Control of Systems with Uncertain Parameters

Åström, Karl Johan

1976

Document Version:
Publisher's PDF, also known as Version of record

Link to publication

Citation for published version (APA):
Åström, K. J. (1976). *Control of Systems with Uncertain Parameters*. (Technical Reports TFRT-7115). Department of Automatic Control, Lund Institute of Technology (LTH).

Total number of authors:
1

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

CODEN: LUTFD2/(TFRT-7115)/1-009/(1976)

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K.J. ÅSTRÖM

Department of Automatic Control
Lund Institute of Technology
January 1977

Dokumentutgivare
Lund Institute of Technology
Dept of Automatic Control

Handläggare

Författare
K J Åström

Dokumentnamn
REPORT

Dokumentbeteckning
LUTFD2/(TFRT-7115)/1-009/(1977)

Utgivningsdatum
Jan. 1977

Ärendebeteckning

Dokumenttitel och undertitel

Control of Systems with Uncertain Parameters

Referat (sammandrag)

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Referat skrivet av
Author

Förslag till ytterligare nyckelord

Klassifikationssystem och -klass(er)

Indextermer (ange källa)

Omfång
9 pages

Övriga bibliografiska uppgifter

Språk
English

Sekretessuppgifter

ISSN

ISBN

Dokumentet kan erhållas från
Department of Automatic Control
Lund Institute of Technology
P O Box 725, S-220 07 LUND 7, Sweden

Mottagarens uppgifter

Pris
10:- Sw Kr

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Abstract

The formulae for linear quadratic control of a system with uncertain parameters are derived in this note. The results are of interest in order to take uncertainties into account in computer aided design and also for adaptive control algorithms.

Contents

1. INTRODUCTION
2. PROBLEM STATEMENT
3. MAIN RESULT
4. AN EXAMPLE
5. REFERENCES

1. INTRODUCTION

This note considers a linear quadratic problem where the parameters of the system model are random variables. The criterion is to minimize the expected loss where the expectation is taken both with respect to process and measurement disturbances and with respect to parameter uncertainties. The solution of the problem is of interest in order to evaluate the effects of uncertainties in design of control laws for systems with uncertain parameters and for adaptive control. Results of this type was derived for single-input single-output systems using transfer function representations in Wieslander and Wittenmark (1971). The state-space formulas which are simpler were used in Peterka and Åström (1973).

2. PROBLEM STATEMENT

Consider the system

$$x(t+1) = \Phi x(t) + \Gamma u(t) + e(t) \tag{1}$$

where $\{e(t)\}$ is a sequence of random vectors with zero mean values and covariances R_1. Assume that the parameters Φ and Γ are random variables with known mean values and known covariances. The elements of Φ and Γ are uncorrelated with $\{e(t)\}$. Assume that the state variable x can be measured without error. Determine a control strategy such that the criterion

$$E\left[\sum_{k=0}^{N} x^T(k)Q_1x(k) + u^T(k)Q_2u(k)\right] \tag{2}$$

is minimal. The matrices Q_1 and Q_2 are assumed known. The matrices Φ and Γ have mean values

$$\begin{aligned} E\,\Phi &= \hat{\Phi} \\ E\,\Gamma &= \hat{\Gamma} \end{aligned} \tag{3}$$

To express the uncertainties of Φ and Γ. We introduce

$$\begin{aligned} \Phi &= [\ \varphi_1 \ \ \varphi_2 \ \cdots \ \varphi_n\ ] \\ \Gamma &= [\ \gamma_1 \ \ \gamma_2 \ \cdots \ \gamma_n\ ] \end{aligned} \tag{4}$$

The uncertainties are defined by

$$\begin{aligned} R_{\varphi_i\varphi_j} &= E\,[\ \varphi_i - \hat{\varphi}_i]\ [\ \varphi_j - \hat{\varphi}_j\ ] = \text{cov}\,[\ \varphi_i,\ \varphi_j\ ] \\ R_{\varphi_i\gamma_j} &= E\,[\ \varphi_i - \overline{\varphi}_i]\ [\ \gamma_j - \hat{\gamma}_j\ ] = \text{cov}\,[\ \varphi_i,\ \gamma_j\ ] \\ R_{\gamma_i\gamma_j} &= E\,[\ \gamma_i - \hat{\gamma}_i]\ [\ \gamma_j - \hat{\gamma}_j\ ] = \text{cov}\,[\ \gamma_i,\ \gamma_j\ ] \end{aligned} \tag{5}$$

3. MAIN RESULT

To solve the problem we procede in the usual manner using Dynamic Programming. Assume that the minimum exists and introduce

$$V(x,t) = \min_u E\left[\sum_{k=t}^{N} x^T(k)\ Q_1 x(k) + u^T(k)\ Q_2 u(k)\right] \tag{6}$$

where E denotes the conditional expectation given x(t). We have

$$V(x,N) = x^T\ Q_1\ x \tag{7}$$

It will now be shown by induction that V(x,t) is a quadratic form, i.e.

$$V(x,t) = x^T S(t)\psi + \alpha(t) \tag{8}$$

This is apparently true for t = N with

$$\begin{cases} S(N) = Q_1 \\ \alpha(N) = 0 \end{cases}$$

Assuming that (8) holds for t + 1 it will be shown by induction that it holds also for t. We have

$$V(x,t) = \min_u E\left[x^T(t)Q_1x(t) + u^T(t)Q_2u(t) + V[x(t+1),t+1]\right] \tag{9}$$

Drop the argument t in x(t) to simplify the writing, introduce (1) and (8) into (6). Then

$$V(x,t) = \min_u E\left[x^TQ_1x + u^TQ_2u + V(\Phi x + \Gamma u + e, t+1)\right] =$$

$$= \min_u E\left[x^TQ_1x + u^TQ_2u + (\Phi x + \Gamma u + e)^TS(t+1)(\Phi x + \Gamma u + e) + \alpha(t+1)\right] =$$

$$= \min_u \left[x^TQ_1x + u^TQ_2u + Ex^T\Phi^TS(t+1)\Phi x + Ex^T\Phi^TS(t+1)\Gamma u + \right.$$

$$\left. + Eu^T\Gamma^TS(t+1)\Phi x + Eu^T\Gamma^TS(t+1)\Gamma u + \mathrm{tr}\ S(t+1)R_1 + \alpha(t+1)\right] \tag{10}$$

To procede it is now necessary to evaluate

$$E \; x^T\Phi^T S(t+1)\Phi x$$

$$E \; x^T\Phi^T S(t+1)\Gamma u$$

$$E \; u^T\Gamma^T S(t+1)\Gamma u$$

This is simple in principle but messy in detail unless we are careful about our notation. Equation (4) gives

$$\Phi^T S\Phi = \begin{pmatrix} \varphi_1^T \\ \varphi_2^T \\ \vdots \\ \varphi_n^T \end{pmatrix} S\,[\varphi_1 \;\; \varphi_2 \ldots \varphi_n] = \begin{pmatrix} \varphi_1^T S\varphi_1 & \varphi_1^T S\varphi_2 & \ldots & \varphi_1^T S\varphi_n \\ \varphi_2^T S\varphi_1 & \varphi_2^T S\varphi_2 & \ldots & \varphi_2^T S\varphi_n \\ \vdots & & & \\ \varphi_n^T S\varphi_1 & \varphi_n^T S\varphi_2 & \ldots & \varphi_n^T S\varphi_n \end{pmatrix}$$

Taking mathematical expectations we find

$$E\Phi^T S\Phi = \hat{\Phi}^T S\hat{\Phi} + A_1(S) \tag{11}$$

$$A_1(S) = \begin{pmatrix} \mathrm{tr}\; SR_{\varphi_1\varphi_1} & \mathrm{tr}\; SR_{\varphi_2\varphi_1} & \ldots & \mathrm{tr}\; SR_{\varphi_n\varphi_1} \\ \mathrm{tr}\; SR_{\varphi_1\varphi_2} & \mathrm{tr}\; SR_{\varphi_2\varphi_2} & \ldots & \mathrm{tr}\; SR_{\varphi_n\varphi_2} \\ \vdots & & & \\ \mathrm{tr}\; SR_{\varphi_1\varphi_n} & \mathrm{tr}\; SR_{\varphi_2\varphi_n} & \ldots & \mathrm{tr}\; SR_{\varphi_n\varphi_n} \end{pmatrix} \tag{12}$$

$$E\; \varphi_i^T S\varphi_j = E\; \varphi_j^T S\varphi_i = E\;\mathrm{tr}\,\varphi_j^T S\varphi_i = E\;\mathrm{tr}\, S\varphi_i\varphi_j^T = \mathrm{tr}\, SR_{\varphi_i\varphi_j}$$

which implies

$$\mathrm{tr}\; SR_{\varphi_j\varphi_i} = \mathrm{tr}\; R_{\varphi_j\varphi_i}S = \mathrm{tr}\; SR_{\varphi_i\varphi_j} = \mathrm{tr}\; R_{\varphi_i\varphi_j}S \tag{14}$$

Similarly we find

$$E\; \Phi^T S\Gamma = \hat{\Phi}^T S\hat{\Gamma} + A_2(S) \tag{15}$$

$$A_2(S) = \begin{pmatrix} \text{tr } SR_{\gamma_1\varphi_1} & \text{tr } SR_{\gamma_2\varphi_1} & \cdots & \text{tr } SR_{\gamma_p\varphi_1} \\ \text{tr } SR_{\gamma_1\varphi_2} & \text{tr } SR_{\gamma_2\varphi_2} & \cdots & \text{tr } SR_{\gamma_p\varphi_2} \\ \vdots & & & \\ \text{tr } SR_{\gamma_1\varphi_n} & \text{tr } SR_{\gamma_2\varphi_n} & \cdots & \text{tr } SR_{\gamma_p\varphi_n} \end{pmatrix} \tag{16}$$

$$E\ \Gamma^T S \Gamma = \hat{\Gamma} S \hat{\Gamma} + A_3(S) \tag{17}$$

$$A_3(S) = \begin{pmatrix} \text{tr } SR_{\gamma_1\gamma_1} & \text{tr } SR_{\gamma_1\gamma_2} & \cdots & \text{tr } SR_{\gamma_1\gamma_p} \\ \text{tr } SR_{\gamma_2\gamma_1} & \text{tr } SR_{\gamma_2\gamma_2} & \cdots & \text{tr } SR_{\gamma_2\gamma_p} \\ \vdots & & & \\ \text{tr } SR_{\gamma_p\gamma_1} & \text{tr } SR_{\gamma_p\gamma_2} & \cdots & \text{tr } SR_{\gamma_p\gamma_p} \end{pmatrix} \tag{18}$$

Introducing (10), (15) and (17) into (8) we get

$$\begin{aligned} V(x,t) = \min_u \Big\{ & u^T[Q_2 + \hat{\Gamma}^T S(t+1)\hat{\Gamma} + A_3(S(t+1))]\, u \; + \\ & + u^T[\hat{\Gamma}^T S(t+1)\hat{\Phi} + A_2^T(S(t+1))]\, x \; + \\ & + x^T[\hat{\Phi}^T S(t+1)\hat{\Gamma} + A_2(S(t+1))]\, u \; + \\ & + x^T[Q_1 + \hat{\Phi}^T S(t+1)\hat{\Phi} + A_1(S(t+1))] \; + \\ & + \text{tr } S(t+1) R_1 + \alpha(t+1) \Big\} \end{aligned}$$

Summarizing the results we find

THEOREM

The optimal control strategy is given by

$$u(t) = -L\, x(t)$$

where

$$L = [Q_2 + \hat{\Gamma}^T S(t+1)\hat{\Gamma} + A_3(S(t+1))]^{-1}[\hat{\Gamma}^T S(t+1)\hat{\Phi} + \hat{A}_2^T(S(t+1))]$$

and

$$S(t) = \hat{\Phi}^T S(t+1)\Phi + Q_1 - L^T[Q_2 + \hat{\Gamma}^T S(t+1)\hat{\Gamma} + A_3(S(t+1))] L + A_1(S(t+1)) =$$

$$= \hat{\Phi}^T S(t+1)[\hat{\Phi} - \hat{\Gamma}L] + Q_1 + A_1(S(t+1)) - A_2(S(t+1)) L =$$

$$= (\hat{\Phi} - \hat{\Gamma}L)^T S(t+1) (\hat{\Phi} - \hat{\Gamma}L) + Q_1 + A_1 + L^T Q_2 L - A_2 L - L^T A_2^T$$

where

$$A_1(S) = \begin{bmatrix} \mathrm{tr}\, R_{\varphi_1\varphi_1}S & \mathrm{tr}\, R_{\varphi_1\varphi_2}S & \cdots & \mathrm{tr}\, R_{\varphi_1\varphi_n}S \\ \mathrm{tr}\, R_{\varphi_2\varphi_1}S & \mathrm{tr}\, R_{\varphi_2\varphi_2}S & \cdots & \mathrm{tr}\, R_{\varphi_2\varphi_n}S \\ \vdots & & & \\ \mathrm{tr}\, R_{\varphi_n\varphi_1}S & \mathrm{tr}\, R_{\varphi_n\varphi_2}S & \cdots & \mathrm{tr}\, R_{\varphi_n\varphi_n}S \end{bmatrix}$$

$$A_2(S) = \begin{bmatrix} \mathrm{tr}\, R_{\varphi_1\gamma_1}S & \mathrm{tr}\, R_{\varphi_1\gamma_2}S & \cdots & \mathrm{tr}\, R_{\varphi_1\gamma_p}S \\ \mathrm{tr}\, R_{\varphi_2\gamma_1}S & \mathrm{tr}\, R_{\varphi_2\gamma_2}S & \cdots & \mathrm{tr}\, R_{\varphi_2\gamma_p}S \\ \vdots & & & \\ \mathrm{tr}\, R_{\varphi_n\gamma_1}S & \mathrm{tr}\, R_{\varphi_n\gamma_2}S & \cdots & \mathrm{tr}\, R_{\varphi_n\gamma_p}S \end{bmatrix}$$

$$A_3(S) = \begin{bmatrix} \mathrm{tr}\, R_{\gamma_1\gamma_1}S & \mathrm{tr}\, R_{\gamma_1\gamma_2}S & \cdots & \mathrm{tr}\, R_{\gamma_1\gamma_p}S \\ \mathrm{tr}\, R_{\gamma_2\gamma_1}S & \mathrm{tr}\, R_{\gamma_2\gamma_2}S & \cdots & \mathrm{tr}\, R_{\gamma_2\gamma_p}S \\ \vdots & & & \\ \mathrm{tr}\, R_{\gamma_p\gamma_1}S & \mathrm{tr}\, R_{\gamma_p\gamma_2}S & \cdots & \mathrm{tr}\, R_{\gamma_p\gamma_p}S \end{bmatrix}$$

where φ_i and γ_i are defined by (4) and $R_{\varphi\gamma}$ by (5)

4. AN EXAMPLE

The equations become especially simple in the case of systems on companion form. Assume

$$\Phi = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ -a_n & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

In this case we get

$$A_1 = \begin{pmatrix} \text{tr } SR_{\varphi_1\varphi_1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \text{tr } SR_{\gamma_1\varphi_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad A_2^T = \begin{pmatrix} \text{tr } SR_{\gamma_1\varphi_1} & 0 & \dots & 0 \end{pmatrix}$$

$$A_3 = \text{tr } SR_{\gamma_1\gamma_1}$$

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