On H2 and H-infinity Optimal Control of Mass-Spring Networks with Power System Applications

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Abstract

Electric power systems are undergoing huge changes due to the shift from conventional power production to more renewable-based generation like solar and wind. This is primarily driven by the need to mitigate climate change by reducing CO_2 emissions. The shift to more generation from solar and wind will affect the dynamical behaviour of power systems, and consequently how they should be controlled. This thesis explores optimal control with respect to disturbance rejection. The systems that are investigated are damped mass-spring systems. The dynamics of AC frequency in power systems can be captured through such models. Further, the implications of the derived optimal control laws are investigated.

In the first paper of this thesis, undamped mass-spring systems (and more generally lossless systems) are investigated. The optimal controllers that achieve the lowest H_2-gain and H_∞-gain from disturbances to performance outputs are derived analytically for a standard setup. An analytical expression of the optimal gains are also presented. Finally, the results are interpreted in the context of electrical power systems. The results show the detrimental effect low inertia, typically associated with renewable generation like solar and wind, can have on H_2 performance. However, it is further shown numerically that under the optimal controller, these effects are mostly isolated to the low inertia regions of the grid.

The second paper of this thesis considers H_2 optimal control for disturbance rejection for a damped mass-spring system with uniform damping. The main contribution is to show that the optimal controller that achieves the smallest gain from disturbances to performance outputs is itself a damped mass-spring system. The optimal controller works both for stable and unstable systems. In the unstable case the H_2-gain becomes larger than the undamped system in the first paper, while for positively damped systems it becomes smaller.

Together the results presented in this thesis offer optimal controllers for undamped and uniformly damped mass-spring systems. These have been applied to simple models of electrical power transmission. Finally, future work detailing how to extend the techniques to cover a broader range of power system control problems is outlined.

Acknowledgements

I would first like to thank my main supervisor Richard. Without your guidance and comments along the way, this thesis would not have been what it is now. Our supervision meetings are always rewarding, and on top of that a lot of fun. Your knowledge and advice have been invaluable to me and to get me here. I also want to thank my co-supervisor Anders for helping out and being there whenever needed.

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Contents

1

Introduction

1.1 Motivation

Sweden and many other countries are currently in the process of transforming their electricity systems. This transition is characterized by the discontinuation of conventional electricity production in favour of renewable electricity production from the sun and wind [*ENTSO-E Vision: A Power System for a Carbon Neutral Europe* 2022; Jones et al., 2023]. There are may reasons for this. One is the effort to curb climate change, by discontinuing fossil fuels power plants. This transition must continue if the goal of stopping climate change is to be achieved [Masson-Delmotte et al., 2018]. At the same time, electricity consumption is expected to increase as more sectors of society are electrified [*Nordic Grid Development Perspective* 2023]. In Sweden and a few other countries, mainly nuclear power plants have been shut down in recent years, while wind power has been expanded [*Nordic Grid Development Perspective* 2023; "Energy in Sweden 2022 - An overview" 2022]. In many other countries, it is primarily electricity production from fossil sources, like coal and oil that is phased out in favour of production from solar and wind [*Statistical Review of World Energy 2023* 2023; *California Electrical Energy generation 2001-Current* 2023].

When more sun and wind are introduced at the same time as conventional production is phased out, the inertia in the system decreases [Ørum et al., 2018]. The inertia has traditionally given the overall system a 'slowness' to changes from to disturbances, which has given the generators time to act before the disturbance has resulted in a big deviation. The lower level of inertia in the system introduces new challenges for the control systems that control the voltage and frequency of power grids [Ørum et al., 2018]. To be able to introduce more renewable production, it is of the utmost importance that the electrical systems can maintain stable voltage and frequency, even with much lower levels of inertia in the system. This shift in how the system behaves requires the development of the control structures and the knowledge from a system perspective of what is required to create a functioning system with lots of sun and wind.

Wind power has an unpredictable nature and there is a fundamental differences in the type of electrical equipment used for wind power generation. This is likely to have a significant impact on how the Swedish electricity grid works. This thesis looks into some of the limits associated with this transition from a system with mostly conventional generation and large inertia to a system with more wind and solar power and thus some parts of the system with very little inertia.

1.2 Outline of the Thesis

The thesis studies the optimal control of systems that can be described as damped mass-spring systems with different requirements on the damping. These types of systems can, among others, describe simplified power systems models. The results are applied to simplified power systems models based on a first approximation of the AC frequency dynamics of an electric power system.

The contributions of this thesis are the development of controllers to minimise the effects of disturbances on the performance outputs of interest. The effect is measured in size through a norm. In this thesis, the H_2-norm and the H_∞-norm are the norms of interest. This field was developed in the 1980s and 1990s. See for example [Zhou et al., 1996] for a more in-depth description of the field. The methodology most often used is to state the problem in the framework and then use software to get a numerical solution. The main contributions in the thesis are the analytical solutions, given to the stated problems.

In the power systems case, the disturbances correspond to a deviation in power added/withdrawn to the grid at points of generation or load with respect to the expected operating situation, as well as disturbances in the measurement of the frequency deviation from nominal. The performance outputs of interest correspond to the AC frequency deviation from the nominal and the deviation of power output from the nominal from the generators. The aim is to find a controller that keeps the size of these performance outputs as small as possible under the given disturbances.

The thesis first looks into the type of control used, namely optimal disturbance rejection using H_2 and H_∞ control. The general model of an undamped and damped mass-spring system is presented. Quite some time is spent on showing the links and similarities between damped mass-spring models and a commonly used simplification of power systems. The interpretations of H_2 and H_∞ optimal control are also discussed in the context of power systems. Finally, the contributions given in two different papers are presented, together with the two papers published.

2

Control Theory Background

In this chapter H_2 and H_∞ optimal control are briefly introduced. The modelling class, which consists of mass-spring systems without damping and with uniform damping, is also introduced.

2.1 Disturbances and Optimal Control

This thesis considers H_2 and H_∞ optimal control. The H_2 and H_∞ norms are defined for the transfer function of a stable system as

$$\|G(s)\|_{H_2} := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{tr}\left(G(j\omega)^* G(j\omega)\right) d\omega \right)^{1/2} \quad \text{and}$$
$$\|G(s)\|_{H_\infty} := \sup_{\omega \in \mathbb{R}} \sigma_{\max}\left(G(j\omega)\right),$$

where j is the imaginary unit, $\operatorname{tr}(\cdot)$ the trace and $\sigma_{\max}$ the largest singular value.

Here follows a simplified description of the H_2 and H_∞ optimal control for disturbance rejection. For a more detailed explanation see for example [Zhou et al., 1996]. Given a linear system that can be described by a state-space representation one can define the performance outputs z of interest and how disturbances w enter the system. This can be described in generalized plant dynamics in state-space form

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}.$$

Define a controller K from y to u in state-space form as

$$\dot{x}_K = A_K x_K + B_K y, \quad x_K(0) = 0,$$
$$u = C_K x_K + D_K y.$$

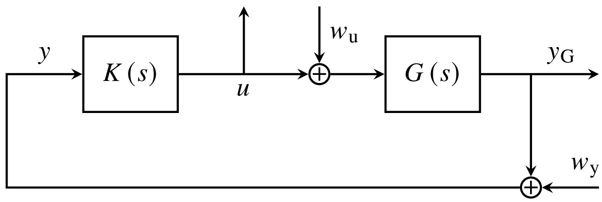


Figure 2.1 Block diagram describing the structure of the problem that is studied in Paper I and II

Equivalently the generalized plant and controller can be defined in transfer function form according to

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} G_{zw}(s) & G_{zu}(s) \\ G_{yw}(s) & G_{yu}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{and} \quad u = K(s)\,y.$$

Call the closed loop transfer function from disturbances to performance outputs $T_{zw}(s)$. Therefore

$$T_{zw}(s) = G_{zw}(s) + G_{zu}(s) K(s) \left(I - G_{yu}(s) K(s)\right)^{-1} G_{yw}(s).$$

Optimal disturbance rejection aims to find a controller $K(s)$ that achieves

$$\gamma_{\bullet}^{*} = \inf\left\{\gamma : \|T_{zw}(s)\|_{\bullet} < \gamma\right\},$$

where $\|\cdot\|_{\bullet}$ denotes either the H_2 or H_∞ norm.

Most often H_2 and H_∞ problems are solved using numerical methods based on Riccati equations [Zhou et al., 1996]. For H_2 an exact solution exists, while for H_∞ there may not exist a controller that achieves $\gamma_{H_\infty}^{*}$. By using for example a bisection algorithm, a K that achieves γ arbitrarily close to $\gamma_{H_\infty}^{*}$ can be found. For more details see for example [Doyle et al., 1989]. In both Paper I and Paper II exact solutions are given by solving Riccati equations.

2.2 Modelling Class

In this thesis, we exploit special problem structures to allow for simplified and analytical solutions to the H_2 and H_∞ optimal control problems. The control structure and where disturbances enter is summarised in Fig. 2.1. The performance outputs are both y_G, which is the output from the plant $G(s)$, without disturbances, and u, which is the control action. The target of the controller is thus to minimize the effects of the disturbances on the output of the process, without using too much control action. We also restrict the process dynamics to be described as damped mass-spring systems. These models can be described by the following differential

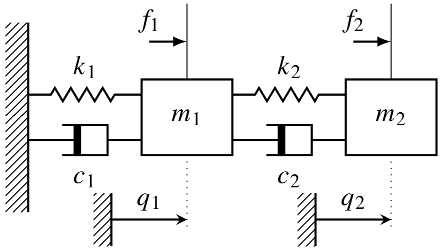


Figure 2.2 Example of a mass-spring network, with damped springs.

equation, where q is often called a generalized position.

$$\begin{aligned} &M\ddot{q} + C\dot{q} + Kq = f \\ &q(0) = \dot{q}(0) = 0, \\ &y_G = \dot{q}, \\ &M \succ 0 \quad K \succeq 0. \end{aligned} \tag{2.1}$$

To grasp the modelling class one can think of mechanical systems like the one in Fig. 2.2. The two different papers in this thesis can in their simplest form be seen as developing optimal controllers under different restrictions on C.

2.2.1 Linear Lossless Systems

Linear lossless systems are an interesting class of models. For the damped mass-spring equations in (2.1) this corresponds to having the damping matrix $C = 0$. Strictly speaking, the class of linear lossless systems is larger than only systems described by (2.1) with $C = 0$ and therefore a slightly more general description is given here. Lossless systems can be used to describe transportation networks of physical commodities, such as the transportation of a fluid when there are no leaks, or power systems with no losses. Some mathematical characteristics of lossless systems can be given both in state-space form and in transfer function form.

Consider a state space representation of a dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = 0, \\ y &= Cx + Du. \end{aligned} \tag{2.2}$$

If (C, A) is observable and (A, B) is controllable, then there exists a positive definite matrix P such that

$$PA + A^\top P = 0,\ PB = C^\top, \text{ and } D + D^\top = 0$$

if and only if the system is lossless. This ensures that all trajectories of (2.2) for which $x(T) = x(0)$ additionally satisfy $\int_0^T y(t)^\top u(t)dt = 0$. That is, any energy that is supplied to the system can then be retrieved, hence the name lossless.

Another characterisation of lossless systems can be found by looking at the transfer function representation,

$$G(s) = C(sI - A)^{-1} B + D.$$

If the transfer function fulfils

$$G(s) + G(-s)^{\mathsf{T}} = 0 \tag{2.3}$$

it means that it is lossless [Hughes, 2017]. All the above hold for MIMO linear systems. If the system is SISO, then (2.3) implies that the Nyquist curve of the system is only on the imaginary axis (for both positive and negative frequencies). A lossless SISO system has its poles and zeros on the imaginary axis. In general, lossless systems are resonant systems.

2.2.2 Uniformly Damped Systems

Uniformly damped systems are systems on the form of (2.1), but with the requirement on C that it fulfils

$$C = M \sum_{j=0}^{n-1} \alpha_j (M^{-1}K)^j, \quad \alpha_j \in \mathbb{R}. \tag{2.4}$$

This makes M, C and K diagonalizable by congruence, meaning that there exists an invertible matrix T such that $T^{\mathsf{T}}MT$, $T^{\mathsf{T}}CT$ and $T^{\mathsf{T}}KT$ are all diagonal for the same T. A special case of uniform damping is Rayleigh damping which is when $C = \alpha_0 M + \alpha_1 K$ [Rayleigh, 1877]. In (2.4) one can see that C can be either positive definite, negative definite or indefinite, depending on the signs of α_j. Thus the class of systems with uniform damping covers both stable and unstable systems. Damping is often difficult to model from first principles. Therefore models of uniform damping are often used as a first approximation. For more discussions on methods on how to estimate uniform damping matrices from data, see for example [Adhikari and Phani, 2007].

3

Power System Modelling

Power systems are very complicated systems often spanning huge distances, covering entire countries and sometimes spanning continents. They include complicated electrical dynamics in generation, transmission, and in the loads. Powers systems are usually AC systems transmitted through 3 phases, which are 120° phase shifted from each other. Due to the complexity and large scale of power systems, they have sometimes been called the largest machines mankind have ever built [Glover et al., 2010].

To be able to work with these systems in a comprehensible way, many types of modelling are used. When looking into transmission systems, the so-called single-line diagram is often used. This corresponds to representing the three phases by only one AC phase and assuming that the three phases are always 120° phase shifted from each other and balanced [Glover et al., 2010]. An example of a single line diagram is seen in Fig. 3.1. Here node i and node j have generators connected to

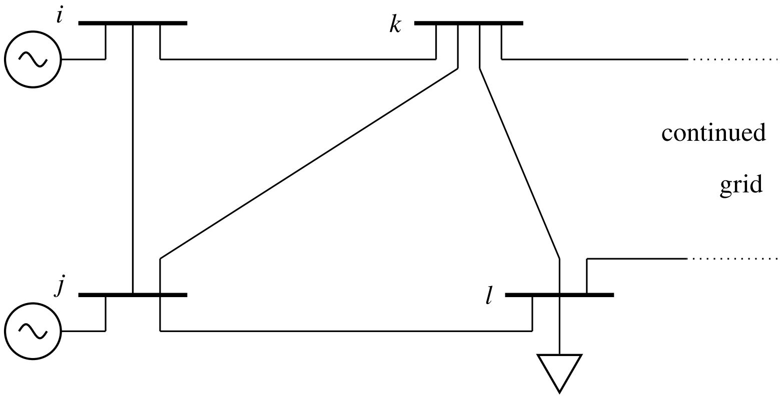


Figure 3.1 Sketch of power system interconnection. The nodes (called buses in power systems) are represented by thick lines, while transmission lines are the thin lines connecting them. Nodes i and j have generators connected to them, node l has a load connected and node k is just a transmission interconnection.

them, node k is just a transmission interconnection and node l has a load connected. In power systems nodes are often called buses, and the buses are interconnected through the transmission lines. Each bus has an associated voltage magnitude and phase angle. The phase angle is the relative difference to a phase defined by the nominal angular frequency of a large generator in the system [Kundur, 1994]. In power systems modelling, only phase differences are of interest.

There are many control objectives when controlling a power system. Two of the most important ones are to keep voltage and the AC frequency close to the desired setpoints and within tight limits. Voltages have different levels in different parts of the system, and transformers are used to interconnect different voltage levels. The nominal frequency differs between systems. In Europe 50 Hz is used, while 60 Hz is used in the US [Glover et al., 2010].

When looking into power transmission, three types of power are often considered, apparent power, active power and reactive power. When modelling power systems complex numbers are often used and the relation between the types of power used is that active power is the real part of apparent power while reactive power is the imaginary part. A very simplified explanation is that active power is what you can use for consumption in electrical appliances, while reactive power is needed in the system for it to work. Reactive power is associated with maintaining the desired voltage levels in the system [Kundur, 1994].

In this thesis, only the dynamics of AC frequency are considered. Voltages are assumed to be constant, and that other controllers have taken care of that. Another simplification is that loads are modelled as constant power sinks. In reality, loads would consume more or less power depending on frequency and voltages, but this is relatively small compared to the changes in generators, and also much harder to model.

When transporting power from generators to consumers the power is transported in different voltage levels, depending on where in the system. He highest voltage levels are in the transmission system, which covers the area of the whole system. Below the transmission system there are sub-transmission systems, which have lower voltage and cover regions. Finally, there is the distribution system, which has the lowest voltage. It is in the distribution system that individual houses are connected to the grid. Industries can be connected to different levels in the system depending on the size and power consumption. Transmission systems and sub-transmission systems are usually operated so-called N-1 security. This means that if any component, transmission line, generator, transformer etc. should be disconnected, the system as a whole should continue to work within voltage and frequency limits. Distribution systems are on the other hand operated as radial networks for easier control of voltages [Kundur, 1994; Glover et al., 2010].

In power systems, there are usually losses associated with resistances in the transmission and distribution networks. To limit the losses, power is often transmitted at high voltages, which reduces the current needed for transporting a given amount of power. This reduces the losses, since resistive losses in power transmission are proportional to the current squared. One of the main purposes of a power transmission network is to transport power with as few losses as possible, so making the power transmission system as close as possible to a lossless system is one of the aims. A distribution network cannot have such high voltages due to the dangers associated with it and the need for insulation at high voltages. Due to the voltage difference, a simplification is to treat the transmission network as lossless and associate all losses with the distribution networks [Kundur, 1994].

3.1 The Swing Equation

When modelling power systems, the so-called swing equation is often used. This captures the simplest dynamics of a generator and is based on Newton's second law. The dynamics for a node i with a generator is defined, with focus on the inertia of rotating synchronous machines, according to

$$J_i\ddot{\theta}_i(t) = T_{\mathrm{m},i}(t) - T_{\mathrm{e},i}(t). \tag{3.1}$$

In the above expression, $\theta_i(t)$ is the rotor angle of the synchronous machine and it is the same as the phase of the node it is associated with. J_i is its moment of inertia, which is a constant, $T_{\mathrm{m},i}(t)$ is the mechanical torque, minus mechanical losses, applied on the generator in node i and $T_{\mathrm{e},i}(t)$ is the electrical torque resulting from the coupling to the electrical grid. Here time dependence have been included to show that everything except the inertia constant can change with time. Multiplying (3.1) with the angular frequency $\dot{\theta}_i(t)$ gives the equation in powers rather than torques according to

$$J_i\dot{\theta}_i(t)\ddot{\theta}_i(t) = (T_{\mathrm{m},i}(t) - T_{\mathrm{e},i}(t))\dot{\theta}_i(t) = P_{\mathrm{m},i}(t) - P_{\mathrm{e},i}(t), \tag{3.2}$$

where $P_{\mathrm{m},i}(t)$ is the mechanical power applied to the generator and $P_{\mathrm{e},i}(t)$ is the electrical power subtracted from the generator to the grid. Equation (3.2) is what is often called the swing equation. The swing equation is used when only looking at frequency dynamics. Thus, frequency dynamics of a power system are often studied through the swing equation in combination with assuming lossless transmissions [Kundur, 1994]. For simplicity and easier reading, the notation of time dependence in $\theta_i(t)$ and mechanical and electric power to a node will hereafter be dropped.

3.2 Transmission

A transmission system consists of nodes, that are connected to each other by transmission lines like the ones in Fig. 3.1. Each node has a phase associated with it.When studying the dynamics of the AC frequency, in a lossless setting, a separation can often be made in the sense that active power affects the AC frequency, while voltage dynamics are mainly affected by reactive power flows. Thus, only active power flow is considered here. The active power transmitted between two nodes i and j in a lossless transmission line is given by

$$P_{ij} = P_{\max,ij} \sin(\theta_i - \theta_j), \tag{3.3}$$

where $P_{\max,ij}$ is the theoretical maximal power transfer between node i and j, which is a non-negative constant. θ_i is the phase of node i and θ_j of node j [Glover et al., 2010]. The value of $P_{\max,ij}$ is given by parameters of the line and the voltage in the grid. For more details see for example [Glover et al., 2010], Chapter 5. From each transmission line, there is an in/outflow to a node, from the connecting nodes. Given a system with n nodes with a generator and m nodes without generators, this means that the electrical power added/withdrawn from a node via the transmission network is given by

$$P_{\mathrm{e},i} = \sum_{j\neq i}^{n+m} P_{\max,ij} \sin\left(\theta_i - \theta_j\right) \qquad \forall i = 1,2,...,n+m. \tag{3.4}$$

3.3 Combining the Swing Equation with Transmission

A power system is built up of many nodes and connections between them. The grid connects generators to each other and to nodes without generation, like loads. In this thesis, loads are modelled as constant power sinks. Since (3.4) only looks at the phase difference of two connecting nodes it can be linearized around the nominal AC frequency ω_0. Given a system with n nodes with a generator and m nodes without generators, inserting the expression for $P_{\mathrm{e},i}$ in the swing equation, together with assuming constant mechanical power, the linearization of (3.2) and (3.4) around the nominal frequency ω_0 results in

$$\begin{aligned} J_i\omega_0\ddot{\bar{\theta}}_i &= \bar{P}_{\mathrm{m},i} - \sum_{j\neq i}^{n+m} P_{\max,ij}\cos\left(\theta_i^0 - \theta_j^0\right)(\bar{\theta}_i - \bar{\theta}_j) \quad \forall i = 1,2,...,n \\ 0 &= \sum_{j\neq i}^{n+m} P_{\max,ij}\cos\left(\theta_i^0 - \theta_j^0\right)(\bar{\theta}_i - \bar{\theta}_j) \quad \forall i = n+1, n+2,...,n+m \end{aligned} \tag{3.5}$$

where $\bar{\theta}_i$ is the deviation of the phase of node i from the one given by nominal angular frequency ω_0 ($\bar{\theta}_i = \theta_i - \omega_0 t$), $\bar{P}_{\mathrm{m},i}$ is the deviation from the mechanical power when in steady state (this will later be used as the input), θ_i^0 is the phase of node i at the linearization point. Equation (3.3) shows that maximum power is transmitted in a lossless line if the phase angle between sending and receiving node is 90°. Usually, phase differences are far below 45° in normal operation due to system security and heating of the lines [Kundur, 1994]. This means that $P_{\max,ij}\cos(\theta_i^0 - \theta_j^0)$ is a positive constant. In a transmission line power can flow in both directions and the direction is only governed by the phase difference, thus $P_{\max,ij} = P_{\max,ji}$. For a lossless transmission line, the power sent and received are equal.

The zero on the left side in the second line of (3.5) comes from looking at all nodes without generation, like those connected only to loads or the interconnection points. Since loads are modelled as constants and the linearization looks at deviations the power flow deviations from the linearization point should sum to zero.

Equation (3.5) has both dynamical and algebraic equations. This can be reduced to a system with only dynamical equations by doing so-called Kron reduction. Call $P_{\max,ij}\cos(\theta_i^0 - \theta_j^0) := k_{ij}$. Define $\bar{\theta}_{1:n}$ as the vector of phases of nodes one to n, and define $\bar{\theta}_{n+1:n+m}$ as the vector of phases for nodes $n+1$ to $n+m$. Equation (3.5) can be written in matrix form according to

$$\begin{bmatrix} J\omega_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\bar{\theta}}_{1:n} \\ \ddot{\bar{\theta}}_{n+1:n+m} \end{bmatrix} = \begin{bmatrix} J\omega_0 \\ 0 \end{bmatrix} \ddot{\bar{\theta}}_{1:n} = \begin{bmatrix} \bar{P}_{\mathrm{m}} \\ 0 \end{bmatrix} - \begin{bmatrix} K_{\mathrm{a}} & K_{\mathrm{b}} \\ K_{\mathrm{b}}^{\top} & K_{\mathrm{c}} \end{bmatrix} \begin{bmatrix} \bar{\theta}_{1:n} \\ \bar{\theta}_{n+1:n+m} \end{bmatrix}, \tag{3.6}$$

where J is a $n \times n$ matrix with J_i on the diagonal and zeros elsewhere, $\bar{P}_{\mathrm{m}}$ is a the vector of $\bar{P}_{\mathrm{m},i}$ of length n. The K-matrix is $-k_{ij}$ on all of-diagonal elements and in the i:th position in the diagonal the sum $\sum_{j\neq i}^{n+m} k_{ij}$, for all i's. The splitting of the K-matrix in submatrices is the following: K_{a} is of size $n \times n$, K_{b} is $n \times m$ and K_{c} is $m \times m$. It also holds that $K_{\mathrm{a}}^{\top} = K_{\mathrm{a}}$ and $K_{\mathrm{c}}^{\top} = K_{\mathrm{c}}$.

Using Kron reduction, which in this case constitutes of solving the zero part of (3.6) and expressing $\bar{\theta}_{n+1:n+m}$ in $\bar{\theta}_{1:n}$ and substituting this back gives

$$J\omega_0 \ddot{\bar{\theta}}_{1:n} = \bar{P}_{\mathrm{m}} - K_{\mathrm{red}} \bar{\theta}_{1:n}, \quad \dot{\bar{\theta}}_{1:n}(0) = \bar{\theta}_{1:n}(0) = 0, \tag{3.7}$$

with K_{red} being the Schur complement, $K_{\mathrm{red}} = K_{\mathrm{a}} - K_{\mathrm{b}} K_{\mathrm{c}}^{-1} K_{\mathrm{b}}^{\top}$. Defining the output as the frequency deviation from nominal of all nodes with with generators ($y_{\mathrm{G}} = \dot{\bar{\theta}}_{1:n}$) the power system modelling is now written on the form of (2.1), with $C = 0$.

3.4 Mechanical Analogue of Electrical Systems

Looking at (3.5) and dropping the bar notation for simplicity, this equation is identical to the mechanical equation of a mass-spring system of masses sliding on a circle of radius 1, connected by springs according to the Fig. 3.2. Instead of mechanical powers acting on the system, forces act on the masses. The equations guiding the mechanical system in Fig. 3.2, can also be derived from Newton's second law and after linearization, we have for each mass i

$$\begin{aligned} m_i \ddot{\theta}_i &= f_i - \sum_{j\neq i}^{n+m} k_{ij}(\theta_i - \theta_j) \quad \forall i = 1, 2, \ldots, n \\ 0 &= \sum_{j\neq i}^{n+m} k_{ij}(\theta_i - \theta_j) \quad \forall i = n+1, n+2, \ldots, n+m \end{aligned} \tag{3.8}$$

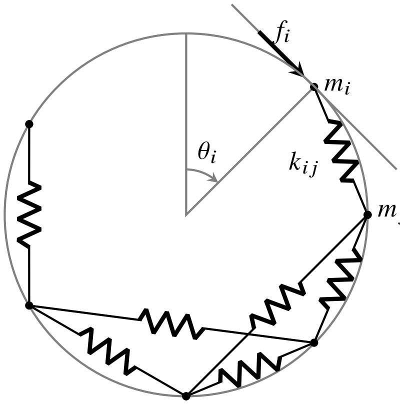


Figure 3.2 Mechanical system that has an equivalent mathematical description as a power system network.

where the constant $k_{ij} = \tilde{k}_{ij}\cos(\theta_i^0 - \theta_j^0)$ and $\tilde{k}_{ij}$ is the spring constant in Hook's law. In (3.8) there are n masses with forces acting on them, and m interconnection points of the springs without any mass.

A type of losses that are studied in mechanical systems is linear damping. Linear damping acts like a force proportional to the velocities of the masses. There can be local damping at each of the masses, acting as if the masses are moving in a medium, and there can be damping resulting from the interconnections. An example of damping through interconnection would be to have dampers in parallel with the springs in Fig. 3.2. For each mass i damping can be applied as a factor proportional to the difference in angular velocities in the system according to

$$\begin{aligned} m_i\ddot{\theta}_i &= f_i - \sum_{j=1}^{n+m} c_{ij}\dot{\theta}_j - \sum_{j\neq i}^{n+m} k_{ij}(\theta_i - \theta_j) \quad \forall i = 1,2,\ldots,n \\ 0 &= \sum_{j=1}^{n+m} c_{ij}\dot{\theta}_j + \sum_{j\neq i}^{n+m} k_{ij}(\theta_i - \theta_j) \quad \forall i = n+1, n+2, \ldots, n+m \end{aligned} \tag{3.9}$$

where c_{ij} is a constant. The equations in (3.9) can be written in a matrix form according to

$$M\ddot{\theta} = f - C\dot{\theta} - K\theta \iff M\ddot{\theta} + C\dot{\theta} + K\theta = f, \quad \dot{\theta}(0) = \theta(0) = 0, \tag{3.10}$$

where M is a diagonal matrix with the individual masses on the diagonal, and zero for the interconnection points without masses. C is a matrix with entries c_{ij}, f is a vector with all forces on nodes 1 to n and zero for in the m later positions. Finally, K is a matrix with $-k_{ij}$ on all of-diagonal elements and in the i:th position in the diagonal the sum $\sum_{j\neq i}^{n+m} k_{ij}$, for all i's.

Equation (3.10) is almost on the form of (2.1), but may not fulfill the requirement $M > 0$ (unless there are no massless interconnection points M will only be semi-definite). The second equation in (3.9) was included to also allow loads and interconnection nodes to be modelled in the mechanical analog. In Paper II mechanical systems with a positive definite mass-matrix are investigated. This assumption was largely made for convenience to allow for a simplified definition of uniform damping to be used (note the inverse of M in (2.4)), and also when writing (3.10) in state-space form. More general definitions of uniform damping are available in the literature [Adhikari and Phani, 2007], and the idea is to in the future look into how damping occurring in powers systems can be approximately modelled by including uniform damping.

3.5 Design Objectives for Powers Systems

Comparing the variables from the problem formulation to those in the previous sections shows that in the power systems case of the problem formulation, the output y_{G} is the AC frequency deviation from nominal. The control action u is the deviation in power applied on the generator from the power plant. The setup is thus to limit the AC frequency deviation as a result of the disturbances without changing the primary mechanical power applied to the generator too much. The disturbance w_{u} is here added/withdrawn active power to the nodes, relative to the linearization point, and w_{y} is measurement errors in measuring the frequency. One example of generation uncertainties on a short time scale is extra power applied from a wind turbine in the event of a gust. Another is a reduction in power produced from a PV-cell due to the shadow from a cloud passing by. Yet another type of uncertainty is the sudden, unplanned disconnection of a generator or a major load as a result of a fault in the system. This could be the security tripping of a generator when a voltage or current exceeds limits, or the tripping of a transmission line in the event of a fault.

3.6 H_2 and H_∞ Control Interpretations in Power Systems

H_2 and H_∞ control have interesting interpretations in power systems.

3.6.1 H_2-control

H_2 control can be interpreted as the way to minimize energy throughput of white noise disturbance to performance outputs [Zhou et al., 1996]. This design objective would be to minimize the effect of small random disturbances like the passing of clouds over PV-cells or the passing of wind gusts through a wind farm, which on a system-wide scale is hard to estimate beforehand. The white noise assumption is perhaps only reasonable as a first approximation but filtered white noise will be considered as future work.

3.6.2 H_∞-control

The transmission system operators (TSOs) always want to make sure that the power system in transmission and sub-transmission levels work with N-1 security. This means that if any component, transmission line, generator, transformer etc. should be disconnected, the system as a whole should continue to work within voltage and frequency limits. When accessing control structures and reserves it is done with the largest possible disturbance in mind. This is often the disconnection of the largest generator. Limiting the effect of the largest possible disturbance is similar in flavour to limiting the effect on the worst possible disturbance, which is the objective of H_∞ control.

4

Contributions

The main scientific contributions of this thesis are presented in two papers.

4.1 Paper I

Johan Lindberg and Richard Pates (2023). "Fundamental Limitations on the Control of Lossless Systems" This paper investigates the optimal control with respect to disturbance rejection of lossless systems. Given a lossless system, which can be expressed in state-space form, the optimal controller for the setup is derived. The controller aims to minimize the gain from disturbances on the process input and the measurement signal to the performance outputs, which are both the output of the process and the control action. An analytical expression for the minimal gain from disturbances to outputs is also derived. Results are given in both H_2 and H_∞ norms.

For a lossless system written in state-space representation according to (2.2), the controllers and gains are the following:

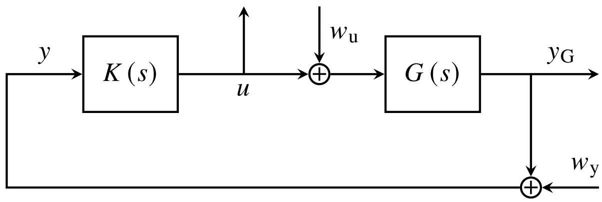
$$K_{H_2}(s) = -C\left(sI - A + 2BC\right)^{-1} B \qquad \gamma^*_{H_2} = \sqrt{2\,\mathrm{tr}\left(CB\right)}$$
$$K_{H_\infty}(s) = -\sqrt{2}I \qquad \gamma^*_{H_\infty} = \sqrt{2}$$

(With the extra condition that the D-matrix has to be zero for the H_∞ case.)

The results are then applied on a simplified model of an electrical transmission network. This simplified model is based on the swing equation expressed in state-space form in the way of (3.6) and (3.7). The gain from disturbances to performance outputs under optimal control for the power systems model is given by

$$\gamma^*_{H_2} = \sqrt{2\left(\frac{1}{J_1\omega_0} + \cdots + \frac{1}{J_n\omega_0}\right)} \qquad \text{and} \qquad \gamma^*_{H_\infty} = \sqrt{2}$$

where $J_k\omega_0$ is the inertia constant of generator k. For renewables such as solar and wind the inertia constant is much lower than for conventional power generation like nuclear and fossil fules. The expression for the H_2 gain suggests that the disturbance rejection capability, with respect to the H_2-norm, is severely degraded when



renewables with low inertia like solar and wind are introduced. This problem with disturbance rejection arises even if only one generator has a low inertia constant. A further investigation shows however, that when the optimal control is applied, the gain is highly local to the generators with low inertia and doesn't spread throughout the system. In the wind turbine case this corresponds to, for example, the case of a gust of wind suddenly changing the power output from a turbine. The local frequency might then change and as a result the turbine will start to deviate from the nominal power output, by a large control signal being applied to it. This will however not change the frequency and control actions in other parts of the system too greatly.

A striking feature of the optimal H_∞-controller is that it is completely decentralized and is just static feedback. In the power systems case, this corresponds to so-called droop control, which is a constant, local feedback term, which is a major mechanism used to regulate AC frequency in conventional grids.

The idea for the paper was developed by both authors. The derivation of the theorem, the application to power systems models, and the numerical simulations were carried out by J. Lindberg. The paper was written by both authors, but primarily by J. Lindberg. The final revision of the paper was mostly carried out by J. Lindberg.

This paper is published in IEEE Control Systems Letters, Volume 7, pp. 157-162.

4.2 Paper II

Johan Lindberg and Richard Pates. "On the H_2 Optimal Control of Uniformly Damped Mass-Spring Systems" The second paper of the thesis looks into a similar problem of disturbance rejection under optimal control, but now $G(s)$ in Fig. 2.1 is no longer lossless. Instead, uniform damping is considered. This paper mainly looks into the damped mass-spring problem with uniform damping described by (2.1) with the process output $y_G = \dot{q}$. For clarity, the differential equations are repeated here again.

$$
\begin{aligned}
&M\ddot{q} + C\dot{q} + Kq = u + w_u \\
&q(0) = \dot{q}(0) = 0, \\
&y_G = \dot{q}, \\
&M \succ 0 \quad K \succeq 0
\end{aligned}
$$

where q is a generalized coordinate, u is an external force acting on the system and C is uniform damping according to (2.4). Here M is called the mass-matrix and K

the spring-matrix.

The main result is to show that the optimal H_2-controller of the damped mass-spring system is itself a damped mass-spring system. That is, the optimal controller is given by a mass spring damper system

$$\begin{aligned} &M_K\ddot{q}_K + C_K\dot{q}_K + K_K q_K = y, \quad q_K(0) = \dot{q}_K(0) = 0, \\ &u = -\dot{q}_K, \end{aligned}$$

where the mass, spring and damper matrices of the controller are given as expressions of the mass, spring and damper matrices of the system that is controlled. An analytical value of the H_2-gain from disturbances to performance output is also given in terms of the original matrices of the damped mass-spring system. When the damping is zero, it coincides with the H_2 solution in Paper I. With positive damping the gain becomes smaller than that with no damping, and with negative damping (i.e. an unstable system), it becomes larger.

The idea for this paper was developed by both J. Lindberg and R. Pates, as a continuation of Paper I. All mathematical derivations were carried out by J. Linberg, as well as the numerical example. The paper was mostly written by J. Lindberg.

This paper is submitted to the 22nd European Control Conference (ECC).

5

Discussions and Future Work

In this thesis, lossless networks and damped mass-spring networks have been studied. Simplified power system can be modelled as these. The study of these type of networks are of great interest due to the transformation of our electrical power systems, as a result of the shift to more renewable electric power production from solar and wind.

The main contributions have been to derive analytical expressions for the optimal controllers for disturbance rejection when the objective is to limit the effect of external disturbances on the output from the system and the control actuation. In the power systems case this corresponds to limiting the frequency deviation and the deviation of active power supplied in relation to the scheduled production. The minimal achievable gains under the optimal controller are also given as analytical expressions. This serves as a lower limit on how well controllers can mitigate disturbances in these networks.

5.1 Discussion

5.1.1 Weights on Disturbances

The theoretical contributions in this thesis are the optimal disturbance rejection for lossless systems and uniformly damped mass-spring systems. The disturbances enter both on the input signal to the system and on the measurement signal. In the papers, the disturbances are assumed to enter in the same way and to be of the same size at all entry points. For the power system case in Paper I, this would for example mean that it assumes an equal size disturbance for a nuclear generator and a small wind turbine. For the measurement noise, this assumption might not be too bad, but for power disturbances, this is not realistic. Simultaneously, renewables like wind and solar are subject to a type of disturbances conventional production has not been, namely the changeability of the weather. In the short time scales, where the problem of storage is not the focus, these can be wind gusts and passing clouds quickly changing the power outputs of the renewable generations. Some of these shortcomings in the theory would be addressed if weights were introduced on the incoming disturbances to try to capture these characteristics. If dynamic weights could be used then optimal controllers could be derived under the specific disturbance profiles of each system of interest.

5.1.2 More Detailed Power System Models

Beyond Swing Equation Models The connection between the results in both Paper I and Paper II and power systems has been the swing equation. The swing equation is good when looking at system-wide phenomena when the inertia of the system plays a crucial role in the dynamics of the system. When using the swing equation as described in this thesis all other dynamics of the generators are disregarded. This works well if the time constants of all other dynamics are significantly shorter than that of the inertia in the system. For conventional generators, this all works well and that is why it has been used so extensively for stability studies and for assessing N-1 security with respect to frequency control.

However, with renewables like wind and solar power, there are no rotating machines. Solar PVs generate DC currents and these are then converted to AC through inverters. For wind turbines, an AC current is generated, but it is variable in frequency due to changes in the rotational speed of the turbine. The AC power is then converted to DC and then back to AC of the correct frequency and phase. This means that there is no mechanical coupling between the mechanical rotations of the turbines and the frequency of the grid. In inverters there are still time constants arising from other dynamics, so one of them could be significantly slower, and a first-order approximation can be reasonable. This is however not clear and a more detailed study would be needed to determine this.

When studying power systems from a control perspective there is often a choice to be made. Either you study the overall system behaviour and assume simplified components, or you study generators and loads in detail but are then seldom able to look into larger systems in detail. This thesis has looked into the first of these two options, with the simplest model, namely the swing equation. One can do more detailed modelling, and it would be interesting to consider lossless approximations of models that include voltage and reactive power dynamics. It is also important to test the findings using simulation software. Then you can, in a comprehensible way, get both system-wide behaviour studies and details of the components.

More Detailed Transmission Line Modelling In the modelling of transmission lines, lossless lines were assumed. A more detailed transmission model is the following. Transmission lines transmit active power (P) and reactive power (Q). The subscript R is for the receiving end and the subscript S is for the sending end of the transmission line. The equations that govern this are for example given in [Glover et al., 2010] and at the receiving end they are

$$\begin{aligned} P_R &= \frac{V_R V_S}{Z'}\cos(\phi_Z - \Delta\theta) - \frac{A_l V_R^2}{Z'}\cos\left(\phi_Z - \phi_{A_l}\right) \quad \text{and} \\ Q_R &= \frac{V_R V_S}{Z'}\sin(\phi_Z - \Delta\theta) - \frac{A_l V_R^2}{Z'}\sin\left(\phi_Z - \phi_{A_l}\right). \end{aligned} \tag{5.1}$$

In the equation above, V_R and V_S are the voltage magnitudes in the receiving and sending ends of the line. $\Delta\theta$ is the difference in the phase angles of the receiving and sending ends of the transmission line. Z' and A_l are line parameters. These are constants and arise from the resistance, conductance and inductance of the transmission line, which in turn depend on the length, material and thickness. In (5.1) all constants are real and ϕ_Z and ϕ_{A_l} are the associated phase angles of Z' and A_l. For more details, see for example [Glover et al., 2010], Chapter 5.

In the previous parts of this thesis lossless transmission lines were used. A lossless transmission line is in (5.1) equivalent to $Z' = X'$, $\phi_{A_l} = 0°$, $\phi_Z = 90°$, Thus the equation for active power in (5.1) is simplified to

$$P_R = \frac{V_R V_S}{X'}\cos(90° - \Delta\theta) - \frac{A V_R^2}{X'}\cos(90°) = \frac{V_R V_S}{X'}\sin(\Delta\theta) = P_{\max}\cdot\sin(\Delta\theta),$$

where X' is the line reactance. This is the same as the transmission relation in (3.3).

5.2 Future Work

There are many interesting directions to continue the work in this thesis. Some of these include:

- Derive the results of Paper I and II with weights on the disturbances and performance outputs, both static and dynamic. This would be of interest to be able to adjust where and how disturbances enter to better represent a real operating scenario of a power system. This would allow someone using the results to weigh in where the disturbances are expected to be larger and to put extra emphasis on the performance outputs of most interest.
- Look into the H_∞ optimal control of the problem in Paper II. Since the H_2 optimal control scheme minimizes the energy throughput of disturbances and H_∞ minimizes the effect of the worst possible disturbance they are both of interest in power systems and beyond.
- Try to apply the results in Paper II in a power systems simulation software. Since Paper II includes losses it is better at representing a full power system that has losses, especially in the distribution part where low voltages lead to significant resistive losses. This simulation task would include finding the

damped mass-spring equivalent of a power system, calculating the optimal control structure, and then implementing an approximation of this given the tools and interconnections available.

- Also include voltage control. When using inverters the share of active power and reactive power can be chosen almost freely, given the apparent power. Thus there are great opportunities when studying renewables to also do voltage control in a similar way as frequency control is done in the swing equation. One way to do this would be to model voltage dynamics resulting from reactive power flows in the framework of lossless systems, similar to frequency and active power.

- Try to include resistances in a transmission line in the framework of Paper II, possibly by using the more detailed transmission line model in (5.1).

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Paper I

Fundamental Limitations on the Control of Lossless Systems

Johan Lindberg **Richard Pates**

Abstract

In this paper we derive fundamental limitations on the levels of H_2 and H_∞ performance that can be achieved when controlling lossless systems. The results are applied to the swing equation power system model, where it is shown that the fundamental limit on the H_2 norm scales with the inverse of the harmonic mean of the inertias in the system. This indicates that power systems may see a degradation in performance as more renewables are integrated, further motivating the need for new control solutions to aid the energy transition.

 IEEE Control Systems Letters, Volume 7, pp. 157-162. doi: 10.1109/LCSYS.2022.3185928.

1. Introduction

The lossless systems form an important class of models. They are frequently used to explain and understand phenomena arising in engineering applications. This is particularly true when describing the transportation of physical quantities, such as electrical power. This is because it is typically desirable to engineer such systems to minimise losses, making the resulting dynamical systems amenable to modelling within the lossless framework. Furthermore lossless systems enjoy rich theoretical properties. For example, factorisations involving lossless transfer functions play a crucial role in H_∞ methods [Kimura, 1997]. In addition, central control theoretic results, such as the Kalman-Yakubovich-Popov Lemma, simplify significantly in the lossless setting [Willems, 1972], and the state-space and circuit theoretic descriptions of lossless systems have a range of appealing structural properties [Hughes, 2017; Pates, 2022].

In this paper, we study the following optimal control problem:

Problem 1 Let

$$\begin{aligned} \dot{x} &= Ax + B\left(u + w_u\right), x\left(0\right) = 0, \\ z &= \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\ y &= Cx + D\left(u + w_u\right) + w_y, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \dot{x}_{\mathrm{K}} &= A_{\mathrm{K}} x_{\mathrm{K}} + B_{\mathrm{K}} y, x_{\mathrm{K}}\left(0\right) = 0, \\ u &= C_{\mathrm{K}} x_{\mathrm{K}} + D_{\mathrm{K}} y, \end{aligned} \tag{2}$$

and denote the closed-loop transfer function from $w = \begin{bmatrix} w_u^\mathsf{T} & w_y^\mathsf{T} \end{bmatrix}^\mathsf{T}$ *to* z *as defined by (1) and (2) as* $T_{zw}\left(s\right)$. *Find*

$$\gamma_\bullet^* = \inf\left\{\gamma : \|T_{zw}\left(s\right)\|_\bullet < \gamma\right\},$$

where $\|\cdot\|_\bullet$ *denotes either the* H_2 *or* H_∞ *norm.* ◊

This is a standard setup, in which the objective is to design a dynamic feedback control law

$$K\left(s\right) = C_{\mathrm{K}}\left(sI - A_{\mathrm{K}}\right)^{-1} B_{\mathrm{K}} + D_{\mathrm{K}}$$

to minimise the effects of process disturbances and sensor noise on the output and control effort of a process with dynamics

$$G\left(s\right) = C\left(sI - A\right)^{-1} B + D. \tag{3}$$

The main theoretical contribution, given as Theorem 1 in Section 2, is to show that when the process $G\left(s\right)$ is lossless, $\gamma_{H_2}^* = \sqrt{2\,\mathrm{tr}\left(CB\right)}$ and if in addition $D = 0$, $\gamma_{H_\infty}^* = \sqrt{2}$.

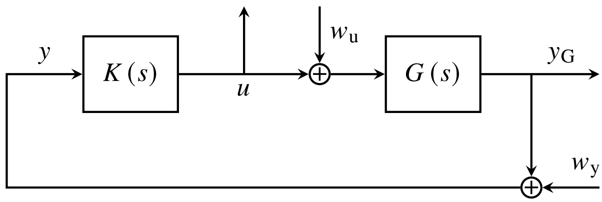


Figure 1. Block diagram representation of Problem 1. The objective is to design a feedback control law to minimise the effects of process disturbances and sensor noise (w_u and w_y) on the output and control effort (y_G and u) of a process.

The utility of this result stems from the fact that it analytically characterises fundamental limitations on the control of lossless systems. For example, since electric power systems at the transmission and sub-transmission level are close to lossless, it shows that *no matter how they are designed*, power system controllers in this part of the grid can never achieve better levels of H_2 or H_∞ performance than $\gamma^*_{H_2}$ and $\gamma^*_{H_\infty}$. Since the derived expressions are analytical, they can be rewritten in terms of the model parameters. In Section 3.1 we use this to highlight that system inertia plays a fundamental role in the control of power systems, by showing that $\gamma^*_{H_2}$ scales with the inverse of the harmonic mean of the inertias in the system. As discussed in Section 3.2, this suggests that the decrease in system inertia and increase in stochastic disturbances that accompanies the introduction of renewables [Jordehi, 2018] can significantly deteriorate performance, as quantified by the H_2 norm. This provides further evidence that new control approaches are required to support the energy transition, perhaps through the use of more advanced measurement tools [Byrne et al., 2014].

These results naturally complement existing results on fundamental limitations on the control of large-scale systems. There the focus has typically been on the performance limits imposed by restrictions on controller structure, such as locality [Bamieh et al., 2002]. However, due to the difficulty of the underlying mathematical problems [Witsenhausen, 1968; Lessard and Lall, 2011], there are few extensions of these results that cover broader classes of controller dynamics [Tegling and Sandberg, 2017]. In contrast, the limitations derived here hold for all causal controllers. Although no locality restrictions are imposed, the optimal control laws associated with Problem 1 are inherently structured. This is discussed in Section 3.3, where it is demonstrated that while the optimal control laws cannot prevent the emergence of undesirable behaviours, they can prevent them spreading throughout the system.

2. Fundamental Limitations

A transfer function $G(s)$ in the form of (3) is said to be lossless if $G(s) = -G(-s)^{\mathsf{T}}$. Other equivalent descriptions of losslessness include the condition given as (4) below, and that the state-space model $\dot{x} = Ax + Bu$, $y = Cx + Du$ defines a (behavioral) description of the driving point behavior of an electrical network constructed with only capacitors, inductors, transformers and gyrators [Hughes, 2017]. The following theorem shows that if the process to be controlled (3) is lossless (and has no direct term in the H_∞ case), then both $\gamma^*_{H_2}$ and $\gamma^*_{H_\infty}$ can be determined analytically. These expressions thus impose fundamental limits on the levels of H_2 and H_∞ performance that can be achieved when controlling lossless systems. These limitations will be interpreted in the context of simple electric power system models in the next section.

THEOREM 1
Assume that A, B, C, and D are as in Problem 1, and that the pair (A,B) is controllable. If there exists a positive definite P such that

$$PA + A^\mathsf{T} P = 0,\ PB = C^\mathsf{T},\ \textit{and}\ D + D^\mathsf{T} = 0, \tag{4}$$

then in the H_2 case of Problem 1,

$$\gamma^*_{H_2} = \sqrt{2\,\mathrm{tr}\,(CB)}.$$

If in addition $D = 0$, then $\gamma^*_{H_\infty} = \sqrt{2}$.

REMARK 1 It is shown in [Hughes, 2017] that the dynamics of lossless systems are always behaviorally controllable, meaning that the controllability assumption in Theorem 1 is essentially without loss of generality.

Proof. The dynamics in (1) are a special case of the generalised plant dynamics

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_w & B \\ C_z & 0 & D_{zu} \\ C & D_{yw} & D_{yu} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \tag{5}$$

where

$$B_w = \begin{bmatrix} B & 0 \end{bmatrix},\ C_z = \begin{bmatrix} C \\ 0 \end{bmatrix},$$

and

$$D_{zu} = \begin{bmatrix} D \\ I \end{bmatrix},\ D_{yw} = \begin{bmatrix} D & I \end{bmatrix},\ D_{yu} = D.$$

Observe that under the conditions of the theorem statement, the pairs (A, B_w) and (A, B) are controllable. To see this, first note that controllability of (A, B) implies controllability of (A, B_w). Since P is invertible, controllability of (A, B) implies that $(-PAP^{-1}, PB)$ is also controllable. Therefore $(B^\mathsf{T} P, -P^{-1}A^\mathsf{T} P)$ is observable, which implies that (C, A) is observable by (4). Observability of (C, A) then implies observability of (C_z, A). Furthermore the matrices D_{zu} and D_{yw} are full rank. Therefore the H_2 and H_∞ solutions to Problem 1 can be tackled within the Riccati equation framework (see for example the requirements on p.383–384 of [Zhou et al., 1996]). We will first prove the result under the simplifying assumption that $D = 0$. In this case the generalised plant in (5) meets the conditions of [Doyle et al., 1989, §III.A]. This simplifies the Riccati equations that must be solved significantly, and illustrates the key steps of the proof. We will then show how to extend the approach for $D \neq 0$ in the H_2 case.

The H_2 case: Let X denote the unique stabilising solution of

$$XA + A^{\mathsf{T}}X - XBB^{\mathsf{T}}X + C_z^{\mathsf{T}}C_z = 0, \tag{6}$$

and Y denote the unique stabilising solution of

$$YA^{\mathsf{T}} + AY - YC^{\mathsf{T}}CY + B_wB_w^{\mathsf{T}} = 0. \tag{7}$$

By [Doyle et al., 1989, Theorem 1], $\gamma^*_{H_2} = \sqrt{\|G_{\text{ctl}}(s)\|^2_{H_2} + \|G_{\text{obs}}(s)\|^2_{H_2}}$, where

$$\begin{aligned} G_{\text{ctl}}(s) &= \left(C_z - D_{zu}B^{\mathsf{T}}X\right)\left(sI - A + BB^{\mathsf{T}}X\right)^{-1}B_w, \\ G_{\text{obs}}(s) &= B^{\mathsf{T}}X\left(sI - A + YC^{\mathsf{T}}C\right)^{-1}\left(B_w - YC^{\mathsf{T}}D_{yw}\right). \end{aligned}$$

By [Zhou et al., 1996, Corollary 13.8] X and Y correspond to the unique positive-semidefinite solutions to the given Riccati equations. Since $B_wB_w^{\mathsf{T}} = BB^{\mathsf{T}}$ and $C_z^{\mathsf{T}}C_z = C^{\mathsf{T}}C$, we then see by comparison with (4) that

$$X = P \text{ and } Y = P^{-1}. \tag{8}$$

Substituting the above into the expressions for $G_{\text{ctl}}(s)$ and $G_{\text{obs}}(s)$ shows that

$$\begin{aligned} G_{\text{ctl}}(s) &= \begin{bmatrix} C \\ -C \end{bmatrix}(sI - A + BC)^{-1}\begin{bmatrix} B & 0 \end{bmatrix}, \\ G_{\text{obs}}(s) &= C(sI - A + BC)^{-1}\begin{bmatrix} B & -B \end{bmatrix}. \end{aligned}$$

Therefore

$$\gamma^*_{H_2} = \sqrt{2\|G_{\text{obs}}(s)\|^2_{H_2}} = \sqrt{2\,\text{tr}\left(CZC^{\mathsf{T}}\right)},$$

where Z is the solution to the Lyapunov equation

$$(A - BC)Z + Z\left(A^{\mathsf{T}} - C^{\mathsf{T}}B^{\mathsf{T}}\right) + 2BB^{\mathsf{T}} = 0.$$

By comparison with (4) we see that $Z = P^{-1}$, which implies that

$$\gamma^*_{H_2} = \sqrt{2\,\text{tr}\left(CP^{-1}C^{\mathsf{T}}\right)} = \sqrt{2\,\text{tr}(CB)}$$

as required.

The H_∞ case: Whenever such solutions exist, let X_γ denote the unique stabilising solution of

$$X_\gamma A + A^\mathsf{T} X_\gamma - X_\gamma \left(BB^\mathsf{T} - \gamma^{-2} B_w B_w^\mathsf{T}\right) X_\gamma + C_z^\mathsf{T} C_z = 0,$$

and Y_γ denote the unique stabilising solution of

$$Y_\gamma A^\mathsf{T} + AY_\gamma - Y_\gamma \left(C^\mathsf{T} C - \gamma^{-2} C_z^\mathsf{T} C_z\right) Y_\gamma + B_w B_w^\mathsf{T} = 0.$$

By [Doyle et al., 1989, Theorem 3], $\|T_{zw}\|_{H_\infty} < \gamma$ if and only if X_γ and Y_γ exist, and the magnitude of the largest eigenvalue of $X_\gamma Y_\gamma$ is less than γ^2. By [Zhou et al., 1996, Corollary 13.8], if $\gamma > 1$ then X_γ and Y_γ exist, and correspond to the unique positive-semidefinite solutions to the given Riccati equations. Comparison with (4) shows that

$$X_\gamma = \frac{\gamma}{\sqrt{\gamma^2-1}} P, \text{ and } Y_\gamma = \frac{\gamma}{\sqrt{\gamma^2-1}} P^{-1}.$$

Therefore the magnitude of the largest eigenvalue of $X_\gamma Y_\gamma$ equals $\gamma^2/\left(\gamma^2-1\right)$, which implies that $\gamma_{H_\infty}^* = \sqrt{2}$.

The case $D \neq 0$: Loop shifting can be used to extend the above arguments to allow for nonzero D. As explained in [Zhou et al., 1996, p.453–454], the H_2 optimal control problem with generalised plant as in (5) is equivalent to the H_2 optimal control problem with generalised plant

$$\begin{bmatrix} \dot{x} \\ \tilde{z} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} A & B_w & BS \\ C_z & 0 & \begin{bmatrix} D \\ I \end{bmatrix} S^{-1} \\ RC & R^{-1}\begin{bmatrix} D & I \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} x \\ \tilde{w} \\ \tilde{u} \end{bmatrix},$$

where $R = \left(I + DD^\mathsf{T}\right)^{\frac{1}{2}}$ and $S = \left(I + D^\mathsf{T} D\right)^{\frac{1}{2}}$. This generalised plant meets the conditions on [Zhou et al., 1996, p.384]. Therefore $\gamma_{H_2}^*$ can be determined using arguments based on Riccati equations as described in the $D = 0$ case, but with the Riccati equations in (6) and (7) replaced with their 'generalised' counterparts, as specified in [Zhou et al., 1996, Theorem 14.7]. These also admit the solutions in (8), resulting in the same optimal value for $\gamma_{H_2}^*$. □

3. Implications for Power System Control

In this section we present and discuss the application of Theorem 1 to simple models of electric power systems at the transmission and sub-transmission level. These results, although preliminary since they lack the support of a detailed numerical study from a realistic power system model, clearly point to the fundamental role played by inertia when considering the control of power systems.

3.1 Applying Theorem 1 to Swing Equation Models

In the absence of damping, after linearisation, the swing equation power system model is described by the equations

$$\begin{aligned}&M_k\ddot{\theta}_k = p_{\mathrm{N},k} + u_k + w_{u,k}, \begin{bmatrix}\theta_k(0)\\ \dot{\theta}_k(0)\end{bmatrix} = 0,\ k \in \{1,\ldots,n\},\\ &\begin{bmatrix}p_{\mathrm{N}}\\ 0\end{bmatrix} = -\begin{bmatrix}K_{\mathrm{a}} & K_{\mathrm{b}}\\ K_{\mathrm{b}}^{\mathsf{T}} & K_{\mathrm{c}}\end{bmatrix}\begin{bmatrix}\theta\\ \theta_{\mathrm{int}}\end{bmatrix}.\end{aligned} \tag{9}$$

In the above, the first equation describes the dynamics at a set of n generator buses. $M_k > 0$ denotes the inertia parameter of the kth generator bus, $p_{\mathrm{N},k}$ the power injection from the transmission network, u_k an adjustable power injection, and $w_{u,k}$ a power disturbance. The variable θ gives the vector of electrical angles at the generator buses, and θ_{int} the angles at the remaining buses. The second equation describes how these relate to the power injections p_{N} as defined by the dynamics of the transmission network. In particular the block matrix with entries $K_{\mathrm{a}} \in \mathbb{R}^{n\times n}$, $K_{\mathrm{b}} \in \mathbb{R}^{n\times m}$ and $K_{\mathrm{c}} \in \mathbb{R}^{m\times m}$, is a weighted Laplacian matrix, with edge weights determined by the line susceptances and load angles across the lines at equilibrium.

We consider the problem of how to select the adjustable power injections u, when a set of noisy angular frequency measurements

$$y_k = \dot{\theta}_k + w_{y,k},\ k \in \{1,\ldots,n\}, \tag{10}$$

where $w_{y,k}$ denotes the measurement noise, are available. In the remainder of this subsection, we will show that under a set of mild assumptions this problem fits naturally into the framework of Problem 1. Furthermore, for an appropriate choice of the system state, all the conditions of Theorem 1 are satisfied, and the expression for $\gamma^*_{H_2}$ simplifies further.

The required assumptions are:

A1 The transmission network is connected.

A2 The load angle across every transmission line has magnitude less than 90°.

A1 is essentially without loss of generality, since if the network is not connected, every connected component can be analysed separately. A2 is required to ensure the weighted edges in the Laplacian are positive. A2 will likely be satisfied in practice, since operational limits on transmission lines typically preclude load angles greater than 45° [Kundur, 1994][p.230].

Under A1–A2, the matrix $K_{\mathrm{red}} = K_{\mathrm{a}} - K_{\mathrm{b}}K_{\mathrm{c}}^{-1}K_{\mathrm{b}}^{\mathsf{T}}$ can be factored as $K_{\mathrm{red}} = LL^{\mathsf{T}}$, where $L \in \mathbb{R}^{n\times n-1}$ has full (column) rank. Letting $M \in \mathbb{R}^{n\times n}$ denote the diagonal matrix with kth diagonal entry equal to M_k and $x = \begin{bmatrix}\dot{\theta}^{\mathsf{T}} & \theta^{\mathsf{T}}L\end{bmatrix}^{\mathsf{T}}$, it then follows that

(9) and (10) can be written as

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & -M^{-1}L \\ L^{\mathsf{T}} & 0 \end{bmatrix} x + \begin{bmatrix} M^{-1} \\ 0 \end{bmatrix} (u + w_u), \, x(0) = 0, \\ y &= \begin{bmatrix} I & 0 \end{bmatrix} x + w_y. \end{aligned} \tag{11}$$

These equations take the form of the first and third equations in (1). Therefore in the context of the swing equation power system model, Problem 1 corresponds to searching for a control law to minimise the effects of power disturbances and measurement noise on the deviations in electrical frequency and control effort.

Setting

$$P = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}$$

shows that the conditions of Theorem 1 are satisfied. The controllability assumption is satisfied since the controllability matrix equals

$$\begin{bmatrix} M^{-1} & 0 & \cdots \\ 0 & L^{\mathsf{T}}M^{-1} & \cdots \end{bmatrix} \in \mathbb{R}^{(2n-1)\times n(2n-1)},$$

which has rank equal to $2n-1$ (both M and L have full rank). Furthermore, the expression for the H_2 norm in Theorem 1 simplifies to

$$\gamma^*_{H_2} = \sqrt{2\left(\frac{1}{M_1} + \cdots + \frac{1}{M_n}\right)}. \tag{12}$$

This implies that $\gamma^{*2}_{H_2} / n = 2 / \mathrm{HM}(M_1, \ldots, M_n)$, where HM denotes the harmonic mean. This indicates that the fundamental limit on H_2 performance scales with the inverse of the harmonic mean of the inertias in the system.

Remark 2 The lossless assumption is reasonably well justified in the power system context when considering generation and consumption at the transmission and sub-transmission level (at least from the perspective of control system design, where qualitative, simple modelling is often more appropriate). Observe also that damping effects that arise from control actions are captured by (9), since this equation includes adjustable power injections (this implies for example that droop-control cannot achieve better performance than the levels specified by Theorem 1, since a droop controller is a special case of (2)).

Nevertheless, the modelling setup considered here is highly simplified, so the given expressions should be used to provide insights, and supplement other analysis approaches. In particular neglecting voltage dynamics (as is implicit in any analysis based on the swing equations) is questionable. That said, high fidelity power system models seem to be close to lossless. For example the resistances in the equivalent circuit descriptions of synchronous machines and transmission lines in [Kundur, *1994, Chapters 5–6] are very small (for reasons of space this line will not be pursued further here, it is interesting to think how (12) would generalise if more sophisticated models were used). It should also be stressed that the model considered is not suitable for distribution networks where losses can be significant.*

3.2 The Impact of Increasing Renewable Generation

In the power system context, the increased use of renewables is typically associated with a reduction in system inertia, an increase in stochastic disturbances, and a larger number of different components. In this subsection we interpret the effects of these trends within the context of (12). These results apply under the tacit assumption that frequency measurements are being used for control (viz. (10)). This is the status quo in practice. However it is being increasingly recognised that different approaches are required to handle the renewable transition. One possible interpretation of the results in this section is to provide further support for this, by revealing the presence of fundamental performance limitations that scale poorly with the reduction and increased heterogeneity of inertia throughout a system.

Reduced inertia and increased stochastics: When conventional power generation is replaced by renewable sources such as wind, the total inertia in a power system is reduced. This is because the synchronous machines used in conventional generation have considerable mass. In contrast, the inertia in a wind turbine is typically electrically decoupled from the grid, so contributes relatively little inertia by comparison, and even if the power electronics in a wind turbine are used to emulate the dynamics of conventional generator (by operating the turbine as a virtual synchronous machine), the level of synthetic inertia that can be realised is typically far smaller.

The effect of reducing the inertia of the generation sources can be captured by reducing the sizes of the constants M_k. Interestingly this does not affect $\gamma^*_{H_\infty}$, which suggests that performance with respect to worst case disturbances may not degrade. However it is easily seen from (12) that reducing the value of any M_k will increase $\gamma^*_{H_2}$. Since the size of the H_2 norm captures how stochastic disturbances are amplified (which will become more prevalent with an increased use of renewables), this suggests that power system performance may be adversely affected by the increased use of renewables. Furthermore these performance limitations depend in a fundamental way on, for example, the sizes of the virtual inertia constants that can be synthesised, emphasising the importance of larger inertia constants in attenuating stochastic disturbances.

Increased system heterogeneity: A secondary effect of increasing the use of renewables is that the inertia parameters in (9) will cover a wider range of values. Jensen's inequality implies that

$$\sum_{k=1}^{n} \frac{1}{M_k} \geq \sum_{k=1}^{n} \frac{1}{M_{\text{tot}}/n}, \text{ where } M_{\text{tot}} = \sum_{k=1}^{n} M_k, \tag{13}$$

and equality is achieved only if $M_1 = M_2 = \cdots = M_n$. Therefore given a fixed total amount of inertia in the system (constant M_{tot}), the more similar the individual inertia parameters are (the closer the M_k's are to M_{tot}/n), the smaller $\gamma^*_{H_2}$ is. Conversely, the more heterogeneous the set of masses, the larger $\gamma^*_{H_2}$ becomes. This indicates that the sensitivity of the system to stochasticity may be further exacerbated as renewable sources are added purely as a result of having components with a wider range of inertia parameters present in the system.

3.3 Optimal Control Structures

Condensing notions of system performance down to numbers such as $\gamma^*_{H_2}$ and $\gamma^*_{H_\infty}$, especially when the system in question is extremely large, rarely tells the full story. In this subsection we investigate performance of the swing equation power system model (11) further by studying the H_2 and H_∞ norms of the sub-matrices of $T_{zw}(s)$ associated with individual disturbances and output signals. In particular, denoting

$$\tilde{w}_k = \begin{bmatrix} w_{u,k} \\ w_{y,k} \end{bmatrix} \text{ and } \tilde{z}_k = \begin{bmatrix} (Cx + Du)_k \\ u_k \end{bmatrix},$$

we study

$$\gamma_{H_2,ik} = \left\|T_{\tilde{z}_i \tilde{w}_k}(s)\right\|_{H_2} \text{ and } \gamma_{H_\infty,ik} = \left\|T_{\tilde{z}_i \tilde{w}_k}(s)\right\|_{H_\infty},$$

where $T_{\tilde{z}_i \tilde{w}_k}(s)$ is the closed loop transfer function from $\tilde{w}_k$ to $\tilde{z}_i$. To do so of course requires a choice of control law. We apply the optimal control laws for Problem 1. It turns out that in both the H_2 and H_∞ case these are highly structured. More specifically (when $D = 0$, as is the case in (11)), in the H_2 case

$$K(s) = -C(sI - A + 2BC)^{-1} B$$

is optimal for Problem 1, and in the H_∞ case $K(s) = -\sqrt{2}I$ is optimal for Problem 1 (for a discussion on how to synthesise these types of control laws in a structure exploiting manner, see [Pates, 2022]).

Model description: In order to investigate performance, swing equation power system models were randomly generated (the code used to do this can be found at https://github.com/Johan-Lindb/L-CSS22). Note that the purpose here is to investigate trends, and the numerical values obtained from the model should not be compared with results from the power system literature. Network topologies were generated by first randomly specifying the locations of 10 clusters, with a random number of generators or loads, on a map. The loads represent a collection of several loads in a distribution system and are considered to be constant. The total number of buses was selected to be 100. Within each cluster, the buses were connected through a minimum spanning tree representing a sub-transmission network. Using eigenvector centrality, the most central bus in each cluster was selected and then connected to the central buses in the other clusters. This represents the transmission network of a power system and was created to give $n-1$ contingency, which ensures that if one of the lines or clusters were taken out, the other clusters would still be connected. An example of a system that was generated is given in Fig. 2.

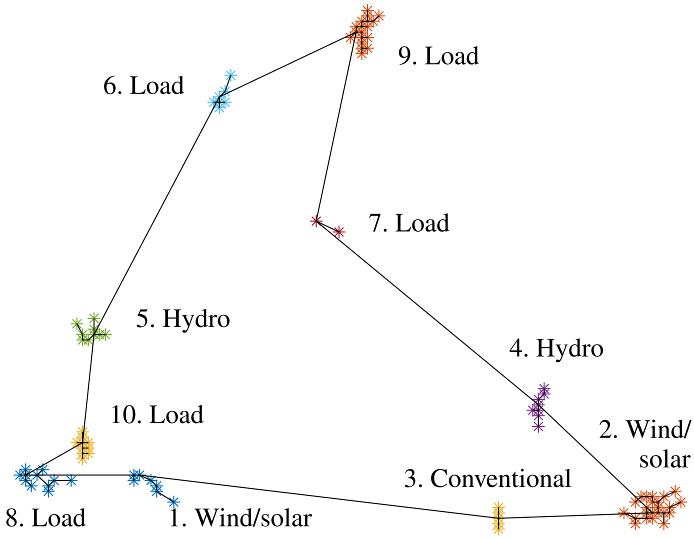
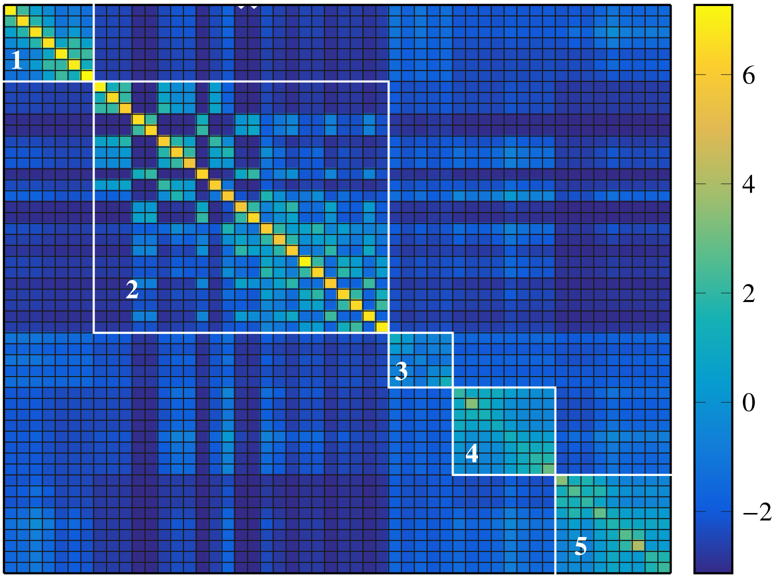


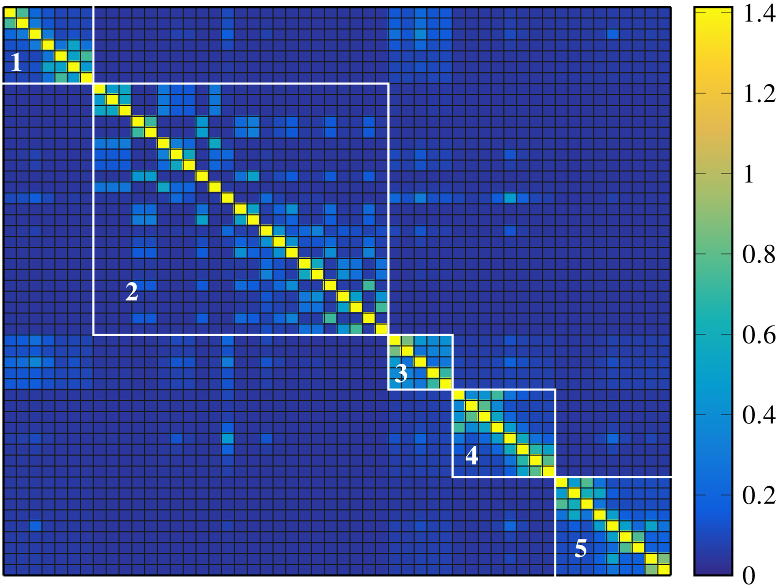
Figure 2. Example of a network with 100 buses.

The power consumption/production of each bus was selected at random, with probability of higher magnitude in the centre of the clusters. The power producing clusters were assigned to be either conventional, hydro or wind/solar. The solar/wind generation areas are large farms with power outputs in comparison with hydro power generators or conventional generators, like large off-shore wind farms. The inertia parameter M_k at each generator bus was set to be proportional to the product of the rated power of the generator, and a constant related to the type of generator. For conventional and hydro buses, these constants were chosen to be 6 and 3 respectively [Ørum et al., 2018]. Renewables such as solar and wind typically have much lower inertias (resulting either from shunt capacitances, or designing power electronics to synthesise inertia). To reflect this, the constant for wind/solar buses was chosen to be one one-thousandth of the conventional generator constant. Other inertias were also tested. For inertias a factor of ten smaller than that of conventional and hydro power, the conclusions below still hold.

The entries of the weighted Laplacian were determined based on the line parameters of the networks. The line parameters of the lines in the sub-transmission and the transmission networks were assigned to handle all power flows, meaning that under normal operation the phase angle would be far below the 45° limit [Kundur, 1994], and for the transmission network also under $n-1$ contingency. Since the reduced Laplacian K_{red} affects neither $\gamma^*_{H_2}$ nor $\gamma^*_{H_\infty}$, highly simplified network modelling seems sufficient.



(a)



(b)

Figure 3. (a)–(b) $\ln(\gamma_{H_2,ik})$ and $\gamma_{H_\infty,ik}$, respectively, for the power systems model in Fig. 2.

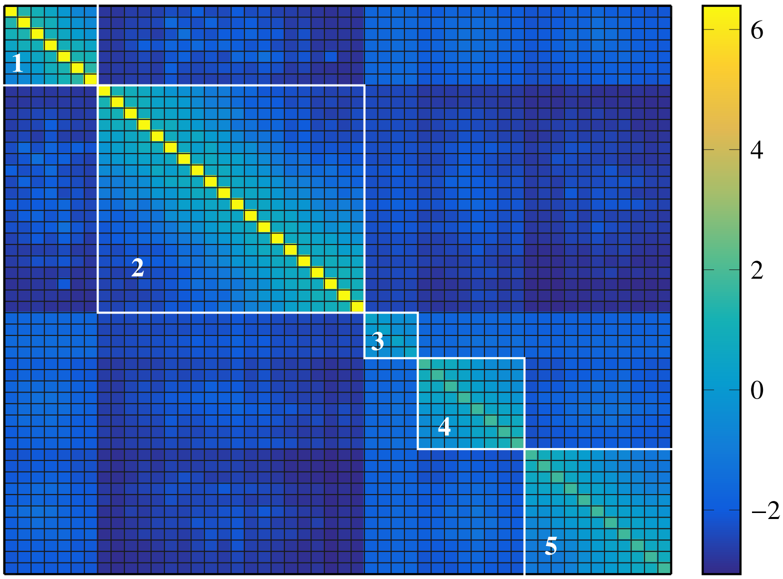


Figure 4. The natural log of the average of $\gamma_{H_2,ik}$ for 100 different power system models with the same number of buses in each cluster.

Performance of the optimal control laws: Consider now the network in Fig. 2, with power generation and inertia assigned to the buses as described above. Fig. 3a shows $\ln(\gamma_{H_2,ik})$. Here the wind/solar generators are in the first two clusters. Observe that there are orders of magnitude differences between different values of $\gamma_{H_2,ik}$. Almost all sensitivity to the disturbances is isolated in the buses where the solar/wind generators are located. There is some disturbance sensitivity between different buses within a cluster, but between different clusters there is almost no disturbance sensitivity. This means that whereas a lossless power system model with renewable power generation is locally sensitive to stochastic disturbances at the buses with very little inertia, the optimal control law keeps the effect of the disturbances local. Fig. 3b shows $\gamma_{H_\infty,ik}$. As expected from Theorem 1 $\gamma_{H_\infty,kk} = \gamma^*_{H_\infty} = \sqrt{2}$. Just as in the H_2 case, the H_∞ optimal control law, despite being completely decentralised, keeps the effects of the disturbances mainly local.

The system in Fig. 2 is just one example of a power system model. To investigate if the same conclusions would hold for other networks, 100 different networks were generated. To make the results comparable, clusters 1–2 always contained wind/solar generation, 3 conventional power generation, 4–5 hydro power generation, and 6–10 constant power loads. The sizes of the clusters were also fixed. In Fig. 4 the natural logarithm of the average of $\gamma_{H_2,ik}$ of the 100 networks is shown. The conclusions from before still hold. This was also true for the H_∞ case, but the plot is omitted

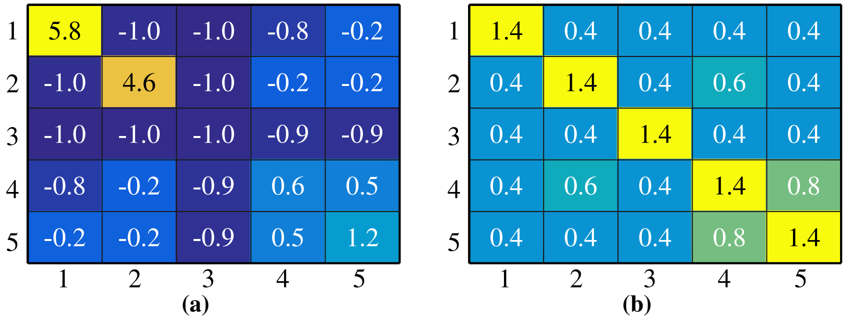


Figure 5. (a) $\ln(\gamma_{H_2,ik})$ for the lumped model of the power system in Fig. 2. (b) $\gamma_{H_\infty,ik}$ for the lumped model of the power system in Fig. 2. In both subfigures, the indexing corresponds to the numbering of the clusters in Fig. 2.

since it is very similar to Fig. 3b.

Lumped models: In power systems engineering it is very common to work with aggregated models, where several buses are lumped together. We now investigate a version of the model from the previous subsection, in which the buses within each cluster are lumped together into a single bus, with inertia and power production/consumption equal to the sum in the given cluster. The weighed Laplacian was determined using only the lines in the transmission network. The resulting gains are shown in Figs. 5a and 5b.

In Fig. 5a it can be seen that a disturbance in a cluster affects that cluster the most. However, the H_2 gain is dramatically decreased from the corresponding levels in Fig. 3a. This is a result of the heterogeneity within the cluster, as explained by (13). The conclusion is that lumping models can give good insight into some aspects of the dynamics, but that the disturbance sensitivity of individual buses can be severely underestimated. Fig. 5b shows that lumping does not affect the performance metric when looking at the H_∞ norm. This suggests that when looking at worst case disturbances, lumping can still give good insights.

4. Conclusions

Analytical solutions to two optimal control problems involving lossless systems have been given, and interpreted in the context of electric power systems. Simplified models were used to demonstrate the presence of a fundamental performance limit that scales poorly with the model parameter changes associated with the increased use of renewables. This provides control theoretic evidence for the need for more sophisticated control methods and techniques to support the renewable energy transition.

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Paper II

On the H_2 Optimal Control of Uniformly Damped Mass-Spring Systems

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Abstract

In this paper we provide an analytical solution to an H_2 optimal control problem, that applies whenever the process corresponds to a uniformly damped network of masses and springs. The solution covers both stable and unstable systems, and illustrates analytically how damping affects the levels of achievable performance. Furthermore, the resulting optimal controllers can be synthesised using passive damped mass-spring systems, allowing for controller implementations without an energy source. We investigate the impact of both positive and negative damping through a small numerical example.

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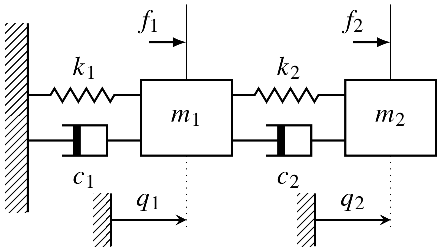

Figure 1. Example of a mass-spring network, with damped springs.

1. Introduction

In this paper we study an H_2 optimal control problem for a process with dynamics modelled by the linear differential equations

$$M\ddot{q}+C\dot{q}+Kq=f,\ q(0)=\dot{q}(0)=0, \tag{1}$$

under the restriction that the matrix M is positive definite, and K is positive semi-definite. This is a prototypical setup for the dynamics of a network of damped masses and springs, such as that illustrated in Fig. 1, when linearised around an equilibrium point. The variable q is a vector of generalised coordinates describing the configuration of the system relative to equilibrium, and f a vector of forces applied at those coordinates. The entries of M and K can typically be determined directly from the expressions for the kinetic and potential energy for the system, and our particular focus is on exploring the impact of the matrix C on the optimal control problem, and the corresponding optimal control law.

The motivation for studying this model class stems from the fact that networks of masses and springs are frequently used to model engineering processes, with applications ranging from electrical power systems, to vehicle suspension systems, to optimization algorithms [Dörfler et al., 2013; Hedrick and Butsuen, 1990; Bhaya and Kaszkurewicz, 2006]. Key to their importance is the balance they strike between simplicity and versatility. On the one hand, models of even very large systems can be systematically built up through simple descriptions of the underlying physics. Yet despite this structural simplicity, the resulting models can still describe a very rich range of behaviours, including resonances across a wide range of time and length scales, and even instability when allowing for negative damping.

In addition to their practical relevance, linear mass-spring networks also have a range of desirable theoretical properties. These are particularly striking in lossless case (namely when $C=0$), where for example central control theoretic results such as the Kalman-Yakubovich-Popov lemma simplify greatly, and the dynamics in (1) can be realised with highly structured state-space realisations [Willems, 1972]. These features can be exploited to simplify optimal control problems for lossless systems, which can result in optimal control laws that can both be described analytically and synthesised with simple passive networks [Pates, 2022; Lindberg and Pates, 2023].

When damping is introduced, the underlying theoretical properties of (1) become significantly more complex (see for example [Tisseur and Meerbergen, 2001; Gohberg et al., 1982] and the references therein for a discussion of the quadratic eigenvalue problem). However if the damping is *uniform*, many of the desirable properties from the lossless case are preserved. Mathematically, uniform damping means that C is symmetric and satisfies $CM^{-1}K = KM^{-1}C$. This condition is equivalent to the existence of a congruence transformation that decouples (1) into a set of orthogonal modes [Caughey and O'Kelly, 1965] (we discuss this transformation further in section 3). Although an assumption of convenience, uniform damping is surprisingly versatile, and will be satisfied by any symmetric C matrix on the form

$$C = Mg\left(M^{-1}K\right),$$

where g is a polynomial (or entire) function. Note in particular that such a C need not be positive semi-definite, and so uniformly damped systems need not be stable. Special cases include Rayleigh damping [Rayleigh, 1877], where $C = \alpha_0 M + \alpha_1 K$ for $\alpha_0, \alpha_1 \in \mathbb{R}$. Given the difficulties in modelling damping phenomena from first principles, uniform models of damping are often adopted, at least as a first approximation, and methods for estimating uniform damping matrices from data are discussed in [Adhikari and Phani, 2007].

In this paper we study a natural H_2 optimal control problem for (1) under the uniform damping assumption. We start by developing analytical results for a highly structured optimal control problem. This is presented as Theorem 2 in section 2. In section 3 we show how to exploit the uniform damping assumption to apply this result to solve optimal control problems for systems described by (1), and discuss and illustrate the structure in the resulting optimal controllers. A striking feature of the obtained optimal controller is that it has the same structure as the problem itself! For the damped mass-spring network in (1) this means that the optimal control law is itself a damped mass-spring network. The theorems give an analytical solution to problems with uniform damping, both positive and negative. Thus the optimal controller obtained works for both stable and unstable damped mass-spring networks. The H_2-gain from disturbances to performance outputs under optimal control is expressed in network matrices. The analytical nature of the result allows it to be well suited for large scale networks with damped mass-spring dynamics, like power system networks.

Notation

$\sqrt{E}$ denotes the unique positive semi-definite matrix square root of a positive semi-definite matrix E. The H_2 norm of a stable transfer function $G(s)$ is defined as

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{tr}\left(G(j\omega)^* G(j\omega)\right) d\omega\right)^{1/2}. \tag{2}$$

2. An Analytical Solution to an H_2 Optimal Control Problem

In this section we study an H_2 optimal control problem for the feedback loop in Fig. 2. This is a natural setup, in which the objective is to design a controller $K(s)$ to minimise the effects of process and sensor disturbances on the control effort and process output, as quantified by the H_2 norm. We will provide an analytical solution to this problem under a set of strict assumptions on the state-space realisation of the process transfer function $G(s)$. However in section 3 we will show how to use this result to obtain an analytical expression for the optimal control law when $G(s)$ instead describes the dynamics of a uniformly damped mass-spring system.

Problem 2 Let

$$\begin{aligned} \dot{x}_{\mathrm{G}} &= A_{\mathrm{G}}x_{\mathrm{G}} + B_{\mathrm{G}}(u + w_{\mathrm{u}}), \; x_{\mathrm{G}}(0) = 0, \\ z &= \begin{bmatrix} y_{\mathrm{G}} \\ u \end{bmatrix} = \begin{bmatrix} C_{\mathrm{G}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_{\mathrm{G}} \\ u \end{bmatrix}, \\ y &= C_{\mathrm{G}}x_{\mathrm{G}} + w_{\mathrm{y}}, \end{aligned} \tag{3}$$

where the matrices A_{G} B_{G} and C_{G} have the following structure

$$A_{\mathrm{G}} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}, \; B_{\mathrm{G}} = \begin{bmatrix} Q \\ 0 \end{bmatrix}, \; C_{\mathrm{G}} = \begin{bmatrix} Q^{\mathsf{T}} & 0 \end{bmatrix}, \tag{4}$$

with the following properties of the sub-matrices:

- *A_{11} is diagonal and square of size $n \times n$;*
- *A_{12} is $n \times m$, where $0 \le m \le n$ and only has entries on the main diagonal and these entries are non-zero;*
- *$A_{21} = -A_{12}^{\mathsf{T}}$;*
- *Q has the property that $QQ^{\mathsf{T}} = I_n$ and is of size $n \times p$;*
- *the 0 in A_{G} is of size $m \times m$, the 0 in B_{G} is $m \times p$ and the 0 in C_{G} $p \times m$, with $p \ge n$.*

Suppose also that the controller $K(s)$ can be described by the state-space system

$$\begin{aligned} \dot{x}_{\mathrm{K}} &= A_{\mathrm{K}}x_{\mathrm{K}} + B_{\mathrm{K}}y, \; x_{\mathrm{K}}(0) = 0, \\ u &= C_{\mathrm{K}}x_{\mathrm{K}} + D_{\mathrm{K}}y. \end{aligned} \tag{5}$$

Define $T_{zw}(s)$ as the closed loop transfer function from $w = \begin{bmatrix} w_{\mathrm{u}}^{\mathsf{T}} & w_{\mathrm{y}}^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}$ to z described by (3) and (5). Find

$$\gamma_{H_2}^{*} = \inf \left\{ \gamma : \|T_{zw}(s)\|_{H_2} < \gamma \right\},$$

where the infimum is taken over A_{K}, B_{K}, C_{K}, and D_{K}. ◊

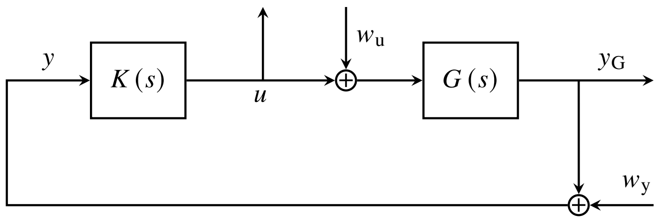

Figure 2. Illustration of Problem 2, with external disturbances w_u and w_y acting on the system. The aim is to minimize the effects of the disturbances on the outputs y_G and u.

Remark 3 In (4) the state vector x_G has length $n+m$, and the length of the input vector u is p. The slightly unconventional naming of the size of the state vector will be explained by the type of problems this setup can describe, in section 3. ◊

Theorem 2
Under the conditions of Problem 2,

$$\gamma^*_{H_2} = \sqrt{\operatorname{tr}\left(Z^3\right) + \operatorname{tr}(Z)}, \tag{6}$$

where $Z = A_{11} + \sqrt{A_{11}^2 + I}$, and an optimal controller is given by:

$$\begin{aligned} \dot{x}_K &= \begin{bmatrix} A_{11} - 2Z & A_{12} \\ A_{21} & 0 \end{bmatrix} x_K + \begin{bmatrix} ZQ \\ 0 \end{bmatrix} y,\ x_K(0) = 0, \\ u &= \begin{bmatrix} -Q^\mathsf{T} Z & 0 \end{bmatrix} x_K. \end{aligned} \tag{7}$$

Proof. Introduce the generalised plant

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A_G & B_w & B_G \\ C_z & 0 & D_{zu} \\ C_G & D_{yw} & D_{yu} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \tag{8}$$

where

$$B_w = \begin{bmatrix} B_G & 0 \end{bmatrix},\ C_z = \begin{bmatrix} C_G \\ 0 \end{bmatrix},$$

and

$$D_{zu} = \begin{bmatrix} 0 \\ I \end{bmatrix},\ D_{yw} = \begin{bmatrix} 0 & I \end{bmatrix},\ D_{yu} = 0.$$

Under the conditions of Problem 2, the pair (A_G, B_w) is controllable and the pair (C_z, A_G) is observable. To see this, first note that since $QQ^\mathsf{T} = I$, thus $\operatorname{rank}(Q) = n$.

Since the diagonal elements in A_{21} are all non-zero, $\operatorname{rank}(A_{21}) = m$. The two first sub-matrices of the controllability matrix are

$$\begin{bmatrix} B_G & A_G B_G \end{bmatrix} = \begin{bmatrix} Q & A_{11}Q \\ 0 & A_{21}Q \end{bmatrix}. \tag{9}$$

Therefore controllability matrix has rank $n+m$, which equals the state dimension. Thus (A_G, B_G) is controllable, which also implies that (A_G, B_w) controllable. The same argument shows that (C_G, A_G) and (C_z, A_G) are observable. Furthermore the matrices D_{zu} and D_{yw} are full rank. Therefore the H_2 solution to Problem 2 can be tackled within the Riccati equation framework of [Doyle et al., 1989].

Let X denote the unique stabilising solution of

$$XA_G + A_G^\mathsf{T} X - XB_G B_G^\mathsf{T} X + C_z^\mathsf{T} C_z = 0, \tag{10}$$

and Y denote the unique stabilising solution of

$$YA_G^\mathsf{T} + A_G Y - YC_G^\mathsf{T} C_G Y + B_w B_w^\mathsf{T} = 0. \tag{11}$$

The solutions X and Y can be used to calculate $\gamma^*_{H_2}$ and define the optimal control laws. We will show under the conditions of Problem 2 that X and Y can be found analytically.

Ansatz: X is a diagonal matrix. Introduce

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

where the dimensions of the sub-matrices of X match those of A_G. Rewriting (10) in terms of the sub-matrices from Problem 2 (recall that $A_{22} = 0$), (10) is reduced to:

$$\begin{aligned} X_1 A_{11} + A_{11}^\mathsf{T} X_1 - X_1^2 + I &= 0, \\ X_1 A_{12} + A_{21}^\mathsf{T} X_2 &= 0, \\ X_2 A_{21} + A_{12}^\mathsf{T} X_1 &= 0. \end{aligned} \tag{12}$$

Since A_{11}, X_1 and X_2 are all diagonal, the first equation reduces to n scalar quadratic equations, with unique positive definite solution $x_{1,i} = a_{11,i} + \sqrt{a_{11,i}^2 + 1}$, where $x_{1,i}$ denotes the ith diagonal element in X_1, and $a_{11,i}$ the ith diagonal element in A_{11}. Further, using that $A_{21} = -A_{12}^\mathsf{T}$, the final two equations in (12) reduce to equations $x_{2,i} = x_{1,i}$ for $i = 1, \ldots, m$. Inserting these solutions into one diagonal matrix gives:

$$X = \begin{bmatrix} Z & 0 \\ 0 & Z_m \end{bmatrix}, \tag{13}$$

where $Z = A_{11} + \sqrt{A_{11}^2 + I}$, and Z_m is Z truncated to the first m rows and columns. Since (A_G, B_w) is stabilisable and (C_z, A_G) is detectable the unique stabilising solution to (10) is equal to the unique positive semi-definite solution to (10) [Zhou and Doyle, 1999, Corollary 12.5]. Since the X we have found is positive definite, it is therefore the sought stabilising solution. Equation (11) can be solved in an analogous manner. Note that $B_G B_G^\top = C_G^\top C_G$ and $C_z^\top C_z = B_w B_w^\top$. This gives the same solution for Y, namely that $Y = X$.

By [Doyle et al., 1989, Theorem 1],

$$\gamma_{H_2}^* = \sqrt{\|G_a(s)\|_{H_2}^2 + \|G_b(s)\|_{H_2}^2},$$

where

$$\begin{aligned} G_a(s) &= B_G^\top X \left(sI - A_G + Y C_G^\top C_G\right)^{-1} \left(B_w - Y C_G^\top D_{yw}\right) \\ &= \begin{bmatrix} Q^\top Z & 0 \end{bmatrix} \left(sI - \begin{bmatrix} A_{11} - Z & A_{12} \\ A_{21} & 0 \end{bmatrix}\right)^{-1} \begin{bmatrix} Q & -ZQ \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} G_b(s) &= \left(C_z - D_{zu} B_G^\top X\right) \left(sI - A_G + B_G B_G^\top X\right)^{-1} B_w \\ &= \begin{bmatrix} Q^\top & 0 \\ -Q^\top Z & 0 \end{bmatrix} \left(sI - \begin{bmatrix} A_{11} - Z & A_{12} \\ A_{21} & 0 \end{bmatrix}\right)^{-1} \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

The H_2-norm of a state-space model can be calculated using either its controllability or its observability gramian. More specifically, if $G_g(s) = C_g(sI - A_g)^{-1} B_g$ is stable, its H_2-norm is given by

$$\|G_g(s)\|_{H_2}^2 = \operatorname{tr}\left(C_g L_g^c C_g^\top\right) = \operatorname{tr}\left(B_g^\top L_g^o B_g\right),$$

where L_g^o is the observability gramian, and L_g^c is the controllability gramian. These matrices are in turn given by the positive definite solutions to the two Lyapunov equations

$$A_g L_g^c + L_g^c A_g^\top + B_g B_g^\top = 0 \text{ and } A_g^\top L_g^o + A_g L_g^o + C_g^\top C_g = 0 \tag{14}$$

We will start by finding the controllability gramian for $G_a(s)$, which we denote L_a^c.

Ansatz: L_a^c is diagonal. Introducing

$$L_a^c = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix},$$

with dimensions of the sub-matrices of L_a^c matching the sub-matrices of A_G, the Lyapunov equation is reduced to:

$$\begin{aligned} (A_{11} - Z)L_1 + L_1(A_{11} - Z)^\top + I + Z^2 &= 0, \\ A_{12} L_2 + L_1 A_{21}^\top &= 0, \\ A_{21} L_1 + L_2 A_{12}^\top &= 0. \end{aligned} \tag{15}$$

Solving the above shows that $L_1 = Z$ and $L_2 = Z_m$.

Solving for $L_{\mathrm{b}}^{\mathrm{o}}$ and making the ansatz that it is diagonal gives the exact same equations as in (15), and thus

$$L_{\mathrm{b}}^{\mathrm{o}} = L_{\mathrm{a}}^{\mathrm{c}} = \begin{bmatrix} Z & 0 \\ 0 & Z_m \end{bmatrix}. \tag{16}$$

Therefore the H_2 gains of $G_{\mathrm{a}}(s)$ and $G_{\mathrm{b}}(s)$ are

$$\begin{aligned} \|G_{\mathrm{a}}(s)\|_{H_2}^2 &= \operatorname{tr}\left(\begin{bmatrix} Q^{\mathsf{T}} Z & 0 \end{bmatrix} L_{\mathrm{a}}^{\mathrm{c}} \begin{bmatrix} ZQ \\ 0 \end{bmatrix} \right) = \operatorname{tr}\left(Z^3\right), \\ \text{and} \\ \|G_{\mathrm{b}}(s)\|_{H_2}^2 &= \operatorname{tr}\left(\begin{bmatrix} Q^{\mathsf{T}} & 0 \\ 0 & 0 \end{bmatrix} L_{\mathrm{b}}^{\mathrm{o}} \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \right) = \operatorname{tr}(Z), \end{aligned} \tag{17}$$

which implies that $\gamma_{H_2}^* = \sqrt{\operatorname{tr}\left(Z^3\right) + \operatorname{tr}(Z)}$ as required.

The realisation of the controller follows from [Doyle et al., 1989, Theorem 1]. In particular

$$\begin{aligned} A_{\mathrm{K}} &= A_{\mathrm{G}} - B_{\mathrm{G}} B_{\mathrm{G}}^{\mathsf{T}} X - Y C_{\mathrm{G}}^{\mathsf{T}} C_{\mathrm{G}} = \begin{bmatrix} A_{11} - 2Z & A_{12} \\ A_{21} & 0 \end{bmatrix}, \\ B_{\mathrm{K}} &= Y C_{\mathrm{G}}^{\mathsf{T}} = \begin{bmatrix} ZQ \\ 0 \end{bmatrix}, \text{ and } C_{\mathrm{K}} = -B_{\mathrm{G}}^{\mathsf{T}} X = \begin{bmatrix} -Q^{\mathsf{T}} Z & 0 \end{bmatrix}, \end{aligned} \tag{18}$$

and the proof is complete. □

3. Applying Theorem 2 to Uniformly Damped Mass-Spring Systems

In this section we will look at an optimal control problem for a damped mass-spring system. We will see that under the assumption of uniform damping, it is possible to convert this on the form of Problem 2. We will further show how to write the optimal H_2-gain for this problem in terms of the mass and damper parameters of the system, and that the resulting H_2-controller is itself a passive, damped, mass-spring system, that inherits many of the structural properties of the system that is being controlled. A small numerical example is provided to illustrate the result.

3.1 The Optimal Control Problem

We first define the problem that is to be studied.

Problem 3 Consider a system described by

$$M\ddot{q}+C\dot{q}+Kq=u+\sqrt{M}w_{\mathrm{u}},\ q(0)=\dot{q}(0)=0,$$
$$y=\dot{q}+\sqrt{M^{-1}}w_{\mathrm{y}}, \tag{19}$$

where M, C, $K \in \mathbb{R}^{n\times n}$, satisfy the following conditions:

1. *M is positive definite;*

2. *K is positive semi-definite;*

3. *C is any symmetric matrix that satisfies $CM^{-1}K = KM^{-1}C$.*

Find a controller on the form of (7) that minimizes the H_2 gain from disturbance w to performance output z, where

$$w=\begin{bmatrix} w_{\mathrm{u}} \\ w_{\mathrm{y}} \end{bmatrix} \quad \text{and} \quad z=\begin{bmatrix} \sqrt{M}\dot{q} \\ \sqrt{M^{-1}}u \end{bmatrix}. \qquad \Diamond \tag{20}$$

◊

In the context of damped mass-spring systems, the first equation in (19) is a statement of Newton's second law, where q is the generalised coordinates, M is the mass-matrix, C is the damper-matrix, and K is the stiffness-matrix. The input u is a force that can be applied to the system by a controller, and w_{u} is a disturbance acting on the system. The second equation in (19) describes the measurements taken, where it is assumed that the velocity of each generalised coordinate is measured subject to measurement noise w_{y}.

We now discuss the implications of the restrictions 1)–3) in Problem 3.

1. This condition makes M invertible, allowing for a simple conversion of (19) into state-space form. This condition can be relaxed at the expense of more complex derivations. For extensions of the concept of uniform damping to this setting, see [Adhikari and Phani, 2007][Theorem 1].

2. This condition is needed to ensure the skew-symmetric structure $A_{21} = -A_{12}^{\mathsf{T}}$ required in Problem 2 appears when converting Problem 3 into the form of Problem 2. In the damped mass-spring interpretation of Problem 3, this corresponds to that all springs having non-negative spring constant.

3. As discussed in the introduction, this is the uniform damping condition. As shown in [Caughey and O'Kelly, 1965], this condition is equivalent to the existence of an invertible matrix S such that $S^{\mathsf{T}}MS = I$, and both $S^{\mathsf{T}}CS$ and $S^{\mathsf{T}}KS$ are diagonal. Note in particular that this implies that $Q = S^{\mathsf{T}}\sqrt{M}$ satisfies $QQ^{\mathsf{T}} = I$. Since Q is square this further implies that $Q^{\mathsf{T}}Q = I$.

REMARK 4 Any matrix C given by a Caughey series

$$C = M \sum_{j=0}^{n-1} \alpha_j (M^{-1}K)^j, \tag{21}$$

where $\alpha_j \in \mathbb{R}$, is uniformly damped. When $\alpha_j = 0$ for $j \geq 2$ this is typically called Rayleigh damping [Rayleigh, 1877]. Conversely whenever K has distinct eigenvalues a uniformly damped C admits a Caughey series. Note that C need not be positive semi-definite (for example when all the α_j's are negative). When this is the case the dynamics in (19) are unstable. ◊

There are further implicit assumptions in Problem 2. Most significantly, the disturbances and performance outputs in Problem 3 are scaled. These scalings are required to transform Problem 3 into Problem 2. However this requirement can likely be significantly relaxed by generalising Problem 2 and Theorem 2. There are a number of ways this could be approached, but we keep these scalings here for simplicity.

From the application point of view, these scaling are not unreasonable as we now discuss. It is likely reasonable that w_u should be scaled by the size of the masses, since larger masses will likely be affected by larger disturbance. Looking at the performance output z, we see that the velocities of large masses incur larger penalties than those of small masses. This is again reasonable, since when larger masses move, they are harder to stop, so it is desirable to prevent this with the control. At the same time, the forces acting on large masses from the controller should be expected to be larger than those of the smaller masses, which is again reflected by the weight on u. The weight on w_y is harder to intuitively explain, but is needed for symmetry. Note also that the scalings in terms of the square root of M are not so unnatural, since the H_2-norm penalises the square of the signals in questions. In the case of the process output y, for example, it means that we are minimising the effect on $\int_0^\infty y(t)^\mathsf{T} M y(t)\,\mathrm{d}t$.

3.2 The Solution to Problem 3

In this subsection we describe and illustrate the solution to Problem 3. The derivation from Problem 2 and Theorem 2 will be given in the next subsection.

The Optimal Cost and Control Law The optimal cost for Problem 3 is given by

$$\gamma_{H_2}^* = \sqrt{\operatorname{tr}\left(Z_C^3\right) + \operatorname{tr}(Z_C)}, \tag{22}$$

where

$$Z_C = -\sqrt{M^{-1}}C\sqrt{M^{-1}} + \sqrt{\sqrt{M^{-1}}CM^{-1}C\sqrt{M^{-1}} + I}. \tag{23}$$

The optimal controller can either be expressed in state-space form, or as a second order differential equation. Taking the later option shows that the control law is given by

$$\begin{aligned} &M_K \ddot{q}_K + C_K \dot{q}_K + K_K q_K = y,\ q_K(0) = \dot{q}_K(0) = 0 \\ &u = -\dot{q}_K, \end{aligned} \tag{24}$$

where the matrices are

- $T_K = -C + \sqrt{\sqrt{M} C M^{-1} C \sqrt{M} + M^2}$,
- $M_K = T_K^{-1} M T_K^{-1}$,
- $C_K = T_K^{-1} \left(-C + 2\sqrt{\sqrt{M} C M^{-1} C \sqrt{M} + M^2} \right) T_K^{-1}$,
- $K_K = T_K^{-1} K T_K^{-1}$.

It is interesting to note that the optimal controller in (24) is itself a damped mass-spring system, with parameters written in terms of the original system. A tedious but straightforward calculation in fact shows that C_K is positive definite (this follows from the fact that the north-west entry in the A_K (7) is negative definite, and the negative of this entry eventually becomes C_K after a sequence of transformations that preserve sign definiteness). This means that the controller can be implemented physically by building a suitable passive damped mass-spring network.

An Illustrative Case If the damping matrix C is proportional to the mass matrix M, the controller expressions simplify considerably. If $C = \alpha_0 M$, then
$T_K = (\sqrt{\alpha_0^2 + 1} - \alpha_0) M$, and (24) becomes

$$\begin{aligned} &\frac{M^{-1} \ddot{q}_K}{(\sqrt{\alpha_0^2+1} - \alpha_0)^2} + \frac{(2\sqrt{\alpha_0^2+1} - \alpha_0) M^{-1} \dot{q}_K}{(\sqrt{\alpha_0^2+1} - \alpha_0)^2} + \frac{M^{-1} K M^{-1} q_K}{(\sqrt{\alpha_0^2+1} - \alpha_0)^2} = y, \\ &u = -\dot{q}_K. \end{aligned} \tag{25}$$

The unstable case corresponds here to that $\alpha_0 < 0$. The more negative α_0 becomes, the more positive the term in front of $\dot{q}_K$ becomes in relation to the terms in front of $\ddot{q}_K$ and q_K. An interpretation of this is that for an unstable system the controller introduces more damping, and the more unstable the original system was, the greater the damping provided by the controller. It should also be noted that the denominator in all terms becomes larger with more negative α_0. This means that the more unstable the system is, the more important the measurement y becomes in the control dynamics.

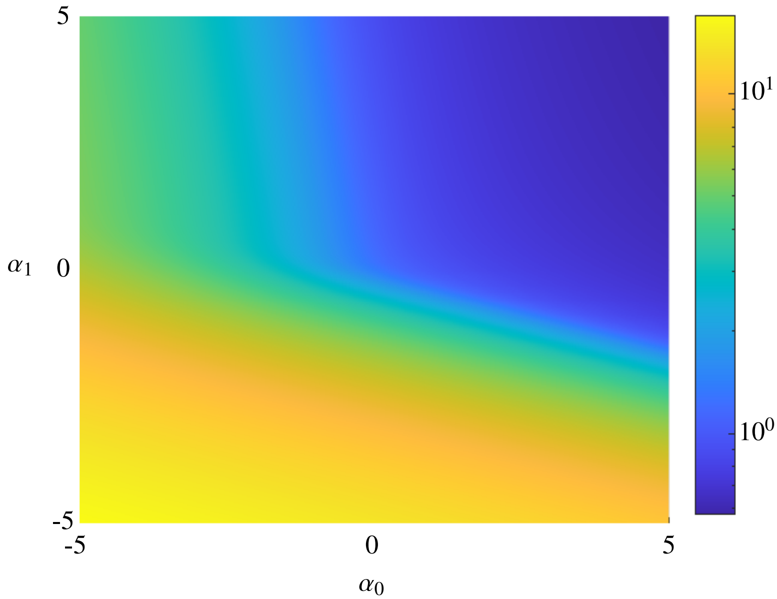

Figure 3. Levels of H_2 performance achieved as Rayleigh damping parameters α_0 and α_1 vary between -5 and 5. Larger values of α_0 and α_1, which correspond to increased levels of damping in the original system, result in improved performance.

A Numerical Example Consider the system in Fig. 1, where $m_1 = 1$ kg, $m_2 = 4$, $k_1 = 1$ N/m, and $k_2 = 2$ N/m. This gives the following mass and stiffness matrices

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}. \tag{26}$$

Suppose that the damping is given as Rayleigh damping, then

$$C = \alpha_0 M + \alpha_1 K.$$

In Fig. 3 the optimal H_2-gain from disturbances to outputs is plotted for different values of α_0 and α_1 on the interval from -5 to 5. Here it can clearly be seen that if both $\alpha_0 > 0$ and $\alpha_1 > 0$, which corresponds to positive damping, $\gamma^*_{H_2}$ is small. For strictly negative damping, corresponding to the third quadrant where both $\alpha_0 < 0$ and $\alpha_1 < 0$, the performance is much poorer, since the uncontrolled system is unstable. In the second and fourth quadrant of Fig. 3 C is positive definite for some combinations of α_0 and α_1, and negative definite or indefinite for others. Depending on the definiteness of C, the performance metric $\gamma^*_{H_2}$ can either be larger or smaller than the case of no damping.

3.3 Solving Problem 3 with Theorem 2

Converting Problem 3 into Problem 2 We now describe the required transformations to convert Problem 3 into Problem 2. We start by introducing a new variable $p = \sqrt{M}q$, thus $q = \sqrt{M^{-1}}p$. Inserting this into (19) and multiplying both sides by $\sqrt{M^{-1}}$ from the left gives

$$\ddot{p} + \sqrt{M^{-1}}C\sqrt{M^{-1}}\dot{p} + \sqrt{M^{-1}}K\sqrt{M^{-1}}p = \sqrt{M^{-1}}u + w_u \tag{27}$$

Define $\tilde{u} = \sqrt{M^{-1}}u$. Since K is positive semi-definite and M is positive definite, $\sqrt{M^{-1}}K\sqrt{M^{-1}}$ is positive semi-definite. Under the uniform damping assumption, $\sqrt{M^{-1}}C\sqrt{M^{-1}}$ and $\sqrt{M^{-1}}K\sqrt{M^{-1}}$ commute. Since in addition $\sqrt{M^{-1}}C\sqrt{M^{-1}}$ and $\sqrt{M^{-1}}K\sqrt{M^{-1}}$ are symmetric, there exists a unitary transformation Q (where $QQ^\mathsf{T} = Q^\mathsf{T}Q = I$) such that

$$\begin{aligned}\sqrt{M^{-1}}C\sqrt{M^{-1}} &= Q^\mathsf{T}\Lambda_C Q,\\ \sqrt{M^{-1}}K\sqrt{M^{-1}} &= Q^\mathsf{T}\begin{bmatrix}\Lambda_K & 0\\ 0 & 0\end{bmatrix}Q,\end{aligned} \tag{28}$$

where Λ_C and Λ_K are both diagonal and Λ_K is positive definite. Now define

$$L = \begin{bmatrix}\sqrt{\Lambda_K}\\ 0\end{bmatrix}.$$

This makes L of size $n \times m$, where n is the number of masses in the system, and m is the number of non-zero eigenvalues of K. Introduce the state variable

$$x = \begin{bmatrix}Q\dot{p}\\ L^\mathsf{T}Qp\end{bmatrix}. \tag{29}$$

Define $\tilde{y} = \sqrt{M}y$. Including the performance output z and measurement $\tilde{y}$, (27) admits the state-space realisation

$$\begin{aligned}\dot{x} &= \begin{bmatrix}-\Lambda_C & -L\\ L^\mathsf{T} & 0\end{bmatrix}x + \begin{bmatrix}Q\\ 0\end{bmatrix}(\tilde{u} + w_u),\\ z &= \begin{bmatrix}[Q^\mathsf{T} \;\; 0] & 0\\ [0 \;\; 0] & I\end{bmatrix}\begin{bmatrix}x\\ \tilde{u}\end{bmatrix},\\ \tilde{y} &= \begin{bmatrix}Q^\mathsf{T} & 0\end{bmatrix}x + w_y.\end{aligned} \tag{30}$$

Problem 3 has now been written on the form of Problem 2, with the sub-matrices fulfilling all conditions in the formulation of Problem 2.

Extracting the Optimal Solution to Problem 3 By Theorem 2, the optimal H_2-gain from disturbances to performance outputs is given by

$$\begin{aligned}\gamma^*_{H_2} &= \sqrt{\mathrm{tr}\left(Z^3\right) + \mathrm{tr}\left(Z\right)}, \text{ where}\\ Z &= -\Lambda_C + \sqrt{\Lambda_C^2 + I}.\end{aligned} \tag{31}$$

With $\Lambda_C = Q\sqrt{M^{-1}}C\sqrt{M^{-1}}Q^\mathsf{T}$, Z can be expressed in terms of the original system matrices according to

$$\begin{aligned} Z = &-Q\sqrt{M^{-1}}C\sqrt{M^{-1}}Q^\mathsf{T} \\ &+\sqrt{Q\sqrt{M^{-1}}CM^{-1}C\sqrt{M^{-1}}Q^\mathsf{T}+I}. \end{aligned} \tag{32}$$

Using the cyclic property of the trace, and the fact that the matrices Q and Q^T can be pulled out of the square root, shows that

$$\operatorname{tr}\left(Z^3\right) = \operatorname{tr}\left(Z_C^3\right) \quad \text{and} \quad \operatorname{tr}(Z) = \operatorname{tr}(Z_C), \tag{33}$$

where Z_C is defined in (23).

From (7), the optimal controller in state-space form is given by

$$\begin{aligned} \dot{x}_\mathrm{K} &= \begin{bmatrix} -\Lambda_C - 2Z & -L \\ L^\mathsf{T} & 0 \end{bmatrix} x_\mathrm{K} + \begin{bmatrix} ZQ \\ 0 \end{bmatrix} \tilde{y}, x_\mathrm{K}(0) = 0, \\ \tilde{u} &= \begin{bmatrix} -Q^\mathsf{T}Z & 0 \end{bmatrix} x_\mathrm{K}, \end{aligned} \tag{34}$$

where Z is defined as in (31). This structure is very similar to the structure of the problem in (30). Reversing the described transformations yields the controller expression in (24). This can be done by first introducing p_K through

$$x_\mathrm{K} = \begin{bmatrix} Q\dot{p}_\mathrm{K} \\ L^\mathsf{T}Qp_\mathrm{K} \end{bmatrix},$$

and then setting $q_\mathrm{K} = T_\mathrm{K}\sqrt{M^{-1}}p_\mathrm{K}$, where

$$T_\mathrm{K} = \sqrt{M}Q^\mathsf{T}ZQ\sqrt{M} = -C + \sqrt{\sqrt{M}CM^{-1}C\sqrt{M} + M^2},$$

and simplifying. T_K is non-singular since M is non-singular. This can most easily be seen in first expression of T_K above where Z is non-singular according to (31) and Q is non-singular due to it being unitary. Equation (34) can, with these transforms, be rewritten and simplified to

$$\begin{aligned} &T_\mathrm{K}^{-1}MT_\mathrm{K}^{-1}\ddot{q}_\mathrm{K} + T_\mathrm{K}^{-1}KT_\mathrm{K}^{-1}q_\mathrm{K} + T_\mathrm{K}^{-1}\left(-C + 2\sqrt{\sqrt{M}CM^{-1}C\sqrt{M} + M^2}\right)T_\mathrm{K}^{-1}\dot{q}_\mathrm{K} = y, \\ &u = -\dot{q}_\mathrm{K}, \end{aligned} \tag{35}$$

which is the result in (24).

4. Conclusions

An analytical solution to a structured optimal control problem has been derived. This was used to analytically solve a corresponding problem for any system that can be modelled as a uniformly damped network of masses and springs. The results illustrate the impact of damping on system performance, and also that such systems can be optimally regulated by passive networks of damped masses and springs.

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