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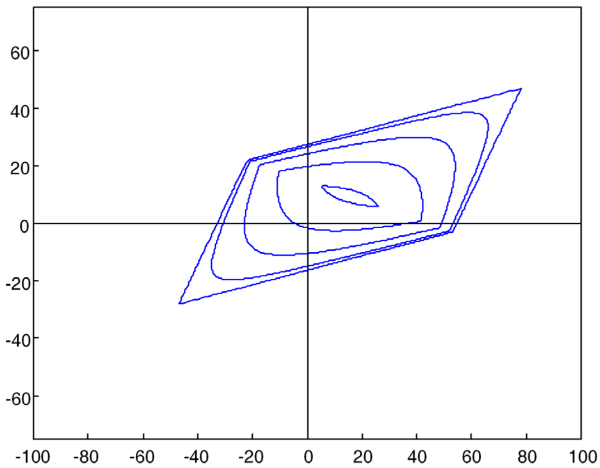

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Introduction

A failure criterion for fibre network materials is presented. Well-known criteria such as the von Mises criterion and the Tsai-Wu criterion are phenomenological. The criterion discussed here has a rational physical base in the sense that the strength properties of the material are related to its microstructure and to the strength of the individual fibres, thus not necessarily determined by strength testing of the material.

The most important fibre network material from a commercial point of view is probably paper. Need for a failure or yield criterion for this material is evident in relation to strength design of packages taking into account non-trivial states of stress, i.e. states of stress other than uni-axial tension or compression in the machine direction (MD) or the cross machine direction (CD) of the paper. Load carrying paper packages include packages made of kraft paper, paper board, multi-material laminations and corrugated paper board. The 2D strength or yield of paper has in previous analyses been defined by criteria such as the Tsai-Wu criterion, a modified Tsai-Wu criterion and a criterion presented by Karafillis, Boyce and Parks, [1,2,3].

The present criterion is here discussed only for 2D states of stress. Application to materials with a 3D fibre orientation distribution will require an analogous 3D analysis. Any experimental verification or calibration is not included in this short report. A numerical example of application relates to the prediction of the strength at pure shear from given strength properties of the material at uniaxial loading. A result of that application was used in an analysis of corrugated board [4]. The criterion is here discussed and treated only as a failure criterion, but it can be extended to criteria for gradual damage and plastic yielding, depending on the assumptions made regarding release of global failure of the material structure and regarding properties of the individual fibre.

Basic assumptions

At the micro-level of the material, i.e. at the fibre level, the failure criterion is

$$\begin{cases} \sigma_f = \sigma_{ft} \\ \sigma_f = -\sigma_{fc} \end{cases} \tag{1}$$

where σ_f is the normal stress in axial direction the fibre, and σ_{ft} and σ_{fc} its corresponding tensile and compressive strength, respectively.

The fibre stress σ_f is linked to the state of stress in the material, σ_{ij}, by using the same assumption as adopted in the Cox's model [5] for calculation of the stiffness properties of a fibre network. Failure of the network is then assumed to coincide with fibre failure as determined by (1).

The fibres are in Cox's model assumed to be linear elastic, straight, uniform, thin, long and with zero bending stiffness. Each fibre is assumed to remain straight during straining of the network material. For the individual fibre, stress and strain are considered only in the axial direction of the fibre.

Notations for fibre and network properties

The cross section area of a fibre is denoted A_f, its modulus of elasticity E_f, its axial stress σ_f and its axial strain ε_f. The network density is denoted ρ, in 2D analysis defined as total fibre length per area. The 2D orientation of a fibre is indicated by θ, θ being the angle from a global x-axis to the fibre and limited by $0 \leq \theta \leq \pi$. The x-axis may for paper be chosen as the MD and the y-axis as the CD.

The fibre orientation distribution is indicated by a density function $f(\theta)$, defined by

$$f(\theta) = \lim_{\Delta\theta \to 0} \frac{[N_f(\Delta\theta + \theta) - N_f(\theta)] / N_{tot}}{\Delta\theta} \quad , \tag{2}$$

where $N_f(\theta)$ is the number of fibres orientated in between 0 and θ. $N_{tot} = N_f(\pi)$ is the total number of fibres in the area under observation. At uniform isotropic distribution, $f(\theta)$ is constant,

$$f(\theta) = 1/\pi \; . \tag{3}$$

An often assumed orthotropic distribution is

$$f(\theta) = 1/\pi + a\cos(2\theta) \, , \tag{4}$$

where a is a parameter. A simple physical interpretation of this distribution is found by the parameter substitution $a=(p-1)/(\pi(p+1))$ by which eq (4) can be written as

$$f(\theta) = \frac{2(1+(p-1)\cos^2\theta)}{\pi(p+1)} \tag{5}$$

[6], where the parameter p has the physical interpretation

$$p = f(0) / f(\pi/2), \tag{6}$$

which for paper is the ratio between the number of fibres oriented in MD and CD. At uniform distribution is $p=1.0$ and $a=0$.

Stiffness matrix of network

The 2D stiffness properties of a network are calculated at the assumptions of Cox's model. At engineering network strain $\boldsymbol{\varepsilon} = [\varepsilon_x, \varepsilon_y, \gamma_{xy}]^T$, the axial strain in a fibre, ε_f, with orientation θ is

$$\varepsilon_f = [\cos^2\theta \quad \sin^2\theta \quad \sin\theta\cos\theta]\,\boldsymbol{\varepsilon}, \tag{7}$$

which is the network strain in the direction θ, [7]. At this strain the fibre force is

$$P_f = A_f\, E_f\, \varepsilon_f\,. \tag{8}$$

The fibre force P_f may at a cross section through the material be divided into one horizontal and one vertical component,

$$\begin{cases} P_{fx} = P_f\,|\cos\theta| \\ P_{fy} = P_f\,\sin\theta \end{cases}, \tag{9a,b}$$

where $0 \le \theta \le \pi$. Positive force indicates tension in the fibre.

Determination of the number of fibres that intersects a vertical or horizontal cross section requires some analysis. The network density of fibres with orientations from θ to $\theta+d\theta$ is $\rho f(\theta)d\theta$. The total length of the fibres with this orientation and located in a narrow vertical strip-shaped area Ldx is $Ldx\rho f(\theta)d\theta$. The length of the individual fibre segment in this area is $dx/|\cos\theta|$. Dividing of the total length by the individual length gives the number of fibre segments: $L\rho f(\theta)|\cos\theta|d\theta$. Hence, the number of fibres with orientation θ that intersects a vertical section of a length dy is

$$\rho f(\theta)\,|\cos\theta|\,d\theta\, dy \tag{10}$$

and for a horizontal section of length dx the number is

$$\rho f(\theta) \sin\theta \, d\theta dx \, . \tag{11}$$

The network stresses σ_x, σ_y and τ_{xy} can now be calculated by summation of the fibre force for all fibres, i.e. for all θ, and then divide by the size of the relevant cross section area, tdx or tdy, t being the thickness of the fibre network material:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \int_0^\pi P_{fx} \rho f(\theta) \left|\cos\theta\right| d\theta dy / tdy \\ \int_0^\pi P_{fy} \rho f(\theta) \sin\theta \, d\theta dx / tdx \\ \int_0^\pi P_{fx} \rho f(\theta) \sin\theta \, d\theta dx / tdx \end{bmatrix} . \tag{12}$$

The shear stress τ_{xy} can alternatively and with the same result be calculated by integration of P_{fy}. Equation (12) can by the notation $\boldsymbol{\sigma}=[\sigma_x, \sigma_y, \tau_{xy}]^T$ be written as

$$\boldsymbol{\sigma} = \mathbf{D}\, \boldsymbol{\varepsilon} . \tag{13}$$

The stiffness matrix **D** can be separated into a constant and a matrix $\mathbf{D}_\theta$, which depends only of the orientation distribution $f(\theta)$. Hence

$$\mathbf{D} = \frac{A_f \, E_f \, \rho}{t} \mathbf{D}_\theta , \tag{14}$$

where

$$\mathbf{D}_\theta = \int_0^\pi f(\theta) \begin{bmatrix} c^4 & s^2c^2 & sc^3 \\ s^2c^2 & s^4 & s^3c \\ sc^3 & s^3c & s^2c^2 \end{bmatrix} d\theta , \tag{15}$$

$c=\cos\theta$ and $s=\sin\theta$. Equation (15) shows that the shear stiffness D_{33} is always equal to the *x-y* coupling terms D_{12} and D_{21}, no matter the fibre orientation distribution. For uniform distribution, i.e. for $f(\theta)=1/\pi$, is found

$$\mathbf{D}_\theta = \begin{bmatrix} 3/8 & 1/8 & 0 \\ 1/8 & 3/8 & 0 \\ 0 & 0 & 1/8 \end{bmatrix} . \tag{16}$$

Also for the orthotropic distributions $f(\theta)=1/\pi+a\cos(2\theta)$ is by integration found an uncomplicated result,

$$\mathbf{D}_\theta = \begin{bmatrix} 3/8+a\pi/4 & 1/8 & 0 \\ 1/8 & 3/8-a\pi/4 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}, \tag{17}$$

with no effect of the orthotropic fibre distribution on the shear stiffness.

Following the analysis of Cox, the stiffness matrix for a general anisotropic orientation distribution,

$$f(\theta) = \frac{1}{\pi}\left(1 + \sum_{i=1}^{\infty}\left(a_i \cos(2i\theta) + b_i \sin(2i\theta)\right)\right), \tag{18}$$

is given in [8] as

$$\mathbf{D}_\theta = \begin{bmatrix} 3/8+a_1/4+a_2/16 & 1/8-a_2/16 & b_1/8+b_2/16 \\ 1/8-a_2/16 & 3/8-a_1/4+a_2/16 & b_1/8-b_2/16 \\ b_1/8+b_2/16 & b_1/8-b_2/16 & 1/8-a_2/16 \end{bmatrix}. \tag{19}$$

It is noteworthy that the higher order orientation coefficients $a_i, i\geq 3$, and $b_i, i\geq 3$, do not effect the stiffness properties.

Failure criterion as a failure surface

A stress based failure criterion may generally be expressed as

$$F(\boldsymbol{\sigma},\boldsymbol{\alpha}) = 0, \tag{20}$$

where $\boldsymbol{\sigma}$ defines the stress and $\boldsymbol{\alpha}$ is set of functions or parameters that define the properties of the material. In 2D plane stress analysis $\boldsymbol{\sigma}=[\sigma_x, \sigma_y, \tau_{xy}]^T$ and for the present model $\boldsymbol{\alpha}=[f(\theta,\varphi), \rho A_f/t, \sigma_{ft}, \sigma_{fc}]^T$. At 2D fibre orientation distribution, dealt with in this report, $f(\theta,\varphi)=f(\theta)$. To find the yield surface, eq (20), $\mathbf{D}$ is first determined from $f(\theta)$ by eq. (14) and (15), then eq (13) is solved for the strains, $\boldsymbol{\varepsilon}=[\varepsilon_x,\varepsilon_y,\gamma_{xy}]^T$, and the principal strains calculated. The principal strains can be calculated as the eigenvalues of the strain tensor. The in-plane principal strains are denoted as ε_1 and ε_2, with $\varepsilon_1 \geq \varepsilon_2$, and for the 2D analysis the strain tensor is

$$\begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 \\ \gamma_{xy}/2 & \varepsilon_y \end{bmatrix}. \tag{21}$$

The principal strains in the material equal the strain in the most strained fibres. Since the stress in the fibres is the strain times E_f, the failure surface is finally obtained by eq. (1), giving

$$\begin{cases} \varepsilon_i E_f - \sigma_{ft} = 0 & if\ \varepsilon_i \geq 0,\ i = 1, 2 \\ \varepsilon_i E_f + \sigma_{fc} = 0 & if\ \varepsilon_i \leq 0,\ i = 1, 2 \end{cases}. \tag{22}$$

Following the above it is possible to derive explicit analytical expressions for the failure surface. The derivation and the result may however be more or less complicated and comprehensive depending on $f(\theta)$. In a general 3D case it may be more convenient to calculate the failure surface numerically than to derive an explicit expression.

The principal strains ε_1 and ε_2 can for uniform 2D fibre orientation distribution, i.e. with $f(\theta)=1/\pi$, be obtained from eq. (16), (14), (13) and (21), giving

$$\varepsilon_i E_f = [t/(A_f \rho)][\sigma_x + \sigma_y \pm 2\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}\ ] \quad i = 1, 2. \tag{23}$$

For the orthotropic orientation distribution $f(\theta)=1/\pi+a\cos(2\theta)$ the principal strains can be obtained from eq. (17), (14), (13) and (21), giving

$$\begin{aligned} &\varepsilon_i E_f = [t/(A_f \rho)][\sigma_x + \sigma_y - a\pi(\sigma_x - \sigma_y) \pm \\ &\pm 2\sqrt{(\sigma_x - \sigma_y)^2 - a\pi(\sigma_x^2 - \sigma_y^2) + (a^2\pi^2/4)(\sigma_x + \sigma_y)^2 + (2 - a^2\pi^2)^2\tau^2}\ ] \\ &/[1 - a^2\pi^2/2] \qquad i = 1, 2. \end{aligned}$$

.........(24)

Failure criterion as an ultimate load multiplier

The stress at failure, $\boldsymbol{\sigma}$, is written as a load multiplier λ, $\lambda \geq 0$, times a stress vector $\boldsymbol{\sigma}_0$. The vector $\boldsymbol{\sigma}_0$ defines sign and ratio $\sigma_x:\sigma_y:\tau_{xy}$ of the stress condition to be studied:

$$\boldsymbol{\sigma} = \lambda \boldsymbol{\sigma}_0. \tag{25}$$

The strain vector $\boldsymbol{\varepsilon}_0$ at stress $\boldsymbol{\sigma}_0$ is determined for a material with the $\mathbf{D}_\theta$ of eq. (15) by solving the equation

$$\mathbf{D}_\theta\,\boldsymbol{\varepsilon}_0 = \boldsymbol{\sigma}_0\,. \tag{26}$$

The principal strains of the strain state $\boldsymbol{\varepsilon}_0$ are denoted ε_{01} and ε_{02}. Replacing stiffness $\mathbf{D}_\theta$ with $\mathbf{D}$ would give principal strains that are $t/(A_f E_f\,\rho)$ times larger. Comparison of these strains with the fibre limit strains σ_{ft}/E_f and σ_{fc}/E_f gives one or two values of λ:

$$\begin{cases} \lambda_1 = \sigma_{ft} A_f \rho / (t\varepsilon_{01}) & if \;\; \varepsilon_{01} > 0 \\ \lambda_2 = -\sigma_{fc} A_f \rho / (t\varepsilon_{02}) & if \;\; \varepsilon_{02} < 0 \end{cases} \tag{27a,b}$$

There are two values of λ if $\varepsilon_{01}>0$ and $\varepsilon_{02}<0$. The decisive λ is then determined by

$$\lambda = Min(\lambda_1, \lambda_2) \geq 0\,. \tag{28}$$

As an example, for uniform fibre orientation distribution, i.e. with $f(\theta)=1/\pi$, is found

$$\begin{cases} \lambda_1 = [\sigma_{ft} A_f \rho / t] \;/\; [\sigma_{0x} + \sigma_{0y} + 2\sqrt{(\sigma_{0x} - \sigma_{0y})^2 + 4\tau_0^2}\;] \\ \lambda_2 = [-\sigma_{fc} A_f \rho / t] \;/\; [\sigma_{0x} + \sigma_{0y} - 2\sqrt{(\sigma_{0x} - \sigma_{0y})^2 + 4\tau_0^2}\;] \end{cases} \tag{29a,b}$$

The corresponding result for the orthotropic distribution $f(\theta)=1/\pi+acos(2\theta)$ is obvious from the analogies between eq. (23) and (24)

$$\begin{cases} \lambda_1 = [\sigma_{ft} A_f \rho / t]\;\; [1 - a^2\pi^2/2] \,/\; [\sigma_{0x} + \sigma_{0y} - a\pi(\sigma_{0x} - \sigma_{0y}) + \\ + 2\sqrt{(\sigma_{0x} - \sigma_{0y})^2 - a\pi(\sigma_{0x}^2 - \sigma_{0y}^2) + (a^2\pi^2/4)(\sigma_{0x} + \sigma_{0y})^2 + (2 - a^2\pi^2)^2 \tau_0^2}\;\;] \\ \lambda_2 = [-\sigma_{fc} A_f \rho / t]\;\; [1 - a^2\pi^2/2] \,/\; [\sigma_{0x} + \sigma_{0y} - a\pi(\sigma_{0x} - \sigma_{0y}) + \\ - 2\sqrt{(\sigma_{0x} - \sigma_{0y})^2 - a\pi(\sigma_{0x}^2 - \sigma_{0y}^2) + (a^2\pi^2/4)(\sigma_{0x} + \sigma_{0y})^2 + (2 - a^2\pi^2)^2 \tau_0^2}\;\;] \end{cases}$$

........(30a,b)

Strength at uniaxial stress and at pure shear

The failure stresses at uniaxial tensile or compressive loading in the *x*- and *y*-directions are considered as well as the failure stress at pure shear stress. All these failure stresses can for the fibre orientation distribution $f(\theta)=1/\pi+a\cos(2\theta)$ be calculated by means of eq (30a,b) and (28). According to these equations failure may be due to fibre tension or fibre compression. As an example, failure at uniaxial tensile loading in the *x*-direction may be due to tensile failure of fibres oriented in the *x*-direction or due to compressive failure of fibres oriented in the *y*-direction. Whether one or the other failure mode is decisive depends on the parameter *a* and the ratio σ_{fc}/σ_{ft}.

In the below presentation of failure stresses, constants K_1 and K_2 defined by

$$\begin{cases} K_1 = [\sigma_{ft} A_f \rho / t] \; [1 - a^2 \pi^2 / 2] \\ K_2 = [-\sigma_{fc} A_f \rho / t] \; [1 - a^2 \pi^2 / 2] \end{cases} \tag{31a,b}$$

are used in order to save space. The failure stress obtained for tension in the *x*-direction is

$$\sigma_x = Min \begin{Bmatrix} K_1 / (3 - 2a\pi) \\ K_2 / (-1) \end{Bmatrix} \geq 0 \tag{32}$$

and for tension in the *y*-direction the failure stress is

$$\sigma_y = Min \begin{Bmatrix} K_1 / (3 + 2a\pi) \\ K_2 / (-1) \end{Bmatrix} \geq 0 \,. \tag{33}$$

For compressive loading in the *x*-direction the failure stress is

$$\sigma_x = Max \begin{Bmatrix} K_1 / (-1) \\ K_2 / (3 - 2a\pi) \end{Bmatrix} \leq 0 \tag{34}$$

and for compression in the *y*-direction is the failure stress

$$\sigma_y = Max \begin{Bmatrix} K_1 / (-1) \\ K_2 / (3 + 2a\pi) \end{Bmatrix} \leq 0 \,. \tag{35}$$

At pure shear is the magnitude of stress at failure obtained by eq (30a,b) independent of the sign of the stress,

$$|\tau| = Min\left\{ \begin{array}{l} K_1/(4-2a^2\pi^2) \\ K_2/(-4+2a^2\pi^2) \end{array} \right\} \geq 0 . \tag{36}$$

This sign independence is due to the symmetry of the fibre orientation distribution.

Parameter data used in numerical illustrations

The material property data used in the below numerical examples are intended to roughly reflect properties that can be typical for a paper. In Table 1 that data is shown. In the numerical examples is the fibre orientation distribution $f(\theta)=1/\pi+a\cos(2\theta)$ used throughout. Figure 1 shows this distribution for some various a-values. Although a figure for fibre modulus of elasticity is included in Table 1, it is not needed for the calculation of failure stress.

Three parameters are sufficient input data for calculation of a plane stress failure envelope. These parameter may be chosen as $(\sigma_{ft}+\sigma_{fc})A_f\rho/t$, σ_{fc}/σ_{ft} and a. The first parameter has the dimension stress and determines the size of the envelop. The second parameter affects the location of the envelope and the third parameter the shape of the envelope. An alternative to the first parameter is $\sigma_{ft}A_f\rho/t$. In Table 2 is shown the parameter values corresponding to the material data in Table 1. To study the effect of the parameters, also other values are used in below calculations.

Table 1. Material property data.

Quantity	Notation	Value
Fibre cross section area	A_f	$2.5\ 10^{-10}$ m^2
Fibre modulus of elasticity	E_f	40000 MPa
Fibre tensile strength	σ_{ft}	250 MPa
Fibre compressive strength	σ_{fc}	150 MPa
Fibre orientation distribution parameter	a	$0.5/\pi$
Network density	ρ	200 mm/mm^2
Paper thickness	t	0.1 mm

Table 2. Input parameters for calculation of plane stress failure envelope.

Parameter	Notation	Value
Failure stress level parameter	$(\sigma_{ft}+\sigma_{fc})A_f\rho/t$	200 MPa
Fibre compression to tension strength ratio	σ_{fc}/σ_{ft}	0.6
Fibre orientation distribution parameter	a	$0.5/\pi$

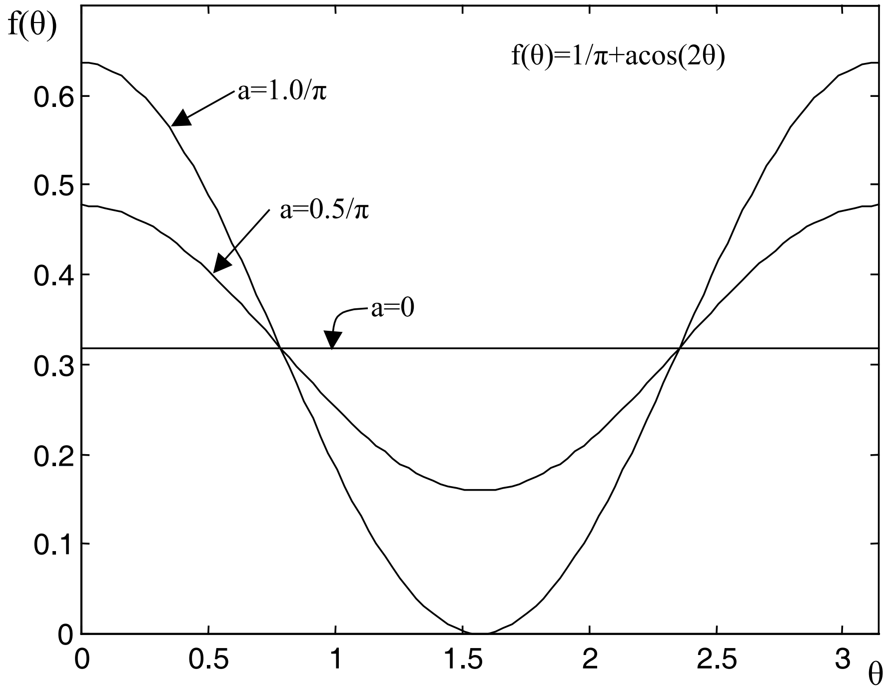

Figure 1. Fibre orientation distribution densities.

Numerical results for strength at uniaxial stress and at pure shear

Table 3 shows the uniaxial and pure shear load strength data obtained by eq (31)-(36) for the material data indicated in Table 2. The strength values obtained seem to reasonably well agree with data that could be expected for a paper.

Table 3. Results obtained for uniaxial normal stress and pure shear.

Loading condition	Material strength
Tension in *x*-direction	27.3 MPa
Compression in *x*-direction	16.4 MPa
Tension in *y*-direction	13.7 MPa
Compression in *y*-direction	8.2 MPa
Shear	9.4 MPa

Numerical results for failure envelope at zero shear stress

Stress states characterised by $\boldsymbol{\sigma}_0=(\sigma_{0x},\sigma_{0y},\tau_0)=(\cos\beta,\sin\beta,0)$ are studied for $0\leq\beta\leq2\pi$. The stress magnitude at failure is calculated by eq (28) and (30) for a large number of β-values, making it possible to draw the failure envelope in a σ_x-σ_y diagram. To illustrate the performance of the failure criterion and how the input parameters effect the envelope, several sets of input data were used.

The results obtained are shown in Figure 2. Figure 2a is valid for isotropic materials and shows the effect of fibre compressive strength. Figure 2b shows the effect of the failure stress level parameter ratio and Figure 2c shows the effect of ratio σ_{fc}/σ_{ft}. Figure d and e shows the effect of the orthotropy parameter a at $\sigma_{fc}/\sigma_{ft}=1.0$ and at $\sigma_{fc}/\sigma_{ft}=0.6$, respectively. Figure e shows that the shape of the failure envelope obtained with the data of Table 2 seems to agree fairly well with a shape that is typical for the failure envelope of paper.

Numerical results for failure envelop at non-zero shear stress

The failure envelope for general in-plane stress states can be found by assigning a constant shear stress, τ_0/λ, in (30a,b) and plot the failure stresses, σ_x and σ_y, for the corresponding shear levels. This was made by a numerical technique for function zero finding, as provided by the computer code Matlab.

In Figure 3a, the stress envelope is plotted for the isotropic case, i.e. $a=0$, and equal tensile and compressive strengths, i.e. $\sigma_{fc}/\sigma_{fc}=1.0$. The rombic shape is seen to disappear as the shear stress increases, producing a convexity around zero values of σ_x and σ_y. The maximum shear strength is found at $\sigma_x=\sigma_y=0$ and its value is 31.25 MPa for $\sigma_{ft}A_f\rho/t$ = 125 MPa.

In Figure 3b, the same orthoptropy as for the midle curve in Figure 2b is used, i.e. a=0.5/π, and the ratio of fibre compressive to tensile strength, σ_{fc}/σ_{fc}, is 0.6. In this case is for $\sigma_{ft}A_f\rho/t$ = 125 MPa maximum shear strength found at $[\sigma_x,\sigma_y]$= [15,10] MPa and its value is about 25 MPa. It can be seen that the shear strength at $[\sigma_x,\sigma_y]$= [0,0] is close to 18 MPa, which agrees with (36), giving the pure shear stress strength 18.75 MPa.

Interesting to note in Figure 3 is also that the level curves scale down non-linearly with increasing shear stress, from the tensile-compressive quadrants, whilst from the tensile-tensile and compressive-compressive quadrants the level curves scale linearly.

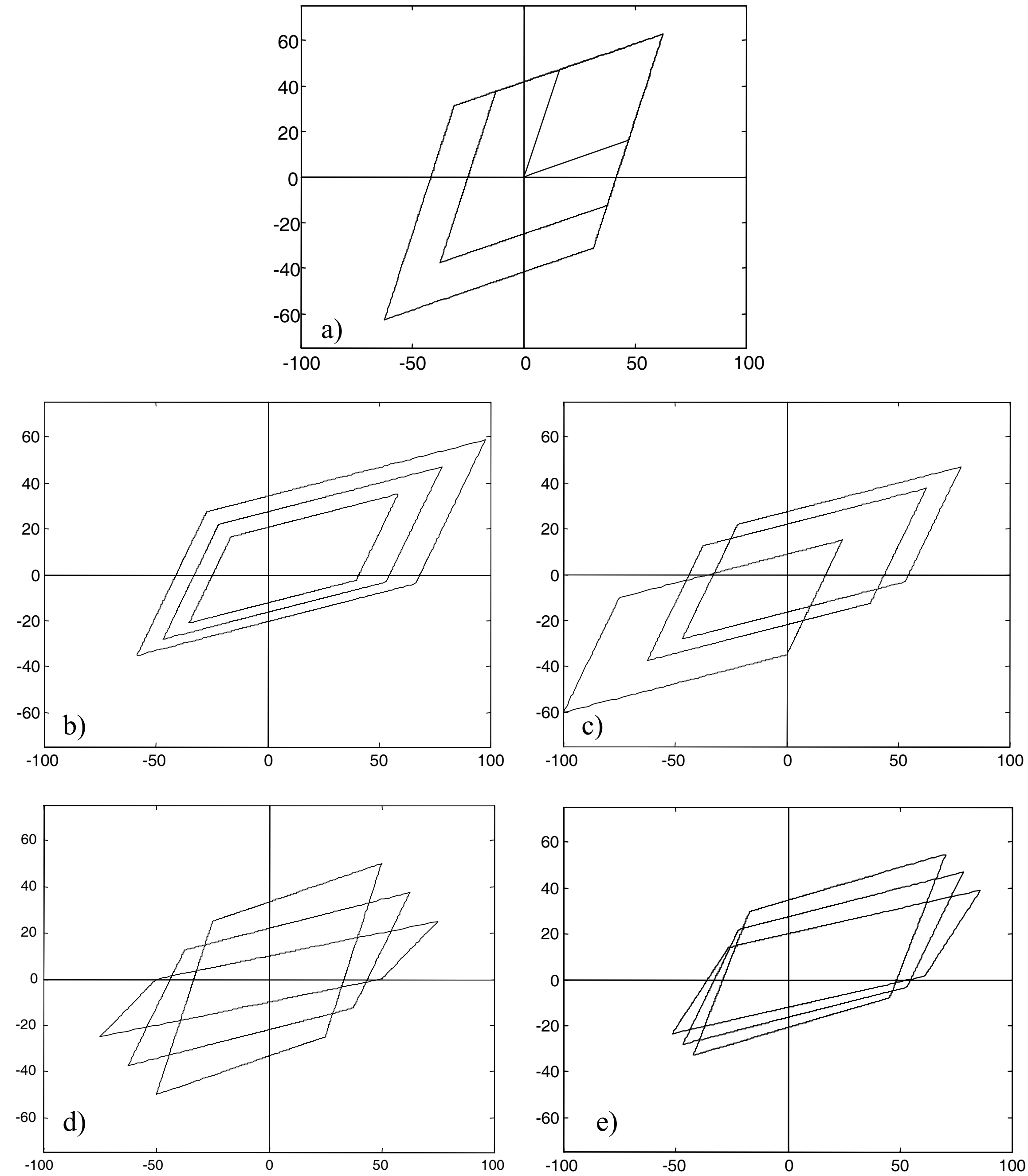

Figure 2. Plane stress failure envelopes, σ_y versus σ_x, at $\tau=0$. Unit: MPa.

a) $\sigma_{ft}A_f\rho/t$ = 125 MPa, σ_{fc}/σ_{ft}=0.0, 0.6 and 1.0, a=0
b) $(\sigma_{ft}+\sigma_{fc})A_f\rho/t$ =150 MPa, 200 MPa and 250 MPa, σ_{fc}/σ_{ft} =0.6, a=0.5/π
c) $(\sigma_{ft}+\sigma_{fc})A_f\rho/t$=200 MPa, σ_{fc}/σ_{ft} =0.6, 1.0 and 4.0, a=0.5/π
d) $(\sigma_{ft}+\sigma_{fc})A_f\rho/t$=200 MPa, σ_{fc}/σ_{ft} =1.0, a=0.00/π, 0.50/π and 1.00/π
e) $(\sigma_{ft}+\sigma_{fc})A_f\rho/t$=200 MPa, σ_{fc}/σ_{ft} =0.6, a=0.25/π, 0.50/π and 0.75/π

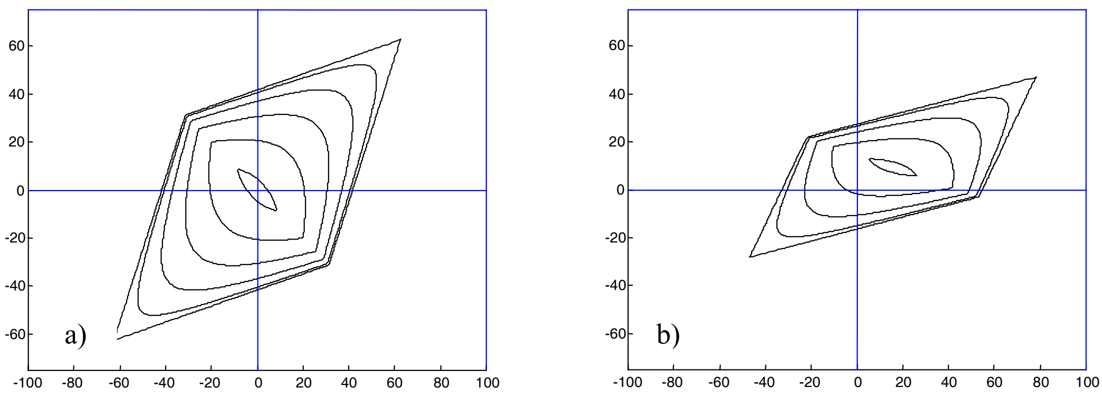

Figure 3. Plane stress failure envelopes at various level of shear stress.

a) τ=0, 6, 12, 18, 24 and 30 MPa, $\sigma_{ft}A_f\rho/t$ = 125 MPa, σ_{fc}/σ_{ft}=1.0, a=0
b) τ=0, 6, 12, 18, and 24 MPa, $\sigma_{ft}A_f\rho/t$ = 125 MPa, σ_{fc}/σ_{ft}=0.6, a=0.5/π

Numerical results for shear strength versus compressive strength

In this section the strength of the material at pure shear stress is denoted f_τ and the uniaxial compressive strengths in the x- and y-directions are denoted f_{cMD} and f_{cCD}, respectively. Since it is difficult to experimentally determine the shear strength of paper, a relation between the shear strength and the compressive strengths has been proposed [9]:

$$f_\tau = \sqrt{f_{cMD}\, f_{cCD}}\ . \tag{37}$$

This relation is here studied by calculation of the ratio

$$\alpha = f_\tau \,/\, \sqrt{f_{cMD}\, f_{cCD}} \tag{38}$$

by eq (33)-(36) for various values of the fibre orientation parameter a and the fibre strength ratio σ_{fc}/σ_{ft}. The ratio α and the paper compressive strength ratio f_{cMD}/f_{cCD} are both determined by the two parameters a and σ_{fc}/σ_{ft}, and in Figure 4 is α shown versus f_{cMD}/f_{cCD}. For $0<\sigma_{fc}/\sigma_{ft}\leq 1.0$, which interval includes all values of σ_{fc}/σ_{ft} that are of practical interest for paper, it is found that the fibre strength ratio σ_{fc}/σ_{ft} does not effect α. The variation in f_{cMD}/f_{cCD} from 1 to 5 corresponds for $0<\sigma_{fc}/\sigma_{ft}\leq 1.0$ to variation of the parameter a from 0 to $1/\pi$. Typically f_{cMD}/f_{cCD} may be in the order

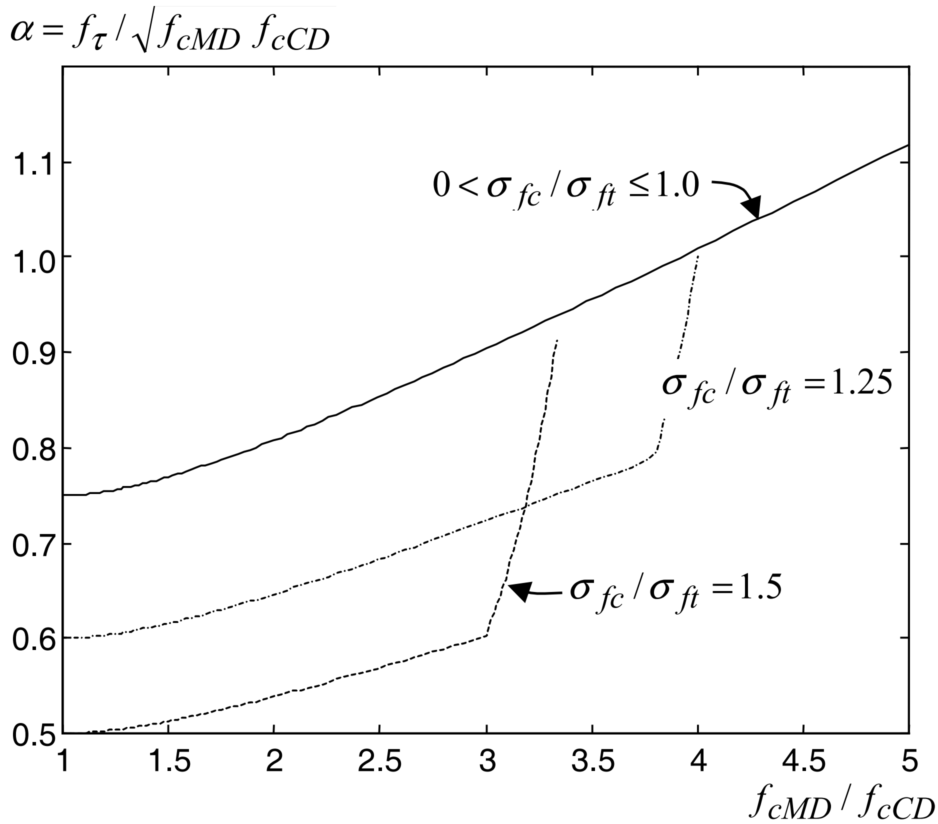

Figure 4. Shear strength coefficient α as a function of material compressive strength ratio f_{cMD}/f_{cCD}.

of 2 or 3 for paper, suggesting that eq (37) in some cases may somewhat overestimate the shear strength of a paper. In order to illustrate the performance of the model two curves for values of σ_{fc}/σ_{ft} greater than 1.0 are included in Figure 4. For such materials, eq (37) is not suitable, but instead the shear strength is better related to tensile strength.

Concluding remarks and possible model extensions

Numerical values of the parameters of the model can be chosen without any attention to their physical interpretation in terms of properties of the micro-structure of the material. Then the criterion can be regarded as phenomenological failure criterion for continuous and homogeneous materials. Treated in such a way, the criterion is similar to a maximum strain criterion for anisotropic materials with different limit strain in tension and compression.

It is obvious that possible future work in relation to the model presented should comprise verification and calibration by comparison to experimental results obtained for some fibrous material. Similar comparison to results obtained by advanced finite element network mechanics fracture modelling, [8,10], may also be of value.

Potential extensions of the model comprise:

- Development of equations and examples of numerical results for 3D fibre orientations and 3D states of stress.

- Leave the assumption that the fibre strength values σ_{ft} and σ_{fc} are deterministic, but instead define the fibre strength values by a mean value and a coefficient of variation together with some strength distribution function such as the Weibull distribution. This would affect the predicted strength in particular when all fibres are strained close to their limits. For zero shear stress this means that the failure envelope would be reduced in particular at the sharp corners where both σ_x and σ_y are large, giving high strain in large portion of the fibres.

- Leave the assumption that material failure coincides with the first fibre failure, but instead reduce the material stiffness as more and more fibre fail, and find the failure state of stress as the maximum stress during the damage process. By this not only a material failure criterion could be obtained but also a damage model for the non-linear stress-strain performance of the material. In such a further model development is might also be worthwhile to assume brittle fibre performance only in tension, but for the axial compression of the fibre assume some a plastic yield.

- Leave the assumption of an elastic-brittle performance of the individual fibre and instead assign elastic-plastic properties to the fibre. This would define a plastic performance of the network material.

- Work in order to find out how properties perpendicular to the fibre can be taken into account. Steps in that direction can be found in [6]. Such extension may be needed for development of some failure criterion that takes into account third direction failure in materials like paper with a 2D fibre orientation.

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