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Abstract

Stored energies of radiating systems have generated research interest for several decades due to their relationship with Q-factor and fractional bandwidth. In this paper material derivatives are used to provide a new interpretation of widely used stored energy expressions. This provides a fundamental relationship between stored energies of radiating systems and their electromagnetic material properties. It is shown that electric stored energy is related to the electric material parameter (permittivity) derivative. Similarly, magnetic stored energy is linked to the derivative of the permeability. Further, as an extension Cauchy-Riemann equations are used to relate stored energy to dissipation.

1 Introduction

The concept of stored electric and magnetic energy of radiating systems has interested researchers for several decades [17]. This interest comes mainly from the fact that stored energies are used to evaluate Q-factors, which provide good approximations of fractional bandwidths. Although much progress has been made in understanding stored energies, alternative methods of interpreting these quantities are still useful in providing further physical insight.

Past research on stored energies and Q-factor of radiating systems include many approaches [17]. One classical technique is based on subtracting the far-field from the total energy [1, 13, 15]. Other approaches include taking frequency derivatives of the antenna input impedance [16, 21] and the method of moments (MoM) impedance matrix [5, 9, 19].

Linking electrostatic and magnetostatic energies to material perturbations is well established. Stratton [18, ch. II] notably proposed this nearly a century ago, see also [12, ch. II], [14, ch. VI]. He related a perturbation of the background permittivity and background permeability of a material to the changes in electrostatic and magnetostatic energies, respectively. This showed that, given constant charge, electrostatic energy is inversely proportional to the permittivity. Similarly, given constant current, magnetostatic energy is directly proportional to the permeability. Using material perturbations to interpret stored energies of radiating systems is more complex, emphasized by the fact that there is still no consensus on its definition [17].

The goal of this paper is to relate stored energies and Q-factors of radiating systems to changes in material parameters. To this end, stored energies are expressed in material derivatives by showing equality with energy expressions proposed by Harrington, Mautz, and Vandenbosch [9, 19] based on frequency derivatives. This shows that electric stored energy is linked to a permittivity derivative and magnetic stored energy is linked to a permeability derivative. Once these relationships between stored energies and material derivatives are established, it follows from Cauchyâ€“Riemann equations that there is a link to material losses. This is the first time that dissipated power has been used to express stored energies. Further, to demonstrate the versatility of the approach, material derivatives of antenna input impedance are related to their Q-factor.

Section 2 provides a background on stored energies of radiating systems and then section 3 derives identities between frequency and material derivatives of the Green's function as well as the MoM impedance matrix. These identities are used in section 4 to express stored energies with MoM matrices in terms of material derivatives. Further, section 5 presents antenna Q-factor in terms of material derivatives of input impedance. The paper is concluded in section 6. Finally, the appendices contain complementary information regarding the required mathematical derivations.

2 Background on stored energies and Q-factor

There have been several attempts over the years to compute stored electric and magnetic energies of antennas for time-harmonic fields in a homogeneous lossless background, see [17] for an overview. Currently, there is no consensus on a definition for stored energies of radiating systems [17]. However, useful methods of approximating stored energies of electrically small antenna in free space are available [17].

One formulation isolates stored energy of a radiating system by subtracting the radiated energy density approximated by the far-field amplitude $|\boldsymbol{F}|^2$ from the time-averaged total electromagnetic energy density [1, 5, 21] as

$$W_{\mathrm{F}} = \frac{1}{4}\int_{\mathbb{R}^3} \varepsilon_0 \left|\boldsymbol{E}(\boldsymbol{r})\right|^2 + \mu_0 \left|\boldsymbol{H}(\boldsymbol{r})\right|^2 - 2\varepsilon_0 \frac{|\boldsymbol{F}|^2}{|\boldsymbol{r}|^2}\,\mathrm{d}V, \tag{2.1}$$

where the electric and magnetic energy densities are given by $\varepsilon_0 \left|\boldsymbol{E}\right|^2/4$ and $\mu_0 \left|\boldsymbol{H}\right|^2/4$, respectively, and $\boldsymbol{r}$ denotes the position vector. The vacuum permittivity ε_0 and permeability μ_0 in (2.1) can easily be generalized to a homogeneous background material. A flaw in (2.1) is that it is coordinate dependent and therefore not a proper definition of stored energies [4, 21]. Nevertheless, this is a good definition of stored energies of electrically small antennas [17].

Vandenbosch [19] proposed expressing the stored energy in the current density instead of fields. These expressions are equivalent to (2.1), except for a coordinate-dependent term [4, eq. 5]. In numerical calculations, the current density is expanded in basis functions and the stored energy is conveniently written using matrix notation producing quadratic forms proposed by Harrington and Mautz [9], where stored electric and magnetic energies are expressed as

$$W_{\mathrm{e}} = \frac{1}{8}\mathbf{I}^{\mathrm{H}}\Big(\frac{\partial \mathbf{X}}{\partial \omega} - \frac{\mathbf{X}}{\omega}\Big)\mathbf{I} \quad \text{and } W_{\mathrm{m}} = \frac{1}{8}\mathbf{I}^{\mathrm{H}}\Big(\frac{\partial \mathbf{X}}{\partial \omega} + \frac{\mathbf{X}}{\omega}\Big)\mathbf{I}, \tag{2.2}$$

respectively. Here, $\mathbf{I}$ is a column matrix representing the current density $\boldsymbol{J}(\boldsymbol{r})$, the Hermitian transpose is denoted by superscript ${}^{\mathrm{H}}$, $\mathbf{X}$ denotes the MoM reactance matrix, and the angular frequency is given by ω.

Generally, practical interest into stored energies concerns the inverse relationship between fractional bandwidth and Q-factor that is defined as [21]

$$Q = \frac{2\omega \max\left\{W_{\mathrm{e}}, W_{\mathrm{m}}\right\}}{P_{\mathrm{d}}}, \tag{2.3}$$

where dissipated power is given by P_d. While there is currently only indirect methods to measure Q-factor of an antenna such as the one in [21] that proposes to estimate the Q-factor, which for a self-resonant input impedance simplifies to

$$Q_{\mathrm{Z_{in}}} = \frac{\omega}{2R_\mathrm{in}}\left|\frac{\partial Z_\mathrm{in}}{\partial\omega}\right| = \omega\left|\frac{\partial\varGamma}{\partial\omega}\right|, \tag{2.4}$$

where Z_in is the input impedance with real part given by R_in and $\varGamma$ is the reflection coefficient for a matched load. This approach has been shown to generally agree well with Q-factor determined from frequency derivatives of the MoM reactance matrix (2.2), see [17].

3 Material perturbations

Dyadic Green's functions are used to express the electromagnetic fields in terms of currents [11]. Therefore, this function can be used to relate the electromagnetic fields in expression (2.1) for stored electric and stored magnetic energies to source currents [4]. In this section angular frequency derivatives of the Green's function and MoM impedance matrix are transformed to permittivity and permeability derivatives. These expressions are used in the next section to provide an alternative interpretation of stored electric and magnetic energies derived from frequency derivatives of the MoM impedance matrix shown in (2.2).

To investigate the effect of material perturbations on an antenna structure, all material parameters are expressed in terms of the background material, serving as a reference point as shown in Fig. 1. The permittivity and permeability of the background material are given by ε_b and μ_b, respectively. The background material can be considered as vacuum, but here is allowed to be any nondispersive permittivity and permeability. A heterogeneous non-magnetic material distribution is expressed in terms of the background permittivity as $\varepsilon = \varepsilon_\mathrm{r}\varepsilon_\mathrm{b}$, where ε_r is a relative value allowing all dielectric parameters to be defined in terms of the background permittivity. Further, a PEC structure with surface current density may be confined withing a dielectric region having contrast current density.

The scalar Green's function (fundamental solution to the Helmhotz equation in a homogeneous background) has derivatives with respect to background material parameters $\varepsilon_\mathrm{b}, \mu_\mathrm{b}$ and angular frequency ω that can be equivalently expressed as

$$\begin{aligned}\varepsilon_\mathrm{b}\frac{\partial}{\partial\varepsilon_\mathrm{b}}\frac{\mathrm{e}^{-\mathrm{j}\omega\sqrt{\varepsilon_\mathrm{b}\mu_\mathrm{b}}R}}{4\pi R} &= \mu_\mathrm{b}\frac{\partial}{\partial\mu_\mathrm{b}}\frac{\mathrm{e}^{-\mathrm{j}\omega\sqrt{\varepsilon_\mathrm{b}\mu_\mathrm{b}}R}}{4\pi R}\\ &= \frac{-\mathrm{j}\omega\sqrt{\varepsilon_\mathrm{b}\mu_\mathrm{b}}}{2}\frac{\mathrm{e}^{-\mathrm{j}\omega\sqrt{\varepsilon_\mathrm{b}\mu_\mathrm{b}}R}}{4\pi} = \frac{\omega}{2}\frac{\partial}{\partial\omega}\frac{\mathrm{e}^{-\mathrm{j}\omega\sqrt{\varepsilon_\mathrm{b}\mu_\mathrm{b}}R}}{4\pi R},\end{aligned} \tag{3.1}$$

where R is the distance between source and observer points and the time convention $\mathrm{e}^{\mathrm{j}\omega t}$ is used. This expression shows that frequency and material perturbations of the scalar Green's function are interchangeable. This can be understood in the lossless case by the fact that both permittivity and permeability affect the velocity of the

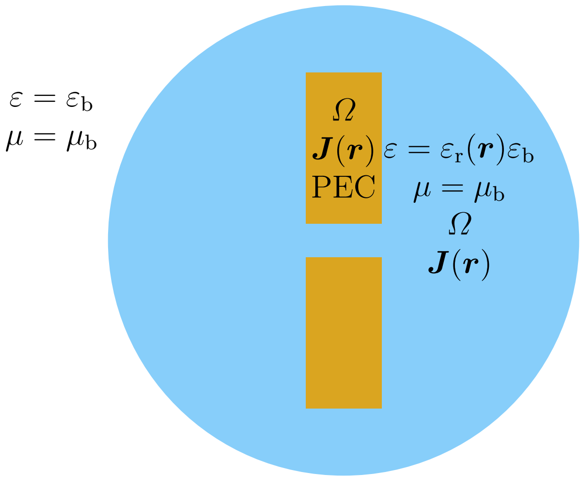

Figure 1: The background material with permittivity ε_b and permeability μ_b encloses an antenna region $\mathit{\Omega}$ with a PEC conducting region and a dielectric region with permittivity $\varepsilon_\mathrm{r}(\boldsymbol{r})\varepsilon_\mathrm{b}$. PEC regions have surface current densities and dielectric regions contrast currents densities both denoted $\boldsymbol{J}(\boldsymbol{r})$.

electromagnetic wave. The resulting decrease in wavelength when the frequency is increased and material kept constant is interchangeable with increasing permittivity or permeability and keeping frequency constant. However, the square root of permittivity and permeability accounts for the factor $1/2$ in (3.1) when taking derivatives with respect to angular frequency of the scalar Green's function.

The dyadic Green's function (based on the scalar Green's function) in a homogeneous background material described by $\varepsilon_\mathrm{b}, \mu_\mathrm{b}$ is [11]

$$\mathbf{G} = \left(\mathbf{1} + \frac{1}{\omega^2\varepsilon_\mathrm{b}\mu_\mathrm{b}}\nabla\nabla\right)\frac{\mathrm{e}^{-\mathrm{j}\omega\sqrt{\varepsilon_\mathrm{b}\mu_\mathrm{b}}R}}{4\pi R}. \tag{3.2}$$

The same relationship (3.1) of material and frequency derivatives as for the scalar Green's function also hold for the dyadic Green's function

$$\varepsilon_\mathrm{b}\frac{\partial\mathbf{G}}{\partial\varepsilon_\mathrm{b}} = \mu_\mathrm{b}\frac{\partial\mathbf{G}}{\partial\mu_\mathrm{b}} = \frac{\omega}{2}\frac{\partial\mathbf{G}}{\partial\omega}. \tag{3.3}$$

To account for different material properties in the antenna region, volumetric MoM [10] is used. The MoM matrix can then be split into two parts

$$\mathbf{Z} = \mathbf{Z}_0 + \mathbf{Z}_\rho, \tag{3.4}$$

where $\mathbf{Z}_0$ is the background term and the non-magnetic material part is given by $\mathbf{Z}_\rho$ (described in Appendix A). The background term is evaluated using the dyadic Green's function (3.2) as

$$Z_{0,mn} = \mathrm{j}\omega\mu_\mathrm{b}\int_{\mathit{\Omega}}\int_{\mathit{\Omega}}\boldsymbol{\psi}_m(\boldsymbol{r}')\cdot\mathbf{G}(\boldsymbol{r}',\boldsymbol{r})\cdot\boldsymbol{\psi}_n(\boldsymbol{r})\,\mathrm{dV}'\,\mathrm{dV}, \tag{3.5}$$

where $\boldsymbol{\psi}_m$ denotes the basis functions [8].

The MoM impedance matrix (3.4) is differentiated with respect to permittivity and permeability using (3.3) for $\mathbf{Z}_0$ and Appendix A for $\mathbf{Z}_\rho$ and then written as

$$\varepsilon_{\mathrm{b}}\frac{\partial \mathbf{Z}}{\partial \varepsilon_{\mathrm{b}}} = \frac{\omega}{2}\frac{\partial \mathbf{Z}}{\partial \omega} - \frac{\mathbf{Z}}{2} \quad \text{and} \quad \mu_{\mathrm{b}}\frac{\partial \mathbf{Z}}{\partial \mu_{\mathrm{b}}} = \frac{\omega}{2}\frac{\partial \mathbf{Z}}{\partial \omega} + \frac{\mathbf{Z}}{2} \tag{3.6}$$

in terms of angular frequency derivatives.

The background material may be lossy and therefore have complex valued permittivity and permeability. Using holomorphic properties of $\mathbf{Z}$ for passive material models leads to Cauchyâ€“Riemann equations [2, 3, 16]

$$\frac{\partial \mathbf{X}}{\partial \operatorname{Re} \varepsilon_{\mathrm{b}}} = -\frac{\partial \mathbf{R}}{\partial \operatorname{Im} \varepsilon_{\mathrm{b}}} \quad \text{and} \quad \frac{\partial \mathbf{X}}{\partial \operatorname{Re} \mu_{\mathrm{b}}} = -\frac{\partial \mathbf{R}}{\partial \operatorname{Im} \mu_{\mathrm{b}}}, \tag{3.7}$$

where only one of the Cauchy-Riemann equations is used. The other containing *e.g.*, $\partial\mathbf{X}/\partial \operatorname{Im} \varepsilon_{\mathrm{b}} = \partial\mathbf{R}/\partial \operatorname{Re} \varepsilon_{\mathrm{b}}$, is less clearly linked to stored energy. The Cauchy-Riemann equation (3.7) demonstrates that the reactance matrix ($\mathbf{X}$), when differentiated with respect to real material parameters, can be expressed in terms of the resistance matrix ($\mathbf{R}$) differentiated with respect to material losses. These matrices are defined in terms of the MoM impedance matrix as

$$\mathbf{X} = \frac{\mathbf{Z} - \mathbf{Z}^{\mathrm{H}}}{2\mathrm{j}} \quad \text{and} \ \mathbf{R} = \frac{\mathbf{Z} + \mathbf{Z}^{\mathrm{H}}}{2}, \tag{3.8}$$

where the reactance matrix ($\mathbf{X}$), also used in (2.2), is the imaginary part of the MoM impedance matrix and relates currents to the difference between magnetic and electric energies. The resistance matrix ($\mathbf{R}$) relates currents to radiated power and dissipated power in the material.

4 Stored energies

In this section, it is shown that stored energies derived from frequency derivatives of the MoM impedance matrix (2.2) can equivalently be expressed in terms of material derivatives of the MoM impedance matrix. For these expressions the limit of no background material losses are assumed. This new relationship of stored electric energy is summarized in the diagram shown in Fig. 2. This relationship identifies that the effect on the MoM impedance matrix of differentiating with respect to permittivity (3.6) is, besides a scaling factor, identical to the matrix multiplied with the current vector in (2.2), where stored electric energy is computed from frequency derivatives of the MoM reactance matrix. This leads to an expression for stored electric energy as a derivative of the reactance matrix with respect to the background permittivity. Similarly, the relationship between stored magnetic energy (2.2) and a derivative with respect to the background permeability can be obtained through the MoM impedance matrix differentiated with respect to the background permeability (3.6) and Cauchy-Riemann equation (3.7), in the limit of no background material losses, *i.e.*, $\operatorname{Im} \varepsilon_{\mathrm{b}} \to 0$ and $\operatorname{Im} \mu_{\mathrm{b}} \to 0$, the derivatives can be applied to the reactance matrix (3.8) as

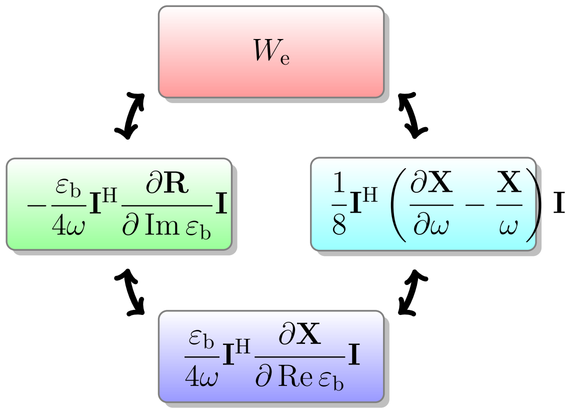

Figure 2: Stored electric energy in the limit of no background material losses expressed in terms of derivatives, with respect to frequency, permittivity and dielectric losses.

$$W_{\mathrm{e}} = \frac{\varepsilon_{\mathrm{b}}}{4\omega}\mathbf{I}^{\mathrm{H}}\frac{\partial \mathbf{X}}{\partial \operatorname{Re}\varepsilon_{\mathrm{b}}}\mathbf{I} \quad \text{and } W_{\mathrm{m}} = \frac{\mu_{\mathrm{b}}}{4\omega}\mathbf{I}^{\mathrm{H}}\frac{\partial \mathbf{X}}{\partial \operatorname{Re}\mu_{\mathrm{b}}}\mathbf{I}. \tag{4.1}$$

The expressions in (4.1) show that the MoM reactance matrix's derivatives with respect to the electric material parameter namely permittivity and the magnetic material parameter namely permeability leads to equivalent expressions for stored energies as ones based on taking frequency derivatives (2.2).

Additionally these expressions show for the first time that electric stored energy depends on the reactance sensitivity to electric material changes. Similarly, the magnetic stored energy depends on the reactance sensitivity to magnetic material changes. This is related to the accepted relation between electric energy and permittivity in electrostatics and magnetic energy and permeability in magnetostatics [12, 18]. However, in the case where there is a radiated field, stored electric energy and stored magnetic energy (4.1) are both functions of permittivity and permeability leading to a complex relationship between these parameters.

The expressions for stored energies from material derivatives (4.1) can further be reformulated using the Cauchy-Riemann equation (3.7), in the limit of no background material losses. These equations express stored electric and magnetic energies (4.1) as

$$W_{\mathrm{e}} = \frac{-\varepsilon_{\mathrm{b}}}{4\omega}\mathbf{I}^{\mathrm{H}}\frac{\partial \mathbf{R}}{\partial \operatorname{Im}\varepsilon_{\mathrm{b}}}\mathbf{I} \quad \text{and } W_{\mathrm{m}} = \frac{-\mu_{\mathrm{b}}}{4\omega}\mathbf{I}^{\mathrm{H}}\frac{\partial \mathbf{R}}{\partial \operatorname{Im}\mu_{\mathrm{b}}}\mathbf{I}, \tag{4.2}$$

respectively. Equality between expressions in (4.2) and (4.1) lead to an interpretation of stored energies in terms of sensitivity (derivative) to background material losses of the resistance matrix. The negative sign in these expressions is due to the choice of time convention. Here, it should be noted that material derivative expressions can be written equal to (2.2), not only in the lossless limit using complex derivatives (3.6) [20] leading to *e.g.*, $W_{\mathrm{e}} = \operatorname{Im}\frac{\varepsilon_{\mathrm{b}}}{4\omega}\mathbf{I}^{\mathrm{H}}\frac{\partial \mathbf{Z}}{\partial \varepsilon_{\mathrm{b}}}\mathbf{I} = \operatorname{Im}\frac{\varepsilon_{\mathrm{b}}}{2\omega}\mathbf{I}^{\mathrm{H}}\frac{\partial \mathbf{X}}{\partial \varepsilon_{\mathrm{b}}}\mathbf{I} = \operatorname{Im}\frac{\varepsilon_{\mathrm{b}}}{2\omega}\mathbf{I}^{\mathrm{H}}\frac{\partial \mathbf{R}}{\partial \varepsilon_{\mathrm{b}}}\mathbf{I}$. Similar expressions for stored magnetic energy are obtained by replacing background permittivity with permeability. However, for lossy background material these expressions are difficult to interpret and may require another approach [7].

The resistance matrix (3.8) used to formulate stored energies in (4.2) is related to dissipated power of an antenna. This power can be split into power leaving a volume (*e.g.*, radiation) and power lost within the volume (*e.g.*, material losses). Stored energies are generally thought to only be related to changes in antenna reactance. However, in (4.2) it is shown that they also relate to changes in dissipated power. This is demonstrated by using Cauchy-Riemann equations (3.7) to express material derivatives of resistance and reactance in terms of one another.

When the derivatives with respect to material loss in (4.2) produce negative definite matrices as is the case for electrically small structures they are a reliable approximation of stored energies. For this case the dissipated power is increased by these material loss perturbations. However, it should be noted that for electrically large structures the expressions (4.2) can produce negative values [6]. One interpretation of this is that the reduction in power leaving a volume may exceed the increase in power lost within the volume when perturbing material losses.

5 Q-factor from input impedance

The material derivative identities of the MoM matrix (3.6) can be used to reinterpret the Q-factor from input impedance (2.4). By doing this, the versatility of material derivative approach is demonstrated.

The derivatives of the MoM matrix (3.6) are related to material derivatives of the input impedance Z_{in} shown in Appendix B leading to

$$\varepsilon_{\mathrm{b}}\frac{\partial Z_{\mathrm{in}}}{\partial \varepsilon_{\mathrm{b}}} = \frac{\omega}{2}\frac{\partial Z_{\mathrm{in}}}{\partial \omega} - \frac{Z_{\mathrm{in}}}{2} \quad \text{and} \quad \mu_{\mathrm{b}}\frac{\partial Z_{\mathrm{in}}}{\partial \mu_{\mathrm{b}}} = \frac{\omega}{2}\frac{\partial Z_{\mathrm{in}}}{\partial \omega} + \frac{Z_{\mathrm{in}}}{2}. \tag{5.1}$$

Using these expressions valid for a lossy background material, Q-factor formulated with frequency derivatives of input impedance (2.4) can equivalently be expressed using (5.1) as

$$Q_{\mathrm{Z}'_{\mathrm{in}}} = \frac{1}{2R_{\mathrm{in}}}\left|\varepsilon_{\mathrm{b}}\frac{\partial Z_{\mathrm{in}}}{\partial \varepsilon_{\mathrm{b}}} + \mu_{\mathrm{b}}\frac{\partial Z_{\mathrm{in}}}{\partial \mu_{\mathrm{b}}}\right|, \tag{5.2}$$

leading to an interpretation of Q-factor in terms of material sensitivity of the input impedance. It should be noted that in (5.2) self resonance is assumed.

6 Conclusion

In this paper a relationship between stored energies of radiating systems and material derivatives is investigated. From this investigation it is shown that widely used expressions to evaluate stored energies can be interpreted in terms of material derivatives. This leads to the fundamental relationship between stored electric energy and electric material properties namely permittivity. Similarly, stored magnetic energy is related to permeability. Further, the MoM resistance matrix is used to interpret stored energies for the first time. This demonstrates a new link between material losses and stored energies of radiating systems. Lastly, it is also shown that frequency derivatives used to compute Q-factor from input impedance can be replaced by material derivatives.

Appendix A Volumetric MoM material derivatives

The MoM material part (3.4) assuming only non-magnetic properties is given by

$$Z_{\rho,mn} = \int_{\Omega} \boldsymbol{\psi}_m(\boldsymbol{r}) \cdot \rho(\boldsymbol{r}) \cdot \boldsymbol{\psi}_n(\boldsymbol{r}) \,\mathrm{dV}, \tag{A.1}$$

where the complex resistivity is $\rho = -\mathrm{j}/(\omega\varepsilon_\mathrm{b}\chi_\mathrm{e})$ and where χ_e is the electric susceptibility when the background permittivity is equal to vacuum. The derivative with respect to background permittivity of $\mathbf{Z}_\rho$ can be expressed as

$$\varepsilon_\mathrm{b}\frac{\partial \mathbf{Z}_\rho}{\partial \varepsilon_\mathrm{b}} = \frac{\omega}{2}\frac{\partial \mathbf{Z}_\rho}{\partial \omega} - \frac{\mathbf{Z}_\rho}{2}, \tag{A.2}$$

since $\partial\mathbf{Z}_\rho/\partial\omega = -\mathbf{Z}_\rho/\omega$. Further, due to the assumption of non-magnetic media, the derivative with respect to background permeability is

$$\mu_\mathrm{b}\frac{\partial \mathbf{Z}_\rho}{\partial \mu_\mathrm{b}} = \frac{\omega}{2}\frac{\partial \mathbf{Z}_\rho}{\partial \omega} + \frac{\mathbf{Z}_\rho}{2} = \mathbf{0}. \tag{A.3}$$

Appendix B MoM matrix related to input impedance

From the MoM impedance matrix (3.4), the admittance matrix is $\mathbf{Y} = \mathbf{Z}^{-1}$ and is related to the input admittance as

$$Y_\mathrm{in} = \frac{1}{Z_\mathrm{in}} = \frac{\mathbf{V}^\mathrm{T}\mathbf{Y}\mathbf{V}}{V_\mathrm{in}^2}, \tag{B.1}$$

where Z_in is the input impedance. Assuming a frequency and material independent input voltage, V_in, frequency differentiation of the input admittance lead to

$$V_\mathrm{in}^2\frac{\partial Y_\mathrm{in}}{\partial \omega} = \mathbf{V}^\mathrm{T}\frac{\partial \mathbf{Y}}{\partial \omega}\mathbf{V} = -\mathbf{V}^\mathrm{T}\mathbf{Z}^{-1}\frac{\partial \mathbf{Z}}{\partial \omega}\mathbf{Z}^{-1}\mathbf{V}. \tag{B.2}$$

Similarly the background permittivity derivative of the admittance matrix can be written as

$$\frac{\partial \mathbf{Y}}{\partial \varepsilon_\mathrm{b}} = -\mathbf{Z}^{-1}\frac{\partial \mathbf{Z}}{\partial \varepsilon_\mathrm{b}}\mathbf{Z}^{-1}. \tag{B.3}$$

Then, using the relation between derivative of background permittivity and frequency of the MoM matrix (3.6) (inverse of admittance matrix) this can be equivalently expressed in terms of a derivative with respect to the background permittivity as

$$\varepsilon_\mathrm{b}\frac{\partial \mathbf{Y}}{\partial \varepsilon_\mathrm{b}} = \frac{\omega}{2}\frac{\partial \mathbf{Y}}{\partial \omega} + \frac{\mathbf{Y}}{2}. \tag{B.4}$$

It follows from the relationship between frequency derivatives of the input admittance and admittance matrix (B.2) that

$$\varepsilon_{\rm b}\frac{\partial Y_{\rm in}}{\partial \varepsilon_{\rm b}} = \frac{\omega}{2}\frac{\partial Y_{\rm in}}{\partial \omega} + \frac{Y_{\rm in}}{2}, \tag{B.5}$$

this leads to

$$\varepsilon_{\rm b}\frac{-\frac{\partial Z_{\rm in}}{\partial \varepsilon_{\rm b}}}{Z_{\rm in}^2} = \frac{\omega}{2}\frac{-\frac{\partial Z_{\rm in}}{\partial \omega}}{Z_{\rm in}^2} + \frac{1}{2Z_{\rm in}} \tag{B.6}$$

and can be rewritten as the left-hand side of (5.1). Similarly, differentiation with respect to background permeability can be shown to lead to the right-hand side of (5.1).

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