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THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Interacting particle systems for opinion dynamics: the Deffuant model and some generalizations

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Interacting particle systems for opinion dynamics: the Deffuant model and some generalizations

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Abstract

In the field of *sociophysics*, various concepts and techniques taken from statistical physics are used to model and investigate some social and political behavior of a large group of humans: their social network is given by a simple graph and neighboring individuals meet and interact in pairs or small groups. Although most of the established models feature rather simple microscopic interaction rules, the macroscopic long-time behavior of the collective often eludes an analytical treatment due to the complexity, which stems from the interaction of the large system as a whole.

An important class of models in the area of opinion dynamics is the one based on the principle of *bounded confidence*: Individuals hold and share opinions with others in random encounters. Their mutual influence will lead to updated opinions approaching a compromise, but only if the distance of opinions was not too large in the first place. A much-studied representative of this class is the model, which was introduced by Deffuant et al. in 2000: Neighboring individuals meet pairwise and symmetrically move towards the average of the two involved opinions if their difference does not exceed a given threshold.

In the first paper of this thesis, we study the Deffuant model with real-valued opinions on integer lattices, using geometric and probabilistic tools as well as concepts from statistical physics. These prove to be very effective in the analysis of the model on the integer lattice in dimension 1, i.e. the two-sidedly infinite path $\mathbb{Z}$, and is adapted to give at least partial results for the lattice in higher dimensions as well as infinite percolation clusters. In papers 2 and 3, we stay on $\mathbb{Z}$ but consider a generalization of the model to higher-dimensional opinion spaces, namely vectors and absolutely continuous probability measures, as well as to more general metrics than the Euclidean, used to measure the distance between two opinions.

The last appended paper deals with "water transport on graphs", a new combinatorial optimization problem related to the possible range of opinions for a fixed individual given an initial opinion configuration. We show that on finite

graphs, the problem is NP-hard in general and prove a dichotomy that is partly responsible for the fact that our methods used in the analysis of the Deffuant model are less effective on the integer lattice $\mathbb{Z}^d,\ d \geq 2$: If the initial values are i.i.d. and bounded, the supremum of values at a fixed vertex – achievable with help of pairwise interactions as in the Deffuant model – depends non-trivially on the initial configuration both for finite graphs and $\mathbb{Z}$, while it a.s. equals the essential supremum of the marginal distribution on higher-dimensional lattices.

Keywords: Deffuant model, bounded confidence, opinion dynamics, sociophysics, consensus formation, general opinion space, percolation, pumpless water transport.

List of Papers

A Olle Häggström and **Timo Hirscher**,
Further results on consensus formation in the Deffuant model,
Electronic Journal of Probability, Vol. 19, 2014.

B **Timo Hirscher**,
The Deffuant model on $\mathbb{Z}$ with higher-dimensional opinion spaces,
Latin American Journal of Probability and Mathematical Statistics, Vol. 11, 2014.

C **Timo Hirscher**,
Overly determined agents prevent consensus in a generalized Deffuant model on $\mathbb{Z}$ with dispersed opinions,
submitted to *Advances in Applied Probability*.

D Olle Häggström and **Timo Hirscher**,
Water transport on graphs,
submitted to *Networks*.

Paper not included in this thesis

E **Timo Hirscher** and Anders Martinsson,
Segregating Markov chains,
submitted to *Journal of Theoretical Probability*.

Acknowledgements

Provided you answer the question “What do you do for a living?” by saying “I’m doing a PhD in math.”, the most usual follow-up question is “How is it like to do a PhD in mathematics?”. Well, I always considered this question quite tricky since mathematics in general and PhD studies in particular are so varied that it is difficult to give a concise and yet satisfactory answer.

Last year in June, I hiked from the village of Kilpisjärvi to ‘Treriksröset’, the point where the borders of Finland, Norway and Sweden meet, and realized striking parallels between the 11km hike and the past nearly five years of study, making it an extremely suitable metaphor.

After having realized that there are no shortcuts (lake Kilpisjärvi was still partly frozen and the boat cutting the hike to 3km not operating yet), you set off for a journey with a clear goal in mind but little to no idea how the path leading there will look like. You know that others have done it and in the beginning you happily follow the trail. Every now and then you pass one of those little orange-tipped poles providing the affirmative feeling that you are still on track. After a while, the first snow fields appear and make you wonder if you started the journey being well-equipped or rather quite naive. Then within minutes, the sun disappears and snowfall sets in; your feet are all wet and you begin to fight the thought of turning around and heading back.

But then you remember your goal, realize that you cleared a considerable part of the way and keep going. Although the sun and the stunning view reward your decision, the rocky and snowy sections of the trail make it difficult not to loose orientation – until a tiny signpost in the distance makes you aware of the fact that you have gone astray and leads you back on track. Surprisingly enough, a Finnish mobile provider gives decent coverage all the way to the tripoint and an acceptable feeling of security that one could try and call for help in the worst case.

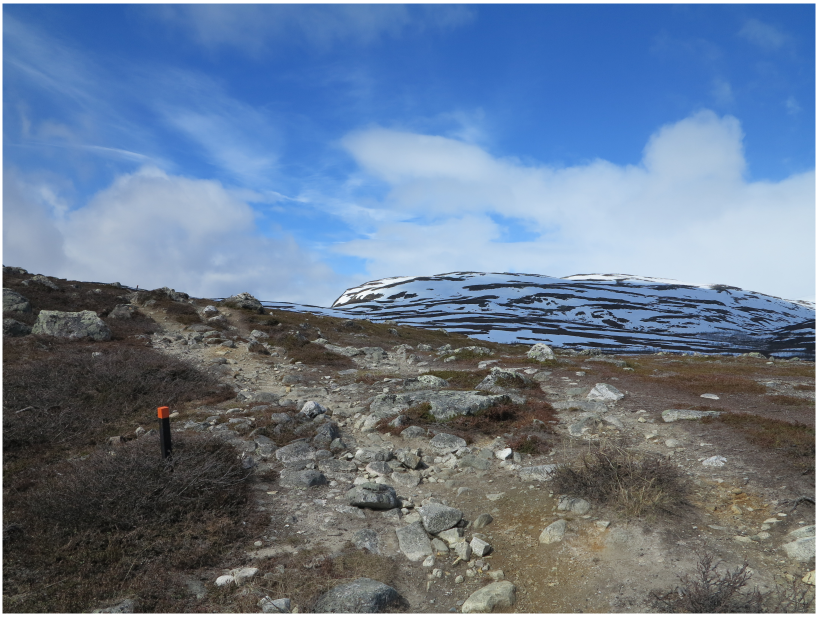
Besides all these similarities, there are a couple of notable dissimilarities: For instance, once the goal is reached, doing a PhD you have to take the next step in your life, not to hike 11km back to where you came from. Furthermore, during five years of study at Chalmers you meet more than two human beings. I now want to take the opportunity and give a few words of thanks to these people.

First and foremost, I would like to thank my advisor Olle Häggström for all his support, motivation and constructive criticism. Figuratively speaking, you have been both, the little poles marking the path and my mobile provider on this journey. Next I want to express my gratitude to Jeff Steif, for co-organizing the graduate course on 'Markov chains and mixing times' together with me, as this gave me the great opportunity to gain experience in and to get his advice on lecturing on graduate level. Furthermore, I would like to thank all my present and former colleagues at the mathematical department for creating this excellent working atmosphere and sharing both a laugh and some good advice in times when I desperately needed the one or the other. Without degrading others, I want to explicitely name Marie, my personal problem solver, as well as Dawan, Matteo and Peter whom I became close friends with.

Speaking of friends, there are a number of people outside the department contributing to this thesis by making my life in the past few years either easier or more exciting or both. I want to extend my deepest gratitude to Ewa, Gustav and Karin, Jan, Jörgen, Mareile, Mathias, Miri, Patricia, Sascha, Tino, Wiebke, all those who made Guldhedens Studiehem my home and many others whose names could make this list the major part of the thesis.

Finally, I want to thank my wonderful family, especially my parents, my sister and her family. You never questioned my decisions but provided me with constant love and support. There are no words adequately expressing my thankfulness to you.

Timo Hirscher
Göteborg, March 2016



"Whether you think you can, or you think you can't – you're right."
Henry Ford (1863 - 1947)

Contents

1
Introduction

Two friends, Jakob and Johan, meet by coincidence at Brunnsparken in central Gothenburg. They haven't seen each other in a long time, so they sit down in a café and have a chat. Since both of them are interested in new technologies, they soon start talking about the changes that the city planners intend to implement before Gothenburg's 400th anniversary in 2021. At some point, the question comes up how many of the busses will be running on renewable energies only by then. While Jakob is convinced that about 30 percent of the busses will be independent of fossil fuels, Johan is more pessimistic. His guess is that the fraction of local busses running on clean energy might be one tenth in 5 years from now. He points out that such changes are expensive and take time, especially in the public sector. Jakob argues that the pilot project ElectriCity in fact shows the city's effort towards such a change and that economic considerations could actually become a driving force away from fossil fuels in near future. During the exchange, they consider each other's arguments as well-founded and valid.

If confronted with the same question after their conversation, Jakob might have adapted his guess down to one fourth, Johan his instead up to 15 percent. Had Johan instead met an excessively optimistic Jakob claiming that all of Gothenburg's busses will be electric by 2021, both of them would have rated the view of the other as unrealistic, his arguments as not worth considering and hence left the café without updating their guesses.

This everyday phenomenon called *selective exposure* – people in general try to avoid new pieces of information likely to challenge their decisions and beliefs all too much – gained substantial attention in the field of psychology when Festinger [21] provided a solid theoretical framework in his book entitled "A Theory of Cognitive Dissonance", which was published in 1958. Following his pioneering work, a considerable number of experiments were conducted in order to describe, understand and explain this defensive behavior, that occasionally gets in the way when people actually try to form a knowledgeable opinion and in many cases accounts for the persistence of faulty beliefs. An extensive synopsis of these studies together with a thorough discussion of the area of conflict between curious open-mindedness and protective stubbornness in the process of information selection can be found in [34].

In an extremely simplified version, these competing principles are implemented in models for opinion formation based on bounded confidence (which will be reviewed in Section 4): On the one hand, people in general tend to assimilate, i.e. to adapt their points of view towards the opinion of others if confronted with their valid arguments. This process, on the other hand, only takes place if there is a certain trust in the position of one's discussion partner; if it is too far off our own standpoint, we are not willing to debate and re-evaluate our beliefs.

The idea to study opinion formation processes in a group of people using models with extremely simplified interaction rules is anything but new. The first attempts, however, were mere reinterpretations of mathematical models, used in statistical physics to describe interactions of elementary particles, and did not feature aspects of reflective behavior such as bounded confidence. Already in the 1930s, the theoretical physicist Ettore Majorana, a student of the famous Enrico Fermi, wrote an article titled "The value of statistical laws in physics and social sciences" [47]. It was originally supposed to be published in a sociology journal, hence to present the beneficial use of methods and ideas from statistics in physics to scholars of a different discipline and in this way to establish a connection between the two fields. This essay, however, was carelessly discarded and kept in a drawer until Majorana mysteriously disappeared on a boat trip from Palermo to Napels in 1938.

The manuscript was found by his brother and finally published in 1942, thanks to the efforts of Giovanni Gentile Jr., a former co-author and friend of Majorana. Despite its novel ideas, the fact that the paper was written in Italian and published posthumously limited its impact considerably. In fact, there was no translation into English until Mantegna [48] presented the article in the journal "Quantitative Finance" as recently as in 2005. Due to the fact that this last publication of Majorana received very little attention and therefore did not cause any notable further research efforts, it was not until the 1970s that theoretical physicists once more got interested in phenomena from social science and finally put Majorana's suggestion into practice: to see opinion dynamics in large groups as interacting particle systems and then exploit the fact that these are amenable to a rigorous mathematical modelling and an analysis based on statistical laws.

As a first step, statistical models – originally designed to describe the dynamic development of an ensemble of interacting particle spins on atomic level – were used to model the opinion formation in a social group of individuals mutually influencing each other. One of the major aims was to reinterpret known phenomena from physics, such as phase transitions or ordered and disordered states, in the new sociological context and by that to relate purely mathematical aspects of the model's dynamics to common social phenomena in group behavior.

During the last two decades more and more physicists and mathematicians started similar attempts to understand the opinion dynamics in a large group of individuals by using simplistic interaction models and to analyze them by applying qualitative and quantitative methods from statistical physics. The fact that new social phenomena which arose with the advancement of the internet, like e-mail correspondences for example, feature large groups of individuals, simple interactions and allow for a computational treatment of the corresponding large datasets contributed substantially to this evolution.

2

Can elementary magnets go on strike? – A historical account

2.1 Statistical mechanics and the Ising model

Taking into consideration that the research area of opinion dynamics is rooted in the discipline of physics, the story really began in the second half of the 19th century, when James Clerk Maxwell, Ludwig Boltzmann and Josiah Willard Gibbs elaborated the ideas of Daniel Bernoulli to describe the kinetic dynamics in gases statistically and in this way launched the branch of statistical mechanics. Their pioneering idea was not to focus on each single particle and its individual movements, but to characterize the whole system with a set of parameters and their distributions among the possible states of the system, the so-called *statistical ensemble*.

The starting point of opinion dynamics based on statistical physics, a field that later became labelled as *sociophysics*, was however not the branch of thermodynamics but the closely related field dealing with ferromagnetism. Just like water changing its state of matter depending on the temperature, ferromagnetic material undergoes a phase transition in the sense that macroscopic properties of the matter are changed. Well above a certain critical temperature, the ferromagnetic material is unmagnetic on a macroscopic scale (if not exposed to a strong external magnetic field); well below this temperature however, a phenomenon that is called *spontaneous magnetization* occurs: the microscopic magnetic dipole moments, originating from atomic spins, start to align and turn the material into a magnet – even in the absence of an external field.

Already in 1907, Pierre Weiss [66] tried to explain this behavior, building on earlier work by Pierre Curie. He used an approach that became known as *mean field theory*: In a large statistical system, the effects of all other particles on one fixed particle is replaced by their statistical average. This approximation turns a many-body problem with interactions, which in general is very difficult to solve exactly, into a one-body problem with external field. Clearly, this is a rather crude simplification as the fluctuating interaction of the considered particle with the rest of the system is approximated by a time-independent effective field. Nevertheless, it made the spin problem tractable and allowed Weiss to draw conclusions explaining the two different phases of ferromagnetic material. The mean field theory approximation is however only qualitatively accurate and fails to give satisfactory answers to questions about the behavior near the phase transition. For temperatures near the critical one, the actual local magnetic fields are rapidly varying in time and consequently turn their statistical average into a quite poor representation of their effect.

A slightly different approach to explain ferromagnetic behavior was the following theoretical model that physicist Wilhelm Lenz invented in 1920 and proposed to his student Ernst Ising for further studies two years later: A collection of n atoms is arranged to form a regular atomic lattice. Their elementary magnetic dipoles, often simply called *spins*, can be either in the state “up” or “down”, represented by the numerical values $+1$ and -1 respectively. All spins taken together form what is called a spin configuration $\sigma \in \{-1, +1\}^n$. If we assume that neighboring spins interact with a certain coupling strength J and that the material is exposed to an external magnetic field h, the configuration σ

is attributed a total energy given by the *Hamiltonian function*

$$H(\sigma) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \mu\, h \sum_i \sigma_i, \tag{2.1}$$

where the first sum is taken over all pairs $\langle i, j \rangle$ of nearest neighbors in the atomic lattice and μ denotes the magnetic moment. While the minus sign of the second term is mere convention (as the magnetic moment actually is antiparallel to the spin), $J > 0$ corresponds to a ferromagnetic interaction. Thus, in the ferromagnetic case, the energy of the configuration decreases with both the number of nearest neighbor pairs having spins pointing into the same direction and spins aligned in accordance with the external field.

Following a basic physical principle, the system will act in a way to minimize the free energy, which makes states of low energy more probable in thermal equilibrium. This is captured by the so-called Gibbs measure attributing probability

$$\mathbb{P}(\sigma) = \frac{1}{Z_\beta}\, \mathrm{e}^{-\beta H(\sigma)} \tag{2.2}$$

to a fixed spin configuration σ, with the *partition function* $Z_\beta = \sum_\sigma \mathrm{e}^{-\beta H(\sigma)}$ being the appropriate normalizing constant. The model parameter β, called the *inverse temperature*, is given by $\beta = \frac{1}{k_{\mathrm{B}}\, T}$, where k_{B} denotes a (positive) physical constant, the so-called *Boltzmann constant*, and T is the temperature (in degree Kelvin). If we consider the case with no external field (i.e. $h = 0$), it is intuitively obvious from (2.2) that for high temperature all possible configurations nearly have the same probability, while for low temperature configurations with high energy (i.e. many opposing nearest neighbor pairs) are almost excluded.

For a finite system, this transition happens smoothly and a phase transition in the sharp (mathematical) sense can only be observed in the case of infinitely many particles, commonly known as *thermodynamic limit*. On the infinite d-dimensional grid $\mathbb{Z}^d$, we can consider the spatial average of spins which is called *magnetization* of the material and defined by

$$\langle\, \sigma \,\rangle = \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \sum_{i \in \Lambda_n} \sigma_i, \tag{2.3}$$

where $\Lambda_n = \{-n, \ldots, n\}^d$. With this notion in hand, we can distinguish between a paramagnetic, disordered phase in which the magnetization is almost surely 0 and a ferromagnetic, ordered phase in which non-zero magnetization has positive probability.

In his PhD thesis, Ising [37] analyzed the one-dimensional case and found that the correlation of spin values decays exponentially with the distance of two sites, which implies that the magnetization equals 0. He erroneously concluded that the model does not feature any phase transition even in higher dimensions. This claim was proven wrong by Rudolf Peierls [54] about one decade later. He investigated the two-dimensional zero-field Ising model (i.e. on the square-lattice $\mathbb{Z}^2$ with $h = 0$) and proved that it has a non-zero magnetization at sufficiently low temperatures. As the model without external field must have zero magnetization at sufficiently high temperatures, he was the first to show that a model from statistical mechanics exhibits a phase transition. A few years later, Lars Onsager [53] computed the critical temperature for the zero-field Ising model on the square-lattice rigorously and found it to be

$$T_{\mathrm{c}} = \frac{2\,J}{k_{\mathrm{B}} \cdot \ln(1+\sqrt{2})}.$$

The Ising model on the square-lattice still is one of the simplest mathematical models that does feature the phenomenon of a phase transition.

To simulate a configuration of the Ising model on a finite graph with given external parameters (T and h), the standard approach is to use the Monte Carlo method based on the well-known algorithm by *Metropolis–Hastings*. In this rejection sampling algorithm, applied to the Ising model, one starts with a random configuration and then performs single spin updates according to the following rule: Pick a site uniformly at random and flip its spin with probability $\min\{\mathrm{e}^{-\beta\,\Delta H}, 1\}$, where ΔH is the invoked change of the total energy. In the ferromagnetic regime without external field, flipping the spin at a chosen site might be rejected only if the majority of its neighbors agrees with the current spin as this implies $\Delta H > 0$. Evidently, a low temperature will considerably favor flips decreasing the energy over flips increasing it and therefore drive the system towards more ordered states with growing patches of aligned spins.

A different way to incorporate the microscopic evolution in a ferromagnet at a fixed temperature with help of the Ising model is the so-called *Glauber dynamics*. In this algorithm, to flip the randomly chosen spin has probability $\frac{1}{1+\mathrm{e}^{\beta\,\Delta H}}$. In contrast to the Metropolis–Hastings algorithm, here even transitions to lower energy states might be rejected, but the tendency to order remains as updates towards lower energy have probability larger than $\frac{1}{2}$, towards higher energy less than $\frac{1}{2}$.

In a long chain of atoms, these alignments at low temperature do take place as well, but for any temperature above absolute zero, thermal fluctuations will consistently break the aligned parts of the chain and in this way prevent a global alignment of the system. This is the reason why the model on $\mathbb{Z}$ does not achieve a global magnetization even for low temperatures. A quite comprehensive expositon of the early years of statistical physics including a more detailed discussion of mean field theory and the Ising model from a slightly more physical point of view can be found in [38].

2.2 Sociophysics

Even though the proposal by Majorana to start treating social phenomena by using statistical models of reduced complexity and to focus on how microscopic interaction rules entail macroscopic properties of the system, that can be compared to global observables, went more or less unheard by the social sciences, the striking similarity between interacting elementary magnets and simplified processes of group behavior led physicists about 30 years later to finally establish this connection.

In a colloquium in 1969, physicist Wolfgang Weidlich suggested to compare the interactions within a group of individuals holding opposing attitudes towards a given yes-no question with ferromagnetism, more precisely the dynamics of the Metropolis–Hastings algorithm applied to the Ising model. Two years later, he published this idea in the article *'The statistical description of polarization phenomena in society'* [64] in which he elaborated how the mathematical model intended to describe and explain ferromagnetism with help of statistical mechanics can be put into a sociological context: In the sociological reinterpretation, the interaction strength J corresponds to the willingness of an individual to adopt the attitude of the majority among its neighbors and the temperature as a model parameter for the social pressure exerted on each individual (low temperature corresponding to high social pressure). An external magnetic field (i.e. $h \neq 0$) is understood to shape some preference of one attitude over the other, shared by all individuals. Weidlich derived the stationary distributions for different values of h and J and even included a section in which a possible comparison between model and real data is discussed. Furthermore, he already suggested natural extensions of this initial link between social dynamics and statistical physics: More than two possible attitudes could be considered, h and J could be replaced by sets of parameters $\{h_i\}$ and $\{J_{ij}\}$ (i.e. chosen to be depending on the individuals and nearest-neighbor pairs respectively) and letting the transition probability to flip the spin at a given site depend not only on the current configuration of its neighbors, but also on its own history could introduce a sense of tradition or stubbornness.

In 1982, Galam et al. [27] used the Ising model on K_n, the complete graph on n vertices, to describe the collective behavior in a plant where dissatisfied workers might start a strike. Using a mean field theory approach, they rediscovered the phase transition described in the foregoing section and interpreted the regime of high temperature as an individual phase (mutual influences are very limited) and low temperature as a collective one (the group behaves coherently), separated by a critical phase in which small changes in the system can lead to drastic changes in the group. In contrast to the physical application of the Ising model, where a collection of atoms is forming a regular lattice, it is reasonable to consider the underlying interaction network among workers in a small plant to be all-to-all, meaning that every worker can actually influence all his fellow workers.

Following these seminal papers, an increasing number of related models were introduced, motivated and analyzed – in the past two decades predominantly with the help of computer simulation. The principle interaction rules diverged slowly but surely from particle physics and today the area of sociophysics comprises an abundance of models for opinion dynamics in groups. The most noted among these will be reviewed in the following chapters.

Just as in any cross-disciplinary application, the question has to be adressed whether these interacting particle systems are suitable to model human group behavior or not. Interestingly enough, already Weidlich [64] and Galam et al. [27] tried to survey the advantages as well as limitations of and possible objections against applying a simplified model from statistical mechanics in a sociological context. Apparently, there are glaring differences between the two fields of application. Possibly most important is the contrasting complexity of the elementary components: In physics, the systems consist of relatively simple objects, usually atoms and molecules, the behavior of which is relatively well understood; hence the complex evolution of the collective arises from the interaction patterns. In social science, however, the collective consists of a large number of human beings and the behavior of each single individual is already the outcome of a complex interplay between physiology and psychology of which only very little is understood. Especially the fact that in all common models for opinion dynamics the individuals are presupposed to behave adaptively (i.e. reacting to external influences) and not strategically (i.e. following a certain plan they have in mind) seems to be an unrealistic assumption. Apart from that, one has to admit that humans differ a great deal from one another in many aspects while it is rather safe to consider atoms of the same kind as perfectly identical. It is doubtful whether the few parameters needed to capture the state of a physical system are sufficient to describe the properties of a collection of human beings.

In a nutshell, the reduction of humans to identical and simplistic elements in a large system is a quite controversial issue and critics might come to the conclusion that reducing the complexity on microscopic level to such an extent that the system makes a treatment using tools from statistical physics possible without changing the essential macroscopic phenomenology is a hopeless task. One could even take this one step further and claim that researchers were tempted by the substantial progress in the study of collective phenomena in the field of physics to apply these models in other contexts, such as social behavior in groups, and established this connection at any sacrifice.

Nevertheless, one cannot deny the fact that there are certain phenomena in the dynamics of group behavior (both animal and human), that show striking structural similarities to ferromagnetism and suggest a meaningful relation between the two. Just like the spins in an ensemble of atoms, the individuals might be in a chaotic state at first – meaning that no large scale structure exists – but then gradually align and finally undergo a transition from disorder to order in the sense that the system exhibits large scale regularities, which in the physical context correspond to a state of low energy. In their article "A theory of social imitation", Callen and Shapero [7] name the collective movement in a school of fish or a flock of birds, the synchronous flashing of fireflies as well as temporary fashion styles as prominent examples: Without any leader, the group becomes increasingly homogeneous through local interaction and alignment until a consistent collective is formed – similarly to spontaneous magnetization of ferromagnetic matter not exposed to an external field.

For prey, being a part of a homogeneous group provides a certain degree of safety against predator attacks. In the context of social interactions and opinion formation in groups, the drive towards order is due to the tendency of interacting individuals to become more alike, an effect called *social influence*. This effect is often intensified by the known psychological phenomena of *selective attention* and *pleasure of recognition*: Our brain is geared towards filtering out relevant information, giving an advantage to things we can relate to. The idea of a selective internal filter was originally proposed by Broadbent [5] in 1958 and later refined and elaborated with help of various experiments investigating human habits and capabilities of handling information input (see also [20]). The pleasure of recognition (which incidentally is an important aspect in the composition of musical and literary work, see [57]) as well as the phenomenon of selective exposure, mentioned in the introduction, are closely intertwined with the inclination of people in general to meet and interact with others that resemble themselves in various aspects and share central attitudes, a behavior referred to as *homophily*. This term was introduced by Lazarsfeld and Merton [42], who considered two forms of homophily: *value homophily*, based on shared values and beliefs, as well as *status homophily*, based on a similar cultural background. The form that is most relevant in the context of opinion dynamics, *induced homophily*, which is based on similarity emerging from regular contact and mutual influence, was added and studied later (see for instance [50]). In this form it is most obvious how homophily can lead to a self-enhancing process and play a central role in the homogenization of a social group.

If we stick to the metaphor, ordered low energy states in statistical mechanics correspond to consensus or uniformity in the context of opinion dynamics and disordered states of higher energy in turn to fragmentation or disagreement. One of the main questions in social dynamics is – similarly to the situation in statistical physics – under which circumstances the microscopic interactions will lead to such a transition, since if there were no interactions, in both contexts heterogeneity would prevail.

Apart from this rather heuristic relation, there are other important arguments that alleviate the problem of reducing humans to elementary particles: In statistical physics most of the qualitative properties of a larger-scale system do not depend on the microscopic details of the dynamics but instead on global properties like symmetries, dimensionality or conservation laws. Diverse models exhibit essentially similar phenomena (e.g. phase transitions) despite their different rules and patterns, making these features in some sense model-invariant, a concept called *universality*. In this respect it is at least justifiable that modelling a few of the most important properties of single individuals will capture the essential driving forces of the evolution and thereby give meaningful results when it comes to qualitative features of the model's large scale behavior. In addition to that, just as many other complex systems, the opinion formation in a large group of humans is of statistical nature, i.e. a large number of comparable microscopic elements compose a macroscopic object, which has properties that are formed by the collective but the contribution of any individual particle is negligible. A statistical approach therefore seems to be quite reasonable. In fact, this argument was brought up already by Majorana [47] in the 1930s.

The lack of analytical means that could be applied to the common models for social dynamics as well as the increasing computational power resulted in numerous simulation-based analyses beginning in the 1990s. On the one hand they surely complement the analytical study of such models based on tools from statistical physics, on the other hand simulation-approaches are limited to a rather small number of individuals. Even if it seems to be sufficient for an examination of the opinion formation in social groups, as mentioned before, the concept of order-disorder phase transitions is rigorously defined only in the limit of a system with infinitely many particles. A number of individuals that is not sufficiently large might therefore cause finite size effects that invalidate conclusions drawn from a comparison with analog systems in physics, in which the number of interacting particles is commonly by far larger than in a social group. In this respect it is of vital importance to be able to figure out which macroscopic features are robust with respect to changes in the number of interacting individuals by analyzing the used model for different orders of magnitude of the system's size.

3
Opinion dynamics

Since there are many situations in everyday life where it is necessary for a group of people to form a point of view with majority appeal in order to make a shared decision (especially in a democratic framework, as discussed in [4]), it has always been a major focus of social science to understand the opinion formation process in a larger group of socially interacting individuals (for a broader introduction of the concept of 'public opinion' and an overview of some early efforts of social scientists in this area of research, see [14]). Inspired by statistical mechanics, in particular Weidlich's sociological reinterpretation of the Ising model for ferromagnetism, various models for opinion dynamics arose in the sequel.

In this chapter, we will shortly introduce a number of models used in the field of opinion dynamics that are either based on or very similar to interacting particle systems from statistical physics. First, we will list commonly used network structures and opinion spaces, then describe the characteristic interaction rules of the most common models. Before we engage in this review, it should be mentioned that not all of the models which appeared in the early years of opinion dynamics were inspired by statistical mechanics.

In 1974, for instance, DeGroot [17] presented a different approach to describe the dynamics of an opinion formation process, reminiscent of a finite Markov chain. In his model, n individuals update their opinions in rounds and compose their new ones as a weighted average of all current opinions:

$$\eta_{t+1}(i) = \sum_{j=1}^{n} p_{ij}\, \eta_t(j), \tag{3.1}$$

where $\eta_t(i)$ is the opinion of individual i after round t and p_{ij} is the weight it attributes to the opinion of individual j. In the definition of the model, DeGroot does however not specify (deliberately) which convex set the initial opinions belong to; could be real numbers, vectors or probability distributions. He considers the weights, which form a row-stochastic matrix $P = (p_{ij})_{i,j}$, to be time-independent. This allows to transfer standard results about the asymptotics of time-homogeneous finite Markov chains to the model: A consensus is reached (starting from a general set of initial opinions), in the sense that all opinions converge to a common limit, if and only if the matrix P, taken as one-step transition matrix, corresponds to a Markov chain in which all recurrent states belong to the same aperiodic communication class. Then the unique stationary distribution gives the weights according to which the common limiting opinion is composed.

Note that the iterated matrix products that represent the array of opinions at later times are multiplications from the left (as apposed to multiplications from the right in the case of a Markov chain). A stochastic process of this kind is commonly known as *repeated averaging*. A few years later, Chatterjee and Senata [11] addressed the more general case in which the weights depend on time. They establish sufficient conditions on the sequence of weight matrices for the opinions to converge to a common limit.

3.1 Underlying social network structures

No matter if we consider the model of DeGroot based on repeated averaging or interacting particle systems based on models from statistical mechanics, the following is apparently true for opinion dynamics in general: When it comes to the question whether the individual opinions will converge to a common limit or not, it is a very important aspect, between which of the individuals there is a potential for mutual influence – in other words the topology of the interaction network. We think of the individuals as nodes that form a social network (given by a simple graph) in which a connection between two individuals that enables them to influence each other is represented by an edge.

Under the assumption that the interaction is all-to-all, often termed *complete mixing*, the mean field approximation becomes particularly useful. In most cases it makes an analytical treatment possible in the sense that solving the corresponding differential equations will give insights about the long-term behavior. However, already in today's globalized companies this assumption is hardly realistic – not to mention the extremely sparse networks of e-mail correspondences and the like. For this reason, all of the models we are about to review were mainly considered on much sparser networks than the complete graph.

Finite graphs

Clearly, all simulation-based analyses are confined to opinion dynamics on finite social networks. A particularly simple example is that of a finite square lattice: It features two dimensions (which as we know from the Ising model can make a crucial difference to dimension 1) and still has comparably few edges. The necessary compromise between the efforts to keep both computation time and boundary effects to a minimum, led to samples comprising a number of individuals roughly ranging from $n = 10^2$ to $n = 200^2$. In some simulations (e.g. in [2], [22] and [49]), the boundary conditions were taken to be periodic in order to remediate their negative impact on the homogeneity of the network. In [16], where both a complete graph and a finite square lattice were used to represent the underlying social network, the authors accentuated the fact that a grid features many short cycles (measured against the relatively small number of edges) just like real social networks do. In respect of its striking regularity it might however be questioned if this makes a square lattice an appropriate candidate to model social relations.

More sophisticated choices for the interaction network that have been studied, among others, are realizations of random graph models such as the three introduced by Erdős–Rényi, Barabási–Albert and Watts–Strogatz: The so-called *Erdős–Rényi graph*, often simply denoted by $G(n, p)$, is a random graph on n nodes, in which each of the $\binom{n}{2}$ possible edges is independently chosen to be present with probability p. If the size of this network is varied, it might be suitable to choose $p = \frac{c}{N-1}$ in order to keep the average degree constant (at the chosen value c).

The *Barabási–Albert model* is one of the most popular algorithms for generating random scale-free networks, i.e. graphs with a degree distribution that follows a power law (at least in the tail)

$$N(d) \sim d^{-\gamma},$$

where $N(d)$ is the fraction of nodes with degree d and γ a parameter typically valued in the range $[2, 3]$.

The model is based on a principle called *preferential attachment*: The network is built incrementally from a core of m fully connected individuals by adding new nodes one by one, each choosing m older nodes to connect to with a probability proportional to their degree. Scale-free networks proved to be realistic models for e-mail networks or friendship graphs, both popular objects of study in the branch of social network analysis.

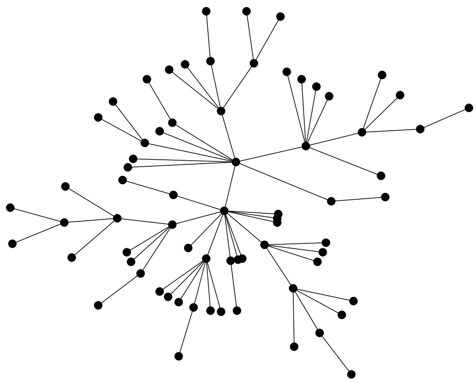


Figure 3.1: *A typical Barabási–Albert network, for $m = 1$, of comparatively small size ($n = 70$).*

Lastly, the algorithm proposed by Watts and Strogatz generates a simple random graph that has two main features found in real social networks: local, strongly connected clusters and short average path lengths. Graphs of this kind are called *small-world networks*. The algorithm features three parameters (the number of nodes n, the mean degree $2m$ as well as the rewire probability β) and proceeds as follows: Given the set of nodes $\mathbb{Z}_n = \{0, \ldots, n-1\}$ placed on a circle, first, a directed ring lattice is constructed by including an arrow from each node i to its m immediate successors, i.e.

$$\vec{E} = \{(i,j);\ i,j \in \mathbb{Z}_n,\ 1 \le j - i \pmod n \le m\}.$$

Then, all of these directed edges are processed in lexicographical order and replaced by undirected ones: With probability $1-\beta$, the arrow (i,j) simply gets transformed into the edge $\langle i,j\rangle$. With probability β, however, it gets rewired and instead the edge $\langle i,k\rangle$ is included, where k is picked uniformly at random from the elements of $\mathbb{Z}_n \setminus \{i,j\}$, that are currently not linked to i (neither by an arrow nor by an undirected edge).

In this way, for β positive but small, a few of the local connections get replaced by long-range relations and a small-world network is formed. For extreme choices of β, this is not the case: $\beta = 0$ corresponds to the regular ring lattice with degree $2m$ and for $\beta = 1$, the algorithm returns a graph with average degree $2m$ in which all edges were placed randomly, see Figure 3.2.

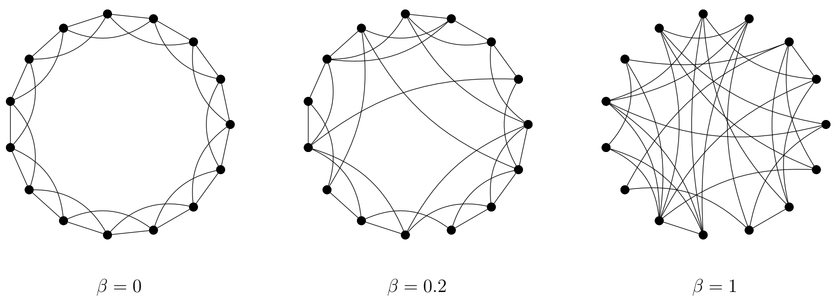


Figure 3.2: *Output of the Watts–Strogatz algorithm for $n = 15$, $m = 2$ and different values of β.*

The same idea can of course be applied to square lattices etc. as well.

It should be mentioned that there have been various efforts to implement opinion dynamics on adaptive random networks. Gil and Zanette [29], for example, proposed a model in which the social network is given by the complete graph initially, but whenever two agents meet and fail to agree on one opinion, the link in between them is deleted with a certain probability. This procedure leads to a gradual thinning of the network until only homogeneous opinion clusters remain.

Although certainly more realistic, the coevolution of opinions and relations adds substantially to the complexity of the problem. A different approach to implement homophily is the one of *bounded confidence*: While the network stays unchanged, neighboring agents only interact if their opinions are reasonably close. Models of this kind are reviewed in Section 4.1.

Infinite graphs

In a probabilistic analysis of opinion formation processes, as opposed to studies that are simulation-based, considering infinite networks becomes feasible and in fact, it often makes both the arguments and results more elegant: Tools like the law of large numbers or ergodicity might be applied and turn phenomena that occur with high probability on finite networks into almost sure events. Apart from that, infinite systems can serve as idealized approximations to finite but very large systems.

Major parts of this thesis deal with opinion dynamics on the, in a way, simplest infinite network: the two-sidedly infinite path. To be more precise, we consider the graph with vertex set $\mathbb{Z}$ and edge set $E = \{\langle v, v+1\rangle;\ v \in \mathbb{Z}\}$, see Figure 3.3 below for an illustration.

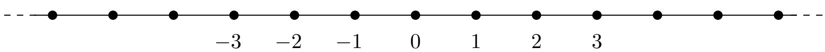


Figure 3.3: *A section of the two-sidedly infinite path* $\mathbb{Z}$.

Since it marks a natural next step, we also looked at its higher-dimensional equivalent: the d-dimensional lattice, i.e. the graph $G = (V, E)$ with $V = \mathbb{Z}^d$ and $E = \{\langle u, v\rangle;\ u, v \in V,\ \|u - v\|_2 = 1\}$, where $d \geq 2$ and $\|\,.\,\|_2$ denotes the Euclidean norm.

Additionally, we investigated opinion dynamics on the infinite cluster of supercritical *i.i.d. bond percolation* on the lattice $\mathbb{Z}^d,\ d \geq 2$, a standard representative for the class of infinite random graphs. The concept of i.i.d. bond percolation is in effect nothing else but the formal procedure to get the Erdős–Rényi graph from the complete graph K_n as described above – applied to more general graphs, in our case the integer lattice: For every edge, we decide independently if it is kept (with probability p) or removed (with probability $1 - p$). A maximal set of vertices linked by kept edges is called a *cluster*. For a more extensive introduction of the model, we refer to the book by Grimmett [31].

Broadbent and Hammersley [6] introduced this model in 1957 and proved that for all $d \geq 2$, there exists a critical probability p_c (depending on and monotonically decreasing with d) that marks a phase transition in the following sense: For $p < p_c$ there will almost surely be only finite clusters, while for $p < p_c$ a.s. a unique infinite cluster exists.

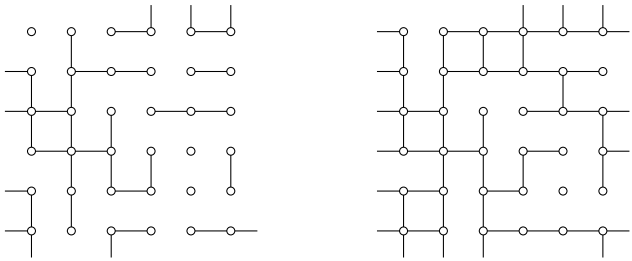


Figure 3.4: *A segment of i.i.d. bond percolation on the square lattice, with parameter* $p = 0.4$ *to the left and* $p = 0.6$ *to the right.*

In 1980, Kesten [39] proved that $p_c = \frac{1}{2}$ for the square lattice, completing an earlier result by Harris [33], who established $p_c \leq \frac{1}{2}$ by showing that there is a.s. no infinite cluster for bond percolation on the square grid with parameter $p = \frac{1}{2}$.

3.2 Opinion spaces

Just as DeGroot noted in the penultimate section of [17], when it comes to the mathematical modeling of opinions there are no rigid limits: They could be represented by numbers, vectors or even probability distributions. One only has to make sure that the opinion space is geared towards the interaction rule of the model, i.e. that it is closed with respect to all possible opinion updates.

Adopted from the Ising model, the first attempts to study opinion dynamics based on statistical mechanics featured $\{+1, -1\}$-valued opinions. As long as the evolution of attitudes towards a single yes-no question is to be modelled, this might seem sufficient, but already allowing an agent to be in the state 'irresolute' makes it necessary to include more than two opinion values i.e. to depart from binary variables. As counterpart to discrete-valued opinions, normally used to represent choices, over time there appeared models featuring opinion variables, continuously distributed on $[0, 1]$ or even the whole set of non-negative real numbers. Besides the fact that in many situations, e.g. estimating a certain unknown, a continuous opinion space is more natural, it simplifies to implement compromising behavior of interacting agents holding different opinions: The restriction to discrete opinions sometimes forces *imitating behavior* (one agent takes on the exact opinon of another).

Actually, there is a rather crucial downside to interaction rules of this kind: During the updates, the aggregate value of opinions changes, which violates the idea of *(mass) conservation* found in many physical systems. Surely, this is not a natural property in a social science setting, where mutual influences in general are asymmetric. However, as mentioned before, global properties of interacting particle systems (like conservation laws) play an important role, not least in the mathematical analysis. As a consequence, updates based on imitation – which are simple taken by themselves but render it impossible to adopt arguments using the principle of mass conservation – potentially make a model more involved from a technical point of view. This is one reason why considering continuous opinions can be quite different; the fact that a concept like 'majority opinion' does not have an equivalent in the continuous setting is another.

Accompanying the advances in the field of opinion dynamics, a growing interest in the natural extension to vector-valued opinions arose. In 1997, Axelrod [1] was one of the first to publish an article focussed on higher-dimensional opinions as opposed to earlier publications considering opinions to be scalar variables. He coined the notion of *cultural dynamics* interpreting the opinion vector as 'culture' of an individual, comprising "the set of individual attributes that are subject to social influence". In his original model, the mindset of an agent comprises 5 features which can take on any one of 10 traits. In short, the opinion space is given by $\{0, 1, \dots, 9\}^5$. Due to the reasons named above, it didn't take long until variants with continuous higher-dimensional opinion spaces emerged.

The border between cultural and scalar opinion dynamics is not sharp and many similarities exist. However, there are models featuring multidimensional opinions that do not have counterparts with scalar opinions and are therefore qualitatively different. In addition to that, as soon as the distance between two opinions matters (as is the case for bounded confidence models, see Section 4.1), the geometry comes into play. Regardless of the fact that there are many more standard metrics to choose from in higher dimensions, there is one very important difference even in Euclidean geometry: Consider a set A, its convex hull $\overline{A}$ and a point $x \notin \overline{A}$. In dimension $d \geq 2$, the distance of x to $\overline{A}$ is in general strictly less than the distance to the set A itself, see Figure 3.5 for an illustration. This is not true for $d = 1$ and makes compromising in some sense more powerful in higher dimensions when it comes to bridging gaps in between different opinions.

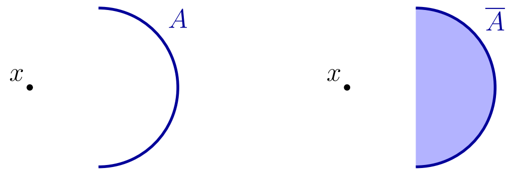


Figure 3.5: *Forming convex combinations can crucially reduce gaps – yet only in dimension $d \geq 2$.*

There have in fact been very few attempts to represent opinions by probability distributions, although this can be seen as a very natural way of modelling indetermination. In 2008, Martins [49] proposed a model in which the individuals are given two choices and internally hold a distribution embodying their preference. When they interact, they only tell each other which of the two options they would prefer and then update their probabilities according to the information received. From a mathematical point of view, a distribution on a finite probability space is nothing but a vector from the simplex of corresponding dimension, hence the opinion space still finite-dimensional. In this thesis, even infinite-dimensional opinion spaces, more precisely a model in which opinions are given by absolutely continuous distributions on $[0, 1]$, will be considered.

3.3 Interaction rules

In what follows, we are going to list some of the standard models used in sociophysics. All of them share similar ideas and they were studied with the common aim to define opinion states of the whole population (e.g. consensus or disagreement) and to determine if and how the range of the model's parameters splits up into different regimes, according to the long-time behavior of the model: In most of the cases, the dynamics tends to reduce the variability compared to the initial opinion values, a trend that can lead to a state of consensus in the long run, depending on the model specifications.

This subsection is dedicated to models that are still very close to those used in statistical mechanics, while in the next chapter we will review models that include rational behavior which might not have a counterpart in elementary physics. For further references and a more detailed discussion of the listed models, we refer the reader to the comprehensive survey article *'Statistical physics of social dynamics'* [9] by Castellano, Fortunato and Loreto.

(a) Voter model

Shortly after Weidlich's sociological reinterpretation of the Ising model, in 1973, this interacting particle system was introduced by Clifford and Sudbury [12] as a model for two spatially competing species and later named for its natural interpretation in the context of opinion dynamics among voters. Its definition is very simple: Each individual holds an opinion given by a $\{-1,+1\}$-valued variable. At every time step, one individual is selected at random and will then adopt the opinion of another agent, picked uniformly among its neighbors.

On regular lattices the evolution of this model is to some extent similar to the Ising model – in one dimension, that is on the two-sidedly infinite path $\mathbb{Z}$, it actually corresponds exactly to the limiting case of the Ising model with zero temperature. Based on well known results about random walks on grids, Clifford and Sudbury were able to conclude that on the integer lattice in dimension $d \in \{1,2\}$ any fixed finite subset of agents will a.s. finally agree (on one of the two opinions), while this does not hold for $d \geq 3$. This behavior comes from the fact that a simple random walk on the lattice is recurrent (i.e. will a.s. return to its starting point) in dimension 1 and 2, but transient (i.e. the event that there is no return to the starting point has non-zero probability) in dimension 3 and higher. A more exhaustive analysis including ergodic theorems and a complete description of all invariant measures was done by Holley and Liggett [36] in 1975. Later, the voter model was studied on various other social networks and qualitatively different behavior was found also on small-world networks for instance (see [10]).

Variants of the model include the *multitype voter model* (introduced by Spitzer [59]), in which more than two opinion values are possible, as well as the *constrained voter model* (introduced by Vazquez et al. [63]) which is defined as follows: Each agent is in one of three states ('left', 'right' or 'center') and interactions as described above can only occur involving at least one centrist (as the extremists, 'left' and 'right', are assumed to ignore each other). This behavior is a discrete analog of the so-called *bounded confidence* principle (see Chapter 4).

(b) Majority rule model
A finite collection of n individuals is considered, a fraction p_+ of which initially holds opinion $+1$, all others the opinion -1. The interaction rule is reminiscent of the one in the voter model, however agents do not necessarily meet in pairs: At each iteration a random group of individuals is chosen, and all group members then adopt the majority opinion inside the group. In the simplest version, the size of the chosen groups is a fixed odd number. But there are various variants with random size and different ways to resolve a tie in a group consisting of an even number of individuals. The model was introduced by Galam [26] and proposed to describe public debates.

Another model based on the majority rule is the so-called *majority-vote model*. Just like in the Ising model, spins are updated one at a time. At each step, the spin to be updated takes on the value of the majority of its neighbors with probability $1 - q$, the minority value with probability q and is chosen uniformly from $\{-1, +1\}$ if there is a tie. For $q = 0$ this corresponds to the Metropolis–Hastings kinetics for the zero-field Ising model at zero temperature (except for the fact that given a tie, the Metropolis–Hastings algorithm will perform a flip with probability 1), for $q = \frac{1}{2}$ to the Glauber dynamics at infinite temperature. The majority-vote model was introduced by Liggett [44], however slightly different from what became standard as he considered an individual to be part of its own neigborhood. Based on simulations, de Oliveira [18] showed that the model, considered on the square lattice, exhibits an order-disorder phase transition when q is increased. More recent studies verified this property also for small-world networks [8] and the Erdős–Rényi graph [55].

(c) Hierarchical majority rule model
A structurally different model based on the majority rule was proposed by Galam [25]: A group of $n = r^k$ individuals ($r, k \in \mathbb{N}$) equipped with identically distributed $\{-1, +1\}$-valued opinions is considered, but no social network is specified. Let p_0 denote the probability for the opinion to be $+1$.

Instead of forming a consensus by interacting, they iteratively elect group-representatives: In the first round, all individuals are randomly divided into groups of size r. In every group a representative is chosen among the members sharing the majority opinion of the group – uniformly among all members if r is even and there is a tie. This procedure is then iterated among the elected representatives until a single leader is chosen in the kth round. If p_i denotes the probability that a representative on hierarchical level i holds opinion $+1$, the recursion is given by

$$p_{i+1} = \sum_{l=\frac{r+1}{2}}^{r} \binom{r}{l} p_i^l (1-p_i)^{r-l} \quad \text{if } r \text{ is odd and}$$

$$p_{i+1} = \frac{1}{2}\binom{r}{\frac{r}{2}} p_i^{\frac{r}{2}} (1-p_i)^{\frac{r}{2}} + \sum_{l=\frac{r}{2}+1}^{r} \binom{r}{l} p_i^l (1-p_i)^{r-l} \quad \text{if } r \text{ is even.}$$

(d) Sznajd model

There are different versions of this model sharing the same basic interaction principle. The following is not the one originally introduced by Sznajd-Weron and Sznajd [62] although the most popular variant. The individuals are again considered to occupy the sites of a graph (forming the interaction network) and to hold $\{-1,+1\}$-valued opinions. A pair of neighboring agents is picked and if they agree, all their neighbors adopt this opinion as well (illustrated in Figure 3.6 below). If they disagree, however, nothing happens.

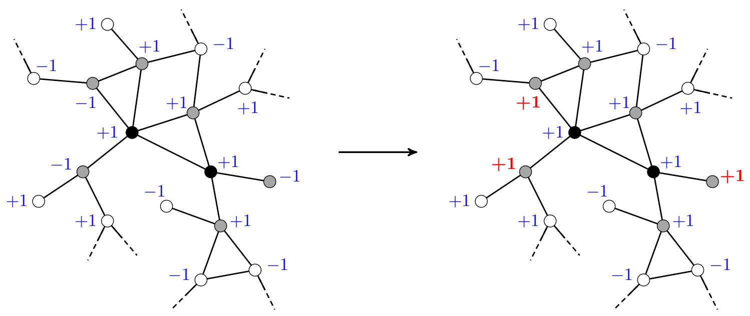


Figure 3.6: *Update rule in the Sznajd model: If the two neighbors picked (black) agree, they impose their opinion on all other individuals linked to them (gray).*

The Sznajd model is designed to incorporate the typical human behavior to be influenced more easily by a group of people that agree on a certain topic, compared to the influence of single individuals. Variants of the model have in fact been applied in order to describe and analyze voting behavior in elections.

(e) CODA model

In 2008, Martins [49] presented a new model featuring binary choices (between options A and B say, again represented by a spin $\sigma \in \{+1, -1\}$) that is based on **c**ontinuous **o**pinions and **d**iscrete **a**ctions (CODA) in the following sense: An opinion is in fact given by a probability distribution (more precisely the odds $\frac{p}{1-p}$ are considered, where p denotes the probability the considered individual attributes to option A being the better choice, $1 - p$ consequently the probability for the complementary option B). When agents interact, they only tell each other their preference (i.e. $\sigma = +1$ corresponding to $p > \frac{1}{2}$ or $\sigma = -1$ corresponding to $p < \frac{1}{2}$) but not the precise value of p.

From this piece of information, the opinions are updated with a Bayesian reasoning: Let $\alpha := \mathbb{P}(\sigma = +1 \,|\, A)$ denote the probability that an agent believes in A if that actually is the better choice and $\beta := \mathbb{P}(\sigma = -1 \,|\, B)$ the analog for B in place of A. Assuming rational agents, one might think of α and β to be larger than $\frac{1}{2}$. When individuals i and j meet and share their preferences, σ_i and σ_j, the prior odds $\frac{p_i}{1-p_i}$ of agent i get updated to

$$\frac{\mathbb{P}(A \,|\, \sigma_j = +1)}{\mathbb{P}(B \,|\, \sigma_j = +1)} = \frac{p_i}{1 - p_i} \cdot \frac{\alpha}{1 - \beta},$$

if $\sigma_j = +1$ and to $\frac{p_i}{1-p_i} \cdot \frac{1-\alpha}{\beta}$ otherwise.

These interaction rules make the model distinct from the ones introduced so far in two different ways: On the one hand, despite binary choices, the agents hold continuous opinions and as a consequence hold back some information when they interact. On the other, despite pairwise interactions, the model does equip the agents with a certain memory of the past, which is normally not the case for adaptive behavior in this setting. Both features can be seen to incorporate traits of human behavior.

4

Incorporation of selective exposure

There are many phenomena in opinion formation processes in groups, that can not be captured by the models based on or closely related to the Ising model. Although contrarian behavior can be incorporated into the Ising model by considering antiferromagnetic material (i.e. $J < 0$), as discussed already by Callen and Shapero at the end of [7], this again leads to conformity even though only on antiparallel sublattices. In order to include phenomena like homophily or individual strong-willed behavior and persisting extremism, additional concepts had to be implemented, such as *bounded confidence* for instance.

As alluded to in the introduction, models that incorporate this principle involve in their interaction rules the mental defense mechanism known as *selective exposure*, a psychological phenomenon which can not be found in the interplay of physical particles: When two individuals meet, they will only influence each another if their current opinion values are not too far apart from each other. More precisely, in most of the models there exists a parameter $\theta \geq 0$ shaping the *tolerance* of the individuals: If the current opinion value of an agent is η, other agents holding opinions at a distance larger than θ from η will just be ignored.

Many of the bounded confidence models listed below have also been reviewed in [45]. Let us continue the list from the foregoing chapter with the interacting particle system that is the core theme of this thesis.

4.1 Bounded confidence models

(f) Deffuant model

Besides the aforementioned tolerance θ, this model features another parameter, $\mu \in (0, \frac{1}{2}]$, that embodies the willingness of an individual to approach the opinion of the other in a compromise. Encounters always happen in pairs, so if agents u and v meet at time t, holding opinions $a, b \in \mathbb{R}$ respectively, the update rule reads as follows:

$$(\eta_t(u), \eta_t(v)) = \begin{cases} (a + \mu(b-a), b + \mu(a-b)) & \text{if } |a-b| \leq \theta, \\ (a, b) & \text{otherwise,} \end{cases}$$

where $\eta_t(u)$ denotes the opinion of agent u at time t. The idea behind this is simple: When two individuals interact and discuss the topic in question, they will only rate the opinion encountered as worth considering if it is close enough to their own personal belief. If this is the case, however, they will have a constructive debate and their opinions will symmetrically get closer to each other – in the special case $\mu = \frac{1}{2}$, they will separate having come to a complete agreement at the average of the opinions they hold before the interaction.

In this manner, groups of compatible agents concentrate more and more around certain opinion values (their initial average) and once each such cluster of individuals is sufficiently far from neighboring ones, the final opinions are formed and all groups will from then on only become more homogeneous by internal interactions.

When Deffuant, Neau, Amblard and Weisbuch introduced this model in [16] (some authors refer to it as Deffuant–Weisbuch model), it was considered on a finite number of agents having i.i.d. initial opinions, distributed uniformly on $[0, 1]$. As social network they chose the complete graph and a finite square lattice respectively. The encounters occurred in discrete time by picking at each time step a pair of agents uniformly at random from the edge set of the underlying interaction network graph. Depending on the values of the model parameters, θ and μ, in their simulation-studies they observed one of the following two long-time scenarios: Either the agents ended up in one opinion cluster (corresponding to a consensus) or split into several clusters (fragmentation or disagreement). A controversial point in this context is the size threshold beyond which a small number of outliers are considered minor clusters.

Stauffer et al. [61] introduced a discretized version of the model, in which the opinions can take on values from the set $\{1, 2, \ldots, q\},\ q \in \mathbb{N}$, and are rounded to the nearest integer after an update of the form described above. There have also been attempts to analyze the model with the tolerance parameter θ varying from individual to individual, revealing that in such a generalization it is the individuals with largest tolerance that ultimately determine the system's behavior.

In a recent publication [58], the idea of variable confidence bounds $\theta_t(v)$ that depend on the current opinion values has been presented: the more extreme the opinion $\eta_t(v)$ of an agent v, the smaller the corresponding value of θ. This extension of the Deffuant model bears resemblance to the relative agreement model (see below).

(g) Hegselmann–Krause model

The model introduced in [35] is quite similar to the Deffuant model, only the rule for encounters (which again happen in discrete time) is different: Given a network graph, at every time step each individual interacts with all its compatible neighbors at once and takes the average as its new opinion. If we let $\sim$ denote the reflexive adjacency relation, i.e. $u \sim v$ if $u = v$ or u and v are neighbors in the graph, and $\eta_t(u)$ once more the opinion of agent u at time t, we can write the update rule as follows:

$$\eta_{t+1}(v) = \frac{1}{N_t(v)} \sum_{\substack{u \sim v \\ |\eta_t(u) - \eta_t(v)| \leq \theta}} \eta_t(u) \quad \text{for all } v, \tag{4.1}$$

where the sum runs over the set of agents that consists of v plus its compatible neighbors and $N_t(v) = \big|\{u;\ u \sim v,\ |\eta_t(u) - \eta_t(v)| \leq \theta\}\big|$ is the size of this set at time t. Note that in contrast to the Deffuant model, the mean opinion is not conserved over time.

When it comes to simulations of the model, the major disadvantage of the Hegselmann–Krause model compared to the one introduced by Deffuant et al. is that for a dense interaction network averages of large groups of agents have to be calculated. This makes the running time until a meaningful pattern – allowing to decide whether the system approaches consensus or fragmentation – emerges rather long. However, for a finite number of individuals the system converges to a stable state in finite time: Once the opinion clusters are formed and all agents in one fixed cluster are compatible with one another, they will completely agree after one more time step making further changes of their opinions impossible.

The two models for opinion dynamics introduced by Deffuant et al. as well as Hegselmann and Krause, as described above, can be transferred to higher-dimensional opinions without any further changes – only the notion of distance has to be specified: We need to replace the absolute value by a suitable metric, which then determines the confidence ranges around a given opinion.

The vectorial version of both models was studied in [24] for instance – on the complete graph with opinion vectors from the unit square $[0, 1]^2$. Both the Euclidean and the supremum norm (i.e. $||\,.\,||_2$ and $||\,.\,||_\infty$) were used as distance metric, shaping circular and square confidence ranges respectively. The generalization of the Deffuant model on the two-sidedly infinite path $\mathbb{Z}$ to higher-dimensional opinion spaces is the object of investigation in two of the appended papers (see below): While in Paper B, vector-valued opinions are considered and the Euclidean as well as other metrics used as notions of distance, Paper C deals with the case of opinions given by absolutely continuous probability measures on $[0, 1]$ and the total variation as distance metric.

(h) Axelrod model

The model proposed by Axelrod [1] in 1997 was actually the first one introducing the concept of bounded confidence. However here, rather than having a sharp threshold, the probability of interaction decays gradually with respect to the distance of the two opinions involved: Think of the individuals again as nodes of a network. Every single one of them is endowed with an opinion vector in $\{1, 2, \dots, q\}^d$, each coordinate of which is understood to represent one of d cultural features and q is the number of possible traits per feature. In that sense, the opinion vector $\boldsymbol{\eta}(i) = (\eta_1(i), \dots, \eta_d(i))$ is modelling the current beliefs and attitudes of agent i with respect to d interrelated topics.

In an elementary step of the dynamics an individual i and a neighboring one, say j, are randomly selected and interact with probability

$$p_{i,j} = \frac{1}{d} \sum_{k=1}^{d} \mathbb{1}_{\{\eta_k(i)=\eta_k(j)\}}, \tag{4.2}$$

which is scaling with the number of shared attitudes. If they interact, one of the features in which they disagree (i.e. k such that $\eta_k(i) \neq \eta_k(j)$) is chosen uniformly at random and individual j assumed to be convinced by the arguments of i, in other words $\eta_k(j)$ is set to be equal to $\eta_k(i)$, just like in the multitype voter model.

The Axelrod model became quite popular among social scientists for the fact that it includes two principles (which we mentioned earlier) that are considered to be typical in cultural assimilation: *social influence*, i.e. interacting makes people more alike, and *homophily* – humans tend to interact more frequently with others that share essential beliefs, attitudes and behaviors. Obviously, this model also features two kinds of absorbing states: Either all opinions are the same (consensus) or any two neighboring opinions do not share one single trait (disagreement).

Following the seminal paper of Axelrod [1] – who focussed on i.i.d. initial opinion vectors being uniform on $\{1, 2, \ldots, q\}^d$ and finite square lattices as network – several analyses based on numerical simulations have been performed and show that the value of q determines whether the final state reached will be consensus or disagreement, for different networks and initial distributions.

In the original model, the actual values of the coordinates are mere labels: It does not make a difference if two neighbors have traits that differ by 1 or $q-1$. In [19] a more metric variant of the model has been considered in the sense that the interaction probability in (4.2) is changed to

$$p_{i,j} = \frac{1}{d} \sum_{k=1}^{d} \Big(1 - \frac{|\eta_k(i) - \eta_k(j)|}{q-1}\Big).$$

A further variant of the Axelrod model was suggested in the paper by Deffuant et al. [16] as a multidimensional counterpart of the Deffuant model: They considered the traits to be binary variables (corresponding to $q = 2$

above) and neighbors interact only if the number of features they disagree on does not exceed a given threshold. So the interaction probability becomes a step function at some given confidence bound. Also the interaction rule itself was defined slightly different: Once the random feature the individuals i and j disagree on is selected, j is not convinced of $\eta_k(i)$ by default but adapts with probability $\mu \in (0, \frac{1}{2}]$.

(i) Relative agreement model

Shortly after introducing the Deffuant model, the authors Deffuant, Amblard, Weisbuch and Faure [15] came up with yet another model shaping opinion formation under bounded confidence: In the so-called *relative agreement model*, the mindset of an individual is characterized not only by a continuous real-valued opinion, but also by an associated uncertainty. Agents start from i.i.d. opinions, uniformly distributed on $[-1, 1]$, and the interaction rules are as follows: Individuals meet pairwise and when agent i (holding opinion x_i and uncertainty u_i) encounters agent j (opinion x_j, uncertainty u_j), they interact only if the intervals $[x_i - u_i, x_i + u_i]$ and $[x_j - u_j, x_j + u_j]$ overlap. Under this premise, let

$$h_{ij} = \min\{x_i + u_i, x_j + u_j\} - \max\{x_i - u_i, x_j - u_j\}$$

denote the overlap. If $h_{ij} > u_j$, i.e. $x_j \in [x_i - u_i, x_i + u_i]$, the opinion and uncertainty value of agent i get updated from (x_i, u_i) to

$$\big(x_i + \mu \cdot (\tfrac{h_{ij}}{u_j} - 1) \cdot (x_j - x_i), u_i + \mu \cdot (\tfrac{h_{ij}}{u_j} - 1) \cdot (u_j - u_i)\big)$$

and analogously for agent j. The parameter $\mu \in (0, \frac{1}{2}]$ plays essentially the same role as in the Deffuant model. Besides the fact that the relative agreement model (just like the Axelrod model) implements a more gradual decay of confidence with distance of opinions, there is another feature that makes it a less idealized simplification of real-life opinion dynamics: the asymmetry in its interactions. Even if $h_{ij} > \max\{u_i, u_j\}$, implying that both agents update their opinion and uncertainty when they meet, the amount of influence agents have on each other differs. Individuals with low uncertainty influence others more compared to those with high uncertainty value.

In [15], the model was simulated with the complete graph as interaction network. In a later work [2], Amblard and Deffuant studied the model additionally on both a regular grid and a small-world network.

Having listed some of the most common models for opinion dynamics, which incorporate the idea of bounded confidence, it should be mentioned that in recent years, there have been first attempts to apply these interacting particle systems to areas outside the field of opinion formation in groups. For example, Morărescu and Girard [51] used a variant of the weighted Hegselmann–Krause model to define a randomized algorithm designed to detect communities in networks: Given the network $G = (V, E)$ and opinion processes $\{\eta_t(v)\}_{v \in V}$, they considered the confidence bound to be decreasing in time – to be more precise, they set $\theta_t = R\,\rho^t$ for appropriately chosen constants $R > 0$ and $\rho \in (0, 1)$ – and defined the set of active links at time t as

$$E(t) = \{(u, v) \in E;\ |\eta_t(u) - \eta_t(v)| \leq \theta_t\}.$$

The weighted version is a generalization of the original Hegselmann–Krause model in the sense that the arithmetic mean in (4.1) gets replaced by a weighted convex combination (to account for the fact that the influences of compatible neighbors contributing to the updated opinion might not be equally strong). Together with the time-dependent confidence bound, the update rule thus reads

$$\eta_{t+1}(v) = \sum_{\substack{u \sim v \\ |\eta_t(u) - \eta_t(v)| \leq \theta_t}} p_t(v, u)\,\eta_t(u) \quad \text{for all } v.$$

The authors chose the weights to be given by doubly stochastic invertible matrices $P(t) = \big(p_t(u, v)\big)_{u,v \in V}$, that depend on $E(t)$ only. They showed that under these technical assumptions, the model started with absolutely continuous opinions converges almost surely in finite time and their algorithm then returns the stable opinion clusters as communities of the graph.

The idea behind it is easy to grasp: Strongly connected local clusters of the graph perform enough updates to become more alike before the confidence bound gets so small that it prevents further assimilation. Sparsely connected parts of the network instead, will most likely not manage to homogenize fast enough and thus freeze with multiple opinions. In fact, this community detection algorithm performed quite well when tested on standard benchmark graphs and compared to more established algorithms in this field, such as the traditional methods of graph partitioning and spectral clustering or the popular one based on edge centrality, which was proposed by Girvan and Newman in 2002. For a detailed introduction to the topic of community detection in graphs as well as a presentation of the standard techniques just named, we refer to [23].

4.2 Disagreement versus consensus – earlier investigations of the Deffuant model

Now that we have put the model, which Deffuant et al. proposed first, into the broader context of other common models for opinion formation processes, we want to give a short overview of the results that have been achieved in earlier analyses of the Deffuant model.

The findings in the original paper [16] were threefold. Starting from i.i.d. initial opinions, uniformly distributed on $[0,1]$, the authors simulated various configurations in order to understand the influence of the parameters θ and μ as well as the underlying network topology in respect of the model's long-term behavior.

For the complete mixing case with $n = 1000$ individuals (i.e. the interaction network is the complete graph K_{1000}), Deffuant et al. noted that a confidence parameter $\theta = \frac{1}{2}$ most likely leads to consensus (pretty much at the expectation $\frac{1}{2}$), whereas $\theta = \frac{1}{5}$ causes a fragmentation into two finally homogeneous groups (with opinion values roughly at $\frac{1}{4}$ and $\frac{3}{4}$ respectively). Besides this dichotomy of regimes, by keeping θ fixed they found that the convergence parameter μ and the model size n influence the convergence time only, not the qualitative behavior, which as a consequence primarily depends on θ. The persistent opinions were arranged equidistantly and their number scaled roughly like $\lfloor \frac{1}{2\theta} \rfloor$.

When they tracked the opinion evolution of single agents from their initial opinions to one of the several persistent ones in the fragmentation case, μ turned out to be influential after all: They observed that the overlap of ranges (in terms of initial opinions) that finally led to one of the persistent opinions strongly depends on μ. For $\mu = \frac{1}{2}$ agents holding initial opinions in regions between two persistent ones could basically end up in either of the two groups, while for smaller values (e.g. $\mu = \frac{1}{20}$) almost every individual joined the cluster, whose final opinion was closest to its initial opinion value. So in a certain sense, the parameter μ determines how conservative the individuals are – both in the microscopic interactions and overall.

In addition to that, they simulated the model also for agents occupying the sites of a square lattice (of size 29×29). Here, essentially the same qualitative behavior was found: for $\theta > 0.3$ a large group consensus around $\frac{1}{2}$ with few extrem-valued outliers and no consensus for smaller values of θ. In the fragmentation case, however, the variety of scattered opinions was way bigger than in the setting of complete mixing, as clusters of individuals that are compatible in terms of opinion values can be separated spatially and in this way be prevented from interacting.

In another article published by almost the same group of authors [65], an investigation concerning heterogeneous confidence bounds was added: They simulated the complete mixing case on 200 individuals with confidence bound $\theta = \frac{1}{5}$, except for 8 individuals among them featuring a larger value ($\theta = \frac{2}{5}$). It is important to note that individual θ-values in the Deffuant model in general violate mass conversation: An encounter of two agents, whose difference in opinions lies in between their different values of θ, leads to the situation where the one with larger θ performs an opinion update, the other one does not.

Nevertheless, an interesting combination of the fragmentation and consensus case over the course of time could be observed in the simulations: In the short run clustering depends on the lower confidence bound, in the long run it depends on the higher bound. First, the majority of agents formed two incompatible opinion clusters at a distance larger than $\frac{1}{5}$, then the few 'open-minded' agents started to act as mediators between these groups and slowly but steadily brought them within talking distance of each other, which finally led to a global consensus – not at $\frac{1}{2}$ though, as asymmetric interactions are not average preserving and can cause such a drift. The transition time from one regime to the other depended very much on the proportion of individuals with larger confidence bound.

In addition to it, Deffuant et al. simulated the model with confidence bounds decreasing in time (which can be seen to describe the reasonable process of positions hardening in the course of time). In the simplest fragmentation case this led to major opinion clusters at values of about 0.60 and 0.42 – closer to each other than in the case of constant confidence bounds. Clearly, this arises from the fact that the opinions gather in a convergence movement first, before the confidence bound becomes too small and they split into two incompatible groups.

A completely different approach to the original model with fully mixed population, i.e. everybody interacts with everybody else, was pursued by Ben-Naim et al. [3]. They did not run any computer simulations of the agent based model, but considered a density based model instead (assuming that the number of individuals is large – a method termed *thermodynamical limit* in statistical physics):

If $P(x,t)$ denotes the density of agents having opinion x at time t and μ is fixed to be $\frac{1}{2}$, the following rate equation arises:

$$\frac{\partial}{\partial t} P(x,t) = \iint\limits_{|x_1-x_2|\le\theta} P(x_1,t)\, P(x_2,t) \Big[\delta(x - \tfrac{x_1+x_2}{2}) - \delta(x-x_1)\Big]\, \mathrm{d}x_1\, \mathrm{d}x_2,$$

where $\delta(.)$ denotes the Dirac delta function.

Given i.i.d. $\mathrm{unif}([0,1])$ initial opinions and $\theta = 1$ (i.e. no bounded confidence restriction), they showed that the density converges to a delta function at the initial mean $\frac{1}{2}$. In the non-trivial cases ($\theta < 1$), however, the rate equation is no longer analytically solvable. Ben-Naim and his co-workers solved it numerically (after having discretized the opinion space into $\frac{200}{\theta}$ equally spaced states) and discovered some further interesting facts about the persistent opinion clusters: In the long term, the density converges to a finite weighted sum of delta functions, i.e.

$$P(x,\infty) = \sum_{i=1}^{r} m_i\, \delta(x - x_i),$$

where $x_1, \ldots, x_r$ are the persistent opinions and $m_i,\ 1 \le i \le r$, the masses of (that means the fraction of agents ending up in) the corresponding clusters. The conservation laws (for mass and mean) obviously force

$$\sum_{i=1}^{r} m_i = 1 \quad \text{as well as} \quad \sum_{i=1}^{r} m_i\, x_i = \tfrac{1}{2}.$$

As could be expected, the behavior in the case of absent confidence restriction (namely $r = 1,\ x_1 = \frac{1}{2}$) was also found for values $\theta > \frac{1}{2}$, while for $\theta < \frac{1}{2}$ the number of clusters (at pairwise distance larger than θ) is larger than 1, in fact $r \ge 3$. In addition to that, they also found that there occur three types of persistent opinion clusters: major (mass $> \theta$), minor (mass $< \frac{\theta}{100}$) and a central cluster located at opinion value $\frac{1}{2}$. All of them are generated (and the central cluster annihilated) in a periodic sequence of bifurcations as θ is decreased. The first major clusters appear for $\theta < \frac{1}{4}$, which coincides well with the findings of Deffuant et al. who only considered major clusters and disregarded single outliers sticking to extrem opinions. Actually Ben-Naim et al. considered $\theta = 1$ to be fixed, the initial opinions instead to be i.i.d. $\mathrm{unif}([-\Delta, \Delta])$ with variable Δ, but a simple rescaling translates their results to the original model.

The heuristics they used and implemented, inspired by the methods in statistical physics, were more rigorously applied and verified in a rather recent work by Gómez-Serrano et al. [30]. They motivate the mean-field approach mathematically and prove that the long-term behavior of the limiting case (infinitely many particles) is similar to that of the model with a very large but finite number of completely mixing agents.

Laguna et al. [40] discovered another feature of the long-term behavior in the Deffuant model with complete mixing which is governed by the convergence parameter μ: The fraction of agents that end up in the two most extreme opinion clusters (which Ben-Naim et al. already showed to be minor but of larger order compared to the other minor clusters) is scaling with μ. For $\theta < \frac{1}{2}$ and larger values of μ, the formation of central opinions is fast enough to seclude many agents holding extreme initial opinions from the unification process. If μ is comparatively small, however, those extremists have enough time to become more moderate in order to be included in one of the major opinion clusters later on. In this sense, even if it may sound counterintuitive, for $\theta < \frac{1}{2}$ the formation of a partial consensus in the population actually benefits from a slower pace in the dynamics.

Stauffer and Meyer-Ortmanns [60] were among the first ones to follow up on the idea by Deffuant et al. to consider the model with an interaction topology other than the complete graph. They used random graphs generated by the Barabási–Albert model as underlying network – the usual undirected version (introduced in Section 3.1) as well as a directed one. The results of their computer simulations suggest that the transition from fragmentation to consensus happens for the value of θ being about 0.4 (on both the directed and undirected network). Unlike the case of a fully mixed population, the number of persistent opinions in the non-consensus case not only depends on θ but also on n, the number of individuals (for the same reason as in the case of a square lattice). The dependence of the number of clusters on n (with θ fixed) was estimated to be linear.

In 2004, Fortunato [22] investigated the threshold for a *complete consensus* among the agents – as opposed to previous notions of consensus describing the formation of a widely adapted main stream opinion neglecting some few outliers (in other words: only one major cluster). He simulated the Deffuant model on a complete graph, a square lattice with perodic boundary conditions as well as two random graphs – those originating from the Barabási–Albert and the Erdős–Rényi model. In the latter, he chose to adapt the probability p (with which an edge is kept) to the number n of agents in such a way that the average degree, $(n-1)p$, stays roughly constant.

Fortunato made two central observations: Firstly, the critical value for θ above which a complete consensus is formed equals $\frac{1}{2}$ in all four social topologies, irrespectively of μ. Secondly, on each of the four networks the probability of complete consensus converges to a step function at the threshold $\theta = \frac{1}{2}$ when the number of individuals is increased.

It has to be mentioned at this point that he performed update steps as ordered sweeps over the population (for the sake of simplicity): In each round every individual gets – one after the other – the opportunity to interact with a randomly selected neighbor. This is different from the original update rule where the edge along which the next potential interaction takes place is picked uniformly at random. For large regular systems, however, this seems unlikely to matter.

The first result for the Deffuant model considered on an infinite graph was published by Lanchier [41] in 2011. He studied the standard Deffuant model (i.i.d. $\mathrm{unif}([0,1])$ initial opinions) on the two-sidedly infinite path $\mathbb{Z}$ using the following geometric idea: Instead of analyzing the opinion profiles $\{\eta_t(v)\}_{v\in\mathbb{Z}}$ directly, where $\eta_t(v)$ denotes the opinion of individual v at time t, he considered what he calls their *broken line representation*, i.e. $\{\xi_t(v)\}_{v\in\mathbb{Z}}$ with

$$\xi_t(0) = 0 \quad \text{and} \quad \xi_t(v) = \begin{cases} \sum\limits_{0\le u\le v-1} (2\eta_t(u)-1), & \text{if } v>0,\\ \sum\limits_{v\le u\le -1} (2\eta_t(u)-1), & \text{if } v<0. \end{cases}$$

Using quite intricate geometric arguments and the concentration inequality due to Azuma–Hoeffding, he verified a set of properties for this concatenation of two symmetric random walks (one evolving forwards, one backwards in time; both starting at the origin) which allowed to prove the following result:

Theorem 4.1. *Consider the Deffuant model on the graph $G = (V,E)$, where $V = \mathbb{Z}$ and $E = \{\langle v, v+1\rangle;\ v \in \mathbb{Z}\}$. If $\mu \in (0, \frac{1}{2}]$ is arbitrary but fixed, the initial opinions are i.i.d. $\mathrm{unif}([0,1])$ and $\{\eta_t(v)\}_{v\in\mathbb{Z}}$ denotes the opinion profile at time t, then the following holds:*

(i) For $\theta > \frac{1}{2}$, all neighbors are eventually compatible in the sense that for all $v \in \mathbb{Z}$:

$$\lim_{t\to\infty} \mathbb{P}(|\eta_t(v) - \eta_t(v+1)| \le \theta) = 1.$$

(ii) For $\theta < \frac{1}{2}$, with probability 1 there will be infinitely many $v \in \mathbb{Z}$ with

$$\lim_{t\to\infty} |\eta_t(v) - \eta_t(v+1)| > \theta.$$

One thing that is quite remarkable about this phase transition in the behavior of the Deffuant model is the fact that it already occurs for the one-dimensional lattice – in marked contrast to the Ising model.

Häggström [32] used different techniques to reprove and slightly sharpen this result – showing that in the consensus regime (i), all opinions actually converge almost surely to the mean $\frac{1}{2}$ of the initial distribution. The crucial idea in his proof resides in the connection of the opinion dynamics of the Deffuant model to a non-random interaction process, which he proposed to call *Sharing a drink* (SAD). The SAD-procedure is dual to the opinion formation in the sense that it keeps track of the opinion genealogy of an individual, i.e. the contributions of all initial opinions to the current composition of its opinion.

This idea could in fact be employed to generalize the result for the Deffuant model on $\mathbb{Z}$ to initial opinion configurations other than i.i.d. $\mathrm{unif}([0,1])$, as was done in Paper A (see below) and by Shang [56] simultaneously and independently.

5

Extreme opinions and water transport

In their analyses of the Deffuant model on $\mathbb{Z}$ featuring i.i.d. $\text{unif}([0,1])$ initial opinions, both Lanchier [41] and Häggström [32] singled out agents that are cast-iron centrists. These agents start with an opinion value close to the mean $\frac{1}{2}$ and will never move far away from it (irrespectively of future interactions), due to the fact that the influences they can possibly be exposed to are – loosely speaking – either close to the mean as well or marginal. The opinion $\eta_t(v)$, of an agent $v \in \mathbb{Z}$ at a later time $t > 0$, is a convex combination of all initial opinions and the maximally possible contributions on $\mathbb{Z}$ decay inversely proportional to the graph distance. Hence, the initial opinion profile $\{\eta_0(v)\}_{v\in\mathbb{Z}}$ can be such that agent v sits well-shielded in a large section of individuals equipped with initial opinions close to $\frac{1}{2}$ and all individuals holding more extreme opinions are too far away to have a significant influence on v.

With this idea in mind (leaving aside the fact that the bounded confidence restriction might actually eliminate possible influences), obvious candidates for vertices of this kind are what Häggström [32] calls two-sidedly ε-flat vertices and Lanchier [41] denotes by the random set

$$\Omega_0 = \Big\{ v \in \mathbb{Z};\ \tfrac{1}{2} - \varepsilon < \tfrac{1}{n+1} \sum_{u=v}^{v+n} \eta_0(u),\ \tfrac{1}{n+1} \sum_{u=v-n}^{v} \eta_0(u) < \tfrac{1}{2} + \varepsilon,\ \forall\, n \geq 0 \Big\}.$$

If the initial opinions are i.i.d. $\mathrm{unif}([0,1])$, it can be verified that the set Ω_0 is almost surely non-empty (in fact of infinite cardinality) for all $\varepsilon > 0$, see Prop. 1.1 in [41] or Lemma 4.3 in [32], and that the opinion at two-sidedly ε-flat vertices will be confined to the interval $[\frac{1}{2} - 6\varepsilon, \frac{1}{2} + 6\varepsilon]$ for all times, see Lemma 6.3 in [32]. This consideration, however, is adjusted to the geometry of the underlying network $\mathbb{Z}$ and does not answer the question whether on more general graphs as well (e.g. higher-dimensional grids), we can find vertices whose opinions are constrained to stay close to the mean by the initial profile already.

In the standard Deffuant model, the existence of agents that will hold an opinion close to the mean $\frac{1}{2}$, no matter how the random interactions take place, force a supercritical behavior of the system (for θ sufficiently large) as they will always be at speaking terms with the whole range of opinions $[0, 1]$ then. Lorenz and Urbig [46] addressed the question, for which values of θ an asymptotic consensus on K_n can be enforced (alternatively prevented) if the interactions are not random but chosen in an elaborate succession, i.e. the agents follow a predefined communication plan, adjusted to the initial opinion profile. More precisely, Lorenz and Urbig define θ_{low} (resp. θ_{high}) as the infimum (resp. supremum) of confidence bounds, such that returning to random encounters after an appropriately chosen finite succession of interactions will lead to consensus (disagreement) with probability 1, and prove

$$\max_{1 \leq k \leq n-1} \Delta x_k \leq \theta_{\mathrm{low}} \leq \max_{1 \leq k \leq n-1} \sum_{j=0}^{k-1} \mu^j\, \Delta x_{k-j} \qquad \text{as well as}$$

$$\theta_{\mathrm{high}} = \max_{1 \leq k \leq n-1} \left(\frac{1}{n-k} \sum_{i=k+1}^{n} x_i - \frac{1}{k} \sum_{j=1}^{k} x_j \right),$$

where $\Delta x_i = x_{i+1} - x_i$, for $1 \leq i \leq n-1$, and $(x_1, \dots, x_n)$ denotes the vector of initial opinions $\{\eta_0(i)\}_{i=1}^n$ in increasing order. These results are verified by exhibiting a communication plan that circumvents (resp. aims for) the creation of large gaps in the opinion range.

In the same way as for gaps, one can try to manipulate the interaction scheme in such a way that the opinion of one fixed agent gets as extreme as possible (which might then answer the question if there are nodes stuck with an opinion close to the mean for all times right from the beginning). If we drop the bounded confidence restriction, this combinatorial optimization problem can be seen as the task of transporting water on a graph without pumps: Agents are reinterpreted as identical water barrels on a plane, their social network as a system of (locked, water-filled) pipes connecting them and the opinion values as the corresponding water levels. Opening pipe $\langle u, v\rangle$ will lead to an update $(\eta(u), \eta(v)) \mapsto ((1-\mu)\,\eta(u)+\mu\,\eta(v), (1-\mu)\,\eta(v)+\mu\,\eta(u))$, where $\mu \in (0, \frac{1}{2}]$ can be chosen arbitrarily. We then want to maximize the water level in a fixed barrel (*target vertex*) by opening and closing the locks in an appropriate order.

If we disregard the option to close locks, the problem turns into finding a connected subset of nodes including the target vertex with maximal average; a concept known as greedy lattice animal, which will be introduced and reviewed in the next section. Its relation to the water transport problem, which is relevant in the analysis of the Deffuant model as outlined above, will be discussed in Section 5.2.

5.1 Greedy lattice animals and site percolation

In 1993, Cox, Gandolfi, Griffin and Kesten [13] introduced the notion of greedy lattice animals: They considered an i.i.d. family of positive random variables $\{X_v;\ v \in \mathbb{Z}^d\}$ and the set of connected subsets comprising n vertices of the grid including the origin, $\Xi_{\mathbf{0}}(n) := \{\xi \subseteq \mathbb{Z}^d;\ \mathbf{0} \in \xi,\ |\xi| = n,\ \xi \text{ is connected}\}$. A set $\xi \in \Xi_{\mathbf{0}}(n)$ with maximal weight $\sum_{v\in\xi} X_v$ is called a *(vertex) greedy lattice animal* (of size n), its weight denoted by N_n.

With respect to the common marginal distribution, represented by $X_{\mathbf{0}}$, the random variable associated to the origin, they established the following asymptotic bound (where $\log^+(x)$ is a short notation for the positive part of the logarithm, i.e. $\max\{\log(x), 0\}$):

Theorem 5.1. *If for some $a > 0$,*

$$\mathbb{E}\big(X_{\mathbf{0}}^d\,(\log^+ X_{\mathbf{0}})^{d+a}\big) < \infty, \tag{5.1}$$

then there exists a constant $M \in \mathbb{R}$ such that

$$\limsup_{n\to\infty} \frac{N_n}{n} \le M \quad \textit{almost surely.}$$

In a subsequent publication, Gandolfi and Kesten [28] improved this result by verifying that the moment condition for the marginal distribution in fact implies a.s. linear growth of N_n in n: They showed that given (5.1), there exists a constant $N \in \mathbb{R}$ such that $\lim_{n\to\infty} \frac{N_n}{n} = N$ almost surely and in L^1.

Recall that in the model of i.i.d. *bond* percolation, which was introduced in Subsection 3.1, we take a graph and toss independent p-biased coins to decide which of the edges are kept and removed respectively. Applying the same thinning procedure not to the edges but to the vertices of a graph instead – i.e. independently, each vertex is chosen to be kept (with probability p) or erased along with all edges it is incident to (with probability $1-p$) – is called i.i.d. *site* percolation. Similarly as for bond percolation, in dimension $d \geq 2$, there exists a critical probability $p_c \in (0,1)$ for i.i.d. site percolation on $\mathbb{Z}^d$ marking the emergence of an infinite cluster. Note that the critical probabilities for bond and site percolation on the integer lattice of dimension at least 2 are related but not equal. For further details we refer once again to Grimmett [31].

Relating both concepts, greedy lattice animals and site percolation, Lee [43] proved among other things the following:

Theorem 5.2. *Fix $d \geq 2$ and consider an i.i.d. family of positive bounded random variables $\{X_v;\ v \in \mathbb{Z}^d\}$, the sets $\Xi_{\mathbf{0}}(n)$ and random variables N_n, for $n \in \mathbb{N}$, as above. Let p_c denote the critical probability of i.i.d. site percolation on $\mathbb{Z}^d$, $R := \inf\{r \in \mathbb{R};\ X_{\mathbf{0}} \leq r$ almost surely$\}$ be the essential supremum of the marginal distribution and N the almost sure limit of $\frac{N_n}{n}$. Then the following holds:*

(i) If $\mathbb{P}(X_{\mathbf{0}} = R) < p_c$, then $N < R$.

(ii) If $\mathbb{P}(X_{\mathbf{0}} = R) \geq p_c$, then $N = R$.

The case $\mathbb{P}(X_{\mathbf{0}} = R) > p_c$ is particularly easy and exhibits the connection to site percolation most obviously: If we disregard all nodes but those $v \in \mathbb{Z}^d$ with $X_v = R$, with probability 1 an infinite cluster remains. The origin can be connected to this cluster through finitely many other nodes, which guarantees a nested sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of connected sets containing the origin with $\lim_{n\to\infty} \frac{1}{n} \left|\{v \in \xi_n;\ X_v = R\}\right| = 1$.

Apart from these results, the idea of a vertex greedy animal (as defined above) can of course be applied to more general graphs than integer lattices.

5.2 Optimizing pumpless water transport

In the context of making a fixed agent's opinion (respectively the water level at a target vertex) most extreme, we don't care about the number of involved vertices, hence with respect to greedy lattice animals the following definition is most appropriate:

For a fixed graph $G = (V, E)$, target vertex v and water levels $\{\eta(u)\}_{u\in V}$, let us call a finite set $C \subseteq V$ a *lattice animal (LA)* for v if C contains v and is connected. C is a *greedy lattice animal* for v if it maximizes the average of water levels over such sets, i.e. if its average equals the value

$$\mathrm{GLA}(v) := \sup_{C \text{ LA for } v} \frac{1}{|C|} \sum_{u\in C} \eta(u).$$

Note that with this altered definition, a greedy lattice animal need not necessarily exist for infinite graphs, as $\mathrm{GLA}(v)$ might not be attained.

If $\kappa(v)$ denotes the supremum of water levels attainable at v by opening *and* closing locks, $\mathrm{GLA}(v)$ can be used as a lower bound on $\kappa(v)$ only. As a consequence, for i.i.d. $\mathrm{unif}([0, 1])$ initial water levels, we can not conclude from Theorem 5.2 and $\mathbb{P}(\eta(\mathbf{0}) = 1) = 0$ that on $\mathbb{Z}^d$, the highest possible water level at the origin $\mathbf{0}$ is bounded away from 1 with positive probability.

In fact, the two problems – greedy lattice animals and water transport – as related as they might seem, are quite different from a technical point of view: The option to shut open locks introduces a temporal dimension and makes it crucial, which moves are performed first. To get the idea, consider the elementary example depicted in Figure 5.1 below.

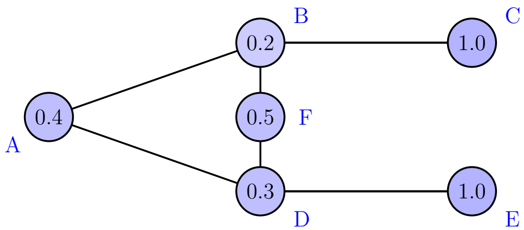


Figure 5.1: *A simple water transport instance on 6 nodes.*

If A is chosen to be the target vertex, the greedy lattice animal is given by the set $\{A, B, C, D, E\}$ with a value of $\mathrm{GLA}(A) = 0.58$. Vertex F can however be used to improve the two bottlenecks B and D. This is done most beneficially, if the pipe $\langle D, F\rangle$ is opened first (until the two water levels have balanced out completely at 0.4), then closed and thereafter the same procedure repeated for the edge $\langle B, F\rangle$, leading to $\kappa(A) = 0.62$. Further, this simple water transport instance exemplifies the enhanced structural complexity of the problem: While it is sufficient to consider spanning trees when searching for a greedy lattice animal, additional edges forming circles might become important in the corresponding water transport problem, as it is the case here.

In Paper D (see below), we address the water transport problem both on finite and infinite graphs and consider its complexity. It turns out that one does not gain from opening several pipes simultaneously or chosing the mixture parameter μ in a move to be less than $\frac{1}{2}$, i.e. closing a pipe before the contents of the two connected barrels have levelled completely. Furthermore, we found that in dimension $d \geq 2$ and given i.i.d. $\mathrm{unif}([0,1])$ initial opinions, the water level of a fixed vertex of the integer lattice $\mathbb{Z}^d$ can almost surely be raised as close to 1 as desired – in contrast to both greedy lattice animals and dimension $d = 1$.

This fact is one of the main obstacles when trying to generalize the results established for the Deffuant model on the two sidedly-infinite path $\mathbb{Z}$ to higher dimensions, as it invalidates one of the most central arguments.

6

Summary of appended papers

Paper A:
Further results on consensus formation in the Deffuant model

(co-authored with Olle Häggström)

The contribution of this paper to the analysis of long-term behavior in the Deffuant model featuring real-valued opinions on infinite graphs can be broken down into three parts.

The first one – as alluded to in Section 4.2 – is the extension of the statement from Theorem 4.1 to more general initial distributions. As was done in [41] and [32], we consider the model on $\mathbb{Z}$ with i.i.d. initial opinions, this time however distributed according to a general law $\mathcal{L}(\eta_0)$ in place of $\mathrm{unif}([0,1])$. Building on the ideas from [32], we were able to settle all cases in which the mean $\mathbb{E}\,\eta_0$ of the initial marginal distribution is well-defined: If $\mathcal{L}(\eta_0)$ is bounded, there exists a critical value θ_c for the parameter θ that marks a sharp phase transition in the long-term behavior from almost sure disagreement (the agents split into finite, incompatible but finally homogeneous segments) to a.s. complete consensus (all opinions converge to the mean $\mathbb{E}\,\eta_0$). The value of θ_c depends on two characteristics of the distribution $\mathcal{L}(\eta_0)$: its mean and its support. More precisely, the critical value turns out to be

$$\theta_c = \max\{\mathbb{E}\,\eta_0 - \text{essinf}(\eta_0), \text{esssup}(\eta_0) - \mathbb{E}\,\eta_0, h\}, \tag{6.1}$$

where the essential infimum and supremum mark the extreme ends of the support and h denotes the length of its largest gap (which means the largest subinterval $I \subseteq [\text{essinf}(\eta_0), \text{esssup}(\eta_0)]$ with $\mathbb{P}(\eta_0 \in I) = 0$). In the case of an unbounded initial distribution – under the assumption that not both $\mathbb{E}\,\eta_0^+$ and $\mathbb{E}\,\eta_0^-$ are infinite – the model a.s. behaves subcritically (disagreement) for any choice of $\theta > 0$. Note that this matches the statement for the bounded case, since θ_c as defined in (6.1) becomes infinite for an unbounded initial distribution.

In addition to that, we point out how these results can be transferred to special cases of dependent initial opinions. For the arguments used to be valid, it is sufficient that the initial configuration is ergodic and fulfils an additional requirement, that is called *finite energy condition* in percolation theory (and was introduced by Newman and Schulman [52]).

In the second part, the model is considered on higher-dimensional integer lattices $\mathbb{Z}^d,\ d \geq 2$. Although the central ideas of proof from dimension one do not transfer to higher dimensions, elaborating some of the arguments allows us to prove at least the following partial result: If the marginal distribution of the i.i.d. initial configuration is bounded and θ sufficiently large (strictly larger than $\frac{3}{4}$ in the case of $\text{unif}([0,1])$ initial opinions for example), the opinion of every agent will still almost surely converge to the mean of the initial distribution. In addition to this, on the one hand we show that the opinions converge in distribution for any value of θ and on the other hand discuss a generalization to transitive, amenable graphs.

In the last part, we consider the Deffuant model on the infinite cluster of supercritical i.i.d. bond percolation on $\mathbb{Z}^d,\ d \geq 2$. In this setting one can retrieve the results derived for the full grid and on top of that, we were able to show that for small values of θ, the opinions of the agents belonging to the infinite cluster cannot converge to one fixed value. Neighboring individuals could, however, still come to a complete agreement in the long run without their opinions converging to a deterministic limit (corresponding to the type of consensus, which Lanchier [41] formulated in part (i) of Theorem 4.1).

Paper B:
The Deffuant model on $\mathbb{Z}$ with higher-dimensional opinion spaces

As mentioned in Section 3.2, this paper deals with the generalization of the Deffuant model on $\mathbb{Z}$ to vector-valued opinions. In the first part, we generalize the findings for univariate opinions from Paper A to multivariate opinions and stick to the Euclidean norm as natural replacement for the absolut value (which was taken to measure the distance between two opinions in the case of real-valued opinions). Using geometric arguments, that are considerably more involved than in the case of scalar opinions, we manage to verify properties of the support of the opinion distribution $\mathcal{L}(\eta_t)$ for times $t > 0$, depending on the initial distribution $\mathcal{L}(\eta_0)$. Especially the notion of a gap in the support of $\mathcal{L}(\eta_0)$ has to be properly defined and analyzed in higher dimensions in order to play the same role as for univariate distributions.

In the second part, we allow for more general metrics ρ to be employed as measures of distance – determining if the opinions of two agents are close enough for them to interact. We are able to transfer the results from the Euclidean case, given that ρ satisfies appropriate extra conditions: weak convexity, local domination by the Euclidean distance and sensitivity to unbounded coordinates. Through several examples, the necessity of these additional assumptions is verified.

Paper C:
Overly determined agents prevent consensus in a generalized Deffuant model on $\mathbb{Z}$ with dispersed opinions

The generalization of the original Deffuant model in terms of opinion spaces is taken one step further in this paper: We consider the model on $\mathbb{Z}$, in which opinions are represented by absolutely continuous probability distributions on $[0, 1]$. In comparison to finite-dimensional opinions, the expectation of $\mathcal{L}(\eta_0)$ corresponds to the so-called *intensity measure* in the context of random probability distributions.

For the sake of concreteness, we consider a model in which the initial opinions are given by symmetric triangular distributions: Initially, for each agent $v \in \mathbb{Z}$ independently, a vector (U, V) from the uniform distribution on $[0, 1]^2$

is drawn. Then v gets attributed the random measure given by the density that is 0 outside $[m, M]$ and linear on both $[m, \frac{m+M}{2}]$ and $[\frac{m+M}{2}, M]$, where $m := \min\{U, V\}$ and $M := \max\{U, V\}$. This way of representing opinions can be seen as an improvement over real-valued opinions introducing the idea of uncertainty (around a favored value).

For this model, we calculate the intensity measure and verify that extremely determined agents (i.e. $|U - V|$ very small) will a.s. prevent consensus for any $\theta \in (0, 1)$, given that the total variation is used to measure the distance between two opinions.

If the determination is bounded, in the sense that the random vector (U, V) is taken from $\mathrm{unif}([0, 1]^2)$ conditioned on $|U - V| \geq \gamma$, for a fixed constant $\gamma \in (0, 1)$, the picture changes. The phase transition in the long-term behavior from a.s. disagreement to a.s. consensus, known from the investigations dealing with finite-dimensional opinion spaces, reappears and we are able to calculate the precise threshold value θ_{c}.

Paper D:
Water transport on graphs

(co-authored with Olle Häggström)

Incited by the impossibility of transferring the ideas used in the analysis of the Deffuant model on $\mathbb{Z}$ to higher-dimensional grids, we defined and analyzed a combinatorial optimization problem that can be seen as pumpless water transport on a graph: The agents holding different opinion values are reinterpreted as identical water barrels that are filled to different levels, the interactions (still taking place along the edges of the network) as opening the lock in the pipe between the two nodes for a certain time span. In this manner, we essentially consider the same interacting particle system and only think of converging water levels instead of compromising individuals, but we drop the randomness of encounters and the confidence bound restriction.

Asking for the maximal amount of water that can be accumulated in a fixed target barrel by opening and closing the locks in an elaborate succession is closely related to the question of how extreme the opinion of an agent possibly can become depending on the initial configuration. First, we provide some tools to describe and analyze optimal strategies to maximize the water in a given barrel and solve the optimization problem for different types of finite graphs.

Then, we consider the problem's complexity and prove by a polynomial reduction of the satisfiability problem 3-SAT to a suitably chosen instance of the water transport problem that the latter is NP-hard.

Finally, we verify a fact that in a manner of speaking accounts for the different challenges faced in the analysis of the Deffuant model on the integer lattice $\mathbb{Z}^d$, depending on the dimension d: Given i.i.d. $\mathrm{unif}([0,1])$ initial water levels, the highest achievable amount in a fixed barrel depends on the initial configuration in a non-deterministic way both for finite graphs and the two-sidedly infinite path $\mathbb{Z}$. For all other quasi-transitive infinite graphs, however, the level can a.s. be increased to a value as close to 1 as desired by opening (and closing) the locks in an appropriate order. The crucial feature of the underlying graph turns out to be, whether or not the graph contains a neighbor-rich half-line, i.e. an infinite path with sufficiently many extra vertices attached to it.

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Further results on consensus formation in the Deffuant model

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Further results on consensus formation in the Deffuant model*

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Abstract

The so-called Deffuant model describes a pattern for social interaction, in which two neighboring individuals randomly meet and share their opinions on a certain topic, if their discrepancy is not beyond a given threshold θ. The major focus of the analyses, both theoretical and based on simulations, lies on whether these single interactions lead to a global consensus in the long run or not. First, we generalize a result of Lanchier for the Deffuant model on $\mathbb{Z}$, determining the critical value for θ at which a phase transition of the long term behavior takes place, to other distributions of the initial opinions than i.i.d. uniform on $[0, 1]$. Then we shed light on the situations where the underlying line graph $\mathbb{Z}$ is replaced by higher-dimensional lattices $\mathbb{Z}^d$, $d \geq 2$, or the infinite cluster of supercritical i.i.d. bond percolation on these lattices.

Keywords: Deffuant model; consensus formation; percolation.
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1 Introduction

Let $G = (V, E)$ be a simple graph, i.e. having undirected edges and neither loops nor multiple edges. The considered graph may either be finite or infinite with bounded maximal degree. Furthermore, without loss of generality we can assume G to be connected, since in what follows one could consider the connected components seperately otherwise. Every vertex is understood to represent an individual and will at each time $t \geq 0$ be assigned a value representing its opinion. All the edges in E are connections between individuals allowing for mutual influence. There are a number of models for what is called *opinion dynamics*, which are qualitatively different but share similar ideas, see [2] for an extensive survey.

The *Deffuant model* (introduced by Deffuant et al. [3]) featuring two model parameters $\mu \in (0, \frac{1}{2}]$ and $\theta \in (0, \infty)$ is defined as follows. At time $t = 0$, the vertices are assigned i.i.d. initial opinions, in the standard case uniformly distributed on the interval $[0, 1]$. In

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addition, serving as a regime for the random encounters, every edge $e \in E$ is assigned a unit rate Poisson process. The latter are independent of each other and the initial distribution of opinion values. Denote the opinion value at $v \in V$ at time t by $\eta_t(v)$, which remains unchanged until at some time t a Poisson event occurs at an edges incident to v, say $e = \langle u, v\rangle$. The opinion values of u and v just before this happens may be $\eta_{t-}(u) = \lim_{s\uparrow t} \eta_s(u) =: a$ and $\eta_{t-}(v) = \lim_{s\uparrow t} \eta_s(v) =: b$ respectively.

If these values are within the confidence bound θ, they come symmetrically closer to each other, if not they stay unchanged, i.e.

$$\eta_t(u) = \begin{cases} a + \mu(b-a) & \text{if } |a-b| \le \theta, \\ a & \text{otherwise} \end{cases}$$

and similarly (1.1)

$$\eta_t(v) = \begin{cases} b + \mu(a-b) & \text{if } |a-b| \le \theta, \\ b & \text{otherwise.} \end{cases}$$

Observe that μ is modelling the willingness of the individuals to step towards other opinions encountered that fall within their interval of tolerance, shaped by θ. In other words, a value of μ close to 0 represents a strong reluctance to change one's mind. For the process to be well-defined, on the one hand one has to make sure that neither two Poisson events occur simultaneously nor that there is a limit point in time for the events occuring on edges incident to one fixed vertex. But since the maximal degree is bounded and we assume the vertex set to be countable, this is almost surely the case. On the other hand, there is a more subtle issue in how the simple interactions shape transitions of the whole system on an infinite graph – is it well-defined there as well? For infinite graphs with bounded degree, this problem is settled by standard techniques in the theory of interacting particle systems, see Thm. 3.9 on p. 27 in [11].

The most natural question to ask seems to be, if the individual opinions will converge to a common consensus in the long run or if they are going to be split up into groups of individuals holding different opinions. In this regard let us define the following types of scenarios for the asymptotic behavior of the Deffuant model on a connected graph as $t \to \infty$:

Definition 1.1.

(i) No consensus
There will be finally blocked edges, i.e. edges $e = \langle u, v\rangle$ s.t.

$$|\eta_t(u) - \eta_t(v)| > \theta,$$

for all times t large enough. Hence the vertices fall into different opinion groups.

(ii) Weak consensus
Every pair of neighbors $\{u, v\}$ will finally concur, i.e.

$$\lim_{t\to\infty} |\eta_t(u) - \eta_t(v)| = 0.$$

(iii) Strong consensus
The value at every vertex converges, as $t \to \infty$, to a common limit l, where

$$l = \begin{cases} \textit{the average of the initial opinion values}, & \textit{if } G \textit{ is finite} \\ \mathbb{E}\, \eta_0, & \textit{if } G \textit{ is infinite.} \end{cases}$$

Let the scenario in which we have weak consensus, but at some vertices v the value $\eta_t(v)$ is not converging be called strictly weak consensus. Whether strictly weak consensus can actually occur (for some graphs and some initial distributions) is an open problem.

On finite graphs, strictly weak consensus is impossible as the opinion average is preserved over time and in general the answer to the question whether we get consensus in the long run or not clearly depends on the initial setting. With independent initial opinions distributed uniformly on $[0, 1]$ even for values of θ close to but smaller than 1 consensus might be prevented, albeit with a small probability, e.g. when we get stuck right from the beginning with all the opinions being close to either 0 or 1 leaving a gap larger than θ in between, preventing any two individuals situated at different ends of the opinion range from compromising. In the interdisciplinary area labelled "sociophysics" some work has been done in simulating the long-term behavior of this model on various types of finite graphs, such as in [15].

On infinite regular lattices however, the picture is different and the minimal example almost settled. For the graph on $\mathbb{Z}$ in which consecutive integers are joined by edges, Lanchier [10] showed for the standard case with i.i.d. $\text{unif}([0, 1])$ distributed initial values that regardless of μ, which is just controlling the speed of convergence, the threshold between no consensus and consensus θ_c is $\frac{1}{2}$, which is the essence of Theorem 2.1.

In this paper, we investigate what happens when this basic setting is generalized, in two different directions. In Section 2 we stay on the one-dimensional lattice, i.e. the line graph on $\mathbb{Z}$, but allow for more general initial distributions and are able to settle most but not all cases of i.i.d. initial configurations (see Theorem 2.2). We also generalize the model slightly to allow for dependent initial opinions given by stationary ergodic sequences that satisfy the so-called *finite energy condition*, known from percolation theory. (The generalization of the Deffuant model to multivariate opinions can be found in the upcoming paper [7].)

In Section 3, $\mathbb{Z}$ is replaced by the general regular lattice $\mathbb{Z}^d$. For $d \geq 2$ most of the techniques developed for the one-dimensional case $\mathbb{Z}$ break down, but we are at least able to show that there won't be disagreement for a sufficiently large confidence bound, larger than $\frac{3}{4}$ in the standard i.i.d. uniform case (see Theorem 3.1). Furthermore, the arguments used transfer with only minor changes to the more general case of an infinite, locally finite, transitive and amenable graph (see Remark 3.6).

Finally, in the last section we consider the Deffuant model on the random subgraph of $\mathbb{Z}^d$ given by supercritical i.i.d. bond percolation independent of the random variables driving the opinion dynamics, i.e. the initial configuration and the Poisson processes. Besides an extension of the result we derived for the full grid to this setting (Theorem 4.2), a lower bound for values of θ allowing for strong consensus on the infinite component is established (Theorem 4.3).

We find it slightly surprising that we can prove this last result for supercritical percolation (with $p < 1$) but not for the full lattice. The more common situation for random processes living on supercritical percolation clusters is that these are easier to handle on the full lattice.

2 Generalized initial configurations on $\mathbb{Z}$

2.1 Independent and identically distributed initial opinion values

Theorem 2.1 (Lanchier). *Consider the Deffuant model on the graph $(\mathbb{Z}, E)$, where $E = \{\langle v, v+1\rangle, v \in \mathbb{Z}\}$ with i.i.d. $\text{unif}([0, 1])$ initial configuration and fixed $\mu \in (0, \frac{1}{2}]$.*

(i) If $\theta > \frac{1}{2}$, the model converges almost surely to strong consensus, i.e. with probability 1 we have: $\lim_{t\to\infty} \eta_t(v) = \frac{1}{2}$ for all $v \in \mathbb{Z}$.

(ii) If $\theta < \frac{1}{2}$ however, the integers a.s. split into (infinitely many) finite clusters of neighboring individuals asymptotically agreeing with one another, but no global consensus is approached.

For the line graph, the critical value θ_c equals thus $\frac{1}{2}$, but what happens at criticality is still an open question. Lanchier's result was reproven by Häggström using somewhat more basic techniques (see [5], Thm. 6.5 and Thm. 5.2).

It turns out that the methods in [5] can be adapted to i.i.d. initial distributions beyond the unif($[0,1]$) case. In the following theorem, we determine θ_c in all cases except when the distribution's positive and negative parts both have infinite expectation (this case remains unsolved). Upon completing this work, we learned that a similar extension was simultaneously and independently done by Shang [14]. Part (a) of our Theorem 2.2 conflicts with Thm. 1 in [14], the discrepancy being due to Shang overlooking the crucial effect that gaps in the support of the distribution of η_0 have, if they are large.

Theorem 2.2. *Consider the Deffuant model on $\mathbb{Z}$ as described earlier with the only exception that the initial opinions are not necessarily distributed uniformly on $[0,1]$ (but still i.i.d.).*

(a) Suppose the initial opinion of all the agents follows an arbitrary bounded distribution $\mathcal{L}(\eta_0)$ with expected value $\mathbb{E}\,\eta_0$ and $[a,b]$ being the smallest closed interval containing its support. If $\mathbb{E}\,\eta_0$ does not lie in the support, there exists some maximal, open interval $I \subset [a,b]$ such that $\mathbb{E}\,\eta_0$ lies in I and $\mathbb{P}(\eta_0 \in I) = 0$. In this case let h denote the length of I, otherwise set $h = 0$.

Then the critical value for θ, where a phase transition from a.s. no consensus to a.s. strong consensus takes place, becomes $\theta_c = \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0, h\}$. The limit value in the supercritical regime is $\mathbb{E}\,\eta_0$.

(b) Suppose the initial opinions' distribution is unbounded but its expected value exists, either in the strong sense, i.e. $\mathbb{E}\,\eta_0 \in \mathbb{R}$, or the weak sense, i.e. $\mathbb{E}\,\eta_0 \in \{-\infty, +\infty\}$. Then the Deffuant model with arbitrary fixed parameter $\theta \in (0,\infty)$ will a.s. behave subcritically, meaning that no consensus will be approached in the long run.

Before embarking on the proof of this generalized result, let us recall some key ingredients of the proof for the standard uniform case in [5]. The arguably most central among these is the idea of *flat points*. A vertex $v \in \mathbb{Z}$ is called *ε -flat to the right* in the initial configuration $\{\eta_0(u)\}_{u\in\mathbb{Z}}$ if for all $n \geq 0$:

$$\frac{1}{n+1}\sum_{u=v}^{v+n} \eta_0(u) \in \left[\tfrac{1}{2} - \varepsilon, \tfrac{1}{2} + \varepsilon\right]. \tag{2.1}$$

It is called *ε-flat to the left* if the above condition is met with the sum running from $v-n$ to v instead. Finally, v is called *two-sidedly ε-flat* if for all $m, n \geq 0$

$$\frac{1}{m+n+1}\sum_{u=v-m}^{v+n} \eta_0(u) \in \left[\tfrac{1}{2} - \varepsilon, \tfrac{1}{2} + \varepsilon\right]. \tag{2.2}$$

In order to grasp the crucial role of flat points another concept has to be mentioned, namely the representation of $\eta_t(v)$ as a weighted average of initial opinions (see La. 3.1 in [5]). This convex combination of initial opinions can be written in a neat form, using as a tool the non-random pairwise averaging procedure Häggström called *Sharing a* *drink* (SAD) in [5]. In the latter, one has an initial profile $\{\xi_0(v)\}_{v\in\mathbb{Z}}$, with $\xi_0(0) = 1$ and $\xi_0(v) = 0$ for all $v \neq 0$, symbolizing a full glass of water at site 0 and empty ones at all other sites. The averaging is now done as in (1.1) but without the threshold θ and the encounters are no longer random, but given by a sequence of edges. Elements of $[0,1]^{\mathbb{Z}}$ that can be obtained by a finite such sequence are called SAD-profiles. An appropriately tailored SAD-procedure will then mimick the dynamics of the corresponding Deffuant model backwards in time in such a way that the state $\eta_t(0)$ in the Deffuant model at any given time $t > 0$ can be written as a weighted average of states at time 0 with weights given by an SAD-profile. In [5], general properties of SAD-profiles and consequences for $\eta_t(0)$ are derived. For example, the opinion value at a vertex which is two-sidedly ε-flat in the initial configuration can throughout time not move further away than 7ε from its initial value (see La. 6.3 in [5]).

Proof of Theorem 2.2. (a) The proof of this part will be subdivided into three steps marked by (i), (ii) and (iii).

(i) At first, let us suppose that the initial opinions are distributed on $[0,1]$ according to $\mathcal{L}(\eta_0)$ having expected value $\mathbb{E}\,\eta_0 = \frac{1}{2}$ and mass around the expectation as well as at least one of the extremes, i.e. for all $\varepsilon > 0$ we have

$$\mathbb{P}\left(\eta_0 < \varepsilon \;\text{ or } \eta_0 > 1-\varepsilon\right) > 0,\; \mathbb{P}\left(\tfrac{1}{2}-\varepsilon \le \eta_0 \le \tfrac{1}{2}+\varepsilon\right) > 0.$$

Then we claim that the result of Theorem 2.1 still holds true.

To prove this generalization of the standard uniform case is in fact to check that the crucial conditions in Häggström's [5] proof are met. First of all, the i.i.d. property guarantees that the distribution of the initial configuration is translation invariant, hence both the left- and right-shift of the system (that is $v \mapsto v-1\ \forall v \in \mathbb{Z}$ and $v \mapsto v+1\ \forall v \in \mathbb{Z}$ respectively) are measure-preserving.

The proof of La. 4.2 in [5] showing that $\mathbb{P}(v \text{ is } \varepsilon\text{-flat to the right}) > 0$ for every $\varepsilon > 0$ and $v \in \mathbb{Z}$ only uses the Strong Law of Large Numbers (SLLN), local modification (which employs that $\mathbb{P}\left(\frac{1}{2}-\varepsilon \le \eta_0(v) \le \frac{1}{2}+\varepsilon\right) > 0$ for all $\varepsilon > 0$, which we assumed) as well as $\mathbb{E}\,\eta_0 = \frac{1}{2}$.

By symmetry the same is true for ε-flatness to the left and the additional assumption that $\mathbb{P}(\eta_0 \notin [\varepsilon, 1-\varepsilon]) > 0$ provides the missing ingredient to mimick Prop. 5.1 and Thm. 5.2 in [5] verbatim: If $\theta < \frac{1}{2}$, pick $\varepsilon > 0$ small enough such that $\theta \le \frac{1}{2} - 2\varepsilon$. With positive probability any given site v is prevented from ever compromising with its neighbors already by the initial configuration, namely if $v-1$ is ε-flat to the left, $v+1$ ε-flat to the right and v itself an outlier in the sense that $\eta_0(v) \notin [\varepsilon, 1-\varepsilon]$. This establishes the subcritical case *(i)* in Theorem 2.1.

To show $\mathbb{P}(v \text{ is two-sidedly } \varepsilon\text{-flat}) > 0$ for all $v \in \mathbb{Z},\ \varepsilon > 0$ (in La. 4.3 in [5]) it is used once more that $\mathbb{P}\left(\frac{1}{2}-\varepsilon \le \eta_0 \le \frac{1}{2}+\varepsilon\right) > 0$. Following the reasoning of Sect. 6 in [5] literally will settle the supercritical case. The only change that has to be made in order to adapt to the generalized setting is that the expected energy at time $t = 0$, i.e. $\mathbb{E}\,(\eta_0(v)^2) \in (0,1]$ in La. 6.2, is no longer $\frac{1}{3}$ as for the uniform distribution. This minor change is not crucial however, since only the value's finiteness is used in the proof of Prop. 6.1.

(ii) Now suppose the initial distribution is as in (i), but fails to have mass around the expectation $\frac{1}{2}$ and leaves a gap of width $h \in (0,1]$, i.e. there exists some maximal (open) interval $I \subset [0,1]$ of length h such that $\frac{1}{2}$ lies inside I and $\mathbb{P}(\eta_0 \in I) = 0$. Then we claim that the critical value becomes $\theta_{\mathrm{c}} = \max\{\frac{1}{2}, h\}$.

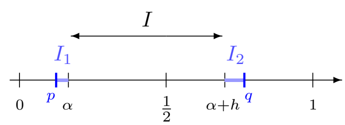
Changing the assumptions concerning the initial distribution of opinions as in (ii) will affect both the sub- and supercritical case as outlined in step (i). Clearly, the limiting behavior a.s. cannot be consensus for $\theta < h$ due to the fact that with probability 1 we will have initial opinion values both below and above $\frac{1}{2}$. Since an update, according to (1.1), can only take place between neighbors that are either both below or both above $\frac{1}{2}$, sites with initial values above the gap I will throughout time stay above it and the same holds for initial values below the gap. In particular, edges that are blocked due to incident values lying on different sides of the gap I in the beginning will stay blocked for ever, making consensus impossible.

For $\theta > h$, however, the behavior is pretty much as in the first case. Nevertheless, when it comes to show that there will be arbitrarily flat points with positive probability, one has to go about somewhat differently due to the fact that for sufficiently small ε, $\mathbb{P}\left(\eta_0 \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]\right) = 0$, which implies that no site can be ε-flat in the initial configuration by the very definition of flatness (taking $n = 0$ in (2.1) and $m = n = 0$ in (2.2) respectively).

Let the gap interval be denoted by $I = (\alpha, \alpha + h)$ and fix $\delta > 0$. Choose two rational numbers in $[0, \frac{1}{2}) \cap [\alpha - \delta, \alpha]$ and $(\frac{1}{2}, 1] \cap [\alpha + h, \alpha + h + \delta]$ respectively, say p and q, and define $I_1 := [p, \alpha]$ and $I_2 := [\alpha + h, q]$. Since I is maximal, one can choose these rationals in such a way that

$$\mathbb{P}(\eta_0 \in I_1) > 0 \text{ as well as } \mathbb{P}(\eta_0 \in I_2) > 0.$$

Clearly, there exist natural numbers m, n s.t. $\frac{m}{m+n}\, p + \frac{n}{m+n}\, q = \frac{1}{2}$. As numbers from I_1 and I_2 differ not more than δ from p and q respectively, the average of m numbers from I_1 and n numbers from I_2 surely lies within $[\frac{1}{2} - \delta, \frac{1}{2} + \delta]$.



Thus, we get that for any fixed $k \in \mathbb{N} = \{1, 2, \dots\}$:

$$\mathbb{P}\left(\frac{1}{k(m+n)} \sum_{v=0}^{k(m+n)-1} \eta_0(v) \in \left[\tfrac{1}{2} - \delta, \tfrac{1}{2} + \delta\right]\right) > 0. \tag{2.3}$$

Now let us consider some fixed time point $t > 0$ and the corresponding configuration $\{\eta_t(v)\}_{v \in \mathbb{Z}}$. There is a.s. an infinite increasing sequence of not necessarily consecutive edges $(\langle v_k, v_k + 1\rangle)_{k \in \mathbb{N}}$ to the right of site 0, on which no Poisson event has occurred up to time t.

Clearly, their positions are random, so let $l_k := v_{k+1} - v_k$, for $k \in \mathbb{N}$, denote the random lengths of the intervals in between and $l_0 := v_1 - v_0 + 1$ the one of the interval including 0, where $\langle v_0 - 1, v_0\rangle$ is the first edge to the left of the origin without Poisson event. Since the involved Poisson processes are independent, it is easy to verify that the l_k, $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, are i.i.d., having a geometric distribution on $\mathbb{N}$ with parameter e^{-t}.

For $\delta > 0$, let A_δ be the event that l_0 is finite and only finitely many of the events $\{l_k \geq k\delta\}$, $k \in \mathbb{N}$, occur. Then their independence and the Borel-Cantelli-Lemma tell us that A_δ has probability 1. On A_δ however the following holds a.s. true:

$$\begin{aligned}\limsup_{v\to\infty}\frac{1}{v+1}\sum_{u=0}^{v}\eta_t(u) &= \limsup_{v\to\infty}\frac{1}{v+1}\sum_{u=v_0}^{v}\eta_t(u)\\ &\le \limsup_{v\to\infty}\frac{1}{v+1}\sum_{u=v_0}^{v}\eta_0(u)+\delta\\ &= \lim_{v\to\infty}\frac{1}{v+1}\sum_{u=0}^{v}\eta_0(u)+\delta=\frac{1}{2}+\delta.\end{aligned}$$

The inequality follows from the fact that the Deffuant model is mass-preserving in the sense that $\eta_t(u)+\eta_t(v)=\eta_{t-}(u)+\eta_{t-}(v)$ in (1.1), hence for all $k\in\mathbb{N}$: $\sum_{u=v_0}^{v_k}\eta_0(u)=\sum_{u=v_0}^{v_k}\eta_t(u)$. For the average at time t running from v_0 to some $v\in\{v_k+1,\dots,v_{k+1}\}$ to differ by more than δ from the one at time 0, the interval has to be of length more than $k\delta$, since $v_k\ge k$ and $\eta_t(u)\in[0,1]$ for all t,u. This, however, will happen only finitely many times. Since δ was arbitrary and mimicking the same argument for the limes inferior, we have established that

$$\lim_{v\to\infty}\frac{1}{v+1}\sum_{u=0}^{v}\eta_t(u)=\frac{1}{2}\text{ almost surely.}\tag{2.4}$$

Now fix $\varepsilon>0$ such that $h+\frac{\varepsilon}{3}<\theta$, choose $\delta=\frac{\varepsilon}{6}$ in (2.3) as well as the rationals p,q and integers m,n accordingly. Due to (2.4) there exists some integer number k s.t. the event

$$A:=\left\{\frac{1}{v+1}\sum_{u=0}^{v}\eta_t(u)\in\left[\tfrac12-\tfrac{\varepsilon}{3},\tfrac12+\tfrac{\varepsilon}{3}\right]\text{ for all }v\ge N\right\}$$

has probability greater than $1-\mathrm{e}^{-2t}$, where $N:=k(m+n)-1$. Let B in turn be the event that there was no Poisson event on $\langle -1,0\rangle$ and $\langle N,N+1\rangle$ up to time t, hence $\mathbb{P}(B)=\mathrm{e}^{-2t}$. Finally, let C be the event that the initial values $\eta_0(0),\dots,\eta_0(N)$ were all in $[p,q]$, km of them below $\frac12$, kn above $\frac12$, and the Poisson firings on the edges $\langle 0,1\rangle,\dots,\langle N-1,N\rangle$ up to time t are sufficiently numerous such that, given B, $\eta_t(u)\in[\frac12-\frac{\varepsilon}{3},\frac12+\frac{\varepsilon}{3}]$ for all $u\in\{0,\dots,N\}$. Note that $q-p\le h+2\delta<\theta$, hence every such Poisson event will lead to an update, and that the independence of the initial configuration and the Poisson processes together with the considerations leading to (2.3) imply that C has positive probability. Furthermore, C is independent of B and $A\cap B$ cannot have probability 0, since

$$\mathbb{P}(A\cap B)=\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A\cup B)>(1-\mathrm{e}^{-2t})+\mathrm{e}^{-2t}-\mathbb{P}(A\cup B)\ge 0.$$

This gives that the conditional probabilities $\mathbb{P}(A|B)$ and $\mathbb{P}(C|B)$ are both strictly greater than 0.

Given B, we can apply the coupling trick, commonly known as *local modification*, precisely as in the proof of La. 4.2 in [5] to find that $\mathbb{P}(A\cap B\cap C)>0$. A one-line calculation shows that $A\cap B\cap C$ implies the ε-flatness to the right of site 0 in the configuration at time t.

Since the distribution of $\{\eta_t(u)\}_{u\in\mathbb{Z}}$ is still translation and left-right reflection invariant, every site $v\in\mathbb{Z}$ is ε-flat to the right (or left) at time t with positive probability on the one hand, and on the other this allows us to follow the argument in (i) settling the subcritical case and forcing $\theta_{\mathrm{c}}\ge\max\{\frac12,h\}$.

A short moment's thought verifies that ε-flatness to the right of site v and ε-flatness to the left of site $v-1$ simultaneously imply two-sided ε-flatness of both, v and $v-1$.

Let A_v^r, B_v^r, C_v^r be the sets appearing above, corresponding to site v and "right", and $A_{v-1}^l, B_{v-1}^l, C_{v-1}^l$ the ones corresponding to $v-1$ and "left". The involved independences lead to

$$\begin{aligned}
&\mathbb{P}(A_v^r \cap B_v^r \cap C_v^r \cap A_{v-1}^l \cap B_{v-1}^l \cap C_{v-1}^l) \\
&\quad = \mathbb{P}(A_v^r \cap C_v^r \cap A_{v-1}^l \cap C_{v-1}^l | B_v^r \cap B_{v-1}^l) \cdot \mathbb{P}(B_v^r \cap B_{v-1}^l) \\
&\quad = \mathbb{P}(A_v^r \cap C_v^r | B_v^r \cap B_{v-1}^l) \cdot \mathbb{P}(A_{v-1}^l \cap C_{v-1}^l | B_v^r \cap B_{v-1}^l) \cdot \mathbb{P}(B_v^r \cap B_{v-1}^l) \\
&\quad = \mathbb{P}(A_v^r \cap C_v^r | B_v^r) \cdot \mathbb{P}(A_{v-1}^l \cap C_{v-1}^l | B_{v-1}^l) \cdot \mathbb{P}(B_v^r \cap B_{v-1}^l) > 0,
\end{aligned}$$

since $\mathbb{P}(B_v^r \cap B_{v-1}^l) = \mathrm{e}^{-3t} > 0$. Hence two-sided ε-flatness at time t has positive probability as well. Following the argument corresponding to the supercritical case in (i), using the preserved translation invariance of the distribution of $\{\eta_t(u)\}_{u\in\mathbb{Z}}$ once more, we find that there will be consensus in the long run, if only $\theta > \max\{\frac{1}{2}, h\}$.

Putting both arguments together, this proves the claim $\theta_{\mathrm{c}} = \max\{\frac{1}{2}, h\}$.

(iii) Finally, suppose that $[a,b]$ is the smallest closed interval containing the support of the initial opinions' distribution and that the latter features a gap of width $h \in [0, b-a]$ around the expected value $\mathbb{E}\,\eta_0 \in [a,b]$. Then we claim that the critical value becomes $\theta_{\mathrm{c}} = \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0, h\}$ and the limit in the case of strong consensus is $\mathbb{E}\,\eta_0$.

Clearly, the dynamics of the Deffuant model are not effected by translations of the initial distribution ($x \mapsto x + c$ for some constant $c \in \mathbb{R}$). A scaling ($x \mapsto \frac{x}{c},\ c \in \mathbb{R}_{>0}$) has the only effect that the value for the parameter θ has to be rescaled too, in order to get identical dynamics.

Let $c := \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0\}$ and consider the linear transformation

$$x \mapsto \tfrac{x - \mathbb{E}\,\eta_0}{2\,c} + \tfrac{1}{2}.$$

The transformed initial distribution satisfies the assumptions in step (ii) and leaves a gap of width $\frac{h}{2\,c}$ around the mean $\frac{1}{2}$. Therefore, the considerations in (ii) allow us to conclude

$$\theta_{\mathrm{c}} = 2\,c \cdot \max\{\tfrac{1}{2}, \tfrac{h}{2\,c}\} = \max\{c, h\} = \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0, h\}.$$

Note that the limit of an individual opinion in the supercritical case is the retransformed equivalent of $\frac{1}{2}$, i.e. $2\,c \cdot \left(\frac{1}{2} + (\frac{\mathbb{E}\,\eta_0}{2\,c} - \frac{1}{2})\right) = \mathbb{E}\,\eta_0$.

(b) To prove the statement on unbounded initial distributions we have to treat two cases, namely the one where $\mathbb{E}\,|\eta_0| < \infty$ and the other where exactly one of both $\mathbb{E}\,\eta_0^+, \mathbb{E}\,\eta_0^-$ is infinite.

(i) In case of an unbounded initial distribution with existing first moment and expectation $\mathbb{E}\,\eta_0 < \infty$, the SLLN reads (for arbitrarily chosen $v \in \mathbb{Z}$):

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{1}{n+1} \sum_{u=v}^{v+n} \eta_0(u) = \mathbb{E}\,\eta_0\right) = 1.$$

Consequently, there exists some number $r > 0$ s.t.

$$\mathbb{P}\left(\frac{1}{n+1} \sum_{u=v}^{v+n} \eta_0(u) \in [\mathbb{E}\,\eta_0 - r, \mathbb{E}\,\eta_0 + r] \text{ for all } n \in \mathbb{N}_0\right) > 0.$$

Slightly abusing the definition (the expectation $\frac{1}{2}$ in (2.1) would have to be replaced by $\mathbb{E}\,\eta_0$), one could say that with positive probability site v is r-flat to the right.

Let the confidence bound θ take on some value in $(0,\infty)$. Strictly along the lines of Prop. 5.1 in [5], it follows that if $v-1$ and $v+1$ are r-flat to the left and right respectively and simultaneously $\eta_0(v) \notin [\mathbb{E}\,\eta_0 - r - \theta, \mathbb{E}\,\eta_0 + r + \theta]$ – an event with positive probability – the values at $v-1$ and $v+1$ will throughout all of time stay within the interval $[\mathbb{E}\,\eta_0 - r, \mathbb{E}\,\eta_0 + r]$ leaving the edges $\langle v-1, v\rangle$ and $\langle v, v+1\rangle$ blocked. Since this happens at every site v with positive probability, ergodic theory tells us that it will almost surely occur at infinitely many sites.

(ii) Now suppose that the expectation of η_0 exists only in the weak sense, i.e. $\mathbb{E}\,\eta_0 \in \{-\infty, +\infty\}$. Once more, symmetry allows us to focus on the case $\mathbb{E}\,\eta_0^+ = \infty$, $\mathbb{E}\,\eta_0^- < \infty$. In this case the SLLN reads

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{1}{n} \sum_{u=v+1}^{v+n} \eta_0(u) = \infty\right) = 1. \tag{2.5}$$

We can assume $\mathbb{P}(\eta_0 < 0) > 0$, otherwise a translation (irrelevant for the dynamics) as in the last step of (a) will reduce the problem to this setting. Some one-sided version of the idea of proof using flatness can then be employed.

Let the confidence bound $\theta \in (0,\infty)$ be arbitrary but fixed. By (2.5), for sufficiently large $N \in \mathbb{N}$ the following event has non-zero probability:

$$A_N := \left\{ \frac{1}{n} \sum_{u=v+1}^{v+n} \eta_0(u) > \theta \text{ for all } n \geq N \right\}.$$

Local modification is again the key step to advance. Let $\xi := \mathcal{L}(\eta_0)$ denote the distribution of η_0 and $\xi|_{(\theta,\infty)}$ its distribution conditioned on the event $\{\eta_0 > \theta\}$. Clearly, ξ is stochastically dominated by $\xi|_{(\theta,\infty)}$, i.e. $\xi \preceq \xi|_{(\theta,\infty)}$, implying

$$\mathcal{L}\big((\eta_0(u))_{u\geq v+1}\big) = \bigotimes_{u\geq v+1} \xi \preceq \left(\bigotimes_{u=v+1}^{v+N} \xi|_{(\theta,\infty)}\right) \otimes \left(\bigotimes_{u>v+N} \xi\right).$$

Let B be the event $\{\eta_0(v+1) > \theta, \ldots, \eta_0(v+N) > \theta\}$, which has non-zero probability, and

$$A_1 := \left\{ \frac{1}{n} \sum_{u=v+1}^{v+n} \eta_0(u) > \theta \text{ for all } n \in \mathbb{N} \right\}.$$

The stochastic domination from above yields:

$$\begin{aligned} \mathbb{P}(A_1) &\geq \mathbb{P}(A_1 \cap B) = \mathbb{P}(A_N \cap B) = \mathbb{P}(A_N | B) \cdot \mathbb{P}(B) \\ &\geq \mathbb{P}(A_N) \cdot \mathbb{P}(B) > 0. \end{aligned}$$

The very same ideas as in the proof of Prop. 5.1 in [5] show that if A_1 occurs and the edge $\langle v, v+1\rangle$ doesn't allow for an update, irrespectively of the dynamics on $\{u \in \mathbb{Z}, u \geq v+1\}$, we have that $\eta_t(v+1) > \theta$ is preserved for all times $t > 0$. By symmetry the same holds for site $v-1$ and the half-line to the left, i.e. $\{u \in \mathbb{Z}, u \leq v-1\}$. Independence of the initial opinions therefore guarantees that with positive probability, the initial configuration can be such that $\eta_0(v) < 0$ and the values at sites $v-1$ and $v+1$ are doomed to stay above θ, blocking the edges adjacent to v once and for all. Ergodicity makes sure that with probability 1 infinitely many sites will get stuck this way. □

Example 2.3. *(a) As a first toy application of the above result, let us consider the Deffuant model on $\mathbb{Z}$ in which the initial values are independently distributed according to a beta distribution Beta(α,β), where the two real numbers $\alpha,\beta>0$ represent the parameters of this family of distributions. That means η_0 has support $[0,1]$ and its distribution the density function*

$$f_{\alpha,\beta}(x)=\frac{1}{\mathrm{B}(\alpha,\beta)}\;x^{\alpha-1}\,(1-x)^{\beta-1},\quad \textit{for } x\in[0,1],$$

where the normalizing factor is given by the beta function

$$\mathrm{B}(\alpha,\beta)=\int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1}\,dt.$$

Since $f_{\alpha,\beta}>0$ on the open interval $(0,1)$, there are no gaps in the support and a simple calculation shows $\mathbb{E}\,\eta_0=\frac{\alpha}{\alpha+\beta}$. Consequently, part (a) of Theorem 2.2 shows that the critical value for the confidence bound separating the regimes of consensus and fragmentation is

$$\theta_{\mathrm{c}}=\begin{cases}\frac{\alpha}{\alpha+\beta}, & \textit{if } \alpha\geq\beta\\ \frac{\beta}{\alpha+\beta}, & \textit{otherwise}\end{cases}=\frac{\max\{\alpha,\beta\}}{\alpha+\beta}.$$

This example appears in [14] as well.

(b) Letting the initial values be independently drawn from a uniform distribution on the discrete set $\{-0.8,-0.3,0.7,0.8\}$, $[-0.8,0.8]$ is the minimal closed interval containing the support of $\mathcal{L}(\eta_0)$. Obviously, there is a gap of width $h=1$ around the mean $\mathbb{E}\,\eta_0=0.1$. Applying part (a) of Theorem 2.2 we can conclude that

$$\theta_{\mathrm{c}}=\max\{\mathbb{E}\,\eta_0-(-0.8),0.8-\mathbb{E}\,\eta_0,h\}=\max\{0.9,0.7,1\}=1.$$

(c) If we take the initial opinions to be i.i.d. and uniform on the set $[0,\frac{1}{8}]\cup[\frac{7}{8},1]$ instead, its expectation is $\mathbb{E}\,\eta_0=\frac{1}{2}$. But even though $\mathbb{P}(|\eta_0-\mathbb{E}\,\eta_0|>\frac{1}{2})=0$, a choice of $\theta\in(\frac{1}{2},\frac{3}{4})$ will a.s. lead to no consensus, as $\theta_{\mathrm{c}}=\frac{3}{4}$, again by part (a) of the above theorem. The next proposition actually shows that even for $\theta=\theta_{\mathrm{c}}$ the limiting scenario will a.s. be no consensus.

For a bounded initial distribution whose support has a large gap around its mean, we can deal with the behavior at criticality:

Proposition 2.4. *Let the initial opinions be again i.i.d. with $[a,b]$ being the smallest closed interval containing the support of the marginal distribution, and the latter feature a gap (α,β) of width $\beta-\alpha>\max\{\mathbb{E}\,\eta_0-a,b-\mathbb{E}\,\eta_0\}$ around its expected value $\mathbb{E}\,\eta_0\in[a,b]$.*

At criticality, that is for $\theta=\theta_{\mathrm{c}}=\max\{\mathbb{E}\,\eta_0-a,b-\mathbb{E}\,\eta_0,\beta-\alpha\}=\beta-\alpha$, we get the following: If both α and β are atoms of the distribution $\mathcal{L}(\eta_0)$, i.e. $\mathbb{P}(\eta_0=\alpha)>0$ and $\mathbb{P}(\eta_0=\beta)>0$, the system approaches a.s. strong consensus. However, it will a.s. lead to no consensus if either $\mathbb{P}(\eta_0=\alpha)=0$ or $\mathbb{P}(\eta_0=\beta)=0$.

Proof. In order to prove this statement, we can follow the arguments in the proof of part (a) of Theorem 2.2. By the translation and scaling invariance of the dynamics as described in step (iii) of the cited proof, we can restrict ourselves to the case in step (ii) and assume that the support of $\mathcal{L}(\eta_0)$ is a subset of $[0,1]$, $\mathbb{E}\,\eta_0=\frac{1}{2}$ and $\mathbb{P}\left(\eta_0<\varepsilon\ \text{ or } \eta_0>1-\varepsilon\right)>0$ for all $\varepsilon>0$. Note that under these further assumptions, we have $\theta=\theta_{\mathrm{c}}=\beta-\alpha>\frac{1}{2}$.

If both ends of the gap are atoms, we can follow the reasoning in the supercritical case in step (ii) of the proof of Theorem 2.2 (a) and for every $\delta > 0$ choose natural numbers m, n such that $\frac{m}{m+n}\,\alpha + \frac{n}{m+n}\,\beta \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$, to get (2.3). Using such a collection of initial opinions, i.e. m times the value α and n times β, all of them will be precisely within the confidence bound, hence allow for the manipulation described above as local modification. Having arbitrarily flat points with positive probability at time $t > 0$, $\theta > \frac{1}{2}$ guarantees a.s. strong consensus.

The negative statement is easy to handle. If without loss of generality $\mathbb{P}(\eta_0 = \alpha) = 0$, with probability 1 there will be no initial value lying in the interval $[\alpha, \beta)$. Since $\theta = \beta - \alpha$, this gap cannot be bridged. We refer once more to step (ii) in the proof of part (a) of Theorem 2.2 for a more detailed reasoning. □

Does Proposition 2.4 constitute progress in the attempt to solve the critical case in the setting of uniformly distributed initial opinions (the open problem mentioned right after Theorem 2.1)? Probably not, since in this setting, due to the large width of the gap $\beta - \alpha > \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0\}$, the criticality comes only from the gap in the distribution, not the distance between the mean and the extreme ends of the initial distribution.

As already mentioned in the introductory section, a next step of generalization in terms of the initial opinions would be vector-valued distributions. Despite the fact that this seems to be a minor modification it invokes major changes and would thus excessively expand this section, which is why it is omitted here and treated as a separate topic in [7].

2.2 Dependent initial opinion values

The definition of the Deffuant model generalizes straightforwardly to dependent initial configurations. Considering that – in our treatment of the model on $\mathbb{Z}$ in the foregoing subsection – the independence of initial opinions was merely used to deduce translation invariance and ergodicity with respect to shifts as well as for the local modification, it is a valid question in how far the results of Theorem 2.2 can be generalized to initial configurations $\{\eta_0(v)\}_{v\in\mathbb{Z}}$ that do not form an i.i.d. sequence. The example below shows that stationarity and ergodicity of the sequence of initial opinions is not enough to retain the results from Subsection 2.1. In order to be able to locally modify the configuration as done in the proof of Theorem 2.2, we have to add an extra condition, which is a natural extension to continuous state spaces of the well-known finite energy condition of percolation theory (see for instance Def. 2 in [1]).

Definition 2.5. *Let $\{\xi_v\}_{v\in\mathbb{Z}}$ be a stationary sequence of random variables. It is said to satisfy the* finite energy condition *if it allows conditional probabilities such that the conditional distribution of ξ_0 given $\{\xi_v\}_{v\in\mathbb{Z}\setminus\{0\}}$ almost surely has the same support as the marginal distribution $\mathcal{L}(\xi_0)$.*

Carefully checking its proof with this extra condition in hand, we can get the following generalization of Theorem 2.2:

Theorem 2.6. *Consider the Deffuant model on $\mathbb{Z}$ with initial opinion values $\{\eta_0(v)\}_{v\in\mathbb{Z}}$. If $\{\eta_0(v)\}_{v\in\mathbb{Z}}$ is a stationary sequence of random variables, ergodic with respect to shifts and satisfying the finite energy condition, the results of Theorem 2.2 still hold true.*

To see that the added assumption that conditioning on the configuration apart from a given site v will not change the support of the distribution at site v is essential and can not be dropped, see the following example.

Example 2.7. *Let U be a random variable, uniformly distributed on $\{-4, -3, \dots, 4\}$. The initial configuration will now be made up of blocks of length 9 centered in the sites*

$\{c_k\}_{k\in\mathbb{Z}} := \{U + 9\,k\}_{k\in\mathbb{Z}}$. Each block will independently be either of the form $\eta_0(c_k) = \frac{1}{2}$ and $\eta_0(v) = 0$ for $v \in \{c_k - 4, \dots, c_k - 1, c_k + 1, \dots, c_k + 4\}$ or $\eta_0(c_k) = \frac{1}{2}$ and $\eta_0(v) = 1$ for $v \in \{c_k - 4, \dots, c_k - 1, c_k + 1, \dots, c_k + 4\}$, both with probability $\frac{1}{2}$.

The initial configuration $\{\eta_0(v)\}_{v\in\mathbb{Z}}$ defined in this way is translation invariant and ergodic with respect to shifts, having the marginal distribution $\mathcal{L}(\eta_0)$ concentrated on $\{0, \frac{1}{2}, 1\}$ with $\mathbb{P}(\eta_0 = 0) = \mathbb{P}(\eta_0 = 1) = \frac{4}{9}$ and $\mathbb{P}(\eta_0 = \frac{1}{2}) = \frac{1}{9}$.

If Theorem 2.1 applied, the critical value should be $\theta_c = \frac{1}{2}$ but it is not hard to see that for $\theta < \frac{4}{5}$ compromises are at first confined to happen within intervals consisting of blocks of the same kind and can thus only lead to values in $[0, \frac{1}{10}] \cup [\frac{9}{10}, 1]$ at sites next to a neighboring block of the other kind, see also Thm. 2.3 in [5]. This means that the edges connecting two blocks of different kind will be blocked throughout time forcing a.s. no consensus.

Due to the fixed block size, the sequence $\{\eta_0(v)\}_{v\in\mathbb{Z}}$ as defined above is obviously not mixing. An easy modification, for instance allowing random block lengths taking values 9 and 11, shows that even an initial configuration which is given by a stationary mixing sequence of random variables does not, in general, allow for the results of the i.i.d. case to be transferred.

3 Upper bound for the critical range of θ on $\mathbb{Z}^d$

3.1 Application of energy arguments

Moving on to higher dimensions as far as the underlying lattice is concerned provides the opportunity to go around blocked edges and there is no handy generalization of the notion of flatness. Among other things, these changes render most of the arguments used in the $\mathbb{Z}$ case void. Enough can be resurrected, however, to establish a lower bound for θ above which consensus is achieved. Throughout Sections 3 and 4 (Theorem 4.3 being an exception) we will only assume that the configuration of initial opinion values $\{\eta_0(v)\}_{v\in\mathbb{Z}^d}$ is stationary and ergodic with respect to shifts of the kind $T_i : v \mapsto v + e_i$, where e_i is the ith standard basis vector of $\mathbb{R}^d$ for $i \in \{1, \dots, d\}$.

Theorem 3.1. *Consider the Deffuant model on the d-dimensional lattice $\mathbb{Z}^d$.*

(a) If the initial values are distributed uniformly on $[0, 1]$ and $\theta > \frac{3}{4}$, the configuration will a.s. approach weak consensus, i.e.

$$\mathbb{P}\big(\lim_{t\to\infty} |\eta_t(u) - \eta_t(v)| = 0\big) = 1$$

for all $u, v \in \mathbb{Z}^d$ s.t. $\langle u, v\rangle$ forms an edge.

(b) For general initial distributions on $[0, 1]$ the range of θ, where final consensus is guaranteed, is non-trivial, i.e. including values smaller than 1, unless the initial values are concentrated on 0 and 1, taking on both values with positive probability.

To prove this, we need first to establish some lemmas, the first one involving the idea of *energy*, introduced in Sect. 6 of [5] (not to be confused with the completely unrelated concept of finite energy from Subsection 2.2).

Assume the initial values $\{\eta_0(v)\}_{v\in\mathbb{Z}^d}$ have a stationary distribution, ergodic with respect to shifts and the marginal distribution has bounded support, without loss of generality we can take $[0, b]$ to be the smallest closed interval containing it. Denote by $W_t(v) = \mathcal{E}(\eta_t(v))$ the energy at vertex v at time t, where $\mathcal{E} : [0, b] \to \mathbb{R}_{\geq 0}$ is some fixed convex function. If a Poisson event occurs at the edge $e = \langle u, v\rangle$ at time t, and the values at u and v, $\eta_{t-}(u)$ and $\eta_{t-}(v)$ respectively, are within θ, energy is transferred and (possibly) lost along the edge. The latter to the amount

$$w_t(e) := (W_{t-}(u) + W_{t-}(v)) - (W_t(u) + W_t(v)). \tag{3.1}$$

Since $\eta_t(u) = (1-\mu)\,\eta_{t-}(u) + \mu\,\eta_{t-}(v)$ and $\eta_t(v) = (1-\mu)\,\eta_{t-}(v) + \mu\,\eta_{t-}(u)$, the convexity of $\mathcal{E}$ gives:

$$\begin{aligned} W_t(u) + W_t(v) &\le (1-\mu)\,W_{t-}(u) + \mu\,W_{t-}(v) + (1-\mu)\,W_{t-}(v) + \mu\,W_{t-}(u) \\ &= W_{t-}(v) + W_{t-}(u), \end{aligned}$$

i.e. the non-negativity of $w_t(e)$. Let T denote the sequence of arrival times of the Poisson events at e and define the accumulated energy loss along e as

$$W_t^{\text{loss}}(e) := \sum_{s\in T\cap[0,t]} w_s(e).$$

Finally, let $E(v)$ denote the set of edges incident to v and define the total energy attributed to vertex v as

$$W_t^{\text{tot}}(v) := W_t(v) + \frac{1}{2}\sum_{e\in E(v)} W_t^{\text{loss}}(e). \tag{3.2}$$

Note that by (3.1) the sum $W_t^{\text{tot}}(v) + W_t^{\text{tot}}(u)$ is preserved when an update along the edge $\langle u,v\rangle$ takes place. Along the lines of La. 6.2 in [5] we can show the following analog:

Lemma 3.2. *For every $v\in\mathbb{Z}^d$ and $t\ge 0$ we have*

$$\mathbb{E}\,[W_t^{\text{tot}}(v)] = \mathbb{E}\,[W_0(\mathbf{0})]. \tag{3.3}$$

Proof. Note first that for fixed time t the process $\{W_t^{\text{tot}}(v)\}_{v\in\mathbb{Z}^d}$ only depends on the initial configuration and the independent Poisson processes attributed to the edges. Its distribution is therefore translation invariant and the process ergodic with respect to shifts.

Let $\Lambda_n = [-n,n]^d$ denote the box of sidelength $2n$ centered at the origin $\mathbf{0}$. It contains $|\Lambda_n| = (2n+1)^d$ vertices of the grid $\mathbb{Z}^d$ and there are $2d\,(2n+1)^{d-1}$ edges linking vertices inside Λ_n to vertices outside of the box. The set of such edges is called *edge boundary* of Λ_n and denoted by $\partial_E\Lambda_n$.

The multivariate version of Birkhoff's Theorem, attributed to Zygmund (see e.g. Thm. 10.12 in [8]), tells us that

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\sum_{v\in\Lambda_n} W_t^{\text{tot}}(v) = \mathbb{E}\,[W_t^{\text{tot}}(\mathbf{0})] \text{ almost surely.} \tag{3.4}$$

Note that the statement of (3.4) is still true if we pass from the original sequence of sets $(\Lambda_n)_{n\in\mathbb{N}}$ to any subsequence.

Translation invariance of the configuration implies $\mathbb{E}\,[W_t^{\text{tot}}(v)] = \mathbb{E}\,[W_t^{\text{tot}}(\mathbf{0})]$ for all sites v and by definition $W_0^{\text{loss}}(e) = 0$ for all edges e since at time 0 no Poisson event has occurred yet, hence $W_0^{\text{tot}}(\mathbf{0}) = W_0(\mathbf{0})$.

Let us now choose a subsequence $(\Lambda_{n_k})_{k\in\mathbb{N}}$ such that

$$\sum_{k=1}^{\infty}\frac{|\partial_E\Lambda_{n_k}|}{|\Lambda_{n_k}|} < \infty. \tag{3.5}$$

As mentioned, (3.4) clearly implies

$$\lim_{k\to\infty}\frac{1}{|\Lambda_{n_k}|}\sum_{v\in\Lambda_{n_k}} W_t^{\text{tot}}(v) = \mathbb{E}\,[W_t^{\text{tot}}(\mathbf{0})] \text{ almost surely.} \tag{3.6}$$

In order to establish the claim it is therefore left to show that the limit in (3.6) is constant over time.

Take $\varepsilon > 0$ small and fix a time interval $[t, t+\varepsilon]$. Note that the energy function $\mathcal{E}$ is bounded on $[0, b]$ by $M := \max\{\mathcal{E}(0), \mathcal{E}(b)\}$, due to its convexity. Let $N_{n,\varepsilon}$ be the number of Poisson events on edges in $\partial_E \Lambda_n$ within the time interval $(t, t+\varepsilon]$, see Figure 1, and A_n be the event

$$A_n := \left\{ N_{n,\varepsilon} \geq \tfrac{1}{M} \left(|\partial_E \Lambda_n| + \sqrt{|\Lambda_n|} \right) \right\}.$$

The number on every single edge is a Poisson distributed random variable with parameter ε, consequently having mean and variance ε.

As those random variables are independent, a choice of ε such that $\varepsilon \leq \frac{1}{M}$ yields using Chebyshev's inequality:

$$\mathbb{P}(A_n) \leq \mathbb{P}\Big(N_{n,\varepsilon} - \mathbb{E}\, N_{n,\varepsilon} \geq \tfrac{1}{M} \sqrt{|\Lambda_n|} \Big) \leq M^2 \, \frac{\mathrm{var}(N_{n,\varepsilon})}{|\Lambda_n|} \leq M \, \frac{|\partial_E \Lambda_n|}{|\Lambda_n|}.$$

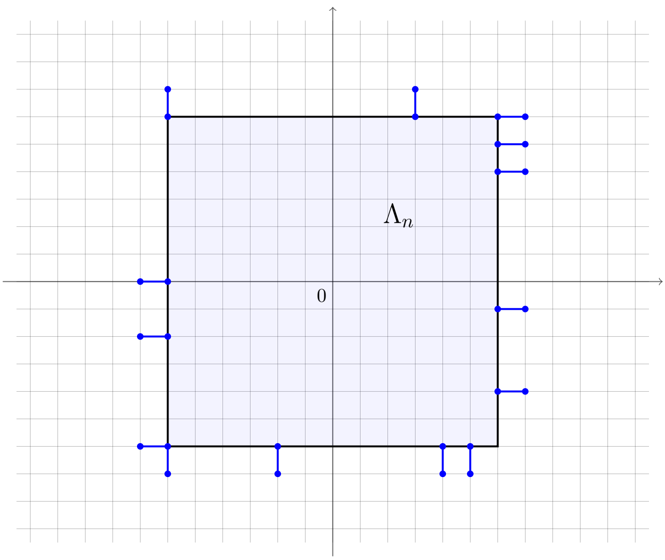


Figure 1: The interactions on the boundary of the box Λ_n in the time interval $[t, t+\varepsilon]$ are few compared to the size of the box for large n.

In view of (3.5), the Borel-Cantelli-Lemma shows that almost surely only finitely many A_{n_k} will occur. In order to conclude, we have to show that this implies

$$\lim_{k\to\infty} \frac{1}{|\Lambda_{n_k}|} \sum_{v \in \Lambda_{n_k}} W^{\mathrm{tot}}_{t+\varepsilon}(v) = \lim_{k\to\infty} \frac{1}{|\Lambda_{n_k}|} \sum_{v \in \Lambda_{n_k}} W^{\mathrm{tot}}_{t}(v), \tag{3.7}$$

which in turn guarantees that the limit in (3.6) is constant over time.

It is not hard to convince yourself that Poisson events off $\partial_E \Lambda_{n_k}$ will not change $\sum_{v \in \Lambda_{n_k}} W^{\mathrm{tot}}_t(v)$ and every single event on $\partial_E \Lambda_{n_k}$ can change the sum of total energies in Λ_{n_k} by at most M. Therefore, on the complement of A_{n_k}, we get that

$$\frac{1}{|\Lambda_{n_k}|}\left|\sum_{v\in\Lambda_{n_k}} W^{\text{tot}}_{t+\varepsilon}(v)-\sum_{v\in\Lambda_{n_k}} W^{\text{tot}}_{t}(v)\right| \le \frac{M}{|\Lambda_{n_k}|}\cdot N_{n_k,\varepsilon} < \frac{|\partial_E\Lambda_{n_k}|}{|\Lambda_{n_k}|}+\frac{1}{\sqrt{|\Lambda_{n_k}|}}.$$

As this converges to 0 when $k\to\infty$, we have shown that (3.7) holds almost surely, which concludes the proof. □

Lemma 3.3. *For the Deffuant model on the lattice $\mathbb{Z}^d$ as above, with threshold parameter $\theta\in(0,b]$, the following holds a.s. for every two neighbors $u,v\in\mathbb{Z}^d$:*

$$\begin{aligned}&\textit{Either } |\eta_t(u)-\eta_t(v)|>\theta \textit{ for all sufficiently large } t\textit{, i.e. the edge } \langle u,v\rangle\\ &\textit{is finally blocked, or}\\ &\lim_{t\to\infty}|\eta_t(u)-\eta_t(v)|=0,\ \textit{i.e. the two neighbors will finally concur.}\end{aligned}\tag{3.8}$$

Proof. The above lemma corresponds to Prop. 6.1 in [5] and the original proof generalizes to the higher-dimensional setting with only minor changes.

As the times between Poisson events on a single edge are exponentially distributed, the memoryless property ensures that given a finite collection of edges and some fixed time s, the edge which experiences the next Poisson event is chosen uniformly at random. Let us take $\mathcal{E}: x\mapsto x^2$ as energy function and fix $e=\langle u,v\rangle$ as well as some $\delta>0$. If there is a Poisson event at e at time t and the opinion values of u and v are not more than θ apart from each other, energy to the amount of $w_t(e)=2\mu\,(1-\mu)(\eta_{t-}(u)-\eta_{t-}(v))^2$ is lost along the edge, see (3.1). If $|\eta_{t-}(u)-\eta_{t-}(v)|\in(\delta,\theta]$, such an increase of $W^{\text{loss}}_t(e)$ would be at least $2\mu\,(1-\mu)\,\delta^2$. The opinion values of u and v can only change if one of the $4d-1$ edges incident to either u or v experiences a Poisson event. Given $|\eta_s(u)-\eta_s(v)|\in(\delta,\theta]$ for some fixed time s, the probability that it is in fact e where the first Poisson event after time s on an edge incident to either u or v occurs is $\frac{1}{4d-1}$.

By the extended version of the Borel-Cantelli-Lemma (involving conditional probabilities, see e.g. Cor. 6.20 in [8]) such an increase will happen infinitely often, if $|\eta_t(u)-\eta_t(v)|\in(\delta,\theta]$ for arbitrarily large t, forcing $(W^{\text{loss}}_t(e))_{t\ge0}$ to diverge. This cannot happen with positive probability, since according to Lemma 3.2 we have

$$\mathbb{E}\,[W^{\text{loss}}_t(e)]\le 2\,\mathbb{E}\,[W^{\text{tot}}_t(v)]=2\,\mathbb{E}\,[W_0(\mathbf{0})]\le 2\,b^2.$$

Hence, it follows that a.s. $|\eta_t(u)-\eta_t(v)|\notin(\delta,\theta]$ for sufficiently large t.

For small values of δ, more precisely $\delta<\frac{\theta}{2}$, the margin $|\eta_t(u)-\eta_t(v)|$ cannot jump back and forth between $[0,\delta]$ and $(\theta,b]$, since single updates can change the value at any site by no more than $\mu\theta\le\frac{\theta}{2}$. Consequently, for $0<\delta<\frac{\theta}{2}$, the following holds almost surely:

$$\limsup_{t\to\infty}|\eta_t(u)-\eta_t(v)|\in[0,\delta]\quad\text{or}\quad\liminf_{t\to\infty}|\eta_t(u)-\eta_t(v)|\in(\theta,b].$$

For δ can be chosen arbitrary small and there are only countably many edges, the claim is established. □

Lemma 3.4. *The probability that there will be finally blocked edges is either 0 or 1.*

Proof. Fix an edge $e=\langle u,v\rangle$ and assume that $\mathbb{P}(e \text{ is finally blocked})=0$. By translation invariance of the process, this has to be true for all edges $e\in E$. The union bound together with the preceeding lemma gives:

$$\mathbb{P}(\lim_{t\to\infty}|\eta_t(u)-\eta_t(v)|=0\ \forall\, u,v\in\mathbb{Z}^d)=1.$$

For $\mathbb{P}(e \text{ is finally blocked}) > 0$, let $N(v)$ denotes the number of edges incident to site v that are finally blocked. Then the ergodicity of $\{\eta_0(v)\}_{v\in\mathbb{Z}^d}$ and the independent Poisson processes attributed to the edges with respect to shifts, forces that almost surely the following holds (using Zygmund's Ergodic Theorem):

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{v\in\Lambda_n} N(v) = \mathbb{E}\left[N(\mathbf{0})\right] = 2d \cdot \mathbb{P}(e \text{ is finally blocked}) > 0.$$

Hence, with probability 1 infinitely many edges will be finally blocked. □

Having derived these auxiliary results, we can proceed to prove the main result of this section:

Proof of Theorem 3.1. (a) Given some confidence bound $\theta \geq \frac{1}{2}$, the value at every vertex which is incident to a finally blocked edge must be finally located in one of the intervals $[0, 1-\theta)$ or $(\theta, 1]$. Due to Lemma 3.3 this holds for every vertex almost surely if there are edges which are finally blocked. The foregoing lemma tells us, that if an edge is finally blocked with positive probability, we get

$$\liminf_{t\to\infty} |\eta_t(v) - \tfrac{1}{2}| \geq \theta - \tfrac{1}{2} \text{ for all } v \in \mathbb{Z}^d \text{ a.s.} \tag{3.9}$$

Choosing the energy function $\mathcal{E} : x \mapsto |x - \frac{1}{2}|$ and applying Lemma 3.2 we find:

$$\begin{aligned}
\mathbb{E}\left[\liminf_{t\to\infty} W_t(v)\right] &= \mathbb{E}\left[\liminf_{t\to\infty} |\eta_t(v) - \tfrac{1}{2}|\right] \\
&\leq \liminf_{t\to\infty} \mathbb{E}\left[|\eta_t(v) - \tfrac{1}{2}|\right] \\
&\leq \liminf_{t\to\infty} \mathbb{E}\left[W_t^{\text{tot}}(v)\right] \\
&= \mathbb{E}\left[W_0^{\text{tot}}(v)\right] = \tfrac{1}{4},
\end{aligned}$$

where Fatou's Lemma was used in the first inequality and the non-negativity of $W_t^{\text{loss}}(e)$ in the second. If we assume $\mathbb{P}(e \text{ is finally blocked}) > 0$ for some, hence any e, the first expectation must be at least $\theta - \frac{1}{2}$ by (3.9), which leads to a contradiction if θ is larger than $\frac{3}{4}$.

(b) Note that no special feature of unif$([0,1])$ was used, but $\mathbb{E}\left[|\eta_0 - \frac{1}{2}|\right] = \frac{1}{4}$. Consequently, the above result still holds if unif$([0,1])$ is replaced by some other distribution $\mathcal{L}(\eta_0)$ on $[0,1]$ and the bound $\frac{3}{4}$ replaced by $\mathbb{E}\left[|\eta_0 - \frac{1}{2}|\right] + \frac{1}{2}$ simultaneously. Furthermore, this bound is non-trivial, i.e. less than 1, provided $\mathbb{P}(\eta_0 \in \{0,1\}) < 1$ for this implies $\mathbb{E}\left[|\eta_0 - \frac{1}{2}|\right] < \frac{1}{2}$. If however $\eta_0 \in \{0,1\}$ almost surely, trivially only $\theta = 1$ will not allow for finally blocked edges, given η_0 is not a.s. constant.

□

Remark 3.5. *(a) There are two major differences to the results on $\mathbb{Z}$. Firstly, even if intuitively appealing it is no longer ensured that weak consensus as described in Theorem 3.1 will lead to consensus in the strong sense, i.e. that every individual value converges to the mean. By ergodicity we know*

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{v\in\Lambda_n} \mathbb{1}_{\{\lim_{t\to\infty} \eta_t(v) \text{ exists}\}} = \mathbb{P}\big(\lim_{t\to\infty} \eta_t(\mathbf{0}) \text{ exists}\big).$$

In the case of consensus, the indicator functions on the left hand side are either all 0 *or all* 1*. In other words, for θ such that weak consensus is guaranteed, the existence of the limits is an event with probability either* 0 *or* 1*. In the latter case* *another application of ergodicity and dominated convergence show that this limit must be the mean of the initial distribution:*

$$\begin{aligned}\lim_{t\to\infty}\eta_t(v) &= \lim_{n\to\infty}\frac{1}{|\Lambda_n|}\sum_{u\in\Lambda_n}\lim_{t\to\infty}\eta_t(u)\\ &= \mathbb{E}\left[\lim_{t\to\infty}\eta_t(v)\right]\\ &= \lim_{t\to\infty}\mathbb{E}\left[\eta_t(v)\right] = \mathbb{E}\left[\eta_0(v)\right],\end{aligned}$$

where the first equality follows from weak consensus, the last is Lemma 3.2 with the identity as energy function.

Secondly, it is no longer clear that we can talk about a critical value for θ separating the parameter space neatly into a sub- and a supercritical regime, since final consensus is not necessarily an increasing event in θ. By Lemma 3.4 it is clear that for fixed θ we have that all neighbors finally concur with probability either 0 or 1. Hence both cases can not occur simultaneously but there might be a range for θ in which they alternate, unlike in the case of $\mathbb{Z}$.

(b) *Let us next consider another example. Taking for instance $\mathrm{unif}(\{0,\frac{1}{2},1\})$ as distribution of the initial values, the reasoning in part (b) of the theorem shows that finally blocked edges are in this case only possible for*

$$\theta \le \mathbb{E}\left[|\eta_0-\tfrac{1}{2}|\right]+\tfrac{1}{2}=\tfrac{1}{3}+\tfrac{1}{2}=\tfrac{5}{6}.$$

For other distributions it might even be beneficial to choose some different convex energy function giving a potentially sharper bound on $\theta\ge\frac{1}{2}$ of the kind: The probability for finally blocked edges can only be non-zero for θ such that

$$\inf\{\mathcal{E}(x),\ x\in[0,1-\theta)\cup(\theta,1]\}\le\mathbb{E}\left[\mathcal{E}(\eta_0)\right].$$

Clearly, this inequality is trivial if the minimal value $\min\{\mathcal{E}(x),\ x\in[0,1]\}$ is attained on $[0,1-\theta)\cup(\theta,1]$. If this is not the case, it reads

$$\min\{\mathcal{E}(1-\theta),\mathcal{E}(\theta)\}\le\mathbb{E}\left[\mathcal{E}(\eta_0)\right], \tag{3.10}$$

due to the convexity of $\mathcal{E}$. Choosing $\mathcal{E}$ such that it vanishes on the support of $\mathcal{L}(\eta_0)$ will only give the trivial bound $\theta\le\frac{1}{2}+\sup\{|x-\frac{1}{2}|,\ x\in\mathrm{supp}(\mathcal{L}(\eta_0))\}$.

In addition, Jensen's inequality tells us that regardless of the chosen convex energy function, from (3.10) we cannot get a bound on θ so sharp that $\mathbb{E}\,\eta_0\notin(1-\theta,\theta)$. Since in this case we trivially have

$$\inf\{\mathcal{E}(x),\ x\in[0,1-\theta)\cup(\theta,1]\}\le\mathcal{E}\big(\mathbb{E}\,\eta_0\big)\le\mathbb{E}\left[\mathcal{E}(\eta_0)\right].$$

Finally, a gap in the distribution of η_0 also reduces the scope of (3.10), since for $\mathbb{P}(\eta_0\in(1-\theta,\theta))=0$ we get:

$$\mathcal{E}(\eta_0)\ge\inf\{\mathcal{E}(x),\ x\in[0,1-\theta)\cup(\theta,1]\}\ \text{a.s.}$$

This trivially implies the above inequality.

In summary, the same factors obstructing consensus in the Deffuant model on $\mathbb{Z}$ reappear in this treatment of the higher-dimensional case (cf. part (a) of Theorem 2.2).

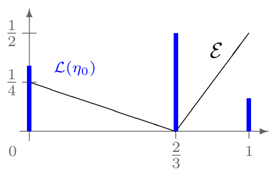
(c) *Next, it is worth noting that the energy function chosen in the proof of Theorem 3.1 is in fact best possible regarding (3.10) for symmetric distributions. If $\mathcal{E}$ is rescaled by some positive factor or translated by adding a constant, the inequality (3.10) stays unchanged. In the case of a symmetric distribution the inequality is symmetric around $\frac{1}{2}$, which is why it holds for the pair $(x \mapsto \mathcal{E}(x), \theta)$ if and only if it holds for $(x \mapsto \mathcal{E}(1-x), \theta)$. A symmetrization of the kind $\tilde{\mathcal{E}}(x) = \frac{1}{2}\left(\mathcal{E}(x) + \mathcal{E}(1-x)\right)$ will thus not change the right-hand side and at most increase the left-hand side if $\mathcal{E}(\theta) \neq \mathcal{E}(1-\theta)$, making the condition only stricter.*

Therefore, an energy function giving the best bound on parameters θ allowing for finally blocked edges through (3.10) can be assumed to be symmetric on $[0,1]$ and having the image set $[0, \frac{1}{2}]$. Set $X := \frac{1}{2} + |\eta_0 - \frac{1}{2}|$, a $[\frac{1}{2}, 1]$-valued random variable, which by the symmetry of η_0 implies $\mathbb{E}\left[\mathcal{E}(X)\right] = \mathbb{E}\left[\mathcal{E}(\eta_0)\right]$. The largest θ satisfying (3.10) is then the unique one (larger than $\frac{1}{2}$) for which $\mathcal{E}(\theta) = \mathbb{E}\left[\mathcal{E}(\eta_0)\right]$. Note that the convexity of the energy function forces it to be strictly monotonous where it is not attaining its minimum, which is 0, and a choice such that $\mathcal{E}(\eta_0) = 0$ a.s. will only give a trivial bound on θ as discussed above.

Another look at Jensen's inequality tells us that $\mathbb{E}\left[\mathcal{E}(X)\right] \geq \mathcal{E}(\mathbb{E}\, X)$, with strict inequality if $\mathcal{E}$ is not linear on $\mathrm{supp}(\mathcal{L}(X))$. If this inequality is strict, larger values for θ than $\mathbb{E}\, X$ will also satisfy (3.10). Being linear on $\mathrm{supp}(\mathcal{L}(X))$ and convex means being linear at least on the smallest interval containing the support, i.e. $I := \mathrm{conv}(\mathrm{supp}(\mathcal{L}(X)))$. How $\mathcal{E}$ is defined on $[\frac{1}{2}, 1] \setminus I$ is irrelevant, so we may assume it to be linear on all of $[\frac{1}{2}, 1]$. The assumptions on symmetry and image set finally force $\mathcal{E}$ to be the function $x \mapsto |x - \frac{1}{2}|$.

(d) *In the case of an asymmetric distribution of η_0 there are actually better choices.*

Consider the example sketched on the right, where $\mathbb{P}(\eta_0 = 0) = \frac{1}{3},\ \mathbb{P}(\eta_0 = \frac{2}{3}) = \frac{1}{2},\ \mathbb{P}(\eta_0 = 1) = \frac{1}{6}$, and the energy function is piecewise linear as shown.



Taking $x \mapsto |x - \frac{1}{2}|$ as energy function shows via (3.10) that finally blocked edges are only possible for

$$\theta \leq \mathbb{E}\left[|\eta_0 - \tfrac{1}{2}|\right] + \tfrac{1}{2} = \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{6}\right) + \tfrac{1}{2} = \tfrac{5}{6}.$$

Taking $\mathcal{E}$ piecewise linear with $\mathcal{E}(0) = \frac{1}{4},\ \mathcal{E}(\frac{2}{3}) = 0$ and $\mathcal{E}(1) = \frac{1}{2}$ gives in turn $\mathbb{E}\left[\mathcal{E}(\eta_0)\right] = \frac{1}{6} = \mathcal{E}(\frac{2}{9}) = \mathcal{E}(\frac{7}{9})$, hence a.s. no blocked edges for $\theta > \frac{7}{9}$, which is slightly better.

Note however that for every convex $\mathcal{E}$ there are always linear functions $l_1,\ l_2$ such that $l_1(1-\theta) = \mathcal{E}(1-\theta),\ l_2(\theta) = \mathcal{E}(\theta)$ and $l_1, l_2 \leq \mathcal{E}$. Taking their maximum will give a convex function leaving the left-hand side of (3.10) unchanged and at most decreasing the right-hand side. By an appropriate affine transformation of the kind $y \mapsto a\,y + c,\ a > 0$ this function can be altered to have image set $[0, \frac{1}{2}]$ without changing the condition on θ that follows from (3.10) as mentioned above. Consequently, the sharpest bound using (3.10) will even in the asymmetric case always be established by some piecewise linear function with only one bend mapping to $[0, \frac{1}{2}]$ as in the example.

(e) *It is worth remarking, that the bounds coming from (3.10) applied to the model with i.i.d. initial opinions on $\mathbb{Z}$ are a lot closer to the truth for centered distributions.*

The best we can come up with for the uniform case is $\frac{3}{4}$ and for $\mathrm{unif}(\{0, \frac{1}{2}, 1\})$ even $\frac{5}{6}$, whereas Theorem 2.2 tells us that on $\mathbb{Z}$ the actual bound on θ to allow for finally *blocked edges is $\frac{1}{2}$ in either case. In the asymmetric example from above, we get the bound $\theta \leq \frac{7}{9}$ which is not too far off its critical value $\theta_c = \frac{2}{3}$ on $\mathbb{Z}$.*

For a distribution of η_0 which is strongly concentrated around the mean, for instance $\mathbb{P}(\eta_0 = 0) = \mathbb{P}(\eta_0 = 1) = \frac{1}{n}$, $\mathbb{P}(\eta_0 = \frac{1}{2}) = \frac{n-2}{n}$, with n large, the bound derived using $x \mapsto |x - \frac{1}{2}|$ as energy function is $\theta \leq \mathbb{E}\left[|\eta_0 - \frac{1}{2}|\right] + \frac{1}{2} = \frac{1}{n} + \frac{1}{2}$. The corresponding critical value on $\mathbb{Z}$ according to Theorem 2.2 is again $\frac{1}{2}$, hence quite well approximated.

That we get the right answer for a non-constant distribution concentrated on $\{0, 1\}$ is due to the huge gap. For a slightly changed symmetric version, for example $\mathbb{P}(\eta_0 = 0) = \mathbb{P}(\eta_0 = 1) = \frac{n-1}{2n}$, $\mathbb{P}(\eta_0 = \frac{1}{2}) = \frac{1}{n}$, again n large, however, the best bound we get following the reasoning of the above theorem is

$$\theta \leq \mathbb{E}\left[|\eta_0 - \tfrac{1}{2}|\right] + \tfrac{1}{2} = \tfrac{1}{2} \cdot \tfrac{n-1}{n} + \tfrac{1}{2} = 1 - \tfrac{1}{2n}$$

and this is far off the true value on $\mathbb{Z}$, which is once more $\theta_c = \frac{1}{2}$.

(f) As in Theorem 2.2, the general case where the initial distribution's support is contained in $[a, b]$, $a < b \in \mathbb{R}$, can be treated by appropriate translation and scaling.

In conclusion, the results from Section 2 show that for $d = 1$ and a sequence of initial values satisfying the finite energy condition (see Definition 2.5), there exists a critical parameter θ_c (which is $\frac{1}{2}$ in the standard uniform case) at which a phase transition from no consensus to strong consensus takes place. Strictly weak consensus could only exist for the unsolved case of $\theta = \theta_c$.

Theorem 3.1 states that the case of no consensus is impossible for initial marginal distributions that attribute a positive probability to $(0, 1)$ and θ large enough ($\frac{3}{4}$ in the uniform case).

Remark 3.6. *The results from Theorem 3.1 can actually be generalized from the grid $\mathbb{Z}^d$ to any infinite, locally finite, transitive and amenable (connected) graph $G = (V, E)$. In this generality, the configuration of initial opinions would have to be ergodic with respect to the graph automorphisms instead of shifts, of course.*

Recall that a graph is called locally finite if every vertex has a finite degree, which together with the regularity of a transitive graph implies bounded degree. A graph is called amenable *if there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of finite sets such that the ratio of boundary and volume $\frac{|\partial_E F_n|}{|F_n|}$ tends to 0 as $n \to \infty$. Such sequences are called* Følner sequences.

In the case of an infinite, locally finite, transitive and amenable connected graph, we can choose the Følner sequence $(F_n)_{n\in\mathbb{N}}$ as an increasing sequence with $\bigcup_{n\in\mathbb{N}} F_n = V$; see the appendix of [6] for further details. As a replacement for Zygmund's ergodic theorem, we can then use the mean ergodic theorem for L^2-functions which can be found as Thm. A.5 in [6], with $(F_n)_{n\in\mathbb{N}}$ stepping in for $(\Lambda_n)_{n\in\mathbb{N}}$:

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{v\in F_n} W_t^{\text{tot}}(v) = \mathbb{E}\left[W_t^{\text{tot}}(\mathbf{0})\right] \quad \text{in } L^2,$$

where $\mathbf{0}$ is some fixed vertex of G. It is not a problem that this result only gives L^2-convergence instead of almost sure convergence, since L^2-convergence is stronger than convergence in probability and the latter implies almost sure convergence of a subsequence, which is enough for our purposes.

3.2 Consequences in terms of stochastic dominance

From the area of probabilistic risk analysis the following orders of stochastic dominance are known, which make it possible to rewrite the results from the foregoing subsection obtained by using energy arguments in a nice way.

Definition 3.7. *Let X, Y be two random variables with finite expectation and $\mathcal{F}_{\text{cx}}$ denote the set of all convex, $\mathcal{F}_{\text{icx}}$ the set of all increasing convex functions on $\mathbb{R}$.*

(i) X is said to be smaller than Y in the usual stochastic order, commonly denoted by $X \leq_{\text{st}} Y$, if for all $a \in \mathbb{R}$:

$$\mathbb{P}(X > a) \leq \mathbb{P}(Y > a).$$

(ii) X is said to be smaller than Y in the convex order, commonly denoted by $X \leq_{\text{cx}} Y$, if for all functions $\varphi \in \mathcal{F}_{\text{cx}}$ for which the corresponding expectations exist:

$$\mathbb{E}\left[\varphi(X)\right] \leq \mathbb{E}\left[\varphi(Y)\right].$$

(iii) X is said to be smaller than Y in the increasing convex order, commonly denoted by $X \leq_{\text{icx}} Y$, if for all functions $\varphi \in \mathcal{F}_{\text{icx}}$ for which the corresponding expectations exist:

$$\mathbb{E}\left[\varphi(X)\right] \leq \mathbb{E}\left[\varphi(Y)\right].$$

It is obvious from the definition that $\leq_{\text{cx}}$ implies $\leq_{\text{icx}}$. Furthermore, the converse is true, if the expectations of both random variables coincide, i.e.

$$X \leq_{\text{cx}} Y \;\Leftrightarrow\; X \leq_{\text{icx}} Y \text{ and } \mathbb{E}\,X = \mathbb{E}\,Y,$$

see for example Thm. 4.A.35 in [13].

An easy coupling argument (using quantile transformation) shows that $\leq_{\text{st}}$ implies $\leq_{\text{icx}}$.

Proposition 3.8. *Let $(\eta_t(v))_{t\geq 0}$ denote the piecewise constant jump process describing the value at some fixed vertex $v \in \mathbb{Z}^d$ throughout time, as before. Furthermore, let the initial values again be distributed on $[0, b]$ and $\mathbb{E}\,\eta_0$ be the corresponding expected value.*

For any two points in time $0 \leq s \leq t$, we have $\eta_t(v) \leq_{\text{cx}} \eta_s(v)$. This in turn directly implies $|\eta_t(v) - \mathbb{E}\,\eta_0| \leq_{\text{icx}} |\eta_s(v) - \mathbb{E}\,\eta_0|$.

Proof. First of all, it is worth remarking that the partial orders $\leq_{\text{cx}}$ and $\leq_{\text{icx}}$ are actually defined on the set of distributions and do therefore not depend on a random variable X itself but rather on $\mathcal{L}(X)$. The distribution of $\eta_t(v)$ is by symmetry the same for every $v \in \mathbb{Z}^d$, hence it is enough to consider one fixed vertex.

Let φ be a convex function on $\mathbb{R}$. For every $t \geq 0$ the random variable $\eta_t(v)$ lies in $[0, b]$ and since convexity implies continuity on closed intervals, φ attains its minimum

$$c := \min\left\{\varphi(x),\ x \in [0, b]\right\}.$$

Hence $\mathcal{E} : x \mapsto \varphi(x) - c$ is a non-negative convex function on $[0, b]$ and therefore a proper choice as energy function as outlined in the beginning of the foregoing subsection.

Let $W_t(v) = \mathcal{E}(\eta_t(v))$ denote the energy attributed to the chosen vertex at time t and $W_t^{\text{tot}}(v) = W_t(v) + \frac{1}{2}\sum_{e\in E(v)} W_t^{\text{loss}}(e)$ its total energy, just as in (3.2). Lemma 3.2 tells us that $\mathbb{E}\left[W_t^{\text{tot}}(v)\right] = \mathbb{E}\left[W_0(v)\right]$ for all $t \geq 0$ and the fact that $(W_t^{\text{loss}}(e))_{t\geq 0}$ is non-decreasing and non-negative for every edge e gives accordingly

$$\begin{aligned}\mathbb{E}\left[W_t(v)\right] &= \mathbb{E}\left[W_t^{\text{tot}}(v)\right] - \frac{1}{2}\sum_{e\in E(v)}\mathbb{E}\left[W_t^{\text{loss}}(e)\right]\\ &\leq \mathbb{E}\left[W_s^{\text{tot}}(v)\right] - \frac{1}{2}\sum_{e\in E(v)}\mathbb{E}\left[W_s^{\text{loss}}(e)\right] = \mathbb{E}\left[W_s(v)\right]\\ &\leq \mathbb{E}\left[W_0(v)\right] \quad \text{for } 0\leq s\leq t.\end{aligned}$$

If we plug in the special form of $\mathcal{E}$ chosen above (and add c along the chain of inequalities) this reads:

$$\mathbb{E}\left[\varphi(\eta_t(v))\right] \leq \mathbb{E}\left[\varphi(\eta_s(v))\right] \quad \Big(\leq \mathbb{E}\left[\varphi(\eta_0(v))\right]\Big).$$

Since $\varphi\in\mathcal{F}_{\text{cx}}$ was arbitrary, this proves the first part of the claim.

To see that $(|\eta_t(v)-\mathbb{E}\,\eta_0|)_{t\geq 0}$ is a non-increasing sequence with respect to $\leq_{\text{icx}}$ one only has to note that the function $x\mapsto|x-\mathbb{E}\,\eta_0|$ is convex. A short moment's thought reveals that the composition of an increasing convex with a convex function is again convex. Thus, for any $\varphi\in\mathcal{F}_{\text{icx}}$ the already proved part applied to the convex function $x\mapsto\varphi(|x-\mathbb{E}\,\eta_0|)$ provides

$$\mathbb{E}\left[\varphi(|\eta_t(v)-\mathbb{E}\,\eta_0|)\right] \leq \mathbb{E}\left[\varphi(|\eta_s(v)-\mathbb{E}\,\eta_0|)\right],$$

which in turn proves $|\eta_t(v)-\mathbb{E}\,\eta_0| \leq_{\text{icx}} |\eta_s(v)-\mathbb{E}\,\eta_0|$. □

This proposition in hand makes it possible to reprove the result from Theorem 3.1: Already in 1979, Meilijson and Nádas [12] showed that $Y\leq_{\text{icx}} X$ implies $Y\leq_{\text{st}} h_{\mathcal{L}(X)}(X)$, where the function h_μ denotes the mean residual life of a random variable with distribution μ, i.e.:

$$\text{For } Z\sim\mu \text{ and } t\in\mathbb{R} \text{ s.t. } \mu\big((t,\infty)\big)>0:\; h_\mu(t):=\mathbb{E}\left[Z\,|\,Z>t\right].$$

Having the initial distribution $\mathcal{L}(\eta_0)=\text{unif}([0,1])$ means $|\eta_0-\frac{1}{2}|\sim\text{unif}([0,\frac{1}{2}])$, which gives

$$h_{\text{unif}([0,1/2])}(t)=\tfrac{1}{4}+\tfrac{t}{2}.$$

Consequently, we get $|\eta_t-\frac{1}{2}|\leq_{\text{st}}\frac{1}{4}+\frac{Z}{2}$, where $Z\sim\text{unif}([0,\frac{1}{2}])$, another contradiction to (3.9) if $\theta>\frac{3}{4}$.

That the processes $(\eta_t(v))_{t\geq 0}$ are non-increasing in the convex order renders it possible to conclude convergence in distribution. This however is far from the almost sure convergence derived in the one-dimensional case.

Proposition 3.9. *Let $(\eta_t(v))_{t\geq 0}$ be as before. There exists a $[0,b]$-valued random variable η_∞ such that $\eta_t(v)\xrightarrow{d}\eta_\infty$ for every $v\in\mathbb{Z}^d$.*

Proof. Again, symmetry ensures that if the statement holds true for some vertex v it is valid for all such. Building on a famous result of Straßen and following ideas of Doob, Kellerer showed in 1972 that for a family of probability measures $\{\mu_t\}_{t\geq 0}$ which is non-decreasing in the increasing convex order there always exists a submartingale with the corresponding marginals, see Thm. 3 in [9]. Therefore, the non-increasing family $\{\mathcal{L}(\eta_t(v))\}_{t\geq 0}$ can be interpreted as the marginal distributions of a supermartingale $(X_t)_{t\geq 0}$. As the mean of these distributions is constant, which follows from Lemma 3.2 as mentioned in the above remark and corresponds to the stronger condition of non-increasing ordering w.r.t. $\leq_{\text{cx}}$, $(X_t)_{t\geq 0}$ actually is a martingale.

Doob's martingale convergence theorem guarantees a random variable X_∞ such that $(X_t)_{t\geq 0}$ converges to X_∞ almost surely, hence in distribution. Writing η_∞ instead of X_∞ establishes the claim. □

4 On the infinite cluster of supercritical bond percolation

In this section we consider the Deffuant opinion dynamics on the random subgraph of $\mathbb{Z}^d$, $d \geq 2$, which is formed by supercritical i.i.d. bond percolation, independent of the initial configuration and the Poisson processes determining the times of potential opinion updates.

That means, each edge of the grid is independently chosen to be open with a fixed probability $p \in (0, 1]$. One of the classical results in percolation theory tells us that for $d \geq 2$, there exists a critical value $p_c(d) \in (0, 1)$ for p above which we will a.s. find an infinite cluster and that this cluster is a.s. unique. The common notation for the event that some vertex v sits in the infinite cluster is $\{v \leftrightarrow \infty\}$. Slightly abusing this notation we will write $\{e \leftrightarrow \infty\}$ for the event that the edge e is part of the infinite cluster.

The fact that ergodicity, one essential element to derive the results from the foregoing section, is preserved when we consider the (random) subgraph of $\mathbb{Z}^d$ formed by i.i.d. bond percolation allows for an immediate transfer of the corresponding results for the whole grid.

Lemma 4.1. *Let the Deffuant model with initial values drawn from a distribution on $[0, b]$ and parameter $\theta \in (0, b]$ be as above, but now take place on the graph of a supercritical i.i.d. bond percolation on $\mathbb{Z}^d$ which is independent of the initial configuration and the Poisson processes. Then the lemmas of the foregoing section extend as follows:*

(a) $\mathbb{E}\,[W_t^{\text{tot}}(v)\,|\,v \leftrightarrow \infty] = \mathbb{E}\,[W_0(\mathbf{0})]$

(b) Given the edge $\langle u, v\rangle$ is open, we get as in Lemma 3.3 that a.s. $|\eta_t(u) - \eta_t(v)| > \theta$ for sufficiently large t or $\lim_{t\to\infty} |\eta_t(u) - \eta_t(v)| = 0$.

(c) The probability that some edges of the infinite cluster will be finally blocked in the Deffuant model is either 0 or 1.

Proof. (a) Using the notation from Lemma 3.2 and its line of reasoning, it is obvious that the process $\{W_t^{\text{tot}}(v) \cdot \mathbb{1}_{\{v\leftrightarrow\infty\}}\}_{v\in\mathbb{Z}^d}$ is ergodic with respect to shifts. Hence instead of (3.4) one has

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{v\in C_\infty\cap\Lambda_n} W_t^{\text{tot}}(v) = \mathbb{E}\,[W_t^{\text{tot}}(\mathbf{0}) \cdot \mathbb{1}_{\{\mathbf{0}\leftrightarrow\infty\}}] \text{ a.s.,} \tag{4.1}$$

where C_∞ denotes the infinite percolation cluster. By the same argument as in the quoted lemma, the left-hand side is constant over time and we thus get

$$\begin{aligned}
\mathbb{P}(v \leftrightarrow \infty) \cdot \mathbb{E}\,[W_t^{\text{tot}}(v)\,|\,v \leftrightarrow \infty] &= \mathbb{E}\,[W_t^{\text{tot}}(v) \cdot \mathbb{1}_{\{v\leftrightarrow\infty\}}] \\
&= \mathbb{E}\,[W_t^{\text{tot}}(\mathbf{0}) \cdot \mathbb{1}_{\{\mathbf{0}\leftrightarrow\infty\}}] \\
&= \mathbb{E}\,[W_0(\mathbf{0}) \cdot \mathbb{1}_{\{\mathbf{0}\leftrightarrow\infty\}}] \\
&= \mathbb{P}(\mathbf{0} \leftrightarrow \infty) \cdot \mathbb{E}\,[W_0(\mathbf{0})],
\end{aligned}$$

using symmetry and independence. Dividing by the probability for percolation of a given vertex $\mathbb{P}(v \leftrightarrow \infty)$, which is non-zero for supercritical percolation, yields the claim.

(b) To get the second statement one simply has to mimick Lemma 3.3. The only things changing are that we have to condition on the event of $e = \langle u, v\rangle$ being open in the realization of the i.i.d. bond percolation and the probability at a given point in time that e will be the next edge incident to either u or v where a Poisson event occurs is no longer precisely $\frac{1}{4d-1}$ but bounded from below by the same value (since some of the other edges might be closed).

(c) Following the proof of Lemma 3.4, let us consider the probability that some given edge e is open, further belongs to the infinite percolation component and is finally blocked in the Deffuant dynamics. If

$$p_{\text{block}} := \mathbb{P}(e \leftrightarrow \infty, e \text{ finally blocked}) = 0,$$

the union bound and part (b) guarantee that a.s. all neighbors in the infinite component will finally concur. If this probability is positive, however, and $N(v)$ denotes the number of edges incident to v, open in the realization of the i.i.d. bond percolation, that will get finally blocked in the Deffuant model, another application of Zygmund's Ergodic Theorem yields:

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{v \in C_\infty \cap \Lambda_n} N(v) = \mathbb{E}\left[N(\mathbf{0}) \cdot \mathbb{1}_{\{\mathbf{0}\leftrightarrow\infty\}}\right] = 2d \cdot p_{\text{block}} > 0 \text{ a.s.}$$

Hence with probability 1, there will be (infinitely many) edges that belong to the infinite percolation component and are finally blocked.

□

Having checked that these auxiliary results transfer appropriately to the setting of supercritical percolation, the following equivalent to Theorem 3.1 can be verified with the very same reasoning as before:

Theorem 4.2. *Consider the Deffuant model on the subgraph of $\mathbb{Z}^d$, $d \geq 2$, formed by an independent supercritical i.i.d. bond percolation as described above.*

(a) If the initial values are distributed uniformly on $[0,1]$ and $\theta > \frac{3}{4}$, a.s. we will finally have weak consensus in the infinite percolation cluster, i.e. for all $u, v \in \mathbb{Z}^d$ given the event $\{u, v \leftrightarrow \infty\}$ we have

$$\mathbb{P}\big(\lim_{t\to\infty} |\eta_t(u) - \eta_t(v)| = 0\big) = 1.$$

(b) For general initial distributions on $[0,1]$, the range of θ, where final consensus of the infinite cluster is guaranteed, is non-trivial, i.e. including values smaller than 1, unless the initial values are concentrated on 0 and 1, taking on both values with positive probability.

Proof. Given the event that $v \in \mathbb{Z}^d$ is in the infinite percolation cluster which contains (open) edges that are finally blocked by the opinion dynamics we get as in (3.9)

$$\liminf_{t\to\infty} |\eta_t(v) - \tfrac{1}{2}| \geq \theta - \tfrac{1}{2} \text{ a.s.}$$

Choosing again $\mathcal{E} : x \mapsto |x - \frac{1}{2}|$ as energy function the above lemma and the conditional version of Fatou's Lemma yield the following chain of inequalities:

$$\begin{aligned}
\theta - \tfrac{1}{2} &\leq \mathbb{E}\left[\liminf_{t\to\infty} |\eta_t(v) - \tfrac{1}{2}| \;\middle|\; v \leftrightarrow \infty\right] \\
&\leq \liminf_{t\to\infty} \mathbb{E}\left[|\eta_t(v) - \tfrac{1}{2}| \;\middle|\; v \leftrightarrow \infty\right] \\
&\leq \liminf_{t\to\infty} \mathbb{E}\left[W_t^{\text{tot}}(v) \;\middle|\; v \leftrightarrow \infty\right] \\
&= \mathbb{E}\left[W_0^{\text{tot}}(v)\right] = \mathbb{E}\left[|\eta_0(v) - \tfrac{1}{2}|\right].
\end{aligned}$$

Consequently, for blocked edges to occur in the infinite percolation cluster we have to have $\theta \leq \frac{3}{4}$ in the standard case of unif($[0,1]$) initial opinions and $\theta \leq \frac{1}{2} + \mathbb{E}\left[|\eta_0(v) - \frac{1}{2}|\right]$ in the general case. □

So far, this seems like just a generalization of Section 3. In the percolation setting however, a coupling argument allows to prove a result concerning the other end of the θ-spectrum, under slightly stronger conditions on the initial opinion configuration (see also Remark 4.5 below).

Theorem 4.3. *Consider again the Deffuant model on the infinite cluster of supercritical percolation, this time with i.i.d. initial opinion values distributed on* $[0,1]$*, s.t.* $[0,1]$ *is the minimal closed interval containing the support of the marginal distribution. In addition, we require the percolation parameter* p *to be less than 1.*

For $\theta < \frac{1}{2}$ *the probability that the opinion dynamics approach strong consensus on the infinite percolation cluster is 0.*

Proof. The line of reasoning to prove this statement is by contradiction. Assuming strong consensus for some fixed value of θ in $(0, \frac{1}{2})$, we are going to show that there will be finally blocked edges in the infinite percolation component with positive probability. This contradicts part (c) of Lemma 4.1.

To that end let us consider two coupled copies of the supercritical i.i.d. bond percolation, see Figure 2. Fix an edge $e = \langle u, v\rangle$ and let the two copies coincide on $E(\mathbb{Z}^d) \setminus \{e\}$. Let $p \in (0,1)$ denote the probability for an edge to be open in the percolation model and A be the event that the edges incident to u other than e are closed and v sits in the infinite component. By a coupling argument using local modification it can easily be seen that this event has positive probability if p is supercritical.

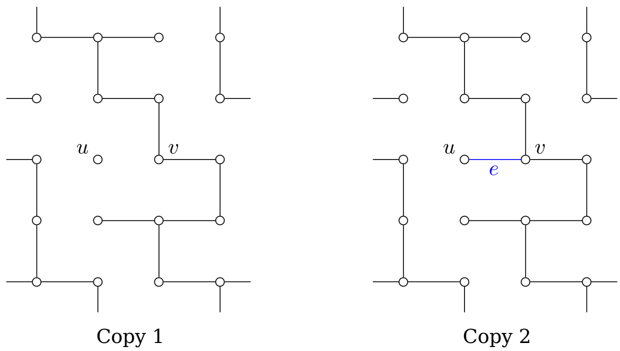


Figure 2: Two appropriately coupled copies of the same i.i.d. percolation process on $\mathbb{Z}^d$ on which the opinion dynamics procedure takes place.

Now we want to couple the two copies in such a way that with positive probability e is closed in copy 1 and open in copy 2 under the event A. Let U be a unif($[0,1]$)-distributed random variable, independent of the percolation process on $E(\mathbb{Z}^d) \setminus \{e\}$. Declare e to be open in copy 1 if $U < p$, closed otherwise, and open in copy 2 if $U > 1-p$ and closed otherwise. This defines two proper i.i.d. bond percolation processes.

If B denotes the event that the edge e is closed in copy 1 and open in copy 2, we get $\mathbb{P}(B) = \min\{p, 1-p\} > 0$. By independence we also have that the event $A \cap B$ has positive probability.

Since the event that there is strong consensus on the infinite percolation cluster is ergodic with respect to shifts, it is a 0-1-event. Due to the assumption it must have probability 1. Define $\delta := \frac{1}{2} - \theta$, which is positive.

Let us now restrict our attention to the event $A \cap B$ and the first copy. Since v lies in the infinite component, there is a time $T < \infty$ s.t.

$$\mathbb{P}\big(|\eta_t(v) - \mathbb{E}\,\eta_0| < \tfrac{\delta}{2} \text{ for all } t \geq T \mid A \cap B\big) > 0. \tag{4.2}$$

Note that given $A \cap B$, in copy 1 the process $(\eta_t(v))_{t\geq 0}$ is independent of $\eta_0(u)$ as well as the Poisson process attributed to e. By the choice of θ and the properties of the initial distribution we get in addition:

$$\mathbb{P}\big(\eta_0(u) \notin [\mathbb{E}\,\eta_0 - (\theta + \tfrac{\delta}{2}), \mathbb{E}\,\eta_0 + (\theta + \tfrac{\delta}{2})]\big) > 0.$$

If we finally define C to be the event that $A \cap B$ occurs, no Poission event occurs at e before T, $|\eta_t(v) - \mathbb{E}\,\eta_0| < \frac{\delta}{2}$ for all $t \geq T$ and $|\eta_0(u) - \mathbb{E}\,\eta_0| \geq \theta + \frac{\delta}{2}$, independence of the latter events conditioned on $A \cap B$ makes sure that C occurs with positive probability.

If we run the opinion dynamics on both copies simultaneously it is obvious that they behave identically as long as no Poisson event occurs for e. Given the event C the values at u and v are further than θ apart from time T on. Hence, even in the second copy, there will never be an interaction between the two since no Poisson event occurs at e before time T. In other words, with probability at least $\mathbb{P}(C) > 0$ there will be no consensus in the infinite percolation cluster of the second copy, to which given $A \cap B$ both u and v belong. Since both copies underly the same distribution, this contradicts the assumption that we have strong consensus. It is worth noting that strictly weak consensus can not be excluded since the argument in (4.2) does not hold for the weak case. □

Remark 4.4. *The two results of Theorem 4.2 and 4.3 put together imply the following: The Deffuant model on the infinite cluster, formed by supercritical i.i.d. bond percolation on $\mathbb{Z}^d$ with non-trivial percolation parameter $p \in (p_c, 1)$, featuring i.i.d. initial opinions having a non-degenerate marginal distribution on $[0,1]$ – in the sense that it attributes positive probability to $(0,1),\ [0,\varepsilon)$ and $(1-\varepsilon, 1]$ for all $\varepsilon > 0$ – either approaches weak consensus for all $\theta \in (0,1]$ or there is a phase transition in this parameter.*

Remark 4.5. *Similarly to the ideas in Subsection 2.2, we can relax the strong condition of independence when it comes to the initial opinion values and still receive the same result. In the proof of Theorem 4.3, the only instance where more than stationarity and ergodicity with respect to shifts of the initial configuration $\{\eta_0(v)\}_{v\in\mathbb{Z}^d}$ was used is in the conclusion that the event C has positive probability. This however can also be guaranteed without the independence of initial opinion values, if only $\{\eta_0(v)\}_{v\in\mathbb{Z}^d}$ additionally satisfies the finite energy condition as laid down in Definition 2.5 but now with $\mathbb{Z}^d$ in place of $\mathbb{Z}$.*

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B

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The Deffuant model on $\mathbb{Z}$ with higher-dimensional opinion spaces

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The Deffuant model on $\mathbb{Z}$ with higher-dimensional opinion spaces

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Abstract. When it comes to the mathematical modelling of social interaction patterns, a number of different models have emerged and been studied over the last decade, in which individuals randomly interact on the basis of an underlying graph structure and share their opinions. A prominent example of the so-called bounded confidence models is the one introduced by Deffuant et al.: Two neighboring individuals will only interact if their opinions do not differ by more than a given threshold θ. We consider this model on the line graph $\mathbb{Z}$ and extend the results that have been achieved for the model with real-valued opinions by considering vector-valued opinions and general metrics measuring the distance between two opinion values. As in the univariate case there turns out to exist a critical value θ_c for θ at which a phase transition in the long-term behavior takes place, but θ_c depends on the initial distribution in a more intricate way than in the univariate case.

1. Introduction

Consider a simple graph $G = (V, E)$ and assume the vertex set V to be either finite or countably infinite with bounded maximal degree. The vertices are assumed to represent individuals and each of them is assigned an opinion value. The edges in E – being connections between individuals – are understood to embody the possibility of mutual influence. For that reason it is no restriction to focus on connected graphs, as the components could be treated individually otherwise. From different directions including social sciences, physics and mathematics, there has been raised interest in various models for what is called *opinion dynamics* and deals with the evolution of such a system under a given set of interaction rules.

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These models are qualitatively different but share similar ideas, see Castellano et al. (2009) for an extensive survey.

The *Deffuant model* – introduced by Deffuant et al. (2000) – is one of those and features two parameters, the confidence bound $\theta > 0$ and the convergence parameter $\mu \in (0, \frac{1}{2}]$, shaping the willingness to approach the other individual's opinion in a compromise. There are two types of randomness in the model: One is the random *initial configuration*, meaning that at time $t = 0$ the vertices are assigned identically distributed opinions, the other are the *random encounters* thereafter. Serving as a regime for the latter, all the edges in E are assigned unit rate Poisson processes, which are independent of one another and the initial configuration. Whenever a Poisson event occurs on an edge, the corresponding adjacent vertices interact in the manner described below. Just like in most of the analyses of this model, we will consider i.i.d. initial opinion values, but comment on how the considerations can be generalized.

By $\eta_t(v)$ we denote the opinion value at vertex $v \in V$ at time $t \geq 0$. The current value will not change until at some future time t a Poisson event occurs at one of the edges incident to v, say $e = \langle u, v\rangle$, which then might cause an update. Let $\eta_{t-}(u) := \lim_{s\uparrow t} \eta_s(u) = a$ and $\eta_{t-}(v) := \lim_{s\uparrow t} \eta_s(v) = b$ be the two opinion values of u and v, just before this happens.

If these opinions lie at a distance less than the confidence bound θ from one another, they will symmetrically take a step, whose size is scaled by μ, towards a common compromise, if not they stay unchanged. Although there is a section on vector-valued binary opinions in the original paper by Deffuant et al. (2000), using a different model, the Deffuant model with the interaction rule just described was originally only defined for opinions being real-valued and the absolute value as notion of distance. In order to broaden the original scope of this model to vector-valued opinions, the natural replacement for the absolute value is the Euclidean distance

$$d(x, y) = \|x - y\|_2 = \sqrt{(x-y)^2}, \text{ for all } x, y \in \mathbb{R}^k.$$

Given this measure of distance, the rule for opinion updates in the Deffuant model reads as follows:

$$\eta_t(u) = \begin{cases} a + \mu(b-a) & \text{if } \|a-b\|_2 \leq \theta, \\ a & \text{otherwise} \end{cases}$$

and similarly (1.1)

$$\eta_t(v) = \begin{cases} b + \mu(a-b) & \text{if } \|a-b\|_2 \leq \theta, \\ b & \text{otherwise.} \end{cases}$$

Note that choosing $k = 1$ gives back the original model.

As the assumptions on the graph force E to be countable, there will almost surely be neither two Poisson events occurring simultaneously nor a limit point in time for the Poisson events on edges incident to one fixed vertex. Yet in addition to that there is a more subtle issue in how the simple pairwise interactions shape transitions of the whole system in the infinite setting, putting it into question whether the whole process is well-defined by the update rule (1.1). For infinite graphs with bounded degree, however, this problem is settled by standard techniques in the theory of interacting particle systems, see Thm. 3.9 on p. 27 in Liggett (1985).

One of the most natural questions in this context – motivated by interpretations coming from social science – seems to be, under what conditions the individual opinions will converge to a common consensus in the long run and under what conditions they are going to split up into groups of individuals holding different opinions instead. In this regard let us define the following types of scenarios for the asymptotic behavior of the Deffuant model on a connected graph as time tends to infinity:

Definition 1.1.

(i) *No consensus*
There will be finally blocked edges, i.e. edges $e = \langle u, v\rangle$ s.t.

$$\|\eta_t(u) - \eta_t(v)\|_2 > \theta,$$

for all times t large enough. Hence the vertices fall into different opinion groups.

(ii) *Weak consensus*
Every pair of neighbors $\{u, v\}$ will finally concur, i.e.

$$\lim_{t\to\infty} \|\eta_t(u) - \eta_t(v)\|_2 = 0.$$

(iii) *Strong consensus*
The value at every vertex converges, as $t \to \infty$, to a common limit l, where

$$l = \begin{cases} \text{the average of the initial opinion values}, & \text{if } G \text{ is finite} \\ \mathbb{E}\,\eta_0, & \text{if } G \text{ is infinite} \end{cases}$$

and $\mathcal{L}(\eta_0)$ denotes the distribution of the initial opinion values.

The first analyses of the Deffuant model and similar opinion dynamics were strongly simulation-based and thus confined to a finite number of agents. In Fortunato (2004) for example, the long-term behavior of the Deffuant model on four different kinds of finite graphs was simulated: Two deterministic examples – the complete graph and the square lattice – as well as two random graphs – those given by the Erdős-Rényi model as well as the Barabási-Albert model. He found strong numerical evidence that, given initial opinions that are independently and uniformly distributed on $[0, 1]$, a confidence threshold θ less than $\frac{1}{2}$ leads to a fragmentation of opinions, $\theta > \frac{1}{2}$ leads to a consensus – irrespectively of the underlying graph structures that were considered. Later, the simulation studies were extended to the generalization of the Deffuant model to higher-dimensional opinion values, see for instance Lorenz (2006).

There are however crucial differences between the interactions on a finite compared to an infinite graph. In the finite case, statements about consensus or fragmentation tend to be valid not with probability 1 but at best with a probability that is close to 1: In the standard case of i.i.d. unif($[0, 1]$) initial opinions for example, any non-trivial confidence bound, i.e. $\theta \in (0, 1)$, can lead to either consensus or fragmentation depending on the initial values and the order of interactions. Furthermore, the fact that the dynamics (1.1) preserves the opinion average of two interacting agents implies that strong consensus follows from weak consensus on a finite graph. This does not have to hold in an infinite setting.

The first major step in terms of a theoretical analysis of the model on an infinite graph was taken by Lanchier (2012), who treated the model on the line graph $\mathbb{Z}$ – similarly with an i.i.d. unif([0, 1]) configuration. His main result implies that there is a phase transition at $\theta = \frac{1}{2}$ from a.s. no consensus to a.s. weak consensus. These findings were reproven and slightly sharpened by Häggström (2012) to the statement of Theorem 2.1 below, using a non-random pairwise averaging procedure on $\mathbb{Z}$ which he termed *Sharing a drink* (SAD) to get a workable representation of the opinion values at times $t > 0$.

Using his line of argument, the results were generalized to initial distributions other than unif([0, 1]) by Häggström and Hirscher (2014) as well as Shang (2013), independently. In Häggström and Hirscher (2014), the analysis of the Deffuant model was in addition to that extended to other infinite graphs, namely higher-dimensional integer lattices $\mathbb{Z}^d$ and the infinite cluster of supercritical i.i.d. bond percolation on these lattices.

In this paper we stay on the infinite line graph, that is the integer numbers $\mathbb{Z}$ with consecutive integers forming an edge. The direction in which we want to broaden the analysis is – as already indicated – the generalization of the Deffuant model on $\mathbb{Z}$ to vector-valued opinions. In Section 2, we give a brief summary of the results for real-valued opinions derived in Häggström and Hirscher (2014), together with the key ideas and tools that were used there.

In Section 3 we establish corresponding results for the case of higher-dimensional opinions sticking, as indicated above, to the Euclidean norm as measure of distance between the opinions of interacting agents. Actually, the main results (Theorem 3.2 and 3.15) in this section match the statement for real-valued opinions (Theorem 2.2) in the sense that the radius of the initial distribution as well as the largest gap in its support – the generalized definitions of which you will find in Definition 3.1 and 3.14 – determine the critical value for θ at which there is a phase transition from a.s. no consensus to a.s. strong consensus. While the concept of a distribution's radius straightforwardly transfers to higher dimensions, the one of a gap has to be properly redefined and investigated. Doing this, we can in fact characterize the support of the opinion values at times $t > 0$, see Proposition 3.13. Even though we will throughout the paper consider the initial opinions to be i.i.d. it is mentioned in the remark after Theorem 3.15, how the arguments can be extended to particular dependent initial configurations in the way it was done in Häggström and Hirscher (2014).

Section 4 finally deals with the generalization of the Deffuant model to distance measures other than the Euclidean, in both one and higher dimensions. We pin down properties a general metric ρ (used to determine whether two opinions are close enough to compromise or not) needs to have in order to allow for the results from Section 3 to be preserved (see Theorem 4.3 and 4.11). Examples are given to illustrate the necessity of the requirements imposed on ρ.

At this point it should be mentioned that the vectorial model that was already introduced in the original paper by Deffuant et al. (2000) and analyzed quite recently by Lanchier and Scarlatos (2014) does not fit the general framework of this paper. Unlike all opinion dynamics considered here, its update rule is different from (1.1) and especially not average preserving, leading to substantial qualitative differences.

2. Background on the univariate case

Theorem 2.1 (**Lanchier**). *Consider the Deffuant model on the graph* $(\mathbb{Z}, E)$, *where* $E = \{\langle v, v+1\rangle, v \in \mathbb{Z}\}$ *with i.i.d.* $\mathrm{unif}([0,1])$ *initial configuration and fixed* $\mu \in (0, \frac{1}{2}]$.

(i) If $\theta > \frac{1}{2}$, *the model converges almost surely to strong consensus, i.e. with probability* 1 *we have:* $\lim_{t\to\infty} \eta_t(v) = \frac{1}{2}$ *for all* $v \in \mathbb{Z}$.

(ii) If $\theta < \frac{1}{2}$ *however, the integers a.s. split into (infinitely many) finite clusters of neighboring individuals asymptotically agreeing with one another, but no global consensus is approached.*

Accordingly, for independent initial opinions that are uniform on $[0,1]$, the critical value θ_c equals $\frac{1}{2}$, with subcritical values of θ leading a.s. to no consensus and supercritical ones a.s. to strong consensus. The case when the confidence bound actually takes on value θ_c is still an open problem. The ideas Häggström (2012) used to reprove the above result were adapted to accommodate more general univariate initial distributions leading to a similar statement for all such having a first moment $\mathbb{E}\,\eta_0 \in \mathbb{R}\cup\{-\infty, +\infty\}$, see Thm. 2.2 in Häggström and Hirscher (2014), which reads as follows:

Theorem 2.2. *Consider the Deffuant model on* $\mathbb{Z}$ *with real-valued i.i.d. initial opinions.*

(a) Suppose the initial opinion of all agents follows an arbitrary bounded distribution $\mathcal{L}(\eta_0)$ *with expected value* $\mathbb{E}\,\eta_0$ *and* $[a,b]$ *being the smallest closed interval containing its support. If* $\mathbb{E}\,\eta_0$ *does not lie in the support, let* $I \subseteq [a,b]$ *be the maximal, open interval such that* $\mathbb{E}\,\eta_0$ *lies in* I *and* $\mathbb{P}(\eta_0 \in I) = 0$. *In this case let* h *denote the length of* I, *otherwise set* $h = 0$.

Then the critical value for θ, *where a phase transition from a.s. no consensus to a.s. strong consensus takes place, becomes* $\theta_c = \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0, h\}$. *The limit value in the supercritical regime is* $\mathbb{E}\,\eta_0$.

(b) Suppose the initial opinions' distribution is unbounded but its expected value exists, either in the strong sense, i.e. $\mathbb{E}\,\eta_0 \in \mathbb{R}$, *or the weak sense, i.e.* $\mathbb{E}\,\eta_0 \in \{-\infty, +\infty\}$. *Then the Deffuant model with arbitrary fixed parameter* $\theta \in (0, \infty)$ *will a.s. behave subcritically, meaning that no consensus will be approached in the long run.*

The situation at criticality is unsolved with the exception of the case when the gap around the mean is larger than its distance to the extremes of the initial distribution's support. Given this condition, however, the following proposition – which is Prop. 2.4 in Häggström and Hirscher (2014) – settles the question about the long-term behavior for critical θ:

Proposition 2.3. *Let the initial opinions be again i.i.d. with* $[a,b]$ *being the smallest closed interval containing the support of the marginal distribution, and the latter feature a gap* (α, β) *of width* $\beta - \alpha > \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0\}$ *around its expected value* $\mathbb{E}\,\eta_0 \in [a,b]$.

At criticality, that is for $\theta = \theta_c = \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0, \beta-\alpha\} = \beta - \alpha$, *we get the following: If both* α *and* β *are atoms of the distribution* $\mathcal{L}(\eta_0)$, *i.e.* $\mathbb{P}(\eta_0 = \alpha) > 0$ *and* $\mathbb{P}(\eta_0 = \beta) > 0$, *the system approaches a.s. strong consensus. However, it will a.s. lead to no consensus if either* $\mathbb{P}(\eta_0 = \alpha) = 0$ *or* $\mathbb{P}(\eta_0 = \beta) = 0$.

Since the same line of reasoning was used in both Häggström (2012) and Häggström and Hirscher (2014) to derive the results we just stated, it is worth taking a closer look on the key concepts involved, especially as they will be the foundation for most of the conclusions drawn in the upcoming sections.

The presumably most central among these is the idea of *flat points.* If $\mathbb{E}\,\eta_0 \in \mathbb{R}$, a vertex $v \in \mathbb{Z}$ is called *ε-flat to the right* in the initial configuration $\{\eta_0(u)\}_{u\in\mathbb{Z}}$ if for all $n \geq 0$:

$$\frac{1}{n+1}\sum_{u=v}^{v+n}\eta_0(u) \in [\mathbb{E}\,\eta_0 - \varepsilon, \mathbb{E}\,\eta_0 + \varepsilon]\,. \tag{2.1}$$

It is called *ε-flat to the left* if the above condition is met with the sum running from $v-n$ to v instead. Finally, v is called *two-sidedly ε-flat* if for all $m, n \geq 0$

$$\frac{1}{m+n+1}\sum_{u=v-m}^{v+n}\eta_0(u) \in [\mathbb{E}\,\eta_0 - \varepsilon, \mathbb{E}\,\eta_0 + \varepsilon]\,. \tag{2.2}$$

However, in order to understand how vertices being one- or two-sidedly ε-flat in the initial configuration play an important role in the further evolution of the configuration another concept is indispensable, namely the non-random pairwise averaging procedure Häggström (2012) called *Sharing a drink* (SAD).

Think of glasses being placed at all integers, the one at site 0 being brimful, all others empty. Just as in the Deffuant model, neighbors interact and share, but this time without randomness and confidence bound. In other words, we start with the initial profile $\{\xi_0(v)\}_{v\in\mathbb{Z}}$, given by $\xi_0(0) = 1$ and $\xi_0(v) = 0$ for all $v \neq 0$, and a finite sequence $(e_n)_{n=1}^N$ of edges along which updates of the form (1.1) are performed, i.e. for the profile $\{\xi_n(v)\}_{v\in\mathbb{Z}}$ after step n and $e_{n+1} = \langle u, u+1\rangle$ we get $\{\xi_{n+1}(v)\}_{v\in\mathbb{Z}}$ by

$$\begin{aligned}\xi_{n+1}(u) &= (1-\mu)\,\xi_n(u) + \mu\,\xi_n(u+1),\\ \xi_{n+1}(u+1) &= \mu\,\xi_n(u) + (1-\mu)\,\xi_n(u+1);\end{aligned} \tag{2.3}$$

all other values stay unchanged.

Elements of $[0,1]^{\mathbb{Z}}$ that can be obtained in such a way are called SAD-profiles. The crucial connection to the Deffuant model is that the opinion value $\eta_t(0)$ at any given time $t > 0$ can be written as a weighted average of values at time $t = 0$ with weights given by an SAD-profile, see La. 3.1 in Häggström (2012). The fact that all SAD-profiles share certain properties (the most important being unimodality) renders it possible to derive characteristics of the future evolution of the Deffuant dynamics given the initial configuration. For instance, the opinion value at a two-sidedly ε-flat vertex in the initial configuration can never move further than 6ε away from the mean, see La. 6.3 in Häggström (2012).

These two vital ingredients – flat points and SAD-profiles – of the line of argument in Häggström (2012) and Sect. 2 in Häggström and Hirscher (2014) can be adapted in order to analyze the Deffuant model with vector-valued opinions, as we will see in the following section.

3. Deffuant model with multivariate opinions and the Euclidean norm as measure of distance

Having characterized the long-term behavior of the Deffuant dynamics on $\mathbb{Z}$ starting from a general univariate i.i.d. configuration, the next step of generalization with regard to the marginal initial distribution is, as indicated in the introduction, to allow for vectors instead of numbers to represent the opinions. Like in the univariate case, we want the initial opinions to be independent and identically distributed, just now with some common distribution $\mathcal{L}(\eta_0)$ on $\mathbb{R}^k$. This will ensure ergodicity of the setting (with respect to shifts) as before.

In this section we will consider $\mathbb{R}^k$ to be equipped with the Borel σ-algebra generated by the Euclidean norm, denoted by $\mathcal{B}^k$.

Definition 3.1. If the distribution of η_0 has a finite expectation, define its *radius* by

$$R := \inf\left\{r > 0,\ \mathbb{P}\big(\eta_0 \in B[\mathbb{E}\,\eta_0, r]\big) = 1\right\},$$

where $B[y, r] := \{x \in \mathbb{R}^k,\ \|x - y\|_2 \le r\}$ denotes the closed Euclidean ball with radius r around y. Note that the radius of an unbounded distribution is infinite.

The notion of ε-flatness easily translates to the new setting by just replacing the intervals by balls: If $\mathbb{E}\,\eta_0 \in \mathbb{R}^k$, a vertex $v \in \mathbb{Z}$ is called ε-flat to the right in the initial configuration $\{\eta_0(u)\}_{u\in\mathbb{Z}}$ if for all $n \ge 0$:

$$\frac{1}{n+1}\sum_{u=v}^{v+n} \eta_0(u) \in B[\mathbb{E}\,\eta_0, \varepsilon], \tag{3.1}$$

similarly for ε-flatness to the left and two-sided ε-flatness – compare with (2.1) and (2.2).

With these notions in hand we can state and prove a higher-dimensional analogue of Theorem 2.2, valid for initial distributions whose support does not feature a substantial gap around the mean. The proof of this result will be a fairly straightforward adaptation of the methods for the univariate case indicated in Section 2. In contrast, the more general case treated in Theorem 3.15 requires invoking more intricate geometrical considerations.

Theorem 3.2. *In the Deffuant model on $\mathbb{Z}$ with the underlying opinion space $(\mathbb{R}^k, \|\,.\,\|_2)$ and an initial opinion distribution $\mathcal{L}(\eta_0)$ we have the following limiting behavior:*

(a) If $\mathcal{L}(\eta_0)$ has radius $R \in [0, \infty)$ and mass around its mean, i.e.

$$\mathbb{P}\big(\eta_0 \in B[\mathbb{E}\,\eta_0, r]\big) > 0 \text{ for all } r > 0, \tag{3.2}$$

the critical parameter is $\theta_c = R$, meaning that for $\theta < R$ we have a.s. no consensus and for $\theta > R$ a.s. strong consensus.

(b) Let $\eta_0 = (\eta_0^{(1)}, \dots, \eta_0^{(k)})$ be the random initial opinion vector. If at least one of the coordinates $\eta_0^{(i)}$ has an unbounded marginal distribution, whose expected value exists (regardless of whether finite, $+\infty$ or $-\infty$), then the limiting behavior will a.s. be no consensus, irrespectively of θ.

Proof: (a) To show the first part is just like in the univariate case (included in part (a) of Theorem 2.2) little more than following the arguments in the last two sections of Häggström (2012): The central arguments go through even for vector-valued opinions as the crucial properties of the absolute value that were used are shared by its replacement in higher dimensions, the Euclidean norm. Because of that, we only sketch the main line of reasoning and refer to Sect. 6 in Häggström (2012) and Sect. 2 in Häggström and Hirscher (2014) for a more thorough presentation of the arguments.

First of all, the (multivariate) Strong Law of Large Numbers – in the following abbreviated by SLLN – tells us that the averages in (3.1) for large n are close to the mean in Euclidean distance. For $\varepsilon > 0$ fixed, choose $N \in \mathbb{N}$ such that the event

$$A := \left\{ \frac{1}{n+1} \sum_{u=1}^{n+1} \eta_0(u) \in B[\mathbb{E}\,\eta_0, \tfrac{\varepsilon}{3}] \text{ for all } n \geq N \right\}$$

has positive probability. Using (3.2) and the fact that the initial opinions are i.i.d., we can locally modify the configuration to conclude that the event $\{\eta_0(v) \in B[\mathbb{E}\,\eta_0, \frac{\varepsilon}{3}] \text{ for } v = 1, \ldots, N+1\} \cap A$ has positive probability, implying the ε-flatness to the right of site 1 – just as it was done in La. 4.2 in Häggström (2012).

For $\theta < R$, the probability of $\{\eta_0 \notin B[\mathbb{E}\,\eta_0, \theta + \varepsilon]\}$ is non-zero for ε small enough, hence a vertex can be at distance larger than θ from $B[\mathbb{E}\,\eta_0, \varepsilon]$ initially. Due to the independence of initial opinions, the event that site -1 is ε-flat to the left, 1 is ε-flat to the right and $\eta_0(0) \notin B[\mathbb{E}\,\eta_0, \theta+\varepsilon]$ has positive probability. Using the SAD representation, it follows – mimicking Prop. 5.1 in Häggström (2012) – that given such an initial configuration the opinion value at site 1 will be a convex combination of averages in (3.1) for all times $t > 0$ and thus in $B[\mathbb{E}\,\eta_0, \varepsilon]$, due to the convexity of Euclidean balls. The same holds for site -1 and the half-line to the left. Consequently, the edges $\langle -1, 0\rangle$ and $\langle 0, 1\rangle$ will stay blocked for ever. Ergodicity of the initial opinion sequence ensures that with probability 1 (infinitely many) vertices will get isolated that way, which settles the subcritical case.

In the supercritical regime, i.e. $\theta > R$, we focus on two-sidedly ε-flat vertices: If site 0 is ε-flat to the left and 1 is ε-flat to the right, both are two-sidedly ε-flat – using again the convexity of $B[\mathbb{E}\,\eta_0, \varepsilon]$. By independence this event has positive probability, by ergodicity we will a.s. have (infinitely many) two-sidedly ε-flat vertices. Mimicking La. 6.3 in Häggström (2012) literally, we find that vertices which are two-sidedly ε-flat in the initial configuration will never move further than 6ε away from the mean, irrespectively of future interactions. Choosing $\varepsilon > 0$ small, such that $7\varepsilon < \theta - R$ say, will ensure that updates along edges incident to two-sidedly ε-flat vertices will never be prevented by the distance of opinions exceeding the confidence bound.

The proof of Prop. 6.1 in Häggström (2012), which states that neighbors will either finally concur or the edge between them be blocked for large t, can be adopted as well: Its central idea – borrowed from physics – that every individual starts with an initial amount of energy that is then partly transferred partly lost in interactions works regardless whether the opinions $\{\eta_t(v)\}_{v\in\mathbb{Z}}$ are shaped by numbers or vectors. Merely in the current setting, the term $W_t(v) = (\eta_t(v))^2$, that defines the energy at vertex v at time t, has to be read as a dot product. Again, if the opinions $\eta_t(u), \eta_t(v)$ of two neighbors are within the confidence bound but $\|\eta_t(u) - \eta_t(v)\|_2 \geq \delta$ for some fixed $\delta > 0$, $W_t(u) + W_t(v)$ decreases by at least $2\mu(1-\mu)\delta^2$ when they compromise. This can not happen infinitely often with positive probability as the expected energy at time $t = 0$ is $\mathbb{E}\,W_0(v) = \mathbb{E}\,(\eta_0^2) < \infty$ and the expectation of $W_t(v)$ is both non-increasing with t and non-negative. For details see Prop. 6.1 and La. 6.2 in Häggström (2012).

Following from the considerations above, two-sidedly ε-flat vertices and their neighbors therefore have to finally concur with probability 1, forcing the opinion values of the neighbors to eventually lie at a distance strictly less than 7ε from the mean as well. By our choice of ε, this conclusion propagates inductively showing that the limiting behavior will a.s. be strong consensus, if we let ε tend to 0.

(b) In order to prove the second claim, we use part (b) of Theorem 2.2, focussing on the ith coordinate only. Fix $\theta \in (0, \infty)$. Since

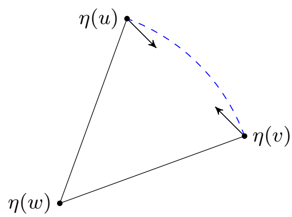
$$|x_i - y_i| \leq \|x - y\|_2 \text{ for all vectors } x, y \in \mathbb{R}^k \text{ and } i \in \{1, \dots, k\},$$

a distance of more than θ in the ith coordinate of the opinion vectors for two neighbors u, v implies that the edge between them is blocked. The arguments used for unbounded distributions in Theorem 2.2 – see Thm. 2.2 in Häggström and Hirscher (2014) – show that under the given conditions, there are a.s. vertices that differ more than θ from both their neighbors in the ith coordinate (with respect to the absolut value) in the initial configuration and this will not change no matter whom their neighbors will compromise with. Consequently, the corresponding opinion vectors will always be at Euclidean distance more than θ.

□

Remark 3.3. Pretty much as in the univariate setting, the case where all unbounded coordinates of η_0 do not have an expected value (neither finite nor $+\infty$ nor $-\infty$) remains unsolved by Theorem 3.2.

When it comes to bounded initial distributions which do have a large gap around the mean, the picture in higher dimensions drastically changes – something that will require several preliminary results before we are ready to state and prove this section's main result, Theorem 3.15. The major difference to the univariate case is that with higher-dimensional opinions the update along some edge $\langle u, v\rangle$ can actually lead to a situation, where both u and v come closer to the opinion corresponding to a third vertex w, which lies within the confidence bound of neither $\eta(u)$ nor $\eta(v)$, see the below picture.



In the case of real-valued opinions this is impossible, because in that setting an update along $\langle u, v\rangle$ always increases $\min\{|\eta(u) - \eta(w)|, |\eta(v) - \eta(w)|\}$, if $\eta(w)$ does not lie in between $\eta(u)$ and $\eta(v)$.

To illustrate how this changes the conditions, let us consider the initial distributions $\text{unif}(S^{k-1})$, where S^{k-1} denotes the Euclidean unit sphere in $\mathbb{R}^k$. For $k = 1$ this is just $\text{unif}(\{-1, 1\})$, which by Theorem 2.2 has the trivial critical value $\theta_c = 2$.

For $k \geq 2$ however, the fact that opinions close to each other can compromise in order to form a central opinion will bring θ_c down to the radius 1 of the distribution as we will see in the sequel.

The statement of the main result in this section, Theorem 3.15, resembles very much the one of Theorem 2.2 (a), only the notion of a gap in the initial distribution has to be reinterpreted in the higher-dimensional setting, making the proof of this generalized result rather technical. However, while establishing auxiliary results, we will gain additional information about the set of opinion values that can occur in the Deffuant model at times $t > 0$ depending on the initial distribution and the confidence bound. When it comes to the initial distribution $\mathcal{L}(\eta_0)$, the most important features besides its expected value are its support and the corresponding radius.

Definition 3.4. Consider an $\mathbb{R}^k$-valued random variable ζ. Its *support* is the following subset of $\mathbb{R}^k$, which is closed with respect to the Euclidean metric:

$$\operatorname{supp}(\zeta) := \left\{x \in \mathbb{R}^k,\ \mathbb{P}\big(\zeta \in B[x,r]\big) > 0 \text{ for all } r > 0\right\}.$$

Observe that this definition corresponds to the standard notion of *spectrum of a measure*, see for example Thm. 2.1 and Def. 2.1 in Parthasarathy (1967) – applied to the distribution of a random variable.

If the initial distribution has a finite expectation, the radius can also be written as

$$R = \sup\left\{\|\mathbb{E}\,\eta_0 - x\|_2,\ x \in \operatorname{supp}(\eta_0)\right\},$$

as the following proposition shows.

Proposition 3.5. *If $\mathbb{E}\,\eta_0 \in \mathbb{R}^k$, we have*

$$\inf\left\{r > 0,\ \mathbb{P}\big(\eta_0 \in B[\mathbb{E}\,\eta_0, r]\big) = 1\right\} = \sup\left\{\|\mathbb{E}\,\eta_0 - x\|_2,\ x \in \operatorname{supp}(\eta_0)\right\}. \tag{3.3}$$

Proof: First, consider a set A which is compact in $(\mathbb{R}^k, \|\,.\,\|_2)$ and a subset of the complement of $\operatorname{supp}(\eta_0)$. We claim that these properties imply $\mathbb{P}(\eta_0 \in A) = 0$. Indeed, for every point $x \in A \subseteq (\operatorname{supp}(\eta_0))^{\mathrm{c}}$ there exists a radius $r_x > 0$ s.t. $\mathbb{P}\big(\eta_0 \in B[x, r_x]\big) = 0$. Let $B(y,r)$ denote the open Euclidean ball with radius r around y, then $\{B(x, r_x),\ x \in A\}$ is an open cover of A, which by compactness has a finite subcover $\{B(x_i, r_{x_i}),\ 1 \leq i \leq n\}$. Consequently

$$\mathbb{P}(\eta_0 \in A) \leq \mathbb{P}\Big(\eta_0 \in \bigcup_{i=1}^{n} B[x_i, r_{x_i}]\Big) = 0.$$

If r is greater than the supremum in (3.3) it follows that $\operatorname{supp}(\eta_0) \subseteq B(\mathbb{E}\,\eta_0, r)$. Since

$$\big(B(\mathbb{E}\,\eta_0, r)\big)^{\mathrm{c}} = \Bigg(B[\mathbb{E}\,\eta_0, r+1] \setminus B(\mathbb{E}\,\eta_0, r)\Bigg) \cup \Bigg(\bigcup_{q \in \mathbb{Q}^k \setminus B[\mathbb{E}\,\eta_0, r+1]} B[q, 1]\Bigg)$$

and the right-hand side is a countable union of nullsets with respect to $\mathcal{L}(\eta_0)$, we get $\mathbb{P}\big(\eta_0 \in B[\mathbb{E}\,\eta_0, r]\big) = 1$, which means that r is greater or equal to the infimum in (3.3).

On the other hand, if r is less than the supremum in (3.3), there exists a point $x \in \operatorname{supp}(\eta_0) \setminus B[\mathbb{E}\,\eta_0, r]$, which consequently has a positive distance δ to the closed ball $B[\mathbb{E}\,\eta_0, r]$. This gives

$$\mathbb{P}\big(\eta_0 \in B[\mathbb{E}\,\eta_0, r]\big) \leq 1 - \mathbb{P}\big(\eta_0 \in B[x, \tfrac{\delta}{2}]\big) < 1.$$

In other words, r does not appear in the set the infimum is taken over. Putting both arguments together proves (3.3). □

Definition 3.6.

(i) For a finite graph $G = (V, E)$ and an edge $e = \langle u, v\rangle \in E$ let the update described in (1.1), considered as a deterministic map on the set of $\mathbb{R}^k$-valued profiles, be denoted by T_e^θ. So if T_e^θ is applied to $\xi = \{\xi(v)\}_{v\in V}$ it just means that all values stay unchanged with the only exception of

$$\begin{pmatrix} T_e^\theta \xi(u) \\ T_e^\theta \xi(v) \end{pmatrix} = \begin{pmatrix} (1-\mu)\,\xi(u) + \mu\,\xi(v) \\ \mu\,\xi(u) + (1-\mu)\,\xi(v) \end{pmatrix} \quad \text{if } \|\xi(u) - \xi(v)\|_2 \le \theta. \tag{3.4}$$

(ii) Consider a finite section $\{1, \dots, n\}$ of the line graph, a finite sequence $(e_i)_{i=1}^N$ of edges $e_i \in \{\langle 1, 2\rangle, \dots, \langle n-1, n\rangle\}$ and some values $x_1, \dots, x_n$ in $\operatorname{supp}(\eta_0)$. Such a triple will from now on be called a *finite configuration*.
To *update the configuration* (with respect to θ) will mean that we take the values $x_1, \dots, x_n$ as initial opinions, i.e. we set $\eta_0(v) = x_v$ for all $v \in \{1, \dots, n\}$, and then apply $T_{e_N}^\theta \circ T_{e_{N-1}}^\theta \circ \dots \circ T_{e_1}^\theta$ to $\{\eta_0(v)\}_{v\in\{1,\dots,n\}}$.
Slightly abusing the notation, let the outcome, i.e. the final opinion values $\{T_{e_N}^\theta \circ \dots \circ T_{e_1}^\theta\, \eta_0(v)\}_{v\in\{1,\dots,n\}}$, be denoted by $\{\eta_N(1), \dots, \eta_N(n)\}$.

(iii) Let ν denote the initial distribution $\mathcal{L}(\eta_0)$. For $\theta > 0$, let $\mathscr{D}_\theta(\nu)$ denote the set of vectors in $\mathbb{R}^k$ which the opinion values of finite configurations can collectively approach, if updated according to confidence bound θ. More precisely, $x \in \mathscr{D}_\theta(\nu)$ if and only if for all $r > 0$, there exist some $n \in \mathbb{N} = \{1, 2, \dots\}$, $x_1, \dots, x_n \in \operatorname{supp}(\eta_0)$ and $(e_i)_{i=1}^N$ as above, such that updating the configuration with respect to θ yields $\eta_N(v) \in B[x, r]$ for all $v \in \{1, \dots, n\}$.

It is worth emphasizing that finite configurations are supposed to mimick the dynamics of the Deffuant model, interpreting $(e_i)_{i=1}^N$ as the locations of the first N Poisson events on the edges $\langle 0, 1\rangle, \langle 1, 2\rangle, \dots, \langle n-1, n\rangle, \langle n, n+1\rangle$ in (strict) chronological order. In this respect, considering θ, we can choose the sequence $(e_i)_{i=1}^N$ such that only Poisson events causing an actual update are considered by simply eliminating all events on edges where the opinions of the two vertices are more than θ apart.

Note that according to the definition, $\mathscr{D}_\theta(\nu)$ depends on $\operatorname{supp}(\eta_0)$ and θ, as well as μ, the latter being less obvious. See Example 3.18 below for an instance where μ actually makes a difference. Let us now turn to various properties of the set $\mathscr{D}_\theta(\nu)$.

Lemma 3.7. *Fix the distribution ν of η_0 and let $\mathscr{D}_\theta(\nu)$ and R be defined as above.*

(a) $\mathscr{D}_\theta(\nu)$ is closed and increases with θ.

(b) $\operatorname{supp}(\eta_0) \subseteq \mathscr{D}_\theta(\nu) \subseteq \overline{\operatorname{conv}(\operatorname{supp}(\eta_0))} \subseteq B[\mathbb{E}\,\eta_0, R]$ for all $\theta > 0$, where $\operatorname{conv}(A)$ denotes the convex hull, $\overline{A}$ the closure of a set A.

Proof: (a) The first claim follows directly from the definition: For a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{D}_\theta(\nu)$ such that $\|x - x_n\|_2 \to 0$ and every $r > 0$, there exists some $x_n \in B[x, \frac{r}{2}]$. Due to $x_n \in \mathscr{D}_\theta(\nu)$, there exists a finite configuration with all final opinion values in $B[x_n, \frac{r}{2}]$. But since $B[x_n, \frac{r}{2}] \subseteq B[x, r]$, this implies $x \in \mathscr{D}_\theta(\nu)$.

As for the second claim, since we are free to choose the edge sequence in finite configurations, it is obvious that making θ larger only allows for more options when we are to come up with a setting that brings the opinion values collectively inside $B[x, r]$ for some given $x \in \mathbb{R}^k$ and $r > 0$.

(b) The first inclusion is trivial, as for $x \in \text{supp}(\eta_0)$ the finite configuration with $n = 1,\ x_1 = x$ will do. The second inclusion is due to the fact that every update of opinions is a convex combination, see (3.4). Consequently, all final opinion values of finite configurations lie within $\text{conv}(\text{supp}(\eta_0))$. The last inclusion, which is meaningful only for $R < \infty$, follows from Proposition 3.5 and the fact that $B[\mathbb{E}\,\eta_0, R]$ is both convex and closed.

□

It should be mentioned that an easy corollary to Carathéodory's Theorem on the convex hull states that the convex hull of a compact set in $\mathbb{R}^k$ is compact as well. If η_0 has a bounded support, this implies that the convex hull of $\text{supp}(\eta_0)$ is actually closed, i.e. $\overline{\text{conv}(\text{supp}(\eta_0))} = \text{conv}(\text{supp}(\eta_0))$.

Example 3.8. To get familiar with the idea behind $\mathscr{D}_\theta(\nu)$, let us consider the discrete real-valued initial distribution given by $\mathbb{P}(\eta_0 = \frac{1}{n}) = \frac{1}{2^n}, n \in \mathbb{N}$. It is not hard to see that this implies $\text{supp}(\eta_0) = \{\frac{1}{n},\ n \in \mathbb{N}\} \cup \{0\}$. Having the Taylor expansion of the logarithm in mind we find

$$\mathbb{E}\,\eta_0 = \sum_{n=1}^{\infty} \frac{1}{n\,2^n} = -\left(-\sum_{n=1}^{\infty} \frac{(\frac{1}{2})^n}{n}\right) = -\ln(1 - \tfrac{1}{2}) = \ln(2).$$

By Theorem 2.2 we get $\theta_c = R = \ln(2)$, since $\mathbb{P}(\eta_0 \in [0,1]) = 1$ and the largest gap in between the point masses is $\frac{1}{2}$.

For two point masses situated at x and y at distance $0 < \|x - y\|_2 \le \theta$, all convex combinations of x, y are in $\mathscr{D}_\theta(\nu)$: For $\alpha \in [0,1]$ and $r > 0$, take $m, n \in \mathbb{N}$ s.t.

$$\left|\frac{m}{m+n} - \alpha\right| \le \frac{r}{4\,\max\{\|x\|_2, \|y\|_2\}}.$$

Let us set up a finite configuration with $m + n$ vertices, $x_1 = \ldots = x_m = x$ and $x_{m+1} = \ldots = x_{m+n} = y$ as well as enough Poisson events on every edge (in an appropriate order) such that – having updated the configuration according to the edge sequence – the outcome $\eta_N(v)$ will be at distance less than $\frac{r}{2}$ from the average $\frac{m}{m+n}\,x + \frac{n}{m+n}\,y$ for all $v \in \{1, \ldots, m+n\}$. Since all the opinion values lie in an interval of length at most θ in the beginning and hence always will, we could choose the edge sequence by always taking the edge with largest current discrepancy next, to see that a finite sequence with the claimed property exists. This will ensure

$$\begin{aligned}\|\eta_N(v) - (\alpha x + (1-\alpha)y)\|_2 &\le \tfrac{r}{2} + \|(\tfrac{m}{m+n}\,x + \tfrac{n}{m+n}\,y) - (\alpha x + (1-\alpha)y)\|_2\\ &\le \tfrac{r}{2} + |\tfrac{m}{m+n} - \alpha| \cdot \|x\|_2 + |\alpha - \tfrac{m}{m+n}| \cdot \|y\|_2\\ &\le r,\end{aligned}$$

hence $\alpha x + (1-\alpha)y \in \mathscr{D}_\theta(\nu)$. This observation together with the fact that gaps of width larger than θ can not be bridged leads to

$$\mathscr{D}_\theta(\nu) = [0, \tfrac{1}{n_\theta}] \cup \{\tfrac{1}{n},\ n < n_\theta\},$$

where $n_\theta := \max\{n \in \mathbb{N},\ \frac{1}{n-1} - \frac{1}{n} > \theta\}$.

Lemma 3.9.

(a) For all $x \in \mathbb{R}^k$ and $0 \le \delta < \frac{\theta}{2}$, the set $\mathscr{D}_\theta(\nu) \cap B[x, \delta]$ is convex.
(b) If $R < \infty$, then $\mathscr{D}_{2R}(\nu) = \overline{\text{conv}(\text{supp}(\eta_0))} = \text{conv}(\text{supp}(\eta_0))$.

(c) *The connected components of $\mathscr{D}_\theta(\nu)$ are convex and at distance at least θ from one another. If $\mathscr{D}_\theta(\nu)$ is connected, then $\mathscr{D}_\theta(\nu) = \overline{\mathrm{conv}(\mathrm{supp}(\eta_0))}$.*

(d) *If $R < \infty$ and ν has mass around its mean, i.e. condition (3.2) holds, then $\mathscr{D}_\theta(\nu) = \mathrm{conv}(\mathrm{supp}(\eta_0))$ already for $\theta > R$.*

(e) *For $R < \infty$, the set-valued mapping*

$$\begin{cases} (0,\infty) \to \mathcal{B}^k \\ \vartheta \mapsto \mathscr{D}_\vartheta(\nu) \end{cases}$$

is piecewise constant with only finitely many jumps on $[\delta, \infty)$ for all $\delta > 0$.

(f) *If $\mathscr{D}_\theta(\nu)$ is connected and $\mathbb{E}\,\eta_0$ finite, then $\mathbb{E}\,\eta_0 \in \mathscr{D}_\theta(\nu)$*

Proof: (a) The proof of the first part of this lemma follows the idea of the above example. Let $y, z \in \mathscr{D}_\theta(\nu)$ and their distance be $0 < \|y - z\|_2 \le 2\delta < \theta$. Let $\varepsilon = \theta - 2\delta > 0$. For any $\varepsilon \ge r > 0$, there exist finite configurations χ_1 and χ_2 with final values in $B[y, \frac{r}{4}]$ and $B[z, \frac{r}{4}]$ respectively. For $\alpha \in [0,1]$ choose again $m, n \in \mathbb{N}$ s.t.

$$\left|\frac{m}{m+n} - \alpha\right| \le \frac{r}{4\,\max\{\|y\|_2, \|z\|_2\}}.$$

We define a new finite configuration by putting m copies of χ_1 and n copies of χ_2 next to each other: Their finite sections of the line graph (together with the assigned initial values) will be concatenated blockwise – the order among the blocks being irrelevant – by adding an edge between two consecutive blocks in order to form the underlying line graph of a larger finite configuration. To get an edge sequence for the whole configuration we will simply string together the edge sequences of the individual copies, again in a blockwise manner and arbitrary order.

Updating according to the edge sequence will then bring all the opinion values within distance θ of one another. Therefore, we can bring the final outcomes arbitrarily close, say at distance at most $\frac{r}{4}$, to the average of the initial values, let's denote it by $\overline{x}$, by just adding a large enough (but finite) number of Poisson events on each edge (appropriately ordered as before). From the properties of the chosen building blocks, χ_1 and χ_2, it readily follows that the initial average is at distance at most $\frac{r}{4}$ from $\frac{m}{m+n}\,y + \frac{n}{m+n}\,z$. This entails for every vertex v of the finite configuration

$$\begin{aligned}
\|\eta_N(v) - (\alpha y + (1-\alpha)z)\|_2 &\le \tfrac{r}{4} + \|\overline{x} - (\alpha y + (1-\alpha)z)\|_2 \\
&\le \tfrac{r}{4} + \tfrac{r}{4} + \|(\tfrac{m}{m+n}\,y + \tfrac{n}{m+n}\,z) - (\alpha y + (1-\alpha)z)\|_2 \\
&\le \tfrac{r}{2} + |\tfrac{m}{m+n} - \alpha| \cdot \|y\|_2 + |\alpha - \tfrac{m}{m+n}| \cdot \|z\|_2 \\
&\le r,
\end{aligned}$$

which shows $\alpha y + (1-\alpha)z \in \mathscr{D}_\theta(\nu)$.

(b) By Lemma 3.7 it is enough to show $\mathscr{D}_{2R}(\nu) \supseteq \mathrm{conv}(\mathrm{supp}(\eta_0))$. Thus, letting $x, y \in \mathrm{supp}(\eta_0) \subseteq B[\mathbb{E}\,\eta_0, R]$, we have to show that $\mathrm{conv}(\{x,y\}) \subseteq \mathscr{D}_{2R}(\nu)$. But since $\|x - y\|_2$ can be at most $2R$, this is done as described in Example 3.8, just the line segment $\mathrm{conv}(\{x,y\})$ plays now the role of the interval considered there.

(c) First of all, the connected components of $\mathscr{D}_\theta(\nu)$ are actually path-connected and moreover the pathes can be chosen to be polygonal chains: Assume that a connected component C contains more than one path-connected component.

Fix one such, say C_1. Due to connectedness of C, a second one C_2 must exist s.t. the Euclidean distance between C_1 and C_2 is 0. But part (a) then implies that also $C_1 \cup C_2$ is path-connected, a contradiction. Moreover, using the statement of part (a) we can transform any curve in $\mathscr{D}_\theta(\nu)$ to a polygonal chain which completely lies in $\mathscr{D}_\theta(\nu)$.

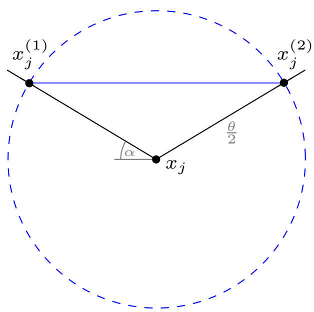
Let us turn to the convexity of connected components. Fix a component C of $\mathscr{D}_\theta(\nu)$ and $x, y \in C$, s.t. $\|x-y\|_2 \geq \theta$, since otherwise (a) guarantees

$$\mathrm{conv}(\{x,y\}) = \{\alpha x + (1-\alpha)y,\ \alpha \in [0,1]\} \subseteq C.$$

By the above, there exists a polygonal chain in $\mathscr{D}_\theta(\nu)$, say

$$l := \begin{cases} [0,1] \to \mathbb{R}^k \\ s \mapsto l(s) \end{cases}$$

such that $l(0) = x,\ l(1) = y$ and l is continuous and piecewise linear. Let us define $x_0 = x,\ x_{j+1} = l(s_j)$, where $s_j := \max\{s \in [0,1],\ \|x_j - l(s)\|_2 = \frac{\theta}{2}\}$, if $\|x_j - y\|_2 \geq \theta$ and $x_{j+1} = y$ otherwise. Using (a) and these intermediate points shows that we can assume without loss of generality a certain sparseness of the chain, namely that its intermediate points $x_1, \dots, x_n$ are s.t. pairwise distances in $\{x = x_0, x_1, \dots, x_n, x_{n+1} = y\}$ are at least $\frac{\theta}{2}$ and hence $n \leq \frac{2L}{\theta}$, where L denotes the length of the original chain. Note that the modification of the polygonal chain as just described will only decrease its length. Given a polygonal chain in $\mathscr{D}_\theta(\nu)$ connecting x and y, let us assume that the minimal angle at an intermediate point is $\pi - 2\alpha < \pi$ at x_j. Considering $B[x_j, \frac{\theta}{2}]$ and using (a) once more, we can replace x_j by the two intersection points of the ball's boundary and the chain – $x_j^{(1)}, x_j^{(2)}$ – and conclude that the new polygonal chain through the nodes $x, x_1, x_2, \dots, x_{j-1}, x_j^{(1)}, x_j^{(2)}, x_{j+1}, \dots, x_n, y$ still lies in $\mathscr{D}_\theta(\nu)$ and is shorter – at least by the amount of $\theta \cdot (1 - \cos(\alpha))$.



We can then sparsify the updated chain as described above and denote the result by l_1. Iterating the whole procedure gives a sequence $(l_m)_{m\in\mathbb{N}}$ of shorter and shorter polygonal chains in $\mathscr{D}_\theta(\nu)$ connecting x and y. Since the length is bounded below by $\|x-y\|_2$, the internal angels must approach π uniformly. Let $\pi - 2\alpha_1, \dots, \pi - 2\alpha_n$ be the angles at $x_1, \dots, x_n$. An easy geometric argument yields that all points on the chain are at distance at most

$$\sum_{j=1}^{n} \tan(2\alpha_1 + \cdots + 2\alpha_j)\, L \leq \tfrac{8nL}{\pi} \sum_{j=1}^{n} \alpha_j \leq \tfrac{16L^2}{\pi\theta} \sum_{j=1}^{n} \alpha_j.$$

from the line through x and x_1, if $\sum_{j=1}^{n} \alpha_j \leq \frac{\pi}{8}$, as $\tan(z) \leq \frac{4}{\pi} z$ for all $z \in [0, \frac{\pi}{4}]$. This also holds for the endpoint y, which is why the maximal distance of a point on the chain to the line segment between x and y is bounded by $\frac{32L^2}{\pi\theta} \sum_{j=1}^{n} \alpha_j$. Let n_m and $(\alpha_j^{(m)})_{j=1}^{n_m}$ correspond to l_m. Then

$$\sum_{j=1}^{n_m} \alpha_j^{(m)} \leq \tfrac{2L}{\theta} \max_{1\leq j\leq n_m} \alpha_j^{(m)} \overset{m\to\infty}{\longrightarrow} 0$$

implies that the sequence $(l_m)_{m\in\mathbb{N}}$ must approach the line segment between x and y, i.e. $\mathrm{conv}(\{x,y\}) = \{\alpha x + (1-\alpha)y,\ \alpha \in [0,1]\}$, uniformly – in the sense that

$$\max_{s\in l_m} \min_{z\in \mathrm{conv}(\{x,y\})} \|s-z\|_2 \to 0 \quad \text{as } m \to \infty.$$

Since C being a component of $\mathscr{D}_\theta(\nu)$ is closed, we find $\mathrm{conv}(\{x,y\}) \subseteq C$, which proves the convexity of C.

Assuming that there are two points in different connected components, say $x \in C_1, y \in C_2$ s.t. $\|x-y\|_2 < \theta$, already implies (by part (a)) that $C_1 \cup C_2$ is connected, as before. Finally, if $\mathscr{D}_\theta(\nu)$ is connected, what we just proved induces that it is convex. Being a closed superset of $\mathrm{supp}(\eta_0)$, this implies

$$\overline{\mathrm{conv}(\mathrm{supp}(\eta_0))} \subseteq \mathscr{D}_\theta(\nu),$$

which by Lemma 3.7 is all that needed to be shown.

(d) Let us now assume that ν has not only a finite radius but also mass around its mean, that is $\mathbb{E}\,\eta_0 \in \mathrm{supp}(\eta_0)$. For $\theta > R$, $\mathscr{D}_\theta(\nu)$ is then connected, which by part (c) implies the claim. Indeed, let $\varepsilon \in (0, \theta - R)$ and choose a point x in $B[\mathbb{E}\,\eta_0, \varepsilon] \cap \mathrm{supp}(\eta_0)$. By the choice of ε, all points in $B[\mathbb{E}\,\eta_0, R]$ are at distance less than θ from x, which by the reasoning in part (a) and $\mathscr{D}_\theta(\nu) \subseteq B[\mathbb{E}\,\eta_0, R]$ (see Lemma 3.7) implies $\mathrm{conv}(\{x,y\}) \subseteq \mathscr{D}_\theta(\nu)$ for all $y \in \mathscr{D}_\theta(\nu)$, hence the connectedness of $\mathscr{D}_\theta(\nu)$.

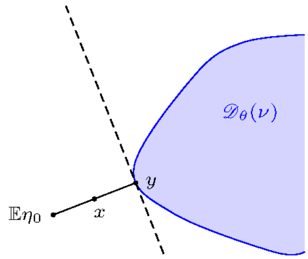
(e) The first thing to notice is that, given $R < \infty$, for all $\theta > 0$ the set $\mathscr{D}_\theta(\nu)$ has finitely many connected components. Indeed, choose a point x_i in each, then the open balls $B(x_i, \theta)$ must be disjoint by (c) and lie within $B(\mathbb{E}\,\eta_0, R+\theta)$. Consequently, there can't be more than $(\frac{R+\theta}{\theta})^k$ of them.

Let $C_1, \ldots, C_n$ be the connected components of $\mathscr{D}_\delta(\nu)$, for some $\delta > 0$, and $d \geq \delta$ the minimal distance between them. When θ is made larger than d, at least two of the components merge. Hence there can be only $n-1$ further jumps. For $\delta \leq \theta < d$ we have $\mathscr{D}_\theta(\nu) = \mathscr{D}_\delta(\nu)$.

(f) The last claim can easily be proved by contradiction. Let us therefore assume that $\mathbb{E}\,\eta_0 \notin \mathscr{D}_\theta(\nu)$. As this set is closed, there exists some $y \in \mathscr{D}_\theta(\nu)$ such that the Euclidean distance from $\mathbb{E}\,\eta_0$ to $\mathscr{D}_\theta(\nu)$ is given by $\|\mathbb{E}\,\eta_0 - y\|_2 > 0$. Let us define $x := \frac{1}{2}(\mathbb{E}\,\eta_0 + y)$. By the convexity of $\mathscr{D}_\theta(\nu)$ we can conclude $(z-y)\cdot(x-y) \leq 0$ for all $z \in \mathscr{D}_\theta(\nu)$: If there existed some $z \in \mathscr{D}_\theta(\nu)$ such that $(z-y)\cdot(x-y) > 0$, y would not be closest to $\mathbb{E}\,\eta_0$ in $\mathscr{D}_\theta(\nu)$. Using this, as well as $\operatorname{supp}(\eta_0) \subseteq \mathscr{D}_\theta(\nu)$, we can conclude

$$\mathbb{E}\left((\eta_0 - x)\cdot(y-x)\right) > 0,$$
$$\text{and } (\mathbb{E}\,\eta_0 - x)\cdot(y-x) < 0,$$

a contradiction.



□

Example 3.10.

(a) To get an impression of how $\mathscr{D}_\theta(\nu)$ grows with θ, let us consider the initial distribution on $\mathbb{R}^3$ given by $\operatorname{unif}(\{(2,1,0),(2,-1,0),(-2,0,1),(-2,0,-1)\})$, i.e. featuring four point masses at the given vertices. It is easy to check that $\mathbb{E}\,\eta_0 = (0,0,0)$ and $R = \sqrt{5}$, see Figure 3.1.

Since all pairwise distances are at least 2, $\mathscr{D}_\theta(\nu) = \operatorname{supp}(\eta_0)$ for $\theta < 2$. For $\theta \geq 2$ the opinion values $(2,1,0)$ and $(2,-1,0)$ can compromise, same for $(-2,0,1)$ and $(-2,0,-1)$. This implies that $\mathscr{D}_\theta(\nu)$ contains both line segments $\{(2,\alpha,0),\ \alpha\in[-1,1]\}$ and $\{(-2,0,\alpha),\ \alpha\in[-1,1]\}$. The latter are at distance 4, hence we can conclude

$$\mathscr{D}_\theta(\nu) = \begin{cases} \{(2,1,0),(2,-1,0),(-2,0,1),(-2,0,-1)\}, & \text{for } \theta < 2\\ \{(2,\alpha,0),(-2,0,\alpha),\ \alpha\in[-1,1]\}, & \text{for } \theta\in[2,4)\\ \operatorname{conv}(\{(2,1,0),(2,-1,0),(-2,0,1),(-2,0,-1)\}), & \text{for } \theta > 4. \end{cases}$$

For $\theta = 4$ it depends on whether the values $(-2,0,0),(2,0,0)$ can be achieved or merely approximated by finite configurations, in other words μ (see also Example 3.18). Note how $\mathscr{D}_\theta(\nu)$ grows by forming local convex hulls.

If we choose $\operatorname{unif}(\{(0.99,1,0),(0.99,-1,0),(-0.99,0,1),(-0.99,0,-1)\})$ to be the initial distribution instead, we can observe a certain chain reaction effect. $\theta\geq 2$ brings the point masses pairwise within the confidence bound as before, but this time also their convex hulls. So for this distribution ν we find

$$\mathscr{D}_\theta(\nu) = \begin{cases} \operatorname{supp}(\eta_0), & \text{for } \theta < 2\\ \operatorname{conv}(\operatorname{supp}(\eta_0)), & \text{for } \theta \geq 2. \end{cases}$$

(b) Example 3.8 already shows that the mapping $\vartheta \mapsto \mathscr{D}_\vartheta(\nu)$ can have infinitely (but still countably) many jumps on $(0,\infty)$. Taking the discrete initial distribution given by

$$\mathbb{P}(\eta_0 = 2^n) = \tfrac{1}{3^n} \text{ and } \mathbb{P}(\eta_0 = -2^n) = \tfrac{1}{3^n}, \text{ for } n\in\mathbb{N},$$

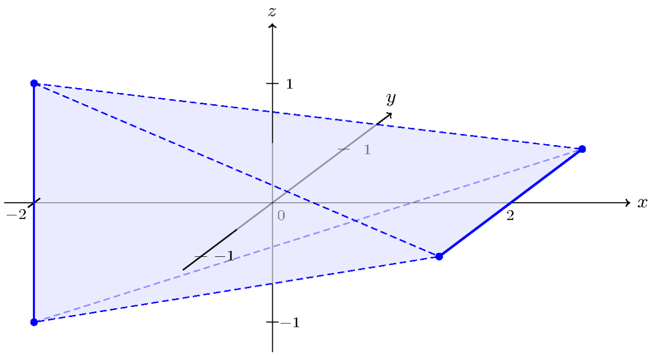


FIGURE 3.1. $\mathscr{D}_\theta(\nu)$ for η_0 being uniformly distributed on the set $\{(2,1,0),(2,-1,0),(-2,0,1),(-2,0,-1)\}$, evolves with growing θ.

shows that part (e) of Lemma 3.9 doesn't hold for the case $R=\infty$, i.e. under the weaker condition that $\mathbb{E}\,\eta_0$ is finite.

(c) Coming back to the example mentioned above, where $\eta_0 \sim \mathrm{unif}(S^{k-1})$ for some $k \geq 2$, it is not hard to see that $\mathscr{D}_\theta(\nu) = B[\mathbf{0},1]$ for all $\theta > 0$. Indeed, since $\mathrm{supp}(\eta_0) = S^{k-1}$ is connected and $\mathrm{supp}(\eta_0) \subseteq \mathscr{D}_\theta(\nu)$, it has to be contained in a connected component of $\mathscr{D}_\theta(\nu)$. All such are convex by Lemma 3.9, hence $\mathrm{conv}(S^{k-1}) = B[\mathbf{0},1] \subseteq \mathscr{D}_\theta(\nu)$. The reverse inclusion follows directly from part (b) of Lemma 3.7.

Definition 3.11. For $\theta > 0$ and $t \geq 0$, let the *support of the distribution of* η_t be denoted by $\mathrm{supp}_\theta(\eta_t)$.

The support of η_t evidently depends on θ. However, for $t=0$ it holds that $\mathrm{supp}_\theta(\eta_0) = \mathrm{supp}(\eta_0)$ irrespectively of θ, as the dynamics of the model is not yet involved. Note that for values of θ where $\mathscr{D}_\theta(\nu)$ increases, $\mathrm{supp}_\theta(\eta_t)$ can actually depend on μ as well, see Example 3.18 below. Let us next derive properties of $\mathrm{supp}_\theta(\eta_t)$ similar to those of $\mathscr{D}_\theta(\nu)$.

Lemma 3.12.

(a) For $0 < s < t$ *we get* $\mathrm{supp}_\theta(\eta_s) = \mathrm{supp}_\theta(\eta_t)$.

(b) $\mathrm{supp}_\theta(\eta_t)$ *increases with* θ *and for all* $\theta > 0$:

$$\mathrm{supp}(\eta_0) \subseteq \mathrm{supp}_\theta(\eta_t) \subseteq \overline{\mathrm{conv}(\mathrm{supp}(\eta_0))} \subseteq B[\mathbb{E}\,\eta_0, R].$$

Proof: (a) $\mathrm{supp}_\theta(\eta_s) \subseteq \mathrm{supp}_\theta(\eta_t)$ readily follows from the fact, that for every set A $\mathbb{P}(\eta_s(v) \in A) > 0$ implies $\mathbb{P}(\eta_t(v) \in A) > 0$, since with positive probability there won't be any Poisson events on the edges $\langle v-1, v\rangle$ and $\langle v, v+1\rangle$ in the time interval $[s,t]$ forcing $\eta_s(v) = \eta_t(v)$.

But the reverse inclusion is also true. To see this we will locally modify the configuration: $x \in \mathrm{supp}_\theta(\eta_t)$ if and only if for all $r > 0$, there exists some $n \in \mathbb{N}$ such that the event that $\eta_t(0) \in B[x,r]$ and at least one of the edges $\langle -n, -n+1\rangle, \dots, \langle -1, 0\rangle$ and $\langle 0, 1\rangle, \dots, \langle n-1, n\rangle$ respectively, has not experienced any Poisson event up to time t has positive probability. That the Poisson events occurring on $\langle -n, -n+1\rangle, \dots, \langle n-1, n\rangle$ up to t already occur in the same order up to time s (and no further events) has positive probability. Due to the fact that the Poisson events are independent of the starting configuration, such a modification of the interactions establishes $\mathbb{P}(\eta_s(0) \in B[x,r]) > 0$.

(b) To prove the monotonicity in θ, we will dissect the event described in part (a) a little more closely. For $x \in \mathrm{supp}_\vartheta(\eta_t)$ and $r > 0$, let us consider the event that $\eta_t(0) \in B[x,r]$ and at least one of the edges between $-n$ and 0 as well as between 0 and n has not experienced any Poisson event up to time t. For sufficiently large n this has positive probability as mentioned before. Fix n to be large enough in this respect and denote the corresponding event by A.

Let again $(e_i)_{i=1}^N$ encode the chronologically ordered locations of the random but finite number of Poisson events occurring up to time t on the edge set $\langle -n, -n+1\rangle, \dots, \langle n-1, n\rangle$. Further, let $(e_{i_j})_{j=1}^{N'}$ be the subsequence of $(e_i)_{i=1}^N$ which contains only those edges on which a difference exceeding the confidence bound prevented the occurring Poisson event from invoking an actual update of opinions. Since there are only finitely many choices for the sequence $(e_i)_{i=1}^N$ and its corresponding subsequence, if $N \in \mathbb{N}$ is fixed, and N is a.s. finite, we can partition the event A into $\{A_m,\ m \in \mathbb{N}\}$ according to the different choices of (e_i) and (e_{i_j}). Note that for the subsequences to be considered equal not only their length and ordered elements must coincide, but also the set of indices $\{i_j,\ 1 \le j \le N'\}$ has to be identical. From $\mathbb{P}(A) > 0$ we can conclude that there must be some A_m which has positive probability. In other words, there exists a set $C \subseteq (\mathbb{R}^k)^{2n-1}$ s.t.

$$\mathbb{P}\big((\eta_0(v))_{v=-n+1}^{n-1} \in C\big) > 0$$

and given a starting configuration in C, Poisson events on the edges given by the fixed sequence $(e_i)_{i=1}^N$ corresponding to A_m will ensure, in the Deffuant model with confidence bound ϑ, that the final value at 0 is in $B[x,r]$.

Let B be the event that the locations of all Poisson events on the edge set $\{\langle -n, -n+1\rangle, \dots, \langle n-1, n\rangle\}$ up to t are given by the subsequence of $(e_i)_{i=1}^N$ which is obtained by removing the elements of (e_{i_j}). Given B and $\{(\eta_0(v))_{v=-n+1}^{n-1} \in C\}$, the dynamics of the Deffuant model with confidence bounds ϑ and $\theta \ge \vartheta$ respectively will coincide up to time t between the two edges without Poisson events shielding 0 from $-n$ and n. Since B has positive probability and the Poisson events are independent of $\{(\eta_0(v))_{v=-n+1}^{n-1} \in C\}$ this implies that $x \in \mathrm{supp}_\vartheta(\eta_t)$ forces $x \in \mathrm{supp}_\theta(\eta_t)$ for all $\theta \ge \vartheta$, hence the claimed monotonicity.

When it comes to the second statement, the first inclusion was actually proved in (a) as the argument used in order to show $\mathrm{supp}_\theta(\eta_s) \subseteq \mathrm{supp}_\theta(\eta_t)$ is also valid for $s = 0$. The second and third inclusion can be verified as in part (b) of Lemma 3.7.

□

The following proposition reveals how the set $\mathscr{D}_\theta(\nu)$ comes into play in the analysis of the long-term behavior of the Deffuant model.

Proposition 3.13. *If $\vartheta \mapsto \mathscr{D}_\vartheta(\nu)$ has no jump in $[\theta-\varepsilon,\theta+\varepsilon]$ for fixed θ and some $\varepsilon > 0$, the following equality holds true for all $t > 0$:*

$$\mathrm{supp}_\theta(\eta_t) = \mathscr{D}_\theta(\nu).$$

Proof: Before proving this result, we want to mention that given $R < \infty$, the continuity assumption can be weakened: If $R < \infty$ and $\vartheta \mapsto \mathscr{D}_\vartheta(\nu)$ has no jump at θ, part (e) of Lemma 3.9, already implies that $\mathscr{D}_\vartheta(\nu)$ is constant on an interval $[\theta-\varepsilon,\theta+\varepsilon]$ for suitably small $\varepsilon > 0$.

Let us first focus on the inclusion $\mathrm{supp}_\theta(\eta_t) \supseteq \mathscr{D}_\theta(\nu)$. For every fixed x in $\mathscr{D}_\theta(\nu) = \mathscr{D}_{\theta-\varepsilon}(\nu)$ and all $r > 0$, there exists a finite configuration with $n \in \mathbb{N}$, $x_1,\dots,x_n \in \mathrm{supp}(\eta_0)$ and edge sequence $(e_i)_{i=1}^N$, such that updating the configuration with respect to the confidence bound $\theta - \varepsilon$ yields $\eta_N(v) \in B[x,r]$ for all $v \in \{1,\dots,n\}$. Let further $t > 0$ be fixed. Due to $x_v \in \mathrm{supp}(\eta_0)$, we get $\mathbb{P}(\eta_0 \in B[x_v,\varepsilon]) > 0$.

Consequently, in the Deffuant model on $\mathbb{Z}$ the following event has positive probability: $\eta_0(v) \in B[x_v,\varepsilon]$ for all $v \in \{1,\dots,n\}$, up to time t Poisson events have occurred on neither $\langle 0,1\rangle$ nor $\langle n,n+1\rangle$ and the locations of the events on $\langle 1,2\rangle,\dots,\langle n-1,n\rangle$ are chronologically ordered given by $(e_i)_{i=1}^N$. Note that every Poisson event which leads to an update in the given finite configuration does the same in this configuration of the whole model with respect to parameter θ, as the margins coming from slightly altered initial values are convex combinations of the initial margins $\eta_0(v) - x_v$ and thus always bounded by ε. This shows $\mathbb{P}(\eta_t(1) \in B[x,r+\varepsilon]) > 0$, hence $x \in \mathrm{supp}_\theta(\eta_t)$.

When it comes to the reverse inclusion, consider again the Deffuant model with confidence bound θ. By definition, $x \in \mathrm{supp}_\theta(\eta_t)$ if and only if for all $r > 0$: $\mathbb{P}(\eta_t(v) \in B[x,r]) > 0$. But every such value $\eta_t(v)$ is formed by (finitely many) convex combinations starting from a finite collection of initial values $\{\eta_0(u)\}_{u=v-k}^{v+l}$. Part (a) of Lemma 3.9 shows that $\eta_{s-}(u), \eta_{s-}(v) \in \mathscr{D}_{\theta+\varepsilon}(\nu)$ immediately implies $\eta_s(u), \eta_s(v) \in \mathscr{D}_{\theta+\varepsilon}(\nu)$ after an update along the edge $\langle u,v\rangle$ at time s, since this can only occur if the former are at distance less than or equal to θ. Due to $\{\eta_0(u)\}_{u=v-k}^{v+l} \subseteq \mathrm{supp}(\eta_0) \subseteq \mathscr{D}_{\theta+\varepsilon}(\nu)$, we can use this consideration in an inductive argument to verify $\eta_t(v) \in \mathscr{D}_{\theta+\varepsilon}(\nu)$ and hence

$$\mathrm{supp}_\theta(\eta_t) \subseteq \overline{\mathscr{D}_{\theta+\varepsilon}(\nu)} = \mathscr{D}_{\theta+\varepsilon}(\nu) = \mathscr{D}_\theta(\nu). \qquad \square$$

Note that if $\vartheta \mapsto \mathscr{D}_\vartheta(\nu)$ has a jump at θ, the subtle issue with critical compromises, as considered in Proposition 2.3, reappears. To make this point clear, let us consider the initial distribution $\nu = \mathrm{unif}(\{\frac{1}{4},\frac{3}{4}\})$, for which we find

$$\mathscr{D}_{\frac{1}{2}}(\nu) = \mathrm{supp}_{\frac{1}{2}}(\eta_t) = [\tfrac{1}{4},\tfrac{3}{4}].$$

Taking $\eta_0 \sim \mathrm{unif}\big([0,\frac{1}{4}] \cup [\frac{3}{4},1]\big)$ instead yields

$$[0,1] = \mathscr{D}_{\frac{1}{2}}(\nu) \supsetneq \mathrm{supp}_{\frac{1}{2}}(\eta_t) = [0,\tfrac{1}{4}] \cup [\tfrac{3}{4},1].$$

Definition 3.14. Given an initial distribution $\mathcal{L}(\eta_0) = \nu$, define the length of the *largest gap* in its support as

$$h := \inf\{\theta > 0,\ \mathscr{D}_\theta(\nu) \text{ is connected}\}.$$

Following this definition we get $h = 0$ for $\nu = \text{unif}(S^{k-1})$ and $k \geq 2$, but $h = 2$ for $\nu = \text{unif}(S^0)$. Considering the other two distributions appearing in the above example, we observe that $\text{unif}(\{(2, 1, 0), (2, -1, 0), (-2, 0, 1), (-2, 0, -1)\})$ has $h = 4$ and $\text{unif}(\{(0.99, 1, 0), (0.99, -1, 0), (-0.99, 0, 1), (-0.99, 0, -1)\})$ instead $h = 2$. In addition, parts (b) and (d) of Lemma 3.9 tell us that $h \leq 2R$ if R is finite and $h \leq R$ if additionally $\mathbb{E}\, \eta_0 \in \text{supp}(\eta_0)$.

Having generalized the notion of a gap in a distribution on $\mathbb{R}$ to higher dimensions finally allows us to formulate and prove a result corresponding to the cases of Theorem 2.2 that were omitted by Theorem 3.2.

Theorem 3.15. *Consider the Deffuant model on $\mathbb{Z}$ with an initial distribution on $(\mathbb{R}^k, \|\,.\,\|_2)$ that is bounded, i.e.*

$$R = \inf \big\{ r > 0,\ \mathbb{P}\big(\eta_0 \in B[\mathbb{E}\, \eta_0, r]\big) = 1 \big\} < \infty,$$

and h being the length of the largest gap in its support. Then the critical value for the confidence bound, where a phase transition from a.s. no consensus to a.s. strong consensus takes place is $\theta_c = \max\{R, h\}$.

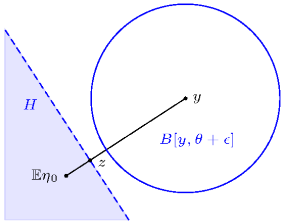
Proof: Having analyzed the qualitative differences invoked by higher-dimensional opinion values, the proof of this theorem is to a large extent similar to the one of part (a) of Thm. 2.2 in Häggström and Hirscher (2014), which is Theorem 2.2 in the foregoing section. Let us consider the following three scenarios:

(i) *For $\theta < h$ we cannot have consensus:*
By definition of h the set $\mathscr{D}_{\theta+\varepsilon}(\nu)$ is not connected for $\varepsilon > 0$ sufficiently small; by Lemma 3.9 (e) we can choose ε such that $\vartheta \mapsto \mathscr{D}_\vartheta(\nu)$ has no jump at $\theta + \varepsilon$ and thus (by Proposition 3.13) get $\mathscr{D}_{\theta+\varepsilon}(\nu) = \text{supp}_{\theta+\varepsilon}(\eta_t)$ for all $t > 0$. In addition, Lemma 3.9 (c) tells us that there exist two connected components, say C_1 and C_2, both being convex and at distance at least $\theta + \varepsilon$ from the corresponding complementary part of $\text{supp}_{\theta+\varepsilon}(\eta_t)$, i.e. $\|x - y\|_2 \geq \theta + \varepsilon$ for all $x \in C_i, y \in \text{supp}_{\theta+\varepsilon}(\eta_t) \setminus C_i$ and $i = 1, 2$.
By Lemma 3.12 we know that $\text{supp}(\eta_0) \subseteq \text{supp}_\theta(\eta_t) \subseteq \text{supp}_{\theta+\varepsilon}(\eta_t)$. In the Deffuant model with confidence bound θ opinions in C_1 cannot compromise with opinions in $\text{supp}_\theta(\eta_t) \setminus C_1 \subseteq \text{supp}_{\theta+\varepsilon}(\eta_t) \setminus C_1$ and thus never leave the convex set C_1. The same holds for C_2.
Consequently, we get $\mathbb{P}(\eta_0(v) \in C_i) = \mathbb{P}(\eta_t(v) \in C_i) > 0$, for $i = 1, 2$. For a fixed vertex v, it follows from the independence of initial opinions that $\mathbb{P}(\eta_0(v) \in C_1, \eta_0(v + 1) \in C_2) > 0$, which dooms the edge $\langle v, v + 1\rangle$ to be blocked for all $t \geq 0$, due to $\|\eta_t(v) - \eta_t(v + 1)\|_2 \geq \theta + \varepsilon$. Ergodicity of the initial configuration ensures that a.s. infinitely many neighboring vertices will be prevented from compromising by holding opinions in C_1 and C_2 respectively, hence no consensus in the long run.

(ii) *For $\theta < R$ we cannot have consensus:* In the regime $\theta < R$, for any fixed $\varepsilon \in \big(0, \frac{R-\theta}{2}\big)$ there exists some point $y \in \text{supp}(\eta_0) \setminus B[\mathbb{E}\, \eta_0, \theta + 2\varepsilon]$. Choose z to be the point on the line segment connecting $\mathbb{E}\, \eta_0$ and y which has Euclidean distance ε to $\mathbb{E}\, \eta_0$, see the below picture. With the help of this point, define the following half-space: $H := \{x \in \mathbb{R}^k,\ (x - z) \cdot (y - z) \leq 0\}$. Clearly, $B[\mathbb{E}\, \eta_0, \varepsilon] \subseteq H$ and according to the same argument as in part (f) of Lemma 3.9 we find $\mathbb{P}(\eta_0 \in H) > 0$, as the contrary would imply

$$\mathbb{E}\,[(\eta_0 - z) \cdot (y - z)] > 0 > (\mathbb{E}\, \eta_0 - z) \cdot (y - z),$$

a contradiction.



Using this auxiliary construction, we can finish the proof of this subcase following the argument in the proof of Theorem 2.2 (b), see Thm. 2.2 in Häggström and Hirscher (2014). As the distribution is bounded, the SLLN states

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{u=v+1}^{v+n}\eta_0(u)=\mathbb{E}\,\eta_0\right)=1. \tag{3.5}$$

Consequently, for sufficiently large $N \in \mathbb{N}$ the following event has non-zero probability:

$$A_N := \left\{\frac{1}{n}\sum_{u=v+1}^{v+n}\eta_0(u)\in H \text{ for all } n\geq N\right\}.$$

Let ξ denote the (real-valued) distribution of $(\eta_0 - z)\cdot(y-z)$ and $\xi|_{(-\infty,0]}$ its distribution conditioned on the event $\{(\eta_0 - z)\cdot(y-z)\leq 0\} = \{\eta_0 \in H\}$. Obviously, $\xi|_{(-\infty,0]}$ is stochastically dominated by ξ, i.e. $\xi|_{(-\infty,0]} \preceq \xi$, which implies

$$\left(\bigotimes_{u=v+1}^{v+N}\xi|_{(-\infty,0]}\right)\otimes\left(\bigotimes_{u>v+N}\xi\right)\preceq\bigotimes_{u\geq v+1}\xi.$$

Let B be the event $\{\eta_0(v+1)\in H,\dots,\eta_0(v+N)\in H\}$, which has non-zero probability by independence, and

$$A_1 := \left\{\frac{1}{n}\sum_{u=v+1}^{v+n}\eta_0(u)\in H \text{ for all } n\in\mathbb{N}\right\}.$$

Rewriting the event A_N as

$$A_N = \left\{\frac{1}{n}\sum_{u=v+1}^{v+n}\big(\eta_0(u)-z\big)\cdot\big(y-z\big)\leq 0 \text{ for all } n\geq N\right\},$$

the stochastic domination from above yields:

$$\begin{aligned}\mathbb{P}(A_1)&\geq\mathbb{P}(A_1\cap B)=\mathbb{P}(A_N\cap B)=\mathbb{P}(A_N|B)\cdot\mathbb{P}(B)\\&\geq\mathbb{P}(A_N)\cdot\mathbb{P}(B)>0.\end{aligned}$$

The very same ideas as in the proof of Prop. 5.1 in Häggström (2012) show that if A_1 occurs and the edge $\langle v, v+1\rangle$ doesn't allow for an update up to time $t>0$, irrespectively of the dynamics on $\{u\in\mathbb{Z}, u\geq v+1\}$, we get that $\eta_t(v+1)$ is a convex combination of the averages $\{\frac{1}{n}\sum_{u=v+1}^{v+n}\eta_0(u),\ n\in\mathbb{N}\}$, hence in H as the latter is convex. By symmetry, the same holds for site $v-1$ and the half-line to the left, i.e. $\{u \in \mathbb{Z}, u \leq v-1\}$. Independence of the initial opinions therefore guarantees that with positive probability, the initial configuration can be such that $\eta_0(v) \in B(y, \varepsilon)$ and the values at sites $v-1$ and $v+1$ are doomed to stay in H, blocking the edges adjacent to v once and for all, as the distance of y to H is at least $\theta + \varepsilon$. Ergodicity makes sure that with probability 1 infinitely many sites will get stuck this way.

(iii) *For $\theta > \max\{R, h\}$ we get a.s. strong consensus:*
Choose β such that $0 < \beta < \theta - \max\{R, h\}$. From the definition of h and Lemma 3.9 (e), we can conclude $\mathbb{E}\,\eta_0 \in \mathscr{D}_{\theta-\beta}(\nu)$. Because of that, for all $\varepsilon > 0$, there exists a finite configuration such that the final opinion values all lie in $B[\mathbb{E}\,\eta_0, \frac{\varepsilon}{6}]$, i.e. $n \in \mathbb{N}$, $x_1, \dots, x_n \in \operatorname{supp}(\eta_0)$ and an edge sequence $(e_i)_{i=1}^N$ from $\{\langle 1,2\rangle, \dots, \langle n-1, n\rangle\}$, s.t. updating the configuration with respect to the confidence bound $\theta - \beta$ yields $\eta_N(v) \in B[\mathbb{E}\,\eta_0, \frac{\varepsilon}{6}]$ for all $v \in \{1, \dots, n\}$, see Definition 3.6. From this point on, we can go about as in step (ii) of the proof of Thm. 2.2 (a) in Häggström and Hirscher (2014):

Let us consider some fixed time point $t > 0$ and the corresponding configuration $\{\eta_t(v)\}_{v\in\mathbb{Z}}$. With probability 1, there exists an infinite increasing sequence of not necessarily consecutive edges $(\langle v_k, v_k+1\rangle)_{k\in\mathbb{N}}$ to the right of site 1, on which no Poisson event has occurred up to time t.

Let $l_k := v_{k+1} - v_k$, for $k \in \mathbb{N}$, denote the random lengths of the intervals in between and $l_0 := v_1 - v_0 + 1$ the one of the interval including 1, where $\langle v_0 - 1, v_0\rangle$ is the first edge to the left of 1 without Poisson event. Since the involved Poisson processes are independent, it is easy to verify that the l_k, $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, are i.i.d., having a geometric distribution on $\mathbb{N}$ with parameter e^{-t}.

For $\delta > 0$, let A_δ be the event that l_0 is finite and only finitely many of the events $\{l_k \geq k\,\frac{\delta}{R}\}$, $k \in \mathbb{N}$, occur. Then their independence and the Borel-Cantelli lemma tell us that A_δ has probability 1. On A_δ however the following holds a.s. true:

$$\begin{aligned}
\limsup_{v\to\infty} \Big\| \frac{1}{v}\sum_{u=1}^{v} \eta_t(u) - \mathbb{E}\,\eta_0 \Big\|_2 &= \limsup_{v\to\infty} \Big\| \frac{1}{v}\sum_{u=1}^{v} \big(\eta_t(u) - \mathbb{E}\,\eta_0\big) \Big\|_2 \\
&= \limsup_{v\to\infty} \Big\| \frac{1}{v}\sum_{u=v_0}^{v} \big(\eta_t(u) - \mathbb{E}\,\eta_0\big) \Big\|_2 \\
&\leq \limsup_{v\to\infty} \Big\| \frac{1}{v}\sum_{u=v_0}^{v} \big(\eta_0(u) - \mathbb{E}\,\eta_0\big) \Big\|_2 + \delta \\
&= \limsup_{v\to\infty} \Big\| \frac{1}{v}\sum_{u=1}^{v} \big(\eta_0(u) - \mathbb{E}\,\eta_0\big) \Big\|_2 + \delta \\
&= \delta.
\end{aligned}$$

The second and second to last equality follow from the finiteness of v_0, the last equality from the SLLN applied to the sequence $(\eta_0(u))_{u\geq 1}$, stating

$$\lim_{v\to\infty} \frac{1}{v}\sum_{u=1}^{v} \eta_0(u) = \mathbb{E}\,\eta_0 \text{ almost surely.}$$

The inequality is due to the fact that the Deffuant model is mass-preserving in the sense that $\eta_t(u)+\eta_t(v)=\eta_{t-}(u)+\eta_{t-}(v)$ in (1.1), hence for all $k\in\mathbb{N}$: $\sum_{u=v_0}^{v_k}\eta_0(u)=\sum_{u=v_0}^{v_k}\eta_t(u)$. For the average at time t running from v_0 to some $v\in\{v_k+1,\dots,v_{k+1}\}$ to differ by more than δ from the one at time 0, the interval has to be of length more than $k\,\frac{\delta}{R}$, since $\|\eta_t(u)-\mathbb{E}\,\eta_0\|_2\in[0,R]$ for all t,u and $v_k\geq k$. This, however, will happen only finitely many times.

Since $\delta>0$ was arbitrary, we have established that even for $t>0$

$$\lim_{v\to\infty}\frac{1}{v}\sum_{u=1}^{v}\eta_t(u)=\mathbb{E}\,\eta_0 \text{ almost surely.} \tag{3.6}$$

Now we are going to use the finite configuration from above and a conditional version of the so-called *local modification*, a technique often used in percolation theory. Due to (3.6), there exists some integer number k s.t. the event

$$A:=\left\{\frac{1}{v}\sum_{u=1}^{v}\eta_t(u)\in B[\mathbb{E}\,\eta_0,\tfrac{\varepsilon}{3}] \text{ for all } v\geq kn\right\}$$

has probability greater than $1-\mathrm{e}^{-2t}$.

Let B in turn be the event that there was no Poisson event on $\langle 0,1\rangle$ and $\langle kn,kn+1\rangle$ up to time t, hence $\mathbb{P}(B)=\mathrm{e}^{-2t}$. Finally, let C be the event that the initial values satisfy

$$\eta_0(ln+i)\in B[x_i,\min\{\beta,\tfrac{\varepsilon}{6}\}],\ \text{for all } 0\leq l\leq k-1 \text{ and } 1\leq i\leq n,$$

and the Poisson firings on the edges $\langle 0,1\rangle,\dots,\langle kn,kn+1\rangle$ up to time t are given by a concatenation of the k finite sequences given by shifting $(e_i)_{i=1}^N$ ln vertices to the right, $0\leq l\leq k-1$. In other words, up to time t there are no Poisson events on the $k+1$ edges $\{\langle 0,1\rangle,\langle n,n+1\rangle,\dots,\langle kn,kn+1\rangle\}$ and the dynamics in the k blocks $\{ln+1,\dots,(l+1)n\}$ resembles the dynamics of the finite configuration, accordingly leading to $\eta_t(v)\in B[\mathbb{E}\,\eta_0,\frac{\varepsilon}{3}]$ for all $v\in\{1,\dots,kn\}$, see also the proof of Proposition 3.13. Note that C has non-zero probability, $C\subseteq B$ and also $A\cap B$ has strictly positive probability as $\mathbb{P}(A\cap B^{\mathsf{c}})\leq\mathbb{P}(B^{\mathsf{c}})=1-\mathrm{e}^{-2t}<\mathbb{P}(A)$.

Consider two configurations $\{\eta_0'(v)\}_{v\in\mathbb{Z}}$ and $\{\eta_0''(v)\}_{v\in\mathbb{Z}}$, independent from each other and having the same distribution as $\{\eta_0(v)\}_{v\in\mathbb{Z}}$ underlying the dynamics of the Deffuant model. Then also the compound configuration

$$\tilde{\eta}_0(v)=\begin{cases}\eta_0'(v), & \text{for } v\in\{1,\dots,kn\}\\ \eta_0''(v), & \text{for } v\notin\{1,\dots,kn\}\end{cases}$$

has the i.i.d. distribution of the initial configuration. With positive probability $A\cap B$ occurs for the initial configuration $\{\eta_0''(v)\}_{v\in\mathbb{Z}}$ and C for the initial configuration $\{\eta_0'(v)\}_{v\in\mathbb{Z}}$. The fact that $(\tilde{\eta}_s(v))_{v\in\mathbb{Z}}$ equals $\{\eta_s'(v)\}_{v\in\mathbb{Z}}$ on $\{1,\dots,kn\}$ and $\{\eta_s''(v)\}_{v\in\mathbb{Z}}$ outside $\{1,\dots,kn\}$ for $s\in[0,t]$ given B, together with the independence of the involved building block configurations, shows that with positive probability $A\cap B\cap C'$ holds for the configuration at time t, where

$$C'=\left\{\eta_t(v)\in B[\mathbb{E}\,\eta_0,\tfrac{\varepsilon}{3}] \text{ for all } v\in\{1,\dots,kn\}\right\}.$$

An easy calculation reveals that $A\cap C'$ implies the ε-flatness to the right of site 1 in the configuration at time t. By symmetry in left and right, the same holds true for the site 0 and ε-flatness to the left with respect to the configuration $\{\eta_t(v)\}_{v\in\mathbb{Z}}$. As the two parts $\{\eta_t(v)\}_{v\leq 0}$ and $\{\eta_t(v)\}_{v\geq 1}$ of the configuration at time t are conditionally independent given there was no Poisson event on the edge $\langle 0,1\rangle$ up to time t, we have actually shown that the origin is two-sidedly ε-flat with respect to the configuration $\{\eta_t(v)\}_{v\in\mathbb{Z}}$ with positive probability.

The supercritical case is now settled as in part (a) of Theorem 3.2. Following the reasoning of Sect. 6 in Häggström (2012), the proof of La. 6.3 there tells us that a two-sidedly ε-flat vertex will never move further than 6ε away from the mean and Prop. 6.1 guarantees that two neighbors will a.s. either finally concur or end up further than θ apart from each other. Choosing $0<\varepsilon<\frac{\theta-R}{6}$ the latter is impossible for vertices neighboring a two-sidedly ε-flat vertex, which means that they will a.s. finally concur and the same holds true for every vertex by induction. Ergodicity of the setting at time t guarantees that there will be a.s. (infinitely many) two-sidedly ε-flat vertices forcing almost sure strong consensus.

□

Remark 3.16. It is worth emphasizing that only the support and expected value of a bounded initial distribution determine the critical value for θ: As long as it does not affect the support, the dependence relations between the coordinates of the random vector η_0 do not influence the critical parameter θ_c.

Furthermore, having proved this result for more general multivariate distributions, part (a) of Theorem 3.2 becomes a special case of Theorem 3.15, since using part (d) of Lemma 3.9 shows that the maximal gap in a distribution of η_0 with mass around its mean cannot be larger than its radius, i.e. $h\leq R$.

Finally, the requirement that the initial opinions are independent is not as vital as it might seem. The independence was merely used to guarantee that we can locally modify initial configurations and still obtain events with positive probability. Consequently, the i.i.d. property can be replaced by the weaker condition that $\{\eta_0(v)\}_{v\in\mathbb{Z}}$ is a stationary sequence, ergodic with respect to shifts and allowing conditional probabilities such that the conditional distribution of $\eta_0(0)$ given $\{\eta_0(v)\}_{v\in\mathbb{Z}\setminus\{0\}}$ almost surely has the same support as the marginal distribution $\mathcal{L}(\eta_0)$, with the above conclusions remaining valid. This last condition is a natural extension to continuous state spaces of the well-known *finite energy condition* from percolation theory – for a more detailed discussion of this extension to dependent initial opinions, see Sect. 2.2 in Häggström and Hirscher (2014).

Example 3.17.

(a) With Theorem 3.15 in hand, we can finally settle the case of $\eta_0\sim\mathrm{unif}(S^{k-1})$. Irrespectively of k, this distribution has radius $R=1$, but for $k=1$, the maximal gap is $h=2$, for $k>1$ instead $h=0$. By the above theorem, we can conclude
$$\theta_c=\max\{R,h\}=\begin{cases}2, & \text{for } k=1\\ 1, & \text{for } k\geq 2.\end{cases}$$
In short, the fact that S^{k-1} is disconnected for $k=1$ but connected for $k\geq 2$ makes all the difference.

(b) If the random vector η_0 has independent coordinates, each being Bernoulli distributed with parameter $p \in (0,1)$, i.e. for all $1 \leq i \leq k$

$$\mathbb{P}\big(\eta_0^{(i)} = 1\big) = 1 - \mathbb{P}\big(\eta_0^{(i)} = 0\big) = p,$$

its support is the hypercube $\{0,1\}^k$ and the expected value $\mathbb{E}\,\eta_0 = p\,\mathbf{e}$, where $\mathbf{e}$ is the k-dimensional vector of all ones. The radius of this initial distribution is $R = \max\{\|\mathbb{E}\,\eta_0 - \mathbf{0}\|_2, \|\mathbb{E}\,\eta_0 - \mathbf{e}\|_2\} = \sqrt{k}\,\max\{p, 1-p\}$. It is not hard to see that a distribution with the hypercube as its support has the maximal gap $h = 1$. Indeed, for $\theta < 1$ no two opinion values can interact, for $\theta > 1$ all neighboring corners get within the confidence bound and their pairwise convex hulls form the edges of the hypercube, hence their union is a connected set giving $\mathscr{D}_\theta(\nu) = [0,1]^k$, for $\theta > 1$, by means of Lemma 3.9.

In conclusion, the Deffuant model with this initial distribution features the critical value

$$\theta_c = \begin{cases} 1, & \text{for } k = 1 \text{ or } k = 2,3 \text{ and } p \in [1 - \frac{1}{\sqrt{k}}, \frac{1}{\sqrt{k}}] \\ \sqrt{k}\,\max\{p, 1-p\}, & \text{for } k \geq 4 \text{ or } k = 2,3 \text{ and } p \notin [1 - \frac{1}{\sqrt{k}}, \frac{1}{\sqrt{k}}]. \end{cases}$$

As stated in the above remark, independence of the individual coordinates is not essential, as long as the support stays unchanged. A relation like $\eta_0^{(1)} = 1 - \eta_0^{(2)}$ in the Bernoulli example with parameter $p = \frac{1}{2}$ however, will influence both $\mathrm{supp}(\eta_0)$ and as a consequence θ_c as well.

Example 3.18. There is one more crucial change when the opinions in the Deffuant model on $\mathbb{Z}$ are given by vectors instead of real numbers. The parameter μ, shaping the size of compromising steps, which was of no particular interest so far, can actually play a crucial role in the critical case.

In order to verify this claim, let us consider the two-dimensional initial distribution given by $\mathrm{unif}(\{(0,0),(1,0),(\frac{1}{\pi},1)\})$, which is depicted below. Given $\theta = 1$ we have

$$[0,1] \times \{0\} \subseteq \mathrm{supp}_\theta(\eta_t) \text{ for all } t > 0,$$

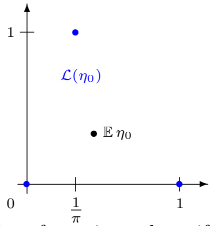
following the reasoning of Example 3.8. But the point $(\frac{1}{\pi}, 0)$ can only be approximated, never attained by $\eta_t(v)$, if μ is rational for example. For $\mu = \frac{1}{\pi}$ on the other hand, the event that $\eta_t(v) = (\frac{1}{\pi}, 0)$ has positive probability for $t > 0$, which leads to $\mathrm{supp}(\eta_t) = \mathrm{conv}(\mathrm{supp}(\eta_0))$.

Note that for this distribution, we have $h = 1 > R$, since $\mathbb{E}\,\eta_0 = \frac{1}{3}\,(1 + \frac{1}{\pi}, 1)$. Similarly to the proof of the above theorem, we can conclude that the Deffuant model on $\mathbb{Z}$ with confidence bound $\theta = \theta_c = 1$ and this initial distribution approaches almost surely no consensus for $\mu \in (0, \frac{1}{2}] \cap \mathbb{Q}$ and almost surely strong consensus for $\mu = \frac{1}{\pi}$:

If μ is rational, vertices holding the initial opinion $(\frac{1}{\pi}, 1)$ can never compromise with such holding an opinion $(a, 0)$ since a is rational and can therefore not be $\frac{1}{\pi}$. Consequently, we will have a.s. no consensus due to blocked edges.

If $\mu = \frac{1}{\pi}$ however, we can come up with a finite configuration allowing for the local modification, which guaratees the existence of two-sidedly ε-flat vertices: Actually $n = 3$ is enough and

$$x_1 = (1,0),\ x_2 = (0,0),\ x_3 = (\tfrac{1}{\pi}, 1)$$



will be an appropriate choice of starting values, if the edge sequence $(e_i)_{i=1}^N$ begins with $e_1 = \langle 1,2\rangle,\ e_2 = \langle 2,3\rangle$, since that will bring the value at site 1 to $(1-\frac{1}{\pi},0)$, the one at 2 to $(\frac{1}{\pi},\frac{1}{\pi})$ and the one at 3 to $(\frac{1}{\pi},1-\frac{1}{\pi})$, all lying in $B[\mathbb{E}\,\eta_0,\frac{1}{2}]$, and thus their pairwise distances are all less than the confidence bound. If the edge sequence contains the edge pair $(\langle 1,2\rangle,\langle 2,3\rangle)$ enough times, the final values of the finite configuration will all lie at Euclidean distance at most $\frac{\varepsilon}{3}$ from the initial average $\frac{1}{3}(x_1+x_2+x_3) = \mathbb{E}\,\eta_0$ for any fixed $\varepsilon > 0$. Note that in the present case, when transforming the finite configuration into a part of the dynamics on the whole line graph, we don't have to worry about taking small balls around the initial values x_i in order to get an event C with positive probability, since the x_i are atoms of the initial distribution. Taking small balls would actually invalidate the argument due to the fact that the parameter θ is pinned to the critical value $\theta_c = 1$ not allowing for small marginals.

Another fact – adding to part (e) of Lemma 3.9 – that can be seen from this example is that the jumps of the mapping $\vartheta \mapsto \mathscr{D}_\vartheta(\nu)$ do not have to be continuous from the right in the sense that

$$\mathscr{D}_\theta(\nu) = \bigcap_{\vartheta>\theta} \mathscr{D}_\vartheta(\nu).$$

Given $\mu \in \mathbb{Q}$ we get for this initial distribution

$$\mathscr{D}_\theta(\nu) = \begin{cases} \mathrm{supp}(\eta_0), & \text{for } \theta < 1 \\ [0,1]\times\{0\} \cup \{(\frac{1}{\pi},0)\}, & \text{for } \theta = 1 \\ \mathrm{conv}(\mathrm{supp}(\eta_0)), & \text{for } \theta > 1, \end{cases}$$

hence there can actually be a double jump.

4. Metrics other than the Euclidean distance

Having investigated the changes that multidimensional opinion values cause in the Deffuant model, another interesting aspect is the impact of the measure of distance between two opinions. What happens if we apply some general metric ρ other than the natural choice given by the Euclidean norm?

Although this generalization does not entirely fit the framework as laid out in Section 1, it is not worth repeating all the definitions as one would simply have to replace all appearing distances $\|x-y\|_2$ by $\rho(x,y)$ correspondingly. Note however that switching to a general metric ρ influences the dynamics of the Deffuant model only in determining which opinion values are within 'speaking distance', that is allowing for an update if neighbors with corresponding opinions interact. Once the two values are close enough in this respect, the updated opinion values will just be the convex combinations described in (1.1), even if the straight line connecting both values might no longer be the geodesic between them (as in the Euclidean case) and the steps taken towards the arithmetic average can be of different length if ρ is not translation invariant.

With respect to the considerations in the foregoing section, the following properties of a distance measure play an important role.

Definition 4.1. Consider a metric ρ on $\mathbb{R}^k$.

(i) Let the metric ρ be called *sensitive to coordinate* i, if there exists a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\lim_{s\to\infty} \varphi(s) = \infty$ and for any two vectors $x, y \in \mathbb{R}^k$ with $|x_i - y_i| > s$, it holds that $\rho(x,y) > \varphi(s)$.

(ii) Call ρ *locally dominated by the Euclidean distance*, if there exist some $\gamma, c > 0$ such that for $x, y \in \mathbb{R}^k$ with $\|x-y\|_2 \le \gamma$ it holds that

$$\rho(x,y) \le c \cdot ||x-y||_2. \tag{4.1}$$

(iii) Finally, let ρ be called *weakly convex* if for all $x, y, z \in \mathbb{R}^k$:

$$\rho(x, \alpha y + (1-\alpha)\, z) \le \max\{\rho(x,y), \rho(x,z)\} \quad \text{for all } \alpha \in [0,1].$$

The convexity of balls $B_\rho(x,r) = \{y \in \mathbb{R}^k,\ \rho(x,y) < r\}$ generated by the metric is a crucial feature. It is not hard to check that the balls generated by ρ are convex if and only if the metric is weakly convex: Sufficiency is obvious, since $y, z \in B_\rho(x,r)$ immediately gives $\operatorname{conv}(\{y,z\}) \subseteq B_\rho(x,r)$. As to necessity, if there are $x, y, z \in \mathbb{R}^k$, $\alpha \in (0,1)$ s.t. $\rho(x, \alpha y + (1-\alpha)\, z) > \max\{\rho(x,y), \rho(x,z)\}$, we can choose $r \in (\max\{\rho(x,y), \rho(x,z)\}, \rho(x, \alpha y + (1-\alpha)\, z))$ and conclude that $B_\rho(x,r)$ can not be convex. It should be mentioned that when talking about the metric space $(\mathbb{R}^k, \rho)$, we will always assume that it is equipped with the Borel σ-algebra generated by the metric ρ.

If ρ is locally dominated by the Euclidean distance, we can find a constant $C = C(\theta)$ such that (4.1) holds in fact for all $x, y \in \mathbb{R}^k$ with $\rho(x,y) \le \theta$ if c is replaced by C: If $\|x-y\|_2 > \gamma$ but $\rho(x,y) \le \theta$, we can conclude that

$$\rho(x,y) \le \theta \le \tfrac{\theta}{\gamma}\, \|x-y\|_2,$$

hence $C := \max\{c, \frac{\theta}{\gamma}\}$ will do.

Definition 4.2. Let the Deffuant model with respect to a general distance measure ρ be defined just as in Section 1, with the only change that the restriction of the confidence bound in (1.1) will now rule that Poisson events cause updates only if $\rho(a,b) \le \theta$, where a, b denote the opinion values at the corresponding vertices. As the convexity of balls is enormously important in the analysis presented in the foregoing section, in what follows ρ will be assumed to be weakly convex.

No consensus still means that we have finally blocked edges, that is some $\langle u, v\rangle$ s.t. $\rho(\eta_t(u), \eta_t(v)) > \theta$ for all t large enough. Similarly, the convergence notion in the definition of consensus is now based on the distance ρ.

As before, the initial opinions are i.i.d. with some common distribution $\mathcal{L}(\eta_0)$ on $\mathbb{R}^k$. If the distribution of η_0 has a finite expectation, we define its radius with respect to ρ as

$$R_\rho := \inf\big\{r > 0,\ \mathbb{P}\big(\eta_0 \in B_\rho(\mathbb{E}\,\eta_0, r)\big) = 1\big\}\,,$$

similarly to the Euclidean case, see Definition 3.1.

Likewise, the notion of ε-flatness transfers to the new setting as follows: A vertex $v \in \mathbb{Z}$ is called *ε-flat (with respect to ρ)* to the right in the initial configuration $\{\eta_0(u)\}_{u\in\mathbb{Z}}$ if for all $n \geq 0$:

$$\frac{1}{n+1}\sum_{u=v}^{v+n}\eta_0(u) \in B_\rho(\mathbb{E}\,\eta_0, \varepsilon), \tag{4.2}$$

similarly for ε-flatness to the left and two-sided ε-flatness.

By imposing appropriate additional restrictions on the weakly convex metric ρ and the initial distribution, we can retrieve the result of Theorem 3.2 also in this generalized setting. The extra restriction on $\mathcal{L}(\eta_0)$ is that $\mathbb{E}\,[\eta_0^2]$ is finite, as this is no longer directly implied by the finiteness of the initial distribution's radius (just think of a bounded metric). The Cauchy-Schwarz inequality implies that this constraint is equivalent to the finiteness of the entries in the covariance matrix corresponding to the distribution of η_0, which is why we will simply refer to it as having a finite second moment, just as in the univariate case.

Finally, note that if we fix an initial distribution $\mathcal{L}(\eta_0)$, due to the update rule (1.1), all possible future opinion values lie in the convex hull of its support, $\mathrm{conv}(\mathrm{supp}\,\eta_0)$. For this reason it will suffice in every respect that ρ is weakly convex (and possibly locally dominated by the Euclidean norm) on $\mathrm{conv}(\mathrm{supp}\,\eta_0)$ only, not the entire $\mathbb{R}^k$.

Theorem 4.3. *In the Deffuant model on $\mathbb{Z}$ with the underlying opinion space $(\mathbb{R}^k, \rho)$ and an initial opinion distribution $\mathcal{L}(\eta_0)$ we have the following limiting behavior:*

(a) If ρ is locally dominated by the Euclidean distance and $\mathcal{L}(\eta_0)$ has a finite second moment, a finite radius $R_\rho \in [0, \infty)$ and mass around its mean, i.e.

$$\mathbb{P}\big(\eta_0 \in B_\rho(\mathbb{E}\,\eta_0, r)\big) > 0 \text{ for all } r > 0, \tag{4.3}$$

the critical parameter is $\theta_c = R_\rho$, meaning that for $\theta < R_\rho$ we have a.s. no consensus and for $\theta > R_\rho$ a.s. strong consensus.

(b) Let $\eta_0 = (\eta_0^{(1)}, \dots, \eta_0^{(k)})$ be the random initial opinion vector. If one of the coordinates $\eta_0^{(i)}$ has an unbounded marginal distribution (with respect to the absolute value), its expected value exists (regardless of whether finite, $+\infty$ or $-\infty$) and ρ is sensitive to this coordinate, the limiting behavior will a.s. be no consensus, irrespectively of θ.

Proof: (a) The proof of this theorem is exactly the same as the proof of Theorem 3.2. One only has to check that the additional requirements on ρ make up for the crucial properties of the Euclidean norm that were used in the cited proof. The (multivariate) SLLN states that the averages in (4.2) for large n are close to the mean in Euclidean distance, hence with respect to ρ due to (4.1). Local modification of the initial profile will then guarantee the existence of one-sidedly ε-flat vertices.

The crucial role of ε-flat vertices is preserved by the weak convexity of ρ: The proof of Prop. 5.1 in Häggström (2012) shows that given an edge $\langle v-1, v\rangle$ along which there have been no updates yet, the opinion value at v is a convex combination of averages as in (4.2), hence lies in $B_\rho(\mathbb{E}\,\eta_0, \varepsilon)$ as well, if v was ε-flat to the right with respect to the initial configuration, due to convexity of the ρ-balls.

As to the supercritical regime, the a.s. existence of two-sidedly ε-flat vertices follows from the a.s. existence of one-sidedly ε-flat vertices and the i.i.d. property of the initial configuration, just as in the Euclidean case. The weak convexity of ρ is needed once more to conclude that the opinion values of two-sidedly ε-flat vertices stay close to the mean, just as in La. 6.3 in Häggström (2012).

When we want to apply the argument of Prop. 6.1 in Häggström (2012), stating that neighbors will a.s. either finally concur or the edge between them be blocked for large t, it is essential that condition (4.1), together with the finite second moment, allows once again to borrow the energy idea. The extra condition of a finite second moment implies the finiteness of the expected initial engergy $\mathbb{E}\,[W_0(v)] = \mathbb{E}\,[\eta_0(v)^2]$, as mentioned just before the theorem. If the opinions $\eta_t(u), \eta_t(v)$ of two neighbors are within the confidence bound with respect to ρ but $\rho(\eta_t(u), \eta_t(v)) \geq \delta$ for some $\delta > 0$, then due to (4.1): $||\eta_t(u) - \eta_t(v)||_2 \geq \frac{\delta}{C}$, where $C = \max\{c, \frac{\theta}{\gamma}\} > 0$, see the comments after Definition 4.1. This will cause an energy loss of at least $2\mu(1-\mu)(\frac{\delta}{C})^2$ when they compromise. Again, this cannot happen infinitely often with positive probability as the expected energy at time $t = 0$ is finite and the expected total energy preserved over time.

(b) Given ρ is sensitive to coordinate i, the idea of proof of the second claim can be reutilized as well. The sensitivity leads to the fact that there is some $s > 0$ s.t. $|x_i - y_i| > s$ implies $\rho(x, y) > \theta$. As alluded in the proof of Theorem 3.2, the arguments used for unbounded distributions in Thm. 2.2 in Häggström and Hirscher (2014) show that under the given conditions, there are a.s. vertices that differ more than s from both their neighbors in the ith coordinate (with respect to the absolut value) in the initial configuration and this will not change no matter whom their neighbors will compromise with. Consequently the corresponding opinion vectors will always be at ρ-distance more than θ.

□

Example 4.4.

(a) The L^p-norm for general $p \in [1, \infty]$ on $\mathbb{R}^k$ is defined as follows:

$$\|x\|_p := \Big(\sum_{i=1}^{k} |x_i|^p\Big)^{\frac{1}{p}} \quad \text{for } p \in [1, \infty) \text{ and } \quad \|x\|_\infty := \max_{1\leq i\leq k} |x_i|.$$

In fact, these norms are all equivalent. More precisely, for $1 \leq q < p \leq \infty$:

$$\|x\|_p \leq \|x\|_q \leq k^{\left(\frac{1}{q}-\frac{1}{p}\right)}\, \|x\|_p.$$

This implies for all $p \in [1, \infty]$:

$$\|x\|_p \leq \sqrt{k}\, \|x\|_2.$$

In other words all induced metrics $\rho(x, y) = \|x-y\|_p$, are – to be precise globally – dominated by the Euclidean distance.

It is easy to check that the norm axioms guarantee the convexity of balls, hence the metric induced by $\|\,.\,\|_p$ is weakly convex for any $p \in [1, \infty]$.

Furthermore, $\|x\|_p \geq k^{\left(\frac{1}{p}-1\right)}\|x\|_1 \geq k^{\left(\frac{1}{p}-1\right)}|x_i|$ for all $1 \leq i \leq k$ implies sensitivity to every coordinate. In conclusion, both parts of Theorem 4.3 can be applied to the Deffuant model with the metric induced by some L^p-norm, i.e. $\rho(x,y) = \|x-y\|_p$, $p \in [1,\infty]$, as distance measure.

(b) If the definition of $\|\,.\,\|_p$ is extended to values for p in $(0,1)$, the corresponding functions are not subadditive, hence do not induce a metric.

Raised to the power p, we get the distance measures

$$\rho_p(x,y) := \big(\|x-y\|_p\big)^p = \sum_{i=1}^k |x_i - y_i|^p,$$

which are in fact metrics for all $p \in (0,\infty)$ and obviously sensitive to every coordinate. For $p \in (0,1)$ these metrics fail to have convex balls. For $p \in [1,\infty)$ however, they are weakly convex which can be seen from the weak convexity of $\|\,.\,\|_p$ as follows:

$$\begin{aligned}\rho_p(x, \alpha y + (1-\alpha)\,z) &= \Big(\big\|x - \big(\alpha y + (1-\alpha)\,z\big)\big\|_p\Big)^p\\ &\leq \big(\max\{\|x-y\|_p, \|x-z\|_p\}\big)^p\\ &= \max\{\rho_p(x,y), \rho_p(x,z)\}.\end{aligned}$$

The metrics ρ_p, $p \in [1,\infty)$ are no longer equivalent to the Euclidean distance, but still locally dominated in the sense of (4.1). In conclusion, Theorem 4.3 equally applies to the Deffuant model where distances are taken with respect to ρ_p.

More generally, given $\varphi = (\varphi_i)_{i=1}^k$ with non-negative functions φ_i defined on $\mathbb{R}_{\geq 0}$ we can consider

$$\rho_\varphi(x,y) := \sum_{i=1}^k \varphi_i\big(|x_i - y_i|\big).$$

For this to be a proper metric, the φ_i have to be convex satisfying $\varphi_i(s) = 0$ if and only if $s = 0$. Defined this way ρ_φ is convex, in particular weakly convex. It will be locally dominated by the Euclidean distance by default and sensitive to coordinate i if and only if $\varphi_i(s)$ is unbounded as $s \to \infty$.

Example 4.5. The extra condition (4.1) cannot be dropped. Let us consider the discrete metric $\rho(x,y) = \mathbb{1}_{\{x \neq y\}}$ – which is weakly convex – on $\mathbb{R}$. Clearly, it is not locally dominated by the Euclidean metric. Let η_0 have the mixed distribution with constant density $\frac{1}{4}$ on $[-1,1]$ and point mass $\frac{1}{2}$ at 0. Hence $\mathcal{L}(\eta_0)$ has expectation 0 and radius 1 (actually both with respect to ρ and the Euclidean distance). Regarding (4.3), we find $\mathbb{P}(\eta_0 \in B_\rho(0,\varepsilon)) \geq \frac{1}{2}$ for all $\varepsilon \geq 0$. Take $\mu \in (0, \frac{1}{2}]$ to be a transcendental number (e.g. $\frac{1}{\pi}$). Furthermore, we choose $\theta \geq 2$ which obviously makes blocked edges impossible.

At every time t, $\eta_t(v)$ is a finite (but random) convex combination of the initial opinions $\{\eta_0(y)\}_{y\in\mathbb{Z}}$, say

$$\eta_t(v) = \sum_{y\in\mathbb{Z}} \xi_{v,t}(y)\,\eta_0(y), \tag{4.4}$$

which is the SAD representation, see La. 3.1 in Häggström (2012). Almost surely, there are two edges that do not experience Poisson events up to time t and enclose v. It is not hard to show – by induction on the (a.s. finitely many) Poisson events occurring up to time t on the edges between those two – that the non-zero factors $\xi_{v,t}(y)$ in the representation of $\eta_t(v)$ are (random) polynomials in μ with integer coefficients. Furthermore, for $y \neq v$ they have no constant term, for $y = v$ the constant term equals 1: At time 0 we find $\xi_{u,0}(y) = \mathbb{1}_{\{u=y\}}$ for all $u, y \in \mathbb{Z}$. With a Poisson event at time s on the edge $\langle u, u+1\rangle$ that actually causes an update, the coefficients change according to

$$\begin{aligned}\xi_{u,s}(y) &= (1-\mu)\,\xi_{u,s-}(y) + \mu\,\xi_{u+1,s-}(y)\\ \xi_{u+1,s}(y) &= \mu\,\xi_{u,s-}(y) + (1-\mu)\,\xi_{u+1,s}(y),\end{aligned}$$

for all $y \in \mathbb{Z}$, compare with (2.3). This establishes the induction step.

Using the representation (4.4) we find for two neighbors u, v:

$$\eta_t(v) - \eta_t(u) = \sum_{y\in\mathbb{Z}} \big(\xi_{v,t}(y) - \xi_{u,t}(y)\big)\,\eta_0(y).$$

As $\xi_{v,t}(v) - \xi_{u,t}(v)$ is a non-zero polynomial in μ with integer coefficients, it cannot be zero. Additionally, due to the fact that $\theta \geq 2$, the ξ-factors only depend on the Poisson events, which implies that the two random variables

$$X := \frac{1}{\xi_{v,t}(v) - \xi_{u,t}(v)} \sum_{y\neq v} \big(\xi_{v,t}(y) - \xi_{u,t}(y)\big)\,\eta_0(y)$$

and $\eta_0(v)$ are independent. Since $\mathbb{P}(\eta_0(v) = 0) = \mathbb{P}(\eta_0(v) \neq 0) = \frac{1}{2}$, we get

$$\mathbb{P}(\eta_t(v) - \eta_t(u) \neq 0) \geq \mathbb{P}(X = 0, \eta_0(v) \neq 0) + \mathbb{P}(X \neq 0, \eta_0(v) = 0) = \tfrac{1}{2}.$$

This leads to

$$\mathbb{P}\Big(\limsup_{t\to\infty} \rho\big(\eta_t(u), \eta_t(v)\big) = 1\Big) \geq \tfrac{1}{2}$$

for all neigbors u, v, which renders even weak consensus impossible.

In fact, with this choice of initial distribution and metric, the Deffuant model exhibits a limiting behavior that is not a.s. approaching one of the scenarios described in Definition 1.1, since it does not feature blocked edges, nor almost sure consensus formation in the long run – instead at any time t the opinions of two neighbors are with probability at least $\frac{1}{2}$ at distance 1, always at speaking terms but not converging.

Since the choice of θ is trivial, we can find out what happens by looking at the Deffuant model employing the Euclidean distance instead. By Theorem 3.2 all opinions will a.s. approach the mean 0, but whenever two of them do not coincide they are at ρ-distance 1.

Example 4.6. To illustrate the importance of the sensitivity in part (b) of Theorem 4.3, let us consider the two metrics $d(x, y) = \|x - y\|_2$, that is the Euclidean metric, and

$$\rho(x, y) = \begin{cases} \|x-y\|_2, & \text{if } \|x-y\|_2 \leq 1\\ 1, & \text{otherwise.}\end{cases}$$

Evidently, ρ is not sensitive to any coordinate and that it is weakly convex is not hard to check either: For $r < 1$ the balls $B_\rho(x, r)$ are the same as the Euclidean balls, for $r \geq 1$ we get $B_\rho(x, r) = \mathbb{R}^k$. So in either case it is a convex set.

For simplicity, let us take k to be 1 – the Euclidean distance is then induced by the absolute value – and choose the standard normal distribution $\mathcal{N}(0, 1)$ as initial distribution. Due to $\rho(x, y) \leq |x - y|$, ρ is locally dominated by the Euclidean distance. As the normal distribution has a finite second moment and mass around its mean, part (a) of Theorem 4.3 shows that in the Deffuant model using ρ as the distance measure, the radius $R_\rho = 1$ marks the critical value for θ at which we have a phase transition from a.s. no consensus to a.s. strong consensus.

In the Deffuant model using the Euclidean distance however, there will a.s. be no consensus irrespectively of θ according to Theorem 2.2 (b).

The final aim will now be to prove a generalization of Theorem 3.15 to the Deffuant model with general metric ρ instead of the Euclidean. In order to be able to do this we have to transfer the necessary auxiliary results leading to Theorem 3.15, essentially by replacing all occurring Euclidean distances by distances with respect to ρ, however it requires small adjustments.

Definition 4.7. Consider a random variable ξ on $(\mathbb{R}^k, \rho)$. The *support* of its distribution is the following subset of $\mathbb{R}^k$, closed with respect to ρ:

$$\operatorname{supp}(\xi) := \big\{x \in \mathbb{R}^k,\ \mathbb{P}\big(\xi \in B_\rho(x, r)\big) > 0 \text{ for all } r > 0\big\}\,. \tag{4.5}$$

Remark 4.8. The last argument in the proof of Proposition 3.5 is still valid in the general case and verifies $\operatorname{supp}(\eta_0) \subseteq B_\rho[\mathbb{E}\,\eta_0, R_\rho]$ for all initial distributions bounded with respect to ρ. The first part of its proof, i.e. showing that $\operatorname{supp}(\eta_0) \subseteq B[\mathbb{E}\,\eta_0, r]$ implies $\mathbb{P}(\eta_0 \in B[\mathbb{E}\,\eta_0, r]) = 1$, is based on the theorem of Heine-Borel – stating that closed and bounded sets are compact in $(\mathbb{R}^k, \|\,.\,\|_2)$ – which does not hold for general metric spaces. For the discrete metric (see Example 4.5) and a probability measure without point masses, the set defined in (4.5) is in fact empty.

If however $(\mathbb{R}^k, \rho)$ is separable, i.e. there exists a countable dense subset, we get $\mathbb{P}(\xi \in \operatorname{supp}(\xi)) = 1$ for any random variable ξ – see e.g. Thm. 2.1, p. 27 in Parthasarathy (1967) – and thus the full statement of Proposition 3.5.

Given ρ is locally dominated by the Euclidean distance, we can immediately conclude that $(\mathbb{R}^k, \rho)$ is separable, since due to (4.1) the set $\mathbb{Q}^k$ is not only dense in $(\mathbb{R}^k, \|\,.\,\|_2)$ but also in $(\mathbb{R}^k, \rho)$.

In conclusion, if $(\mathbb{R}^k, \rho)$ is separable and η_0 has a finite expectation, its distribution's radius can be written as $R_\rho = \sup\{\rho(\mathbb{E}\,\eta_0, x),\ x \in \operatorname{supp}(\eta_0)\}$.

Adjusting the definition of $\mathscr{D}_\theta(\nu)$ (see Definition 3.6) to the general setting by substituting ρ-balls for Euclidean balls – let us denote the resulting set by $\mathscr{D}_\theta^\rho(\nu)$ – allows to reuse the arguments in the lemmas dealing with its properties. Although referencing to Proposition 3.5, in order to prove Lemma 3.7 only $\operatorname{supp}(\eta_0) \subseteq B[\mathbb{E}\,\eta_0, R]$ was needed, hence its statement is true for any weakly convex ρ – with the terms related to closure now referring to the topology generated by ρ.

As the final conclusions similar to Theorem 3.15 will require ρ to be locally dominated by the Euclidean distance, let us assume for the remainder of this section that ρ is not only weakly convex but also (4.1) holds.

When it comes to the central Lemma 3.9, the claims that can be modified to hold for such ρ as well without major efforts read as follows (again connectedness and closure refer to the topology generated by ρ):

Lemma 4.9. *Let ρ be a weakly convex metric locally dominated by the Euclidean distance.*

(a) For all $x \in \mathbb{R}^k$ and $0 \le \delta < \frac{\theta}{2}$, the set $\mathscr{D}_\theta^\rho(\nu) \cap B_\rho[x, \delta]$ is convex.

(b) The connected components of $\mathscr{D}_\theta^\rho(\nu)$ are convex and at ρ-distance at least θ from one another. If $\mathscr{D}_\theta^\rho(\nu)$ is connected, then $\mathscr{D}_\theta^\rho(\nu) = \overline{\operatorname{conv}(\operatorname{supp}(\eta_0))}$.

(c) If $R_\rho < \infty$ and ν has mass around its mean, i.e. condition (4.3) holds, then $\mathscr{D}_\theta^\rho(\nu) = \overline{\operatorname{conv}(\operatorname{supp}(\eta_0))}$ for all $\theta > R_\rho$.
(d) If $\mathscr{D}_\theta^\rho(\nu)$ is connected and $\mathbb{E}\,\eta_0$ finite, then $\mathbb{E}\,\eta_0 \in \mathscr{D}_\theta^\rho(\nu)$

Proof: The proof is essentially identical to the one of Lemma 3.9. In part (a) we only have to choose $m, n \in \mathbb{N}$ such that

$$\left|\frac{m}{m+n} - \alpha\right| \leq \frac{\min\{\frac{r}{4c}, \frac{\gamma}{2}\}}{\max\{\|y\|_2, \|z\|_2\}}.$$

Then

$$\|(\tfrac{m}{m+n}\,y + \tfrac{n}{m+n}\,z) - (\alpha y + (1-\alpha)z)\|_2 \leq |\tfrac{m}{m+n} - \alpha| \cdot \|y\|_2 + |\alpha - \tfrac{m}{m+n}| \cdot \|z\|_2 \leq \gamma,$$

which together with (4.1) implies

$$\begin{aligned} \rho\big(\eta_N(v), \alpha y + (1-\alpha)z\big) &\leq \tfrac{r}{2} + \rho\big(\tfrac{m}{m+n}\,y + \tfrac{n}{m+n}\,z, \alpha y + (1-\alpha)z\big) \\ &\leq \tfrac{r}{2} + c\,\|(\tfrac{m}{m+n}\,y + \tfrac{n}{m+n}\,z) - (\alpha y + (1-\alpha)z)\|_2 \\ &\leq \tfrac{r}{2} + c\,(|\tfrac{m}{m+n} - \alpha| \cdot \|y\|_2 + |\alpha - \tfrac{m}{m+n}| \cdot \|z\|_2) \leq r. \end{aligned}$$

As to part (b), we can follow the first part of the proof of Lemma 3.9 (c) replacing every Euclidean distance by ρ until the angles are considered. Since $B_\rho[x_j, \frac{\theta}{2}]$ might be oddly shaped, we can define $r := \min\{\frac{\theta}{2c}, \gamma\} > 0$ and consider the Euclidean ball $B[x_j, r]$ which by (4.1) is contained in $B_\rho[x_j, \frac{\theta}{2}]$. Cutting short an angle α as described there, will now reduce the (Euclidean) length of the polygonal chain by at least $2r \cdot (1 - \cos(\alpha))$ and the argument goes through yielding that the Euclidean closure of the component C connected with respect to ρ contains $\operatorname{conv}(\{x, y\})$. It follows from the generalized statement of Lemma 3.7 that being a component of $\mathscr{D}_\theta^\rho(\nu)$, C is ρ-closed. This in turn implies that C is also closed with respect to the Euclidean distance, using (4.1), and hence containing $\operatorname{conv}(\{x, y\})$. The rest of the claim easily follows, again by replacing $\|x - y\|_2$ by $\rho(x, y)$.

Part (c) is an easy consequence of the arguments leading to (a) and (b) that can be verified just as in the proof of Lemma 3.9 (d).

Finally, the only insight needed to accept the proof of Lemma 3.9 (f) as proof of claim (d) above is that $\mathscr{D}_\theta^\rho(\nu)$, being closed in $(\mathbb{R}^k, \rho)$, is also closed in the Euclidean space $(\mathbb{R}^k, \|\,.\,\|_2)$, due to (4.1). □

Definition 4.10. Corresponding to Definition 3.11, let the support of the distribution of η_t in the Deffuant model with parameter θ and distance measure ρ be denoted by $\operatorname{supp}_\theta^\rho(\eta_t)$.

Respectively, the length of the largest gap in $\operatorname{supp}(\eta_0)$ with respect to ρ will be given by

$$h_\rho := \inf\{\theta > 0,\ \mathscr{D}_\theta^\rho(\nu) \text{ is connected in } (\mathbb{R}^k, \rho)\},$$

compare with Definition 3.14.

Following the arguments in the proof of Lemma 3.12 with scrutiny reveals that the corresponding statements are also true for $\operatorname{supp}_\theta^\rho(\eta_t)$ in place of $\operatorname{supp}_\theta(\eta_t)$ and $B_\rho[\mathbb{E}\,\eta_0, R_\rho]$ substituting $B[\mathbb{E}\,\eta_0, R]$ – actually even for metrics which are only weakly convex and not locally dominated by the Euclidean distance for only the convexity of $B_\rho[\mathbb{E}\,\eta_0, R_\rho]$ is needed. Concerning Proposition 3.13 however, we will not bother with the proof of a similar statement for the Deffuant model with general ρ. The only fact needed in the upcoming theorem is

$$\mathrm{supp}_\theta^\rho(\eta_t) \subseteq \mathscr{D}_{\theta+\varepsilon}^\rho(\nu) \quad \text{for } \varepsilon > 0,$$

which readily follows from the last argument in the proof of this very proposition. Having followed up the crucial intermediate steps makes it possible to slightly modify the proof of Theorem 3.15 in order to get an argument establishing the following result:

Theorem 4.11. *Consider the Deffuant model on* $\mathbb{Z}$ *with opinion values in* $(\mathbb{R}^k, \rho)$, *where the corresponding distance measure* ρ *is a weakly convex metric, locally dominated by the Euclidean distance. Assume it features an initial opinion distribution which has a finite second moment and is bounded with respect to* ρ, *i.e.*

$$R_\rho = \inf\big\{r > 0,\ \mathbb{P}\big(\eta_0 \in B_\rho[\mathbb{E}\,\eta_0, r]\big) = 1\big\} < \infty.$$

If h_ρ *denotes the length of the largest gap in its support, then the critical value for the confidence bound, where a phase transition from a.s. no consensus to a.s. strong consensus takes place is* $\theta_\mathrm{c} = \max\{R_\rho, h_\rho\}$.

Proof: As mentioned, the reasoning follows closely the proof of Theorem 3.15. In case (i), where $\theta < h_\rho$ we can conclude from Lemma 3.12 and the above remarks that for $\varepsilon > 0$ such that $\theta + \varepsilon < h_\rho$ it follows that

$$\mathrm{supp}(\eta_0) \subseteq \mathrm{supp}_\theta^\rho(\eta_t) \subseteq \mathscr{D}_{\theta+\varepsilon}^\rho(\nu).$$

The set $\mathscr{D}_{\theta+\varepsilon}^\rho(\nu)$ is not connected (with respect to ρ) by definition of h_ρ, hence comprises convex components C_1 and C_2 at ρ-distance at least $\theta + \varepsilon$ (see Lemma 4.9). Again, we can choose the components such that $\mathbb{P}(\eta_0 \in C_i) > 0$ for $i = 1, 2$, since if we had $\mathbb{P}(\eta_0 \in C_1) = 1$, the fact that C_1 is closed with respect to ρ would give $\mathrm{supp}(\eta_0) \subseteq C_1$ and so (using its convexity and the generalization of Lemma 3.7)

$$\mathscr{D}_{\theta+\varepsilon}^\rho(\nu) \subseteq \overline{\mathrm{conv}(\mathrm{supp}(\eta_0))} \subseteq C_1.$$

But $C_1 = \mathscr{D}_{\theta+\varepsilon}^\rho(\nu)$ contradicts the disconnectedness.

Consequently, for a fixed vertex v independence of the initial opinions guarantees that the event $\{\eta_0(v) \in C_1, \eta_0(v+1) \in C_2\}$ has positive probability, which dooms the edge $\langle v, v+1\rangle$ to be blocked by $\rho(\eta_t(v), \eta_t(v+1)) \geq \theta + \varepsilon$ for all $t \geq 0$. Indeed, in the Deffuant model with parameter θ, $\eta_t(v)$ can not leave the convex set C_1 since $\mathrm{supp}_\theta^\rho(\eta_t) \setminus C_1$, being a subset of $\mathscr{D}_{\theta+\varepsilon}^\rho(\nu) \setminus C_1$, is at distance at least $\theta + \varepsilon$ to C_1 for all t. The same holds for $\eta_t(v+1)$ and C_2 respectively. Due to ergodicity, the existence of blocked edges is therefore an almost sure event.

The analysis of case (ii), $\theta < R_\rho$, requires likewise only minor adjustments of the argument in the proof of Theorem 3.15. To begin with, the finite second moment of η_0 implies $\mathbb{E}\,\eta_0 \in \mathbb{R}^k$, which is not ensured by $R_\rho < \infty$ itself. Let this time y be an element of $\mathrm{supp}(\eta_0) \setminus B_\rho[\mathbb{E}\,\eta_0, \theta + 2\varepsilon]$, which is non-empty for $\varepsilon \in (0, \frac{R-\theta}{2})$. Since both $B_\rho[y, \theta + \varepsilon]$ and $B_\rho[\mathbb{E}\,\eta_0, \varepsilon]$ are convex and closed – with respect to ρ and thus $\|\,.\,\|_2$ due to (4.1) – as well as disjoint, we can choose $z_1 \in B_\rho[y, \theta+\varepsilon]$ and $z_2 \in B_\rho[\mathbb{E}\,\eta_0, \varepsilon]$ such that

$$\|z_1 - z_2\|_2 = \min\{\|a - b\|_2,\ a \in B_\rho[y, \theta + \varepsilon] \text{ and } b \in B_\rho[\mathbb{E}\,\eta_0, \varepsilon]\} > 0$$

and then define $z = \frac{1}{2}(z_1 + z_2)$ and the half-space H with respect to this point z accordingly. Note that H contains $B_\rho[\mathbb{E}\,\eta_0, \varepsilon]$ and is disjoint from $B_\rho[y, \theta+\varepsilon]$, just as in the Euclidean setting, because of the convexity of ρ-balls and the choice of z_1, z_2. Moreover, the local domination property (4.1) forces $B_\rho[\mathbb{E}\,\eta_0, \varepsilon]$ to be a superset of $B[\mathbb{E}\,\eta_0, \delta]$, where $\delta = \min\{\frac{\varepsilon}{c}, \gamma\}$, and thus that $\mathbb{E}\,\eta_0$ lies in the Euclidean interior of H. Having established this, we can follow the rest of the argument (beginning with (3.5), which again follows from the finite second moment of η_0) literally, having in mind that y has ρ-distance larger than $\theta + \varepsilon$ to H.

Finally, in the supercritical case (iii), i.e. $\theta > \max\{R_\rho, h_\rho\}$, we only have to take Lemma 4.9 as a replacement for Lemma 3.9 and again write ρ for the appearing Euclidean distances. It is crucial to notice, that limits with respect to the Euclidean distance as in the SLLN and (3.6) are also limits with respect to ρ, once again using (4.1). Furthermore, in several places either the triangle inequality or the convexity of Euclidean balls was used, but being a weakly convex metric, ρ has the corresponding properties. Using the idea of energy to conclude that two neighbors will a.s. either finally concur or end up with opinions further than θ apart from each other, the fact that ρ is locally dominated by the Euclidean distance is indispensable and employed as in the proof of Theorem 4.3 (a). This is also where the finiteness of the second moment is needed. $\square$

Example 4.12. In order to discern in how far the results of this section do actually add to the univariate case as well, let us finally consider a metric on $\mathbb{R}$ which is not translation invariant. One can take for example $\rho(x, y) = |x^3 - y^3|$ for all $x, y \in \mathbb{R}$. This metric ρ obviously generates convex balls, in other words is weakly convex. However, since

$$\frac{|x^3 - y^3|}{|x - y|} = |x^2 + xy + y^2| \to \infty \quad \text{as } x, y \to \infty$$

it is not locally dominated by the absolut value. Nevertheless, as long as we consider a fixed bounded distribution this problem can be overcome – as was pointed out just before Theorem 4.3 – since on any bounded interval (4.1) holds for ρ and some properly chosen $c > 0$.

If we consider the initial distribution $\nu = \text{unif}\{-\frac{1}{2}, \frac{1}{2}\}$, which has radius $R_\rho = \frac{1}{8}$, we can conclude from Theorem 4.11, that the critical value for the confidence bound is $\theta_\text{c} = \rho(-\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$. Unlike the Euclidean case, this value will change with a translation of the initial distribution: Taking $\eta_0 + \frac{3}{2}$ instead of η_0, in other words $\nu = \text{unif}\{1, 2\}$ as marginal distribution for the initial configuration, we find $R_\rho = \frac{37}{8}$ and $\theta_\text{c} = \rho(1, 2) = 7$.

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Abstract

During the last decades, quite a number of interacting particle systems have been introduced and studied in the border area of mathematics and statistical physics. Some of these can be seen as simplistic models for opinion formation processes in groups of interacting people. In the one introduced by Deffuant et al. agents, that are neighbors on a given network graph, randomly meet in pairs and approach a compromise if their current opinions do not differ by more than a given threshold value θ. We consider the two-sidedly infinite path $\mathbb{Z}$ as underlying graph and extend former investigations to a setting in which opinions are given by probability distributions. Similar to what has been shown for finite-dimensional opinions, we observe a dichotomy in the long-term behavior of the model, but only if the initial narrow-mindedness of the agents is restricted.

1 Introduction

The research field that became known as *opinion dynamics* originated from simple models for interacting elementary particles established in statistical physics, introduced to figure out how microscopic interaction rules lead to macroscopic properties of the whole system. Due to the strong link between statistical mechanics and spatial stochastic processes, interest among mathematicians was raised and in the course of a few decades an abundance of new models with similar but qualitatively different interaction schemes was introduced and analyzed, primarily by computer-based stochastic simulations. The survey article [1] gives a broad overview of the different models and their analyses and applications.

Despite their radical limitations in terms of complexity, these models attracted more and more the attention of the social sciences and were used to

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describe group behavior on an elementary level and to explain real life phenomena. In 2000, Deffuant et al. [3] suggested a simple model that features a bounded confidence restriction: Neighbors talk to each other in pairs and their opinions are updated towards a compromise only if the opinions they hold just before they meet are not further apart than a given threshold. This is meant to capture the realistic phenomenon that people tend to modify their attitude on a specific topic when talking to others, but not if they consider the views of their discussion partner as so alien as to seem like complete nonsense.

In mathematical terms, the *Deffuant model* is structured as follows: First, we are given a simple connected graph $G = (V, E)$, that shapes the underlying network. The vertices are understood to represent agents holding individual opinions on a certain topic. The edges of the graph are supposed to represent the connections between these individuals and entail a possible mutual influence among neighbors. The vertex set V can be either finite or countably infinite. In the latter case the maximal degree in G is commonly assumed to be bounded.

Then there are two model parameters: the already mentioned confidence bound $\theta > 0$ and the convergence parameter $\mu \in (0, \frac{1}{2}]$, shaping the step size towards a compromise when two opinions are updated. Opinions usually take values in $\mathbb{R}$. A higher-dimensional analog was considered in [7] and here we will extend the model further. For these generalizations, we need to specify a metric d that is used to measure the distance of two opinions and takes over the task of the absolute value in the original model.

The first source of randomness is the configuration of initial opinions. Even though there have been attempts to look at settings with dependent initial opinions (see for example Section 2.2 in [6]) the usual setting is to take i.i.d. initial opinions which then evolve dependencies by interacting. The opinion value at vertex $v \in V$ and time $t \geq 0$ will be denoted by $\eta_t(v)$.

The second source of randomness in the model is the succession of pairwise encounters. On finite graphs, the next pair of neighbors to meet is picked uniformly at random. On infinite graphs, the corresponding equivalent is to assign i.i.d. Poisson processes on all edges of the graph: Whenever a *Poisson event* occurs on the edge $e = \langle u, v \rangle$, i.e. a jump in the Poisson process associated with e, the agents u and v interact in the following way: Assume the event happens at time t and the opinions of u and v just before are given by $\eta_{t-}(u) = \lim_{s \uparrow t} \eta_s(u) =: a$ and $\eta_{t-}(v) = \lim_{s \uparrow t} \eta_s(v) =: b$ respectively. Then, depending on the distance of a and b, there might be an update according to the following rule:

$$\eta_t(u) = \begin{cases} a + \mu(b - a) & \text{if } d(a, b) \leq \theta, \\ a & \text{otherwise} \end{cases}$$

and similarly (1)

$$\eta_t(v) = \begin{cases} b + \mu(a - b) & \text{if } d(a, b) \leq \theta, \\ b & \text{otherwise.} \end{cases}$$

Given our assumptions, E is countable, so there will almost surely be neither two simultaneous Poisson events nor a limit point in time for the Poisson events on edges incident to one fixed vertex. This guarantees the well-definedness of the process by (1) for finite G. The extension to infinite graphs with bounded degree is not immediately obvious but a standard argument, see Thm. 3.9 in [11].

When it comes to the long term behavior of the system, it is quite natural to ask whether the agents will form a consensus to which all the opinions converge or not. Let us properly define and distinguish the following two opposing asymptotics of the Deffuant model as time tends to infinity:

Definition 1

(i) *Disagreement*
There will be finally blocked edges, i.e. edges $e = \langle u, v\rangle \in E$ s.t.

$$d(\eta_t(u), \eta_t(v)) > \theta,$$

for all times t large enough. Hence the vertices fall into different opinion groups, that are incompatible with neighboring ones.

(ii) *Consensus*
The value at every vertex converges, as $t \to \infty$, to a common limit l, where

$$l = \begin{cases} \frac{1}{|V|} \sum\limits_{v \in V} \eta_0(v), & \text{if } G \text{ is finite} \\ \mathbb{E}\,\eta, & \text{if } G \text{ is infinite} \end{cases}$$

and $\mathcal{L}(\eta)$ denotes the distribution of the initial opinion values.

Even though these two regimes intuitively seem to be complementary, for infinite graphs it is far from obvious that the asymptotics of the model is necessarily given by one or the other (cf. Def. 1.1 in [6] and also the remark at the end of Section 5).

In this paper, we are going to consider the two-sidedly infinite path $\mathbb{Z}$ as underlying graph, i.e. $V = \mathbb{Z}$ and $E = \{\langle v, v+1\rangle, v \in \mathbb{Z}\}$. The first result for this setting was published in 2011 and is due to Lanchier [10], who showed a sharp phase transition from almost sure disagreement to almost sure consensus at $\theta = \frac{1}{2}$, given initial opinions, that are i.i.d. unif([0, 1]). Shortly thereafter, Häggström [5] reproved Lanchier's findings using a quite different approach; then Häggström and Hirscher [6] extended them to general univariate distributions for $\mathcal{L}(\eta)$. In [7], the case of vector-valued opinions and different distance measures was examined.

One aspect that could be considered unrealistic in these models is the fact that even though opinions are random, for a fixed realization they were given by numbers or vectors, hence entirely determined – not doing justice to the extremely common phenomenon of uncertainty in people's opinions. In what follows, we are going to introduce and analyze a variant of the Deffuant model on $\mathbb{Z}$, where the opinions are given by random absolutely continuous measures on $[0,1]$. The support of these measure-valued opinions can be seen to represent uncertainty: the more concentrated the measure, the more determined the agent.

As a general preparation for an extension of the model in this direction, in Section 2 we will introduce the total variation distance (which will be used as distance measure) and recall a Strong law of large numbers (SLLN) for continuous densities, due to Rubin, replacing the common SLLN which was a crucial ingredient in the case of finite-dimensional opinions.

The model with measure-valued opinions is outlined in Section 3. We consider random symmetric triangular distributions as initial opinions and find that for this setting overly determined agents (i.e. agents whose initial opinion is concentrated on sufficiently short intervals) prevent consensus for all $\theta \in [0,1)$, cf. Theorem 3.1.

The results for finite-dimensional opinion spaces listed above will be sketched in more detail in Section 4. Central ideas from [5] will be presented as they prove to be useful in our setting as well.

In Section 5 the main result for the setting with unrestricted symmetric triangular distributions, Theorem 3.1, is proved: We show that the behavior of the model is trivial in this case as extremely determined agents will have and keep a total variation distance close to 1 to their neighbors' opinions.

If we put a restriction on the initial determination of the agents by disallowing triangular distributions that have a support of length less than a fixed value γ, the familiar phenomenon of a phase transition in θ from a.s. disagreement to a.s. consensus reappears. This case, as well as the precise dependency of the threshold value θ_c on the parameter γ are elaborated in Section 6. In the final section, we breifly discuss possible other initial configurations and earlier attempts to incorporate inhomogeneous open-mindedness of the agents.

2 Distance and convergence of absolutely continuous random measures

As indicated, we want to generalize the Deffuant model on $\mathbb{Z}$ further by looking at opinions that are no longer numbers or vectors but probability distributions instead. These random distributions can be seen to shape indeterminacy in the agents: Even with initial opinion profile and sequence of encounters fixed, the opinion of an individual at a given time is not a fixed value but a probability measure. Initially, the agents are independently assigned random measures from a common distribution. When they meet and their current opinion measures do not differ by more than θ, with respect to a fixed metric on probability measures, the new opinions will be given by convex combinations of the old ones, just as described in (1).

In order to quantify the difference between two distributions there are quite a few metrics to choose from. The so-called total variation distance is among the most common ones.

Definition 2
Let μ and ν be two probability distributions on a set S. The *total variation distance* between the two measures is then defined as

$$\|\mu - \nu\|_{\mathrm{TV}} := \sup_{A \subseteq S} |\mu(A) - \nu(A)|.$$

As the total variation distance of two probability distributions is a number in $[0,1]$, the non-trivial values for θ lie in $(0,1)$ for this model. Further, note that the total variation distance of two probability distributions μ and ν, that are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and have densities f and g respectively, is given by

$$\|\mu - \nu\|_{\mathrm{TV}} = \frac{1}{2} \int_{\mathbb{R}} \big|f(x) - g(x)\big| \,\mathrm{d}x.$$

In addition, if μ and ν are distributions on $[0,1]$, we can immediately conclude $\|\mu - \nu\|_{\mathrm{TV}} \leq \frac{1}{2}\,\|f - g\|_\infty$, where $\|f\|_\infty = \sup_{x\in[0,1]} \big|f(x)\big|$ denotes the supremum norm on $[0,1]^{\mathbb{R}}$.

To be able to transfer the findings from the Deffuant model on $\mathbb{Z}$ featuring real- or vector-valued opinions, we further need an equivalent for the Strong law of large numbers (SLLN) geared towards the densities of random measures. The following result of Rubin [13] serves our purposes.

Let $U, U_1, U_2, \dots$ denote a sequence of independent, identically distributed random variables with values in an arbitrary space Y. Given a compact topological space X, consider a map $f : X \times Y \to \mathbb{R}$, that is measurable in the second argument for each $x \in X$.

Theorem 2.1 (SLLN for continuous densities)
If there exists an integrable function g on Y such that $|f_y(x)| < g(y)$ for all $x \in X$ and $y \in Y$, as well as a sequence of measurable sets $(S_i)_{i\in\mathbb{N}}$ with

$$\mathbb{P}\Big(U \in \bigcap_{i\in\mathbb{N}} S_i^{\,\mathrm{c}}\Big) = 0,$$

and the property that $\{f_y(\,.\,),\ y \in S_i\}$ is equicontinuous on X for all $i \in \mathbb{N}$, then with probability 1,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f_{U_i}(x) = \mathbb{E}\, f_U(x)$$

uniformly in $x \in X$ and the limit function is continuous.

The measure with density $\mathbb{E}\, f_U$ is commonly called *intensity* or *intensity measure*, see e.g. Section 1.2 in [8] for a more general introduction.

3 Initial opinions given by random triangular distributions

For concreteness, let us pick the initial opinions from a specific class of absolutely continious distributions. A rather natural choice, departing from real-valued opinions (which can be seen as Dirac delta measures), are symmetric triangular distributions on random subintervals of $[0,1]$, the endpoints of which are chosen uniformly from $[0,1]$.

More precisely, let us consider the initial opinions $\{\eta_0(v),\ v\in\mathbb{Z}\}$ to be picked in the following way: Consider $\{U(v),\ v\in\mathbb{Z}\}$ to be an i.i.d. sequence of unif($[0,1]^2$) random vectors. The node v will be assigned an initial opinion given by the random absolutely continuous probability measure with density

$$f_0^{(v)}(x) = \begin{cases} 0, & x\notin(m,M) \\ \left(\frac{2}{M-m}\right)^2\cdot(x-m), & x\in(m,\frac{m+M}{2}] \\ -\left(\frac{2}{M-m}\right)^2\cdot(x-M), & x\in(\frac{m+M}{2},M) \end{cases} \tag{2}$$

$$= \tfrac{2}{|y-z|}\cdot\left(1-\tfrac{2}{|y-z|}\cdot\left|x-\tfrac{y+z}{2}\right|\right)^+,\quad x\in[0,1], \tag{3}$$

where $U(v)=(y,z)$ and $m:=\min\{y,z\},\ M:=\max\{y,z\}$, see Figure 1.

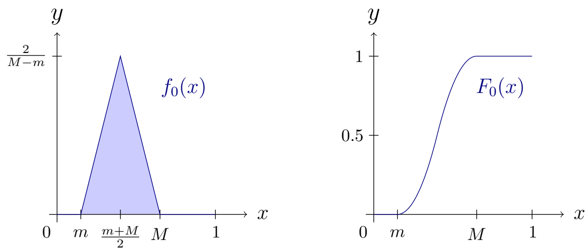

Figure 1: The density and distribution function of a symmetric triangular distribution on $[m,M]$.

Seen from a different angle, to get the initial opinion of a fixed agent, we first choose a central opinion value C uniformly from $[0,1]$ and then a spread for the support of the distribution uniformly among $[0,\min\{C,1-C\}]$. That this procedure is equivalent to the one described above is an immediate consequence of the change of variable formula, see the proof of Lemma 5.1 (especially Figure 3) for more details.

Note that this model features two qualitatively different forms of extreme initial opinions: One the one hand – as in the original model – the agents can have opinions lying at the edges of the spectrum (i.e. concentrated close to 0 or 1 in this case), on the other an individual opinion can be very determined in the sense that $U(v)$ is close to the diagonal (i.e. $|y - z|$ very small), which provokes a highly concentrated density and necessarily a large distance to a vast majority of possible initial opinions.

This effect is quite realistic: Irrespectively of their opinion being exceptional or mainstream, people that are extremely narrow-minded or determined are usually neither willing to consider the opinion nor to accept the arguments of others, let alone to compromise. In this sense, even though the mathematics are closely related to the case of finite-dimensional opinions, the extension of the model to measure-valued opinions introduces an additional real life phenomenon.

For symmetric triangular distributions on $[0, 1]$ without any restriction on the minimal length of their support, we are going to show that the model exhibits a trivial behavior:

Theorem 3.1
Consider the Deffuant model on $\mathbb{Z}$, where the initial opinions are given by independently assigned random triangular distributions as described in (3). Then for all $\theta \in [0, 1)$, the system almost surely approaches disagreement in the long run if the total variation distance is used to measure the distance between two opinions.

For the proof of this result, we refer the reader to Section 5. Since this setting allows to reuse results from the finite-dimensional case, we first want to give a brief overview of these in the following section.

4 Background

As mentioned in the introduction, the first analytic result about consensus formation in the Deffuant model on $\mathbb{Z}$ was established by Lanchier [10] and deals with opinion profiles that are initially given by an i.i.d. sequence of $\text{unif}([0, 1])$ random variables. The distance between two opinions was taken to be the absolute value of their difference. Häggström [5] used different techniques to reprove and slightly sharpen this result. His arguments were later adapted to accommodate other univariate initial distributions as well, leading to an analog covering all marginal distributions that have a first moment $\mathbb{E}\,\eta_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$, see Thm. 2.2 in [6]:

Theorem 4.1
Consider the Deffuant model on the graph $(\mathbb{Z}, E)$, where $E = \{\langle v, v+1\rangle, v \in \mathbb{Z}\}$, with fixed parameter $\mu \in (0, \frac{1}{2}]$. Let the initial configuration be given by an i.i.d. sequence of real-valued random variables, having the common distribution $\mathcal{L}(\eta_0)$, and the distance of two opinions by the absolute value of their difference.

- *(i) Given a bounded distribution $\mathcal{L}(\eta_0)$ with expected value $\mathbb{E}\,\eta_0$, let $[a, b]$ denote the smallest closed interval containing its support. If $\mathbb{E}\,\eta_0$ does not lie in the support, let $I \subset [a, b]$ denote the maximal, open interval with $\mathbb{E}\,\eta_0 \in I$ and $\mathbb{P}(\eta_0 \in I) = 0$. In this case, set h to be the length of I, otherwise set $h = 0$.*

Then the critical value for θ, marking the phase transition from a.s. disagreement to a.s. consensus, becomes $\theta_c = \max\{\mathbb{E}\,\eta_0 - a, b - \mathbb{E}\,\eta_0, h\}$. The common limit value in the supercritical regime is $\mathbb{E}\,\eta_0$.

(ii) *Suppose the distribution $\mathcal{L}(\eta_0)$ is unbounded but its expected value exists, i.e. $\mathbb{E}\,\eta_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$. Then the Deffuant model with arbitrary fixed parameter $\theta \in (0, \infty)$ will a.s. behave subcritically, meaning that disagreement will be approached in the long run.*

With an appropriate adaptation to the more involved geometry of vector-valued opinions, the main ideas in [5] further served to establish similar results for the Deffuant model on $\mathbb{Z}$ with opinion space $\mathbb{R}^d$, $d > 2$, and more general distance measures, see Thm. 3.15 and Thm. 4.11 in [7].

Since the same line of reasoning was used in both [5], [6] and [7] to derive the results for finite-dimensional opinion spaces we just mentioned, let us now take a closer look on the used key concepts and crucial auxiliary results. They form the base for most of the conclusions we will be able to draw in the case of infinite-dimensional opinions.

In [5], Häggström presents two central ideas, whose effect turns out to be highly limited to paths, but combined they prove to be quite powerful in the analysis of the Deffuant model on the infinite path $\mathbb{Z}$. The first one is the notion of *flat points*:

Definition 3
Consider the initial i.i.d. configuration $\{\eta_0(u)\}_{u\in\mathbb{Z}}$ with common marginal distribution $\mathcal{L}(\eta_0)$ on an opinion space X (e.g. $\mathbb{R}$ or $\mathbb{R}^d$), which we consider to be equipped with the metric ρ. Under the premise that the mean $\mathbb{E}\,\eta_0$ of the initial distribution exists and given $\varepsilon > 0$, a vertex $v \in \mathbb{Z}$ is called *ε-flat to the right* (with respect to the initial configuration), if for all $n \geq 0$:

$$\frac{1}{n+1}\sum_{u=v}^{v+n} \eta_0(u) \in B_\varepsilon\big(\mathbb{E}\,\eta_0\big), \tag{4}$$

where $B_r(x) := \{y \in X,\ \rho(x,y) \leq \varepsilon\}$ denotes the (closed) ρ-ball around $x \in X$ with radius $r > 0$. A vertex v is called *ε-flat to the left* if the above condition is met with the sum running from $v-n$ to v instead. Finally, v is called *two-sidedly ε-flat* if for all $m, n \geq 0$

$$\frac{1}{m+n+1}\sum_{u=v-m}^{v+n} \eta_0(u) \in B_\varepsilon\big(\mathbb{E}\,\eta_0\big). \tag{5}$$

The crucial role vertices, that are one- or two-sidedly ε-flat with respect to the initial configuration, can play in the further evolution of the configuration becomes more obvious in the light of the second key idea, the non-random pairwise averaging procedure Häggström [5] proposed to call *Sharing a drink* (SAD) on $\mathbb{Z}$.

Glasses are put along the infinite path at all integers; the one at site 0 is full, all others are empty. Similarly to the Deffuant model, neighbors interact and share, but now we skip randomness and confidence bound: The procedure starts with the initial profile $\{\xi_0(v)\}_{v\in\mathbb{Z}}$, given by $\xi_0(0) = 1$ and $\xi_0(v) = 0$ for all $v \neq 0$. In each step, we choose an edge, along which an update of the form (1) is executed; more precisely, if we are given the profile $\{\xi_n(v)\}_{v\in\mathbb{Z}}$ after step n and choose $\langle u, u+1\rangle$ for the next round, we get

$$\begin{aligned} \xi_{n+1}(u) &= (1-\mu)\,\xi_n(u) + \mu\,\xi_n(u+1), \\ \xi_{n+1}(u+1) &= \mu\,\xi_n(u) + (1-\mu)\,\xi_n(u+1), \\ \xi_{n+1}(v) &= \xi_n(v) \qquad \text{for all } v \notin \{u, u+1\}. \end{aligned} \tag{6}$$

The resulting profiles, after we have performed this procedure a finite number of rounds, will be called *SAD-profiles*. Besides the facts that they feature only finitely many non-zero elements, the elements are all positive and sum to 1, there are less obvious properties that these profiles share which we will collect in the following lemma (for proofs, see Lemmas 2.2, 2.1 and Thm. 2.3 in [5]):

Lemma 4.2
Consider the SAD-procedure on the infinite path $\mathbb{Z}$, started in vertex v, i.e. with $\xi_0(u) = \delta_v(u),\ u \in V$. Then we get the following:

(i) All achievable SAD-profiles are unimodal.

(ii) If the vertex v only shares the water to one side, it will remain a mode of the SAD-profile.

(iii) The supremum over all achievable SAD-profiles started with δ_v at another vertex w equals $\frac{1}{d+1}$, where d is the graph distance between v and w.

The connection to the Deffuant model is established in Lemma 3.1 in [5]: The opinion value $\eta_t(0)$ at any given time $t > 0$ can be written as a weighted average of the initial opinions, where the weights are given by the (random) SAD-profile which is dual to the dynamics in the Deffuant model in the sense that the order of updates has to be reversed.

Combining this link with the concept of ε-flatness makes it possible to derive the following crucial auxiliary results (which are obvious generalizations of intermediate results, established in the proofs of Prop. 5.1, as well as of Lemma 6.3 in [5]):

Lemma 4.3
Consider the Deffuant model on $\mathbb{Z}$ with initial configuration be given by an i.i.d. sequence of random variables having the common distribution $\mathcal{L}(\eta_0)$.

(i) If vertex v is ε-flat to the right with respect to the initial configuration and does not interact with vertex $v-1$, its opinion stays inside $B_\varepsilon\big(\mathbb{E}\,\eta_0\big)$. The same holds for ε-flatness to the left and $v+1$ in place of $v-1$.

(ii) If vertex v is two-sidedly ε-flat with respect to the initial configuration, its opinion value will stay inside $B_{6\varepsilon}\big(\mathbb{E}\,\eta_0\big)$, irrespectively of the dynamics.

Without much further work, these findings can be used to analyze the behavior of the model featuring unrestricted symmetric triangular distributions, as we will see in the following section.

5 Overly determined agents prevent consensus

As the expectation of the initial distribution played a central role in the model featuring real- or vector-valued opinions, we first have to get our hands on its counterpart in the context of random measures, the intensity, before we can set about proving Theorem 3.1.

Lemma 5.1
Consider the absolutely continuous random measure η to be given by the density

$$f_U(x) = \tfrac{2}{|y-z|} \cdot \big(1 - \tfrac{2}{|y-z|} \cdot \big|x - \tfrac{y+z}{2}\big|\big)^+, \quad x \in [0,1], \tag{7}$$

where $U = (y,z)$ is taken uniformly from the unit square $[0,1]^2$, as introduced in (3). Then its intensity measure (commonly denoted $\mathbb{E}\,\eta$) is given by the density

$$\varphi(x) = \begin{cases} -8\,\big[(1-x)\,\ln(1-x) + x\,\big(1-\ln(2)\big)\big], & x \in [0,\frac{1}{2}] \\ -8\,\big[x\,\ln(x) + (1-x)\,\big(1-\ln(2)\big)\big], & x \in [\frac{1}{2},1] \end{cases}.$$

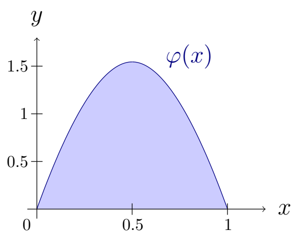

Figure 2: Density of the intensity measure corresponding to random symmetric triangular distributions on $[0,1]$.

Proof: Fix $x \in (0,1)$. First of all, by symmetry, we can take U to be uniform on the set $A := \{(y,z) \in \mathbb{R}^2,\ 0 \le z \le y \le 1\}$. To further simplify the calculations, let us consider the simple linear transform $T((y,z)) = \frac{1}{2}\,(y+z, y-z)$, depicted in Figure 3 below. From the change of variable formula we know that $T(U)$ is uniform on the set $B := \{(u,v),\ v \in [0,\frac{1}{2}], u \in [v, 1-v]\}$.

Given the random density f_U as in (7), we can write

$$f_{T(U)}(x) = \tfrac{1}{v} \cdot \big(1 - \tfrac{1}{v} \cdot |x-u|\big)^+, \quad x \in [0,1]$$

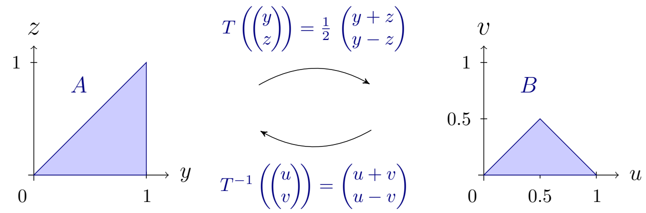

Figure 3: Using transform T to consider the arithmetic mean and half the distance of two independent unif([0, 1]) random variables instead.

and conclude that $f_{(u,v)}(x)$ is non-zero for (u, v) in $B_x = B_1 \cup B_2 \cup B_3$, where

$$\begin{aligned} B_1 :=& \{(u,v),\ v \in [0, \tfrac{x}{2}], u \in [x-v, x+v]\} \\ B_2 :=& \{(u,v),\ v \in [\tfrac{x}{2}, \tfrac{1-x}{2}], u \in [v, x+v]\} \\ B_3 :=& \{(u,v),\ v \in [\tfrac{1-x}{2}, \tfrac{1}{2}], u \in [v, 1-v]\}. \end{aligned}$$

Hence, for $x \in [0, \frac{1}{2}]$, tedious but elementary calculations lead to

$$\begin{aligned} \varphi(x) := \mathbb{E}\left[f_U(x)\right] &= 4 \cdot \iint\limits_{B_x} \frac{1}{v} - \frac{|x-u|}{v^2}\, \mathrm{d}u\, \mathrm{d}v \\ &= -8\,(1-x)\cdot \ln(1-x) - 8x\,\big(1-\ln(2)\big) \end{aligned}$$

By symmetry around $x = \frac{1}{2}$, the claim follows. $\square$

Proof of Theorem 3.1: As usual, let $\eta_t(v)$ denote the opinion of individual $v \in \mathbb{Z}$ at time $t > 0$ and further let $f_t^{(v)}$ be the density corresponding to this random measure. For any fixed $\delta > 0$, let us define the random variables

$$F_t^{(v)}(\delta) := \eta_t(v)([0,\delta]) = \int_0^\delta f_t^{(v)}(x)\, \mathrm{d}x, \quad \text{for all } t > 0,\ v \in \mathbb{Z}.$$

Their values lie in the interval $[0, 1]$, which actually coincides with the support of their distributions. Furthermore, we know that $\{F_0^{(v)}(\delta),\ v \in \mathbb{Z}\}$ are i.i.d. random variables and Fubini's theorem gives

$$\mathbb{E}\left[F_0^{(v)}(\delta)\right] = \int_0^\delta \varphi(x)\, \mathrm{d}x. \tag{8}$$

We can disregard the case $\theta = 0$, since there won't be any dynamics and hence a.s. disagreement. Given $\theta \in (0, 1)$, define $\varepsilon := \frac{1}{2}(1-\theta) > 0$ and choose $\delta > 0$ such that $\int_0^\delta \varphi(x)\, \mathrm{d}x < \varepsilon$.

As mentioned above, the support of the distribution of $F_0^{(v)}(\delta)$ is $[0,1]$ (without gaps), hence we can conclude as in Lemma 4.2 in [5] that any vertex is (one-sidedly) ε-flat with positive probability, with respect to the sequence $\{F_0^{(v)}(\delta),\ v\in\mathbb{Z}\}$. Due to $\mathbb{P}(F_0^{(v)}(\delta)=1)=\delta^2>0$ and independence, the coincidence of the following two events occurs with positive probability for any $v\in\mathbb{Z}$:

(a) Vertex $v-1$ is ε-flat to the left and vertex $v+1$ ε-flat to the right w.r.t. $\{F_0^{(v)}(\delta),\ v\in\mathbb{Z}\}$.

(b) $F_0^{(v)}(\delta)=1$

Using part (i) of Lemma 4.3 and the same line of reasoning as in the proof of Prop. 5.1 in [5], we can conclude that the edge $\langle v-1,v\rangle$ – and similarly $\langle v,v+1\rangle$ – will be blocked forever, as

$$\begin{aligned}\|\eta_t(v)-\eta_t(v-1)\|_{\mathrm{TV}} &\geq \left|F_t^{(v)}(\delta)-F_t^{(v-1)}(\delta)\right|\\ &= F_0^{(v)}(\delta)-F_t^{(v-1)}(\delta)\\ &\geq 1-(\mathbb{E}\ F_0^{(v-1)}(\delta)+\varepsilon)\\ &> 1-2\varepsilon=\theta.\end{aligned}$$

From the fact that approaching disagreement is shift-invariant, hence a 0-1-event, we can conclude that for $\theta\in[0,1)$ there will a.s. be disagreement. □

Remark
The trivial case $\theta=1$ in Theorem 3.1 will surely not lead to blocked edges, so disagreement can be ruled out. However, this does not necessarily imply a consensus formation. The standard energy argument (as we will also use it in Lemma 6.4) fails, since the random symmetric triangular distribution does not have a finite second moment, i.e. $\mathbb{E}\,(f_U(x))^2=\infty$ for all $x\in(0,1)$.

Using the results for univariate opinions once more, we can however conclude consensus for $\theta=1$ if we change to a different distance measure: the so-called Lévy-distance. Consider two probability distributions μ and ν on $[0,1]$. Their *Lévy-distance* $\rho(\mu,\nu)$ is defined as the infimum of the set

$$\{\varepsilon>0 \text{ s.t. } \mu([0,x-\varepsilon])-\varepsilon\leq\nu([0,x])\leq\mu([0,x+\varepsilon])+\varepsilon \text{ for all } x\in[0,1]\}.$$

To settle the case with $\theta=1$ and ρ as distance measure, let us consider the univariate case, where $\{F_t^{(v)}(\delta)=\eta_t(v)([0,\delta]),\ t>0,\ v\in\mathbb{Z}\}$ are the opinions assigned to the agents: As there is no bounded confidence restriction, any encounter leads to an update and the update rule (1) applies to both $\{\eta_t(v)\}_{v\in\mathbb{Z}}$ and $\{F_t^{(v)}(\delta)\}_{v\in\mathbb{Z}}$.

Hence, for any fixed $\delta\in[0,1]$, from Theorem 4.1 and (8) we know that $F_t^{(v)}(\delta)$ converges to $\Phi(\delta):=\int_0^\delta\varphi(x)\ \mathrm{d}x$ almost surely. Consequently, with probability 1, it holds

$$\lim_{t\to\infty}F_t^{(v)}(\delta)=\Phi(\delta)\quad\text{for all } v\in\mathbb{Z},\ \delta\in[0,1]\cap\mathbb{Q}.$$

Since all $F_t^{(v)}$ and Φ are continuous and increasing, this implies almost sure pointwise (in fact even uniform) convergence. In other words, for any $v \in \mathbb{Z}$ the opinion measure $\eta_t(v)$ converges with probability 1 *vaguely* to the intensity measure $\mathbb{E}\,\eta$, having density φ. As vague convergence of measures on a compact interval is metrized by the corresponding Lévy-metric (cf. for example Lemma 2 in [4]), this implies $\lim_{t\to\infty} \rho(\eta_t(v), \mathbb{E}\,\eta) = 0$ almost surely for all $v \in \mathbb{Z}$.

Note that a.s. consensus for $\theta = 1$ and the Lévy-metric does not immediately imply a result for the total variation case, as $\rho(\mu, \nu) \le \|\mu - \nu\|_{\mathrm{TV}}$ for two probability measures μ and ν.

6 Agents with bounded determination

In order to get a non-trivial phase transition in the parameter θ, let us now consider a situation in which all the agents feature at least a certain minimum of open-mindedness. This will be incorporated in our model by disallowing the initial random measure to be concentrated on a subinterval of length less than γ, for a fixed constant $\gamma \in (0, 1)$. We will refer to these as random *restricted triangular distributions.*

Before we can show the main result, Theorem 6.7, which states that there is a phase transition and the precise threshold value for the parameter θ, we need to study the altered intensity measure and verify a few auxiliary results, needed to guarantee the existence of ε-flat vertices (cf. Lemma 6.6).

Lemma 6.1
For fixed $\gamma \in (0, 1)$, consider the absolutely continuous random measure η_γ to be given by the density

$$f_U(x) = \tfrac{2}{|y-z|} \cdot \left(1 - \tfrac{2}{|y-z|} \cdot \left|x - \tfrac{y+z}{2}\right|\right)^+, \quad x \in [0,1], \tag{9}$$

where $U = (y, z)$ is taken uniformly from the set $\{y, z \in [0, 1],\ |y - z| \ge \gamma\}$ and note that this corresponds to the expression in (3), conditional on the support being an interval of length at least γ. Then the density of its intensity measure $\mathbb{E}\,\eta_\gamma$ is given by the following expressions (assuming $0 \le x \le \frac{1}{2}$):

1) for $x \ge \gamma$

$$\varphi_\gamma(x) = -\frac{8}{(1-\gamma)^2}\left[(1-x)\,\ln(1-x) + x\,(1 - \ln(2)) + \frac{\gamma}{4}\right],$$

2) for $x \ge 1 - \gamma$

$$\varphi_\gamma(x) = \begin{cases} -\frac{8}{(1-\gamma)^2}\left[(1-x)\,\ln\gamma + x + \frac{1-2x}{2\gamma} - \frac{\gamma}{2}\right], & x \le \frac{\gamma}{2} \\ -\frac{8}{(1-\gamma)^2}\left[-x\,\ln(2x) + \ln\gamma + x + \frac{(1-x)^2+x^2}{2\gamma} - \frac{3}{4}\gamma\right], & x \ge \frac{\gamma}{2} \end{cases},$$

3) for $x \le \gamma,\ x \le 1-\gamma$

$$\varphi_\gamma(x) = \begin{cases} -\frac{8}{(1-\gamma)^2}\Big[(1-x)\,\ln(1-x) + x - \frac{x^2}{2\gamma}\Big], & x \le \frac{\gamma}{2} \\ -\frac{8}{(1-\gamma)^2}\Big[(1-x)\,\ln(1-x) - x\,\ln\big(\frac{2x}{\gamma}\big) + x + \frac{x^2}{2\gamma} - \frac{\gamma}{4}\Big], & x \ge \frac{\gamma}{2} \end{cases}.$$

The corresponding expressions for $x \in [\frac{1}{2}, 1]$ are obtained by replacing x by $1-x$.

Proof: As in the proof of Lemma 5.1, we can take U to be uniform on the set $A_\gamma := \{(y,z) \in \mathbb{R}^2,\ \gamma \le y \le 1,\ 0 \le z \le y-\gamma\}$ and consider the very same linear transform T, see Figure 4 below.

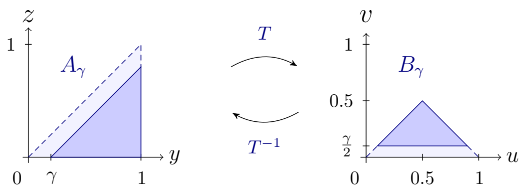

Figure 4: The restricted set A_γ forces a minimum amount of open-mindedness.

Then, $T(U)$ is uniform on the set $B_\gamma := \{(u,v),\ v \in [\frac{\gamma}{2}, \frac{1}{2}], u \in [v, 1-v]\}$ and the corresponding random density is still

$$f_{T(U)}(x) = \tfrac{1}{v} \cdot \big(1 - \tfrac{1}{v} \cdot \big|x-u\big|\big)^+, \quad x \in [0,1].$$

Depending on the values of $x \in [0, \frac{1}{2}]$ and $\gamma \in (0,1)$ – see Figure 5 for an illustration – quite cumbersome but nevertheless elementary calculations in the same vein as in the proof of Lemma 5.1 (which we will leave to the reader to perform) lead to the formulas stated above.

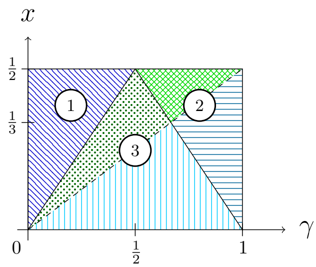

Figure 5: Different regimes for the form of $\varphi_\gamma(x)$.

The last claim follows again by the symmetry in x. □

Lemma 6.2
Consider $\mathbb{E}\,\eta_\gamma$ as in the previous lemma. Irrespectively of the value of $\gamma \in (0,1)$, the density function $\varphi_\gamma(x),\ x \in [0,1]$, corresponding to the intensity measure, is (strictly) increasing on $[0,\frac{1}{2}]$ and (strictly) decreasing on $[\frac{1}{2},1]$.

Proof: In principle, one could simply check the expressions for φ_γ given in Lemma 6.1. However, this simple fact can also be seen directly from the construction: Let us consider the random density in (9) to be generated by a vector $T=(U,V)$ that is chosen uniformly from B_γ, as described in the proof of Lemma 6.1. After having picked $V \sim \text{unif}([\frac{\gamma}{2},\frac{1}{2}])$, we take U to be uniform on $[V,1-V]$.

Consider $x_1,x_2 \in [0,\frac{1}{2}]$, such that $x_1 < x_2$, and $V=v$ to be already fixed. First note that $f_T(x_1) < f_T(x_2)$ for $U > \frac{x_1+x_2}{2} \in (0,\frac{1}{2})$. If $v \leq \frac{x_1+x_2}{2}$, symmetry around $\frac{x_1+x_2}{2}$ shows that $f_{(U,v)}(x_1)$ and $f_{(U,v)}(x_2)$ have the same distribution for U conditioned on $[v, x_1+x_2-v]$. In conclusion, we found $f_T(x_1) \prec f_T(x_2)$ and especially

$$\varphi_\gamma(x_1) = \mathbb{E}\, f_T(x_1) < \mathbb{E}\, f_T(x_2) = \varphi_\gamma(x_2).$$

Symmetry of φ_γ around $x=\frac{1}{2}$ implies the second part of the claim. □

Note that φ_γ can not be arbitrarily well approximated by the density of a restricted triangular distribution. Consequently, for $\varepsilon > 0$ sufficiently small, there can't be any ε-flat vertices with respect to the initial configuration as all triangular distributions have a positive distance to the intensity measure $\mathbb{E}\,\eta_\gamma$ bounded away from 0. For this reason, we have to go the same detour as in the proof of part (ii) of Thm. 2.2 in [6].

We need to verify that the density of the intensity measure actually can appear at a later time, more precisely be arbitrarily well approximated by the opinions that form when agents have interacted. This happens in fact for all positive values of the model parameter θ:

Lemma 6.3
Consider the Deffuant model on $\mathbb{Z}$ with arbitrary parameter $\theta \in (0,1]$ in which the initial opinions are i.i.d. absolutely continuous measures given by the random densities described in (9). Then, at any time $t > 0$ and for any $\varepsilon > 0$, a (sufficiently long) fixed finite section of the infinite path will hold opinions that are less than ε away from $\mathbb{E}\,\eta_\gamma$ in total variation distance (and be bounded by edges on which no Poisson events occurred up to time t) with positive probability.

Proof: Fix $\theta \in (0,1]$, $\varepsilon > 0$ and $t > 0$. The idea is to show that a set of agents with suitably assigned initial opinions can interact in such a way that at time t opinions close to $\mathbb{E}\,\eta_\gamma$ are formed.

Let us consider an i.i.d. sequence $(U_n)_{n\in\mathbb{N}}$ of random variables uniformly distributed on $A_\gamma = \{(y,z) \in \mathbb{R}^2,\ \gamma \leq y \leq 1,\ 0 \leq z \leq y-\gamma\}$, see Figure 4. Then we get the density corresponding to the initial opinion for agent $v \in \mathbb{N}$ by

$$f_0^{(v)} = f_{U_v} \tag{10}$$

where f_{U_v} is taken to be as in (9) and $f_t^{(v)}$ denotes the random density corresponding to the opinion of agent v at time t.

From Theorem 2.1 we know that with probability 1

$$\lim_{n\to\infty} \frac{1}{n} \sum_{v=1}^{n} f_0^{(v)}(x) = \varphi_\gamma(x) \tag{11}$$

uniformly in $x \in [0,1]$.

It is not hard to check that $\max\{|y_1 - y_2|, |z_1 - z_2|\} \leq \delta$ entails

$$\frac{1}{2} \int_0^1 \left| f_{(y_1,z_1)}(x) - f_{(y_2,z_2)}(x) \right| \mathrm{d}x \leq \frac{2\delta}{\gamma}, \tag{12}$$

in other words: If the coordinates of two vectors, (y_1, z_1) and (y_2, z_2), shaping restricted triangular distributions in the sense of (9) do not differ by more than δ, the total variation distance between the corresponding measures is at most $\frac{2\delta}{\gamma}$.

Fix $m \geq \frac{16}{\gamma\theta}$ and subdivide $[0,1]^2$ into m^2 squares. The standard SLLN implies that the fraction of $(U_n)_{n\in\mathbb{N}}$ landing in a square completely contained in A_γ a.s. tends to $\frac{2}{m^2(1-\gamma)^2} > 0$ as $n \to \infty$. We can therefore choose $N \in \mathbb{N}$ large enough such that, with positive probability, every square that is a subset of A_γ contains at least one of $(U_n)_{n=1}^N$ and

$$\left\| \frac{1}{N} \sum_{v=1}^{N} f_0^{(v)} - \varphi_\gamma \right\|_\infty \leq \varepsilon \tag{13}$$

Note that by symmetry under permutations, there is at least a chance of $\frac{1}{N!}$ that the agents $\{1, \dots, N\}$ are assigned these values from A_γ in such a way that those of neighboring agents do not differ much in both coordinates; more precisely, matching the values in a serpentine fashion as depicted in Figure 6 will keep discrepancies in the y-coordinate below $\frac{4}{m}$ and in the z-coordinate below $\frac{3}{m}$.

Putting things together, we found that for large enough N with non-zero probability the agents 1 through N have an initial configuration with a mean at total variation distance at most $\frac{\varepsilon}{2}$ to $\mathbb{E}\,\eta_\gamma$ and distance at most $\frac{8}{\gamma m} \leq \frac{\theta}{2}$ between neighbors.

Assume that there are no updates on the edges $\langle 0, 1\rangle$ and $\langle N, N+1\rangle$ up to time t. It is easy to check (by induction) that in this case, updates on the considered section in sweeps from left to right, i.e. first on $\langle 1, 2\rangle$, then $\langle 2, 3\rangle$ etc. until $\langle N-1, N\rangle$ repetitively, will keep the total variation distance of neighbors inside the section always below θ.

The following lemma finally verifies that a sufficiently large number of such sweeps will eventually bring the considered opinions within total variation distance $\frac{\varepsilon}{2}$ of their mean, due to the fact that the mean is preserved given that there are no updates on neither $\langle 0, 1\rangle$ nor $\langle N, N+1\rangle$.

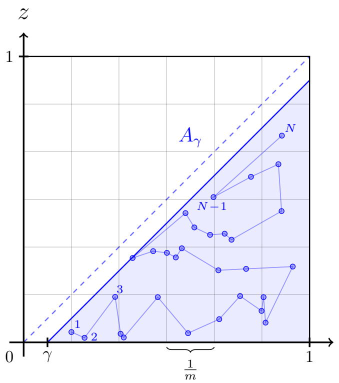

Figure 6: Finding suitable values for the initial opinions that can generate opinions close to the intensity measure $\mathbb{E}\,\eta_\gamma$ on a finite section.

Since the Poisson clocks and the initial configuration are independent, these two events coincide with positive probability and the claim is verified. $\square$

Lemma 6.4

If there are infinitely many (performed) updates along an edge, the total variation distance of the corresponding neighbors' opinions a.s. converges to 0.

Proof: This statement follows immediately using the energy idea used in the proofs of Thm. 2.3 and 5.3 in [5]: Consider

$$W_t(v) := \int_0^1 \left[f_t^{(v)}(x)\right]^2 \mathrm{d}x$$

to be the energy of vertex v at time t. When an update along the edge $\langle u, v\rangle$ is actually performed, i.e. the opinion values $\big(\eta_{t-}(u), \eta_{t-}(v)\big)$ get replaced by $\big(\eta_t(u), \eta_t(v)\big) = \big((1-\mu)\cdot\eta_{t-}(u) + \mu\cdot\eta_{t-}(v), (1-\mu)\cdot\eta_{t-}(v) + \mu\cdot\eta_{t-}(u)\big)$ energy is lost to the amount of

$$\begin{aligned}2\mu(1-\mu)\int_0^1 \left[f_{t-}^{(u)}(x) - f_{t-}^{(v)}(x)\right]^2 \mathrm{d}x &\geq 2\mu(1-\mu)\Big(\int_0^1 \left|f_{t-}^{(u)}(x) - f_{t-}^{(v)}(x)\right| \mathrm{d}x\Big)^2\\ &= 8\mu\,(1-\mu)\,\big\|\eta_{t-}(u) - \eta_{t-}(v)\big\|_{\mathrm{TV}}^2.\end{aligned}$$

As in Lemma 6.2 in [5], define the total energy $W_t^{\mathrm{tot}}(v)$ at v to be $W_t(v)$ plus the energy lost on $\langle v, v+1\rangle$ until time t. If we let $X(v)$ denote the random variable consisting of $\eta_0(v)$ and the Poisson process associated with the edge $\langle v, v{+}1\rangle$, $\{X(v),\ v\in\mathbb{Z}\}$ is an i.i.d. sequence. $W_t^{\mathrm{tot}}(0)$ is a measurable function of

$\{X(v),\ v \in \mathbb{Z}\}$ and $\{W_t^{\text{tot}}(v),\ v \in \mathbb{Z}\}$ its corresponding shifted equivalents. The well-known Pointwise Ergodic Theorem due to Birkhoff-Khinchin thus implies

$$\mathbb{E}\left[W_t^{\text{tot}}(0)\right] = \lim_{\substack{y\to-\infty \\ z\to\infty}} \frac{1}{z-y+1} \sum_{u=y}^{z} W_t^{\text{tot}}(u) \quad \text{a.s.}$$

Note that there are a.s. infinitely many edges on which no Poisson event has occured up to time t and that on a section between two such edges, the sum of total energies is preserved until t. Putting things together, we find

$$\begin{aligned}\mathbb{E}\left[W_t^{\text{tot}}(v)\right] &= \lim_{\substack{y\to-\infty \\ z\to\infty}} \frac{1}{z-y+1} \sum_{u=y}^{z} W_t^{\text{tot}}(u) \quad \text{a.s.} \\ &= \lim_{\substack{y\to-\infty \\ z\to\infty}} \frac{1}{z-y+1} \sum_{u=y}^{z} W_0^{\text{tot}}(u) \quad \text{a.s.} \\ &= \mathbb{E}\left[W_0^{\text{tot}}(v)\right] = \mathbb{E}\left[W_0(v)\right] = -\frac{8}{3\,(1-\gamma)}\left(1+\frac{\ln(\gamma)}{1-\gamma}\right).\end{aligned}$$

If we assume for contradiction that with positive probability for some $\delta > 0$ the total variation distance $\|\eta_t(v) - \eta_t(v+1)\|_{\text{TV}}$ lies in $[\delta, \theta]$ for arbitrarily large t, the conditional Borel-Cantelli lemma (see e.g. Cor. 6.20 in [9]) forces infinitely many performed updates with the total variation distance being at least δ. As the total energy is always non-negative, this implies $\lim_{t\to\infty} \mathbb{E}\left[W_t^{\text{tot}}(v)\right] = \infty$, a contradiction. □

Before we can use Lemma 6.3 to guarantee the existence of flat vertices at time $t > 0$, we need to check that (11) also holds for time $t > 0$.

Lemma 6.5
Given the Deffuant model as described in Lemma 6.3, for all $t \geq 0$, it holds that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{v=1}^{n} f_t^{(v)} = \varphi_\gamma$$

almost surely with respect to the supremum norm on $[0,1]$.

Proof: Fix $t > 0$. From Theorem 2.1 we know that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{v=1}^{n} f_0^{(v)} = \varphi_\gamma \quad \text{a.s.}$$

with respect to the supremum norm. Using the fact that the densities are uniformly bounded by $\frac{2}{\gamma}$ we can conclude that this convergence holds even for $t > 0$, by the same token as in [6]:

To the right of site 1, a.s. there is an infinite increasing sequence of nodes $(v_k)_{k\in\mathbb{N}}$, such that there was no Poisson event up to time t on the collection of edges $\{\langle v_k, v_k+1\rangle,\ k\in\mathbb{N}\}$. We denote the random lengths of the intervals in between by $L_k := v_{k+1} - v_k$, for $k\in\mathbb{N}$. In addition, let $L_0 := v_1 - v_0$ be the length of the interval including agent 1, where $\langle v_0, v_0+1\rangle$ is the first edge to the left of site 1 without Poisson event. Independence of the involved Poisson processes entails that $(L_k)_{k\in\mathbb{N}_0}$ is an i.i.d. sequence of random variables having geometric distribution on $\mathbb{N}$ with parameter e^{-t}.

Fix $\delta > 0$. Using the Borel-Cantelli lemma we find that the event

$$E_\delta := \{L_0 = \infty\} \cup \limsup_{k\to\infty} \left\{ L_k \geq k\cdot \tfrac{\delta\gamma}{2} \right\}$$

has probability 0.

The Deffuant model is mass-preserving in the sense that the sum of opinions of two interacting agents is always preserved. Therefore it holds for all $k\in\mathbb{N}$:

$$\sum_{u=v_0+1}^{v_k} f_0^{(v)} = \sum_{u=v_0+1}^{v_k} f_t^{(u)}.$$

Furthermore, for some $v\in\{v_k+1,\dots,v_{k+1}\}$, the event

$$\left\| \frac{1}{v-v_0} \sum_{u=v_0+1}^{v} f_0^{(u)} - \frac{1}{v-v_0} \sum_{u=v_0+1}^{v} f_t^{(u)} \right\|_\infty \geq \delta$$

forces $L_k \geq k\cdot\frac{\delta\gamma}{2}$, since $v_k \geq k$ and the density $f_s^{(u)}$ is non-negative and uniformly bounded by $\frac{2}{\gamma}$ for all $u\in\mathbb{Z}$ and times $s\geq 0$.

In conclusion, given ${E_\delta}^{\mathrm{c}}$, it holds that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{v=1}^{n} f_t^{(v)}(x) &= \lim_{n\to\infty} \frac{1}{n}\sum_{v=v_0+1}^{n} f_t^{(v)}(x) \leq \lim_{n\to\infty} \frac{1}{n}\sum_{v=v_0+1}^{n} f_0^{(v)}(x) + \delta \\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{v=1}^{n} f_0^{(v)}(x) + \delta = \varphi_\gamma(x) + \delta\end{aligned}$$

uniformly in $x\in[0,1]$. In the same way we get $\varphi_\gamma(x)-\delta$ as a lower bound and letting δ go to 0 finally verifies

$$\lim_{n\to\infty} \frac{1}{n}\sum_{v=1}^{n} f_t^{(v)} = \varphi_\gamma \quad \text{a.s.} \tag{14}$$

w.r.t. the supremum norm. □

Lemma 6.6
Given the Deffuant model as described in Lemma 6.3 and $\varepsilon > 0$, the following holds for all $t > 0$:

(i) With non-zero probability, there has been no Poisson event on the edge $\langle 0,1\rangle$ until time t and site 1 is ε-flat to the right with respect to the configuration $\{\eta_t(v)\}_{v\in\mathbb{Z}}$ and distance measure $\|\,.\,\|_{\mathrm{TV}}$.

(ii) With non-zero probability, site 0 is two-sidedly ε-flat with respect to the configuration $\{\eta_t(v)\}_{v\in\mathbb{Z}}$ and distance measure $\|\,.\,\|_{\mathrm{TV}}$.

PROOF: In order to verify these claims, we only have to put together the ingredients established in Lemmas 6.3 and 6.5. As in the proof of Thm. 2.2 in [6], we will do this by using a conditional variant of the coupling technique introduced in [12], that became known as *local modification* in percolation theory.

Fix $\varepsilon > 0$. Recall that for absolutely continuous measures μ and ν on $[0,1]$, with densities f and g respectively, we get $\|\mu-\nu\|_{\mathrm{TV}} \le \frac{1}{2}\cdot\|f-g\|_\infty$ and let B denote the following event:

$$\Big\{\Big\|\frac{1}{n}\sum_{v=1}^{n}\eta_t(v)-\mathbb{E}\,\eta_\gamma\Big\|_{\mathrm{TV}} \le \frac{\varepsilon}{3} \text{ and } \Big\|\frac{1}{n}\sum_{v=-n}^{-1}\eta_t(v)-\mathbb{E}\,\eta_\gamma\Big\|_{\mathrm{TV}} \le \frac{\varepsilon}{3}, \text{ for all } n\ge N\Big\}.$$

From Lemmas 6.3 and 6.5, we know that $N\in\mathbb{N}$ can be chosen sufficiently large such that $\mathbb{P}(B) > 1-\mathrm{e}^{-2t}$ and $\mathbb{P}(C\cap D) > 0$, where C denotes the event of no Poisson events on the two edges $\langle 0,1\rangle$ and $\langle N,N+1\rangle$ up to time t,

$$D := \big\{\|\eta_t(v)-\mathbb{E}\,\eta_\gamma\|_{\mathrm{TV}} \le \tfrac{\varepsilon}{3} \text{ for all } 1\le v\le N\big\}.$$

Additionally, since $\mathbb{P}(C) = \mathrm{e}^{-2t}$, we must have $\mathbb{P}(B\cap C) > 0$.

Now let $\boldsymbol{\eta}_t := \{\eta_t(v)\}_{v\in\mathbb{Z}}$ and $\boldsymbol{\eta}'_t := \{\eta'_t(v)\}_{v\in\mathbb{Z}}$ be the configurations at time t originated from two independent copies of the considered model. There is a strictly positive probability that $B\cap C$ happens for $\boldsymbol{\eta}_t$ as well as that $C\cap D$ happens for $\boldsymbol{\eta}'_t$. Given C, the hybrid process $\tilde{\boldsymbol{\eta}}_t$ defined by

$$\tilde{\eta}_t(v) = \begin{cases} \eta_t(v) & \text{if } v\notin\{1,\dots,N\} \\ \eta'_t(v) & \text{if } v\in\{1,\dots,N\}\end{cases}$$

is a perfectly fine copy of the model as well, showing that the event $B\cap D$ has non-zero probability. It is an easy exercise to check that $B\cap D$ actually implies the ε-flatness to the right of site 1.

In fact, the same argument applies to the second setting. Here, however, we choose C to be the event that there were no Poisson events on $\langle -N-1,-N\rangle$ and $\langle N,N+1\rangle$ as well as $D := \big\{\|\eta_t(v)-\mathbb{E}\,\eta_\gamma\|_{\mathrm{TV}} \le \frac{\varepsilon}{3} \text{ for all } -N\le v\le N\big\}$. Then the two-sidedly ε-flatness of site 0 follows from $B\cap D$. □

Let us now use Lemmas 6.4 and 6.6 to prove the main statement about the model featuring restricted random triangular distributions.

Theorem 6.7
Consider the Deffuant model on $\mathbb{Z}$, in which the total variation distance is used to measure the difference between two opinions and in which the initial opinions are given by independently assigned random restricted triangular distributions with fixed $\gamma\in(0,1)$ as described in (9). Then there is a sharp phase transition in the following sense: for $\theta\in[0,\theta_c)$, the system almost surely approaches disagreement in the long run; for $\theta\in(\theta_c,1]$, it almost surely approaches consensus.

The threshold θ_c is given by

$$\begin{aligned}\theta_c &= \frac{1}{2}\int_0^1 \Big|f_{(\gamma,0)}(x) - \varphi_\gamma(x)\Big|\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 \Big|\tfrac{2}{\gamma}\cdot\big(1-\tfrac{2}{\gamma}\cdot\big|x-\tfrac{\gamma}{2}\big|\big)^+ - \varphi_\gamma(x)\Big|\,\mathrm{d}x.\end{aligned} \tag{15}$$

PROOF: As mentioned in Section 4, we will closely follow the ideas in [5] to establish this result, just now the opinions are given by (random) absolutely continuous measures, or rather their density functions. In fact, most of the work has already been done by showing Lemma 6.6. Let us define

$$\theta_c := \frac{1}{2}\int_0^1 \Big|f_{(\gamma,0)}(x) - \varphi_\gamma(x)\Big|\,\mathrm{d}x$$

and $\varepsilon := \frac{1}{7}\cdot|\theta_c - \theta|$. In the sequel, we will consider the two regimes: the subcritical one ($\theta < \theta_c$) and the supercritical one ($\theta > \theta_c$).

In the subcritical regime, fix $t > 0$ and let B denote the event that there are no Poisson events neither on $\langle -1, 0\rangle$ nor on $\langle 0, 1\rangle$ during $[0, t]$ and site -1 is ε-flat to the left, site 1 is ε-flat to the right with respect to the configuration $\{\eta_t(v)\}_{v\in\mathbb{Z}}$. By Lemma 6.6 part (i), the obvious symmetry and conditional independence, we know that B occurs with positive probability. Let the initial opinion of agent 0 be given by f_{U_0} in the sense of (10) and C be the event that the first coordinate of U_0 is less than $\gamma + \frac{\varepsilon\gamma}{2}$, which by the shape of A_γ (see Figure 4) and (12) entails

$$\big\|\eta_0(0) - \mathbb{E}\,\eta_\gamma\big\|_{\mathrm{TV}} \geq \theta_c - \varepsilon.$$

Given that there are no Poisson events on edges incident to site 0, we can (again by local modification) conclude that $B\cap C$ has positive probability. From Lemma 4.3, we know that the total variation distance between $\eta_s(1)$ and the intensity measure $\mathbb{E}\,\eta_\gamma$ will not exceed ε for $s \geq t$ due to its one-sided ε-flatness if there is no interaction with site 0 (same for $\eta_s(-1)$). However, given $B\cap C$ the opinions $\eta_t(1)$ and $\eta_t(0) = \eta_0(0)$ are at distance larger than $\theta_c - 2\varepsilon > \theta$ and hence they never will be close enough to interact, since the same holds for $\eta_t(-1)$, which leaves the opinion at site 0 unchanged for all time. In other words, with non-zero probability the edge $\langle 0, 1\rangle$ will be finally blocked (same for $\langle -1, 0\rangle$).

To conclude the claimed almost sure behavior, we apply the ergodicity argument used in the proof of Lemma 6.4 once again: Whether the configuration approaches disagreement or not can be checked given the initial configuration plus all Poisson processes associated to the edges. The sequence $\{X(v),\ v\in\mathbb{Z}\}$ (as defined in the proof of Lemma 6.4) is i.i.d., hence ergodic with respect to shifts. Thus the translation-invariant event "disagreement" necessarily has to be trivial, i.e. must have probability either 0 or 1. Since we already showed that its probability is non-zero, the event has to be an almost sure one in the subcritical regime.

In the supercritical case, we know that at time $t > 0$ any fixed site is two-sidedly ε-flat with positive probability (part (ii) of Lemma 6.6). Lemma 6.2 implies that the largest total variation distance of a restricted triangular distribution as defined in (9) to the intensity measure $\mathbb{E}\,\eta_\gamma$ is given by what we defined as θ_c. Since all opinions at a later time are convex combinations of the initial ones, this distance can not be exceeded. If site 0 is two-sidedly ε-flat with respect to the configuration $\{\eta_t(v)\}_{v\in\mathbb{Z}}$, the opinion $\eta_s(0)$ will be at total variation distance at most 6ε to $\mathbb{E}\,\eta_\gamma$, for all $s \geq t$ (part (ii) of Lemma 4.3). This implies that its neighboring opinions differ by not more than $\theta_c + 6\varepsilon < \theta$, hence Lemma 6.4 forces the differences to converge to 0. By induction, this is actually true for any pair of neighbors. Again, ergodicity ensures that this consensus behavior occurs with probability 1.

It further implies the almost sure existence of two-sidedly ε-flat vertices for any strictly positive value of ε. For this reason, the measure, the opinions converge to, must be the intensity measure, which concludes the proof. □

Example 6.8
Let us consider the Deffuant model on $\mathbb{Z}$ with opinions being absolutely continuous probability distributions and the initial ones given by random restricted triangular distributions with parameter $\gamma = \frac{1}{3}$. From Lemma 6.1 we know that the corresponding intensity measure has the somewhat cumbersome density

$$\varphi_{\frac{1}{3}}(x) = \begin{cases} -18\left[(1-x)\,\ln(1-x) - \frac{3}{2}\,x^2 + x\right], & 0 \leq x \leq \frac{1}{6} \\ -18\left[(1-x)\,\ln(1-x) - x\,\ln(6x) + x + \frac{3}{2}\,x^2 - \frac{1}{12}\right], & \frac{1}{6} \leq x \leq \frac{1}{3} \\ -18\left[(1-x)\,\ln(1-x) + x\,(1-\ln(2)) + \frac{1}{12}\right], & \frac{1}{3} \leq x \leq \frac{1}{2} \\ -18\left[x\,\ln(x) + (1-x)\,(1-\ln(2)) + \frac{1}{12}\right], & \frac{1}{2} \leq x \leq \frac{2}{3} \\ -18\left[x\,\ln(x) - (1-x)\,\ln(6-6x) + \frac{3}{2}\,x^2 - 4x + \frac{29}{12}\right], & \frac{2}{3} \leq x \leq \frac{5}{6} \\ -18\left[x\,\ln(x) - \frac{3}{2}\,x^2 + 2x - \frac{1}{2}\right], & \frac{5}{6} \leq x \leq 1. \end{cases}$$

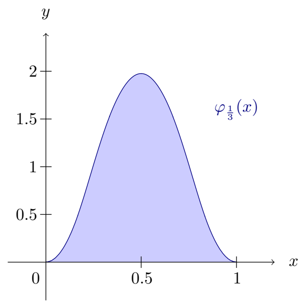

Figure 7: The intensity measure is concentrated more towards the center if the random triangular distributions are restricted to a minimum width.

If the total variation distance is used to measure the disparity of two opinions, we can conclude from Theorem 6.7 that the threshold θ_c for this model takes on the value

$$\theta_c = \frac{1}{2} \int_0^1 \left| f_{\left(0,\frac{1}{3}\right)}(x) - \varphi_{\frac{1}{3}}(x) \right| \, dx \approx 0.83172.$$

7 Alternative choices (concerning initial configuration and distance measure) and inhomogeneous open-mindedness

No doubt that the extension of the Deffuant model on $\mathbb{Z}$ to measure-valued opinions leaves a wide range of possible laws for the initial configuration to be examined. We saw trivial behavior for triangular distributions and a non-trivial phase transition for restricted triangular distributions.

If we want to stick to absolutely continuous measures on a compact support S and $\|\,.\,\|_{\mathrm{TV}}$ as distance measure, the line of argument from Section 6 will in principle carry over on condition that

$$\mathbb{E} \int_S [f_\omega(x)]^2 \, dx < \infty,$$

where f_ω, $\omega \in \Omega$, denotes the random initial density shaped by a probability space $(\Omega, \mathbb{P})$, and that the total variation distance of an initial opinion to the intensity measure is a.s. bounded away from 1. The first condition is needed to prove Lemma 6.4, the latter will in fact give the threshold θ_c, as essential supremum of the total variation distance of initial opinions to the intensity measure. There is however one issue, that must not be overlooked: In order to establish Lemma 6.3, we needed that there are no major gaps in the support of the initial opinions – just as in the case of finite-dimensional opinions. To examine this problem more closely, elaborate geometric considerations as in [7] seem to be necessary and we will thus leave this for future studies.

If one wants to include point processes as opinions, in many situations the total variation distance will not work as a meaningful measure for the discrepancy of two opinions, at least if the support of two such processes is disjoint with positive probability. Using the Lévy-distance instead could however lead to interesting models in such a setting. In fact, even in the case of triangular distributions, it seems to be unrealistic that two determined agents are at maximal distance, whether the intervals, on which their opinions are concentrated, are in close proximity or at different ends of the spectrum. From this point of view, albeit more difficult to handle, the Lévy-distance appears to be a more suitable choice.

Finally, it might be interesting to point out the new feature of the model (compared to finite-dimensional opinions) that was mentioned in the beginning of Section 5 once more and put it into a broader context: In the Deffuant model with triangular distributions, besides the common tolerance parameter θ and willingness to compromise μ, we have a diversified scale of open-mindedness of the agents shaped by the random support of their initial opinion.

There have been attempts, e.g. by Deffuant et al. in [14], to simulate the long-term behavior of a variant of the model in which the agents have different θ-values. On the analytical side, this is quite a crucial change since it brings along situations in which the opinion of only one of two interacting neighbors is updated. Then the sum of opinions is no longer preserved, which renders void many of our central arguments. Another advance in the same direction is the so-called *relative agreement model*, introduced in [2]. There, the bounded confidence rule is dropped and replaced by a continuous counterpart: Agents feature both a real-valued opinion and a separate value corresponding to their individual uncertainty, which taken together shape a dispersed opinion in the form of a symmetric interval of length two times the uncertainty around the opinion value. If an agent gets influenced by another, the impact depends on the overlap of the two opinion intervals relative to the length of the interval corresponding to the influenced agent. Again, the asymmetric way of updating opinion values makes the relative agreement model, although based on the same principles and ideas, qualitatively quite different.

On the modelling side, the way inhomogeneous open-mindedness or uncertainty is incorporated in our extension of the model does not only avoid this issue, but also lead to the realistic property that agents themselves become more open-minded by interacting with open-minded neighbors.

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Water transport on graphs

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Water transport on graphs

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Abstract

If the nodes of a graph are considered to be identical barrels – featuring different water levels – and the edges to be (locked) water-filled pipes in between the barrels, one might consider the optimization problem of how much the water level in a fixed barrel can be raised with no pumps available, i.e. by opening and closing the locks in an elaborate succession. This problem originated from the analysis of an opinion formation process and proved to be not only sufficiently intricate in order to be of independent interest, but also algorithmically complex, namely NP-hard. We deal with both finite and infinite graphs as well as deterministic and random initial water levels and find that an infinite path, due to its leanness, behaves much more like a finite graph in this respect.

Keywords: Water transport, graph algorithms, optimization, complexity, infinite path.

1 Introduction

Imagine a plane on which rainwater is collected in identical rain barrels, some of which are connected through pipes (that are already water-filled). All the pipes feature locks that are normally closed. If a lock is opened, the contents of the two barrels which are connected via this pipe start to level, see Figure 1. If one waits long enough, the water levels in the two barrels will be exactly the same, namely lie at the average $\frac{a+b}{2}$ of the two water levels (a and b) before the pipe was unlocked.

After a rainy night in which all of the barrels accumulated a certain amount of precipitation we might be interested in maximizing the water level in one fixed barrel by opening and closing some of the locks in carefully chosen order.

In order to mathematically model the setting, consider an undirected graph $G = (V, E)$, which is either finite or infinite with bounded maximum degree.

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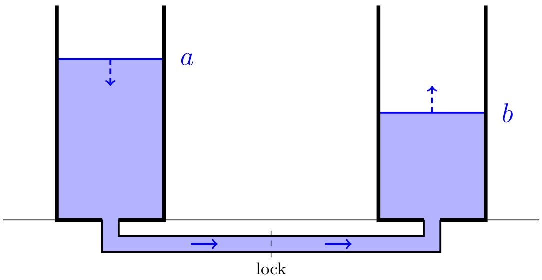


Figure 1: Levelling water stages after just having opened a lock.

Furthermore, we can assume without loss of generality that G is connected and simple, that means having neither loops nor multiple edges. Every vertex is understood to represent one of the barrels and the pipes correspond to the edges in the graph. The barrels themselves are considered to be identical, having a fixed capacity $C > 0$.

Given some initial profile $\{\eta_0(u)\}_{u\in V} \in [0, C]^V$, the system is considered to evolve in discrete time and in each round we can open one of the locked pipes and transport water from the fuller barrel into the emptier one. If we stop early, the two levels might not have completely balanced out giving rise to the following update rule for the water profile: If in round k the pipe $e = \langle x, y\rangle$ connecting the two barrels at sites x and y, with levels $\eta_{k-1}(x) = a$ and $\eta_{k-1}(y) = b$ respectively, is opened and closed after a certain period of time, we get

$$\begin{aligned} \eta_k(x) &= a + \mu_k\,(b - a) \\ \eta_k(y) &= b + \mu_k\,(a - b) \end{aligned} \tag{1}$$

for some $\mu_k \in [0, \frac{1}{2}]$, which we assume can be chosen freely by appropriate choice of how long the pipe is left open. All other levels stay unchanged, i.e. $\eta_k(w) = \eta_{k-1}(w)$ for all $w \in V \setminus \{x, y\}$.

The quantity of interest is then defined as follows:

Definition 1
For a graph $G = (V, E)$, an initial water profile $\{\eta_0(u)\}_{u\in V}$ and a fixed vertex $v \in V$ (the *target vertex*), let a *move sequence* be given by a list of edges and time spans that determines which pipes are opened (in chronological order) and for how long. Let then $\kappa(v)$ be defined as the supremum over all water levels that are achievable at v with move sequences consisting of finitely many rounds, i.e.

$$\kappa(v) := \sup\{r \in \mathbb{R}, \text{ there exists } T \in \mathbb{N}_0 \text{ and a move sequence s.t. } \eta_T(v) = r\}.$$

Readers familiar with mathematical models for social interaction processes might note that (1) basically looks like the update rule in the opinion formation process given by the so-called *Deffuant model* for consensus formation in social networks (as described in the introduction of [5]), only μ can change from update to update and the bounded confidence restriction is omitted. This however is no coincidence: The situation described in the context above arises naturally in the analysis of the Deffuant model where the question is how extreme an opinion can a fixed agent possibly get given an initial opinion profile on a specified network graph, if the interactions take place appropriately.

In order to tackle this question, Häggström [4] invented a non-random pairwise averaging procedure, which he proposed to call *Sharing a drink* (SAD). This procedure – which is the main focus of the second section – was originally considered on the (two-sidedly) infinite path only, i.e. the graph $G = (V, E)$ with $V = \mathbb{Z}$ and $E = \{\langle v, v+1\rangle,\ v \in \mathbb{Z}\}$, but can immediately be generalized to any graph (see Definition 2) and is dual to the water transport described above in a sense to be made precise in Lemma 2.1.

In Section 3, we will deal with the water transport problem on finite graphs. After formally introducing the idea of optimal move sequences, we investigate both their essential building blocks and the effect of simultaneously opened pipes. In subsection 3.3, being a collection of examples, we will in fact deal with both situations – the one in which we consider the initial water levels to be deterministic and the other in which they are random. In the latter case $\kappa(v)$ obviously becomes a random variable as well since it strongly depends on the initial profile. On non-transitive graphs (see Definition 8) its distribution can moreover depend on the chosen vertex v – even for i.i.d. initial water levels, see Example 3.2.

In the fourth section, we extend the complexity consideration touched upon in some of the examples from Section 3. We show that it is an NP-hard problem to determine $\kappa(v)$ for a given finite graph, target vertex v and initial water profile in general, something that might be considered as a valid excuse for the fact that we are unable to give a neat general solution when it comes to optimal move sequences in the water transport problem on finite graphs, as dealt with in Section 3.

As opposed to the two precedent sections, Section 5 is devoted to infinite graphs. We consider i.i.d. initial water levels (with a non-degenerate marginal distribution) and detect a remarkable change of behavior: On an infinite path, the highest achievable water level at a fixed vertex depends on the initial profile in the sense that it has a non-degenerate distribution, just like on any finite graph. If the infinite graph contains a neighbor-rich half-line (see Definition 7), however, this dependence becomes degenerate: For any vertex $v \in V$, the value $\kappa(v)$ almost surely equals the essential supremum of the marginal distribution. This fact makes the two-sidedly infinite path quite unique: It constitutes the only exception among all infinite quasi-transitive graphs, to the effect that $\kappa(v)$ is a non-degenerate random variable – an observation which is captured in the last theorem: the nonetheless central Theorem 5.3.

2 Connection to the SAD-procedure

Let us first repeat the formal definition of the SAD-procedure:

Definition 2
For a graph $G = (V, E)$ and some fixed vertex $v \in V$, define $\{\xi_0(u)\}_{u\in V}$ by setting

$$\xi_0(u) = \delta_v(u) := \begin{cases} 1 & \text{for } u = v \\ 0 & \text{for } u \neq v. \end{cases}$$

In each time step, an edge $\langle x, y\rangle$ is chosen and the profile $\{\xi_0(u)\}_{u\in V}$ updated according to the rule (1) with $\{\xi_k(u)\}_{u\in V}$ in place of $\{\eta_k(u)\}_{u\in V}$. One can interpret this procedure as a full glass of water initially placed at vertex v (all other glasses being empty), which is then repeatedly shared among neighboring vertices by each time step choosing a pair of neighbors and pouring a μ_k-fraction of the difference from the glass containing more water into the one containing less. Let us refer to this interaction process as *Sharing a drink (SAD)*.

Just as in [4], the SAD-procedure can be used to describe the composition of the contents in the water barrels after finitely many rounds of opening and closing pipe locks. The following lemma corresponds to Lemma 3.1 in [4], but since the two dual processes (water transport and SAD) evolve in discrete time in our setting, the proof simplifies somewhat.

Lemma 2.1
Consider an initial profile of water levels $\{\eta_0(u)\}_{u\in V}$ on a graph $G = (V, E)$ and fix a vertex $v \in V$. For $T \in \mathbb{N}_0$ define the SAD-procedure that starts with $\xi_0(u) = \delta_v(u)$ (see Definition 2) and is dual to the chosen move sequence in the water transport problem in the following sense: If in round $k \in \{1, \dots, T\}$ the water profile is updated according to (1), the update in the SAD-profile at time $T-k \in \{0, \dots, T-1\}$ takes place along the same edge and with the same choice of μ_k. Then we get

$$\eta_T(v) = \sum_{u\in V} \xi_T(u)\, \eta_0(u). \tag{2}$$

Proof: We prove the statement by induction on T. For $T = 0$, the statement is trivial and there is nothing to show. For the induction step fix $T \in \mathbb{N}$ and assume the first pipe opened to be $e = \langle x, y\rangle$. According to (1) we get

$$\eta_1(u) = \begin{cases} \eta_0(u) & \text{if } u \notin \{x, y\} \\ (1-\mu_1)\,\eta_0(x) + \mu_1\,\eta_0(y) & \text{if } u = x \\ (1-\mu_1)\,\eta_0(y) + \mu_1\,\eta_0(x) & \text{if } u = y. \end{cases}$$

Let us consider $\{\eta_1(u)\}_{u\in V}$ as some initial profile $\{\eta'_0(u)\}_{u\in V}$. By induction hypothesis we get

$$\begin{aligned}\eta'_{T-1}(v) &= \sum_{u\in V} \xi'_{T-1}(u)\,\eta'_0(u)\\ &= \sum_{u\in V\setminus\{x,y\}} \xi'_{T-1}(u)\,\eta_0(u) + \Big((1-\mu_1)\,\xi'_{T-1}(x) + \mu_1\,\xi'_{T-1}(y)\Big)\,\eta_0(x)\\ &\quad + \Big((1-\mu_1)\,\xi'_{T-1}(y) + \mu_1\,\xi'_{T-1}(x)\Big)\,\eta_0(y),\end{aligned}$$

where $\eta'_{T-1}(v) = \eta_T(v)$ and $\{\xi'_k(u)\}_{u\in V}$, $0 \le t \le T-1$, is the SAD-procedure corresponding to the move sequence after round 1. As by definition the SAD-procedure ξ arises from ξ' by adding an update at time $T-1$ along edge e with parameter μ_1, we find $\xi_k(u) = \xi'_k(u)$ for all $k \in \{0,\dots,T-1\}$ and $u \in V$ as well as

$$\xi_T(u) = \begin{cases} \xi_{T-1}(u) = \xi'_{T-1}(u) & \text{if } u \notin \{x,y\}\\ (1-\mu_1)\,\xi_{T-1}(x) + \mu_1\,\xi_{T-1}(y) & \text{if } u = x\\ (1-\mu_1)\,\xi_{T-1}(y) + \mu_1\,\xi_{T-1}(x) & \text{if } u = y,\end{cases}$$

which establishes the claim. □

In the following sections, we want to consider not only deterministic but also random initial profiles of water levels. Having this mindset already, it might be useful to halt for a moment and realize that the statement of Lemma 2.1 deals with a deterministic duality that does not involve any randomness (once the initial profile and the move sequence are fixed).

Before we turn to the task of rising water levels, let us prepare two more auxiliary results. The first one follows directly from the energy argument that was used in the proof of Thm. 2.3 in [4]:

Lemma 2.2
Given an initial profile of water levels $\{\eta_0(u)\}_{u\in V}$ on a graph $G = (V, E)$, fix a finite set $A \subseteq V$ and a set $E_A \subseteq E$ of edges inside A that connects A. If we open the pipes in E_A – and no others – in repetitive sweeps for times long enough such that $\mu_k \ge \varepsilon$ for some fixed $\varepsilon > 0$ in each round (cf. (1)), then the water levels inside the set A approach a balanced average, i.e. converge to the value $\frac{1}{|A|}\sum_{v\in A}\eta_0(v)$. The corresponding dual SAD-profiles started with $\xi_0(u) = \delta_v(u)$, $u \in V$, converge uniformly to $\frac{1}{|A|}\,\delta_A$ for all $v \in A$.

Proof: Let us define the energy after round k inside A by

$$W_k(A) = \sum_{v\in A} \big(\eta_k(v)\big)^2.$$

A short calculation reveals that an update of the form (1) reduces the energy by $2\mu_k^2\,(b-a)^2$, where the updated water levels were a and b respectively. If μ_k is bounded away from 0, the fact that $W_k(A) \ge 0$ for all k entails that the difference in water levels $|b - a|$ before a pipe is opened can be larger than any fixed positive value only finitely many times. In effect, since any pipe in E_A is opened repetitively we must have $|\eta_k(u) - \eta_k(v)| \to 0$ as $k \to \infty$ for all edges $\langle u, v\rangle \in E_A$. As the updates are average preserving, the first part of the claim follows from the fact that E_A connects A.

The second part of the lemma follows by applying the same argument to the dual SAD-procedure. □

The following lemma constitutes an extremely narrowed variant of Thm. 2.3 in [4] which applies to graphs more complex than paths as well and will come in useful in Example 3.3:

Lemma 2.3
Fix a (connected) graph $G = (V, E)$ and a vertex $v \in V$. For any $w \in V \setminus \{v\}$, the supremum of $\xi_k(w)$ taken over all times k and SAD-procedures started with $\xi_0(u) = \delta_v(u)$, $u \in V$, is less than or equal to $\frac{1}{2}$.

PROOF: If the SAD-procedure is started with a full glass of water at $v \neq w$, the assumption that the amount at w can rise above $\frac{1}{2}$ leads to the following contradiction: Assume k to be the first time s.t. $\xi_k(w) > \frac{1}{2}$. Then in round k node w necessarily pooled the water with some neighbor u, that had more water than w. But since this relation is preserved by an update, it implies

$$\xi_k(w) + \xi_k(u) \geq 2\,\xi_k(w) > 1,$$

which is impossible as the amount of water shared always sums to 1. □

To round off these preliminary considerations, let us collect some results about SAD-profiles from [4] – partly already mentioned – into a single lemma for convenience.

Lemma 2.4
Consider the SAD-procedure on a path, started in vertex v, i.e. with $\xi_0(u) = \delta_v(u)$, $u \in V$.

(a) The SAD-profiles achievable on paths are all unimodal.

(b) If the vertex v only shares the water to one side, it will remain a mode of the SAD-profile.

(c) The supremum over all achievable SAD-profiles started with δ_v at another vertex w equals $\frac{1}{d+1}$, where d is the graph distance between v and w.

The results in [4] actually all deal with the two-sidedly infinite path, but it is evident how the arguments used immediately transfer to finite paths. Part (a) hereby corresponds to Lemma 2.2 in [4], part (b) to Lemma 2.1 and part (c) to Thm. 2.3. The argument Häggström [4] used to prove the statement in (c) for the two-sidedly infinite path can in fact be generalized to prove the result for trees without much effort, as was done by Shang (see Prop. 6 in [7]).

In fact, we believe that not only the cut back statement from Lemma 2.3 but also the natural generalization of Thm. 2.3 in [4] holds true for general graphs. Our attempts to prove the generalization to non-tree graphs have, however, turned out unsuccessful.

3 Water transport on finite graphs

In this section, we consider the underlying network to be finite, i.e. $|V| = n \in \mathbb{N}$. In order to increase the water level at our fixed site v one could in principle start by greedily trying to connect the barrels with the highest water levels to the one at v. However, optimizing this idea is far from being trivial. Let us first define optimal move sequences and then reveal some properties and building blocks that they share.

Definition 3
For fixed $v \in V$ and a given initial water profile $\{\eta_0(u)\}_{u\in V}$ let $\varphi \in (E\times[0,\frac{1}{2}])^T$, where $\varphi_k = (e_k, \mu_k)$, be called a *finite move sequence* if $T \in \mathbb{N}_0$. φ is a *finite optimal move sequence* if opening the pipes $e_1, \dots, e_T$ in chronological order, each for the period of time that corresponds to μ_k in (1), will lead to the final value $\eta_T(v) = \kappa(v)$.

For any move sequence $\varphi \in (E \times [0, \frac{1}{2}])^T$, we will denote by $\{\xi_T(u)\}_{u\in V}$ the SAD-profile that corresponds to φ via the duality laid down in Lemma 2.1.

If no finite optimal move sequence exists, let us call $\Phi = \{\varphi^{(m)},\ m \in \mathbb{N}\}$ an *optimal meta-sequence of moves*, provided that $\varphi^{(m)} \in (E \times [0, \frac{1}{2}])^{T_m}$ is a finite move sequence for each $m \in \mathbb{N}$, achieving $\eta_{T_m}(v) > \kappa(v) - \frac{1}{m}$ and the SAD-profiles $\{\xi_{T_m}(u)\}_{u\in V}$ dual to $\varphi^{(m)}$ converge pointwise to a limit $\{\xi(u)\}_{u\in V}$ as $m \to \infty$.

It is tempting to assume that in the case where no finite optimal move sequence exists, we could get away with an infinite move sequence instead of a sequence of finite move sequences Φ as described above. However this is not the case, see Example 3.6.

Lemma 3.1
Take the network $G = (V, E)$ to be finite, and fix the target vertex v as well as the initial water profile. Then the existence of an optimal move (meta-)sequence is guaranteed and the following simplification will not change its performance: In an optimal move (meta-)sequence, without loss of generality we can assume $\mu_k = \frac{1}{2}$ for all k.

Proof: By the very definition of $\kappa(v)$, the existence of an optimal finite or meta-sequence of moves is guaranteed: Let $A \subseteq [0,1]^V$ denote the set of achievable SAD-profiles. Its closure $\overline{A}$ in $([0,1]^V, \|\,.\,\|_2)$ is bounded and therefore compact. Given the initial water profile $\{\eta_0(u)\}_{u\in V}$, the function

$$f := \begin{cases} [0,1]^V \to [0, C] \\ \{\xi(u)\}_{u\in V} \mapsto \sum\limits_{u\in V} \xi(u)\,\eta_0(u) \end{cases}$$

is continuous. Hence there exists a closed subset F of $\overline{A}$ on which f achieves its maximum $\kappa(v)$ over $\overline{A}$. The SAD-profiles dual to finite optimal move sequences are given by $F \cap A$. If $F \cap A = \emptyset$ and $\Phi = \{\varphi^{(m)},\ m \in \mathbb{N}\}$ is a collection of finite move sequences s.t. $\varphi^{(m)} \in (E \times [0, \frac{1}{2}])^{T_m}$ and $\eta_{T_m}(v) > \kappa(v) - \frac{1}{m}$ for all $m \in \mathbb{N}$, we can assume without loss of generality that the corresponding SAD-profiles $\{\xi_{T_m}(u)\}_{u \in V}$ have a limit $\{\xi(u)\}_{u \in V}$ (by passing on to a subsequence if necessary) as $\overline{A}$ is compact. This turns Φ into an optimal meta-sequence of moves and the limit of its dual SAD-profiles necessarily lies in F.

Assume now that the first move in a finite sequence $\varphi \in (E \times [0, \frac{1}{2}])^T$ is to open the lock on pipe $e_1 = \langle x, y \rangle$ for a time corresponding to $\mu_1 \in [0, \frac{1}{2}]$ in (1).

Without loss of generality we can assume $\eta_0(x) \geq \eta_0(y)$ (which in turn implies $\eta_1(x) \geq \eta_1(y)$). If we look at the SAD-profile $\{\xi'_{T-1}(u)\}_{u \in V}$ corresponding to $\varphi' := (\varphi_2, \ldots, \varphi_T) \in (E \times [0, \frac{1}{2}])^{T-1}$ – in effect we look at the outcome of the move sequence after the first step applied to the new initial water profile $\{\eta_1(u)\}_{u \in V}$ – we can distinguish two cases: either $\xi'_{T-1}(x) \geq \xi'_{T-1}(y)$ or $\xi'_{T-1}(x) < \xi'_{T-1}(y)$. In the first case changing μ_1 to 0, i.e. erasing the first move will not decrease the water level finally achieved at v, see (2). In the second case the same holds for changing μ_1 to $\frac{1}{2}$. Since we can consider any step in the move sequence to be the first one applied to the intermediate water profile achieved so far, this establishes the claim for finite optimal move sequences.

As any finite move sequence can be simplified in this way without worsening its outcome, the argument applies to the elements of a sequence $\Phi = \{\varphi^{(m)},\ m \in \mathbb{N}\}$ of finite move sequences and thus to an optimal meta-sequence as well. □

3.1 Macro moves

When it comes to the opening and closing of pipes, it is not self-evident how far things change if we allow pipes to be opened simultaneously. First of all one has to properly extend the model laid down in (1) by specifying how the water levels behave when more than two barrels are connected at the same time. In order to keep things simple, let us assume that the pipes are all short enough and of sufficient diameter such that we can neglect all kinds of flow effects. Moreover, let us take the dynamics to be as crude as can be by assuming that the water levels of the involved barrels approach their common average in a linear and proportional fashion, which is made more precise in the following definition.

Definition 4
Given a graph $G = (V, E)$, let $A \subseteq V$ be a set of at least 3 nodes and $E_A \subseteq E$ a set of edges inside A that connects A. A *macro move* on E_A (or simply A) will denote the action of opening all pipes that correspond to edges in E_A in some round k simultaneously and will – analogously to (1) – change the water levels for all vertices $u \in A$ to

$$\eta_k(u) = (1 - 2\mu_k)\, \eta_{k-1}(u) + 2\mu_k\, \overline{\eta}_{k-1}(A), \quad \text{where } \overline{\eta}_{k-1}(A) = \frac{1}{|A|} \sum_{w \in A} \eta_{k-1}(w)$$

is the average over the set A after round $k-1$ and $\mu_k \in [0, \frac{1}{2}]$.

First of all, Lemma 2.1 transfers immediately and almost verbatim to move sequences including macro moves: In a move sequence with a macro move on the set A in the first round, we get the water levels

$$\eta_1(u) = \begin{cases} \eta_0(u) & \text{if } u \notin A \\ (1-2\mu_1)\,\eta_0(u) + 2\mu_1\,\overline{\eta}_0(A) & \text{if } u \in A. \end{cases}$$

If $\{\xi_{T-1}(u),\ u \in V\}$ and $\{\xi_T(u),\ u \in V\}$ are such that

$$\eta_T(v) = \sum_{u \in V} \xi_T(u)\,\eta_0(u) = \sum_{u \in V} \xi_{T-1}(u)\,\eta_1(u),$$

we find by comparing the coefficient of $\eta_0(u)$

$$\xi_T(u) = \begin{cases} \xi_{T-1}(u) & \text{if } u \notin A \\ (1-2\mu_1)\,\xi_{T-1}(u) + \sum_{w \in A} 2\mu_1\,\frac{\xi_{T-1}(w)}{|A|} & \text{if } u \in A, \end{cases}$$

which is the SAD-profile originating from the very same macro move applied to $\{\xi_{T-1}(u),\ u \in V\}$. With this tool in hand, we can prove the following extension of Lemma 3.1:

Lemma 3.2
Take the network $G = (V, E)$ to be finite, and fix the target vertex v as well as the initial water profile.

(a) Even if we allow macro moves, the statement of Lemma 3.1 still holds true, i.e. reducing the range of μ_k from $[0, \frac{1}{2}]$ to $\{0, \frac{1}{2}\}$ in each round k does not worsen the outcome of optimal move (meta-)sequences.

(b) The sharp upper bounds on achievable water levels are not changed if we allow for pipes to be opened simultaneously. In other words, the supremum $\kappa(v)$ of water levels achievable at a vertex v, as characterized in Definition 1, stays unchanged if we allow move sequences to include macro moves.

Proof:

(a) Just as in Lemma 3.1, we consider a move sequence consisting of finitely many (macro) moves – say again $T \in \mathbb{N}$ – and especially the SAD-profile dual to the moves after round 1, denoted by $\{\xi_{T-1}(u),\ u \in V\}$. If the first action is a macro move on the set A, let us divide its nodes into two subsets according to whether their initial water level is above or below the initial average across A:

$$A_a := \{u \in A,\ \eta_0(u) \geq \overline{\eta}_0(A)\} \quad \text{and} \quad A_b := \{u \in A,\ \eta_0(u) < \overline{\eta}_0(A)\}.$$

If $\sum_{u \in A_a} \xi_{T-1}(u) \leq \sum_{u \in A_b} \xi_{T-1}(u)$, changing μ_1 to $\frac{1}{2}$ will not decrease the final water level achieved at v. If instead $\sum_{u \in A_a} \xi_{T-1}(u) \geq \sum_{u \in A_b} \xi_{T-1}(u)$, the same holds for erasing the first move (i.e. setting $\mu_1 = 0$).

(b) Obviously, allowing for pipes to be opened simultaneously can if anything increase the maximal water level achievable at v. However, any such macro move can be at least approximated by opening pipes one after another. Levelling out the water profile on a set of more than 2 vertices completely will correspond to the limit of infinitely many single pipe moves on the edges between them (in a sensible order).

Let us consider a finite move sequence φ including macro moves on the sets $A_1 \ldots, A_l$ (in chronological order). From part (a) we know that with regard to the final water level achievable at v we can assume w.l.o.g. that all moves are complete averages (i.e. $\mu_k = \frac{1}{2}$ for all k). Fix $\varepsilon > 0$ and let us define a finite move sequence $\overline{\varphi}$ including no macro moves in the following way: We keep all the rounds in φ in which pipes are opened individually. For the macro move on $A_i,\ i \in \{1, \ldots, l\}$, we insert a finite number of rounds in which the pipes of an edge set E_{A_i} connecting A_i are opened in repetitive sweeps such that the water level at each vertex $u \in A_i$ is less than $\frac{\varepsilon}{2^i}$ away from the average across A_i after these rounds. Note that Lemma 2.2 guarantees that this is possible.

As opening pipes leads to new water levels being convex combinations of the ones before, the differences of individual water levels caused by replacing the macro moves add up to $\sum_{i=1}^{l} \frac{\varepsilon}{2^i} < \varepsilon$ in the worst case. Consequently, the final water level achieved at v by $\overline{\varphi}$ is at most ε less than the one achieved by φ. Since $\varepsilon > 0$ was arbitrary, this proves the claim.

Note however that the option of macro moves can make a difference when it comes to the attainability of $\kappa(v)$, see Example 3.6.

□

Remark

Lemma 3.2 (a) states that even for macro moves, there is nothing to be gained by closing the pipes before the water levels have balanced out completely. A macro move on the edge set E_A with $\mu_k = \frac{1}{2}$ can be seen as the limit of infinitely many single edge moves on E_A in the sense of Lemma 2.2 – a connection that does not exist for macro moves with $\mu_k \in (0, \frac{1}{2})$. In fact, it is not hard to come up with an initial waterprofile on a path consisting of three nodes, where an incomplete macro move, i.e. with $\mu_k \in (0, \frac{1}{2})$, can not be achieved or even approximated by single edge moves.

But then again, we believe that there always exists a finite optimal move sequence if macro moves are allowed. We state this as an open problem.

Due to Lemmas 3.1 and 3.2 we can assume w.l.o.g. that the parameters μ_k in optimal move (meta-)sequences are always equal to $\frac{1}{2}$ in each round, hence omit them and consider a move sequence to be a list of pipes (i.e. $\varphi \in E^T$) only. We can incorporate a move sequence in which more than one pipe is opened at a time into Definition 3 by either allowing φ_k, for $k \in \{0, \ldots, T\}$, to be a subset of E with more than one element on which the levelling takes place or by viewing φ as a limiting case of move sequences $\{\varphi^{(m)},\ m \in \mathbb{N}\}$, in which pipes are opened separately, that form a meta-sequence of moves Φ – as just described in the proof of the lemma.

In the sequel however – if not otherwise stated – we will stick to the initial regime where pipes are opened one at a time.

3.2 Optimizing move sequences

Closely related to the water transport idea is the concept of *greedy lattice animals* as introduced by Cox, Gandolfi, Griffin and Kesten [2]. The vertices of a given graph G are associated with an i.i.d. sequence of non-negative random variables and a greedy lattice animal of size n is then defined to be a connected subset of n vertices containing the target vertex v and maximizing the sum over the associated n random variables. Since we do not care about the size of the lattice animal, let us slightly change this definition:

Definition 5
For a fixed graph $G = (V, E)$, target vertex v and water levels $\{\eta(u)\}_{u\in V}$, let us call $C \subseteq V$ a *lattice animal (LA)* for v if C is connected and contains v. C is a *greedy lattice animal (GLA)* for v if it maximizes the average of water levels over such sets. This average will be considered as its value

$$\mathrm{GLA}(v) := \frac{1}{|C|}\sum_{u\in C}\eta(u).$$

By Lemma 2.2, it is clear that $\mathrm{GLA}(v) \le \kappa(v)$. In fact, for the majority of settings – consisting of a graph G, a target vertex v and an initial water profile $\{\eta_0(u)\}_{u\in V}$ – strict inequality holds and we can do better than just pooling the amount of water collected in an appropriately chosen connected set of barrels including the one at v.

Furthermore we know from Lemma 3.2 (a) that w.l.o.g. the last move of any finite optimal move sequence will be to pool the amount of water allocated in a connected set of vertices including v. This greedy lattice animal for v in the intermediate water profile created up to that point in time can be more advantageous than the one in the initial water profile if we apply the following improving steps first:

1) Improving bottlenecks

Let us call a vertex u a *bottleneck* of the GLA C for v if $u \in C \setminus \{v\}$ and $\eta(u) < \mathrm{GLA}(v)$. Clearly, each bottleneck u has to be a cut vertex for C (otherwise we could just remove it to improve the GLA). If there exists a connected subset of vertices C_u including u which has a higher average water level than $C_u \cap C$, the value of the GLA for v is improved if the water collected in C_u is pooled first (see Figure 2). Note that C_u might involve more vertices from C than just u, see Example 3.5.

2) Enlargement

The second option to raise the value of the GLA C for v is to apply the idea above to a vertex u in the vertex boundary of C in order for the original GLA to be enlarged to a set of vertices in which u is a bottleneck. For this to be beneficial, there has to exist a connected set of vertices C_u in $V \setminus C$ including u with the following property: The average water level in C_u is smaller than GLA(v) – otherwise it would be part of C – but is raised above this value after improving the potential bottleneck u using water located in $V \setminus C$ (see Figure 2 below).

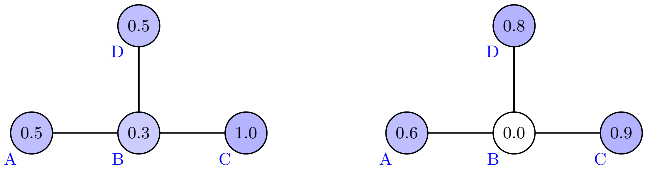


Figure 2: If A is the target vertex, the GLA on the left is $\{A, B, C\}$ (having value 0.6) and the bottleneck B can be improved by first opening the pipe $\langle B, D\rangle$. The GLA for A with respect to the water profile on the right is $\{A\}$, but can be enlarged to $\{A, B, C\}$ if the potential bottleneck B is improved by opening the pipe $\langle B, D\rangle$ first.

3) Choose optimal chronological order

When applying the improving techniques just described, it is essential to choose the optimal chronological order of doing things. Besides the fact that improving bottlenecks and enlarging the GLA has to be done before the final averaging, situations can arise in which different sets of vertices can improve the same bottleneck or the other way round that more than one bottleneck can be improved using non-disjoint sets of vertices, see the set-ups in Figure 3.

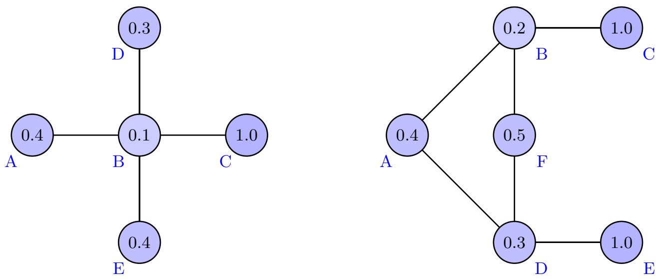


Figure 3: If A is the target vertex, the GLA on the left is $\{A, B, C\}$ (having value 0.5). Improving the bottleneck B can be done using D or E and is most effective if the pipe $\langle B, D\rangle$ is opened first, then $\langle B, E\rangle$.
The GLA for A with respect to the graph on the right is $\{A, B, C, D, E\}$. The water from F can be used to improve both bottlenecks B and D. It is optimal to open pipe $\langle D, F\rangle$ first and then $\langle B, F\rangle$.

Finally, it is worth noticing that lattice animals with lower average than GLA(v) in the initial water profile sometimes can be improved by the techniques just described to finally outperform the initial GLA and its possible improvements and enlargements (see Example 3.5 and especially Figure 5).

3.3 Examples

Example 3.1
The minimal graph which is non-trivial with respect to water transport is a single edge, in other words the complete graph on two vertices:

$$G = K_2 = (\{1, 2\}, \{\langle 1, 2\rangle\}).$$

By the considerations in the previous subsection, we get

$$\kappa(1) = \begin{cases} \eta_0(1) & \text{if } \eta_0(1) \geq \eta_0(2) \\ \frac{\eta_0(1)+\eta_0(2)}{2} & \text{if } \eta_0(1) < \eta_0(2). \end{cases} \tag{3}$$

Let the initial water levels be given by the two random variables U_1 and U_2. From (3) it immediately follows that

$$U_1 \leq \kappa(1) \leq \max\{U_1, U_2\}.$$

If we assume U_1 and U_2 to be independent and uniformly distributed on $[0, 1]$, a short calculation reveals the distribution function

$$F_{\kappa(1)}(x) = \begin{cases} \frac{3}{2}x^2 & \text{for } 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{2}(1-x)^2 & \text{for } \frac{1}{2} \leq x \leq 1, \end{cases}$$

which indeed lies in between $F_{U_1}(x) = x$ and $F_{\max\{U_1,U_2\}}(x) = x^2$, see Figure 4.

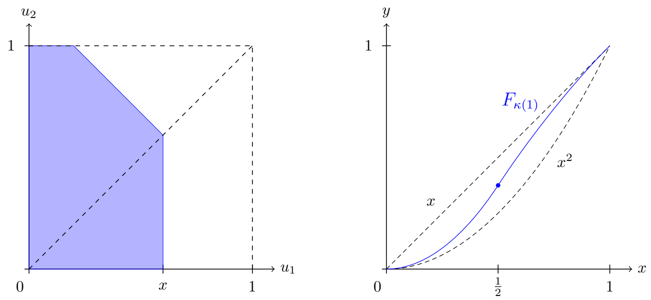
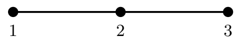


Figure 4: On the left a visualization of $\mathbb{P}(\kappa(1) \leq x)$, on the right the distribution function of $\kappa(1)$.

By symmetry, the exact same considerations hold for $\kappa(2)$.

Example 3.2

The simplest non-transitive graph (i.e. having vertices of different kind, see Definition 8) is the path on three vertices:



$$G = (\{1, 2, 3\}, \{\langle 1, 2\rangle, \langle 2, 3\rangle\}).$$

Again by the above considerations, we find the supremum of achievable water levels at vertex 1 to be

$$\kappa(1) = \max\left\{\eta_0(1), \tfrac{\eta_0(1)+\eta_0(2)}{2}, \tfrac{\eta_0(1)+\eta_0(2)+\eta_0(3)}{3}\right\},$$

which is obviously achieved by a properly chosen greedy lattice animal.

Consider the case in which the initial water levels satisfy

$$\eta_0(3) \geq \eta_0(2) \geq \eta_0(1) \quad \text{and} \quad \eta_0(3) > \eta_0(1) \tag{4}$$

Then $\kappa(1) = \frac{\eta_0(1)+\eta_0(2)+\eta_0(3)}{3}$ and there exists no finite optimal move sequence. This can be seen from the fact that any single move will preserve the inequalities in (4) and thus we have $\eta_T(1) < \kappa(1) < \eta_T(3)$ for all finite move sequences $\varphi \in E^T$.

If we consider the initial water levels to be independent and identically distributed, the (random) supremum of achievable water levels at vertex 2 is stochastically larger than the one at vertex 1: As $\eta_0(1)$ and $\eta_0(2)$ have the same distribution so do

$$\kappa(1) \quad \text{and} \quad \max\left\{\eta_0(2), \tfrac{\eta_0(1)+\eta_0(2)}{2}, \tfrac{\eta_0(1)+\eta_0(2)+\eta_0(3)}{3}\right\}.$$

The latter is less than or equal to $\kappa(2)$. The maximal value achievable by greedy lattice animals at vertex 2 is

$$\mathrm{GLA}(2) = \max\left\{\eta_0(2), \tfrac{\eta_0(1)+\eta_0(2)}{2}, \tfrac{\eta_0(2)+\eta_0(3)}{2}, \tfrac{\eta_0(1)+\eta_0(2)+\eta_0(3)}{3}\right\}.$$

The fact that we can average across one pipe at a time and choose the order of updates allows us to improve over this and gives

$$\kappa(2) = \max\left\{\mathrm{GLA}(2), \tfrac{1}{2}\left(\eta_0(1) + \tfrac{\eta_0(2)+\eta_0(3)}{2}\right), \tfrac{1}{2}\left(\eta_0(3) + \tfrac{\eta_0(1)+\eta_0(2)}{2}\right)\right\}. \quad (5)$$

To see this, we can take a closer look on the SAD-profiles that can be created by updates along the two edges $\langle 1,2\rangle$ and $\langle 2,3\rangle$ starting from the initial profile $\xi_0 = (0,1,0)$: After one update – depending on the chosen edge – the profile is given by $\xi_1 = (\frac{1}{2},\frac{1}{2},0)$ or $(0,\frac{1}{2},\frac{1}{2})$. After the second step we end up with either $\xi_2 = (\frac{1}{2},\frac{1}{4},\frac{1}{4})$ or $(\frac{1}{4},\frac{1}{4},\frac{1}{2})$. All of the corresponding convex combinations appear in the right hand side of (5). By Lemma 2.2, we know that continuing like this will finally result in the limiting profile $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$. It is not hard to check that any sequence of two or more updates will lead to an SAD-profile of type either $(x,\frac{1-x}{2},\frac{1-x}{2})$ or $(\frac{1-x}{2},\frac{1-x}{2},x)$, with $x \in [\frac{1}{4},\frac{1}{2}]$. Hence, it can be written as a convex combination of either $(\frac{1}{2},\frac{1}{4},\frac{1}{4})$ and $(0,\frac{1}{2},\frac{1}{2})$ or $(\frac{1}{4},\frac{1}{4},\frac{1}{2})$ and $(\frac{1}{2},\frac{1}{2},0)$. Consequently, it cannot correspond to a final water level at vertex 2 exceeding the value in (5).

In fact, when maximizing the water level for the middle vertex we can neglect the option of levelling out the profile completely, since for any initial water profile there is a *finite* optimal move sequence $\varphi \in E^T$ achieving

$$\eta_T(2) \geq \tfrac{1}{3}\left(\eta_0(1)+\eta_0(2)+\eta_0(3)\right),$$

as the next example will show.

Example 3.3
Given an initial water profile $\{\eta_0(u)\}_{u\in V}$ and the complete graph K_n as underlying network, we get for any $v \in V$:

$$\kappa(v) = 2^{-l+1}\,\eta_0(v) + \sum_{i=1}^{l-1} 2^{-i}\,\eta_0(v_i),$$

where V is ordered such that $\eta_0(v_1) \geq \eta_0(v_2) \geq \cdots \geq \eta_0(v_n)$ with $v = v_l$. Furthermore, this optimal value can be achieved by a finite move sequence.

To see this is not hard having Lemmas 2.1 and 2.3 in mind. If $v = v_1$, the highest water level is already in v and the best strategy is to stay away from the pipes. For $v \neq v_1$, the contribution of vertex v_1 – i.e. the share $\xi_T(v_1)$ in the convex combination of $\{\eta_0(u)\}_{u\in V}$ optimizing $\eta_T(v)$, see (2) – can not be more than $\frac{1}{2}$ by Lemma 2.3. However, this can be achieved by opening the pipe $\langle v, v_1\rangle$. According to the duality between water transport and SAD, this is what we do last. The argument just used can be iterated for the remaining share of $\frac{1}{2}$ giving that v_2 can contibute at most $\frac{1}{4}$ (given that v_1 contributes most possible) and so on. Obviously, involving vertices holding water levels below $\eta_0(v)$ can not be beneficial, as all vertices are directly connected, so we do not have intermediate vertices being potential bottlenecks.

The optimal finite move sequence $\varphi \in E^T$, where $T = l - 1$, is then given by

$$\varphi_k = \langle v, v_{l-k} \rangle,\ k = 1, \ldots, l-1$$

leading to

$$\eta_k(v) = 2^{-k}\, \eta_0(v) + \sum_{i=1}^{k} 2^{-k+i-1}\, \eta_0(v_{l-i})$$

and consequently $\eta_T(v) = \eta_{l-1}(v) = \kappa(v)$. Note that the option to open several pipes simultaneously is useless on the complete graph. Furthermore, the above move sequence only includes edges to which v is incident, so the very same reasoning holds for the center v of a star graph on n vertices as well.

To determine the optimal achievable value at v we have to sort the n initial water levels first. This can be done using the deterministic sorting algorithm 'heapsort' which makes $\mathrm{O}(n\ \log(n))$ comparisons in the worst case. The calculation of $\kappa(v)$ given the sorted list of initial water levels needs at most $n-1$ additions and $n-1$ divisions by 2.

Example 3.4

Expanding Example 3.2, let us reconsider a finite path – this time not on three but n vertices. Let the vertices be labelled 1 through n and let vertex 1 (sitting at one end of the path) be the target vertex. Given an initial water profile $\{\eta_0(i)\}_{i=1}^n$, $\kappa(1)$ can be determined by $2n-2$ arithmetic operations ($n-1$ additions, $n-1$ divisions) as it turns out to be

$$\kappa(1) = \max_{1 \le l \le n} \frac{1}{l} \sum_{i=1}^{l} \eta_0(i). \tag{6}$$

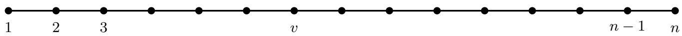
In other words, $\kappa(1)$ equals GLA(1), with respect to the initial water profile (see Definition 5).

This easily follows from Lemma 2.4, as any achievable SAD-profile $\{\xi(u)\}_{u=1}^n$ will be non-increasing in u. Hence the water level at 1 will always be a convex combination of averages over its n lattice animals and thus bounded from above by the right hand side of (6). This value in turn can be at least approximated by averaging over a greedy lattice animal for vertex 1 in the sense of Lemma 2.2.

If we allowed macro moves (opening several pipes simultaneously), the first (and only) move would be to open the pipes $\langle 1, 2\rangle, \ldots, \langle L-1, L\rangle$, where $L \in \{1, \ldots, n\}$ is chosen such that $\{1, \ldots, L\}$ is a GLA for vertex 1.

Example 3.5

Finally, let us consider the path on n vertices, with the target vertex v not sitting at one end.



Given the initial water levels $\{\eta_0(u),\ 1 \le u \le n\}$, let us consider the final SAD-profile $\{\xi(u)\}_{1\le u\le n}$ corresponding to an optimal move (meta-)sequence (for a meta-sequence, it is the limit of its dual SAD-profiles we are talking about, cf. Lemmas 2.1 and 3.1).

First of all, from Lemma 2.4 (a) we know that any achievable SAD-profile on a path is unimodal (which therefore holds for a pointwise limit of SAD-profiles as well). Let us denote the leftmost maximizer of $\{\xi(u)\}_{1\le u\le n}$ by q and set

$$l := \min\{1 \le u \le n,\ \xi(u) > 0\} \quad \text{and} \quad r := \max\{1 \le u \le n,\ \xi(u) > 0\}.$$

By symmetry, we can assume without loss of generality $l \le v \le q \le r$ – if $q < v$, the set-up is merely mirrored. Furthermore, let us pick the optimal move (meta-)sequence such that $\{\xi(u)\}_{1\le u\le n}$ minimizes the distance $q - v$.

The contribution from the nodes $\{q, q+1, \ldots, n\}$ can be seen as a scaled-down version of the problem treated in the previous example: This time the drink to be shared does not amount to 1 but to $\sum_{q\le u\le r} \xi(u)$ instead. From Example 3.4 we can therefore conclude that a flat SAD-profile i.e.

$$\xi(q) = \xi(q+1) = \ldots = \xi(r) \tag{7}$$

is optimal. The same holds for the contribution coming from $\{1, 2, \ldots, v-1\}$, i.e.

$$\xi(l) = \xi(l+1) = \ldots = \xi(v-1). \tag{8}$$

In addition to that, from Lemma 2.4 (c) we know $\xi(r) \le \frac{1}{r-v+1}$.

If $l = v$, part (b) of Lemma 2.4 in turn implies $v = q$. The SAD-profile then features only one non-zero value (namely $\frac{1}{r-v+1}$) and corresponds to the greedy lattice animal for v consisting of the vertices $v, v+1, \ldots, r$. If instead $l < v$ – compared to the balanced average across $\{v, v+1, \ldots, r\}$ just described – the contribution to the final water level at v (cf. (2)) given by

$$\sum_{u=l}^{v-1} \xi(u)\,\eta_0(u) \quad \text{replaced the contribution} \quad \sum_{u=v}^{r} \big(\tfrac{1}{r-v+1} - \xi(u)\big)\,\eta_0(u), \tag{9}$$

where necessarily $\sum_{u=l}^{v-1} \xi(u) = \sum_{u=v}^{r} \big(\frac{1}{r-v+1} - \xi(u)\big) =: M$. As q is a mode and due to (7) we have

$$\tfrac{1}{r-v+1} - \xi(v) \ge \ldots \ge \tfrac{1}{r-v+1} - \xi(q) = \ldots = \tfrac{1}{r-v+1} - \xi(r). \tag{10}$$

The aforementioned replacement is most beneficial if the weighted average to the right in (9) is made as *small* as possible, keeping M fixed. In view of (10) we can conclude, applying again the ideas from the foregoing example – this time think of the initial profile $C - \eta_0(u)$ considered for $v \le u \le r$ only – that this is achieved once more by a balanced average. Hence $l < v$ implies $v < q$

and the just mentioned balanced average has to stretch to the right as far as $q-1$, i.e. $\xi(v) = \ldots = \xi(q-1) < \frac{1}{r-v+1} = \xi(q)$, since otherwise q would not be the leftmost mode. From this and (8) we find $M = (v-l) \cdot \xi(l) = (q-v) \cdot \left(\frac{1}{r-v+1} - \xi(v)\right)$.

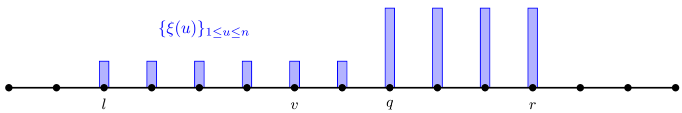
The assumption that $q - v$ was minimized when picking the optimal move (meta-)sequence considered, forces

$$\sum_{u=l}^{v-1} \xi(u)\, \eta_0(u) \;>\; \sum_{u=v}^{q-1} \left(\tfrac{1}{r-v+1} - \xi(u)\right) \eta_0(u),$$

since otherwise the balanced average across $\{v, v+1, \ldots, r\}$ would have been at least as good. Connecting v to barrels to the left consequently yields an improvement of the final water level at v (in comparison to $\frac{1}{r-v+1}\sum_{u=v}^{r} \eta_0(u)$) to the amount of

$$\sum_{u=l}^{v-1} \xi(u)\, \eta_0(u) - \sum_{u=v}^{q-1} \left(\tfrac{1}{r-v+1} - \xi(u)\right) \eta_0(u) = M \cdot \Big(\tfrac{1}{v-l} \sum_{u=l}^{v-1} \eta_0(u) - \tfrac{1}{q-v} \sum_{u=v}^{q-1} \eta_0(u)\Big).$$

As a consequence, for an optimal move (meta-)sequence M must be as large as possible, which means $\xi(l) = \xi(q-1)$ and makes $\{\xi(u)\}_{1 \le u \le n}$ a piecewise constant profile taking on two non-zero values, $\xi(l)$ and $\xi(r)$, as depicted below.



Note that the value $\xi(r) = \frac{1}{r-v+1}$ (and so even $\xi(l)$) is already determined by the choice of l, q and r. In Figure 5 below, a set of initial water levels on the path comprising 15 nodes is shown, for which the SAD-profile corresponding to an optimal move meta-sequence is the one shown above. Furthermore, it can be seen from this instance that the GLA with respect to the initial water profile and its possible enhancements can be outperformed by improving another lattice animal as mentioned at the end of Subsection 3.2.

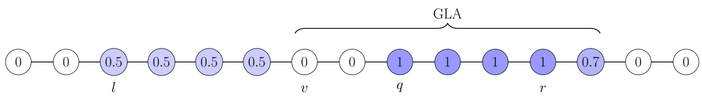


Figure 5: Even for a graph as simple as a finite path, the initial GLA sometimes has little to do with the optimal move (meta-)sequence.

When it comes to the complexity of finding $\kappa(v)$, we can greedily test all choices for l, q, r – of which there are less than n^3. For each choice at most $n+3$

additions/subtractions and four multiplications/divisions have to be made to calculate either

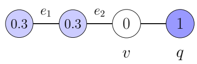
$$\begin{aligned} &\tfrac{q-v}{(q-l)\,(r-v+1)} \sum_{u=l}^{q-1} \eta_0(u) + \tfrac{1}{r-v+1} \sum_{u=q}^{r} \eta_0(u) \quad \text{or} \\ &\tfrac{1}{v-l+1} \sum_{u=l}^{\hat{q}} \eta_0(u) + \tfrac{v-\hat{q}}{(r-\hat{q})\,(v-l+1)} \sum_{u=\hat{q}+1}^{r} \eta_0(u), \end{aligned} \tag{11}$$

depending on whether $v \leq q$ or $\hat{q} \leq v$, where $\hat{q}$ is the rightmost mode of $\{\xi(u)\}_{1 \leq u \leq n}$. Even if there might exist SAD-profiles with $q < v < \hat{q}$ corresponding to optimal move (meta-)sequences, by the above we know that there has to be one with either $v \leq q$ or $\hat{q} \leq v$ as well. The maximal value among those calculated in (11) equals $\kappa(v)$, so the complexity is $\mathcal{O}(n^4)$. In fact, if we calculate and store all $\binom{n}{2}$ sums over intervals of the array of initial water levels $\{\eta_0(u)\}_{1 \leq u \leq n}$, this running time can be reduced to $\mathcal{O}(n^3)$.

Example 3.6

The preceding example can serve to give a concrete instance in which even an infinite sequence of single edge moves can not achieve the supremum as mentioned after Definition 3.

Consider the path on four vertices, the target vertex not to be one of the end vertices and initial water levels as depicted to the right.



From Example 3.5 we know that the optimal SAD-profile will allocate $\frac{1}{6}$ of the shared glass of water to each of the vertices to the left of v and v itself, the maximal amount of $\frac{1}{2}$ to the rightmost vertex q – showing that $\kappa(v) = 0.6$: First, recall that any SAD-profile on a path is unimodal. If q is not the (only) mode, the contribution of v and q has an average of at most 0.5 and thus the SAD-profile in question yields a water level at v of at most 0.5 – see (2). If q is the mode, the SAD-profile is non-decreasing from left to right and thus a flat profile on the vertices other than q uniquely optimal. Finally, to achieve the optimum, the contribution of q has to be maximal, i.e. $\frac{1}{2}$ (see Lemma 2.3).

From the considerations in Thm. 2.3 in [4] it is clear that this SAD-profile, more precisely the value $\frac{1}{2}$ at q, can only be established if the first move is v sharing the drink with q (which corresponds to the last move in the water transport – see Lemma 2.1). Once v starts to share the drink to the left, any other interaction with q will decrease the contribution of the latter and thus put a water level of 0.6 at v out of reach.

To get a flat profile on three vertices, we need however infinitely many single-edge moves (here on e_1 and e_2). An optimal meta-sequence of moves is for example given by

$$\begin{aligned} \Phi &= \{\varphi^{(m)},\, m \in \mathbb{N}\}, \quad \text{where} \quad \varphi^{(m)} \in E^{T_m},\ T_m = 2m+1 \quad \text{and} \\ \varphi^{(m)} &= (\underbrace{e_1, e_2, e_1, e_2, \ldots,}_{m \text{ pairs}} \langle v, q \rangle), \end{aligned}$$

achieving

$$\lim_{m \to \infty} \eta_{T_m}(v) = 0.6 = \kappa(v),$$

a value that can not be approached by any stand-alone (finite or) infinite sequence of moves.

If we allow macro moves, however, there is a two-step move sequence achieving the water level 0.6 at v: First we open the pipes e_1 and e_2 simultaneously to pool the water of the vertices other than q and in the second round, we open the pipe $\langle v, q\rangle$.

4 Complexity of the problem

In this section, we want to build on the complexity considerations for the water transport on finite graphs from the examples in Section 3. In fact, we want to show that the task of determining whether $\kappa(v)$ is larger or smaller than a given constant – for a generic set-up, consisting of a graph, target vertex and initial water profile – is an NP-hard problem. This is done by establishing the following theorem:

Theorem 4.1
The NP-complete problem 3-SAT can be polynomially reduced to the decision problem of whether $\kappa(v) > c$ or not, for an appropriately chosen water transport instance and constant c.

Before we deal with the design of an appropriate water transport instance in order to embed the satisfiability problem *3-SAT*, let us provide the definition of Boolean satisfiability problems as well as known facts about their complexity.

Definition 6
Let $X = \{x_1, x_2, \dots, x_k\}$ denote a set of *Boolean variables*, i.e. taking on logic truth values 'TRUE' (T) and 'FALSE' (F). If x is a variable in X, x and $\overline{x}$ are called *literals* over X. A *truth assignment* for X is a function $t : X \to \{T, F\}$, where $t(x) = T$ means that the variable x is set to 'TRUE' and $t(x) = F$ means that x is set to 'FALSE'. The literal x is true under t if and only if $t(x) = T$, $\overline{x}$ is true under t if and only if $t(x) = F$.

A *clause* C over X is a disjunction of literals and *satisfied* by t if at least one of its literals is true under t. A logic formula F is in *conjunctive normal form (CNF)* if it is the conjunction of (finitely many) clauses. It is called *satisfiable* if there exists a truth assignment t such that all its clauses are satisfied under t.

The standard Boolean satisfiability problem (often denoted by *SAT*) is to decide whether a given formula in CNF is satisfiable or not. If we restrict to the case where all the clauses in the formula consist of at most 3 literals it is called *3-SAT* .

3-SAT was among the first computational problems shown to be NP-complete, a result published in a pioneering article by Cook in 1971, see Thm. 2 in [1].

Let us now turn to the task of embedding 3-SAT into an appropriately designed water transport problem that is in size polynomial in n, the number of clauses of the given 3-SAT problem:

Given the logic formula $F = C_1 \wedge C_2 \wedge \ldots \wedge C_n$ in which each of the clauses C_i consists of at most 3 distinct literals, let us define the comb-like graph depicted in Figure 6. All the white nodes, plus the target vertex v, represent empty barrels. The other ones that are shaded in blue contain water to the amount specified.

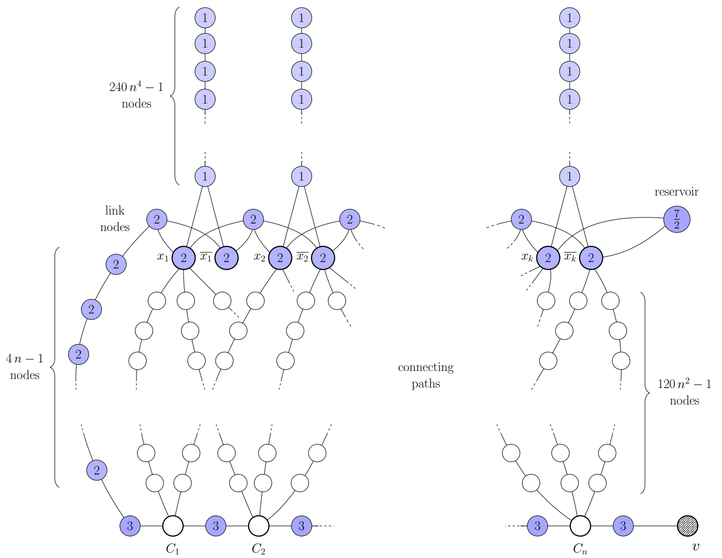


Figure 6: A polynomial reduction of 3-SAT to the water transport problem.

The comb has k teeth, where k is the number of variables appearing in F. Each individual tooth is formed by a path on $240\,n^4 - 1$ vertices with water level 1 each. The lower endvertex of the ith tooth is connected to two vertices representing the literals x_i and $\overline{x_i}$, having water level 2 respectively. In between the teeth there are $k-1$ *link nodes*, each of which features itself a water level of 2 and is connected to the four nodes representing literals of consecutive variables – more precisely, the link node in between tooth i and $i+1$ is connected to the vertices $x_i, \overline{x_i}, x_{i+1}, \overline{x_{i+1}}$, for $i \in \{1, \ldots, k-1\}$. The vertices representing $x_k, \overline{x_k}$ are connected to the rightmost link node as well as to an additional vertex featuring a water reservoir of level $\frac{7}{2}$. Left of the first tooth, there is another link node (with water level 2 as well) connected to x_1 and $\overline{x_1}$ as well as by a path to the shaft of the comb, which is described next.

The comb's shaft is made up of a path on $2n+2$ vertices, with the target vertex v to the very right. To the left of v there is a vertex representing a barrel with water level 3 followed by n (empty) barrels that stand for the clauses

$C_1, \ldots, C_n$ and are seperated by a vertex with water level 3 respectively. The left endvertex (connected to C_1) features a water level of 3 as well and is connected to the leftmost link node as mentioned before, namely via a path consisting of $4n - 1$ nodes with water level 2 each.

Finally, the teeth are connected to the shaft through (disjoint) *connecting paths* from nodes representing literals to nodes representing clauses, where for example $\overline{x_2}$ is linked to C_2 by a path if it appears in this clause. Each of these paths consists of $120\,n^2 - 1$ vertices representing empty barrels. Note that each clause-node is linked to at most 3 connecting paths, whereas the number of connecting paths originating from a vertex representing a literal can vary between 0 and n.

In connection with the water transport problem originating from a 3-SAT formula F as depicted in Figure 6, we claim the following:

Proposition 4.2
Consider the water transport problem based on the logical formula F, given by the graph, target vertex and initial water profile as depicted in Figure 6.

(a) If F is satisfiable, then the water level at v can be raised to a value strictly larger than 2, i.e. $\kappa(v) > 2$.

(b) If F is not satisfiable, then this is impossible, i.e. $\kappa(v) \leq 2$.

Before we deal with the proof of the proposition, note how it implies the statement of Theorem 4.1: First of all, if F is a 3-SAT formula consisting of n clauses, k cannot exceed $3\,n$. Given this, it is not hard to check that the graph in Figure 6 has no more than $720\,n^5 + 360\,n^3 + 9\,n + 2$ vertices and maximal degree at most $n + 3$ (or 5 if $n = 1$). As the initial water levels are all in $\{0, 1, 2, 3, \frac{7}{2}\}$, the size of this water transport instance is clearly polynomial in n. Due to the fact that the value of $\kappa(v)$ can be used to decide whether the given formula F is satisfiable or not – as claimed by Proposition 4.2 – Theorem 4.1 follows.

PROOF OF PROPOSITION 4.2 (a): To prove the first part of the proposition, let us assume that F is satisfiable. Then there exists a truth assignment t with the property that all clauses $C_1, \ldots, C_n$ contain at least one of the k literals that are set true by t. Those can be used to let the water trickle down from the teeth to the path at the bottom in an effective way: We assign each clause to one of the true literals under t which it contains. Then, we average the water over k (disjoint) star-shaped trees. Each such tree has a literal $x \in \{x_1, \overline{x_1}, \ldots, x_k, \overline{x_k}\}$ that is true under t as its center and the top node of the tooth above x as well as the nodes representing the clauses assigned to x as leaves (where the clause-nodes are connected to x in the tree via the corresponding connecting paths). If m clauses chose x, there are $240\,n^4 + m \cdot 120\,n^2$ vertices in the tree and the water accumulated amounts to $240\,n^4 + 1$.

By pooling the water along those trees, all the nodes corresponding to clauses can simultaneously be pushed to a water level as close to the average of the corresponding trees as we like (see Lemma 2.2). As $m \le n$, we can bound these averages from below by

$$\frac{240\,n^4+1}{240\,n^4+120\,mn^2} \ge \frac{240\,n^4+1}{240\,n^4+120\,n^3} > 1-\tfrac{1}{2n}.$$

So after this procedure, each clause-node will have a water level strictly larger than $1-\frac{1}{2n}$. Note that only one of each pair $\{x_i, \overline{x_i}\}$ was used as a water passage, so there is still a path – let us call it *linking path* – consisting of vertices with water level 2 exclusively, from the leftmost link node to the vertex with initial water level $\frac{7}{2}$ through all link nodes and the untouched literals (the ones that are false under t).

By another complete averaging – this time over the path that consists of the shaft (i.e. the path at the bottom in Figure 6), the path to the very left connecting the leftmost link node to the shaft, the linking path just described, as well as the reservoir with level $\frac{7}{2}$ at the other end – will push the water level at v beyond

$$\tfrac{1}{6n+2k+2}\left(\tfrac{7}{2}+(2\,k+4\,n-1)\cdot 2+(n+1)\cdot 3+n\,(1-\tfrac{1}{2n})\right) = \frac{12\,n+4\,k+4}{6\,n+2\,k+2} = 2.$$

Consequently, for the case of satisfiable F we verified for the graph depicted in Figure 6: $\kappa(v) > 2$. □

In the proof of the second part of the proposition, we need a rough estimate of how much the water level in a vertex representing a clause can be raised, if only accessed via connecting paths. This is done in the following lemma.

Lemma 4.3
In the comb-like graph depicted in Figure 6, it is impossible to push the water level in a clause-vertex above the value of $1+\frac{1}{2n}$ *without opening the pipes to its left or right neighbor.*

Proof: The proof of this claim is a simple comparison with a tree similar to the structure above the node corresponding to some clause C_l. Originating from C_l, there are at most 3 connecting paths that lead to three nodes representing literals. Initially, the node corresponding to C_l and the ones on the connecting paths are empty. Their water level can be raised to almost 1 using water from the teeth of the comb and further using nodes with initial water level higher than 1. The fact that opening pipes always produces convex combinations of the involved water levels (see (2)) guarantees that the total amount of water above a fixed level – cumulated over all barrels – is non-increasing when pipes are opened. Initially, the cumulated amount of water above level 1 in the whole graph is

$$(\tfrac{7}{2}-1)+3\,k\cdot(2-1)+(4\,n-1)\cdot(2-1)+(n+1)\cdot(3-1) \le 15\,n+\tfrac{7}{2}. \quad (12)$$

For $n \in \mathbb{N}$, this is clearly less than $20n-1$.

We can mimick any attempt raising the water level at C_l in the comb-graph via its connecting paths in the tree depicted to the right in Figure 7 in such a way, that the water levels at C_l and on its connecting paths are at any point in time at most as high as the ones in the corresponding part of the comparison tree: If water is routed into the connecting paths above C_l but water levels do

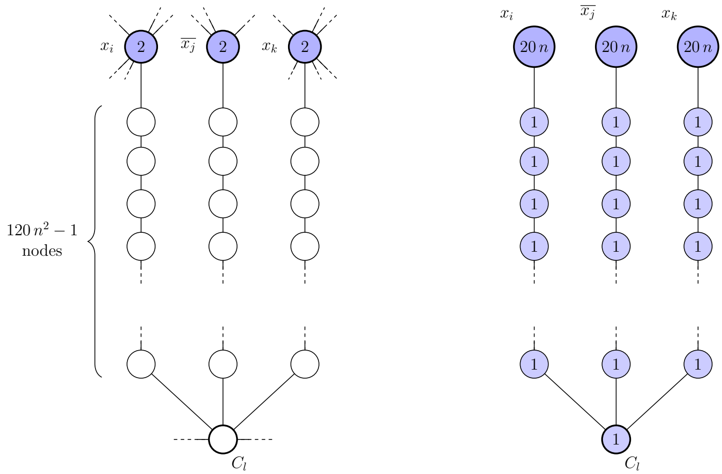


Figure 7: Comparison of the structure above the node representing a clause C_l in the comb-graph with an appropriately tailored tree.

not exceed 1 (e.g. when routing water down from the teeth) we do nothing in the comparison tree. If water from the vertices with initial water level above 1 is introduced into the connecting paths, we introduce the same amount to the corresponding connecting paths in the tree (note that this is possible, as the total amount of water above level 1 in the comb-graph is available in all three leaves of the tree). Every move involving only nodes from the connecting paths depicted and C_l is copied in the tree. This retains the property that the water levels in the tree are not less than the ones in corresponding nodes of the comb-graph and shows that the highest water level achievable at C_l in the tree is an upper bound on the level achievable in the comb-graph. If there are less than 3 connecting paths above C_l in the comb-graph we can either modify the comparison tree accordingly or just not use the extra branches.

By the generalization of Thm. 2.3 in [4] to trees, see the comment after Lemma 2.4, we know that the contribution to the convex combination at C_l from the leaves in the tree is at most 1 divided by the graph distance plus one, i.e. $\frac{1}{120\,n^2+1}$. The water level at C_l in the tree can therefore not exceed

$$1 + 3 \cdot \frac{20\,n}{120\,n^2+1} \leq 1 + \frac{1}{2n},$$

which induces the claim. □

Note that the same argument with only water to the amount of $5\,n-1$ above level 1 available in the leaves would give the upper bound of $1+\frac{15\,n}{120\,n^2} = 1+\frac{1}{8n}$, which will be used in the proof of Proposition 4.2 (b) as well.

Proof of Proposition 4.2 (b): To check that in case F is not satisfiable we get $\kappa(v) \leq 2$ is a bit more involved than the first part: Let us assume the contrary. Then there exists a finite move sequence (involving macro moves say) that achieves a final water level $\eta_T(v) > 2$. By the idea in part (a) of Lemma 3.2 we can assume the last move to be the complete average over a connected vertex set A including v. The only barrels with initial water level larger than 2 are the ones left of each clause-node and of v plus the reservoir. Including any node apart from these into the set A, when trying to achieve $\eta_T(v) > 2$, can therefore only be beneficial if it is a bottleneck (see the discussion after Definition 5).

Structurally speaking, there are three potential candidates for such a set A:

- a set containing some vertex from a connecting path
- a set containing only vertices from the bottom path or
- a set containing the reservoir vertex but no connecting path.

Note that the set we used in the case of satisfiable F was of the third type. We will see in a moment that this is in fact the only relevant candidate for the set A in the sense that the other two do not allow to raise the water level at v above the value of 2, even for a satisfiable formula F.

The first candidate listed is ruled out rather easily: If A contains a vertex from a connecting path, the bottleneck argument forces A to contain the whole corresponding connecting path (recall: a bottleneck has to be a cut vertex between barrels with water levels above average and the target vertex). Then A is of size at least $120\,n^2$ and the amount of water above the water level of 1 is just not sufficient to fill up so many vertices to a level of two: From (12) we know that the water available above level 1 in the whole graph is at most $15\,n+\frac{7}{2}$ initially and non-increasing. The amount in a whole connecting path with water level 2 would be $120\,n^2-1$, so this can definitively not be achieved.

Next, let us assume that A is a subset of the vertices of the bottom path – including vertex v and m clauses. Again by the bottleneck argument, we can assume that the leftmost node in A is not a clause-node, i.e. has initial water level 3 (see Figure 8).

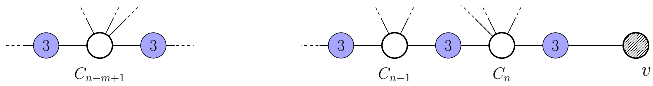


Figure 8: Vertices of the set A considered in the second case.

With the intention to increase the total amount of water inside A before the final averaging, one can try to fill up the clause-nodes. However, from Lemma 4.3 we know that the water level at the clause-nodes (being bottlenecks) in A cannot be pushed much above the level of 1, if accessed via connecting paths only. Further, this makes accessing vertex v through a connecting path and C_n unfavorable. Note that opening the pipes in the bottom path in order to connect barrels representing clauses inside A to the ones with water level 2 or 3 outside A might increase the amount of water in A as well, but will raise the water level at the involved clause-nodes to a level that can not be further improved by using links via connecting paths, so it is most beneficial to fill up the clause-nodes with water routed through connecting paths first.

Let us assume that after this first phase, we managed to achieve a water level of $1 + \frac{1}{2n}$ at $C_{n-m+1}, \ldots, C_n$. This might be technically impossible, but surely dominates the water levels achievable using the connecting paths only and simplifies our further considerations. Staying away from the connecting paths, water can only be routed into A via C_{n-m}. If we average over all nodes in A once while doing so, the final averaging is meaningless (because then the effect of any move between this and the last move will be increased if again all pipes inside A are opened). However, since the last move has to involve v we can assume that any move before leaves the pipe on the edge incident to v closed – and thus w.l.o.g. the pipe from C_n towards v as well.

This in turn requires that the connected subset of nodes outside A (incident to the leftmost node in A) that pools its water with a connected subset inside A, including C_{n-m+1} but not v, has an average of at least $2 + \frac{1}{4n}$, as the amount of water inside A would decrease otherwise. In view of Lemma 4.3, the only useful move is therefore to open the pipes along the shaft and through the nodes with initial water level 2 (which they actually might have lost during the first phase) in order to connect the vertex with water reservoir $\frac{7}{2}$ to A. No matter which of the nodes representing literals we include, the water levels in the path connecting the leftmost link node to the shaft and the shaft itself will be dominated by the ones obtained if we pretend that the water above level 2 from the reservoir can be transferred to the leftmost link node without any losses.

Starting with a water level of $\frac{7}{2}$ in the leftmost link node instead, we might increase the amount of water inside A by at most another $\frac{3}{2} \cdot \frac{2m}{2m+4n} \leq \frac{1}{2}$ (as the path has to involve at least $4\,n$ nodes outside A, the subset inside A is of size at most $2\,m \leq 2\,n$ and already has an average of at least 2). Along a path, the contribution of the water level from an endvertex to the ones formed as convex combinations along the path is decreasing with the graph distance (see Lemma 2.4 (b)).

Despite our greatest efforts, the total amount of water in A will consequently not exceed the value $3\,(m+1) + m\,(1 + \frac{1}{2n}) + \frac{1}{2} \leq 4\,(m+1)$. Since A consists of $2m+2$ vertices, leveling out across this set will possibly raise the amount of water in the barrel at v to the level of 2, but not beyond.

Finally, consider A to contain all of the shaft as well as the vertex with water reservoir $\frac{7}{2}$, but no vertex from a connecting path. Then A, being connected, has to contain the path that consists of $4\,n-1$ vertices, connecting the shaft to the leftmost link node, as well as a linking path through link nodes and vertices representing literals as described above. However, this time – with F being not satisfiable – it is impossible to fill up all the clause-nodes to a level of about 1, leaving at least one path between the reservoir and the leftmost link node unaffected: In order to reach all clause-nodes, we have to use both x_i and $\overline{x_i}$ as water passage for at least one $i \in \{1, \ldots, k\}$.

In comparison to the case of satisfiable F, we will lose an amount of at least $1 - \frac{1}{2n}$ for each clause-node that is not reached before the final averaging – but likewise an amount of at least 1 in the linking path for each pair $\{x_i, \overline{x_i}\}$ in which both nodes were used as water passage – since their water level of 2 reduces to something less than 1 when water from the tooth above is routed through the node all the way down to a clause-node. By the same token as in Lemma 4.3, the clause-nodes can be filled up to a level of at most $1 + \frac{1}{8n}$ through the connecting paths, as the water available outside A above level 1 is k (from the literals not part of the linking path) plus 1 from a vertex representing a literal on the linking path if we need to route through such (and $k+1 < 5\,n-1$). Note that moving water from inside A through a connecting path to a clause will in fact reduce the amount of water in A. Consequently, the set A (which is the same as the one chosen for satisfiable F) still contains $6n+2k+2$ vertices, but the amount of water we can allocate in A is at most

$$\begin{aligned} & \tfrac{7}{2} + (2\,k + 4\,n - 1)\cdot 2 + (n+1)\cdot 3 + n\,(1 + \tfrac{1}{8n}) - (1 - \tfrac{1}{2n}) \\ = \; & 12\,n + 4\,k + \tfrac{29}{8} + \tfrac{1}{2n} \\ < \; & 12\,n + 4\,k + 4, \end{aligned}$$

for $n \geq 2$. Thus, even in this manner we can not raise the water level at vertex v to a level of 2 or above if F is not satisfiable which contradicts the above assumption and in consequence verifies $\kappa(v) \leq 2$ for this case. □

As already mentioned, this shows that solving the decision problem "$\kappa(v) > 2$ or $\kappa(v) \leq 2$" for the comb-like graph depicted in Figure 6 solves the corresponding 3-SAT problem as well. Since 3-SAT is an NP-complete problem, we hereby established that any problem in NP can be polynomially reduced to a decision problem minor to the computation of $\kappa(v)$ in a suitable water transport instance – showing that computing $\kappa(v)$ in general is indeed an NP-hard problem.

5 On infinite graphs

This last section is devoted to the water transport problem on infinite graphs. We consider an infinite, connected, simple graph $G = (V, E)$ with bounded maximal degree. The initial water levels $\{\eta_0(u)\}_{u \in V}$ are considered to be i.i.d. with a (non-degenerate) common marginal distribution concentrated on $[0, C]$, for some $C > 0$. The supremum $\kappa(v)$ of achievable water levels at a fixed target vertex $v \in V$ depends on the initial water levels of course, which makes it a random variable as well.

When the vertices of an infinite graph are assigned individual values, the most natural definition of a *global average* across the graph is to look at a fixed sequence of nested subsets of the vertex set, with the property that every vertex is included eventually, and then consider the limit of averages across those subsets (if it exists).

Given i.i.d. initial water levels, the strong law of large numbers tells us that the randomness of the global average – which is non-degenerate on finite graphs – becomes degenerate if we consider infinite graphs, where it will a.s. equal the expectation of the marginal distribution. $\kappa(v)$ however shows a slightly different behavior: In order to determine whether the supremum of achievable water levels at a given vertex v is a.s. constant or not, we have to investigate the global structure of the infinite graph a bit more closely.

If the graph contains a half-line with sufficiently many extra vertices attached to it, the distribution of $\kappa(v)$ becomes degenerate for all $v \in V$ – as stated in Theorem 5.1 and the final remark: One can in fact, with probability 1, push the water level at v to the essential supremum of the marginal distribution. The two-sidedly infinite path, however, is too lean to feature such a substructure and behaves therefore much more like a finite graph, in the sense that the distribution of $\kappa(v)$ is non-degenerate – see Theorem 5.3. In order to develop these two main results of this section, let us first properly define what we mean by "sufficiently many extra vertices".

Definition 7
Let $G = (V, E)$ be an infinite connected simple graph. It is said to contain a *neighbor-rich half-line*, if there exists a subgraph of G consisting of a half-line

$$H = \big(\{v_k,\ k \in \mathbb{N}\}, \{\langle v_k, v_{k+1}\rangle,\ k \in \mathbb{N}\}\big)$$

and distinct vertices $\{u_k,\ k \in \mathbb{N}\}$ from $V \setminus \{v_k,\ k \in \mathbb{N}\}$ such that there is an injective function $f : \mathbb{N} \to \mathbb{N}$ with the following two properties (cf. Figure 9):

(i) For all $k \in \mathbb{N}$: $\langle u_k, v_{f(k)}\rangle \in E$, i.e. the vertices u_k and $v_{f(k)}$ are neighbors in G.

(ii) The function f is growing slowly enough in the sense that $\sum_{k=1}^{\infty} \frac{1}{f(k)}$ diverges.

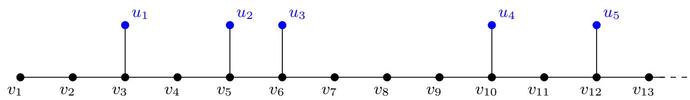


Figure 9: The beginning part of a neighbor-rich half-line.

Note that – by a renumbering of $\{u_k,\ k \in \mathbb{N}\}$ – we can always assume the function f to be (strictly) increasing. Furthermore, if G is connected and contains a neighbor-rich half-line, we can choose any vertex $v \in V$ to be its beginning vertex: If v_l is the vertex with highest index at shortest distance to v in H, replace $(v_1, \dots, v_l)$ by a shortest path from v to v_l in H. The altered half-line will still be neighbor-rich, since for all $M, N \in \mathbb{N}$ and f as above:

$$\sum_{k=1}^{\infty} \frac{1}{f(k)} = \infty \quad \Longleftrightarrow \quad \sum_{k=M}^{\infty} \frac{1}{f(k)+N} = \infty.$$

With this notion in hand, we can state and prove the following result:

Theorem 5.1
Consider an infinite (connected) graph $G = (V, E)$ and the initial water levels to be i.i.d. unif$([0,1])$*. Let $v \in V$ be a fixed vertex of the graph. If G contains a neighbor-rich half-line, then $\kappa(v) = 1$ almost surely.*

Before embarking on the proof of this theorem, we are going to show a standard auxiliary result which will be needed in the proof:

Lemma 5.2
For $\varepsilon > 0$, let $(Y_k)_{k\in\mathbb{N}}$ be an i.i.d. sequence having Bernoulli distribution with parameter ε. If the function $f : \mathbb{N} \to \mathbb{N}$ is strictly increasing and such that $\sum_{k=1}^{\infty} \frac{1}{f(k)}$ diverges, then

$$\sum_{k=1}^{\infty} \frac{Y_k}{f(k)} = \infty \quad \textit{almost surely.}$$

Proof: Let us define

$$X_n = \sum_{k=1}^{n} \frac{Y_k - \varepsilon}{f(k)} \quad \text{for all } n \in \mathbb{N}.$$

As the increments are independent and centered, this defines a martingale with respect to the natural filtration. Furthermore,

$$\mathbb{E}\,(X_n^2) = \sum_{k=1}^{n} \frac{\mathbb{E}\,(Y_k - \varepsilon)^2}{f(k)^2} = (\varepsilon - \varepsilon^2) \cdot \sum_{k=1}^{n} \frac{1}{f(k)^2} \le \varepsilon\, \frac{\pi^2}{6}.$$

By the L^p-convergence theorem (see for instance Thm. 5.4.5 in [3]) there exists a random variable X such that X_n converges to X almost surely and in L^2. Having finite variance, X must be a.s. real-valued and due to

$$\sum_{k=1}^{n} \frac{Y_k}{f(k)} - X_n = \varepsilon \cdot \sum_{k=1}^{n} \frac{1}{f(k)},$$

the divergence of $\sum_{k=1}^{\infty} \frac{1}{f(k)}$ forces $\sum_{k=1}^{\infty} \frac{Y_k}{f(k)} = \infty$ almost surely. □

Proof of Theorem 5.1: Given a graph G with the properties stated and a vertex v, we can choose a neighbor-rich half-line H with $v = v_1$ and the set of extra neighbors $\{u_n\}_{n\in\mathbb{N}}$ as described in and after Definition 7. The initial water levels at $\{u_n\}_{n\in\mathbb{N}}$ are i.i.d. unif$([0,1])$, of course.

Depending on the random initial profile, let us define the following SAD-procedure starting at v: Fix $\varepsilon, \delta > 0$ and let $\{N_l\}_{l\in\mathbb{N}}$ be the increasing (random) sequence of indices chosen such that the initial water level at u_{N_l} is at least $1-\varepsilon$ for all l. Then define the SAD-procedure – starting with $\xi_0(v) = 1,\ \xi_0(u) = 0$ for all $u \in V\setminus\{v\}$ – such that first all vertices along the path $(v_1, v_2, \dots, v_{f(N_1)}, u_{N_1})$ exchange liquids sufficiently often to get

$$\xi_{k_1}(u_{N_1}) \geq \frac{1}{f(N_1)+2} \quad \text{for some } k_1 > 0,$$

and never touch u_{N_1} again. Note that by Lemma 2.2, $\xi_k(u_{N_1})$ can be pushed as close to $\frac{1}{f(N_1)+1}$ as desired in this way. At time k_1, the joint amount of water in the glasses at $v_1, v_2, \dots, v_{f(N_1)}$ equals $1 - \xi_{k_1}(u_{N_1})$ and we will repeat the same procedure along $(v_1, v_2, \dots, v_{f(N_2)}, u_{N_2})$ to get

$$\xi_{k_2}(u_{N_2}) \geq \frac{1}{f(N_2)+2} \cdot \big(1 - \xi_{k_1}(u_{N_1})\big) \quad \text{for some } k_2 > k_1$$

and iterate this.

After m iterations of this kind, the joint amount of water localized at vertices of the half-line H equals $1 - \sum_{l=1}^{m} \xi_{k_l}(u_{N_l})$, which using $1 - x \leq \mathrm{e}^{-x}$ can be bounded from above as follows:

$$\begin{aligned} 1 - \sum_{l=1}^{m} \xi_{k_l}(u_{N_l}) &\leq \prod_{l=1}^{m} \left(1 - \frac{1}{f(N_l)+2}\right) \\ &\leq \exp\left(-\sum_{l=1}^{m} \frac{1}{f(N_l)+2}\right). \end{aligned} \tag{13}$$

Defining $Y_k := \mathbb{1}_{\{\eta_0(u_k) \geq 1-\varepsilon\}}$ for all $k \in \mathbb{N}$ we get $(Y_k)_{k\in\mathbb{N}}$ i.i.d. Ber(ε) and can rewrite the limit of the sum in the exponent as follows:

$$\sum_{l=1}^{\infty} \frac{1}{f(N_l)+2} = \sum_{k=1}^{\infty} \frac{Y_k}{f(k)+2}.$$

This allows us to conclude from Lemma 5.2 that the exponent in (13) tends a.s. to $-\infty$ as $m \to \infty$. Consequently, $m, T \in \mathbb{N}$ can be chosen large enough such that with probability $1-\delta$ it holds that

$$\sum_{l=1}^{m} \xi_{k_l}(u_{N_l}) \geq 1-\varepsilon \quad \text{and} \quad k_m \leq T.$$

Given this event, the move sequence corresponding to the SAD-procedure just described – adding no further updates after time k_m, i.e. $\mu_k = 0$ for $k > k_m$, if $k_m < T$ – then ensures (see Lemma 2.1) that

$$\eta_T(v) \geq \sum_{l=1}^{m} \xi_T(u_{N_l})\, \eta_0(u_{N_l}) \geq (1-\varepsilon)^2,$$

forcing $\kappa(v) \geq (1-\varepsilon)^2$ with probability at least $1-\delta$. Since $\delta > 0$ was arbitrary, this implies $\kappa(v) \geq (1-\varepsilon)^2$ a.s. and letting ε go to 0 then establishes the claim. □

Let us now take a look at how this result can be used to crystallize the outstanding leanness of the two-sidedly infinite path among all infinite quasi-transitive graphs. To this end, let us first recall the definition of quasi-transitivity.

Definition 8
Let $G = (V, E)$ be a simple graph. A bijection $f : V \to V$ with the property that $\langle f(u), f(v)\rangle \in E$ if and only if $\langle u, v\rangle \in E$ is called a *graph automorphism*. G is said to be *(vertex-) transitive* if for any two vertices $u, v \in V$ there exists a graph automorphism f that maps u on v, i.e. $f(u) = v$.

If the vertex set V can be partitioned into finitely many classes such that for any two vertices u, v belonging to the same class there exists a graph automorphism that maps u on v, the graph G is called *quasi-transitive*.

Note that the notion of quasi-transitivity becomes meaningful only for infinite graphs as all finite graphs are quasi-transitive by definition.

Theorem 5.3
Consider an infinite (connected) quasi-transitive graph $G = (V, E)$ and the initial water levels to be i.i.d. unif$([0,1])$*. Let $v \in V$ be a fixed vertex of the graph. If G is the two-sidedly infinite path, that is $V = \mathbb{Z}$ and $E = \{\langle u, u+1\rangle,\ u \in \mathbb{Z}\}$, then $\kappa(v)$ depends on the initial profile. If G is not the two-sidedly infinite path, then $\kappa(v) = 1$ almost surely.*

Proof: Given i.i.d. unif$([0,1])$ initial water levels, we can immediately conclude two things: If G is an infinite (connected) graph, the strong law of large numbers guarantees $\kappa(v) \geq \frac{1}{2}$ almost surely.

If G is the two-sidedly infinite path, there is a positive probability that the vertex v is what Häggström [4] calls two-sidedly ε-flat with respect to the initial profile (see Lemma 4.3 in [4]), i.e.

$$\frac{1}{m+n+1} \sum_{u=v-m}^{v+n} \eta_0(u) \in \left[\tfrac{1}{2}-\varepsilon, \tfrac{1}{2}+\varepsilon\right] \quad \text{for all } m, n \in \mathbb{N}_0. \tag{14}$$

Lemma 6.3 in [4] states that in this situation, the water level at v is bound to stay within the interval $[\frac{1}{2} - 6\varepsilon, \frac{1}{2} + 6\varepsilon]$ irrespectively of future updates. Together with the simple observation $\kappa(v) \geq \eta_0(v)$, it implies that $\kappa(v)$ is a random variable with non-degenerate distribution on $[\frac{1}{2}, 1]$.

In view of Theorem 5.1, to prove the second part, we only have to verify, that an infinite, connected, quasi-transitive graph that is not the two-sidedly infinite path contains a neighbor-rich half-line. Since G is infinite (and by our general assumptions both connected and having finite maximal degree) a compactness argument guarantees the existence of a half-line H on the vertices $\{v_k,\ k \in \mathbb{N}\}$ such that $v_1 = v$ and the graph distance from v_k to v is $k-1$ for all k.

Let us consider the function $d : V \to \mathbb{N}_0$, where $d(u)$ is the graph distance from the node u to a vertex of degree at least 3 being closest to it. Since G is quasi-transitive, connected and not the two-sidedly infinite path, d is finite and can take on only finitely many values, which is why it has to be bounded, by a constant $c \in \mathbb{N}$ say. Consequently, G can not contain stretches of more than $2c$ linked vertices of degree 2. For this reason, there must be a vertex among $v_3, \ldots, v_{2c+3}$, say $v_{f(1)}$, having a neighbor u_1 outside of H. In the same way, we can find a vertex u_2 outside H having a neighbor $v_{f(2)}$ among $v_{2c+6}, \ldots, v_{4c+6}$ and in general some u_k not part of H but linked to a vertex $v_{f(k)} \in \{v_k,\ k \in \mathbb{N}\}$ with

$$(k-1)\,(2c+3) + 3 \leq f(k) \leq k\,(2c+3) \quad \text{for all } k \in \mathbb{N}.$$

This choice makes sure that $v_{f(j)}$ and $v_{f(k)}$ are at graph distance at least 3 for $j \neq k$, which forces the set $\{u_k,\ k \in \mathbb{N}\}$ to consist of distinct vertices. Due to

$$\sum_{k=1}^{\infty} \frac{1}{f(k)} \geq \frac{1}{2c+3} \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

H is a neighbor-rich half-line in the sense of Definition 7 as desired, which concludes the proof. □

Remark

(a) Note that the essential property of the initial water levels, needed in the proof of Theorem 5.1, was independence. The argument can immediately be generalized to the situation where the initial water levels are independently (but not necessarily identically) distributed on $[0, C]$ and we have some weak form of uniformity, namely:

For every $\delta > 0$, there exists some $\varepsilon > 0$ such that for all $v \in V$:

$$\mathbb{P}\big(\eta_0(v) > C - \delta\big) \geq \varepsilon.$$

The sequence $Y_k := \mathbb{1}_{\{\eta_0(u_k) \geq C - \delta\}},\ k \in \mathbb{N}$, similar to the one defined in the proof of Theorem 5.1 will no longer be i.i.d. $\text{Ber}(\varepsilon)$, but an appropriate coupling will ensure that

$$\sum_{k=1}^{\infty} \frac{Y_k}{k} \geq \sum_{k=1}^{\infty} \frac{Z_k}{k}$$

almost surely, where $(Z_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of $\mathrm{Ber}(\varepsilon)$ random variables. Accordingly, we get $\kappa(v) = C$ a.s. even in this generalized setting.

(b) As alluded to in the introduction, the statement of Theorem 5.3 can be interpreted in the following way: When it comes to the qualitative behavior of $\kappa(v)$ for a fixed vertex v in the graph, the radical change does not happen between finite and infinite graphs but rather between the two-sidedly infinite path $\mathbb{Z}$ and all other quasi-transitive infinite graphs, which is why the results for the Deffuant model on $\mathbb{Z}$ can not immediately be transferred to higher-dimensional grids – as discussed in the introduction of Sect. 3 in [5].

(c) Finally, it is worth emphasizing that Theorem 5.3 does not capture the full statement of Theorem 5.1: If we take the two-sidedly infinite path $\mathbb{Z}$ and add an extra neighbor to every node that corresponds to a prime number, the only quasi-transitive subgraph contained is the two-sidedly infinite path itself. However, since it contains a neighbor-rich half-line, Theorem 5.1 states that $\kappa(v) = 1$ for i.i.d. $\mathrm{unif}([0,1])$ initial water levels and any target vertex v.

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