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Åström, Karl Johan

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LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

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K.J. ÅSTRÖM

Department of Automatic Control
Lund Institute of Technology

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LIMITATIONS OF SYSTEM PERFORMANCE DUE TO TIME DELAYS, INSTABILITY AND NON-MINIMUM-PHASE CHARACTERISTICS - AN EXAMPLE

K.J. ÅSTRÖM

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1. INTRODUCTION

It is intuitively clear that the dynamical properties of a system determines how difficult it is to control the system. The purpose of this note is to demonstrate, how the performance is influenced by time delays, instability and non-minimum-phase properties. A simple linear system with a minimum variance criterion is used for the purpose of illustration. The analysis therefore also reflects some properties of the minimum variance regulator.

2. THE MODEL OF THE PROCESS AND ITS ENVIRONMENT

Consider the single-input single-output system whose block diagram is shown in Fig. 1.

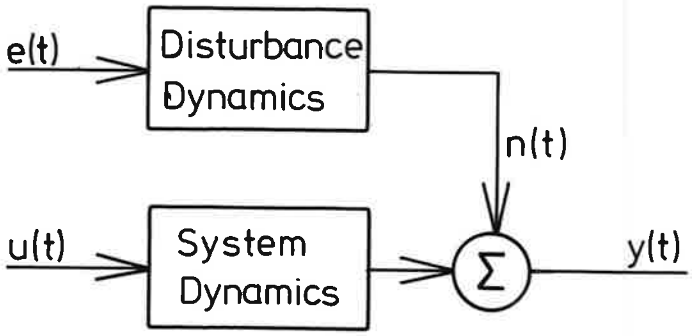


Fig. 1 - Block diagram of the system to be analysed.

The output y is the sum of a disturbance n and the output of a linear deterministic system. The disturbance n is assumed to be a sampled Wiener process. Such a process can be represented as

$$n(t+1) = n(t) + v(t) \tag{1}$$

where $\{v(t)\}$ is a sequence of independent, equally distributed, normal (0,1) random variables. The error in predicting the process over k-steps is thus equal to k. Systems having the following properties will be considered:

1. stable minimum phase systems,
2. unstable minimum phase systems,
3. stable non-minimum phase systems,
4. unstable non-minimum phase systems.

The effect of variations in the time delay will also be considered. This requires special attention in the cases 1 and 2 but is obtained automatically in cases 3 and 4.

The simplest model for the process dynamics which make it possible to illustrate the cases given above is a system with the pulse transfer function

$$H(z) = \frac{bz + 1}{z(z-a)} \tag{2}$$

The input-output relation for the system can be written as

$$y(t) = \frac{bq^{-1} + q^{-2}}{1 - aq^{-1}} u(t) + \frac{1}{1 - q^{-1}} e(t) \tag{3}$$

or

$$(1-aq^{-1})(1-q^{-1})y(t) = (b+q^{-1})(1-q^{-1})u(t-1) +$$

$$+ (1-aq^{-1})e(t) \tag{3'}$$

Notice that the chosen disturbance model implies that the control signal will always appear as $(1-q^{-1})u(t)$. This reflects the fact that an integrating control is necessary in order to eliminate the influence of the drifting disturbance. Technically this effect is conveniently handled by choosing the increment in u as the control variable. The model then becomes

$$y(t) - (1+a)y(t-1) + ay(t-z) = b\nabla u(t-1) + \nabla u(t-2) +$$

$$+ e(t) - ae(t-1) \tag{4}$$

The model is thus in standard form, see Åström [1970, p. 173], with

$A(x) = x^2 - (1+a)x + a$

$B(x) = bx + 1$

$C(x) = x - a$

The criterion is taken as minimizing the expected value of y^2 in the steady state.

The system described by (4) is stable if $|a| < 1$ and minimum phase if $|b| > 1$. The time delay can be increased by one unit by choosing b equal to zero. Controllability is lost for $b = -1/a$ and $b = -1$. When $ab = -1$ there is a pole-zero cancellation and the dynamics of the system (2) is reduced to a simple delay. The pulse transfer function is then

$$H(z) = z^{-1}$$

When $b = -1$ the system has a zero at $z = 1$. This means that it becomes very difficult to eliminate the drifting disturbance. The minimum variance control strategies and the minimal values of the loss functions will now be determined in the different cases.

3. STABLE MINIMUM PHASE SYSTEM

This case corresponds to $|a| < 1$ and $|b| > 1$. The system will also be non-minimum phase if $b = 0$. This case which corresponds to an extra time-delay in the system will also be discussed.

To determine the control strategies we will use the fundamental identity Åström [1970, p. 170] which in this particular case reduces to

$$1 - ax = [1 - (1+a)x + ax^2] \cdot 1 + g_0 x + g_1 x^2$$

Equating coefficients of equal power of x we get

$$-a = -(1+a) + g_0 \qquad g_0 = 1$$

$$0 = a + g_1 \qquad g_1 = -a$$

The minimum variance regulator becomes

$$\nabla u(t) = - \frac{G^*(q^{-1})}{F^*(q^{-1})B^*(q^{-1})} y(t) = - \frac{1 - aq^{-1}}{b + q^{-1}} y(t) \tag{5}$$

and the minimal value of the loss function is

$$Ey^2(t) = 1 \tag{6}$$

This is the variance of the one step predictor for the disturbance n. Under minimum variance control the variance of the control signal is

$$E[\nabla u(t)]^2 = \frac{2a + b + a^2 b}{b(b^2-1)} \tag{7}$$

A graph of the control variance is shown in Fig. 2. Notice that the variance of the control signal increases as b approaches +1 or -1, which corresponds to the limits when the system has a zero on the unit circle.

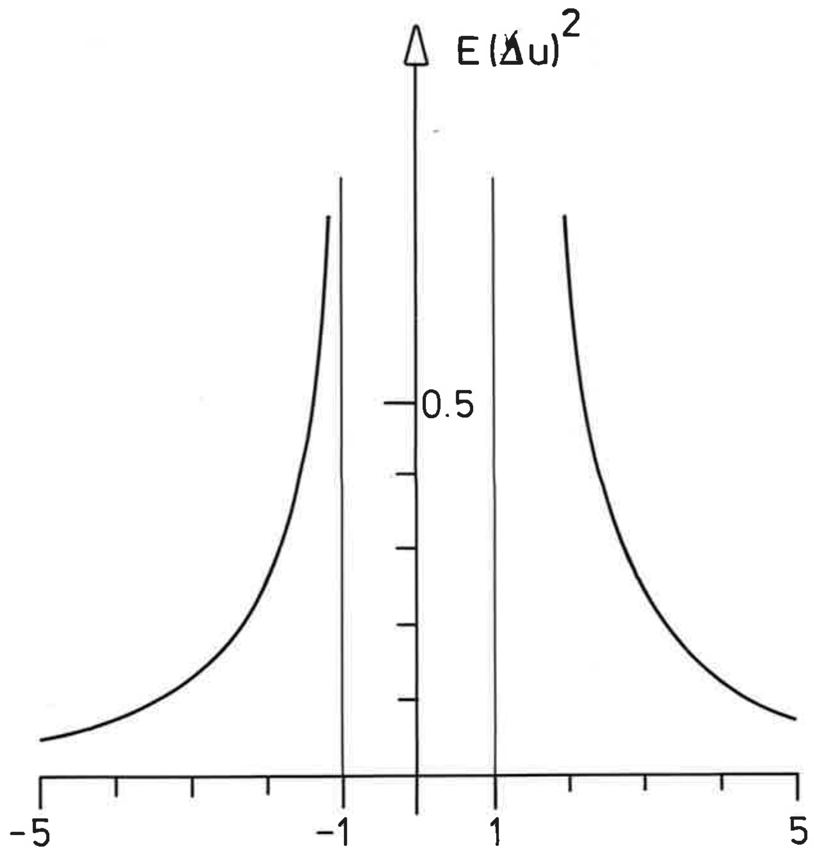


Fig. 2 - Graph of the variance of the control signal for a stable minimum phase system with a = 0.7.

An Extra Time Delay in the System.

The case $b = 0$ which corresponds to an extra time delay in the system will now be considered. The fundamental identity becomes

$$1 - ax = [1 - (1+a)x + ax^2](1+f_1x) + g_0x^2 + g_1x^3$$

Equating coefficients of equal powers of x gives

$$-a = -(1+a) + f_1$$
$$0 = a - (1+a)f_1 + g_0$$
$$0 = af_1 + g_1$$

Hence

$$f_1 = 1$$
$$g_0 = 1$$
$$g_1 = -a$$

The minimum variance regulator becomes

$$u(t) = -\frac{G^*(q^{-1})}{F^*(q^{-1})B^*(q^{-1})}\,y(t) = -\frac{1 - aq^{-1}}{1+q^{-1}}\,y(t) \qquad (8)$$

and the minimal value of the loss function becomes

$$\min Ey^2 = 1 + f_1^2 = 2 \qquad (9)$$

This is the variance of the two step predictor of the disturbance n.

To illustrate the results we will show some simulations of the system. The interactive simulation language SIMNON developed by Elmqvist [1975] was used in the simulations. The programs are given in Appendix A. Fig. 3 shows the disturbance $\{n(t)\}$

that was used in all simulations. It follows from Fig. 1 that the uncontrolled output is the same as the disturbance. Fig. 4 shows the output and the control signal when the control strategy (5) is used.

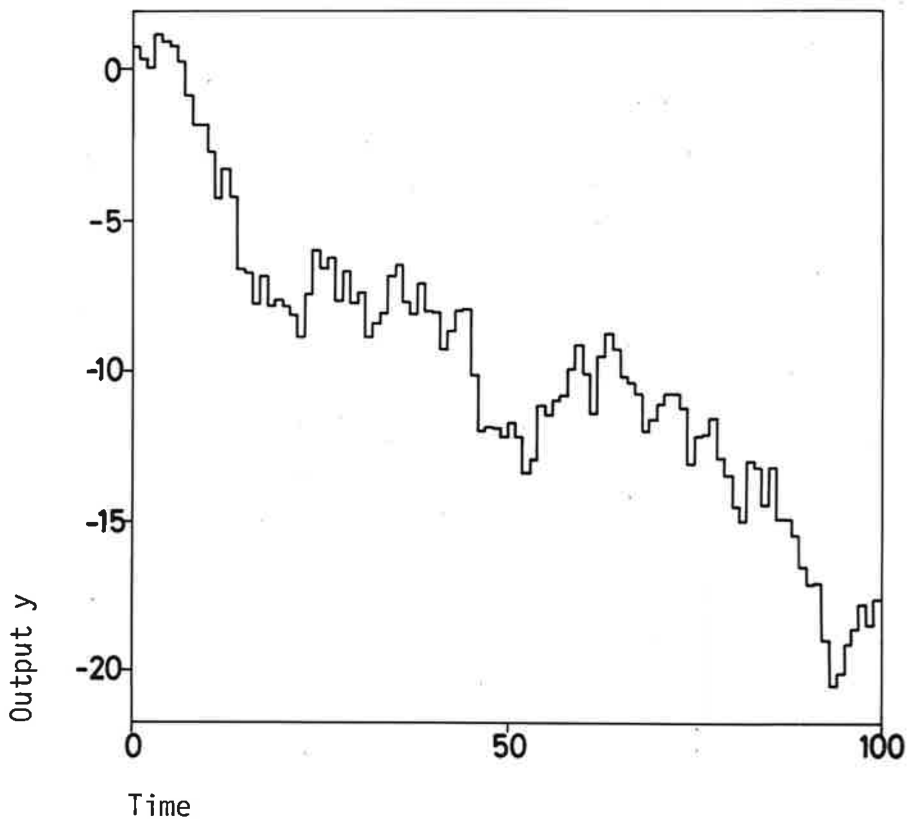
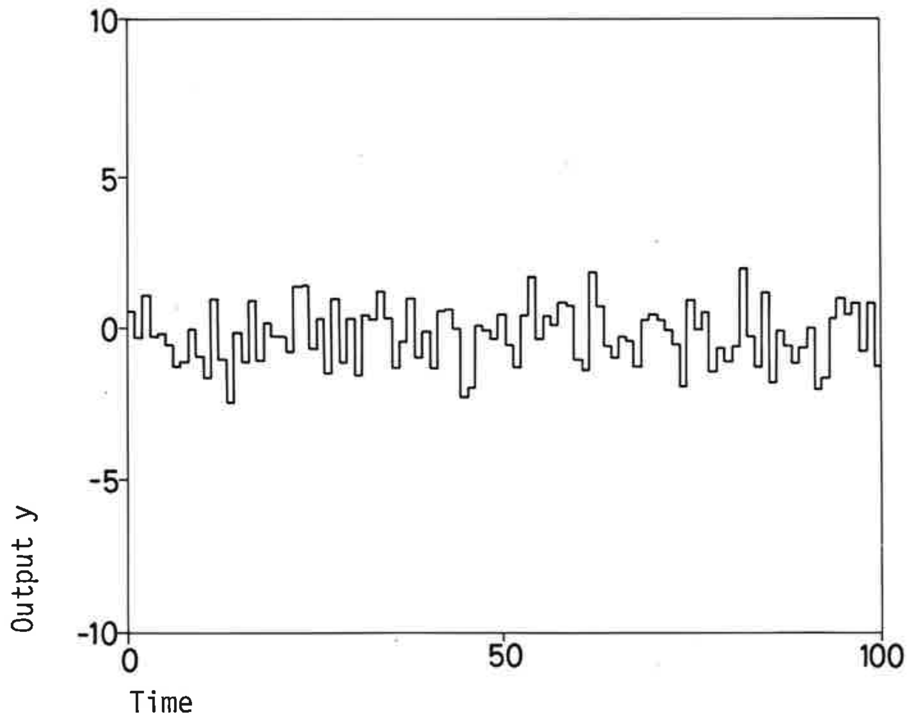


Fig. 3 - Uncontrolled output.



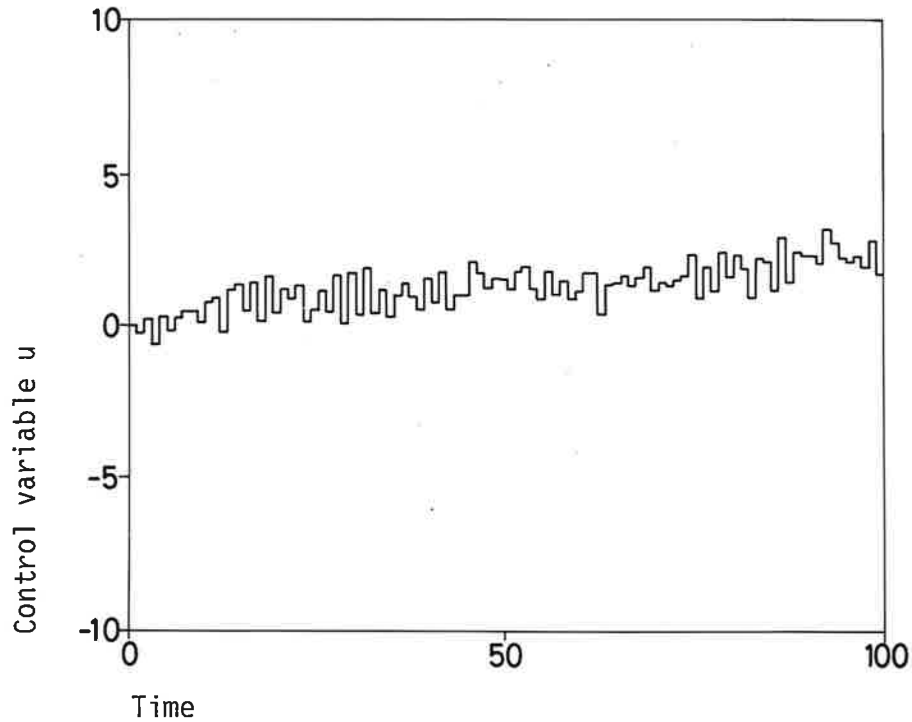


Fig. 4 - Simulation of the system (3) with the regulator (5). The parameter values are a = 0.7 and b = 2, which correspond to a stable minimum phase system. The sample variance of the output is $\overline{y^2} = 0.98$.

4. UNSTABLE MINIMUM PHASE SYSTEM.

This case corresponds to $|a| > 1$ and $|b| > 1$, or $|a| > 1$ and $b = 0$. Notice that when $|a| > 1$ then the polynomial

$$C(x) = x - a$$

has a zero outside the unit disc. This means that formulas in Åström (1970) do not apply directly because they require that the polynomial $C(x)$ has all its zeros inside the unit disc. To solve the problem it is then necessary to transform the system model. Before doing this let us, however, analyse what would happen if the control algorithm of section 3 was used.

Analysis

Consider the system

$$[1 - (1+a_0)q^{-1} + a_0q^{-2}]y(t) = [b_0q^{-1} + q^{-2}]\nabla u(t) + $$

$$+ [1 - a_0q^{-1}]e(t)$$

with the control law (5) i.e.

$$\nabla u(t) = -\frac{1 - aq^{-1}}{b + q^{-1}} y(t) \tag{5}$$

Notice that the control law has a zero $z = a$ which cancels the system pole $z = a$ of the process. The closed loop system is described by

$$[(1-a_0q^{-1})(1-q^{-1})(b+q^{-1}) + q^{-1}(1-aq^{-1})(b_0+q^{-1})]y(t) =$$

$$= (1-a_0q^{-1})(b+q^{-1})e(t)$$

The characteristic equation associated with the closed loop system is

$$x(x-a_0)(b_0x+1) + (b-b_0)(x-a_0)(x-1) + (a-a_0)(b_0x+1) = 0$$

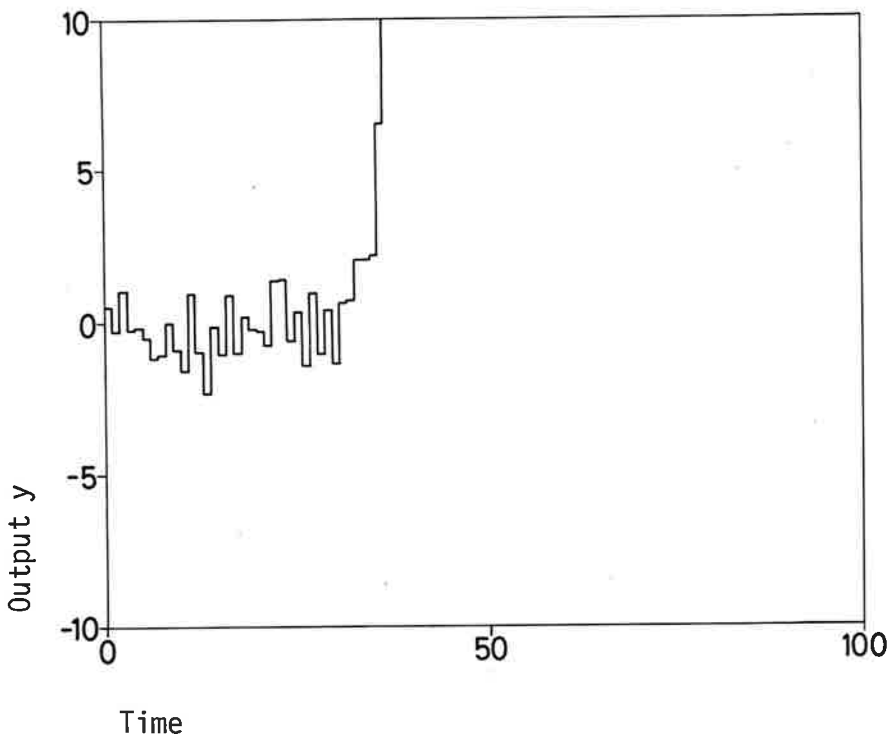
For $a = a_0$ and $b = b_0$ the characteristic equation reduces to

$$x(x-a_0)(b_0x+1) = 0$$

The closed loop system thus has three poles $x = 0$, $x = a_0$ and $x = 1/b_0$. Since $|a_0| > 1$ one pole is thus unstable. Because of the pole-zero cancellation this pole is not excited by the disturbance $e(t)$. If the unstable mode of the process is excited, the output will of course grow exponentially. Because of the zero at $z = a$ in the controller transfer function this component of the output will, however, not generate any control actions.

When the parameters a and b deviate slightly from their nominal values, there is not necessarily a cancellation and the unstable modes may be excited. The system will thus be extremely sensitive to parameter variations. The reason is clearly the fact that the polynomial $C(x)$ has a zero outside the unit disc.

Fig. 5 shows the results of a simulation. The output will grow exponentially without bounds, because the unstable mode is excited. The output exceeds the value 10 at time $t = 37$ and continues to grow exponentially. The output signal will remain small for a while because the growing component of the output is cancelled by the zero $z = a$ in the regulator transfer function. At time $t = 66$ the output has, however, grown so large that round-off errors are noticable and the control signal will also grow exponentially. The regulator is obviously useless in practice.



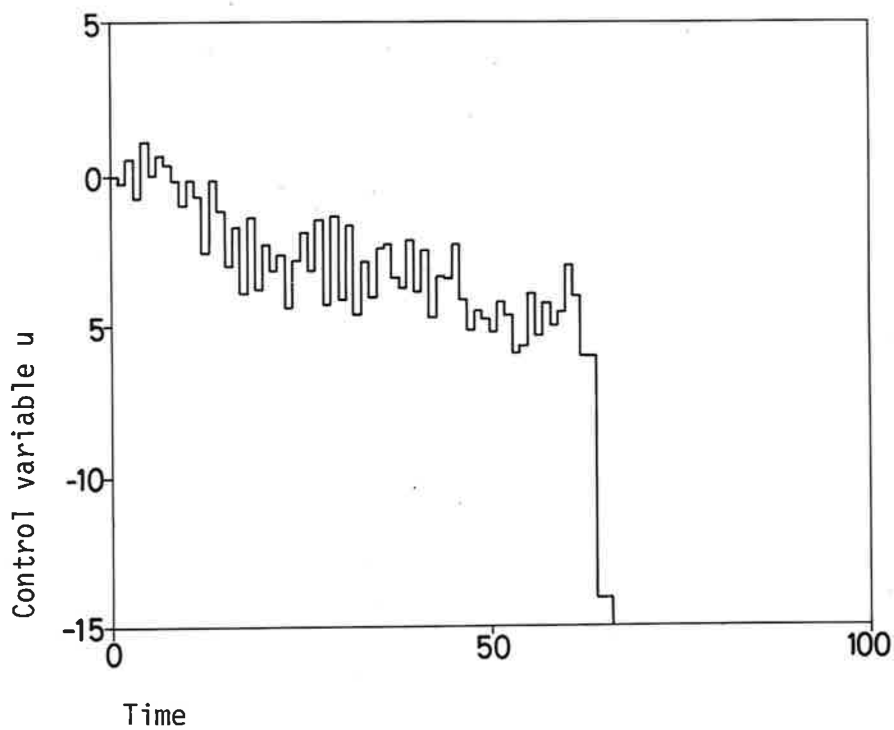


Fig. 5 - Results of simulation of the system with the regulator (5). The parameter values are a = 2 and b = 2 which correspond to an unstable minimum phase system.

Minimum Variance Control of an Unstable Plant

The previous analysis demonstrates clearly the necessity in assuming that the polynomial $C(x)$ has all its zeros inside the unit circle when deriving the minimum variance control law. If a system model, which has a polynomial $C(x)$ with zeros outside the unit circle, is obtained it is necessary to transform the problem. This can be done as follows.

It follows from the representation theorem for random processes that a stochastic process given by

$$y(t) = C^*(q^{-1})e(t)$$

where $\{e(t)\}$ is white noise and the polynomial $C(x)$ has zeroes outside the unit disc can also be represented by

$$y(t) = C_1^*(q^{-1})e(t)$$

where $C_1(x)$ has all its zeroes inside the unit disc. To carry out the construction in the specific case consider the stochastic process

$$\upsilon(t) = e(t) - ae(t-1)$$

where $\{e(t)\}$ is a sequence of independent normal $(0,1)$ random variables. The stochastic process $\{\upsilon(t)\}$ has the property

$$E\upsilon(t+\tau)\upsilon(t) = \begin{cases} 1 + a^2 & \tau = 0 \\ -a & |\tau| = 1 \\ 0 & |\tau| > 1 \end{cases}$$

Such a process can be represented by

$$\upsilon(t) = \varepsilon(t) - \alpha\varepsilon(t-1)$$

where $\{\varepsilon(t)\}$ is a sequence of independent normal $(0,\sigma^2)$ random variables. Evaluating the covariance function of $\{\upsilon(t)\}$ and equating with the previous equation we find

$$\sigma^2(1+\alpha^2) = 1 + a^2$$

$$-\sigma^2\alpha = -a$$

Hence

$$\alpha^2 - \left(\frac{1 + a^2}{a}\right)\alpha + 1 = 0$$

or

$$\alpha = \frac{1 + a^2}{2a} \pm \sqrt{\left(\frac{1 + a^2}{2a}\right)^2 - 1} = \frac{1 + a^2}{2a} \pm \frac{1 - a^2}{2a}$$

$$\alpha = \begin{cases} a \\ 1/a \end{cases}$$

We thus find that there are two values of α which give a process with the desired covariance structure. There is always one value such that the polynomial C has all zeroes inside the unit circle. In the particular case where $|a| > 1$ we get $\alpha = 1/a$ and $\sigma^2 = a^2$. This argument can apparently be carried out in the general case. Hence to apply the minimum variance control strategies we first transform the disturbances and obtain the model

$$A(x) = x^2 - x(1+a) + a$$

$$B(x) = bx + 1$$

$$C(x) = z - 1/a$$

$$\sigma^2 = a^2$$

Proceeding as before we now get

$$1 - x/a = [1 - (1+a)x + ax^2]1 + g_0 x + g_1 x^2$$

Hence

$$-1/a = -(1+a) + g_0 \qquad g_0 = 1 + a - 1/a = (a^2+a-1)/a$$

$$0 = a + g_1 \qquad g_1 = -a$$

The minimum variance strategy becomes

$$\nabla u(t) = -\frac{G^*(q^{-1})}{F^*(q^{-1})B^*(q^{-1})}\, y(t) = -\frac{(a^2+a-1-a^2q^{-1})}{a(b+q^{-1})}\, y(t) \qquad (10)$$

and the minimum value of the loss function is

$$\min Ey^2(t) = a^2 \qquad |a| \geq 1 \qquad (11)$$

A comparison with the previous case shows that the loss increases with increasing magnitude of a.

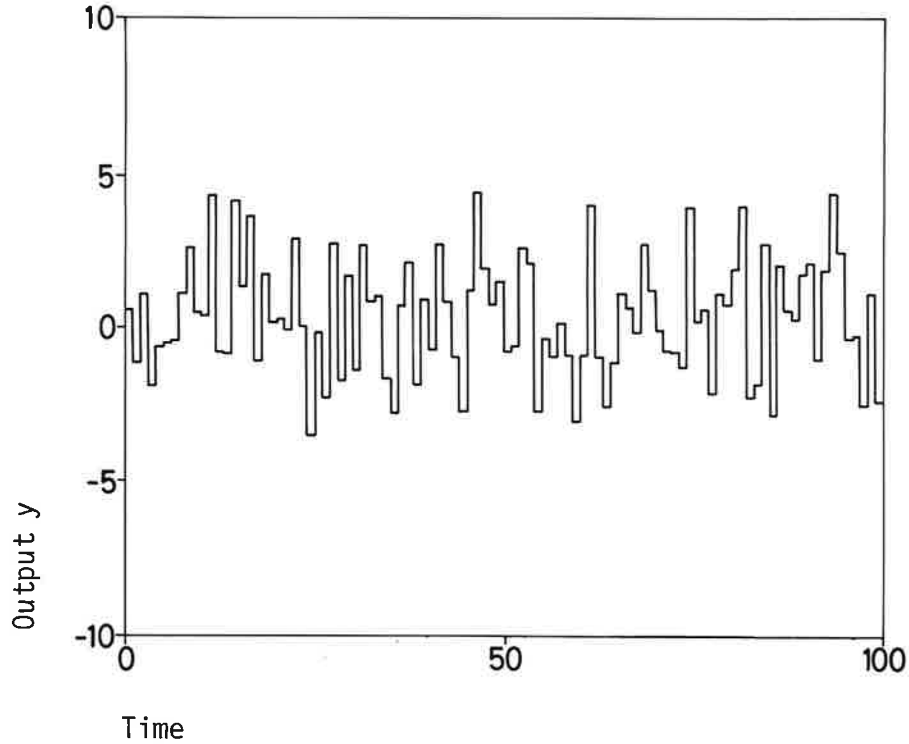
Fig. 6 shows the results of a simulation of the system. A comparison with Fig. 4 shows that the empirical variance of the output has increased from 0.98 to 3.8 which is in good agreement with (11).

An Extra Time Delay in the System.

The case b = 0 which corresponds to an extra time delay in the system will now be considered. We get

$$1 - x/a = [1 - (1+a)x + ax^2](1+f_1 x) + g_0 x^2 + g_1 x^3$$

Hence



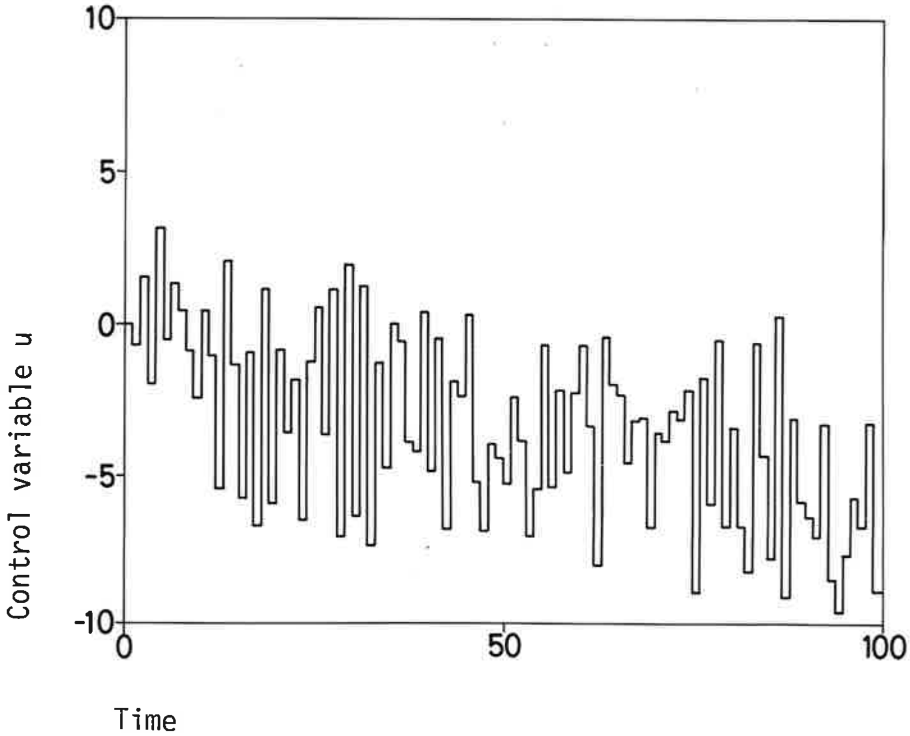


Fig. 6 - Results of simulation of the system with the control law (10). The parameter values are a = 2 and b = 2, which correspond to an unstable minimum-phase system. The empirical variance of the output obtained in the simulation is $\overline{y^2} = 3.8$.

$$- 1/a = - (1+a) + f_1$$

$$0 = a - (1+a)f_1 + g_0$$

$$0 = af_1 + g_1$$

Solving these equations for f_1, g_0 and g_1 we get

$$f_1(a^2+a-1)/a$$

$$g_0 = (1+a)(a^2+a-1)/a = \frac{a^3 + a^2 - 1}{a} = a(a+1) - 1/a$$

$$g_1 = a^2 + a - 1$$

The minimum variance strategy is thus given by

$$\nabla u(t) = - \frac{G^*(q^{-1})}{F^*(q^{-1})B^*(q^{-1})} y(t) = \frac{a^2(a+1) - 1 - a(a^2+a-1)q^{-1}}{a + (a^2+a-1)q^{-1}} \cdot y(t) \qquad (12)$$

and the minimal value of the loss function is given by

$$\min Ey^2 = a^2(1+f_1^2) = a^4 + 2a^3 - 2a + 1 \qquad |a| \geq 1 \qquad (13)$$

A comparison with the corresponding stable system shows that the penalty for an unstable system is even higher in this case. For example when $a = 2$ the loss becomes 4 when $|b| > 1$ but it becomes 29 if b also is zero.

5. STABLE NON-MINIMUM PHASE SYSTEM.

The case $|b| < 1$ will now be discussed. In this case the minimum variance strategy (5) is extremely sensitive to parameter variations because the closed loop system has poles outside the unit circle. See Åström (1970). These poles are not excited by the disturbances if the parameters are exact. If the parameters of the real process differ from the parameters of the model used for design the unstable poles will, however, be excited. The simulation shown in Fig. 7 illustrates what happens. The output signal is apparently well behaved in the simulation. The unstable mode $z = 1/b$ of the closed loop system is, however, coupled to the control signal and the magnitude of the control actions will therefore grow without bounds. This is clearly shown in Fig. 8 which shows the control signal in a different scale. Notice, however, that there is no noticable effect on the output even when the control actions have the magnitude 10^4. If the simulation is continued the control signal will, however, finally be so large that round-off errors will influence the output. In the particular simulation this happens at $t = 185$. See Fig. 9.

A Constrained Minimum Variance Strategy.

Since the minimum variance strategy (5) is extremely sensitive for non-minimum-phase systems we will consider a minimum variance strategy which is constrained in such a way that all closed loop poles are inside the unit circle. Such a strategy is derived in Peterka (1972). To derive the strategy we use the identity

$$B^-(x)C^*(x) = A^*(x)F^*(x) + x^kG^*(x)B^{-*}(x) \qquad (14)$$

where $B = B^-B^+$ is a factorisation of the polynomial B such that B^+ has all its zeroes inside the unit circle and B^- has all its zeroes outside the unit circle. See Peterka (1972). The polynomials are normalised in such a way that $B^-(0) = 1$.

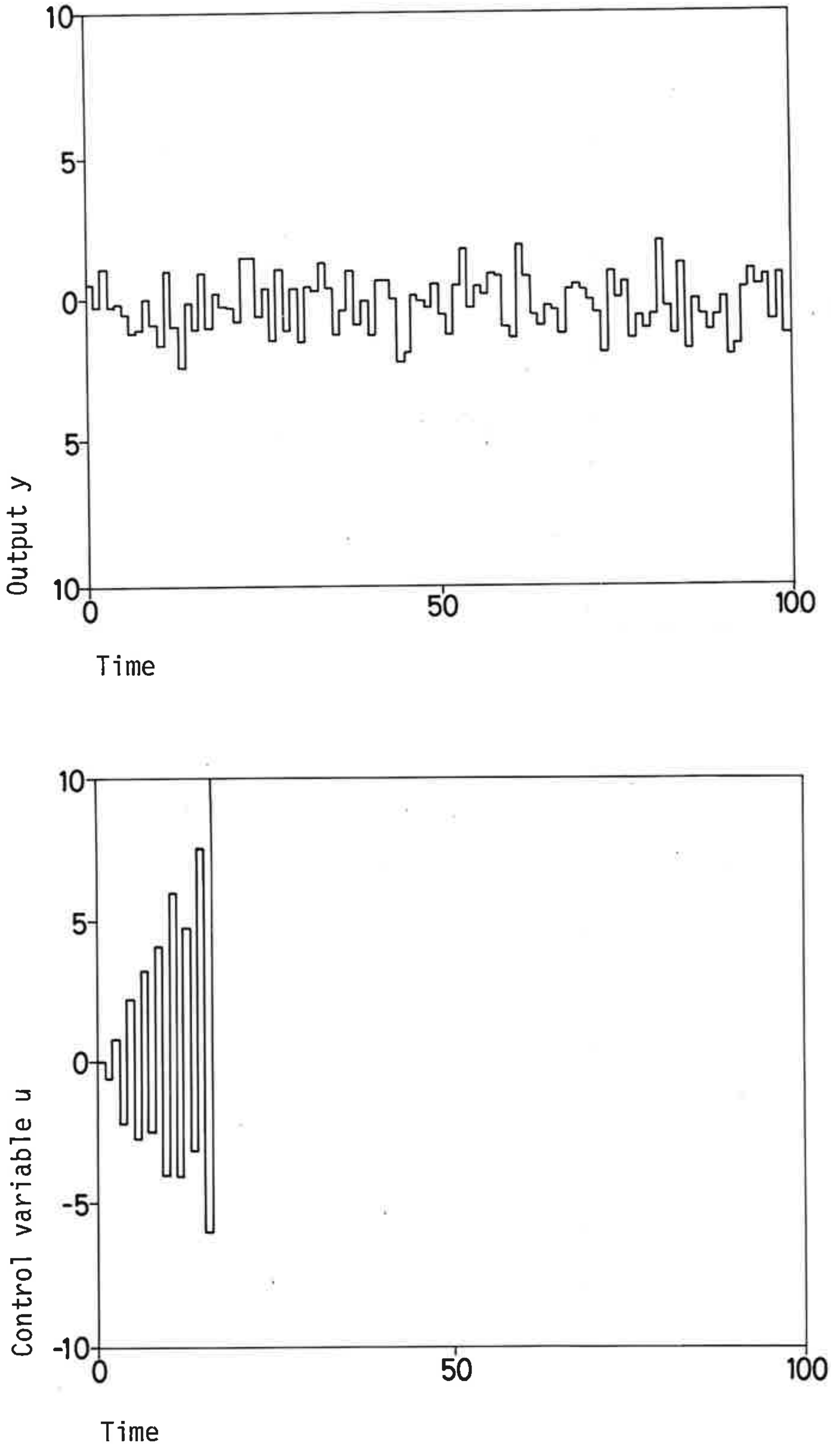


Fig. 7 - Results of simulation of the system (3) with the control law (5). The parameter values are a = 0.7 and b = 0.9, which correspond to a stable non-minimum--phase system.

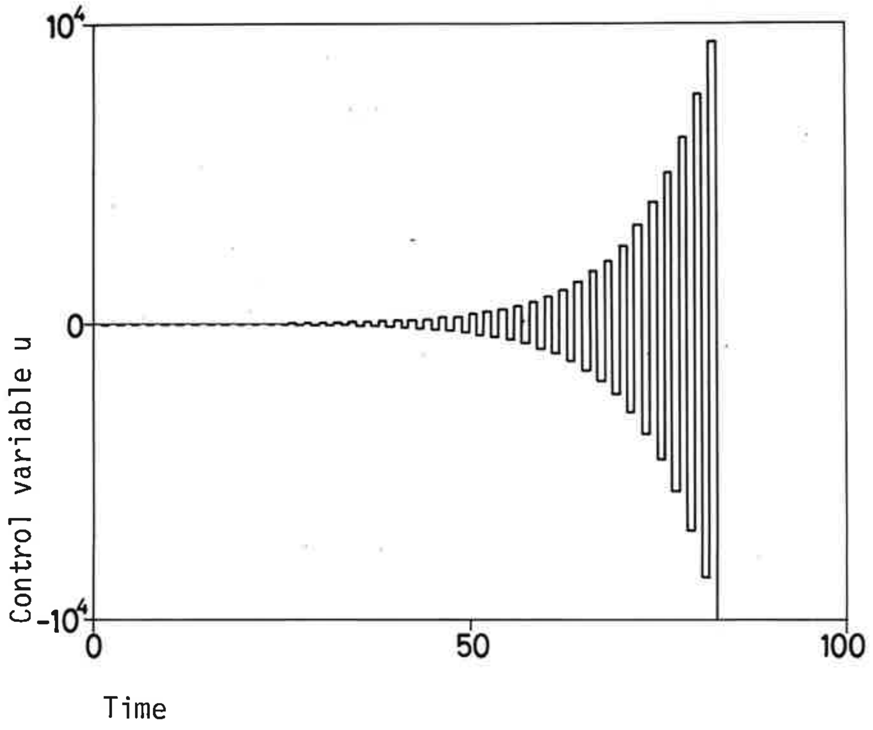


Fig. 8 - Control signal of Fig. 7 in different scale.

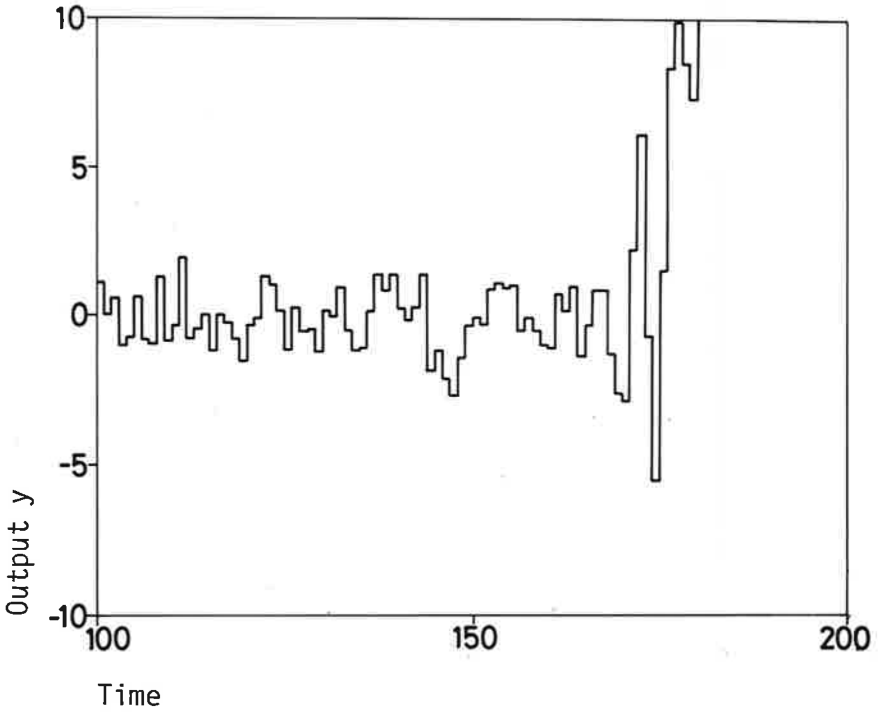


Fig. 9 - Continuation of the simulation shown in Fig. 7.

In the particular example we get

$$B(x) = bx + 1 = 1(bx+1) = B^+B^-$$

that is, $B^- = bx + 1$, $B^{-*} = b + x$ and $B^+ = 1$. The identity (14) then reduces to

$$(1+bx)(1-ax) = [1 - (1-a)x + ax^2](1+f_1x) + x(b+x)(g_0+g_1x)$$

Hence

$$b - a = - 1 - a + f_1 + bg_0$$
$$- ab = - (1+a)f_1 + a + bg_1 + g_0$$
$$0 = af_1 + g_1$$

Elimination of g_1 gives

$$f_1 + bg_0 = 1 + b$$
$$(1+a+ab)f_1 - g_0 = a(1+b)$$

Hence

$$(1-b+ab+ab^2)f_1 = 1 + b + ab + ab^2$$
$$g_0 = 1, \quad g_1 = - a$$

The control strategy becomes

$$\nabla u(t) = - \frac{G^*(q^{-1})}{F^*(q^{-1})B^{+*}(q^{-1})} y(t) = - \frac{1 - aq^{-1}}{1 + q^{-1}} y(t) \tag{15}$$

Notice that the control law does not depend on b. In this particular case the control law is in fact identical to the control law determined in section 3 for the case b = 0. The output under minimum variance control becomes

$$y(t) = \frac{F^*(q^{-1})}{B^-(q^{-1})}\,e(t) = \frac{1+q^{-1}}{1+bq^{-1}}\,e(t) = \left|\frac{(1-b)q^{-1}}{1+bq^{-1}}\right| e(t) \tag{16}$$

The variance of the output becomes

$$Ey^2(t) = 1 + \frac{(1-b)^2}{1-b^2} = \frac{2}{1+b} \tag{17}$$

Notice that the variance is infinite for $b = -1$. This is very natural because $b = -1$ means that dynamics contains a differentiator. Combining the results of this section with those of section 3 we find that the variance of the output as a function of b has the character indicated in Fig. 10.

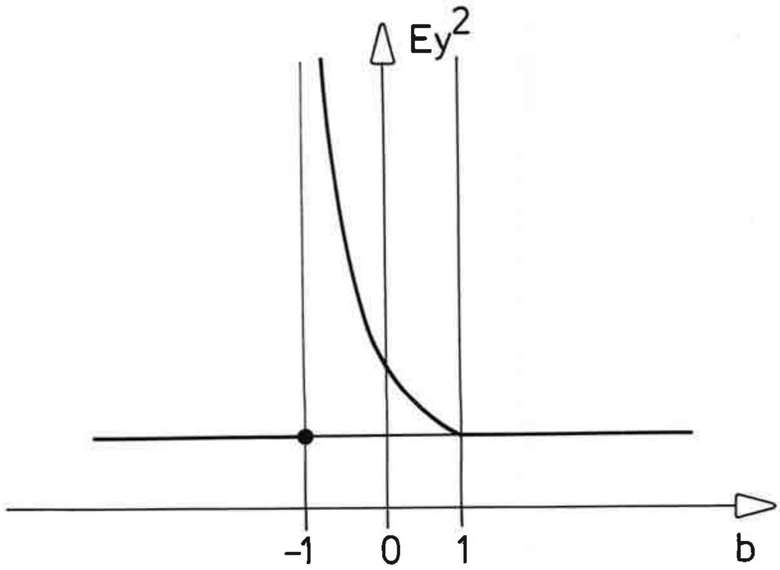


Fig. 10 - Variance of the output of the system with constrained minimum variance control for different values of b. Notice the discontinuity of the function at $b = -1$.

Under minimum variance control the control variable becomes

$$\nabla u(t) = - \frac{1 - aq^{-1}}{1 + bq^{-1}} e(t) = - \left| 1 - \frac{(a+b)q^{-1}}{1 + bq^{-1}} \right| e(t)$$

The variance of the control signal becomes

$$E[\nabla u(t)]^2 = 1 + \frac{(a+b)^2}{1 - b^2} = \frac{1 + 2ab + a^2}{1 - b^2} \tag{18}$$

The results of a simulation of the closed loop system with the control law (15) are shown in Fig. 11. A comparison with Fig. 4 shows that the fluctuations in the output are of the same magnitude. Equation (17) also indicates that $b = 0.9$ only gives 5% increase of the variance compared to the minimum-phase case.

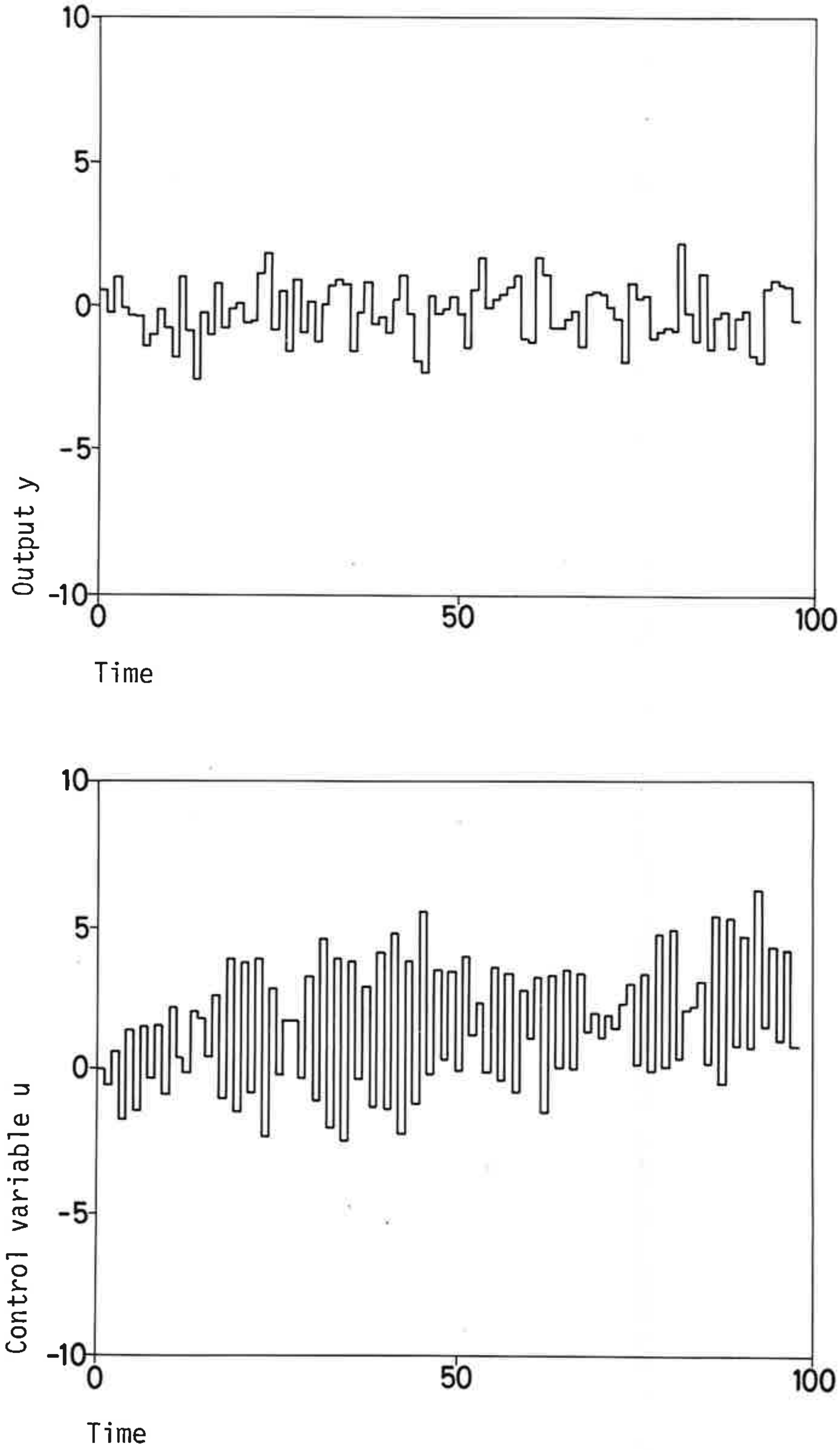


Fig. 11 - Results of simulation of the system (3) with the control law (15). The parameter values are a = 0.7 and b = 0.9 which correspond to a stable non-minimum-phase system. The empirical variance of the output is $\overline{y^2} = 0.97$.

6. UNSTABLE NON-MINIMUM PHASE SYSTEM.

The case $|a| > 1$ and $|b| < 1$ will now be analysed. The identity (14) becomes

$$(1+bx)(1-x/a) = [1 - (1+a)x + ax^2](1+f_1x) + x(b+x)(g_0+g_1x)$$

Hence

$$b - 1/a = -(1+a) + f_1 + bg_0$$
$$-b/a = -(1+a)f_1 + a + g_0 + bg_1$$
$$0 = af_1 + g_1$$

Elimination of g_1 gives

$$f_1 + bg_0 = 1 + a + b - 1/a$$
$$(1+a+ab)f_1 - g_0 = a + b/a$$

Hence

$$(1+b+ab+ab^2)f_1 = 1 + a - 1/a + b + ab + b^2/a$$

or

$$f_1 = \frac{a + a^2 - 1 + ab(a+1) + b^2}{a[1+b+ab+ab^2]}$$

$$g_0 = \frac{(1+b)(a^2+a^3+a^2b-1)}{a(1+b+ab+ab^2)}$$

$$g_1 = -af_1$$

The control strategy is given by

$$\nabla u(t) = -\frac{G^*(q^{-1})}{B^{+*}(q^{-1})F^*(q^{-1})}y(t) = -\frac{g_0 + g_1q^{-1}}{1 + f_1q^{-1}}y(t) \qquad (19)$$

and the output under minimum variance control becomes

$$y(t) = \frac{F^*(q^{-1})}{B^-(q^{-1})} e(t) = \frac{1 + f_1 q^{-1}}{1 + bq^{-1}} e(t) = \left| 1 + \frac{(f_1 - b) q^{-1}}{1 + bq^{-1}} \right| e(t)$$

Hence

$$Ey^2 = a^2 \left| 1 + \frac{(f_1 - b)^2}{1 - b^2} \right| \tag{20}$$

Results of a simulation of the system is shown in Fig. 12. A comparison with Fig. 4 shows that the output signal has much higher variance when the system is unstable and non-minimum-phase.

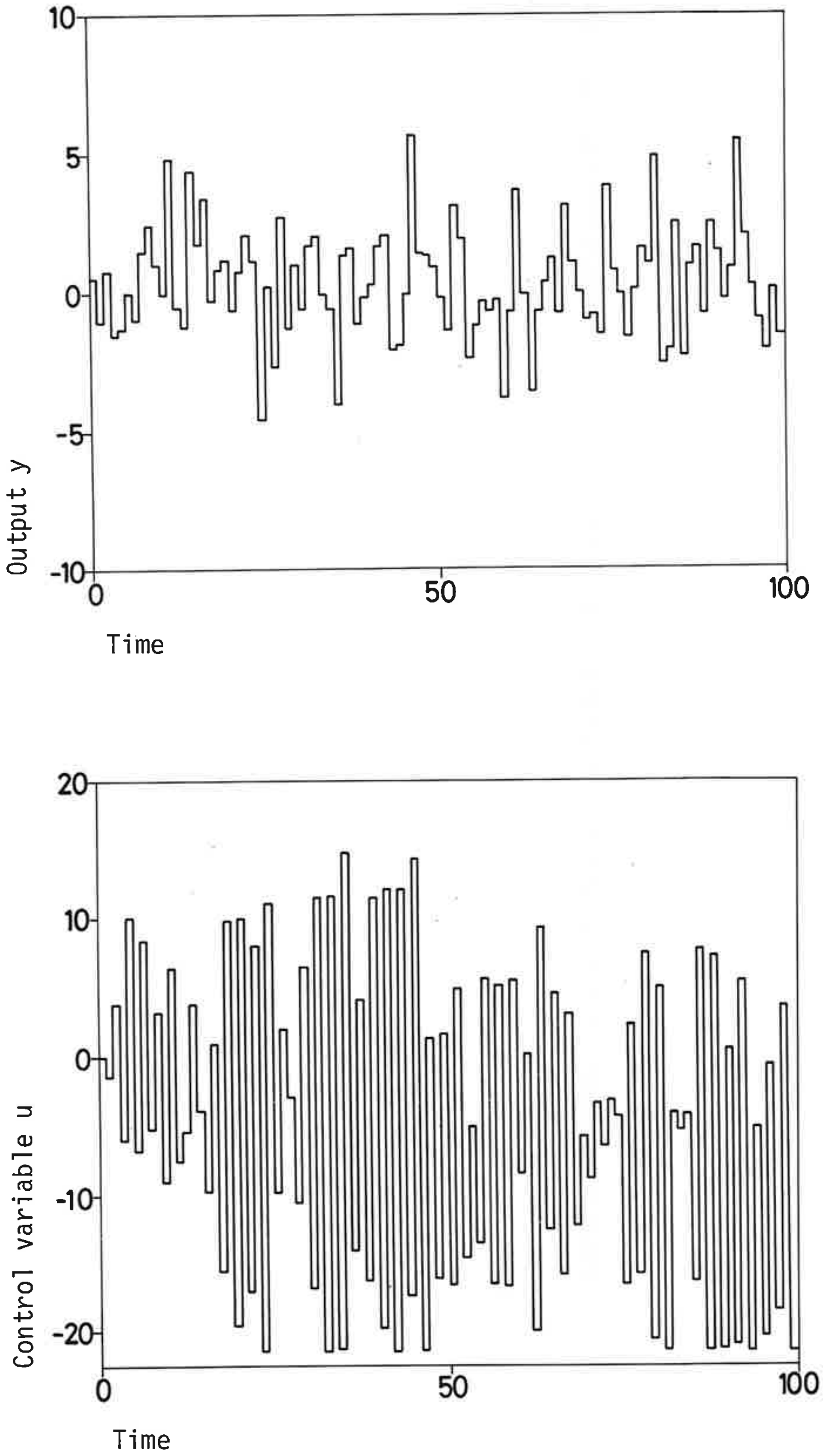


Fig. 12 - Results of simulation of the system (3) with the regulator (19). The parameter values are a = 2 and b = 0.9, which correspond to an unstable non-minimum-phase system. The empirical variance of the output is $y^2 = 4.15$.

7. CONCLUSIONS

The results of the calculations will now be summarised. The minimal variances obtained in the different cases are summarised in the Table below.

Case	$\|a\|$	Ey^2
1a. Stable minimum phase	$\|a\| < 1,\ \|b\| > 1$	1
1b. Stable minimum phase with extra time delay	$\|a\| < 1,\ b = 0$	2
2a. Unstable minimum phase	$\|a\| > 1,\ \|b\| > 1$	a^2
2b. Unstable minimum phase with extra time delay	$\|a\| > 1,\ b = 0$	$a^4 + 2a^3 - 2a + 1$
3. Stable non-minimum phase	$\|a\| < 1,\ \|b\| < 1$	$\frac{2}{1+b}$
4. Unstable non-minimum phase	$\|a\| > 1,\ \|b\| < 1$	$a^2\left[1 + \frac{(f_1 - b)^2}{1 - b^2}\right]$

Compare for example cases 1 and 2, that is, stable and unstable systems. For stable systems the control error equals the prediction error for the process over 1 ($b \neq 0$) and 2 steps ($b = 0$) respectively. When the system is unstable the error increases to a^2 which should be compared with 1 for stable systems. Hence the more unstable the system is the larger is the loss. When $b = 0$ the loss increases from 2 to $a^4 + 2a^3 - 2a + 1$. Next compare cases 1 and 3, that is, stable systems. In the non-minimum phase case $|b| < 1$ the loss increases from 1 to $2/(1+b)$. For $b = -1$ the loss becomes infinite due to loss of controllability of the unstable mode.

8. REFERENCES

Åström, K.J.: Introduction to Stochastic Control Theory, Academic Press, 1970.

Elmqvist, H.: Simnon - An Interactive Simulation Program for Nonlinear Systems - User's Manual. Report 7502, April 1975, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

Peterka, V.: On Steady State Minimum Variance Control Strategy, Kybernetica, 8 (1972), pp. 219-232.

APPENDIX

The SIMNON programs used in the simulation of the closed loop system are listed in this appendix.

```
DISCRETE SYSTEM PROC
"FIRST ORDER PROCESS WITH DRIFTING NOISE

INPUT DU E
OUTPUT Y
TIME T
STATE X1 X2 U N VE VU VY VN
NEW NX1 NX2 NU NN NVE NVU NVY NVN
TSAMP TS

INITIAL
X1:0
X2:0
U:0
N:1
VE:0
VU:0
VY:0
VN:0

OUTPUT
Y=X1+N+E

DYNAMICS
NX1=A*X1+X2+B*(U+DU)
NX2=U+DU
NU=U+DU
NN=N+E
NVE=VE+E*E
NVU=VU+U*U
NVY=VY+Y*Y
NVN=VN+N*N
TS=T+H

A:0.7
B:2
H:1

END
```

```
DISCRETE SYSTEM REG1
"MINIMUM VARIANCE REGULATOR

INPUT Y
OUTPUT DU
TIME T
STATE X
NEW NX
TSAMP TS

INITIAL
X:0

OUTPUT
DU=-(Y-X)/B

DYNAMICS
NX=-X/B+(A+1/B)*Y
TS=T+H

A:0.7
B:2
H:1

END
```

```
DISCRETE SYSTEM REG1
"MINIMUM VARIANCE REGULATOR

INPUT Y
OUTPUT DU
TIME T
STATE X
NEW NX
TSAMP TS

INITIAL
X:0

OUTPUT
DU=-(Y-X)/B

DYNAMICS
NX=-X/B+(A+1/B)*Y
TS=T+H

A:0.7
B:2
H:1

END
```

- -

```
DISCRETE SYSTEM REG2
"CONSTRAINED MIN VAR REG FOR UNSTABLE SYSTEM

INPUT Y
OUTPUT DU
TIME T
STATE X
NEW NX
TSAMP TS

INITIAL
X:0
G0=(A+1-1/A)/B
G1=-A/B

OUTPUT
DU=-G0*Y-X

DYNAMICS
NX=-X/B+(G1-G0/B)*Y
TS=T+H

A:2
B:2
H:1

END
```

- -

```
DISCRETE SYSTEM REG3
"CONSTRAINED MIN VARIANCE REG FOR UNSTABLE NON MIN-PHASE SYSTEM

INPUT Y
OUTPUT DU
```

```
TIME T
STATE X
NEW NX
TSAMP TS

INITIAL
X:0
D=A*(1+B+A*B+A*B*B)
F1=(A+A*A-1+A*B*(A+1)+B*B)/D
G0=((1+A+A*B)*(A+A*A+A*B-1)-A*A-B)/D
G1=-A*F1
VAR=A*A*(1+(F1-B)*(F1-B)/(1-B*B))

OUTPUT
DU=-G0*Y-X

DYNAMICS
NX=-F1*X+(G1-G0*F1)*Y
TS=T+H

A:2
B:0.9
H:1

END
```

```
DISCRETE SYSTEM REG3
"CONSTRAINED MIN VARIANCE REG FOR UNSTABLE NON MIN-PHASE SYSTEM

INPUT Y
OUTPUT DU
TIME T
STATE X
NEW NX
TSAMP TS

INITIAL
X:0
D=A*(1+B+A*B+A*B*B)
F1=(A+A*A-1+A*B*(A+1)+B*B)/D
G0=((1+A+A*B)*(A+A*A+A*B-1)-A*A-B)/D
G1=-A*F1
VAR=A*A*(1+(F1-B)*(F1-B)/(1-B*B))

OUTPUT
DU=-G0*Y-X

DYNAMICS
NX=-F1*X+(G1-G0*F1)*Y
TS=T+H

A:2
B:0.9
H:1

END
```

```
CONNECTING SYSTEM LINK
E[PROC]=E1[NOISE]+DRIFT
Y[REG1]=Y[PROC]
DU[PROC]=DU[REG1]

DRIFT:0.05
END
```