,\xi) = \mathcal{F}_0(x,\xi) - \int_x^\infty \mathcal{G}(x,y)V(y)\mathcal{F}(y,\xi)dy,$$

$\mathcal{G}(x,y)$ is the Green function, which is obtained such that each column of $\mathcal{G}(\cdot,t)$ satisfies the unperturbed equation

$$-F'' + Q_0 F = -\xi^2 F,$$

and the conditions

$$\mathcal{G}(x,x) = 0, \qquad \frac{\partial}{\partial x}\mathcal{G}(x,y)_{|y=x} = I. \tag{3.63}$$

Explicitly the Green function is given by (see [19])

$$\begin{aligned} \mathcal{G}(x,y) &= \mathcal{A}(x)\frac{\sin((x-y)q_P)}{q_P} + \mathcal{B}(y)\frac{\sin((x-y)q_S)}{q_S} \\ &+ \mathcal{C}\frac{\cos((x-y)q_S) - \cos((x-y)q_P)}{\omega^2} \end{aligned} \tag{3.64}$$

where

$$\begin{aligned} \mathcal{A}(x) :&= \begin{pmatrix} G_{12}^H\left[\frac{c_I}{2}G_{11}^H(x-H) - G_{21}^H\right] & G_{11}^H\left[-\frac{c_I}{2}G_{11}^H(x-H) + G_{21}^H\right] \\ G_{12}^H\left[\frac{c_I}{2}G_{12}^H(x-H) - G_{22}^H\right] & G_{11}^H\left[-\frac{c_I}{2}G_{12}^H(x-H) + G_{22}^H\right] \end{pmatrix} \\ &= \begin{pmatrix} -G_{12}^H G_{21}(x) & G_{11}^H G_{21}(x) \\ -G_{12}^H G_{22}(x) & G_{11}^H G_{22}(x) \end{pmatrix}, \\ \mathcal{B}(y) :&= \begin{pmatrix} G_{11}^H\left[\frac{c_I}{2}(-y+H)G_{12}^H + G_{22}^H\right] & -G_{11}^H\left[\frac{c_I}{2}(-y+H)G_{11}^H + G_{21}^H\right] \\ G_{12}^H\left[\frac{c_I}{2}(-y+H)G_{12}^H + G_{22}^H\right] & -G_{12}^H\left[\frac{c_I}{2}(-y+H)G_{11}^H + G_{21}^H\right] \end{pmatrix} \\ &= \begin{pmatrix} G_{11}^H G_{22}(y) & -G_{11}^H G_{21}(y) \\ G_{12}^H G_{22}(y) & -G_{12}^H G_{21}(y) \end{pmatrix}, \\ \mathcal{C} :&= \begin{pmatrix} \hat{\mu}_I G_{12}^H G_{11}^H & -\hat{\mu}_I (G_{11}^H)^2 \\ \hat{\mu}_I (G_{12}^H)^2 & -\hat{\mu}_I G_{12}^H G_{11}^H \end{pmatrix}. \end{aligned} \tag{3.65}$$

The term $\mathcal{A}(x)$ is a first order matrix-valued polynomial in x, $\mathcal{B}(y)$ is a first order matrix-valued polynomial in y and $\mathcal{C}$ is a constant matrix. We define the Jost function as

$$\mathcal{F}_\Theta(\xi) := \mathcal{F}'(0,\xi) + \Theta\mathcal{F}(0,\xi), \tag{3.66}$$

with Θ as in (3.49).

3.8.2 Relation between boundary matrix and Jost function

After performing the Pekeris-Markushevich transform, it is easy to obtain analytical properties of the Jost function. However, all the results obtained in this framework need eventually to be converted back into the original framework. In the following Lemma, we show the relation between those quantities in the two frameworks before and after the Pekeris-Markushevich transform (see [19]).

Lemma 3.8.1. *Let $\mathcal{F}_\Theta(\xi)$ be as above and $B(\xi)$ as in* (3.24). *Then*

$$\mathcal{F}_\Theta(\xi) = \frac{1}{2\xi\hat{\mu}_I\hat{\mu}(0)}\begin{pmatrix} \hat{\mu}(0) & 0 \\ 2\hat{\mu}_I\frac{\hat{\mu}'(0)}{\hat{\mu}(0)} & 2i\hat{\mu}_I\xi \end{pmatrix}\begin{pmatrix} a(f_P) & a(f_S) \\ b(f_P) & b(f_S) \end{pmatrix}\xi\frac{\hat{\mu}_I}{\omega^2}\begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$B(\xi) = A_1(\xi)\mathcal{F}_\Theta(\xi)A_2(\xi) \tag{3.67}$$

where

$$A_1(\xi) = \begin{pmatrix} 2\xi\hat{\mu}_I & 0 \\ 2i\hat{\mu}_I\frac{\hat{\mu}'(0)}{\hat{\mu}(0)} & -i\hat{\mu}(0) \end{pmatrix}, \qquad A_2(\xi) = \begin{pmatrix} \frac{\omega^2}{i\xi\hat{\mu}_I} & 0 \\ 0 & -\frac{\omega^2}{\xi\hat{\mu}_I} \end{pmatrix}. \tag{3.68}$$

Proof. We saw in (3.24) that

$$B(\xi) = \begin{pmatrix} a(f_P^-) & a(f_S^-) \\ b(f_P^-) & b(f_S^-) \end{pmatrix}.$$

Then after passing from f to w we get

$$B(w) = \begin{pmatrix} b(w_P) & b(w_S) \\ a(w_P) & a(w_S) \end{pmatrix} = \begin{pmatrix} ib(f_P) & ib(f_S) \\ -a(f_P) & -a(f_S) \end{pmatrix}\xi\frac{\hat{\mu}_I}{\omega^2}\begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & i \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a(f_P^-) & a(f_S^-) \\ b(f_P^-) & b(f_S^-) \end{pmatrix} \xi \frac{\hat{\mu}_I}{\omega^2} \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}$$

which is the Neumann operator associated to the matrix-valued solution

$$w = \begin{pmatrix} w_{P,1} & w_{S,1} \\ w_{P,2} & w_{S,2} \end{pmatrix}.$$

We have

$$(D^{\mathrm{a}})^{-1} B(w)\big|_{x=0} = \mathcal{F}' + \Theta \mathcal{F}\big|_{x=0}, \qquad \mathfrak{M}(w) = \mathcal{F}.$$

Then, $\mathcal{F}_\Theta(\xi) = (D^{\mathrm{a}})^{-1} B(w)\big|_{x=0}$, so

$$\mathcal{F}_\Theta(\xi) = \frac{1}{2\hat{\mu}_I \hat{\mu}(0)\xi} \begin{pmatrix} 0 & -\hat{\mu}(0) \\ 2\hat{\mu}_I \xi & -2\hat{\mu}_I \dfrac{\hat{\mu}'(0)}{\hat{\mu}(0)} \end{pmatrix} \begin{pmatrix} 0 & i \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a(f_P^-) & a(f_S^-) \\ b(f_P^-) & b(f_S^-) \end{pmatrix}$$
$$\cdot\, \xi \frac{\hat{\mu}_I}{\omega^2} \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix} = A_1^{-1}(\xi) B(\xi) A_2^{-1}(\xi) \tag{3.69}$$

where

$$A_1^{-1}(\xi) = \frac{1}{2\hat{\mu}_I \hat{\mu}(0)\xi} \begin{pmatrix} \hat{\mu}(0) & 0 \\ 2\hat{\mu}_I \frac{\hat{\mu}'(0)}{\hat{\mu}(0)} & 2i\hat{\mu}_I \xi \end{pmatrix}, \qquad A_2^{-1}(\xi) = \xi \frac{\hat{\mu}_I}{\omega^2} \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we obtain also the inverse relation between $B(\xi)$ and $\mathcal{F}_\Theta(\xi)$

$$B(\xi) = A_1(\xi) \mathcal{F}_\Theta(\xi) A_2(\xi),$$

where

$$A_1(\xi) = \begin{pmatrix} 2\xi\hat{\mu}_I & 0 \\ 2i\hat{\mu}_I \frac{\hat{\mu}'(0)}{\hat{\mu}(0)} & -i\hat{\mu}(0) \end{pmatrix}, \qquad A_2(\xi) = \begin{pmatrix} \frac{\omega^2}{i\xi\hat{\mu}_I} & 0 \\ 0 & -\frac{\omega^2}{\xi\hat{\mu}_I} \end{pmatrix}. \qquad \square$$

3.8.3 The Faddeev solution

In this subsection we introduce the Faddeev solutions as it is simpler to work with them rather than with the Jost solution because we get rid of some of the exponential factors that are present in the latter. We define the Faddeev solution to be

$$
\begin{aligned}
H_P^{\pm}(x) &= e^{\mp i x q_P} F_P^{\pm}, \qquad & H_S^{\pm}(x) &= e^{\mp i x q_P} F_S^{\pm}, \\
H_{P,0}^{\pm}(x) &= e^{\mp i x q_P} F_{P,0}^{\pm}, \qquad & H_{S,0}^{\pm}(x) &= e^{\mp i x q_P} F_{S,0}^{\pm},
\end{aligned}
$$

and define $\mathcal{H} \equiv \mathcal{H}^+$, where

$$
\mathcal{H}(x) = \begin{pmatrix} [H_P(x)]_1 & [H_S(x)]_1 \\ [H_P(x)]_2 & [H_S(x)]_2 \end{pmatrix}.
$$

We consider the matrix composed of all the Faddeev solutions with $+$ sign[4]. Then

$$
\begin{aligned}
H_P(x) &= H_{P,0}(x) - \int_x^{\infty} \widetilde{\mathcal{G}}(x,y) V(y) H_P(y) dy, \\
H_{P,0}(x) &= \begin{pmatrix} G_{21}(x) + i q_P \frac{\hat{\mu}_I}{\omega^2} G_{11}^H \\ G_{22}(x) + i q_P \frac{\hat{\mu}_I}{\omega^2} G_{12}^H \end{pmatrix}, \qquad \widetilde{\mathcal{G}}(x,y) = e^{i(y-x)q_P} \mathcal{G}(x,y),
\end{aligned}
$$

$$
\begin{aligned}
H_S(x) &= H_{S,0}(x) - \int_x^{\infty} \widetilde{\mathcal{G}}(x,y) V(y) H_S(y) dy, \\
H_{S,0}(x) &= -\frac{\hat{\mu}_I \xi}{\omega^2} e^{i(q_S - q_P)x} \begin{pmatrix} G_{11}^H \\ G_{12}^H \end{pmatrix},
\end{aligned}
$$

so the unperturbed Faddeev solution is

$$
\mathcal{H}_0^+(x) = \begin{pmatrix} G_{21}(x) + i q_P \frac{\hat{\mu}_I}{\omega^2} G_{11}^H & -\frac{\hat{\mu}_I \xi}{\omega^2} G_{11}^H e^{-i(q_P - q_S)x} \\ G_{22}(x) + i q_P \frac{\hat{\mu}_I}{\omega^2} G_{12}^H & -\frac{\hat{\mu}_I \xi}{\omega^2} G_{12}^H e^{-i(q_P - q_S)x} \end{pmatrix}. \tag{3.70}
$$

The Volterra-type equation

$$
\mathcal{F}(x) = \mathcal{F}_0 - \int_x^{\infty} \mathcal{G}(x,y) V(y) \mathcal{F}(y) dy, \tag{3.71}
$$

[4] We can always swap to the other cases by applying the mappings (3.6) defined in the Riemann surface section.

after multiplying both sides by e^{-ixq_P} becomes

$$\mathcal{F}(x)e^{-ixq_P} = \mathcal{F}_0 e^{-ixq_P} - \int_x^\infty \mathcal{G}(x,y)V(y)\mathcal{F}(y)e^{-ixq_P}dy,$$

and in terms of the Faddeev solution

$$\begin{aligned}\mathcal{H}(x) &= \mathcal{H}_0(x) - \int_x^\infty \mathcal{G}(x,y)e^{-i(x-y)q_P}V(y)\mathcal{F}(y)e^{-iyq_P}dy \\ &= \mathcal{H}_0(x) - \int_x^\infty \mathcal{G}(x,y)e^{-i(x-y)q_P}V(y)\mathcal{H}(y)dy. \qquad (3.72)\end{aligned}$$

We get a Volterra-type equation for the Faddeev solution $\mathcal{H}$ which we will use to derive the analytical properties of the Jost solution $\mathcal{F}$. The first terms of this Volterra-type equation are

$$\begin{aligned}\mathcal{H}(x) = \mathcal{H}_0(x) &- \int_x^\infty \widetilde{\mathcal{G}}(x,y)V(y)\mathcal{H}_0(y)dy \\ &+ \int_x^\infty \widetilde{\mathcal{G}}(x,y)V(y)\int_y^\infty \widetilde{\mathcal{G}}(y,t)V(t)\mathcal{H}(t)dydt \qquad (3.73)\\ &= \mathcal{H}_0(x,\xi) + \sum_{l=1}^\infty \mathcal{H}^{(l)}(x,\xi), \qquad (3.74)\end{aligned}$$

where for $l \geq 1$

$$\begin{aligned}\mathcal{H}^{(l)}(x,\xi) :=& (-1)^l \int_x^H \cdots \int_{t_{l-1}}^H \widetilde{\mathcal{G}}(x,t_1)\cdots\widetilde{\mathcal{G}}(t_{l-1},t_l) \\ &\cdot V(t_1)\cdots V(t_l)\mathcal{H}_0(t,\xi)dt_1\cdots dt_l. \qquad (3.75)\end{aligned}$$

3.8.4 Analytical properties of Jost solutions and Jost function

In this subsection, the goal is to obtain an asymptotic expansion of the entire function $F(\xi)$, as in Theorem 3.8.21, and an exponential type estimate as in Theorem 3.8.22. In order to achieve this, we need to find the asymptotic expansion of the determinants of the Jost function, as in (3.66), in the four different sheets and then convert the results into the framework before the Pekeris-Markushevich transform. This translates into asymptotics of the Rayleigh determinant $\Delta(\xi)$ in the four different sheets and eventually into asymptotics of $F(\xi)$ (see (3.28)).

First, we define a class of potentials for which all the following results will hold.

Definition 3.8.2 (Class of potentials). *We denote by $\mathcal{V}_H$ the class of V such that $V \in L^1(\mathbb{R}_+;\mathbb{C}^{2\times 2})$, continuous and $\operatorname{supp} V \subset [0,H]$ for some $H > 0$ and for each $\epsilon > 0$ the set $(H-\epsilon,H) \cap \operatorname{supp} V_{ij}$, for $i,j = 1,2$, has positive Lebesgue measure.*

For such class of potentials, we have the following results.

Theorem 3.8.3 (Jost solutions). *For $V \in \mathcal{V}_H$ and any fixed $x \geq 0$, the Jost solution $\mathcal{F}(x,\xi)$ is analytic in ξ on each sheet $\Xi_{\pm,\pm}$, of exponential type, and for $\xi \in \Xi$ satisfying*

$$\mathcal{F}(x,\xi) \equiv \mathcal{F}^+(x,\xi) = \mathcal{F}_0^+(x,\xi) - \int_x^H \mathcal{G}(x,y)V(y)\mathcal{F}_0^+(y,\xi)dy + \sum_{k=2}^{\infty} \mathcal{F}_k(x,\xi),$$

$$||\mathcal{F}_k(x,\xi)|| \leq C\frac{|\xi|}{k!}e^{\gamma(\xi)(H-x)}e^{H\frac{\zeta_{P-S}}{2}}e^{-x\left(\frac{\zeta_P}{2}\right)}(\mathfrak{a}(x))^k, \tag{3.76}$$

where

$$\zeta_{P-S} := \operatorname{Im}(q_P - q_S) + |\operatorname{Im}(q_P - q_S)|, \qquad \zeta_P := \operatorname{Im} q_P + |\operatorname{Im} q_P|,$$

$$\mathfrak{a}(x) := \frac{\int_x^H ||V(t)||dt}{\max\{1,|\xi|\}},$$

$$\gamma(\xi) = \begin{cases} 0 & \textit{for } \xi \in \Xi_{+,\pm} \\ -2\operatorname{Im} q_P & \textit{for } \xi \in \Xi_{-,\pm} \end{cases}. \tag{3.77}$$

Proof. Using trigonometric formulas we can write

$$\begin{aligned}&\cos((x-y)q_S) - \cos((x-y)q_P)\\ &= -2\sin\left(\frac{1}{2}(x-y)(q_S+q_P)\right)\sin\left(\frac{1}{2}(x-y)(q_S-q_P)\right)\end{aligned}$$

hence the third term of the Green kernel (3.64) becomes

$$-\frac{2}{\omega^2}\mathcal{C}\sin\left(\frac{1}{2}(x-y)(q_S+q_P)\right)\sin\left(\frac{1}{2}(x-y)(q_S-q_P)\right).$$

We want to calculate the term $\widetilde{\mathcal{G}}(x,y) := \mathcal{G}(x,y)e^{-i(x-y)q_P}$ and looking at (3.64) we see that its sine and cosine terms can be written as

$$\begin{aligned}\frac{\sin[q_P(x-y)]}{q_P}e^{-i(x-y)q_P} &= \frac{1-e^{-2iq_P(x-y)}}{2iq_P} = \frac{1-e^{2iq_P(y-x)}}{2iq_P},\\ \frac{\sin[q_S(x-y)]}{q_S}e^{-i(x-y)q_P} &= \frac{e^{i(x-y)(q_S-q_P)} - e^{-i(x-y)(q_P+q_S)}}{2iq_S}\\ &= \frac{e^{i(y-x)(q_P-q_S)} - e^{i(y-x)(q_P+q_S)}}{2iq_S},\end{aligned}$$

and

$$
\begin{aligned}
&\sin\left[\frac{(x-y)(q_P+q_S)}{2}\right]\sin\left[\frac{(x-y)(q_S-q_P)}{2}\right]e^{-i(x-y)q_P}\\
&=-\sin\left[\frac{(y-x)(q_P+q_S)}{2}\right]\sin\left[\frac{(y-x)(q_P-q_S)}{2}\right]e^{-i(x-y)q_P}\\
&=-\left(\frac{e^{\frac{i}{2}(y-x)(q_P+q_S)}-e^{\frac{-i}{2}(y-x)(q_P+q_S)}}{2i}\right)\\
&\cdot\left(\frac{e^{\frac{i}{2}(y-x)(q_P-q_S)}-e^{\frac{-i}{2}(y-x)(q_P-q_S)}}{2i}\right)e^{-i(x-y)q_P}\\
&=-\left(\frac{e^{i(y-x)q_P}-e^{i(y-x)q_S}-e^{-i(y-x)q_S}+e^{-i(y-x)q_P}}{-4}\right)e^{i(y-x)q_P}\\
&=\frac{e^{2i(y-x)q_P}-e^{i(y-x)(q_P+q_S)}-e^{i(y-x)(q_P-q_S)}+1}{4},
\end{aligned}
$$

so

$$
\begin{aligned}
&\widetilde{\mathcal{G}}(x,y)=\mathcal{G}(x,y)e^{-i(x-y)q_P}=\mathcal{A}(x)\left[\frac{1-e^{2iq_P(y-x)}}{2iq_P}\right]\\
&+\mathcal{B}(y)\left[\frac{-e^{i(y-x)(q_P+q_S)}+e^{i(y-x)(q_P-q_S)}}{2iq_S}\right]\\
&+\mathcal{C}\left[\frac{-e^{2i(y-x)q_P}+e^{i(y-x)(q_P+q_S)}+e^{i(y-x)(q_P-q_S)}-1}{2\omega^2}\right].
\end{aligned}
$$

We take the maximum norm of $\widetilde{\mathcal{G}}(x,y,\xi)$, which is the maximum[5] for fixed x, y and ξ of the absolute value of the components of $\widetilde{\mathcal{G}}(x,y,\xi)$. Taking into account that $0\leq x\leq y\leq H$, which implies $y-x\geq 0$, we get

$$
||\widetilde{\mathcal{G}}(x,y,\xi)||\leq\frac{C}{\max\{1,|\xi|\}}e^{(y-x)\gamma(\xi)}
$$

where $C>0$ is a constant which does not depend on ξ, x and y, and where $\gamma(\xi)$ is defined as

$$
\gamma(\xi)=\max\left\{|\operatorname{Im}q_P|-\operatorname{Im}q_P,\frac{|\operatorname{Im}(q_P-q_S)|-\operatorname{Im}(q_P-q_S)}{2},\right.
$$

[5]Note that the ξ dependence is implicit inside the quasi momenta $q_S(\xi)$ and $q_P(\xi)$

$$\frac{|\operatorname{Im}(q_P+q_S)|-\operatorname{Im}(q_P+q_S)}{2}\Bigg\}. \tag{3.78}$$

From (3.78), we can see that $\gamma(\xi)=0$ in $\Xi_{+,\pm}$, because $\operatorname{Im} q_P>0$, $\operatorname{Im}(q_P+q_S)>0$ and $\operatorname{Im}(q_P-q_S)>0$ (see Lemma 3.4.1). While in the sheets $\Xi_{-,\pm}$, $\gamma(\xi)=-2\operatorname{Im} q_P$ as $\operatorname{Im} q_P>0$, $\operatorname{Im}(q_P+q_S)<0$ and $\operatorname{Im}(q_P-q_S)<0$ (see Lemma 3.4.1) which imply

$$\begin{aligned}\operatorname{Im}(q_P-q_S)<0 &\iff \operatorname{Im} q_P<\operatorname{Im} q_S \iff 2\operatorname{Im} q_P<\operatorname{Im} q_P+\operatorname{Im} q_S\\ &\iff -2\operatorname{Im} q_P>-\operatorname{Im}(q_P+q_S),\end{aligned} \tag{3.79}$$

and

$$\begin{aligned}&\operatorname{Im}(q_P+q_S)<0 \iff \operatorname{Im} q_P<-\operatorname{Im} q_S\\ &\iff -\operatorname{Im} q_P>\operatorname{Im} q_S \iff -2\operatorname{Im} q_P>-\operatorname{Im}(q_P-q_S)\end{aligned}$$

leading to formula (3.77). From (3.70) we can calculate the norm of the unperturbed Faddeev solution $\mathcal{H}_0(x)$, which is

$$||\mathcal{H}_0(x)||\leq C|\xi|e^{x\frac{\zeta_{P-S}}{2}}.$$

Then, the norm of (3.72) after one iteration can be estimated as follows:

$$||\mathcal{H}(x)||\leq||\mathcal{H}_0||+\int_x^\infty\frac{C}{\max\{1,|\xi|\}}e^{(y-x)\gamma(\xi)}||V(y)||\;||\mathcal{H}(y)||dy.$$

Starting from (3.72) and iterating the equation we get the series

$$\mathcal{H}(x,\xi)=\sum_{l=0}^{\infty}\mathcal{H}^{(l)}(x,\xi),$$

where

$$\mathcal{H}^{(0)}(x,\xi)=\mathcal{H}_0(x,\xi)$$

and any l-term is uniformly bounded by

$$\begin{aligned}||\mathcal{H}^{(l)}(x,\xi)||&\leq\int_x^\infty\int_{t_1}^\infty\cdots\int_{t_{l-1}}^\infty C\frac{e^{\gamma(\xi)[t_1-x+(t_2-t_1)+\cdots(t_l-t_{l-1})]}}{(\max\{1,|\xi|\})^l}\\ &\qquad\cdot||V(t_1)||\cdots||V(t_l)||\;||\mathcal{H}_0(t_l,\xi)||dt_l\cdots dt_1\\ &=C|\xi|e^{H\frac{\zeta_{P-S}}{2}}\int_x^\infty\int_{t_1}^\infty\cdots\int_{t_{l-1}}^\infty\frac{e^{\gamma(\xi)(t_l-x)}}{(\max\{1,|\xi|\})^l}||V(t_1)||\cdots||V(t_l)||dt_l\cdots dt_1\end{aligned}$$

$$\leq C\frac{e^{\gamma(\xi)(H-x)}}{(\max\{1,|\xi|\})^l}|\xi|e^{H\frac{\zeta_{P-S}}{2}}\frac{1}{l!}\left(\int_x^H ||V(t)||dt\right)^l$$

$$\leq C\frac{e^{\gamma(\xi)H}}{(\max\{1,|\xi|\})^l}|\xi|e^{H\frac{\zeta_{P-S}}{2}}\frac{1}{l!}\left(\int_0^H ||V(t)||dt\right)^l$$

with the convention $t_0 = x$ and since $V(t) = 0$ for $t \geq H$. Hence the series is absolutely and uniformly convergent on every compact set. Then the Faddeev solution $\mathcal{H}^{(l)}(x,\xi)$ is analytic in each sheet $\Xi_{\pm,\pm}$. □

For ξ in the physical sheet, the Faddeev solution is complex analytic, hence continuous. We can see that

$$||\mathcal{H}(x)-\mathcal{H}_0(x)|| \leq \sum_{l=1}^{\infty} ||\mathcal{H}^{(l)}(x,\xi)|| \leq |\xi|e^{\gamma(\xi)(H-x)}e^{H\frac{\zeta_{P-S}}{2}}e^{\frac{\int_x^H ||V(t)||dt}{\max\{1,|\xi|\}}}.$$

Then, if we want to obtain those estimates in terms of the Jost solution $\mathcal{F}(x)$, we have

$$||\mathcal{H}(x)-\mathcal{H}_0(x)|| = ||\mathcal{F}(x)e^{-ixq_P} - \mathcal{F}_0(x)e^{-ixq_P}|| = ||\mathcal{F}(x)-\mathcal{F}_0(x)||e^{x\left(\frac{\zeta_P}{2}\right)}$$

which leads to

$$||\mathcal{F}(x)-\mathcal{F}_0(x)|| \leq |\xi|e^{\gamma(\xi)(H-x)}e^{H\frac{\zeta_{P-S}}{2}}e^{-x\left(\frac{\zeta_P}{2}\right)}e^{\frac{\int_x^H ||V(t)||dt}{\max\{1,|\xi|\}}}$$

as in (3.76).

A theorem similar to Theorem 3.8.3 can also be found in [19] but for ξ only in the physical sheet, with a different Riemann surface and without proof.

Remark 3.8.4. *From equation* (3.76)*, we see that if we want to get an estimate on the Jost solution minus the zeroth and the first order expansion of the Volterra-type equation, we have*

$$||\mathcal{F}(x,\xi)-\mathcal{F}_0(x,\xi)-\int_x^H \mathcal{G}(x,y)V(y)\mathcal{F}_0(y,\xi)dy||$$

$$\leq \sum_{k=2}^{\infty}\frac{|\xi|}{k!}e^{\gamma(\xi)(H-x)}e^{H\left(\frac{\zeta_{P-S}}{2}\right)}e^{-x\left(\frac{\zeta_P}{2}\right)}(\mathfrak{a}(x))^k$$

$$= |\xi|e^{\gamma(\xi)(H-x)}e^{H\left(\frac{\zeta_{P-S}}{2}\right)}e^{-x\left(\frac{\zeta_P}{2}\right)}\left(e^{\mathfrak{a}(x)}-1-\mathfrak{a}(x)\right)$$

$$\leq |\xi|e^{\gamma(\xi)(H-x)}e^{H\left(\frac{\zeta_{P-S}}{2}\right)}e^{-x\left(\frac{\zeta_P}{2}\right)}\frac{(\mathfrak{a}(x))^2}{2}e^{\mathfrak{a}(x)}.$$

Similarly, we can get an estimate on the Jost solution minus the first two orders of the Volterra-type equation

$$
\begin{aligned}
&||\mathcal{F}(x,\xi) - \mathcal{F}_0(x,\xi) - \int_x^H \mathcal{G}(x,y)V(y)\mathcal{F}_0(y,\xi)dy \\
&+ \int_x^\infty \int_y^\infty \mathcal{G}(x,y)V(y)\mathcal{G}(y,t)V(t)\mathcal{F}_0(t,\xi)dtdy|| \\
&\leq |\xi| e^{\gamma(\xi)(H-x)} e^{H\left(\frac{\zeta_{P-S}}{2}\right)} e^{-x\left(\frac{\zeta_P}{2}\right)} \frac{(\mathfrak{a}(x))^3}{6} e^{\mathfrak{a}(x)}.
\end{aligned}
$$

Remark 3.8.5. *We get different estimates on the Jost solution depending on the sheet, in particular*

- *Sheets* $\Xi_{+,+}$, $\Xi_{+,-}$. *We have that* $\operatorname{Im}(q_P + q_S) > 0$, $\operatorname{Im}(q_P - q_S) > 0$ *and* $\operatorname{Im} q_P > 0$ *which imply that* $\gamma(\xi) = 0$ *and*

$$
||\mathcal{F}(x) - \mathcal{F}_0(x)|| \leq |\xi| e^{(H-x)\operatorname{Im} q_P} e^{-H \operatorname{Im} q_S} e^{\frac{\int_x^H ||V(t)||dt}{\max\{1,|\xi|\}}}. \tag{3.80}
$$

- *Sheets* $\Xi_{-,-}$, $\Xi_{-,+}$. *We have that* $\operatorname{Im}(q_P + q_S) < 0$, $\operatorname{Im}(q_P - q_S) < 0$ *and* $\operatorname{Im} q_P < 0$ *which imply that* $\gamma(\xi) = -2\operatorname{Im} q_P$, *hence*

$$
||\mathcal{F}(x) - \mathcal{F}_0(x)|| \leq |\xi| e^{-2(H-x)\operatorname{Im} q_P} e^{\frac{\int_x^H ||V(t)||dt}{\max\{1,|\xi|\}}}. \tag{3.81}
$$

In the next lemma we show a similar result to Theorem 3.8.3 for the derivative of the Jost solution.

Lemma 3.8.6. *For any fixed* $x \geq 0$, *the derivative of the Jost solution* $\mathcal{F}'(x,\xi) := \frac{\partial}{\partial x}\mathcal{F}(x,\xi)$ *is analytic in* ξ *in each sheet* $\Xi_{\pm,\pm}$ *and satisfies*

$$
\mathcal{F}'(x,\xi) = \mathcal{F}_0'(0,\xi)\mathcal{Q}(\xi) + \mathcal{P}(0,\xi) - \int_x^H \mathcal{G}_x(x,y)V(y)\mathcal{F}_0^+(y)dy \tag{3.82}
$$

$$
+ |q_S||\xi| \sum_{k=2}^{\infty} \mathcal{F}_k(x,\xi), \tag{3.83}
$$

where

$$
\mathcal{F}_0'(0,\xi)\mathcal{Q}(\xi) + \mathcal{P}(0,\xi) =
$$

$$
\begin{pmatrix}
\left(\frac{c_I}{2} G_{11}^H H + i q_P G_{21}(x) - q_P^2 \frac{\hat{\mu}_I}{\omega^2} G_{11}^H\right) e^{ixq_P} & -\frac{i q_S \hat{\mu}_I \xi}{\omega^2} G_{11}^H e^{ixq_S} \\
\left(\frac{c_I}{2} G_{12}^H H + i q_P G_{22}(x) - q_P^2 \frac{\hat{\mu}_I}{\omega^2} G_{12}^H\right) e^{ixq_P} & -\frac{i q_S \hat{\mu}_I \xi}{\omega^2} G_{12}^H e^{ixq_S}
\end{pmatrix},
$$

and terms $\mathcal{Q}(x)$, $\mathcal{P}(x,\xi)$ are defined as

$$\mathcal{Q}(x) = \begin{pmatrix} q_P & 0 \\ 0 & q_S \end{pmatrix}, \qquad \mathcal{P}(x,\xi) = iq_P \frac{c_I}{2} H e^{ixq_P} \begin{pmatrix} G_{11}^H & 0 \\ G_{12}^H & 0 \end{pmatrix},$$

with $c_I := \frac{\hat{\lambda}_I + \hat{\mu}_I}{\hat{\lambda}_I + 2\hat{\mu}_I}$.

Proof. To find an estimate for the derivative of the Jost solution, we differentiate (3.71) with respect to x and we get

$$\begin{aligned} \mathcal{F}'(x,\xi) &= i\mathcal{F}_0(x,\xi)\mathcal{Q}(\xi) + \mathcal{P}(x,\xi) + \mathcal{G}(x,x)V(x)\mathcal{F}(x) \\ &\quad - \int_x^\infty \mathcal{G}_x(x,y)V(y)\mathcal{F}(y)dy \\ &= i\mathcal{F}_0(x,\xi)\mathcal{Q}(\xi) + \mathcal{P}(x,\xi) - \int_x^\infty \mathcal{G}_x(x,y)V(y)\mathcal{F}(y)dy, \end{aligned} \tag{3.84}$$

where in the first passage we used the property (3.63) of the Green kernel which made the second term zero. The x derivative of the Green kernel is

$$\begin{aligned} \mathcal{G}_x(x,y) =& \mathcal{A}_0 \frac{\sin((x-y)q_P)}{q_P} + \mathcal{A}(x)\cos((x-y)q_P) + \mathcal{B}(y)\cos((x-y)q_S) \\ &+ \mathcal{C}\frac{q_P\sin((x-y)q_P) - q_S\sin((x-y)q_S)}{\omega^2}, \end{aligned}$$

where

$$\mathcal{A}_0 = \frac{c_I}{2} \begin{pmatrix} G_{11}^H G_{12}^H & -\left[G_{11}^H\right]^2 \\ \left[G_{11}^H\right]^2 & -G_{11}^H G_{12}^H \end{pmatrix}.$$

As for the Jost solution, if we want to calculate $\widetilde{\mathcal{G}}_x(x,y) := \mathcal{G}_x(x,y)e^{i(x-y)q_P}$, we need to look at all the sine and cosine terms:

$$\begin{aligned} &\frac{\sin\left[q_P(x-y)\right]}{q_P} e^{-i(x-y)q_P} = \frac{1 - e^{2iq_P(y-x)}}{2iq_P}; \\ &\cos((x-y)q_P)e^{-i(x-y)q_P} = \frac{1 + e^{2iq_P(y-x)}}{2}; \\ &\cos((x-y)q_S)e^{-i(x-y)q_P} = \frac{e^{i(y-x)(q_P-q_S)} + e^{i(y-x)(q_P+q_S)}}{2}; \\ &\left[q_P\sin((x-y)q_P) - q_S\sin((x-y)q_S)\right] e^{-i(x-y)q_P} \\ &= \frac{1}{2i}\left[q_P\left(1 - e^{2iq_P(y-x)}\right) - q_S\left(e^{i(y-x)(q_P-q_S)} - e^{i(y-x)(q_P+q_S)}\right)\right]. \end{aligned}$$

These expressions imply the following estimate on the $\widetilde{\mathcal{G}}_x(x,y)$

$$||\widetilde{\mathcal{G}}_x(x,y)|| \leq C|q_S|e^{\gamma(\xi)(y-x)},$$

multiplying both sides of (3.84) by $e^{-i(x-y)q_P}$, iterating the Faddeev solution $\mathcal{H}$ and taking the norm we find

$$\begin{aligned}
&||\mathcal{F}'(x,\xi)e^{-ixq_P}|| \leq ||\left(i\mathcal{F}_0(x,\xi)\mathcal{Q}(x) + \mathcal{P}(x,\xi)\right)e^{-ixq_P}|| + \sum_{l=1}^{\infty}||\mathcal{M}^{(l)}(x,\xi)||;\\
&||\mathcal{M}^{(l)}(x,\xi)|| \leq C\frac{e^{\gamma(\xi)(H-x)}}{(\max\{1,|\xi|\})^{l-1}}|q_S||\xi|e^{H\frac{\zeta_{P-S}}{2}}e^{-x\left(\frac{\zeta_P}{2}\right)}\\
&\cdot\frac{1}{l!}\left(\int_x^H ||V(t)||dt\right)^l = C\frac{e^{\gamma(\xi)(H-x)}}{(\max\{1,|\xi|\})^{l}}|q_S||q_P||\xi|e^{H\frac{\zeta_{P-S}}{2}}\\
&\cdot e^{-x\left(\frac{\zeta_P}{2}\right)}\frac{1}{l!}\left(\int_x^H ||V(t)||dt\right)^l,
\end{aligned}$$

so we recover (3.82). $\mathcal{M}(x,\xi)$ is bounded by a uniformly convergent series, then $\mathcal{F}'(x,\xi)$ is analytic in each sheet $\Xi_{\pm,\pm}$. □

From Theorem 3.8.3 and Lemma 3.8.6 we can obtain estimates of the Jost function and of the entire function $F(\xi)$, defined in Section 3.6.1, which shows that $F(\xi)$ is of exponential type. We present this result in the following lemma.

Corollary 3.8.7. *The function $F(\xi)$ is of exponential type with order one and type at most $12H$. In particular, it satisfies the inequality*

$$|F(\xi)| \leq C|\xi|^{20}e^{12H|\operatorname{Re}\xi|}. \tag{3.85}$$

Proof. From Theorem 3.8.3 we see that the Jost solution at $x=0$ is of exponential type

$$||\mathcal{F}(0,\xi)|| \leq C|\xi|e^{2H\mathfrak{b}(\xi)} \tag{3.86}$$

where

$$\mathfrak{b}(\xi) := \begin{cases} 0 & \xi \in \Xi_{+,+} \\ \operatorname{Re}\xi & \xi \in \Xi\setminus\Xi_{+,+} \end{cases}.$$

The inequality (3.86) is obtained from (3.5), Theorem 3.8.3 and Lemma 3.4.1 that imply

$$\gamma(\xi) = \begin{cases} 0 & \text{for } \xi \in \Xi_{+,\pm} \\ -2\operatorname{Im} q_P & \text{for } \xi \in \Xi_{-,\pm}, \end{cases}$$

and

$$\zeta_{P-S} = \begin{cases} 2\operatorname{Im}(q_P - q_S) & \text{for } \xi \in \Xi_{+,\pm} \\ 0 & \text{for } \xi \in \Xi_{-,\pm} \end{cases}.$$

From Lemma 3.8.6, we can see that it holds

$$||\mathcal{F}'(0)|| \leq C|\xi|^2 e^{2H\mathfrak{b}(\xi)},$$

and since $\mathcal{F}_\Theta := \mathcal{F}'(0) + \Theta\mathcal{F}(0)$ with

$$\Theta = \xi^2 \frac{2\hat{\mu}_I}{\hat{\mu}(0)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -\theta_3 & \theta_2 \\ -\theta_1 & 0 \end{pmatrix},$$

we have

$$||\mathcal{F}_\Theta(\xi)|| \leq C|\xi|^3 e^{2H\mathfrak{b}(\xi)}.$$

We know that $\Delta(\xi) = \det\left(A_1(\xi)\mathcal{F}_\Theta(\xi))A_2(\xi)\right)$ and from (3.68) we get

$$||\Delta(\xi)|| \leq C|\xi|^5 \qquad \xi \in \Xi_{+,+}$$

and

$$||\Delta(w_\bullet(\xi))|| \leq C|\xi|^5 e^{4H\operatorname{Re}\xi}, \qquad \bullet = P, S, PS.$$

Since $F(\xi) := \Delta(\xi)\Delta(w_P(\xi))\Delta(w_S(\xi))\Delta(w_{PS}(\xi))$, we have an estimate on the entire function $F(\xi)$:

$$||F(\xi))|| \leq C|\xi|^{20} e^{12H\operatorname{Re}\xi}, \qquad \xi \in \mathbb{C}. \qquad \square$$

Remark 3.8.8. *In Corollary 3.8.7 the term ξ^{20} and the power of the exponential $12H\operatorname{Re}\xi$ are not sharp estimates as we could have cancellations in the computation of the determinant. For example, from the definition of Θ we can see that the second row is of order ξ^2 while the first row is of lower order, so the determinant of $\mathcal{F}_\Theta(\xi)$ can never be, say, of polynomial order ξ^4. At the end of the section, we will obtain a sharp estimate on the type of the exponential.*

3.8.5 Estimates of the Jost solution and Jost function

The goal is to obtain an asymptotic expansion of the terms in the formula (3.73). Therefore, we compute the asymptotic expansion of $\mathcal{H}_0(x,\xi)$ in Lemma 3.8.9, then the asymptotics of the term $\int_x^\infty \widetilde{\mathcal{G}}(x,y)V(y)\mathcal{H}_0(y)dy$ in Lemma 3.8.10 and the asymptotics of the second iterate of the Volterra equation in (3.73) in Lemma 3.8.11. From the result of these three lemmas we obtain the asymptotic expansion of the Jost solution in Lemma 3.8.12 and of the Jost function in Proposition 3.8.13.

Lemma 3.8.9. *The unperturbed Faddeev solution $\mathcal{H}_0(x,\xi)$ for $|\xi| \to \infty$ and ξ on the physical sheet $\Xi_{+,+}$ admits the asymptotic expansion*

$$\mathcal{H}_0(x,\xi) = -\,\xi\frac{\hat{\mu}_I}{\omega^2}\begin{pmatrix} G_{11}^H & G_{11}^H \\ G_{12}^H & G_{12}^H \end{pmatrix} + \begin{pmatrix} G_{21}(x) & -\frac{c_I x}{2} G_{11}^H \\ G_{22}(x) & -\frac{c_I x}{2} G_{12}^H \end{pmatrix}$$
$$+\,\xi^{-1}\begin{pmatrix} \frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)} G_{11}^H & -\frac{\omega^2 c_I^2 x^2}{8\hat{\mu}_I} G_{11}^H \\ \frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)} G_{12}^H & -\frac{\omega^2 c_I^2 x^2}{8\hat{\mu}_I} G_{12}^H \end{pmatrix} + o(|\xi|^{-1}). \tag{3.87}$$

Proof. The unperturbed Faddeev solution can be written as

$$\mathcal{H}_0^{\pm}(x,\xi) = \begin{pmatrix} G_{21}(x) \pm iq_P \frac{\hat{\mu}_I}{\omega^2} G_{11}^H & -\frac{\hat{\mu}_I \xi}{\omega^2} G_{11}^H e^{-i(q_P - q_S)x} \\ G_{22}(x) \pm iq_P \frac{\hat{\mu}_I}{\omega^2} G_{12}^H & -\frac{\hat{\mu}_I \xi}{\omega^2} G_{12}^H e^{-i(q_P - q_S)x} \end{pmatrix}.$$

When $|\xi| \to \infty$ and $\xi \in \Xi_{+,+}$, we can expand the quasi-momenta in powers of ξ, namely $q_S = i\xi - i\frac{\omega^2}{2\hat{\mu}_I \xi} + O(|\xi|^{-3})$ and $q_P = i\xi - i\frac{\omega^2}{2(\hat{\lambda}_I+2\hat{\mu}_I)\xi} + O(|\xi|^{-3})$ which imply

$$q_P - q_S = \frac{i\omega^2 c_I}{2\xi} + O(|\xi|^{-3})$$
$$e^{-i(q_P-q_S)x} = 1 + \frac{\omega^2 c_I x}{2\hat{\mu}_I \xi} + \frac{\omega^4 c_I^2 x^2}{8\hat{\mu}_I^2 \xi^2} + O(\xi^{-3}).$$

Hence, plugging the asymptotic expansions into the definition of the unperturbed Faddeev solution we get an asymptotic expansion in powers of ξ

$$\mathcal{H}_0(x,\xi) =$$
$$\begin{pmatrix} G_{21}(x) - \frac{\hat{\mu}_I G_{11}^H}{\omega^2}(\xi - \frac{\omega^2}{2\sigma_I \xi} + O(\xi^{-3})) & -\frac{\hat{\mu}_I G_{11}^H \xi}{\omega^2}(1 + \frac{\omega^2 c_I x}{2\hat{\mu}_I \xi} + \frac{\omega^4 c_I^2 x^2}{8\hat{\mu}_I^2 \xi^2} + O(\xi^{-3})) \\ G_{22}(x) - \frac{\hat{\mu}_I G_{12}^H}{\omega^2}(\xi - \frac{\omega^2}{2\sigma_I \xi} + O(\xi^{-3})) & -\frac{\hat{\mu}_I G_{12}^H \xi}{\omega^2}(1 + \frac{\omega^2 c_I x}{2\hat{\mu}_I \xi} + \frac{\omega^4 c_I^2 x^2}{8\hat{\mu}_I^2 \xi^2} + O(\xi^{-3})) \end{pmatrix}$$
$$= -\xi\frac{\hat{\mu}_I}{\omega^2}\begin{pmatrix} G_{11}^H & G_{11}^H \\ G_{12}^H & G_{12}^H \end{pmatrix} + \begin{pmatrix} G_{21}(x) & -\frac{c_I G_{11}^H x}{2} \\ G_{22}(x) & -\frac{c_I G_{12}^H x}{2} \end{pmatrix} + \xi^{-1}\begin{pmatrix} \frac{\hat{\mu}_I G_{11}^H}{2(\hat{\lambda}_I+2\hat{\mu}_I)} & -\frac{\omega^2 c_I^2 G_{11}^H x^2}{8\hat{\mu}_I} \\ \frac{\hat{\mu}_I G_{12}^H}{2(\hat{\lambda}_I+2\hat{\mu}_I)} & -\frac{\omega^2 c_I^2 G_{12}^H x^2}{8\hat{\mu}_I} \end{pmatrix}$$
$$+\, o(|\xi|^{-1})$$

with $\sigma_I := \hat{\lambda}_I + 2\hat{\mu}_I$. $\square$

We can simplify the notation by defining

$$G^H := \begin{pmatrix} G_{11}^H & G_{11}^H \\ \\ G_{12}^H & G_{12}^H \end{pmatrix}$$

and

$$G_H(y) := \begin{pmatrix} G_{21}(y) & -\frac{c_I y}{2} G_{11}^H \\ \\ G_{22}(y) & -\frac{c_I y}{2} G_{12}^H \end{pmatrix}$$

which appear very often in the following.

The goal is to obtain an asymptotic expansion of the Jost solution and the Jost function. We compute this from the asymptotic expansion of the Faddeev solution. In the previous lemma we obtained the asymptotic expansion of the first term $\mathcal{H}_0$ in (3.73). In the next lemma we compute the asymptotic expansion of the first Volterra iterate $\mathcal{H}^{(1)}(x,\xi)$ in (3.73).

Lemma 3.8.10. *For $V \in \mathcal{V}_H$, the first Volterra iterate $\mathcal{H}^{(1)}(x,\xi)$ of (3.73) for $|\xi| \to \infty$ and $\operatorname{Re}\xi > 0$ in the physical sheet $\Xi_{+,+}$ admits the asymptotic expansion*

$$\int_x^H \widetilde{\mathcal{G}}(x,y) V(y) \mathcal{H}_0(y,\xi) dy = \int_x^H \frac{\hat{\mu}_I}{2\omega^2} V(y) G^H dy$$
$$- \frac{1}{4\xi} \frac{\hat{\mu}_I}{\omega^2} \int_x^H \left(\mathcal{B}(y) \frac{\omega^2 c_I (y-x)}{\hat{\mu}_I} + \mathcal{C} \frac{\omega^2 c_I^2 (y-x)^2}{4\hat{\mu}_I^2} \right) V(y) G^H dy$$
$$- \frac{1}{2\xi} \int_x^H V(y) G_H(y) dy - \frac{1}{4\xi} \frac{\hat{\mu}_I}{\omega^2} V(x) G^H + o(\xi^{-1}).$$

Proof. We know that $\widetilde{\mathcal{G}}(x,y)$ is the transformed kernel defined as

$$\widetilde{\mathcal{G}}(x,y) = \mathcal{A}(x) \left[\frac{1 - e^{2iq_P(y-x)}}{2iq_P} \right] + \mathcal{B}(y) \left[\frac{-e^{i(y-x)(q_P+q_S)} + e^{i(y-x)(q_P-q_S)}}{2iq_S} \right]$$
$$+ \mathcal{C} \left[\frac{-e^{2i(y-x)q_P} + e^{i(y-x)(q_P+q_S)} + e^{i(y-x)(q_P-q_S)} - 1}{2\omega^2} \right] = \widetilde{\mathcal{G}}_1(x,y) + \widetilde{\mathcal{G}}_2(x,y),$$

with

$$\widetilde{\mathcal{G}}_1(x,y) := \frac{\mathcal{A}(x)}{2iq_P} + \mathcal{B}(y) \frac{e^{i(y-x)(q_P-q_S)}}{2iq_S} + \mathcal{C} \frac{e^{i(y-x)(q_P-q_S)}}{2\omega^2} - \frac{\mathcal{C}}{2\omega^2}$$

and

$$\widetilde{\mathcal{G}}_2(x,y) := - \frac{\mathcal{A}(x)}{2iq_P} e^{2iq_P(y-x)} - \mathcal{B}(y) \frac{e^{i(y-x)(q_P+q_S)}}{2iq_S}$$

$$+\mathcal{C}\frac{e^{i(y-x)(q_P+q_S)}}{2\omega^2}-\frac{\mathcal{C}}{2\omega^2}e^{2iq_P(y-x)}.$$

We can divide the proof into three steps: in the first step we compute the contribution to the integral $\int_x^H \widetilde{\mathcal{G}}(x,y)V(y)\mathcal{H}_0(y,\xi)dy$ given by $\widetilde{\mathcal{G}}_1(x,y)$; in the second step we calculate the whole contribution coming from the term $\widetilde{\mathcal{G}}_2(x,y)$; and in the third step we sum up the results.

- *Step 1.* When $|\xi|\to\infty$ and $\operatorname{Re}\xi>0$ on the physical sheet $\Xi_{+,+}$, we can use the expansion of the quasi-momenta in terms of powers of ξ and we get the following form of $\widetilde{\mathcal{G}}_1(x,y)$

$$\widetilde{\mathcal{G}}_1(x,y)=\frac{\mathcal{A}(x)}{-2\xi}\left(1+O(\xi^{-2})\right)+\frac{\mathcal{B}(y)}{-2\xi}\left(1+O(\xi^{-2})\right)\left(1-\frac{\omega^2c_I(y-x)}{2\hat{\mu}_I\xi}\right.$$
$$\left.+O(\xi^{-2})\right)+\frac{\mathcal{C}}{2\omega^2}\left(1-\frac{\omega^2c_I(y-x)}{2\hat{\mu}_I\xi}+\frac{\omega^4c_I^2(y-x)^2}{8\hat{\mu}_I^2\xi^2}+O(\xi^{-3})\right)-\frac{\mathcal{C}}{2\omega^2}$$
$$=\frac{1}{-2\xi}\left(\mathcal{A}(x)+\mathcal{B}(y)+\frac{c_I(y-x)}{2\hat{\mu}_I}\mathcal{C}\right)+\frac{1}{4\xi^2}\left(\mathcal{B}(y)\frac{\omega^2c_I(y-x)}{\hat{\mu}_I}\right.$$
$$\left.+\mathcal{C}\frac{\omega^2c_I^2(y-x)^2}{4\hat{\mu}_I^2}\right)=\frac{1}{-2\xi}+\frac{1}{4\xi^2}\left(\mathcal{B}(y)\frac{\omega^2c_I(y-x)}{\hat{\mu}_I}+\mathcal{C}\frac{\omega^2c_I^2(y-x)^2}{4\hat{\mu}_I^2}\right)$$

where we can simplify

$$\mathcal{A}(x)+\mathcal{B}(y)+\frac{c_I(y-x)}{2\hat{\mu}_I}\mathcal{C}=\left(G_{11}^HG_{22}^H-G_{12}^HG_{21}^H\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}$$
$$=\det G\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\tag{3.88}$$

using (3.65). Then the integral term can be written as

$$\int_x^H\widetilde{\mathcal{G}}_1(x,y)V(y)\mathcal{H}_0(y,\xi)dy=\int_x^H\frac{\hat{\mu}_I}{2\omega^2}V(y)G^Hdy$$
$$-\frac{1}{4\xi}\frac{\hat{\mu}_I}{\omega^2}\int_x^H\left(\mathcal{B}(y)\frac{\omega^2c_I(y-x)}{\hat{\mu}_I}+\mathcal{C}\frac{\omega^4c_I^2(y-x)^2}{4\hat{\mu}_I^2}\right)V(y)G^Hdy$$
$$-\frac{1}{2\xi}\int_x^H V(y)G_H(y)dy+O\left(\frac{1}{|\xi|^2}\right).\tag{3.89}$$

- *Step 2.* In $\widetilde{\mathcal{G}}_2(x,y)$, the two exponentials for $|\xi| \to \infty$ and $\operatorname{Re}\xi > 0$ on the physical sheet $\Xi_{+,+}$ become

$$e^{2iq_P(y-x)} = e^{-2(y-x)\xi}\left(1+\tfrac{(y-x)\omega^2}{(\hat{\lambda}_I+2\hat{\mu}_I)\xi}+\tfrac{(y-x)^2\omega^4}{2(\hat{\lambda}_I+2\hat{\mu}_I)^2\xi^2}+O(|\xi|^{-3})\right);$$
$$e^{i(y-x)(q_P+q_S)} = e^{-2(y-x)\xi}\left(1+\tfrac{(y-x)\rho\omega^2}{2\hat{\mu}_I\xi}+\tfrac{(y-x)^2\rho^2\omega^4}{8\hat{\mu}_I^2\xi^2}+O(|\xi|^{-3})\right),$$

because $q_P + q_S = 2i\xi - i\frac{\omega^2\rho}{2\hat{\mu}_I\xi} + O(\xi^{-3})$ and where $\rho := \frac{\hat{\lambda}_I+3\hat{\mu}_I}{\hat{\lambda}_I+2\hat{\mu}_I}$. Then expanding the terms of $\widetilde{\mathcal{G}}_2(x,y)$ we get

$$-\frac{\mathcal{A}(x)}{2iq_P}e^{2iq_P(y-x)} = -e^{-2(y-x)\xi}\frac{\mathcal{A}(x)}{-2\xi}\left(1+O(\xi^{-2})\right)\left(1+O(\xi^{-1})\right);$$
$$-\mathcal{B}(y)\frac{e^{i(y-x)(q_P+q_S)}}{2iq_S} = -e^{-2(y-x)\xi}\frac{\mathcal{B}(y)}{-2\xi}\left(1+O(\xi^{-2})\right)\left(1+O(\xi^{-1})\right);$$
$$\mathcal{C}\frac{e^{i(y-x)(q_P+q_S)}}{2\omega^2} = e^{-2(y-x)\xi}\frac{\mathcal{C}}{2\omega^2}\left(1+\frac{(y-x)\rho\omega^2}{2\hat{\mu}_I\xi}+O(|\xi|^{-2})\right);$$
$$-\frac{\mathcal{C}}{2\omega^2}e^{2iq_P(y-x)} = -e^{-2(y-x)\xi}\frac{\mathcal{C}}{2\omega^2}\left(1+\frac{(y-x)\omega^2}{(\hat{\lambda}_I+2\hat{\mu}_I)\xi}+O(|\xi|^{-2})\right).$$

Summing up those terms we obtain

$$\widetilde{\mathcal{G}}_2(x,y) = \frac{e^{-2(y-x)\xi}}{2\xi}\left[\mathcal{A}(x)+\mathcal{B}(y)+\frac{c_I(y-x)}{2\hat{\mu}_I}\mathcal{C}+O(\xi^{-1})\right]$$
$$= \frac{e^{-2(y-x)\xi}}{2\xi}\left(1+O(\xi^{-1})\right) \tag{3.90}$$

using (3.88). Plugging (3.90) into $\int_x^H \widetilde{\mathcal{G}}_2(x,y)V(y)\mathcal{H}_0(y,\xi)dy$ and recalling Lemma 3.8.9 we get

$$\int_x^H \widetilde{\mathcal{G}}_2(x,y)V(y)\mathcal{H}_0(y,\xi)dy = \int_x^H \frac{e^{-2(y-x)\xi}}{2\xi}V(y)\left(-\xi\frac{\hat{\mu}_I}{\omega^2}\right)\begin{pmatrix} G_{11}^H & G_{11}^H \\ G_{12}^H & G_{12}^H \end{pmatrix}$$
$$= O(|\xi|^{-\infty}) \tag{3.91}$$

for $\operatorname{Re}\xi > 0$ by the dominated convergence theorem. The symbol $O(|\xi|^{-\infty})$ means that the quantity is $O(|\xi|^{-N})$ for any $N \in \mathbb{N}$.

- *Step 3.* Summing up (3.89) and (3.91) we obtain

$$\int_x^H \widetilde{\mathcal{G}}(x,y)V(y)\mathcal{H}_0(y,\xi)dy = \int_x^H \frac{\hat{\mu}_I}{2\omega^2}V(y)G^H dy$$

$$
-\frac{1}{4\xi}\frac{\hat{\mu}_I}{\omega^2}\int_x^H\left(\mathcal{B}(y)\frac{\omega^2 c_I(y-x)}{\hat{\mu}_I}+\mathcal{C}\frac{\omega^2 c_I^2(y-x)^2}{4\hat{\mu}_I^2}\right)V(y)G^H dy
$$

$$
-\frac{1}{2\xi}\int_x^H V(y)G_H(y)dy+o(\xi^{-1}). \qquad \square
$$

In the next lemma we compute the asymptotic expansion of the first Volterra iterate $\mathcal{H}^{(2)}(x,\xi)$ in (3.73).

Lemma 3.8.11. *The second Volterra iterate $\mathcal{H}^{(2)}(x,\xi)$ in* (3.73) *given by* (3.75) *for $|\xi|\to\infty$ and $\operatorname{Re}\xi>0$ in the physical sheet $\Xi_{+,+}$ admits the following asymptotic expansion*

$$
\begin{aligned}
\mathcal{H}^{(2)}(x,\xi) &= \int_x^H\int_y^H \widetilde{\mathcal{G}}(x,y)V(y)\widetilde{\mathcal{G}}(y,t)V(t)\mathcal{H}_0(t,\xi)dydt \\
&= \int_x^H\int_y^H \frac{e^{-\xi x}}{4\xi}\frac{\hat{\mu}_I}{\omega^2}V(y)V(t)G^H dydt+o(|\xi|^{-1}).
\end{aligned}
$$

Proof. From Lemma 3.8.10 we have that

$$
\widetilde{\mathcal{G}}(y,t)V(t)\mathcal{H}_0(t,\xi)=\frac{\hat{\mu}_I}{2\omega^2}V(t)G^H+O(\xi^{-1})
$$

while $\widetilde{\mathcal{G}}(x,y)=-\frac{1}{2\xi}+o(|\xi|^{-1})$ by Lemma 3.8.10 for $\xi\in\Xi_{+,+}$ and $\operatorname{Re}\xi>0$. So

$$
\widetilde{\mathcal{G}}(x,y)V(y)=-\frac{1}{2\xi}V(y)+o(\xi^{-1}),
$$

hence

$$
\begin{aligned}
&\int_x^H\int_y^H \widetilde{\mathcal{G}}(x,y)V(y)\widetilde{\mathcal{G}}(y,t)V(t)\mathcal{H}_0(t,\xi)dydt \\
&= -\int_x^H\int_y^H \frac{1}{4\xi}\frac{\hat{\mu}_I}{\omega^2}V(y)V(t)G^H dydt+o(|\xi|^{-1}). \qquad \square
\end{aligned}
$$

In the next lemma we use the results of Lemma 3.8.9, Lemma 3.8.10 and Lemma 3.8.11 in order to obtain the asymptotic expansion of the Jost solution in the physical sheet $\Xi_{+,+}$.

Lemma 3.8.12. *Let $V\in\mathcal{V}_H$, then the Jost solution $\mathcal{F}(0,\xi)$ has the following asymptotic expansion for $|\xi|\to\infty$ and $\operatorname{Re}\xi>0$ in $\Xi_{+,+}$:*

$$
\mathcal{F}(0,\xi)=-\xi\frac{\hat{\mu}_I}{\omega^2}G^H+G_H(y)-\frac{\hat{\mu}_I}{2\omega^2}\int_0^H V(y)G^H dy+\xi^{-1}\begin{pmatrix}\frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)}G^H_{11} & 0\\ \frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)}G^H_{12} & 0\end{pmatrix}
$$

$$
+\frac{1}{4\xi}\int_0^H c_I\mathcal{B}\left(\frac{y}{2}\right)yV(y)G^H dy+\frac{1}{2\xi}\int_0^H V(y)G_H(y)\,dy
$$
$$
-\frac{1}{4\xi}\int_0^H\int_y^H \frac{\hat{\mu}_I}{\omega^2}V(y)V(t)G^H dydt+o(\xi^{-1}).
$$

Proof. From the three previous lemmas we have that

$$
\begin{aligned}
&\mathcal{F}(x,\xi)e^{-ixq_P}=\mathcal{H}(x,\xi)=-\xi\frac{\hat{\mu}_I}{\omega^2}G^H+G_H(x)\\
&+\xi^{-1}\begin{pmatrix}\frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)}G^H_{11} & -\frac{\omega^2c_I^2x^2}{8\hat{\mu}_I}G^H_{11}\\ \\ \frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)}G^H_{12} & -\frac{\omega^2c_I^2x^2}{8\hat{\mu}_I}G^H_{12}\end{pmatrix}-\int_x^H\frac{\hat{\mu}_I}{2\omega^2}V(y)G^H dy\\
&+\frac{1}{4\xi}\frac{\hat{\mu}_I}{\omega^2}\int_x^H\left(\mathcal{B}(y)\frac{\omega^2c_I(y-x)}{\hat{\mu}_I}+\mathcal{C}\frac{\omega^2c_I^2(y-x)^2}{4\hat{\mu}_I^2}\right)V(y)G^H dy\\
&+\frac{1}{2\xi}\int_x^H V(y)G_H(y)\,dy-\frac{1}{4\xi}\int_x^H\int_y^H\frac{\hat{\mu}_I}{\omega^2}V(y)V(t)G^H dydt+o(\xi^{-1}).
\end{aligned}
\tag{3.92}
$$

Hence evaluating the Jost solution at $x=0$ we get

$$
\begin{aligned}
&\mathcal{F}(0,\xi)=\mathcal{H}(0,\xi)=-\xi\frac{\hat{\mu}_I}{\omega^2}G^H+G_H(y)-\frac{\hat{\mu}_I}{2\omega^2}\int_0^H V(y)G^H dy\\
&+\xi^{-1}\begin{pmatrix}\frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)}G^H_{11} & 0\\ \\ \frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)}G^H_{12} & 0\end{pmatrix}+\frac{1}{4\xi}\int_0^H\left(c_I\mathcal{B}(y)y+\mathcal{C}\frac{c_I^2y^2}{4\hat{\mu}_I}\right)V(y)G^H dy\\
&+\frac{1}{2\xi}\int_0^H V(y)G_H(y)\,dy-\frac{1}{4\xi}\int_0^H\int_y^H\frac{\hat{\mu}_I}{\omega^2}V(y)V(t)G^H dydt+o(\xi^{-1}),
\end{aligned}
$$

and using (3.65), we infer that

$$
\mathcal{B}(y)+\mathcal{C}\frac{c_Iy}{4\hat{\mu}_I}=\mathcal{B}\left(\frac{y}{2}\right).
$$

Thus, we can write

$$
\mathcal{F}(0,\xi)=-\xi\frac{\hat{\mu}_I}{\omega^2}G^H+G_H(0)-\frac{\hat{\mu}_I}{2\omega^2}\int_0^H V(y)G^H dy
$$

$$
+\xi^{-1}\begin{pmatrix} \frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)}G^H_{11} & 0 \\ \\ \frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)}G^H_{12} & 0 \end{pmatrix} + \frac{1}{4\xi}\int_0^H c_I\mathcal{B}\left(\frac{y}{2}\right) yV(y)G^H dy
$$

$$
+\frac{1}{2\xi}\int_0^H V(y)G_H(y)\,dy - \frac{1}{4\xi}\int_0^H\int_y^H \frac{\hat{\mu}_I}{\omega^2}V(y)V(t)G^H dydt + o(\xi^{-1}). \quad \square
$$

In the first proposition we use Lemma 3.8.12 and (3.92) to get an asymptotic expansion on the Jost function $\mathcal{F}_\Theta(\xi)$ defined in (3.66).

Proposition 3.8.13. *For* $V \in \mathcal{V}_H$*, the Jost function in the physical sheet for* $|\xi| \to \infty$ *and* $\operatorname{Re}\xi > 0$ *admits the asymptotic expansion*

$$
\mathcal{F}_\Theta(\xi) = \xi^3\chi_3 + \xi^2\chi_2 + \xi\chi_1 + \chi_0 + \begin{pmatrix} O(|\xi|^{-1}) & O(|\xi|^{-1}) \\ O(1) & O(1) \end{pmatrix}
$$

where

$$
\chi_3 = -\frac{\hat{\mu}_I}{\omega^2}\frac{2\hat{\mu}_I}{\hat{\mu}(0)}G^H_{11}\begin{pmatrix} 0 & 0 \\ \\ 1 & 1 \end{pmatrix},
$$

$$
\begin{aligned}
\chi_2 =& \frac{\hat{\mu}_I}{\omega^2}G^H + \frac{2\hat{\mu}_I}{\hat{\mu}(0)}\begin{pmatrix} 0 & 0 \\ \\ G_{21}(0) & 0 \end{pmatrix} \\
&- \frac{\hat{\mu}_I^2}{\hat{\mu}(0)\omega^2}\int_0^H \left(V_{11}(y)G^H_{11} + V_{12}(y)G^H_{12}\right)\begin{pmatrix} 0 & 0 \\ \\ 1 & 1 \end{pmatrix} dy,
\end{aligned}
$$

$$
\begin{aligned}
\chi_1 =& -G_H(0) + \frac{\hat{\mu}_I}{2\omega^2}\int_0^H V(y)G^H dy + \frac{\hat{\mu}_I}{\omega^2}\begin{pmatrix} \theta_3 G^H_{11} - \theta_2 G^H_{12} & \theta_3 G^H_{11} - \theta_2 G^H_{12} \\ \\ \theta_1 G^H_{11} & \theta_1 G^H_{11} \end{pmatrix} \\
&+ \frac{\hat{\mu}_I^2}{\hat{\mu}(0)(\hat{\lambda}_I + 2\hat{\mu}_I)}G^H_{11}\begin{pmatrix} 0 & 0 \\ \\ 1 & 0 \end{pmatrix} + \frac{\hat{\mu}_I}{\hat{\mu}(0)}\begin{pmatrix} 0 & 0 \\ \\ 1 & 0 \end{pmatrix}\int_0^H V(y)G_H(y)\,dy \\
&- \frac{\hat{\mu}_I^2}{2\hat{\mu}(0)\omega^2}\begin{pmatrix} 0 & 0 \\ \\ 1 & 0 \end{pmatrix}\int_0^H\int_y^H V(y)V(t)G^H dydt \\
&+ \frac{\hat{\mu}_I}{2\hat{\mu}(0)}\begin{pmatrix} 0 & 0 \\ \\ 1 & 0 \end{pmatrix}\int_0^H \left(c_I y\mathcal{B}(y/2)\right)V(y)G^H dy,
\end{aligned}
$$

and

$$\chi_0 = -\frac{1}{2}G^H - \frac{\hat{\mu}_I}{2(\hat{\lambda}_I + 2\hat{\mu}_I)}\begin{pmatrix} G^H_{11} & 0 \\ G^H_{12} & 0 \end{pmatrix}$$
$$-\frac{1}{2}\int_0^H V(y)G_H(y)\,dy - \frac{1}{4}\int_0^H (c_I y\mathcal{B}(y/2))\,V(y)G^H dy$$
$$+\frac{1}{4}\frac{\hat{\mu}_I}{\omega^2}\int_0^H\int_y^H V(y)V(t)G^H dydt$$
$$+\begin{pmatrix} -\theta_3 & \theta_2 \\ -\theta_1 & 0 \end{pmatrix} G_H(0) - \frac{\hat{\mu}_I}{\omega^2}\frac{1}{2}\begin{pmatrix} -\theta_3 & \theta_2 \\ -\theta_1 & 0 \end{pmatrix}\int_0^H V(y)G^H dy.$$

Proof. Since

$$\mathcal{F}(x,\xi) = \mathcal{H}(x,\xi)e^{ixq_P},$$

we have

$$\mathcal{F}'(x,\xi) = (iq_P)\mathcal{H}(x,\xi)e^{ixq_P} + \mathcal{H}'(x,\xi)e^{ixq_P}$$

and

$$\mathcal{F}'(0,\xi) = (iq_P)\mathcal{H}(0,\xi) + \mathcal{H}'(0,\xi).$$

We want to compute the derivative of the Jost solution up to the ξ^0 order. Differentiating (3.92) we get that

$$\mathcal{H}_0'(x,\xi) = \begin{pmatrix} G'_{21}(x) & -\frac{c_I}{2}G^H_{11} \\ G'_{22}(x) & -\frac{c_I}{2}G^H_{12} \end{pmatrix} + O(\xi^{-1})$$

where $G^H_{21}(x) := -\frac{c_I}{2}G^H_{11}(x-H) + G^H_{21}$ and $G_{22}(x) = -\frac{c_I}{2}G^H_{12}(x-H) + G^H_{22}$, so

$$\mathcal{H}_0'(0,\xi) = -\frac{c_I}{2}G^H + O(\xi^{-1}).$$

We know that $iq_P = -\xi + \frac{\omega^2}{2(\hat{\lambda}_I+2\hat{\mu}_I)\xi} + O(|\xi|^{-3})$, so

$$(iq_P)\mathcal{H}(0,\xi) = -\xi\mathcal{H}(0,\xi) + \frac{\omega^2}{2(\hat{\lambda}_I + 2\hat{\mu}_I)\xi}\mathcal{H}(0,\xi) + o(\xi^{-1}) =$$
$$\xi^2\frac{\hat{\mu}_I}{\omega^2}G^H - \xi G_H(0) + \xi\frac{\hat{\mu}_I}{2\omega^2}\int_0^H V(y)G^H dy$$

$$
- \begin{pmatrix} \frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)} G^H_{11} & 0 \\ \frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)} G^H_{12} & 0 \end{pmatrix} - \frac{1}{4}\int_0^H \left(c_I y \mathcal{B}(y/2)\right) V(y) G^H dy
$$
$$
- \frac{1}{2}\int_0^H V(y) G_H(y) dy + \frac{1}{4}\int_0^H \int_y^H \frac{\hat{\mu}_I}{\omega^2} V(y)V(t) G^H dy dt
$$
$$
+ \frac{\omega^2}{2(\hat{\lambda}_I + 2\hat{\mu}_I)\xi}(-\xi)\frac{\hat{\mu}_I}{\omega^2} G^H + o(\xi^{-1}).
$$

Finally, we have

$$
\begin{aligned}
\mathcal{F}'(0,\xi) =& \xi^2 \frac{\hat{\mu}_I}{\omega^2} G^H - \xi G_H(0) + \xi \frac{\hat{\mu}_I}{2\omega^2}\int_0^H V(y) G^H dy \\
& - \frac{1}{4}\int_0^H \left(c_I y \mathcal{B}(y/2)\right) V(y) G^H dy - \frac{1}{2}\int_0^H V(y) G_H(y) dy \\
& + \frac{1}{4}\int_0^H \int_y^H \frac{\hat{\mu}_I}{\omega^2} V(y)V(t) G^H dy dt - \begin{pmatrix} \frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)} G^H_{11} & 0 \\ \frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)} G^H_{12} & 0 \end{pmatrix} \\
& - \frac{\hat{\mu}_I}{2(\hat{\lambda}_I + 2\hat{\mu}_I)} G^H - \frac{c_I}{2} G^H + \frac{\hat{\mu}_I}{\omega^2} V(x) G^H + o(\xi^{-1}).
\end{aligned}
$$

Adding together the last two matrices we get

$$
- \frac{\hat{\mu}_I}{2(\hat{\lambda}_I + 2\hat{\mu}_I)} G^H - \frac{\hat{\lambda}_I + \hat{\mu}_I}{2(\hat{\lambda}_I + 2\hat{\mu}_I)} G^H = -\frac{1}{2} G^H
$$

and thus

$$
\begin{aligned}
\mathcal{F}'(0,\xi) =& \xi^2 \frac{\hat{\mu}_I}{\omega^2} G^H - \xi G_H(0) + \xi \frac{\hat{\mu}_I}{2\omega^2}\int_0^H V(y) G^H dy \\
& - \frac{1}{2} G^H - \frac{\hat{\mu}_I}{2(\hat{\lambda}_I + 2\hat{\mu}_I)} \begin{pmatrix} G^H_{11} & 0 \\ G^H_{12} & 0 \end{pmatrix} - \frac{1}{4}\int_0^H \left(c_I y \mathcal{B}(y/2)\right) V(y) G^H dy \\
& - \frac{1}{2}\int_0^H V(y) G_H(y)\, dy + \frac{1}{4}\int_0^H \int_y^H \frac{\hat{\mu}_I}{\omega^2} V(y)V(t) G^H dy dt + o(\xi^{-1}).
\end{aligned} \tag{3.93}
$$

Now we are able to calculate the expansion of $\mathcal{F}_\Theta(\xi) = \mathcal{F}'(0,\xi) + \Theta\mathcal{F}(0,\xi)$, but we should reflect on the fact that we calculated $\mathcal{F}(0,\xi)$ up to ξ^{-1} order and $\mathcal{F}'(0,\xi)$ up to

ξ^0 order. Since

$$\Theta := \begin{pmatrix} -\theta_3 & \theta_2 \\ 2\frac{\hat{\mu}_I}{\hat{\mu}(0)}\xi^2 - \theta_1 & 0 \end{pmatrix} = \xi^2 \frac{2\hat{\mu}_I}{\hat{\mu}(0)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -\theta_3 & \theta_2 \\ -\theta_1 & 0 \end{pmatrix},$$

$\mathcal{F}_\Theta(\xi)$ assumes the form

$$\mathcal{F}_\Theta(\xi) = \begin{pmatrix} \xi^2\chi_{2,11} + \xi\chi_{1,11} + \chi_{0,11} + o(1) & \xi^2\chi_{2,12} + \xi\chi_{1,12} + \chi_{0,12} + o(1) \\ \xi^3\chi_{3,21} + \xi^2\chi_{2,21} + \xi\chi_{1,21} + o(\xi) & \xi^3\chi_{3,22} + \xi^2\chi_{2,22} + \xi\chi_{1,22} + o(\xi) \end{pmatrix}$$

where $\chi_{i,jk}$ is the jk component of the matrix χ_i. Then, to know the terms of order ξ^0 in the second row, we should have calculated the Jost solution $\mathcal{F}(0,\xi)$ up to the order ξ^{-2}. The matrix Θ can be decomposed into $\Theta = \xi^2 M^1 + M^2$. Multiplying M^1 by any matrix leads to a matrix whose only non-zero terms are in the second row. Hence, we multiply $\xi^2 M^1$ by terms of $\mathcal{F}(0,\xi)$ of order up to ξ^{-1}, whereas we multiply M^2 by terms of $\mathcal{F}(0,\xi)$ of order up to ξ^0. Then we have

$$\begin{aligned}
\Theta\mathcal{F}(0,\xi) = &-\xi^3 \frac{\hat{\mu}_I}{\omega^2}\frac{2\hat{\mu}_I}{\hat{\mu}(0)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} G^H \\
&+ \xi^2 \frac{2\hat{\mu}_I}{\hat{\mu}(0)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} G_H(0) - \xi^2 \frac{\hat{\mu}_I}{2\omega^2}\frac{2\hat{\mu}_I}{\hat{\mu}(0)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \int_0^H V(y)G^H dy \\
&- \xi\frac{\hat{\mu}_I}{\omega^2} \begin{pmatrix} -\theta_3 & \theta_2 \\ -\theta_1 & 0 \end{pmatrix} G^H + \xi\frac{\hat{\mu}_I}{\hat{\mu}(0)}\frac{\hat{\mu}_I}{\hat{\lambda}_I + 2\hat{\mu}_I} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} G^H_{11} & 0 \\ G^H_{12} & 0 \end{pmatrix} \\
&+ \xi\frac{\hat{\mu}_I}{\hat{\mu}(0)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \int_0^H V(y)G_H(y)dy \\
&- \xi\frac{\hat{\mu}_I}{2\hat{\mu}(0)}\frac{\hat{\mu}_I}{\omega^2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \int_0^H \int_y^H V(y)V(t)G^H dydt \\
&+ \xi\frac{\hat{\mu}_I}{2\hat{\mu}(0)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \int_0^H (c_I y\mathcal{B}(y/2))\, V(y)G^H dy \\
&+ \begin{pmatrix} -\theta_3 & \theta_2 \\ -\theta_1 & 0 \end{pmatrix} G_H(0) - \frac{\hat{\mu}_I}{\omega^2}\frac{1}{2} \begin{pmatrix} -\theta_3 & \theta_2 \\ -\theta_1 & 0 \end{pmatrix} \int_0^H V(y)G^H dy + \begin{pmatrix} O(1) & O(1) \\ O(|\xi|) & O(|\xi|) \end{pmatrix}.
\end{aligned} \tag{3.94}$$

Adding (3.94) and (3.93) yields the statement of the proposition. □

A similar result to Proposition 3.8.13 can be found in [19] but with only two orders of expansion and without proof.

In the following, we will compute the asymptotic expansion of the determinant of the Jost function for ξ in the physical sheet $\Xi_{+,+}$ (Lemma 3.8.14) and then we do the same for the sheet $\Xi_{-,-}$ (Lemma 3.8.15) and the other sheets (Lemma 3.8.18 and Lemma 3.8.19). The goal is to find an asymptotic expansion for the entire function $F(\xi)$ as the product of all the Rayleigh determinants (see Theorem 3.8.21) and an exponential type estimate for $F(\xi)$ (see Theorem 3.8.22).

Lemma 3.8.14. *Let $V \in \mathcal{V}_H$, then the determinant of the Jost function for $|\xi| \to \infty$ and $\operatorname{Re}\xi > 0$ on the physical sheet $\Xi_{+,+}$ satisfies*

$$\det \mathcal{F}_\Theta(\xi) = \xi^3 \frac{\hat{\mu}_I}{\omega^2} c(0) + O(|\xi|^2)$$

where $c(0) := \frac{\hat{\lambda}(0)+\hat{\mu}(0)}{\hat{\lambda}(0)+2\hat{\mu}(0)}$.

Proof. We can write $\det \mathcal{F}_\Theta(\xi) = a\xi^5 + b\xi^4 + c\xi^3 + o(\xi^2)$ and we can see that

$$a = \frac{\hat{\mu}_I}{\omega^2} G_{11}^H \left(-\frac{\hat{\mu}_I}{\omega^2} \frac{2\hat{\mu}_I}{\hat{\mu}(0)} G_{11}^H \right) + \frac{\hat{\mu}_I}{\omega^2} G_{11}^H \left(\frac{\hat{\mu}_I}{\omega^2} \frac{2\hat{\mu}_I}{\hat{\mu}(0)} G_{11}^H \right) = 0.$$

The coefficient b is obtained by multiplication of the first row of the terms of order ξ^2 with the second row of the terms of order ξ^2 and by the second row of the term of order ξ^3 multiplied by the first row of the term of order ξ. Hence, only the first three terms play a role in the computation of b, as the other three terms have zero $11-$ and $12-$components. Thus,

$$b = -\frac{\hat{\mu}_I}{\omega^2} G_{11}^H \frac{2\hat{\mu}_I}{\hat{\mu}(0)} G_{21}^H(0) - \frac{\hat{\mu}_I}{\omega^2} \frac{2\hat{\mu}_I}{\hat{\mu}(0)} G_{11}^H \left[-G_{21}^H(0) \right] = 0.$$

Calculating the determinant it is important to know that

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} G_{11}^H & G_{11}^H \\ G_{12}^H & G_{12}^H \end{pmatrix} = \begin{pmatrix} a_1 G_{11}^H + a_2 G_{12}^H & a_1 G_{11}^H + a_2 G_{12}^H \\ a_3 G_{11}^H + a_4 G_{12}^H & a_3 G_{11}^H + a_4 G_{12}^H \end{pmatrix},$$

which means that multiplying a matrix with the same row to any matrix, the result is a matrix with the same rows. Moreover, matrix with the same rows have zero determinant. Keeping this in mind, we can calculate the coefficient c. First, we calculate the contribution to c given by the elements in the first row of order ξ^0 multiplied by the elements of the second row of order ξ^3

$$\begin{aligned}
c^{(03)} =& \frac{2\hat{\mu}_I^2}{\omega^2\hat{\mu}(0)}\frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)}(G_{11}^H)^2 + \frac{2\hat{\mu}_I^2}{\omega^2\hat{\mu}(0)}G_{11}^H\left[\theta_3 G_{21}^H(0)-\theta_2 G_{22}^H(0)\right] \\
&+ \frac{\hat{\mu}_I^2}{\omega^2\hat{\mu}(0)}G_{11}^H\int_0^H \left[V_{11}(y)G_{21}^H(y)+V_{12}(y)G_{22}^H(y)\right]dy \\
&+ \frac{\hat{\mu}_I^2}{\omega^2\hat{\mu}(0)}G_{11}^H\int_0^H \frac{c_I y}{2}\left[V_{11}(y)G_{11}^H+V_{12}(y)G_{12}^H\right]dy
\end{aligned}$$

where $G_{21}^H(y) := -\frac{c_I}{2}G_{11}^H(y-H)+G_{21}^H$ and $G_{22}(y) = -\frac{c_I}{2}G_{12}^H(y-H)+G_{22}^H$. Then we calculate the determinant considering the elements in the first row of order ξ^2 multiplied by the elements of the second row of order ξ^1

$$\begin{aligned}
c^{(21)} =& \frac{\hat{\mu}_I}{\omega^2}G_{11}^H G_{22}^H(0) - \frac{\hat{\mu}_I}{\omega^2}\frac{\hat{\mu}_I^2}{\hat{\mu}(0)(\hat{\lambda}_I+2\hat{\mu}_I)}(G_{11}^H)^2 \\
&+ \frac{\hat{\mu}_I^2}{\omega^2\hat{\mu}(0)}G_{11}^H\int_0^H -\frac{c_I y}{2}\left[V_{11}(y)G_{11}^H+V_{12}(y)G_{12}^H\right]dy \\
&- \frac{\hat{\mu}_I^2}{\omega^2\hat{\mu}(0)}G_{11}^H\int_0^H \left[V_{11}(y)G_{21}^H(y)+V_{12}(y)G_{22}^H(y)\right]dy.
\end{aligned}$$

Finally, we calculate the determinant considering the elements in the first row of order ξ^1 multiplied by the elements of the second row of order ξ^2:

$$\begin{aligned}
c^{(12)} =& (-G_{21}^H(0))\frac{\hat{\mu}_I}{\omega^2}G_{12}^H + (-G_{21}^H(0))\frac{-\hat{\mu}_I^2}{\omega^2\hat{\mu}(0)}\int_0^H \left[V_{11}(y)G_{11}^H+V_{12}(y)G_{12}^H\right]dy \\
&+ \left[-\frac{\hat{\mu}_I}{2\omega^2}\int_0^H \left[V_{11}(y)G_{11}^H+V_{12}(y)G_{12}^H\right]dy\right]\frac{2\hat{\mu}_I}{\hat{\mu}(0)}G_{21}^H(0) \\
&- \left[\theta_3 G_{11}^H-\theta_2 G_{12}^H\right]\frac{\hat{\mu}_I}{\omega^2}\frac{2\hat{\mu}_I}{\hat{\mu}(0)}G_{21}^H(0) = (-G_{21}^H(0))\frac{\hat{\mu}_I}{\omega^2}G_{12}^H \\
&- \left[\theta_3 G_{11}^H-\theta_2 G_{12}^H\right]\frac{\hat{\mu}_I}{\omega^2}\frac{2\hat{\mu}_I}{\hat{\mu}(0)}G_{21}^H(0).
\end{aligned}$$

Hence, summing $c^{(03)}+c^{(21)}+c^{(12)}$, we get $c^{(3)}$

$$\begin{aligned}
c =& \frac{2\hat{\mu}_I^2}{\omega^2\hat{\mu}(0)}G_{11}^H\left[\theta_3 G_{21}^H(0)-\theta_2 G_{22}^H(0)\right] + \frac{\hat{\mu}_I}{\omega^2}G_{11}^H G_{22}^H(0) - \frac{\hat{\mu}_I}{\omega^2}G_{12}^H G_{21}^H(0) \\
&- \frac{2\hat{\mu}_I^2}{\omega^2\hat{\mu}(0)}G_{21}^H(0)\left[\theta_3 G_{11}^H-\theta_2 G_{12}^H\right] = \frac{\hat{\mu}_I}{\omega^2} - \frac{2\theta_2\hat{\mu}_I^2}{\omega^2\hat{\mu}(0)}.
\end{aligned}$$

Now we recall that $\theta_2 := \frac{1}{2\hat{\mu}_I}\frac{\hat{\mu}(0)}{\hat{\lambda}(0)+2\hat{\mu}(0)}$, so $c = \frac{\hat{\mu}_I}{\omega^2}(1 - \frac{\hat{\mu}(0)}{\hat{\lambda}(0)+2\hat{\mu}(0)})$. Then, the determinant of the Jost function $\det \mathcal{F}_\Theta(\xi)$ for ξ in the physical sheet and $|\xi| \to \infty$, $\operatorname{Re}\xi > 0$ satisfies

$$\begin{aligned}\det \mathcal{F}_\Theta(\xi) &= \xi^3 \left(\frac{\hat{\mu}_I}{\omega^2} \left[1 - \frac{\hat{\mu}(0)}{\hat{\lambda}(0) + 2\hat{\mu}(0)} \right] \right) + O(|\xi|^2) \\ &= \xi^3 \left(\frac{\hat{\mu}_I}{\omega^2} c(0) \right) + O(|\xi|^2).\end{aligned}$$

□

We define

$$\begin{aligned}\widetilde{G}^H &= \begin{pmatrix} G^H_{11} & -G^H_{11} \\ G^H_{12} & -G^H_{12} \end{pmatrix} \\ \widetilde{G}_H(y) &= \begin{pmatrix} G_{21}(y) & \frac{c_I y}{2} G^H_{11} \\ G_{22}(y) & \frac{c_I y}{2} G^H_{12} \end{pmatrix},\end{aligned}$$

where, $\widetilde{G}^H$ is obtained from G^H by inverting the sign in the second column, while $\widetilde{G}_H(y)$ is obtained from $G_H(y)$ by inverting the sign in the second column.

In the following lemma we compute the asymptotics of $\det \mathcal{F}_\Theta(w_{PS}(\xi))$ in the sheet $\Xi_{+,+}$ of the Riemann surface Ξ, that is equal to $\det \mathcal{F}_\Theta(\xi)$ in the sheet $\Xi_{-,-}$.

Lemma 3.8.15. *Let $V \in \mathcal{V}_H$, then the determinant of the Jost function $\mathcal{F}_\Theta(w_{PS}(\xi))$ for $\operatorname{Re}\xi > 0$ and as $\operatorname{Re}\xi \to \infty$ in $\Xi_{+,+}$ is*

$$\det \mathcal{F}_\Theta(w_{PS}(\xi)) = \xi^3 \frac{\hat{\mu}_I}{\omega^2} c(0) + \xi^3 \mathscr{A}(\xi) + \xi^2 \mathscr{B}(\xi) + \mathscr{R}(\xi),$$

where

$$\mathscr{A}(\xi) := \frac{2\hat{\mu}_I^2}{\hat{\mu}(0)\omega^2} \int_0^H e^{2\xi y} V_{12}(y) dy$$

and $\mathscr{B}(\xi)$ can be written as

$$\mathscr{B}(\xi) = C_1 \left(\int_0^H f_1(y) e^{2y\xi} dy \right) \left(\int_0^H f_2(t) e^{2t\xi} dt \right)$$

with f_1 and f_2 being in L^1. The term $\mathscr{R}(\xi)$ is a remainder term containing all the terms polynomially smaller than $\xi^3 \mathscr{A}(\xi)$ and all the other terms dominated by $\xi^2 \mathscr{B}(\xi)$.

For simplicity $\mathscr{R}(\xi)$ denotes a remainder as in the statement that will be allowed to change between occurrences.

Proof. It is clear that $\mathcal{F}_\Theta(w_{PS}(\xi))$ for $\xi \in \Xi_{+,+}$ is equal to $\mathcal{F}_\Theta(\xi)$ for $\xi \in \Xi_{-,-}$. When we consider the Jost solution in the unphysical sheet $\Xi_{-,-}$, what changes is the expansion of q_P and q_S, which, for $\operatorname{Re}\xi \to +\infty$ and $\xi \in \Xi_{-,-}$, is

$$
\begin{aligned}
iq_P &= \xi - \frac{\omega^2}{2(\hat{\lambda}_I + 2\hat{\mu}_I)\xi} + O(|\xi|^{-3});\\
iq_S &= \xi - \frac{\omega^2}{2\hat{\mu}_I\xi} + O(|\xi|^{-3});\\
e^{-ix(q_P - q_S)} &= \left(1 - \frac{xc_I\omega^2 x}{2\hat{\mu}_I\xi} + \frac{c_I^2\omega^4 x^2}{8\hat{\mu}_I^2\xi^2} + O(|\xi|^{-3})\right).
\end{aligned}
$$

From the expansions above and the definition of $\mathcal{H}_0(x,\xi)$, it is clear that the unperturbed Faddeev solution in the sheet $\Xi_{-,-}$ is the same as in the physical sheet but with the first column with an opposite sign for odd powers of ξ and with the second column with an opposite sign for even powers of ξ as below

$$
\begin{aligned}
\mathcal{H}_0(x,\xi) =& \xi\frac{\hat{\mu}_I}{\omega^2}\begin{pmatrix} G_{11}^H & -G_{11}^H \\ G_{12}^H & -G_{12}^H \end{pmatrix} + \begin{pmatrix} G_{21}(x) & \frac{c_I x}{2}G_{11}^H \\ G_{22}(x) & \frac{c_I x}{2}G_{12}^H \end{pmatrix}\\
&+ \xi^{-1}\begin{pmatrix} -\frac{\hat{\mu}_I}{2(\hat{\lambda}_I + 2\hat{\mu}_I)}G_{11}^H & -\frac{\omega^2 c_I^2 x^2}{8\hat{\mu}_I}G_{11}^H \\ -\frac{\hat{\mu}_I}{2(\hat{\lambda}_I + 2\hat{\mu}_I)}G_{12}^H & -\frac{\omega^2 c_I^2 x^2}{8\hat{\mu}_I}G_{12}^H \end{pmatrix} + o(|\xi|^{-1}).
\end{aligned}
$$

For the Green function $\widetilde{\mathcal{G}}_1(x,y)$ instead there is invariance under change of sign on q_S and q_P (even function with respect to q_P and q_S). Then, $\int_x^H \widetilde{\mathcal{G}}_1(x,y)V(y)\mathcal{H}_0(y,\xi)dy$ has the same property as $\mathcal{F}_0(0,\xi)$ and assumes the following form

$$
\begin{aligned}
&\int_x^H \widetilde{\mathcal{G}}_1(x,y)V(y)\mathcal{H}_0(y,\xi)dy = \int_x^H \frac{\hat{\mu}_I}{2\omega^2}V(y)\widetilde{G}^H dy\\
&- \frac{1}{4\xi}\frac{\hat{\mu}_I}{\omega^2}\int_x^H \left(\mathcal{B}(y)\frac{\omega^2 c_I(y-x)}{\hat{\mu}_I} + \mathcal{C}\frac{\omega^4 c_I^2(y-x)^2}{4\hat{\mu}_I^2}\right)V(y)\widetilde{G}^H dy\\
&- \frac{1}{2\xi}\int_x^H V(y)\widetilde{G}_H(y)dy + O\left(\frac{1}{|\xi|^2}\right).
\end{aligned}
$$

We know also that in $\Xi_{-,-}$ the following asymptotic expansions hold

$$
e^{2iq_P(y-x)} = e^{2(y-x)\xi}\left(1 - \frac{(y-x)\omega^2}{(\hat{\lambda}_I + 2\hat{\mu}_I)\xi} + \frac{(y-x)^2\omega^4}{2(\hat{\lambda}_I + 2\hat{\mu}_I)^2\xi^2} + O(|\xi|^{-3})\right);
$$

$$e^{i(y-x)(q_P+q_S)} = e^{2(y-x)\xi}\left(1 - \frac{(y-x)\rho\omega^2}{2\hat{\mu}_I\xi} + \frac{(y-x)^2\rho^2\omega^4}{8\hat{\mu}_I^2\xi^2} + O(|\xi|^{-3})\right),$$

because $q_P + q_S = 2i\xi - i\frac{\omega^2\rho}{2\hat{\mu}_I\xi} + O(\xi^{-3})$ and where $\rho := \frac{\hat{\lambda}_I+3\hat{\mu}_I}{\hat{\lambda}_I+2\hat{\mu}_I}$. Then $\widetilde{\mathcal{G}}_2(x,y)$ has the same asymptotic expansion as in the physical sheet after replacing ξ with $-\xi$, namely

$$\widetilde{\mathcal{G}}_2(x,y) = -\frac{e^{2\xi(y-x)}}{2\xi} + \frac{e^{2\xi(y-x)}}{2\xi^2}\frac{(y-x)\omega^2}{2\hat{\hat{\mu}}_I\sigma_I}\left(\mathfrak{d}(x,y) + O(|\xi|^{-1})\right)$$

where

$$\mathfrak{d}(x,y) := (\hat{\lambda}+5\hat{\mu})\left(\mathcal{A}(x)+\mathcal{B}(y)\right) + \frac{(y-x)}{2\hat{\mu}\sigma}\mathcal{C}\left(\hat{\lambda}^2+5\hat{\mu}^2+6\hat{\lambda}\hat{\mu}\right).$$

Then, the integral term $\int_x^H \widetilde{\mathcal{G}}_2(x,y)V(y)\mathcal{H}_0(y,\xi)dy$ becomes

$$\begin{aligned}
&\int_x^H \widetilde{\mathcal{G}}_2(x,y)V(y)\mathcal{H}_0(y,\xi)dy = -\frac{\hat{\mu}_I}{2\omega^2}\int_x^H e^{2(y-x)\xi}V(y)\widetilde{G}^H dy \\
&- \frac{1}{2\xi}\int_x^H e^{2(y-x)\xi}V(y)\widetilde{G}_H(y)dy \\
&+ \frac{1}{2\xi}\int_x^H e^{2(y-x)\xi}\frac{y-x}{2\sigma}\mathfrak{d}(x,y)V(y)\widetilde{G}^H\left(1+O(|\xi|^{-1})\right)dy.
\end{aligned}$$

Adding all the terms, the Faddeev solution becomes

$$\begin{aligned}
\mathcal{H}(x,\xi) &= \xi\frac{\hat{\mu}_I}{\omega^2}\widetilde{G}^H + \widetilde{G}_H(x) \\
&+ \xi^{-1}\begin{pmatrix} -\frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)}G^H_{11} & -\frac{\omega^2 c_I^2 x^2}{8\hat{\mu}_I}G^H_{11} \\ -\frac{\hat{\mu}_I}{2(\hat{\lambda}_I+2\hat{\mu}_I)}G^H_{12} & -\frac{\omega^2 c_I^2 x^2}{8\hat{\mu}_I}G^H_{12} \end{pmatrix} - \int_x^H \frac{\hat{\mu}_I}{2\omega^2}V(y)\widetilde{G}^H dy \\
&- \frac{1}{4\xi}\int_x^H c_I(y-x)\left(\mathcal{B}(y) + \mathcal{C}\frac{c_I(y-x)}{4\hat{\mu}_I}\right)V(y)\widetilde{G}^H dy \\
&- \frac{1}{2\xi}\int_x^H V(y)\widetilde{G}_H(y)\,dy + O(|\xi|^{-2}) \\
&+ \frac{\hat{\mu}_I}{2\omega^2}\int_x^H e^{2(y-x)\xi}V(y)\widetilde{G}^H dy + \frac{1}{2\xi}\int_x^H e^{2(y-x)\xi}V(y)\widetilde{G}_H(y)dy \\
&- \frac{1}{2\xi}\int_x^H e^{2(y-x)\xi}\frac{y-x}{2\sigma}\mathfrak{d}(x,y)V(y)\widetilde{G}^H\left(1+O(|\xi|^{-1})\right)dy
\end{aligned}$$

which is the same as in the physical sheet inverting the first column of the terms with odd powers of ξ (G^H and $G_H(y)$ to $-\widetilde{G}^H$ and $-\widetilde{G}_H(y)$) and the second column of the terms with even powers of ξ (G^H and $G_H(y)$ to $\widetilde{G}^H$ and $\widetilde{G}_H(y)$), plus the last two terms, which are exponentially large and that, instead, were vanishing in the physical sheet. Let $\mathcal{D}_1(x,\xi)$, $\mathcal{D}_2(x,\xi)$ and $\mathcal{D}_3(x,\xi)$ be

$$\mathcal{D}_1(x,\xi) := \frac{\hat{\mu}_I}{2\omega^2}\int_x^H e^{2(y-x)\xi}V(y)\widetilde{G}^H dy$$
$$\mathcal{D}_2(x,\xi) := \frac{1}{2\xi}\int_x^H e^{2(y-x)\xi}V(y)\widetilde{G}_H(y)dy$$
$$\mathcal{D}_3(x,\xi) := -\frac{1}{2\xi}\int_x^H e^{2(y-x)\xi}\frac{y-x}{2\sigma}\mathfrak{d}(x,y)V(y)\widetilde{G}^H,$$

then the Jost function $\mathcal{F}_\Theta(w_{PS}(\xi))$ is the same as in the physical sheet, after the sign replacement explained above, plus all the contributions coming from the exponentially large terms $\mathcal{D}_1(x,\xi)$, $\mathcal{D}_2(x,\xi)$ and $\mathcal{D}_3(x,\xi)$. These $\mathcal{D}_i(x,\xi)$ terms are exponentially large since V is continuous and non-zero in the set $(H-\epsilon,H)$ and thus it has a definite sign in $(H-\epsilon,H)$. Then the part of the integral close to H has no cancellation effects and dominate the rest. For the Jost function, we need to calculate

$$\sum_{i=1}^{3}(iq_P)\mathcal{D}_i(0,\xi)+\mathcal{D}_i'(0,\xi)+\Theta\mathcal{D}_i(0,\xi).$$

On the one hand

$$\sum_{i=1}^{3}(iq_P)\mathcal{D}_i(0,\xi)+\mathcal{D}_i'(0,\xi) = -\xi\frac{\hat{\mu}_I}{2\omega^2}\int_0^H e^{2y\xi}V(y)\widetilde{G}^H$$
$$-\frac{1}{2}\int_0^H e^{2y\xi}V(y)\widetilde{G}_H(y)dy+\frac{1}{2}\int_0^H e^{2y\xi}\frac{y}{2\sigma}\mathfrak{d}(0,y)V(y)\widetilde{G}^H$$

while on the other hand

$$\sum_{i=1}^{3}\Theta\mathcal{D}_i(0,\xi) = \xi^2\frac{2\hat{\mu}_I}{\hat{\mu}(0)}\frac{\hat{\mu}_I}{2\omega^2}\begin{pmatrix}0&0\\1&0\end{pmatrix}\int_0^H e^{2y\xi}V(y)\widetilde{G}^H dy$$
$$+\xi\frac{\hat{\mu}_I}{\hat{\mu}(0)}\begin{pmatrix}0&0\\1&0\end{pmatrix}\int_0^H e^{2y\xi}V(y)\widetilde{G}_H(y)dy$$
$$-\frac{\xi}{2}\frac{2\hat{\mu}_I}{\hat{\mu}(0)}\begin{pmatrix}0&0\\1&0\end{pmatrix}\int_0^H e^{2y\xi}\frac{y}{2\sigma}\mathfrak{d}(0,y)V(y)\widetilde{G}^H$$
$$+\frac{\hat{\mu}_I}{2\omega^2}\begin{pmatrix}-\theta_3&\theta_2\\-\theta_1&0\end{pmatrix}\int_0^H e^{2y\xi}V(y)\widetilde{G}^H dy+\frac{1}{2\xi}\begin{pmatrix}-\theta_3&\theta_2\\-\theta_1&0\end{pmatrix}\int_0^H e^{2y\xi}V(y)\widetilde{G}_H(y)dy$$

$$-\frac{1}{2\xi}\begin{pmatrix}-\theta_3 & \theta_2\\ -\theta_1 & 0\end{pmatrix}\int_0^H e^{2y\xi}\frac{y}{2\sigma}\mathfrak{d}(0,y)V(y)\widetilde{G}^H.$$

Then $\mathcal{F}_\Theta(w_{PS}(\xi))$ admits the following form

$$\mathcal{F}_\Theta(w_{PS}(\xi)) = \xi^3\chi_3^{PS} + \xi^2\chi_2^{PS} + \xi\chi_1^{PS} + \chi_0^{PS} + E(\xi) + \mathscr{R}(\xi)$$

with χ_i^{PS}, for $i = 0, \dots, 3$, being the same as in the physical sheet after inverting the sign, according to the rules mentioned above. The term $E(\xi)$ contains the exponentially large terms and the remainder $\mathscr{R}(\xi)$ contains the polynomially lower order terms and the polynomially lower order terms of the terms containing the exponential $e^{2\xi y}$. In particular

$$\chi_3^{PS} = \frac{\hat{\mu}_I}{\omega^2}\frac{2\hat{\mu}_I}{\hat{\mu}(0)}G_{11}^H\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix},$$

$$\begin{aligned}\chi_2^{PS} =& \frac{\hat{\mu}_I}{\omega^2}\widetilde{G}^H + \frac{2\hat{\mu}_I}{\hat{\mu}(0)}G_{21}(0)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\\ &-\frac{\hat{\mu}_I^2}{\hat{\mu}(0)\omega^2}\int_0^H\left(V_{11}(y)G_{11}^H + V_{12}(y)G_{12}^H\right)\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix}dy,\end{aligned}$$

$$\begin{aligned}\chi_1^{PS} =& \widetilde{G}_H(0) - \frac{\hat{\mu}_I}{2\omega^2}\int_0^H V(y)\widetilde{G}^H dy - \frac{\hat{\mu}_I}{\omega^2}\begin{pmatrix}\theta_3 G_{11}^H - \theta_2 G_{12}^H & \theta_2 G_{12}^H - \theta_3 G_{11}^H\\ \theta_1 G_{11}^H & -\theta_1 G_{11}^H\end{pmatrix}\\ &-\frac{\hat{\mu}_I^2}{\hat{\mu}(0)(\hat{\lambda}_I + 2\hat{\mu}_I)}G_{11}^H\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \frac{\hat{\mu}_I}{\hat{\mu}(0)}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\int_0^H V(y)\widetilde{G}_H(y)dy\\ &-\frac{\hat{\mu}_I}{2\hat{\mu}(0)}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\int_0^H c_I y\mathcal{B}(y/2)V(y)\widetilde{G}^H dy,\end{aligned}$$

$$\begin{aligned}\chi_0^{PS} =& -\frac{\hat{\mu}_I}{2(\hat{\lambda}_I + 2\hat{\mu}_I)}\begin{pmatrix}G_{11}^H & 0\\ G_{12}^H & 0\end{pmatrix} - \frac{1}{2}\widetilde{G}^H - \frac{1}{2}\int_0^H V(y)\widetilde{G}_H(y)dy\\ &-\frac{1}{4}\int_0^H c_I y\mathcal{B}(y/2)V(y)\widetilde{G}^H dy\end{aligned}$$

$$+\begin{pmatrix}-\theta_3 & \theta_2\\ -\theta_1 & 0\end{pmatrix}\widetilde{G}_H(0)-\frac{\hat{\mu}_I}{\omega^2}\frac{1}{2}\begin{pmatrix}-\theta_3 & \theta_2\\ -\theta_1 & 0\end{pmatrix}\int_0^H V(y)\widetilde{G}^H dy,$$

and

$$\begin{aligned}E(\xi)=\;&\xi^2\frac{\hat{\mu}_I^2}{\omega^2\hat{\mu}(0)}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\int_0^H e^{2y\xi}V(y)\widetilde{G}^H dy-\xi\frac{\hat{\mu}_I}{2\omega^2}\int_0^H e^{2y\xi}V(y)\widetilde{G}^H dy\\ &+\xi\frac{\hat{\mu}_I}{\hat{\mu}(0)}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\int_0^H e^{2y\xi}V(y)\widetilde{G}_H(y)dy\\ &-\frac{\xi}{2}\frac{2\hat{\mu}_I}{\hat{\mu}(0)}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\int_0^H e^{2y\xi}\frac{y}{2\sigma}\mathfrak{d}(0,y)V(y)\widetilde{G}^H\\ &+\frac{\hat{\mu}_I}{2\omega^2}\begin{pmatrix}-\theta_3 & \theta_2\\ -\theta_1 & 0\end{pmatrix}\int_0^H e^{2y\xi}V(y)\widetilde{G}^H dy-\frac{1}{2}\int_0^H e^{2y\xi}V(y)\widetilde{G}_H(y)dy\\ &+\frac{1}{2}\int_0^H e^{2y\xi}\frac{y}{2\sigma}\mathfrak{d}(0,y)V(y)\widetilde{G}^H+\frac{1}{2\xi}\begin{pmatrix}-\theta_3 & \theta_2\\ -\theta_1 & 0\end{pmatrix}\int_0^H e^{2y\xi}V(y)\widetilde{G}_H(y)dy\\ &-\frac{1}{2\xi}\begin{pmatrix}-\theta_3 & \theta_2\\ -\theta_1 & 0\end{pmatrix}\int_0^H e^{2y\xi}\frac{y}{2\sigma}\mathfrak{d}(0,y)V(y)\widetilde{G}^H.\end{aligned}$$

The determinant of $\mathcal{F}_\Theta(w_{PS}(\xi))$ is then equal to

$$\det\mathcal{F}_\Theta(w_{PS}(\xi))=\xi^3\left(\frac{\hat{\mu}_I}{\omega^2}c(0)+\mathscr{A}(\xi)\right)+\xi^2\mathscr{B}(\xi)+\mathscr{R}(\xi)$$

where $\mathscr{A}(\xi)$ is

$$\mathscr{A}(\xi):=\frac{2\hat{\mu}_I^2}{\hat{\mu}(0)\omega^2}\int_0^H e^{2\xi y}V_{12}(y)dy.$$

The part of the determinant obtained from the multiplication of the exponentially large terms with each other is included in the term $\mathscr{B}(\xi)$ and they are zero up to order ξ^3. In the case that also the term of order $\xi^2\mathscr{B}(\xi)$ is zero, we get a worse estimate than the previous one. The term $\mathscr{B}(\xi)$ can be written as

$$\mathscr{B}(\xi)=C_1\left(\int_0^H f_1(y)e^{2y\xi}dy\right)\left(\int_0^H f_2(t)e^{2t\xi}dt\right)$$

where $f_1(y)$ and $f_2(y)$ are functions of the form $\sum_{i,j}C_{ij}yV_{ij}(y)$ which are in L^1 as $V\in\mathcal{V}_H$; hence

$$|\mathscr{B}(\xi)|\le Ce^{4H|\xi|}. \tag{3.95}$$

The remainder $\mathscr{R}(\xi)$ contains also all the terms coming from the iterates $\mathcal{H}^{(l)}(x,\xi)$ defined in (3.75), which are all dominated by the term $\xi^2\mathscr{B}(\xi)$ as $\widetilde{\mathcal{G}}(x,y)$ is of polynomial order ξ^{-1}, hence, by iteration, it keeps decreasing. □

In the following lemma we collect the asymptotic expansions of $\det \mathcal{F}_\Theta(\xi)$ and $\det \mathcal{F}_\Theta(w_{PS}(\xi))$ of Lemma 3.8.14 and 3.8.15 and we turn them into asymptotic expansions of the Rayleigh determinants $\Delta(\xi)$ and $\Delta(w_{PS}(\xi))$ using the relations of Lemma 3.8.1.

Lemma 3.8.16. *Let $V \in \mathcal{V}_H$, then the product of the Rayleigh determinants $\Delta(w_{PS}(\xi))\Delta(\xi)$ for $\operatorname{Re}\xi > 0$ and as $\operatorname{Re}\xi \to \infty$ in $\Xi_{+,+}$ admits the form*

$$\begin{aligned}\Delta(w_{PS}(\xi))\Delta(\xi) =&\xi^4\left[4\hat{\mu}^2(0)c^2(0)\omega^4\right]\\ &+\xi^4\left(\frac{2\omega^6\hat{\mu}^2(0)c(0)}{\hat{\mu}_I}\right)\mathscr{A}(\xi)+\xi^3\mathscr{B}(\xi)+\mathscr{R}(\xi).\end{aligned}$$

Proof. We know that $\det \mathcal{F}_\Theta(\xi) = \xi^3\left(\frac{\hat{\mu}_I}{\omega^2}c(0)\right)+O(|\xi|^2)$ and using Lemma 3.8.1 we have

$$\begin{aligned}\Delta(\xi) &= \det A_1(\xi)\det\mathcal{F}_\Theta(\xi)\det A_2(\xi)\\ &= (-2i\hat{\mu}(0)\hat{\mu}_I\xi)\left(\xi^3\left(\frac{\hat{\mu}_I}{\omega^2}c(0)\right)+O(|\xi|^2)\right)\left(\frac{i\omega^4}{\xi^2\hat{\mu}_I^2}\right)\\ &= \xi^2\left(2\hat{\mu}(0)c(0)\omega^2\right)+o(|\xi|^2)\end{aligned}\tag{3.96}$$

and

$$\begin{aligned}\Delta(w_{PS}(\xi)) &= \det A_1(\xi)\det\mathcal{F}_\Theta(w_{PS}(\xi))\det A_2(\xi)\\ &= -2i\hat{\mu}(0)\hat{\mu}_I\xi\left(\xi^3\left(\frac{\hat{\mu}_I}{\omega^2}c(0)\right)+\xi^3\mathscr{A}(\xi)+\xi^2\mathscr{B}(\xi)+\mathscr{R}(\xi)\right)\left(\frac{i\omega^4}{\xi^2\hat{\mu}_I^2}\right)\\ &= \xi^2\left(2\hat{\mu}(0)c(0)\omega^2\right)+\xi^2\left(\frac{2\omega^4\hat{\mu}(0)}{\hat{\mu}_I}\right)\mathscr{A}(\xi)+\xi\left(\frac{2\omega^4\hat{\mu}(0)}{\hat{\mu}_I}\right)\mathscr{B}(\xi)+\mathscr{R}(\xi).\end{aligned}\tag{3.97}$$

Then

$$\begin{aligned}\Delta(w_{PS}(\xi))\Delta(\xi) =&\xi^4\left[4\hat{\mu}^2(0)c^2(0)\omega^4\right]\\ &+\xi^4\left(\frac{2\omega^6\hat{\mu}^2(0)c(0)}{\hat{\mu}_I}\right)\mathscr{A}(\xi)+\xi^3\mathscr{B}(\xi)+\mathscr{R}(\xi).\end{aligned}$$ □

Remark 3.8.17. *In Lemma 3.6.10 we showed that* $\Delta(-\xi) = -\Delta(w_{PS}(\xi))$*. This is confirmed by the formulas* (3.96) *and* (3.97)*, where the polynomial term is identical, while in* (3.97) *a term arises which comes from all the exponentials which are no longer decaying in the limit* $|\xi| \to \infty$*.*

In the following lemma we compute the asymptotic expansion of the Rayleigh determinant $\Delta(w_P(\xi))$, from the asymptotic expansion of $\det \mathcal{F}_\Theta(w_P(\xi))$.

Lemma 3.8.18. *Let* $V \in \mathcal{V}_H$*, then the Rayleigh determinant* $\Delta(w_P(\xi))$ *for* $\operatorname{Re}\xi > 0$ *and as* $\operatorname{Re}\xi \to \infty$ *in* $\Xi_{+,+}$ *satisfies*

$$\Delta(w_P(\xi)) = \xi^4 \left(8\hat{\mu}_I^2 \left(G_{11}^H\right)^2\right) + \xi^3 \left(\frac{2\hat{\mu}(0)\omega^4}{\hat{\mu}_I}\right) \mathscr{A}^P(\xi) + \mathscr{R}^P(\xi), \tag{3.98}$$

where

$$\mathscr{A}^P(\xi) := \frac{2\hat{\mu}_I^3}{\hat{\mu}(0)\omega^4} G_{11}^H \int_0^H e^{2\xi y} a(y)dy \tag{3.99}$$

and

$$a(y) := V_{11}(y)G_{11}^H + V_{12}(y)G_{12}^H.$$

The remainder $\mathscr{R}^P(\xi)$ *contains the terms which are polynomially or exponentially smaller than* $\mathscr{A}^P(\xi)$*.*

Proof. By Lemma 3.8.1 we get

$$\begin{aligned}\Delta(w_P(\xi)) &= \det A_1(\xi) \det \mathcal{F}_\Theta(w_P(\xi)) \det A_2(\xi)\\ &= (-2i\hat{\mu}(0)\hat{\mu}_I\xi) \det \mathcal{F}_\Theta(w_P(\xi)) \left(\frac{i\omega^4}{\xi^2\hat{\mu}_I^2}\right)\\ &= \left(\frac{2\hat{\mu}(0)\omega^4}{\hat{\mu}_I\xi}\right) \det \mathcal{F}_\Theta(w_P(\xi)).\end{aligned} \tag{3.100}$$

As before, $\mathcal{F}_\Theta(w_P(\xi))$ for $\xi \in \Xi_{+,+}$ is equal to $\mathcal{F}_\Theta(\xi)$ for $\xi \in \Xi_{-,+}$. After some lengthy and tedious computation, as in the previous lemma, we obtain

$$\det \mathcal{F}_\Theta(w_{PS}(\xi)) = \xi^5 \left(-\frac{4\hat{\mu}_I^3}{\omega^4\hat{\mu}(0)} \left(G_{11}^H\right)^2\right) + \xi^4 \mathscr{A}^P(\xi) + \mathscr{R}^P(\xi)$$

with

$$\mathscr{A}^P(\xi) := \frac{2\hat{\mu}_I^3}{\hat{\mu}(0)\omega^4} G_{11}^H \int_0^H e^{2\xi y} \left[G_{11}^H V_{11}(y) + G_{12}^H V_{12}(y)\right] dy$$

and the remainder $\mathscr{R}^P(\xi)$ containing all the other terms which are dominated by the term $\xi^4\mathscr{A}^P(\xi)$. In the determinant, in contrast to the previous case, there are no terms containing the product of two integrals containing exponentials, as all the exponentially large terms only appear in the first column. Substituting the result into (3.100) we obtain

$$\Delta(w_P(\xi)) = \xi^4\left(-8\hat{\mu}_I^2\left(G_{11}^H\right)^2\right) + \xi^3\left(\frac{2\hat{\mu}(0)\omega^4}{\hat{\mu}_I}\right)A^P(\xi) + R^P(\xi). \qquad \square$$

In the following lemma we compute the asymptotic expansion of the Rayleigh determinant $\Delta(w_S(\xi))$, from the asymptotic expansion of $\det \mathcal{F}_\Theta(w_S(\xi))$.

Lemma 3.8.19. *Let $V \in \mathcal{V}_H$, then the Rayleigh determinant $\Delta(w_S(\xi))$ for $\operatorname{Re}\xi > 0$ and as $\operatorname{Re}\xi \to \infty$ in $\Xi_{+,+}$ admits the form*

$$\Delta(w_S(\xi)) = \xi^4\left(-8\hat{\mu}_I^2\left[G_{11}^H\right]^2\right) + \xi^2\left(\frac{2\hat{\mu}(0)\omega^4}{\hat{\mu}_I}\right)\mathscr{A}^S(\xi) + \mathscr{R}^S(\xi) \tag{3.101}$$

where

$$\mathscr{A}^S(\xi) := \frac{\hat{\mu}_I^3}{\hat{\mu}(0)\omega^4}\int_0^H e^{2\xi y}\left[\left(\theta_2 G_{12}^H - \frac{\omega^2}{\hat{\mu}_I}G_{21}(0)\right)a(y) - \theta_2 G_{11}^H b(y)\right]dy \tag{3.102}$$

and

$$\begin{aligned} a(y) &:= V_{11}(y)G_{11}^H + V_{12}(y)G_{12}^H \\ b(y) &:= V_{21}(y)G_{11}^H + V_{22}(y)G_{12}^H. \end{aligned}$$

Proof. By Lemma 3.8.1 we get

$$\begin{aligned} \Delta(w_S(\xi)) &= \det A_1(\xi)\det\mathcal{F}_\Theta(w_S(\xi))\det A_2(\xi) \\ &= (-2i\hat{\mu}(0)\hat{\mu}_I\xi)\det\mathcal{F}_\Theta(w_S(\xi))\left(\frac{i\omega^4}{\xi^2\hat{\mu}_I^2}\right) \\ &= \left(\frac{2\hat{\mu}(0)\omega^4}{\hat{\mu}_I\xi}\right)\det\mathcal{F}_\Theta(w_S(\xi)). \end{aligned} \tag{3.103}$$

As before, $\mathcal{F}_\Theta(w_S(\xi))$ for $\xi \in \Xi_{+,+}$ is equal to $\mathcal{F}_\Theta(\xi)$ for $\xi \in \Xi_{+,-}$. After some long computations, we obtain

$$\det\mathcal{F}_\Theta(w_S(\xi)) = \xi^5\left(-\frac{4\hat{\mu}_I^3}{\omega^4\hat{\mu}(0)}\left(G_{11}^H\right)^2\right) + \xi^3\mathscr{A}^S(\xi) + \mathscr{R}^S(\xi)$$

where

$$\mathscr{A}^S(\xi) := \frac{\hat{\mu}_I^3}{\hat{\mu}(0)\omega^4} \int_0^H e^{2\xi y} \left[\left(\theta_2 G_{12}^H - \frac{\omega^2}{\hat{\mu}_I} G_{21}(0) \right) a(y) - \theta_2 G_{11}^H b(y) \right] dy$$

and

$$\begin{aligned} a(y) &:= V_{11}(y) G_{11}^H + V_{12}(y) G_{12}^H \\ b(y) &:= V_{21}(y) G_{11}^H + V_{22}(y) G_{12}^H. \end{aligned}$$

The remainder $\mathscr{R}^S(\xi)$ contains all the rest of the terms, which are dominated by $\xi^3 \mathscr{A}^S(\xi)$. In the determinant, as opposed to the case of the sheet $\Xi_{-,-}$, there are no terms containing the product of two integrals containing exponentials, as all the exponential terms are present only in the second column. Substituting the result in (3.103) we obtain

$$\Delta(w_S(\xi)) = \xi^4 \left(-8\hat{\mu}_I^2 \left[G_{11}^H \right]^2 \right) + \xi^2 \left(\frac{2\hat{\mu}(0)\omega^4}{\hat{\mu}_I} \right) \mathscr{A}^S(\xi) + \mathscr{R}^S(\xi). \qquad \square$$

Remark 3.8.20. *In Lemma 3.6.10, we showed that* $\Delta(w_P(-\xi)) = \Delta(w_S(\xi))$. *This is confirmed by the formulas* (3.98) *and* (3.101), *where the polynomial term, that is of even power, is identical, whereas the next order is different because in the limit* $|\xi| \to \infty$ *different terms are decaying in the two sheets.*

Now, we use the results of Lemma 3.8.16, Lemma 3.8.18 and Lemma 3.8.19, in order to obtain an asymptotic expansion of the entire function $F(\xi)$ defined in (3.28) in Section 3.6.1, as shown in the following theorem.

Theorem 3.8.21. *Let* $V \in \mathcal{V}_H$, *then the entire function* $F(\xi)$, *product of all Rayleigh determinants, admits the following form in the complex plane for* $\operatorname{Re}\xi > 0$ *and as* $\operatorname{Re}\xi \to +\infty$*:*

$$\begin{aligned} F(\xi) &= \Delta(w_S(\xi))\Delta(w_P(\xi))\Delta(w_{PS}(\xi))\Delta(\xi) \\ &= \xi^{12} \left(128\hat{\mu}^2(0)c(0)\omega^6\hat{\mu}_I^3 \left[G_{11}^H \right]^4 \right) \mathscr{A}(\xi) - \xi^8 C \mathscr{B}(\xi) \mathscr{A}^P(\xi) \mathscr{A}^S(\xi) + \mathscr{R}(\xi). \end{aligned}$$

Proof. From Lemma 3.8.18 and Lemma 3.8.19 we get

$$\Delta(w_S(\xi))\Delta(w_P(\xi)) = \xi^8 \left(64\hat{\mu}_I^4 \left[G_{11}^H \right]^4 \right) - \xi^5 \left[\frac{4\omega^8\hat{\mu}^2(0)}{\hat{\mu}_I^2} \right] \mathscr{A}^P(\xi)\mathscr{A}^S(\xi) + \mathscr{R}(\xi)$$

where we included all the other terms of $\Delta(w_{PS}(\xi))\Delta(\xi)$ in the remainder $\mathscr{R}(\xi)$ as they are dominated by $\xi^5 \left[\frac{4\omega^8\hat{\mu}^2(0)}{\hat{\mu}_I^2} \right] \mathscr{A}^P(\xi)\mathscr{A}^S(\xi)$. Using Lemma 3.8.16 we have

$$\Delta(w_{PS}(\xi))\Delta(\xi) = \xi^4 \left[4\hat{\mu}^2(0)c^2(0)\omega^4 \right]$$

$$+\,\xi^4\left(\frac{2\omega^6\hat{\mu}^2(0)c(0)}{\hat{\mu}_I}\right)\mathscr{A}(\xi)+\xi^3\mathscr{B}(\xi)+\mathscr{R}(\xi).$$

Then, the entire function F, for $\operatorname{Re}\xi \to +\infty$, is

$$\begin{aligned} F(\xi) &= \Delta(w_S(\xi))\Delta(w_P(\xi))\Delta(w_{PS}(\xi))\Delta(\xi) \\ &= \xi^{12}\left(128\hat{\mu}^2(0)c(0)\omega^6\hat{\mu}_I^3\left[G_{11}^H\right]^4\right)\mathscr{A}(\xi)-\xi^8 C\mathscr{B}(\xi)\mathscr{A}^P(\xi)\mathscr{A}^S(\xi)+\mathscr{R}(\xi). \end{aligned} \tag{3.104}$$

In (3.104) we kept the largest polynomial order in ξ and the largest term, which is the one obtained by the product of four exponentials. All the other terms of the remainder have smaller exponential order than the term containing $\xi^8\mathscr{B}(\xi)\mathscr{A}^P(\xi)\mathscr{A}^S(\xi)$, or smaller polynomial order than this term, hence they are dominated by it. □

In Corollary 3.8.7 we found a first exponential type estimate of $F(\xi)$. In the next theorem we show an improved exponential type estimate of $F(\xi)$ after having computed the determinants of the Jost function in the different sheets. This make sense because in the computation of the determinants there has been several cancellations and the type of the exponentially large terms arising in the unphysical sheets is different from one sheet to another, as we have seen previously.

Theorem 3.8.22. *Let $V \in \mathcal{V}_H$, then the entire function $F(\xi)$ is of exponential type and for $\operatorname{Re}\xi > 0$ and as $\operatorname{Re}\xi \to \infty$ in the complex plane*

$$|F(\xi)| \le C\xi^8 e^{8H|\operatorname{Re}\xi|} \tag{3.105}$$

Proof. The proof follows from Theorem 3.8.21 as the second term dominates the first one and all the terms inside the remainder $\mathscr{R}(\xi)$. Moreover, we have seen in (3.95) that

$$|\mathscr{B}(\xi)| \le Ce^{4H|\xi|}$$

and from (3.99) and (3.102), it holds

$$\begin{aligned} |\mathscr{A}^P(\xi)| &\le Ce^{2H|\xi|} \\ |\mathscr{A}^S(\xi)| &\le Ce^{2H|\xi|} \end{aligned}$$

as $V \in \mathcal{V}_H \subset L^1$. Thus there exists a constant $C > 0$ so that (3.105) is satisfied. □

3.9 Direct results

In this section, we present the direct results on the number of resonances and the resonance-free regions, which are implied by the asymptotic expansion of $F(\xi)$ (see Theorem

3.8.21) and by the exponential type estimate of $F(\xi)$ (see Theorem 3.8.22). In Theorem 3.6.12 we proved that $F(\xi)$ is entire and together with the result of Theorem 3.8.21, we show in Theorem 3.9.1 that $F(\xi)$ is in the Cartwright class (Definition 2.4.6) with indices $\rho_{\pm}(F) \leq 8H$.

Theorem 3.9.1. *The function $F(z)$ is in the Cartwright class with*

$$\rho_{\pm}(F) \leq 8H.$$

Remark 3.9.2. *The exact value of the indices $\rho_{\pm}$ of the function F might be obtained in the same way as in Chapter 2 in Theorem 2.5.44. However, this would require solving the inverse resonance problem, which is beyond the scope of this work.*

Proof. In order to prove that $F(z)$ is in the Cartwright class, we define $z := i\xi = x+iy$, so $x = -\operatorname{Im}\xi$ and $y = \operatorname{Re}\xi$. We need to prove that

$$\int_{\mathbb{R}} \frac{\log^+|F(x)|dx}{1+x^2} < \infty, \quad \rho_+(F) \leq 8H,\ \rho_-(F) \leq 8H,$$

where $\rho_{\pm}(F) = \limsup_{y\to\infty} \frac{\log|F(\pm iy)|}{y}$. In Theorem 3.8.21 we have seen that

$$\begin{aligned} F(\xi) &= -\xi^8 C\mathscr{B}(\xi)\mathscr{A}^P(\xi)\mathscr{A}^S(\xi) + \mathscr{R}(\xi) \\ &= -\xi^8 C\mathscr{B}(\xi)\mathscr{A}^P(\xi)\mathscr{A}^S(\xi)\left(1 + \frac{\mathscr{R}(\xi)}{\xi^8 C\mathscr{B}(\xi)\mathscr{A}^P(\xi)}\right), \end{aligned}$$

where the last fraction that tends to zero as $\operatorname{Re}\xi \to +\infty$. Then, we have

$$\int_{\mathbb{R}} \frac{\log^+|F(x)|}{1+x^2}dx \leq \int_{\mathbb{R}} \frac{\log\left(C|x|^{12}(1+o(1))\right)}{1+x^2}dx < \infty.$$

For the index ρ_+ we have that $\xi = i(+iy) = -y$, so $\operatorname{Re}\xi = -y$ and $|\operatorname{Re}\xi| = y$, and thus

$$\rho_+(F) \leq \limsup_{y\to\infty} \frac{8\log|y| + 8Hy}{y} = 8H.$$

While for the index ρ_- we have that, $\xi = i(-iy) = y$, so, $\operatorname{Re}\xi = y$ and $|\operatorname{Re}\xi| = y$, whence

$$\rho_-(F) \leq \limsup_{y\to\infty} \frac{20\log|y| + 8Hy}{y} = 8H. \qquad \square$$

The following result is an application of the Levinson theorem (see Theorem 2.4.7), once we determined, in Theorem 3.9.1, $F(\xi)$ to be in the Cartwright class.

Corollary 3.9.3. *Let $V \in \mathcal{V}_H$, then*

$$\mathcal{N}_-(r,F) \le \frac{8Hr}{\pi}(1+o(1)), \qquad r\to\infty,$$

and

$$\mathcal{N}_+(r,F) \le \frac{8Hr}{\pi}(1+o(1)), \qquad r\to\infty.$$

Moreover, for each $\delta > 0$ the number of complex resonances with real part with modulus $\le r$ lying outside both of the two sectors $|\arg\xi - \frac{\pi}{2}| < \delta$, $|\arg\xi - \frac{3\pi}{2}| < \delta$ is $o(r)$ for large r.

As explained in Chapter 2, the previous result tells us how many resonances we are expected to have in a ball of radius r in the complex plane, for large values of r.

In the following theorem, we obtain some estimates of the resonances, which tell us where they are localized on the complex plane and vice-versa the forbidden domain for them. These are obtained from the asymptotics of $F(\xi)$ in Theorem 3.8.21 and from the fact that the resonances are the zeros of $F(\xi)$.

Theorem 3.9.4. *Let $V \in \mathcal{V}_H$, then for any zero ξ_n of the function $F(\xi)$ the following estimate is fulfilled:*

$$|\xi_n| \le Ce^{2H|\operatorname{Re}\xi_n|}.$$

Proof. From Theorem 3.8.21 we know that the asymptotic expansion of $F(\xi)$ is

$$\left|F(\xi) - \xi^{12}\left(128\hat{\mu}^2(0)c(0)\omega^6\hat{\mu}_I^3\left[G_{11}^H\right]^4\right)\mathscr{A}(\xi)\right| \le$$
$$\le |-\xi^8 C\mathscr{B}(\xi)\mathscr{A}^P(\xi)\mathscr{A}^S(\xi) + \mathscr{R}(\xi)| \le C|\xi^8|e^{8H|\operatorname{Re}\xi|}.$$

Evaluating this at a resonance $\xi = \xi_n$, as ξ_n is a zero of F, we get

$$|\xi_n^{12}|\left|\int_0^H e^{2\xi y}V_{12}(y)dy\right| \le C|\xi_n^8|e^{8H|\operatorname{Re}\xi_n|}, \tag{3.106}$$

since, we recall

$$\mathscr{A}(\xi) := \frac{2\hat{\mu}_I^2}{\hat{\mu}(0)\omega^2}\int_0^H e^{2\xi y}V_{12}(y)dy.$$

The term $\left|\int_0^H e^{2\xi y}V_{12}(y)dy\right|$ is bounded from below, because V_{12} is continuous and non zero in $(H-\epsilon, H)$ for $\epsilon > 0$, hence with definite sign. Then the part of the integral in $(H-\epsilon, H)$ has no cancellation effects and dominate the rest. Thus (3.106) becomes

$$|\xi_n|^4 \leq Ce^{8H|\operatorname{Re}\xi_n|}$$

and hence

$$|\xi_n| \leq Ce^{2H|\operatorname{Re}\xi_n|}.$$

□

Remark 3.9.5. *We notice that the constant C in Theorem 3.9.4 is not made explicit, because it is not so important. Indeed, C would give a bound on the $\operatorname{Im}\xi_n$ for small values of $\operatorname{Re}\xi_n$, but since the estimate is obtained for large values of $\operatorname{Re}\xi_n$, the constant C is not important.*

Appendix

Appendix A

A.1 Semiclassical pseudodifferential operators

The following theorems and corollary refer to [36].

Definition A.1.1. *Let $a \in \mathcal{S}\left(\mathbb{R}^{2n}\right)$ and $u \in \mathcal{S}\left(\mathbb{R}^{n}\right)$, then the integral*

$$Op_\epsilon\left(a\right)u(x) = \frac{1}{\left(2\pi\epsilon\right)^n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{\epsilon}\langle x-y,\xi\rangle}a(x,\xi)u(y)dyd\xi \tag{A.1}$$

converges and defines $Op_\epsilon(a)u \in \mathcal{S}\left(\mathbb{R}^{n}\right)$. The quantity $Op_\epsilon(a)$ is called semi-classical differential operator and if is $a \in \mathcal{S}\left(\mathbb{R}^{2n}\right)$, then it is a mapping from $\mathcal{S}'\left(\mathbb{R}^{n}\right)$ to $\mathcal{S}\left(\mathbb{R}^{n}\right)$.

Theorem A.1.2. *Assume that $a, b \in C_c^\infty\left(\mathbb{R}^{2n}\right)$. Then*

$$Op_\epsilon(a)Op_\epsilon(b) = Op_\epsilon(a\#b)$$

where $a\#b(x,\xi;\epsilon) \in S\left(\mathbb{R}^{2n}\right)$ uniformly in ϵ and satisfies the expansion in $S\left(\mathbb{R}^{2n}\right)$ as $\epsilon \to 0$

$$a\#b(x,\xi;\epsilon) \sim \sum_{j=0}^{\infty}(-i\epsilon)^j \sum_{\alpha,|\alpha|=j}\frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi),$$

where $a\#b$ is called Moyal product of a and b and we used the multi-index notation where $\alpha = (\alpha_1, \dots, \alpha_n)$ and for example $\alpha! = \alpha_1!, \dots, \alpha_n!$.

Corollary A.1.3. *It holds that*

$$a\#b = ab + O(\epsilon)_{S(\mathbb{R}^{2n})}$$

that can also be written more informally as

$$Op_\epsilon(a)Op_\epsilon(b) = Op_\epsilon(ab) + O(\epsilon)$$

.

Theorem A.1.4. *Let* $\Lambda_\alpha(x,\xi)$ *be an eigenvalue of* $H_0(x,\xi)$*, and assume* $U \subset T^*\mathbb{R}^2\backslash\{0\}$ *to be open. Assume that* $\Lambda_\alpha(x,\xi)$ *has constant multiplicity* m_α *for all* $(x,\xi) \in U$*. There exist* $\Phi_{\alpha,m}(x,\xi) \in \mathcal{L}\left(\mathcal{D}, L^2\left(\mathbb{R}^-\right)\right)$ *and* $a_{\alpha,m}(x,\xi) \in \mathcal{L}\left(L^2\left(\mathbb{R}^-\right), \mathcal{D}\right)$ *which admit asymptotic expansions*

$$\Phi_{\alpha,\epsilon}(x,\xi) \sim \sum_{m=0}^{\infty} \Phi_{\alpha,m}(x,\xi)\epsilon^m,$$

$$a_{\alpha,\epsilon}(x,\xi) \sim \sum_{m=0}^{\infty} a_{\alpha,m}(x,\xi)\epsilon^m,$$

and satisfy

$$H \circ \Phi_{\alpha,\epsilon}(x,\xi) = \Phi_{\alpha,\epsilon} \circ a_{\alpha,\epsilon}(x,\xi) + O(\epsilon^\infty)$$

where $\circ$ *denotes the composition of symbols. Moreover,* $a_{\alpha,0}(x,\xi)\Lambda_\alpha(x,\xi)I$ *and* $\Phi_{\alpha,0}(x,\xi)$ *is the projection onto the eigenspace associated with* $\Lambda_\alpha(x,\xi)$*.*

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