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A UNIFIED APPROACH TO

MODEL REFERENCE ADAPTIVE SYSTEMS

AND

SELF-TUNING REGULATORS

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Abstract

An outline of model reference adaptive regulators in input-output description is given. The widespread concept of augmented error is given an interpretation and it is shown that there is no essential difference between the model reference adaptive algorithms and the self-tuning regulators. A general self-tuning algorithm is defined for continuous and discrete time systems. It is shown that the algorithm contains both model reference adaptive schemes and self-tuning regulators as special cases.

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TABLE OF CONTENTS

1. INTRODUCTION

The special class of adaptive control problems, where the process is assumed to have constant but unknown parameters, has received much attention since the midsixties. The main approaches are closely related to two different problems of classical control theory:

- the servo problem, i.e. the control object is to make the plant output follow a given reference signal;
- the regulator problem, i.e. to keep the output as close as possible to a constant level, regardless of the disturbances acting on the system.

Two different approaches to adaptive control have arisen as attempts to solve the corresponding adaptive problems. The adaptive servo problem has mostly been formulated for continuous time, deterministic systems. A typical problem statement is to find a control input u(t) such that the plant output y(t) asymptotically equals the output $y_m(t)$ of a model with input $u_m(t)$. This formulation leads to the model reference adaptive systems (MRAS). It has been treated by many authours during the past fifteen years [1-26]. There are many different schemes presented and it is difficult to give a general description. The configuration seen in fig 1.1 is however a typical MRAS system. In fig 1.1 x_m and x denotes the state vectors of the model and plant. A detailed account of the different MRAS configurations is given in Landau [15].

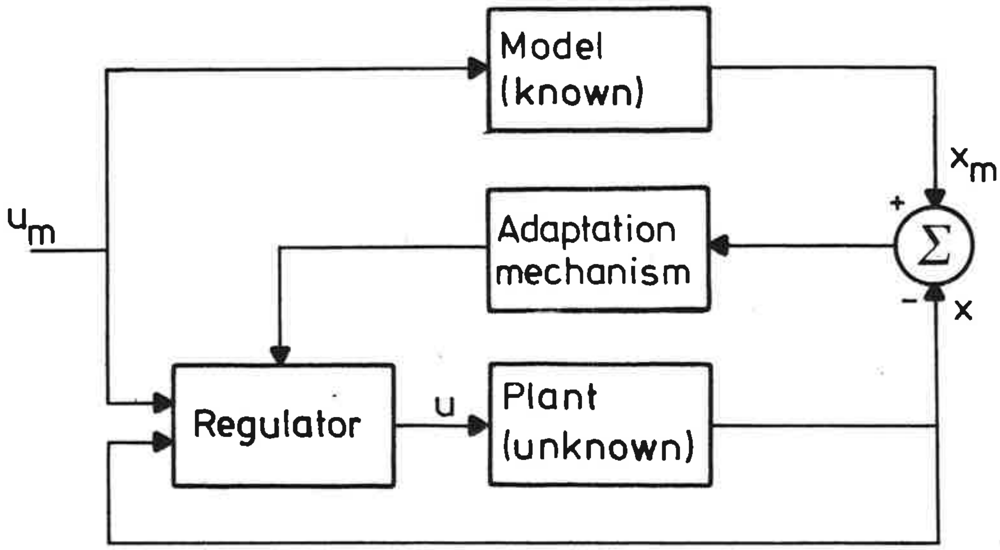

Fig 1.1. MRAS configuration.

When solving <u>the adaptive regulator problem</u> it is natural to model the disturbance also and use stochastic control theory. A fairly natural way to attack the problem of unknown parameters is the following one: firstly estimate the parameters of the unknown plant and then use these estimates to design a regulator as if the parameters were correct. If these two steps are done iteratively in time, the result is an adaptive regulator, which is often referred to as a <u>self-tuning regulator</u> [27-36]. The underlying synthesis and parameter estimation procedures could be of different types. The approach is thus quite flexible. The scheme is depicted in fig 1.2.

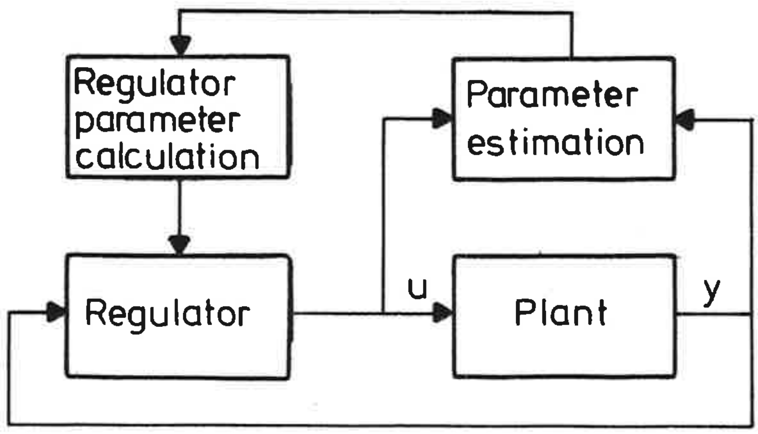

Fig 1.2. Self-tuning regulator configuration.

As will be described later on, the two approaches mentioned have close relationships. The original development showed some significant differences however. A few are listed in the table below.

	Servo problem	Regulator problem
System description	deterministic continuous time	stochastic discrete time
Method of synthesis	minimization of criterion, stability theory	identification + regulator for known parameters
Method of analysis	stability theory	stochastic convergence results

It should be noted that many results available today do not fit into this simple classification. For example, the results for the continuous time servo problem have been extended to the discrete time case too.

A short review of the main ideas related to MRAS schemes is given in chapter 2. Some algorithms proposed are described for easy reference. A corresponding survey of the ideas in the field of self-tuning regulators is given in [35]. In chapter 3 it is shown how several of the continuous time MRAS schemes can be derived from a somewhat different viewpoint. This makes it possible to give a unified description and also to relate them with some self-tuning schemes. The same development is done for discrete time systems in chapter 4. Finally, chapter 5 contains a summary and discussion.

2. SOME MRAS SCHEMES

In the early work on model reference adaptive systems, the most common methods dealt with minimization of some performance index. The approach is known as the gradient or sensitivity approach, and the famous 'MIT-rule' belongs to this class of solutions [1]. The MIT-rule is based on the criterion $\int e^2(t)dt$, where $e(t)$ is the error. The adaptation rule adjusts the gains according to

$$\dot{K}(t) = -e(t)\left(\frac{de(t)}{dK}\right).$$

The MIT-rule became very popular in the beginning due to its simplicity. The scheme has however the important disadvantage that it may lead to an unstable closed loop, Parks [3]. Moreover, it is often very difficult to conclude from theoretical analysis, if instability actually will occur in a given situation.

One possibility to get around the instability problem associated with the sensitivity approach, is to base the design directly on stability theory. The interest was soon directed towards this technique and Lyapunovs second method became the major design tool. This technique will be the major subject of this paper.

Some of the main ideas in the field are illustrated by examples in this chapter. The difficulties encountered are pointed out and some of the most recent schemes are described in some detail.

The case with measurable state vector is treated in section 2.1. A partial solution to the more practical case with non-measurable state is described in section 2.2. Some ideas from the adaptive observers are exploited to proceed to the general solution. These results are given in section 2.3. The general solution is contained in section 2.4. Finally some characteristics of the discrete time problem are pointed out in section 2.5.

2.1. Measurable state.

Most of the early solutions to the adaptive servo problem, using Lyapunov theory, assume that the whole state vector is measurable. This assumption is a considerable simplification. Although the case with non-measurable state is more interesting, two simple examples with measurable state will be given. The solutions are relatively straightforward and they illustrate techniques that are useful for more complicated situations.

Example 2.1. First order all-pole plant with unknown gain [3].

The configuration is shown in fig 2.1. The plant is a first order system which differs from the model only by an unknown constant. The object is to adjust the gain K such that $e(t)=y_m(t)-y(t)$ tends to zero. The solution uses a Lyapunov function

$$V(e,\tilde{K}) = \frac{1}{2}(e^2+c\tilde{K}^2)$$

where $c>0$, $\tilde{K}=K_m-KK_p$. Using

$\dot{e} = -\frac{1}{T}e + \frac{1}{T}\tilde{K}u$, the derivative becomes

$$\dot{V} = e\dot{e} + c\tilde{K}\dot{\tilde{K}} = -\frac{1}{T}e^2 + \frac{1}{T}\tilde{K}ue + c\tilde{K}\dot{\tilde{K}}.$$

If the gain is adjusted according to

$$\dot{\tilde{K}} = -\frac{1}{cT}ue,$$

the derivative of V is

$$\dot{V} = -\frac{1}{T}e^2.$$

The error e thus tends to zero. It can be shown that this implies that $\tilde{K}$ also tends to zero if the input u(t) is sufficiently rich, i.e. contains enough many frequencies.

□

<u>Remark.</u> The properties of the input signal are in general important for the parameter convergence. The exact conditions will however not be stated for the different algorithms.

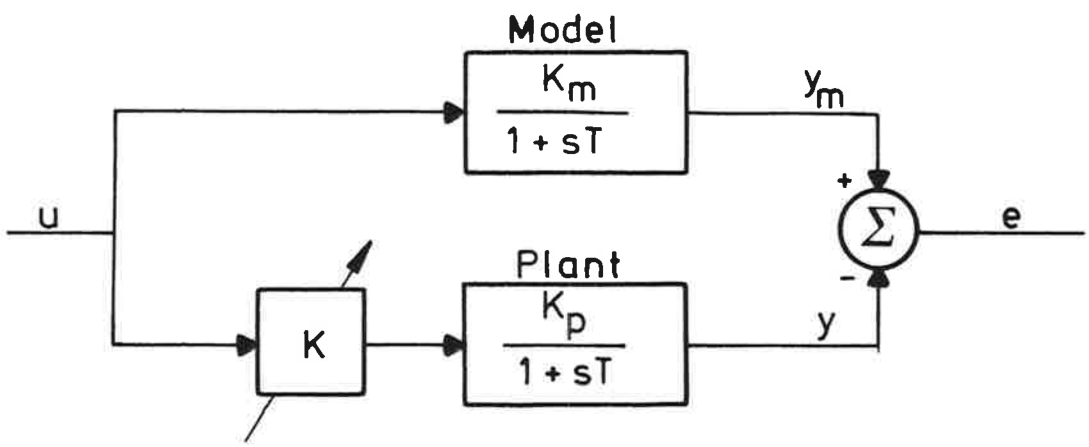

Fig 2.1. Configuration of example 2.1.

<u>Example 2.2.</u> Model and plant with the same zeros; poles unknown.

The plant is described by the state equations

$$\dot{x} = Ax + bu = \begin{bmatrix} a^T \\ \hline 1 & & \\ & \ddots & \\ & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \qquad a^T = [a_1 \; \ldots \; a_n]$$

$$y = c^T x = [c_1 \; \ldots \; c_n] \; x.$$

It is assumed that all state variables in the representation above can be measured. The plant output should follow the output from the model

$$\dot{x}_m = A_m x_m + bu_m = \begin{bmatrix} a_m^T \\ \hline 1 & & \\ & \ddots & \\ & 1 & 0 \end{bmatrix} x_m + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_m \qquad a^T = [a_1^m \; \ldots \; a_n^m]$$

$$y_m = c^T x_m = [c_1 \; \ldots \; c_n] \; x_m.$$

The zeros of the plant and the model are thus assumed to be the same.

As in the known parameter case, the input is chosen to be

$$u = u_m + l^T x,$$

where l is a column vector of adjustable gains. Define $\tilde{l} = \tilde{a} - l = a_m - a - l$. A Lyapunov function candidate is then

$$V = \tilde{x}^T P \tilde{x} + c\tilde{l}^T \tilde{l},$$

where P is positive definite and $c > 0$. The state error $\tilde{x} = x_m - x$ satisfies

$$\dot{\tilde{x}} = \dot{x}_m - \dot{x} = A_m \tilde{x} + (A_m - A)x + b(u_m - u).$$

The derivative of V becomes

$$\dot{V} = \tilde{x}^T(A_m^T P + PA_m)\tilde{x} + 2\tilde{x}^T(P(A_m-A)x + Pb(u_m-u)) + 2c\tilde{l}^T\dot{\tilde{l}} =$$

$$= -\tilde{x}^T Q\tilde{x} + 2(\tilde{x}^T p_1 \tilde{a}^T x - \tilde{x}^T p_1 l^T x + c\tilde{l}^T\dot{\tilde{l}}) =$$

$$= -\tilde{x}^T Q\tilde{x} + 2(\tilde{l}^T x\tilde{x}^T p_1 + c\tilde{l}^T\dot{\tilde{l}})$$

where the matrix Q is given by the well known Lyapunov equation $A_m^T P + PA_m = -Q$ and p_1 is the first column of P. Thus, if the parameter updating is chosen as

$$\dot{\tilde{l}} = -\frac{1}{c}p_1^T\tilde{x}x, \qquad (2.1)$$

then the derivative of the Lyapunov function becomes negative definite. This means that x tends to zero and consequently that y tends to y_m.

□

Admittedly the two examples given describe cases with fairly strong assumptions. All the same they show that for a restricted class of problems the Lyapunov design gives a straightforward and simple solution that guaranties over-all stability. However, the above examples also illustrate two serious drawbacks of most of the algorithms presented till 1969.

Firstly, the zeros are assumed to be unaltered or known. It is difficult to see how the technique used in the examples should be extended to handle the case of unknown zeros which are different from the zeros in the model. The adaptation rule generally has the form of (2.1). The parameter derivative is thus set equal to an error (in example 2.2 $p_1^T\tilde{x}$)

times a state vector (x). This means that the adaptation rule allows adjustment of n (=the order of the system) parameters only. A generalization to adjust more than n parameters thus requires some sort of non-minimal state vector.

Secondly, it is a serious restriction that all the state variables are assumed to be measurable. There are two major techniques available to handle the problem:

a) The state variables can be obtained as linear combinations of input and output derivatives. In the adaptive formulation, the linear combinations have to be determined by adaptation. In practise the differentiators have also to be replaced by approximative differentiators. The result is known as the 'state variable filter' technique [37,38].

b) The state variables can be estimated by some observer. Since the plant is unknown, the parameters of the observer must also be adjusted. This approach thus leads to the problem of making an adaptive observer.

It is clear that the two techniques are basically equivalent. The state estimates are calculated as linear combinations of filtered input and output signals. The solution a) will be the one of main interest here. A partial solution of the problem with non-measurable state which uses method a) will be described in the next section. The general solution however uses some ideas from the technique in b). These are described in section 2.3. Finally, section 2.4 returns to method a) in the general case.

2.2. Non-measurable state - a special case.

The first attempt to solve the adaptive servo problem using the state variable filter concept as described in a) above, was given by Gilbart, Monopoli and Price [7]. Their solution was later reformulated by Monopoli [14]. Monopoli's approach is described below.

Example 2.3. Monopoli's [14] solution in a simple case. The plant is assumed to be governed by

$$y(t) = \frac{B(p)}{A(p)}u(t) = \frac{b_1 p + b_2}{p^2 + a_1 p + a_2}u(t)$$

where p denotes the differential operator. The model is

$$y_m(t) = \frac{B^m(p)}{A^m(p)}u_m(t) = \frac{b_1^m p + b_2^m}{p^2 + a_1^m p + a_2^m}u_m(t)$$

and the object is to make y tend to y_m. The error $e=y_m-y$ satisfies

$$A^m(p)e(t) = B^m(p)u_m(t) - A^m(p)y(t) =$$
$$= B^m(p)u_m(t) + (A(p)-A^m(p))y(t) - B(p)u(t). \tag{2.2}$$

Guided by the idea of the state variable filters which uses filtered input and output signals, define

$$C(p) = p+c ;$$

$$\tilde{u}(t) = \frac{u(t)}{C(p)} ; \quad \tilde{u}_m(t) = \frac{u_m(t)}{C(p)} ; \quad \tilde{y}(t) = \frac{y(t)}{C(p)} .$$

The equation (2.2) can then be rewritten as

$$A^m(p)e(t)=C(p)[-B(p)\tilde{u}(t)+B^m(p)\tilde{u}_m(t)+(A(p)-A^m(p))\tilde{y}(t)] =$$

$$=C(p)[-(b_1p+b_2)\tilde{u}(t)+(b_1^mp+b_2^m)\tilde{u}_m(t)+((a_1-a_1^m)p+(a_2-a_2^m))\tilde{y}(t)] \tag{2.3}$$

It will be clear below that it is possible to include an unknown, nonzero constant with known sign in C(p) without influencing the conclusions. Therefore, if the sign of b_1 is known, (2.3) can be simplified into

$$A^m(p)e(t)=$$

$$=C(p)[-(p+b)\tilde{u}(t)+(b_1^mp+b_2^m)\tilde{u}_m(t)+((a_1-a_1^m)p+(a_2-a_2^m))\tilde{y}(t)] \tag{2.4}$$

where the parameters are different from those in (2.3). It must however be assumed that $b_1 \neq 0$ to obtain (2.4) from (2.3). This means that the difference between the number of poles and the number of zeros (the 'pole excess') is equal to one.

A natural way to try to make e(t) zero is to choose the control signal u(t) according to

$$(p+\hat{b}(t))\tilde{u}(t)=(b_1^mp+b_2^m)\tilde{u}_m(t)+((\hat{a}_1(t)-a_1^m)p+(\hat{a}_2(t)-a_2^m))\tilde{y}(t) \tag{2.5}$$

where $\hat{b}(t)$ etc. stands for estimates of the unknown parameters. Insertion of the control law (2.5) into the error equation (2.4) gives

$$A^m(p)e(t) =$$

$$= C(p)[(\hat{b}(t)-b)\tilde{u}(t)-(\hat{a}_1(t)-a_1)p+(\hat{a}_2(t)-a_2))\tilde{y}(t)] =$$

$$= C(p)[\tilde{b}(t)\tilde{u}(t) - \tilde{a}_1(t)\dot{\tilde{y}}(t) - \tilde{a}_2(t)\tilde{y}(t)] =$$

$$= C(p)\tilde{\theta}^T(t)\varphi(t), \tag{2.6}$$

where $\tilde{b}(t)=\hat{b}(t)-b$ etc. are the parameter errors, $\tilde{\theta}^T(t) = [\tilde{b}(t)\ \tilde{a}_1(t)\ \tilde{a}_2(t)]$ and $\varphi^T(t) = [\tilde{u}(t)\ -\dot{\tilde{y}}(t)\ -\tilde{y}(t)]$.

It is seen from (2.6) that the error depends linearly on the parameter errors. This is an important point which will be used below.

It now remains to choose the parameter adjustments so that overall stability is achieved. Let x be the state vector when realizing (2.6) as

$$\dot{x}(t) = Ax(t) + B\tilde{\theta}^T(t)\varphi(t) \tag{2.7}$$

$$e(t) = C^T x(t) = [1\ 0\ \dots\ 0]x(t).$$

A suitable Lyapunov function is

$$V = x^T P x + \tilde{\theta}^T R \tilde{\theta} \tag{2.8}$$

where P and R are positive definite. The key result to be used now is the Kalman-Yakobovich lemma, which can be stated as

<u>Lemma [39].</u> Let A be asymptotically stable and (A,B) controllable. If $G(s)=C^T(sI-A)^{-1}B$ is strictly positive real (SPR) there exists matrices $P>0$, $Q>0$ such that

$$A^T P + PA = -Q \tag{2.9}$$

$$PB = C$$

[]

This lemma was first exploited by Parks [3].
Differentiation of (2.8) gives:

$$\dot{V} = x^T(A^TP + PA)x + 2x^TPB\tilde{\theta}^T\varphi + 2\tilde{\theta}^TR\dot{\tilde{\theta}}. \qquad (2.10)$$

If the transfer function $C(p)/A^m(p)$ is strictly positive real the lemma can be used to simplify (2.10) as

$$\dot{V} = -x^TQx + 2(x^TC\tilde{\theta}^T\varphi + \tilde{\theta}^TR\dot{\tilde{\theta}}) = -x^TQx + 2(\tilde{\theta}^T\varphi e + \tilde{\theta}^TR\dot{\tilde{\theta}}).$$

By choosing the parameter updating

$$\dot{\tilde{\theta}} = -R^{-1}\varphi e, \qquad (2.11)$$

the derivative becomes negative definite and x tends to zero. This implies that e tends to zero. The updating (2.11) is given in terms of parameter errors, but it could also be written as

$$\dot{\hat{\theta}}(t) = -R^{-1}\varphi(t)e(t), \qquad (2.12)$$

where $\hat{\theta}(t)$ is a vector containing the parameter estimates themselves.

The resulting configuration, which is a typical 'MRAS-configuration', is shown in fig 2.2. The block diagram reflects one part of the closed-loop behaviour, namely the generation of the error $e(t)$. The importance of this 'MRAS-configuration' is that it consists of two positive operators [39], connected with a negative feedback. This fact implies that $e(t)$ tends to zero. It should however be noted that the block diagram does not give any information about other important properties of the MRAS.

□

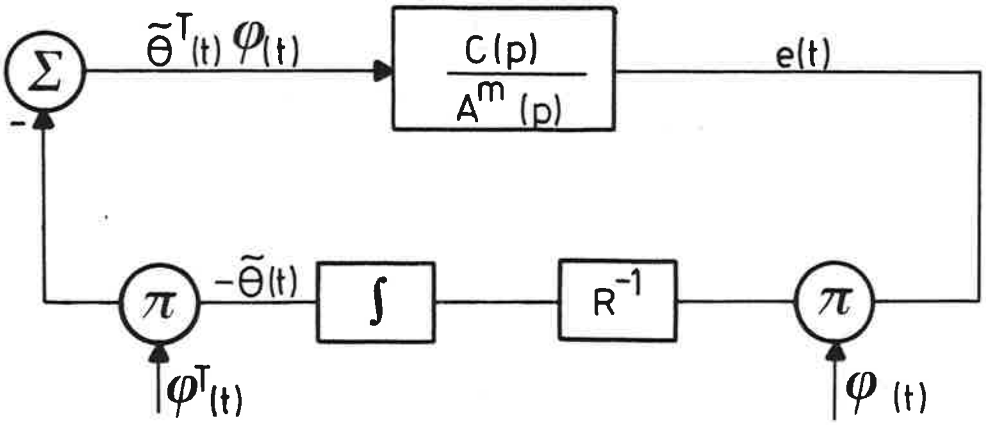

Fig 2.2. 'MRAS-configuration' of example 2.3.

Remark 1. There is a 'compatibility condition' that has to be fulfilled in the above solution. This is the assumption, that the regulator contains as many parameters as in the known parameter case. If this assumption is not introduced, the analysis becomes very difficult.

Remark 2. The requirement on C/A^m to be positive real is easy to fulfill, because $A^m(p)$ is known and $C(p)$ can be chosen freely.

Remark 3. The adaptive law (2.12) is similar to the one used in example 2.2. However, the vector φ has three elements. According to the discussion in connection with example 2.2 φ can be interpreted as a non-minimal state vector.

The above example can be generalized and the adaptive servo problem can be solved under the important assumption that the pole excess is equal to one ($b_1 \neq 0$). The result is a relatively simple adaptive regulator, which has the desired stability properties. It is seen in the example that the control law will contain derivatives of the output if $b_1=0$. It is this point which motivates further efforts in order to solve the general problem. New ideas are required to proceed. Historically these entered via the theory of adaptive observers, which will come next in our presentation.

2.3. Adaptive observers.

The adaptive observers deal with the problem of estimating the state vector of the unknown plant (here in phase variable form):

$$\dot{x} = Ax + bu = \begin{bmatrix} -a_1 & 1 & & \\ \vdots & & \ddots & \\ & & & 1 \\ -a_n & & & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} u \tag{2.13}$$

$$y = c^T x = [1\ 0\ \dots\ 0]x.$$

An observer, or a Kalman filter, for a known system can have the form

$$\dot{\hat{x}} = A\hat{x} + bu + a(y - c^T\hat{x}) = (A - ac^T)\hat{x} + bu + ay. \tag{2.14}$$

Here $\hat{x}$ denotes the state estimate and a is a column vector, which determines the matrix

$$K = A - ac^T = \begin{bmatrix} -k_1 & 1 & & \\ \vdots & & \ddots & \\ & & & 1 \\ -k_n & & & 0 \end{bmatrix}$$

and thereby the observer dynamics. The convergence rate of the estimation error $\tilde{x}=x-\hat{x}$ is determined by K:

$$\dot{\tilde{x}} = Ax + bu - (A\hat{x} + bu + a(y-c^T\hat{x})) = (A-ac^T)\tilde{x} = K\tilde{x}. \tag{2.15}$$

An adaptive observer can be implemented straightforwardly by fixing the observer dynamics and using (2.14) with estimates of the unknown parameters. Hence

$$\dot{\hat{x}} = K\hat{x} + \hat{a}y + \hat{b}u, \quad K = \begin{bmatrix} -k_1 & 1 & & \\ \vdots & & \ddots & \\ & & & 1 \\ -k_n & & & 0 \end{bmatrix}. \tag{2.16}$$

The error equation is then given by

$$\dot{\tilde{x}} = Ax + bu - (K\hat{x}+\hat{a}y+\hat{b}u) = K\tilde{x} + (A-K)x + (b-\hat{b})u - \hat{a}y =$$

$$= K\tilde{x} + \begin{bmatrix} k_1-a_1 \\ \cdot \\ k_n-a_n \end{bmatrix} y - \hat{a}y + (b-\hat{b})u = K\tilde{x} + \tilde{\theta}_1 y + \tilde{\theta}_2 u, \qquad (2.17)$$

where $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are vectors of parameter errors.

Guided by the solution to the regulator problem, the intention now is to get an overall structure as described in fig 2.2. Introducing

$$\tilde{y} = y - \hat{y} = c^T x - c^T\hat{x} = c^T\tilde{x},$$

the following conditions should be satisfied:

1) The output error $\tilde{y}$ should be governed by

   $$\tilde{y}(t) = G(p)(\tilde{\theta}^T(t)\varphi(t)),$$

   where $G(p)$ is strictly positive real, $\tilde{\theta}^T(t) = (\tilde{\theta}_1^T(t) \quad \tilde{\theta}_2^T(t))$ and $\varphi(t)$ is a vector whose components are known signals.
2) The parameter updating should be given by

   $$\dot{\tilde{\theta}}(t) = -R^{-1}\varphi(t)\tilde{y}(t).$$

   Compare with equation (2.12).

The condition 1) can also be reformulated as to find vectors d and , such that

$$\tilde{y} = c^T(pI-K)^{-1}(\tilde{\theta}_1 y + \tilde{\theta}_2 u) = c^T(pI-K)^{-1}d(\tilde{\theta}^T\varphi) \qquad (2.18)$$

and $c^T(pI-K)^{-1}d$ is SPR.

There is no immediate solution to this problem, and the first papers dealing with adaptive observers [40-45] made use of a trick, which consists of feeding an extra input signal into the observer. This makes the structure general enough to accomplish 1) above. The idea will be demonstrated in the following example.

<u>Example 2.4.</u> Observer with known b-vector.
When b is known, the error equation (2.17) is simplified into $\dot{\tilde{x}} = K\tilde{x} + \tilde{\Theta}y$, where $\tilde{\Theta}^T(t) = [c_1(t)\ c_2(t)]$ for a second order system. Adding an extra input signal $w^T(t) = [w_1(t)\ w_2(t)]$ as mentioned above now gives

$$\tilde{y}(t) = c^T(pI-K)^{-1}(\tilde{\Theta}(t)y(t) + w(t)).$$

With $\varphi^T(t) = [v_1(t)\quad v_2(t)]$ and $d = [d_1\quad d_2]$, the equation (2.18) can be written as

$$c^T(pI-K)^{-1}\begin{bmatrix} c_1(t)y(t)+w_1(t) \\ c_2(t)y(t)+w_2(t) \end{bmatrix} =$$

$$= c^T(pI-K)^{-1}\begin{bmatrix} d_1(c_1(t)v_1(t)+c_2(t)v_2(t)) \\ d_2(c_1^1(t)v_1^1(t)+c_2^2(t)v_2^2(t)) \end{bmatrix}$$

or, equivalently,

$$\emptyset = p[c_1(t)y(t)+w_1(t)] + [c_2(t)y(t)+w_2(t)] -$$

$$-pd_1[c_1(t)v_1(t)+c_2(t)v_2(t)]-d_2[c_1(t)v_1(t)+c_2(t)v_2(t)]=$$

$$= c_1(t)[\dot{y}(t)-(pd_1+d_2)v_1(t)]+c_2(t)[y(t)-(pd_1+d_2)v_2(t)]+$$

$$+ \dot{c}_1(t)[y(t)-d_1v_1(t)] + \dot{c}_2(t)d_1v_2(t) + \dot{w}_1(t) + w_2(t)$$

The equation is satisfied if the signals are chosen to be

$$v_1 = \frac{p}{pd_1+d_2}y \ ; \quad v_2 = \frac{1}{pd_1+d_2}y \ ;$$

$$w_1 = 0 \qquad ; \quad w_2 = (c_2\dot{d}_1-c_1\dot{d}_2)v$$

Now choose d_1, d_2 such that $c^T(pI-K)^{-1}d$ is SPR and let the parameter updating be given by

$$\dot{\hat{\theta}}(t) = -R^{-1}\varphi(t)\tilde{y}(t)$$

It then follows that y tends to zero. Under additional assumptions on the input it can then be shown that $\tilde{x}$ tends to zero.

□

Remark 1. Analogously to the regulator problem in example 2.3, there is a φ-vector, consisting of filtered data.
Remark 2. The solution for the general case uses the same idea as above, but there will be more extra signals.
Remark 3. The form of the parameter adjustment guarantees that the signal $w_2(t)$ can be generated without differentiation.

The important point in the above solution is the transformation into the configuration shown in fig 2.2 and the necessity to introduce an exogenous signal to achieve this. It has been shown later that it is possible to use a suitable, nonminimal representaion of the system instead of introducing the extra signal [46,47]. The basic property of these solutions is the use of a description, where the outputs (or errors) are bilinear in states and unknown parameters ('linearity in parameters'). Related problems are treated in [43,48,49].

After this short review of the adaptive observer technique, attention is once again focussed on the adaptive servo problem.

2.4. Non-measurable state - the general case.

Two different approaches to the problem of unaccesible state were described in section 2.1. The adaptive servo problem was solved in section 2.2 for the special case when the pole excess is equal to one. The concept of state variable filter was used. Ideas from the adaptive observer solution given in section 2.3 will now be used to solve the general problem with pole excess greater than or equal to one.

As in example 2.3 it is assumed that the plant is described by the differential equation

$$y(t) = \frac{B(p)}{A(p)} u(t) = \frac{b_\emptyset p^m + \ldots + b_m}{p^n + a_1 p^{n-1} + \ldots + a_n} u(t)$$

and the desired performance is specified with the model

$$y_m(t) = \frac{B^m(p)}{A^m(p)} u_m(t) = \frac{b_\emptyset^m p^m + \ldots + b_m^m}{p^n + a_1^m p^{n-1} + \ldots + a_n^m} u_m(t).$$

The numerator degree m is assumed smaller than n. The solution for the case m=n-1 was given in example 2.3. As before the error $e = y_m - y$ is given by

$$A^m(p)e(t) = C(p)[-B(p)\tilde{u}(t) + B^m(p)\tilde{u}_m(t) + (A(p) - A^m(p))\tilde{y}(t)] \qquad (2.19)$$

where $\tilde{u}$, $\tilde{u}_m$ and $\tilde{y}$ are filtered values of u, u_m and y (cf. equation 2.3). It was noted in example 2.3 that the paranthesis on the right hand side of (2.19) can be written as $\theta^T(t)\varphi(t)$. If m<n-1 the control law will however contain derivatives. This can be avoided by using the methods suggested by the observer solutions. This idea is due to Monopoli [18] and has been used in various forms

in [19,24-26]. The main idea is as follows. An extra signal w(t) (to be specified later on) can be added to the right hand side of (2.19) in order to rewrite the paranthesis as was done in the observer solution. There is however an important difference between the observer and regulator cases. In the observer case, addition of the extra signal just means that the estimates are calculated in a somewhat different manner. In the regulator case, the error e(t) in (2.19) is defined once for all, and a change in the r.h.s. must be accompanied by a similar one in the l.h.s. This means that a new quantity, below denoted $\eta(t)$, will replace e(t) in (2.19). Thus, introduce

$$e_1(t) = \frac{C(p)}{A^m(p)} w(t) \quad \text{and}$$

$$\eta(t) = e(t) + e_1(t).$$

The quantity $\eta(t)$ is called the 'augmented error' in Monopoli [18]. It satisfies

$$A^m(p)\eta(t) =$$
$$= C(p)[-B(p)\tilde{u}(t)+B^m(p)\tilde{u}_m(t)+(A(p)-A^m(p)\tilde{y}(t)+w(t)]. \tag{2.20}$$

In analogy with the observer problem, the signals $\tilde{u}(t)$ and w(t) can be chosen so that (2.20) transforms into

$$A^m(p)\eta(t) = C(p)(\tilde{\theta}^T(t)\varphi(t))$$

and $\eta(t)$ can thus be made to tend to zero.

The extra signal w(t) is composed of products of derivatives of parameter estimates and elements of the φ-vector as in the observer solution. This means that w(t) only influences the transients. It is zero when the parameter estimates move very slowly and φ is bounded. However, the boundedness of φ has to be proved in order to conclude that e(t)->0 from the fact that η(t)->0. It seems that this is not quite rigorously done in [18] as also mentioned in [24,25]. An alternative way to achieve e(t)->0 is to modify the choice of input and extra signal, i.e. not relying on the observer solution. This has been done by Feuer and Morse [24], but the solution is unfortunately complex. For easy reference, the major steps of the solutions by Monopoli [18] and Feuer and Morse [24] will be described in two examples.

Example 2.5. Monopoli's solution [18].
To be consistent with the rest of this paper, the notations are different Monopoli's. A cross-reference is therefore given in table 1.

The plant is described by the differential equation

$$y(t) = \frac{b_0 B(p)}{A(p)} u(t) = \frac{b_0(p^m+b_1p^{m-1}+\ldots+b_m)}{p^n+a_1p^{n-1}+\ldots+a_n} u(t) \qquad (2.21)$$

where the parameters are unknown. It is necessary to know the degree m of the numerator to write the equation in this form. It is assumed that m<=n-1. The model is given by

$$y_m(t) = \frac{B^m(p)}{A^m(p)} u_m(t) = \frac{b_0^m p^m+\ldots+b_m^m}{p^n+a_1^m p^{n-1}+\ldots+a_n^m} u_m(t). \qquad (2.22)$$

It is thus assumed that the degree of $B^m(p)$ is less than or equal to the degree of B(p). This assumption is natural to avoid differentiators in the control law.

As before the goal is to force the error $e = y_m - y$ to zero. Introducing the filter

$$Q(p) = p^{n-1} + q_1 p^{n-2} + \ldots + q_{n-1},$$

the error satisfies the equation

$$A^m(p)e(t)=Q(p)\left[-\frac{b_0 B(p)}{Q(p)}u(t)+\frac{B^m(p)}{Q(p)}u_m(t)+\frac{A(p)-A^m(p)}{Q(p)}y(t)\right] \tag{2.23}$$

Compare with (2.19). Define the augmented error as

$$\eta(t) = e(t) + e_1(t), \text{ where}$$

$$e_1(t) = \frac{Q(p)}{A^m(p)} w(t)$$

and $w(t)$ will be specified later. Then $\eta(t)$ satisfies

$$A^m(p)\eta(t) =$$

$$=Q(p)\left[-\frac{b_0 B(p)}{Q(p)}u(t)+\frac{B^m(p)}{Q(p)}u_m(t)+\frac{A(p)-A^m(p)}{Q(p)}y(t)+w(t)\right] \tag{2.24}$$

as in (2.20). In order to specify the control signal explicitly define an arbitrary but stable polynomial

$$C(p) = p^{n-m-1} + c_1 p^{n-m-2} + \ldots + c_{n-m-1},$$

and solve the identity

$$Q(p) = B(p)C(p) + D(p)/b_0$$

for the polynomial $D(p)$ of degree $n-2$,

$$D(p) = d_0 p^{n-2} + d_1 p^{n-3} + \ldots + d_{n-2}.$$

The equation (2.24) can then be written as

$$A^m(p)\eta(t) =$$

$$= Q(p)\left[-\frac{b_0}{C(p)}u(t) + \frac{D(p)}{C(p)Q(p)}u(t) + \frac{B^m(p)}{Q(p)}u_m(t) + \frac{A(p)-A^m(p)}{Q(p)}y(t) + \right.$$

$$\left. + w(t)\right]. \qquad (2.25)$$

The operator $C(p)$ will later be applied to this equation to determine the control signal. To avoid differentiating the output y, it is then necessary to rewrite the 4:th term on the r.h.s. of (2.25). The object is to make all the transfer operators in the r.h.s. have pole excesses greater than or equal to $n-m-1$. This means that they are still proper after multiplication with $C(p)$.

For this reason, define the polynomials F and G to be respectively the quotient and the rest when dividing $(A(p)-A^m(p))C(p)$ by $A(p)$, i.e.

$$(A(p)-A^m(p))C(p) = A(p)F(p) + G(p), \text{ where}$$

$$F(p) = f_0 p^{n-m-2} + f_1 p^{n-m-3} + \ldots + f_{n-m-2},$$

$$G(p) = g_0 p^{n-1} + g_1 p^{n-2} + \ldots + g_{n-2}.$$

Inserting this identity into (2.25) now gives

$$A^m(p)\eta(t) = Q(p)\left[-\frac{b_0}{C(p)}u(t) + \frac{D(p)+b_0 B(p)F(p)}{C(p)Q(p)}u(t) + \right.$$

$$\left. + \frac{B^m(p)}{Q(p)}u_m(t) + \frac{G(p)}{C(p)Q(p)}y(t) + w(t)\right]. \qquad (2.26)$$

Collecting the unknown parameters of the numerators (divided by $b_\emptyset$) into the vector θ and defining the vector $\varphi(t)$ as

$$\varphi^T(t)=[\frac{B^m(p)}{Q(p)}u_m(t),\frac{p^{n-2}}{C(p)Q(p)}u(t),\ldots,\frac{1}{C(p)Q(p)}u(t),$$

$$\frac{p^{n-1}}{C(p)Q(p)}y(t),\ldots,\frac{1}{C(p)Q(p)}y(t)]$$

gives (2.26) the alternative form

$$A^m(p)\eta(t) = Q(p)[-\frac{b_\emptyset}{C(p)}u(t) + w(t) + b_\emptyset\theta^T\varphi(t)]. \qquad (2.27)$$

Now, let $\hat{b}_\emptyset(t)$, $\hat{\theta}(t)$ denote estimates of $b_\emptyset$ and θ. Let the extra signal $w(t)$ be chosen as

$$w(t) = -\hat{b}_\emptyset(t)w_1(t)$$

and determine $w_1(t)$ and $u(t)$ so that

$$\frac{u(t)}{C(p)} + w_1(t) = \hat{\theta}^T(t)\varphi(t). \qquad (2.28)$$

Equation (2.27) then transforms into

$$A^m(p)\eta(t)=Q(p)[(b_\emptyset-\hat{b}_\emptyset(t))w_1(t) + b_\emptyset(\theta-\hat{\theta}(t))^T\varphi(t)]. \qquad (2.29)$$

This is the well-known form used before. Therefore, if the sign of $b_\emptyset$ is known and if $b_\emptyset Q(p)/A^m(p)$ is SPR, it follows that $\eta(t) \to \emptyset$ with a suitable parameter updating. The exact choices of $u(t)$ and $w_1(t)$ in (2.28) are postponed until the next chapter.

Table 1. Monopolis notation compared to the present one (notations not listed are the same).

Monopolis	This paper
$x(t)$	$y(t)$
$x_m(t)$	$y_m(t)$
$r^1(t)$	$u_m(t)$
$r(t)$	$B^m(p)u_m(t)$
$y(t)$	$e_1(t)$
$D_p(p)$	$A(p)$
$D_u(p)$	$b_\emptyset B(p)$
$D_m(p)$	$A^m(p)$
$D_r(p)$	$B^m(p)$
$D_w(p)$	$Q(p)$
$D_f(p)$	$C(p)$
$A(p)$	$-D(p)$
$B(p)$	$G(p)$
$C(p)$	$b_\emptyset B(p)F(p)$

□

Example 2.6. Feuer and Morse's solution [24].
In this example there will also be some changes compared to the original presentation. These are listed in table 2. As in example 2.5 the plant is given by

$$y(t) = \frac{b_\emptyset B(p)}{A(p)} u(t) = \frac{b_\emptyset (p^m+b_1p^{m-1}+\ldots+b_m)}{p^n+a_1p^{n-1}+\ldots+a_n} u(t). \tag{2.30}$$

The model is written somewhat differently.

$$y_m(t) = \frac{B^m(p)}{A^m(p)} u_m(t) = \frac{1}{\gamma_\emptyset(p)\gamma_1(p)} h(p)u_m(t), \tag{2.31}$$

where $\gamma_\emptyset(p)$, $\gamma_1(p)$ are monic polynomials of degree 1 and n-m-1 respectively. This implies that h(p) is a proper transfer operator. Once again introduce a stable filter

$$T(p) = p^n + t_1 p^{n-1} + \ldots + t_n,$$

and define F and G to be the quotient and the rest when dividing $\gamma_\emptyset(p)\gamma_1(p)T(p)$ by $A(p)$, i.e.

$$\gamma_\emptyset(p)\gamma_1(p)T(p) = A(p)F(p) + G(p), \text{ where}$$

$$F(p) = p^{n-m} + f_1 p^{n-m-1} + \ldots + f_{n-m},$$

$$G(p) = g_\emptyset p^{n-1} + g_1 p^{n-2} + \ldots + g_{n-1}.$$

Using these relations the error equation can be rewritten in a suitable form

$$\gamma_\emptyset(p)\gamma_1(p)e(t) = h(p)u_m(t) - \frac{\gamma_\emptyset(p)\gamma_1(p)b_\emptyset B(p)}{A(p)} u(t) =$$

$$= h(p)u_m(t) - \frac{b_\emptyset B(p)F(p)}{T(p)} u(t) - \frac{G(p)}{T(p)} y(t).$$

Division by $\gamma_1(p)$ gives

$$\gamma_\emptyset(p)e(t) = \frac{h(p)}{\gamma_1(p)} u_m(t) - \frac{b_\emptyset B(p)F(p)}{\gamma_1(p)T(p)} u(t) - \frac{G(p)}{\gamma_1(p)T(p)} y(t) \tag{2.32}$$

Introduce the augmented error defined by

$$\eta(t) = e(t) + e_1(t), \text{ where}$$

$$e_1(t) = -\frac{\hat{b}_\emptyset(t)}{\gamma_\emptyset(p)} w_1(t)$$

and $\hat{b}_\emptyset(t)$ is the estimate of $b_\emptyset$. Insertion into (2.32) gives the following equation for the augmented error

$$\gamma_0(p)\,\eta(t) = \frac{h(p)}{\gamma_1(p)}\,u_m(t) - \frac{b_0(p)B(p)F(p)}{\gamma_1(p)T(p)}\,u(t) -$$

$$- \frac{G(p)}{\gamma_1(p)T(p)}\,y(t) - \hat{b}_0(t)w_1(t) =$$

$$= \frac{h(p)}{\gamma_1(p)}\,u_m(t) - \frac{b_0}{\gamma_1(p)}\,u(t) - \frac{b_0(B(p)F(p)-T(p))}{\gamma_1(p)T(p)}\,u(t) -$$

$$- \frac{G(p)}{\gamma_1(p)T(p)}\,y(t) - \hat{b}_0(t)w_1(t) =$$

$$= -b_0\left(\frac{u(t)}{\gamma_1(p)} + w_1(t)\right) + (b_0-\hat{b}_0(t))w_1(t) +$$

$$+ b_0\left[\frac{h(p)}{b_0\gamma_1(p)}u_m(t) - \frac{B(p)F(p)-T(p)}{\gamma_1(p)T(p)}u(t) - \frac{G(p)/b_0}{\gamma_1(p)T(p)}y(t)\right]$$

(2.33)

Once again form the parameter vector θ from the numerator coefficients in the last paranthesis and define the vector

$${}^{T}(t) = \left[\frac{h(p)}{\gamma_1(p)}u_m(t),\ \frac{p^{n-1}}{\gamma_1(p)T(p)}u(t),\ldots,\ \frac{1}{\gamma_1(p)T(p)}u(t),\right.$$

$$\left.\frac{p^{n-1}}{\gamma_1(p)T(p)}y(t),\ldots,\ \frac{1}{\gamma_1(p)T(p)}y(t)\right].$$

Equation (2.33) can then be written as

$$\gamma_0(p)\,\eta(t) = -b_0\left(\frac{u(t)}{\gamma_1(p)} + w_1(t)\right) + (b_0-\hat{b}_0(t))w_1(t) + b_0\theta^T\varphi(t)$$

(2.34)

If now, as in Monopoli's scheme, the control signal and the extra signal $w_1(t)$ are chosen to satisfy

$$\frac{u(t)}{\gamma_1(p)} + w_1(t) = \bar{\theta}^T(t)\varphi(t), \tag{2.35}$$

the error equation (2.34) becomes

$$\gamma_0(p)\eta(t) = (b_0-\hat{b}_0(t))w_1(t) + b_0(\theta-\hat{\theta}(t))^T\varphi(t). \tag{2.36}$$

This equation is of the desired type. Notice that the stable transfer function $1/\gamma_0(s)$ is always SPR and so the only requirements for $\eta(t)$ to tend to zero are that $b_0>0$ and that a parameter updating similar to the previous ones is used.

Table 2. Feuer and Morse's notation compared to the present one.

Feuer and Morse	This paper
$r(t)$	$u_m(t)$
$x_0(t)$	$e_1(t)$
$x_1(t)$	$w_1(t)$
$\bar{e}(t)$	$\eta(t)$
q	θ
$\zeta(t)$	$\varphi(t)$
g_p	b_0
$\alpha_p(p)$	$B(p)$
$\beta_p(p)$	$A(p)$
$\alpha(p)$	$B^m(p)$
$\beta(p)$	$A^m(p)$
$\pi(p)$	$T(p)$
$\delta(p)$	$F(p)$
$\rho(p)$	$G(p)$

□

2.5. Discrete time systems.

So far all the results of this paper have been given for continuous time systems. Since the algebra for continuous time and discrete time systems have much in common, it is clear that a similar development can be given for discrete time systems. In fact, schemes which are just discrete time translations of the algorithms described have been presented by Ionescu and Monopoli [19,26].

However, as pointed out by Landau and Bėthoux [20], the discrete time problem suffers from an additional difficulty compared to the continuous time case. The reason is basically that for discrete time systems a positive real transfer function must have a feedthrough term (i.e. the impulse response contains a constant term), in contrast to the continuous time case. The implication is that there is no 'easy' case corresponding to the case of the pole excess equal to one for continuous time systems. The problems will be discussed in more detail in the next chapter.

3. A UNIFIED DESCRIPTION - CONTINUOUS TIME

The MRAS schemes are motivated from stability considerations only. This has the advantage that overall stability, which is of primary interest, is automatically assured. On the other hand, it has the implication that the development is not at all points easy to interpret. This is in contrast to the self tuning approach, where the algorithms can naturally be divided into identification and control using a separation principle.

To understand the MRAS schemes more fully, there are in particular two items which require careful investigations.

- The augmented error is introduced ad hoc in order to treat the problem with pole excess greater than one (section 2.4). It is desirable to give an interpretation.
- There are no connections between the adaptive solution and the corresponding known parameter situation.

The first question, how to interpret the augmented error, is relatively easy to answer. Consider the algorithm by Monopoli given in example 2.5. Combining the definitions of $e_1(t)$, $w(t)$ and $w_1(t)$, it is easily verified that

$$e_1(t) = \frac{Q(p)}{A^m(p)} [\hat{b}_0(t)\frac{u(t)}{C(p)} - \hat{b}_0(t)\hat{\theta}^T(t)\varphi(t)].$$

Compare this with the identity

$$e(t) = \frac{Q(p)}{A^m(p)} [-b_0 \frac{u(t)}{C(p)} + b_0\theta^T\varphi(t)].$$

It follows that $e_1(t)=-\hat{e}(t)$, where $\hat{e}(t)$ is the estimate of $e(t)$ using the model

$$e(t) = \frac{Q(p)}{A^m(p)} [-\hat{b}_0 \frac{u(t)}{C(p)} + \hat{b}_0\hat{\theta}^T\varphi(t)].$$

This means that the augmented error $\eta(t)$ is simply the difference between the error $e(t)$ and the corresponding estimate $\hat{e}(t)$. This is an important observation. It makes possible to generate the algorithms in a more systematic way. This will be done in two steps in the following. To obtain a suitable model structure for the algorithms, some results for the known parameter case are summarized in section 3.1. At the same time an answer to the second question raised above is given. With the results for the known parameter case and the interpretation of the augmented error as motivation, a class of adaptive algorithms is defined in section 3.2. The interesting property of this class of algorithms is that it contains several of the earlier proposed MRAS schemes as special cases. This fact is demonstrated in some examples in section 3.3.

3.1. The known parameter case.

The plant is assumed to satisfy

$$y(t) = \frac{b_0 B(p)}{A(p)} u(t) = \frac{b_0 (p^m + b_1 p^{m-1} + \ldots + b_m)}{p^n + a_1 p^{n-1} + \ldots + a_n} u(t), \tag{3.1}$$

where $A(p)$ and $B(p)$ are relatively prime. The problem consists of designing a controller which makes the closed-loop transfer function equal to a given model transfer function

$$\frac{B^m(p)}{A^m(p)} = \frac{b_0^m p^m + \ldots + b_m^m}{p^n + a_1 p^{n-1} + \ldots + a_n^m} \tag{3.2}$$

where $A^m(p)$ and $B^m(p)$ are relatively prime. Furthermore, it is assumed that $B(p)$ and $B^m(p)$ are relatively prime. This means that the plant is in practice restricted to be minimum phase, in order to keep the control signal bounded. The approach taken here is to consider the general controller configuration depicted in fig. 3.1, where R', S' and T' are polynomials in the differential operator. This structure will be seen to include interesting special cases. It should be noted that the realization of the controller does not look exactly as in fig. 3.1, because this would incorporate differentiators in the control law. Instead the feedforward transfer function T'/S' and the feedback transfer function R'/S' are implemented.

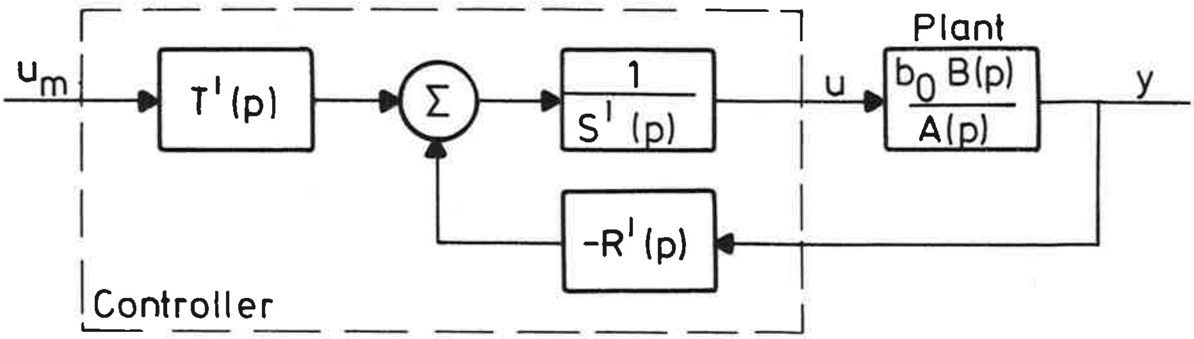

Fig 3.1. Controller configuration.

The desired closed-loop transfer function is obtained if the polynomials R', S' and T' are chosen to satisfy the equation

$$\frac{B^m(p)}{A^m(p)} = \frac{b_0 B(p) T'(p)}{A(p) S'(p) + b_0 B(p) R'(p)}$$

or, equivalently,

$$b_0 B(p) T'(p) A^m(p) = B^m(p) [A(p) S'(p) + b_0 B(p) R'(p)]. \qquad (3.3)$$

It is possible to simplify this polynomial equation. $B^m(p)$ divides $T'(p)$ since $B^m(p)$ is relatively prime to $B(p)$ and $A^m(p)$. In the same way $b_0B(p)$ divides $S'(p)$ because $b_0B(p)$ is relatively prime to $B^m(p)$ and $A(p)$. Thus, introduce new polynomials R, S and T defined by:

$$R'(p) = R(p)$$

$$S'(p) = b_0B(p)S(p) \qquad (3.4)$$

$$T'(p) = B^m(p)T(p)$$

The identity (3.3) is then reduced to

$$T(p)A^m(p) = A(p)S(p) + R(p) \qquad (3.5)$$

and the controller structure is now as shown in fig. 3.2.

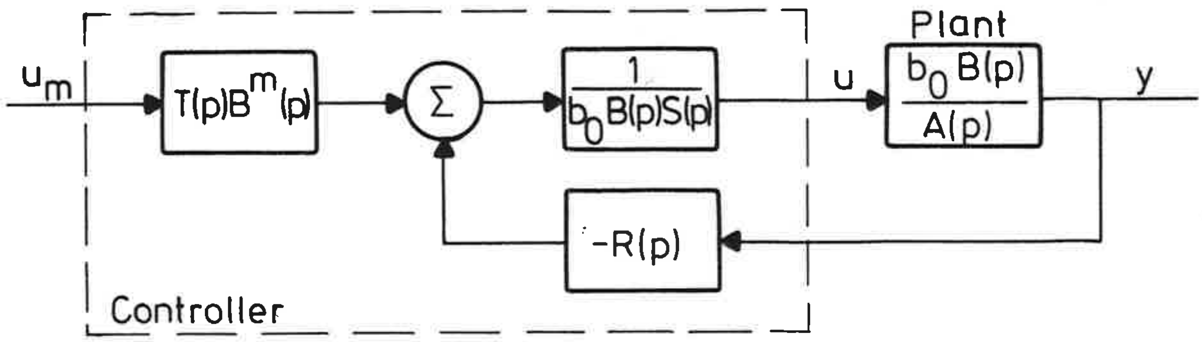

Fig 3.2. Controller configuration with polynomials in (3.4).

The polynomials $b_0B(p)$ and $T(p)$ are cancelled in the closed loop transfer function. These polynomials represent unobservable or uncontrollable parts. The polynomial $T(p)$ can thus be chosen freely without changing the closed-loop transfer function. When $T(p)$ has been determined, the equation (3.5) has many solutions $S(p)$ and $R(p)$. However, in all the algorithms considered it is assumed that the degree of R(p) is less or equal n-1, which assures that the equation has a unique solution, Åström [50]. Since the polynomials A(p) and $A^m(p)$ both have degree n, T(p) and S(p) will also have the same degree. They are also assumed to be monic. There is however one additional condition on T(p) and S(p). For the control law not to contain derivatives of the output, it is necessary to assume that the degree of S(p) (and therefore the degree of T(p)) is greater or equal n-m-1.

To summarize, the controller polynomials R, S and T are determined in the following way:

1) choose the monic polynomial T(p)

$$T(p) = p^k + t_1 p^{k-1} + \ldots + t_k, \quad k >= n-m-1 \tag{3.6}$$

2) solve the equation

$$T(p)A^m(p) = A(p)S(p) + R(p) \tag{3.7}$$

for the unique solutions R(p) and S(p), defined by:

$$R(p) = r_0 p^{n-1} + r_1 p^{n-2} + \ldots + r_{n-1} \tag{3.8}$$

$$S(p) = p^k + s_1 p^{k-1} + \ldots + s_k, \text{ same k as in (3.6)} \tag{3.9}$$

The first step, the choice of T(p) (including its degree), is quite arbitrary in the above formulation. However, this choice is of importance in the presence of noise, and this fact is clear when considering a special case. If k in (3.6) and (3.9) is chosen to be equal to n, it can be shown, see Åström [50], that the described controller is nothing but a standard solution consisting of a Kalman filter and state feedback, augmented with a zero placement. It also follows that T(p) can be interpreted as the characteristic polynomial of the Kalman filter. In the same way the choice k=n-1 makes the controller a frequency domain counterpart to a state space solution with a Luenberger observer and state feedback.

3.2. A class of adaptive controllers.

As described in the beginning of this chapter, the augmented error used in the MRAS approach has an important interpretation. It consists of the difference of a quantity, the error, and its estimate. This difference is used in the parameter estimation. The implication is that there is a strong relationship between the MRAS and the self-tuning regulators.

In order to make this relationship even more clear, the self-tuning principle will be used to define a general class of adaptive regulators. This class of algorithms will later be shown to include the described MRAS schemes as special cases. Apart from the parameter estimation, a suitable model structure for the algorithms is also needed. This is provided by the results for the known parameter case discussed in section 3.1. The first step in the development is to use these results to obtain expressions for the error or, more generally, the error filtered by some transfer function.

The polynomial identity (3.7) is used to get the following expression for the error $e(t) = y(t)-y_m(t)$:

$$
\begin{aligned}
TA^m e(t) &= TA^m y(t) - TA^m y_m(t) = \\
&= (AS + R)y(t) - TA^m y_m(t) = \\
&= b_0 BSu(t) + Ry(t) - TB^m u_m(t) \qquad (3.10)
\end{aligned}
$$

Now define the filtered error

$$e_f(t) = \frac{Q(p)}{P(p)} e(t), \tag{3.11}$$

where Q(p) and P(p) are stable monic polynomials of degree n+k-1 with the same k as in (3.6) and (3.9). Assume that P can be written as

$$P = P_1P_2, \tag{3.12}$$

where P_1 is of degree n-m-1 and P_2 of degree m+k. The filtered error $e_f(t)$ can then be written as

$$e_f(t) = \frac{Q}{P} e(t) =$$

$$= \frac{Q}{TA^m} [\frac{b_\emptyset BS}{P} u(t) + \frac{R}{P} y(t) - \frac{TB^m}{P} u_m(t)] =$$

$$= \frac{Q}{TA^m} [\frac{b_\emptyset(P_2+BS-P_2)}{P} u(t) + \frac{R}{P} y(t) - \frac{TB^m}{P} u_m(t)] =$$

$$= \frac{Q}{TA^m} [b_\emptyset\frac{u(t)}{P_1} + b_\emptyset(BS-P_2)\frac{u(t)}{P} + R\frac{y(t)}{P} - \frac{TB^m}{P}u_m(t)] \tag{3.13}$$

Define filtered u as

$$u_f(t) = \frac{u(t)}{P_1} \tag{3.14}$$

and let θ be a vector containing the unknown parameters of the polynomials $(BS-P_2)$ (degree m+k-1) and $R/b_\emptyset$ (degree n-1) and the constant $1/b_\emptyset$ as the last element.

Furthermore, define the vector

$$\varphi^T(t) = [\frac{p^{m+k-1}}{P}u(t),\ldots,\frac{1}{p}u(t),\frac{p^{n-1}}{P}y(t),\ldots,\frac{1}{p}y(t),$$

$$-\frac{TB^m}{P}u_m(t)]. \tag{3.15}$$

It is then possible to rewrite the expression (3.13) for the filtered error $e_f(t)$ as

$$e_f(t) = \frac{Q}{TA^m}[b_\emptyset u_f(t) + b_\emptyset \theta^T \varphi(t)]. \tag{3.16}$$

This identity suggests the model structure for the algorithm that was mentioned above, namely a model of the form

$$\hat{e}_f(t) = \frac{Q}{TA^m}[\hat{b}_\emptyset u_f(t) + \hat{b}_\emptyset \hat{\theta}^T \varphi(t)]. \tag{3.17}$$

Define the difference between the filtered error and the corresponding model error (cf. the augmented error)

$$\epsilon(t) = e_f(t) - \hat{e}_f(t). \tag{3.18}$$

The following equation is then obtained for $\epsilon(t)$:

$$\epsilon(t) = \frac{Q}{TA^m}[(b_\emptyset-\hat{b}_\emptyset(t))(u_f(t)+\hat{\theta}^T(t)\varphi(t))+b_\emptyset(\theta-\hat{\theta}(t))^T\varphi(t)] \tag{3.19}$$

This equation has the special form used several times in chapter 2 for the augmented error. It is therefore easy to choose the parameter updating in such a way that $\epsilon(t)$ tends to zero, provided that the transfer operator Q/TA^m is strictly positive real.

The parameter estimation described above, which is used in the MRAS schemes, is by no means the only possible solution. In presence of noise it could e.g. be suitable to have a gain of the parameter adjustment which decreases with time. The development done so far thus proposes a class of adaptive algorithms, consisting of two parts:

- a parameter estimator using the model (3.17);
- a control law based on the estimated parameters.

Parameter estimation

In the first step, the estimation part, the solution for the MRAS:s are based on Lyapunov theory or positivity concepts, whereas the self tuning schemes often use a least squares identification in the presence of noise. The LS identification is obviously derived from the minimization of a quadratic error criterion. It is in fact possible to derive an alternative to the MRAS estimation scheme in the same way.

Thus, write (3.16) as

$$e_f(t) = b_0 G(p) u_f(t) + b_0 \theta^T G(p) \varphi(t), \tag{3.20}$$

where

$$G(p) = \frac{Q(p)}{T(p)A^m(p)}.$$

Consider a model of the form

$$\hat{e}_f(t) = \hat{b}_0 G(p) u_f(t) + \hat{b}_0 \hat{\theta}^T G(p) \varphi(t) \tag{3.21}$$

instead of (3.17).

Then $\epsilon(t)$ in (3.18) satisfies the equation

$$\epsilon(t) = (b_0-\hat{b}_0(t))[G(p)u_f(t) + \hat{\theta}^T(t)(G(p)\varphi(t))] +$$

$$+ b_0(\theta-\hat{\theta}(t))^T(G(p)\varphi(t)) \qquad (3.22)$$

which is written as

$$\epsilon(t) = \tilde{\theta}^T(t)(G(p)\varphi(t)). \qquad (3.23)$$

Now choose $\epsilon^2(t)$ as a criterion. Regarding it as a function of $\tilde{\theta}$, the gradient w.r.t. $\tilde{\theta}$ is $2\epsilon(t)(G(p)\varphi(t))$. It is natural to make the parameter adjustment in a modified steepest descent direction, i.e.

$$\dot{\tilde{\theta}}(t) = -R^{-1}(G(p)\varphi(t))\epsilon(t), \text{ R pos definite.} \qquad (3.24)$$

It is possible to verify that this estimation scheme has the desired stability property. Choose the Lyapunov function

$$V = \tilde{\theta}^T R\tilde{\theta}. \qquad (3.25)$$

Its derivative becomes

$$\dot{V} = 2\tilde{\theta}^T(t)R\dot{\tilde{\theta}}(t) = -2\tilde{\theta}^T(t)(G(p)\varphi(t))\epsilon(t) = -2\epsilon^2(t)$$

and it follows that $\epsilon(t) \to 0$. It is thus possible to use an identification algorithm, similar to the MRAS schemes, but without requiring any transfer function to be SPR.

Choice of control law

The choice of control signal contains one difficulty. It is natural to determine the control law such that the estimate

$\hat{e}_f(t)$ of the error is equal to zero. According to equation (3.17) this means that $u_f(t)+\hat{\theta}^T(t)\ \varphi(t)=0$. It can be seen from (2.28) that this corresponds to the choice $w(t)=0$, i.e. no extra signal is used. This control law however uses derivatives of parameter estimates. This means that in practice the control law must be modified. There are different solutions proposed in the literature. Monopoli [18] choses the control signal for the algorithm in example 2.5 as

$$u(t) = \hat{\theta}^T(t)[C(p)\varphi(t)] \tag{3.26}$$

and $w_1(t)$ is chosen to satisfy (2.28). Feuer and Morse [24] propose a different control law. It is desirable to use a control law which has the property that $e_f(t)$ tends to zero if $e_f(t)-\hat{e}_f(t)$ tends to zero. A trivial case is when m=n-1, i.e. the pole excess is equal to one. Then the filtered u, $u_f(t)$ is simply u(t). It is therefore possible to achieve $\hat{e}_f(t)\to 0$ without differentiators.

A class of adaptive algorithms has thus been proposed. The interpretation of the augmented error for the MRAS was the motivation. The augmented error, which is identical to $\epsilon(t)$ in (3.18, 3.19), is used in the parameter estimation part. One special version of the parameter estimation makes the algorithm equivalent to the MRAS schemes. However, sofar the relations between the proposed algorithm and the MRAS schemes have not been shown in detail. This will be done in a few examples in the next section.

3.3. Examples of the general algorithm.

It will now be shown how the algorithms by Monopoli (ex.2.5) and Feuer and Morse (ex.2.6) fit into the general algorithm given in section 3.2. It will also be shown that an algorithm by Narendra and Valavani [25] can be put into the general description.

Example 3.1. Monopoli's scheme (example 2.5).
The definition of the error e(t) is at first changed compared to the one in example 2.5, namely $e(t)=y(t)-y_m(t)$. It then follows from equation (2.26) that

$$e(t) = \frac{Q'}{A^m}\left[b_\emptyset\frac{u(t)}{C} - b_\emptyset\frac{D/b_\emptyset+BF}{CQ'}u(t) - \frac{G}{CQ'}y(t) - \frac{B^m}{Q'}u_m(t)\right]. \quad (3.27)$$

Here Q', which replaces Q in (2.26), is of degree n-1, C of degree n-m-1, $(D/b_\emptyset+BF)$ of n-2 and G of degree n-1. Now compare this with the identity (3.13):

$$e_f(t) = \frac{Q}{A^mT}\left[b_\emptyset\frac{u(t)}{P_1} + b_\emptyset\frac{BS-P_2}{P}u(t) + \frac{R}{P}y(t) - \frac{TB^m}{P}u_m(t)\right]. \quad (3.28)$$

If k in (3.6), (3.9) and (3.11) is chosen to be equal n-m-1, the polynomial degrees are: Q: 2n-m-2; T: n-m-1; P_1:n-m-1; P_2: n-1; P: 2n-m-2; $BS-P_2$: n-2; R: n-1. It is straightforward to verify that (3.27) coincides with (3.28) with the following relations between the polynomials:

$P_1 = T = C$
$P_2 = Q'$
$Q = P = P_1P_2 = TQ' = CQ'$
$R = -G$
$S = C - F$

Because Q = P, it follows that in this case the filtered error $e_f(t)$ is simply equal to the error e(t) itself. The conclusion is that Monopolis scheme is a special case of the general algorithm, namely the case $e_f(t)=e(t)$ and k=n-m-1. The parameter estimation is chosen to be the one usually used in the MRAS schemes.

□

Example 3.2. Feuer and Morse's scheme.
As in the above example the error is redefined compared to example 2.6, i.e. $e(t)=y(t)-y_m(t)$. A major simplification is done in this scheme in eq. (2.31), and the effect is that the development proceeds as if $B^m=1$, $A^m=\nu_0 \nu_1$. For simplicity, thus assume h(p)=1. It then follows from (2.32) that the error e(t) obeys (T' is used in stead of T):

$$e(t) = \frac{1}{\nu_0(p)}\left[b_0\frac{u(t)}{\nu_1} + b_0(BF-T')\frac{u(t)}{\nu_1 T'} + G\frac{y(t)}{\nu_1 T'} - \frac{u_m(t)}{\nu_1 T'}\right] \tag{3.29}$$

The polynomial degrees are: ν_0: 1; ν_1: n-m-1; T': n; (BF-T'): n-1; G: n-1. This identity shall once again be compared with the identity (3.13):

$$e_f(t) = \frac{Q}{\nu_0 \nu_1 T}\left[b_0\frac{u(t)}{P_1} + b_0\frac{BS-P_2}{P}u(t) + \frac{R}{P}y(t) - \frac{T}{P}u_m(t)\right] \tag{3.30}$$

In this case it is relevant to note the polynomial degrees in the case k=n-m: Q: degree 2n-m-1; T: n-m; P_1: n-m-1; P_2: n; P: 2n-m-1; $BS-P_2$: n-1; R: n-1. Once again a comparison of (3.29) and (3.30) shows that they are in fact identical with the following relations between the polynomials:

$P_1 = \nu_1$

$P_2 = T = T'$

$Q = P = P_1 P_2 = \nu_1 T = \nu_1 T'$

$R = G$

$S = F$

It has thus been shown that the algorithm of Feuer and Morse also fits into the general description. Similar to Monopolis scheme the filtered error $e_f(t)$ is equal to the error $e(t)$ and the estimation part is the ordinary MRAS algorithm.

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Example 3.3. Narendra and Valavani's scheme.
Since this scheme is not described in this paper, no details are given. However, it can be shown that this MRAS also is possible to generate within the general framework. The algorithm has $k=n-m-1$ and once again the filtered error is equal to the error itself. The error is given by (3.13), where the polynomials are chosen as:

$P_1 = L$ (degree $n-m-1$)

$P_2 = B^m T$ (degree $n-1$)

$Q = P = P_1 P_2$ (degree $2n-m-2$)

The polynomial L is chosen to make the transfer function LB^m/A^m SPR and the ordinary MRAS estimation scheme is used.

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4. A UNIFIED DESCRIPTION - DISCRETE TIME

The MRAS filosophy was originally developed for continuous time systems but can also be applied to discrete time. Stability is still the major consideration. The interpretation of the augmented error and the relations to the known parameter case will be considered for the discrete time case too.

The augmented error can be interpreted in the same way as for the continuous time case. It is easy to verify that the augmented error is equal to the difference between the error e(t) and $\hat{e}(t|t-1)$, the prediction of e(t) using estimates from time t-1. This important observation provides a bridge to the self-tuning regulators. The connections with the known parameter case are treated in the same way as earlier.

As in chapter 3, a class of algorithms will be defined in two steps. The results for the known parameter case are summarized in section 4.1. The algorithm is defined in section 4.2 using the self-tuning idea. The results for the known parameter case suggests a model structure for the parameter estimation, where the augmented error enters in a natural way. A MRAS algorithm is described as a special case in section 4.3. It is also shown that two self-tuning controllers are special cases. This makes it possible to relate the MRAS and the self-tuning regulators closely.

4.1. The known parameter case.

The following discussion is analogous to the corresponding discussion for continuous time systems in section 3.1.

The plant is assumed to be given by

$$y(t) = \frac{q^{-(k+1)} b_\emptyset B(q^{-1})}{A(q^{-1})} u(t) =$$

$$= \frac{q^{-(k+1)} b_\emptyset (1 + b_1 q^{-1} + \ldots + b_m q^{-m})}{(1 + a_1 q^{-1} + \ldots + a_n q^{-n})} u(t) \qquad (4.1)$$

where q^{-1} is the backward shift operator and k is an extra time delay. The polynomials $A(q^{-1})$ and $B(q^{-1})$ are assumed to be relatively prime. The desired closed loop transfer function is given by

$$\frac{q^{-(k+1)} B^m(q^{-1})}{A^m(q^{-1})} = \frac{q^{-(k+1)} (b_\emptyset^m + b_1^m q^{-1} + \ldots + b_{n-1}^m q^{-(n-1)})}{1 + a_1^m q^{-1} + \ldots + a_n^m q^{-n}} \qquad (4.2)$$

where $A^m(q^{-1})$ and $B^m(q^{-1})$ are assumed to be relatively prime. As in the continuous time case, it is also assumed that $B(q^{-1})$ and $B^m(q^{-1})$ are relatively prime, with the same implication that only non-minimum phase systems can be treated. There is a time delay in the model which is greater than or equal to the original one. This is a natural assumption to avoid non-causal control laws.

As in the continuous time case, the proposed solution consists of a controller with feedforward T'/S' and feedback -R'/S', where R', T' and S' are polynomials in the backward shift operator q^{-1}. It is possible to simplify the controller to the configuration seen in fig. 4.1. The assumptions on relatively primeness above are used and the arguments are the same as for the continuous time case. It should be noted that there is no realizability problem with this configuration, because the polynomials are defined in q^{-1}.

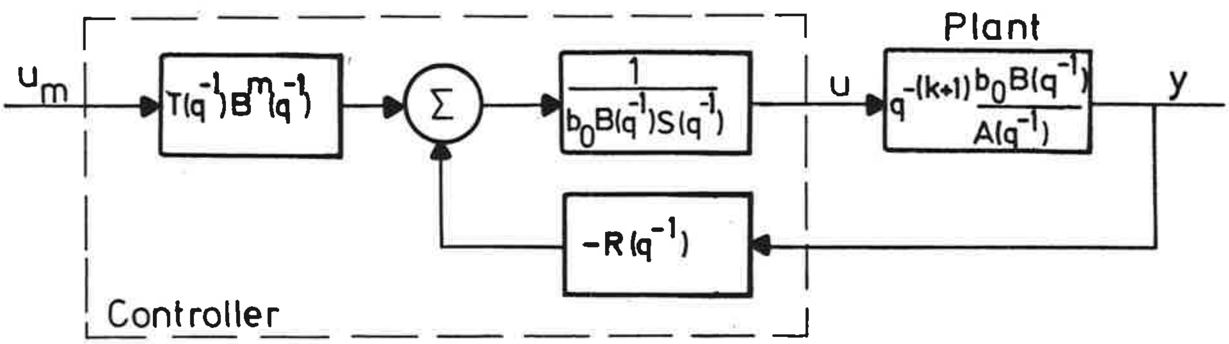

Fig 4.1. Controller configuration.

The following identity must be satisfied

$$T(q^{-1})A^m(q^{-1}) = A(q^{-1})S(q^{-1}) + q^{-(k+1)}R(q^{-1}) \tag{4.3}$$

if the closed-loop transfer function should be equal to the desired one. The T-polynomial can be chosen arbitrarily without affecting the closed-loop transfer function. When T is determined, the equation (4.3) has many solutions S and R but for all the considered algorithms the solution is unique because one of the following conditions is satisfied:

Case 1. Degree(R) < degree(A) <=n

or

Case 2. Degree(S) <= k.

The requirement for the controller to be causal does not imply any conditions on the polynomial degrees. In stead, it must be required that $S(0) \neq 0$, which from (4.3) is equivalent to $T(0) \neq 0$. Furthermore the S- and T-polynomials are scaled so that $T(0)=S(0)=1$. The resulting design procedure is thus as follows:

1) Choose the polynomial $T(q^{-1})$ defined by

$$T(q^{-1}) = 1 + t_1 q^{-1} + \ldots + t_{n_T} q^{-n_T} \tag{4.4}$$

2) Solve the polynomial equation

$$T(q^{-1}) A^m(q^{-1}) = A(q^{-1}) S(q^{-1}) + q^{-(k+1)} R(q^{-1}) \tag{4.5}$$

for the unique solutions $R(q^{-1})$ and $S(q^{-1})$, defined by

$$R(q^{-1}) = r_0 + r_1 q^{-1} + \ldots + r_{n_R} q^{-n_R} \tag{4.6}$$

$$S(q^{-1}) = 1 + s_1 q^{-1} + \ldots + s_{n_S} q^{-n_S} \tag{4.7}$$

with one of the following conditions

n_R < degree(A) (case 1) or $n_S <= k$ (case 2).

The arbitraryness in the choice of the polynomial $T(q^{-1})$ can be commented in the same way as for continuous time systems. Thus, for the case $n_T=n$ and $n_R=n-1$ (case 1) it is easy to interpret the controller as a Kalman filter and state feedback, together with a zero placement. Also in this case the T-polynomial is the characteristic polynomial for the Kalman filter and so it has importance when noise is affecting the system. Analogously, the choice $n_T=n_R=n-1$ corresponds to the situation above but with the Kalman filter replaced by a Luenberger observer.

4.2. A class of adaptive controllers.

The interpretation of the augmented error given in the beginning of this chapter suggests that the MRAS and the self-tuners have much in common. Using this interpretation and the results for the known parameter case given in section 4.1, it is possible to carry through a general development parallell to the one for continuous time systems. The first step towards the definition of a general algorithm is to obtain expressions for a filtered error.

Use the identity (4.5) to write for the error $e(t) = y(t) - y_m(t)$:

$$TA^m e(t) = TA^m y(t) - TA^m y_m(t) =$$

$$= (AS + q^{-(k+1)}R)y(t) - TA^m y_m(t)$$

$$= q^{-(k+1)}[b_0 BSu(t) + Ry(t) - TB^m u_m(t)] \tag{4.8}$$

Define the filtered error $e_f(t)$ as

$$e_f(t) = \frac{Q(q^{-1})}{P(q^{-1})}e(t) = \frac{Q(q^{-1})}{P_1(q^{-1})P_2(q^{-1})}e(t) \tag{4.9}$$

where the stable polynomials Q and P are defined by

$$Q(q^{-1}) = 1 + q_1 q^{-1} + \ldots + q_{n_Q} q^{-n_Q}$$

$$P(q^{-1}) = 1 + p_1 q^{-1} + \ldots + p_{n_P} q^{-n_P}. \tag{4.10}$$

The partition of P into P_1P_2 is done, if possible, with P_1 of degree k. The following equation for $e_f(t)$ is now obtained:

$$e_f(t)=\frac{Q}{P}e(t) = \frac{Q}{TA^m}q^{-(k+1)}[\frac{b_\emptyset BS}{P}u(t) + \frac{R}{P}y(t) - \frac{TB^m}{P}u_m(t)] =$$

$$=\frac{Q}{TA^m}\, q^{-(k+1)}[b_\emptyset\frac{u(t)}{P_1} + b_\emptyset(BS-P_2)\frac{u(t)}{P} + R\frac{y(t)}{P} - \frac{TB^m}{P}u_m(t)] \tag{4.11}$$

Introduce the filtered input

$$u_f(t) = \frac{u(t)}{P_1} \tag{4.12}$$

and define the vector θ, consisting of the unknown parameters of the polynomials $BS-P_2$ and R and the constant $1/b_\emptyset$ as the last element. Also define the vector $\varphi(t)$,

$$\varphi^T(t) = [\frac{u(t-1)}{P}, \frac{u(t-2)}{P},\ldots,\frac{y(t)}{P}, \frac{y(t-1)}{P},\ldots,-\frac{TB^m}{P}u_m(t)]$$

Equation (4.11) can then be written as

$$e_f(t) = \frac{Q}{TA^m}\, q^{-(k+1)}[b_\emptyset u_f(t) + b_\emptyset\theta^T\varphi(t)] \tag{4.13}$$

Guided by the discussion of the continuous time case, it seems resonable to consider a model of (4.13) in the following form:

$$\hat{e}_f(t) = \frac{Q}{TA^m}\, q^{-(k+1)}[\hat{b}_\emptyset u_f(t) + \hat{b}_\emptyset\hat{\theta}^T\varphi(t)]. \tag{4.14}$$

For purposes of identification, it is of interest to calculate the prediction error using the model (4.14). The latest available parameter estimates are used in this calculation. Thus define the prediction error

$$\epsilon(t) = e_f(t) - \hat{e}_f(t|t-1), \tag{4.15}$$

where

$$\hat{e}_f(t|t-1) = \frac{Q}{TA^m}[\hat{b}_0(t-1)u_f(t-k-1) + \hat{b}_0(t-1)\hat{\theta}^T(t-1)\,\varphi(t-k-1)] \tag{4.16}$$

Equations (4.13),(4.15) and (4.16) give

$$\epsilon(t) = \frac{Q}{TA^m}\,[(b_0-\hat{b}_0(t-1))(u_f(t-k-1)+\hat{\theta}^T(t-1)\,\varphi(t-k-1)) +$$

$$+ b_0(\theta-\hat{\theta}(t-1))^T\varphi(t-k-1)]. \tag{4.17}$$

It is now possible to define a class of adaptive controllers consisting of two parts:

- a parameter estimator using the model (4.14);
- a control law based on the estimated parameters.

In the estimation part, the discrete time MRAS:s use a simple translation of the common continuous time algorithm. The assumption that Q/TA^m is SPR plays an important role. There is however one additional difficulty in the discrete time case. It was pointed out in section 3.2 that it is not always possible to obtain $u_f(t)+\hat{\theta}^T(t)\,\varphi(t)=0$ without differentiators. The corresponding situation now is seen in (4.17). Only with k=0 is it possible to choose the control so that $u_f(t-k-1)+\hat{\theta}^T(t-1)\,\varphi(t-k-1)=0$, provided that the control law should be non-anticipative. The extra difficulty mentioned above means that not even in the case k=0 is the problem solved. The reason is the following one. For simplicity, write (4.17) with k=0 as

$$\epsilon(t) = H(q^{-1})\,[\tilde{\theta}^T(t-1)\,\varphi(t-1)] \tag{4.18}$$

with $\tilde{\theta}$ being the parameter error vector as before. Compare this with the continuous time equation

$$\epsilon(t) = G(p)[\tilde{\theta}^T \varphi(t)] \tag{4.19}$$

as written in section 3.2. The extra time delay in (4.18) implies that we cannot conclude, with the same technique as in the continuous time case, that $\epsilon(t)$ tends to zero with the ordinary estimation approach, even if $H(q^{-1})$ is SPR. The extra time delay cannot be included in $H(q^{-1})$ because of the requirement that $H(q^{-1})$ is SPR. The transfer function $H(q^{-1})$ must contain a feedthrough term to be SPR. This is different from the continuous time case, where G(p) is strictly proper. Methods which try to resolve this difficulty have been presented e.g. in Ionescu and Monopoli [26]. They will not be discussed further.

In addition to the problem with the time delay, there is one point in the estimation part to be discussed. With the simplified notation in (4.18) and the assumption k=0 the ordinary MRAS estimation scheme is characterized by the parameter updating

$$\hat{\theta}(t) = \hat{\theta}(t-1) - R^{-1}\varphi(t-1)\,\epsilon(t), \text{ R positive definite.} \tag{4.20}$$

Even if the problem with the time delay is neglected, the positive realness of $H(q^{-1})$ in (4.18) can be shown to play a crucial role for $\epsilon(t)$ to converge to zero. However, if a modified model

$$\hat{e}_f(t) = \hat{\theta}^T(H(q^{-1})\varphi(t-1)) \tag{4.21}$$

is considered and the parameter updating is done according to the formula

$$\hat{\theta}(t) = \hat{\theta}(t-1) - R^{-1}(H(q^{-1})\varphi(t-1))\,\epsilon(t), \tag{4.22}$$

it is not necessary to require $H(q^{-1})$ to be SPR. In contrast to the continuous time case, it is not possible to prove convergence in a straightforward way even with this modification. With a boundedness condition and the stochastic convergence results of Ljung, it is shown in Ljung and Landau [51] that the positive realness condition may be dispensed with if the estimation algorithm includes filtering of $\varphi(t)$ by $H(q^{-1})$ as in (4.22). The problem formulation in [51] also contains noise. The estimation scheme is therefore a variant of (4.22) with a decreasing gain. The important point is that the positive realness condition can be eliminated by choosing an estimation algorithm that differs from the MRAS algorithm.

A general algorithm, motivated by the interpretation of the augmented error, has thus been defined. It can be interpreted as composed of two parts, identification and control and is therefore similar to the self tuning schemes. Explicit relations with both MRAS and self tuners are shown in the next section.

4.3. Examples of the general algorithm.

As in chapter 3 for continuous time systems, some special cases of the general algorithm defined in section 4.2 will now be considered. One algorithm mentioned before, the MRAS by Ionescu and Monopoli [26], will be shown to fit into the prototype algorithm. The basis for the defined algorithm is a 'self-tuning principle' and it is natural that some of the proposed self-tuners have a relationship with the present one. This is also demonstrated in some examples in this section.

Example 4.1. Ionescu and Monopoli's scheme [26].
Since this scheme is just a translation of the one in ex.3.1 into discrete time, no details are given. It is however

straightforward to show that the scheme falls into the description for case 1 and 2 with $e_f(t)=e(t)$. It is also noted that the polynomial degrees are $n_R=n-1$, $n_S=n_T=k$, $n_Q=n_P=n+k-1$ and that the polynomials are related in the same way as for the continuous time version (see ex.3.1).

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In the continuous time case, $e_f(t)$ was defined as $e_f(t)= Q(p)/P(p)e(t)$, where Q and P are of the same degree. This is important, because derivatives of e(t) are not wanted in order to generate $e_f(t)$. In the discrete time case, Q and P are polynomials in q^{-1} with the constant term equal to one. This means that the coefficients in either polynomial could be zero without causing any trouble when computing $e_f(t)$. This is utilized in the following example.

<u>Example 4.2.</u> Åström and Wittenmark's self-tuning controller [36].

Motivated by the discussion above, it is natural to choose $Q=TA^m$ and $P=1$, i.e. $e_f(t)=TA^m e(t)$. This implies that (4.13) has a simple form:

$$e_f(t) = q^{-(k+1)} b_\emptyset [u(t) + \theta^T \varphi(t)]$$

where the elements of φ are lagged input and output values. There are two interesting aspects of this structure. Firstly, because $Q/TA^m=1$, the positive realness condition does not enter at all, and the MRAS estimation scheme (4.2∅) coincides with the modified one in (4.22). Secondly, if the error e(t) is the object of main interest, it is easy to conclude asymptotic stability of e(t) from asymptotic stability of $e_f(t)$ because both T and A^m are asymptotically stable polynomials.

The structure described above is used in [36] for some algorithms which correspond to the solutions with Kalman filter or Luenberger observer (see section 4.1). The problem formulation includes noise and therefore a least squares identification is used. The implication is that the proposed algorithms are special cases of the general algorithm described above.

□

Example 4.3. Self-tuning regulator [27,28].
The basic self-tuning regulator described in Åström and Wittenmark [27] and Wittenmark [28] is based on a minimum variance strategy. This means that the desired closed loop transfer operator is just $q^{-(k+1)}$, i.e. both A^m and B^m are equal to 1. In this case the identity (4.5) simplifies into:

$$AS + q^{-(k+1)}R = T \tag{4.23}$$

It is assumed that the conditions both for case 1, i.e. the degree of R is n-1, and for case 2, i.e. the degree of S is k, are fulfilled. Furthermore, it is assumed that T is of degree n. With P=Q=1 and $e_f(t)=e(t)$, the identity (4.11) now yields:

$$e(t) = \frac{1}{T}q^{-(k+1)}[b_\emptyset u(t) + b_\emptyset(BS-1)u(t) + Ry(t) - Tu_m(t)] \tag{4.24}$$

This is however not the model used in the self tuning regulator. Firstly, the latter is based on a minimum variance strategy for the stochastic system

$$A(q^{-1})y(t) = q^{-(k+1)}b_\emptyset B(q^{-1})u(t) + C(q^{-1})e(t), \tag{4.25}$$

where e(t) is white noise. Since the goal is to minimize the output variance, $u_m(t)$ is equal to zero. Furthermore, it can be shown that the optimal choice of observer characteristic polynomial is $C(q^{-1})$, which is unknown (in fact, the identity (4.23) with T=C is the same as the identity used to derive the minimum variance strategy). In the self tuning regulator it is therefore assumed that C=1 in the model.

Summarizing, the identity (4.24) for the self tuning regulator is

$$e_f(t) = e(t) = y(t) =$$

$$= \frac{1}{C} q^{-(k+1)} [b_\emptyset u(t) + b_\emptyset (BS-1)u(t) + Ry(t)] =$$

$$= \frac{1}{C} q^{-(k+1)} [b_\emptyset u(t) + b_\emptyset \theta^T \varphi(t)] \qquad (4.26)$$

whereas the corresponding model does not include the unknown C-polynomial:

$$\hat{e}_f(t) = \hat{e}(t) = \hat{y}(t) = q^{-(k+1)} [\hat{b}_\emptyset u(t) + \hat{b}_\emptyset \hat{\theta}^T \varphi(t)] \qquad (4.27)$$

In the stochastic framework noise should be added in the r.h.s. of (4.26). The fact that the C-polynomial is included in (4.26) but not in the model (4.27) makes it somewhat unexpected that the algorithm really converges to the optimal minimum variance regulator. In Ljung [34] it is shown that the scheme (with least squares identification) converges if 1/C - 1/2 is SPR. This result once again demonstrates the close relationship between the model reference adaptive approach and the self tuning approach.

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It is interesting to compare the convergence result above with the discussion in section 4.2. There it was suggested that the estimation algorithm should include filtering by $H(q^{-1})$, (4.22). If this is not done, but in stead the MRAS algorithm (4.20) is used, the positive realness condition on $H(q^{-1})$ enters automatically. Applied to the self tuning regulator, this result implies that filtering by 1/C would improve the algorithm. However, C is not known and therefore the filtering cannot be done. The implication is that the estimation scheme becomes similar to the MRAS scheme, and the positive realness condition on 1/C enters. The use of a least squares algorithm slightly changes the condition to the one cited above. A natural modification in order to weaken the condition on C is to filter with $1/\hat{C}$, where $\hat{C}$ is an estimate of C. This modification, as well as other aspects of the convergence problem of MRAS and self tuning schemes, including the positive realness condition, are discussed by Ljung and Landau [51]. They also investigate the connections between the MRAS and self-tuners, mainly in the state space.

<u>Example 4.4.</u> Clarke and Gawthrop's self-tuning controller [32].

Generalizing the ideas of the self tuning regulator, as described in example 4.3, Clarke and Gawthrop consider a 'generalized output'

$$\phi(t) = P(q^{-1})y(t) + Q(q^{-1})u(t-k-1) - R(q^{-1})u_m(t-k-1) \tag{4.28}$$

and applies the basic self tuner to the system generating this output. However, for the special case with Q=0 it is possible to derive the algorithm in another way. Thus change the notation in (4.28) into:

$$\phi(t) = A^m(q^{-1})y(t) - q^{-(k+1)}B^m(q^{-1})u_m(t) \tag{4.29}$$

Then it follows that $\phi(t)$ equals $e_f(t) = A^m e(t)$ with the present notation. This relation between the filtered error and the error is obtained with the choices P=1 and $Q=A^m$. As in example 4.3 (but now with general A^m and B^m) the identity (4.11) simplifies into

$$e_f(t)=\frac{1}{T}q^{-(k+1)}[b_0u(t) + b_0(BS-1)u(t) + Ry(t) - TB^m u_m(t)] \quad (4.30)$$

where the polynomials satisfy the identity

$$TA^m = AS + q^{-(k+1)}R \quad (4.31)$$

As in example 4.3 the identity (4.31) with T=C is the same as the identity used to derive the minimum variance strategy for $e_f(t)$, provided that the degree of S is k (case 2). This is easily seen as follows. The process is still assumed to be governed by (4.25). Then the following is obtained, using (4.31):

$$e_f(t+k+1) = A^m y(t+k+1) - B^m u_m(t) =$$

$$= \frac{AS}{C} y(t+k+1) + \frac{R}{C} y(t) - B^m u_m(t) =$$

$$= \frac{b_0 BS}{C} u(t) + Se(t+k+1) + \frac{R}{C} y(t) - B^m u_m(t) =$$

$$= \frac{1}{C} (b_0 BSu(t) + Ry(t) - CB^m u_m(t)) + Se(t+k+1) \quad (4.32)$$

Because the degree of S is k, the term Se(t+k+1) contains noise that is independant of the left term in brackets. Thus the minimum variance strategy clearly is to choose the control signal according to

$$b_0 BSu(t) + Ry(t) - CB^m u_m(t) = 0 \quad (4.33)$$

This result is a special case of the one derived by Clarke and Gawthrop and their notation is compared to the present one in table 3.

The conclusion is that Clarke and Gawthrop obtain the identity (4.30) with T=C for the generalized output that equals $e_f(t)$, i.e.

$$e_f(t)=\frac{1}{C}q^{-(k+1)}[b_0u(t) + b_0(BS-1)u(t) + Ry(t) - CB^m u_m(t)]$$

$$=\frac{1}{C}\, q^{-(k+1)}[b_0u(t) + b_0\theta^T\varphi(t)] \qquad (4.34)$$

where the unknown C-polynomial is included in the θ-vector and of course noise should be added in the stochastic formulation. The C-polynomial still has to be replaced by 1 in the model:

$$\hat{e}_f(t) = q^{-(k+1)}[\hat{b}_0u(t) + \hat{b}_0\hat{\theta}^T\varphi(t)] \qquad (4.35)$$

The algorithm by Clarke and Gawthrop essentially uses this model structure and the identification part consists of a least squares algorithm.

Table 3. Clarke and Gawthrop's notation compared to the present one.

Clarke/Gawthrop	This paper
P	A^m
R	B^m
F	R
G	b_0BS
H	CB^m

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5. CONCLUSIONS

A short summary of the techniques used in the theory of model reference adaptive systems was given in chapter 2. Some recent schemes, which illustrate current ideas, were described in some detail. It could be concluded from the review that the MRAS schemes have two major disadvantages. The schemes do not relate to the case of known parameters and it is difficult to interpret the augmented error.

In chapters 3 and 4 the MRAS were examined for the continuous and discrete time cases. It resulted in a fairly natural interpretation of the MRAS schemes. They can be thought as composed of two parts. The first is a parameter estimator based on a model structure obtained from analysis of the known parameter case. The second part consists of a feedback law based on the estimated parameters. A general class of adaptive algorithms with this two-step structure was defined. Apart from the MRAS schemes it was shown that some other algorithms of the so called "self-tuning" type can be incorporated into the same description. The most important implication of the analysis is that there are in principle no differences betweenthe MRAS and self-tuning approaches.

There have been arguments in the literature why one should prefer MRAS or self-tuners. The result mentioned above however shows that the differences appear only because of minor changes in the general structure of the algoritms, which is the same for the two approaches. The identification method is perhaps the most striking difference. It has been shown in sections 3.2 and 4.2 that the important assumption for the MRAS:s, that a transfer function should be strictly positive real, is possible to eliminate. This condition essentially depends on a specific choice of estimation algorithm and a minor change of the latter makes it unnecessary to introduce the condition.

There is another interesting problem within the general structure, namely the choice of control signal. The goal is to achieve over-all stability and to make the error itself, not only the augmented error ($\epsilon(t)$), tend to zero. As mentioned earlier a partial answer to this problem is given in Feuer and Morse [24].

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